9-2009

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Technical Report UTD/EE/4/2009

September 2009

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Abstract

Interference alignment, while optimum in its achievement of the maximum degrees of freedom for the K user interference channel, does so only at high SNR and for large numbers of dimensions over which to align the interference, n. A sizable SNR penalty is paid in order to approach the theoretical outerbound and only grows increasingly higher for larger n. For the single antenna, $K = 3$ interference channel, an efficient means of drastically reducing the required power to approach the outerbound of 3/2 is presented. By no longer using a vector of all ones in the creation of the transmit beamforming vectors as originally proposed, a new weighted vector $\mathbf{w}$ is designed in order to distribute the power across the precoding vectors more evenly. Furthermore, we introduce a new structure of beamforming vectors that also provides significant savings in SNR independently of vector $\mathbf{w}$. By doing so, our proposed designs achieve the same degrees of freedom of the original scheme at only a fraction of the SNR.

Keywords: Interference alignment, interference channel, degrees of freedom, coordinated interference mitigation, high-SNR offset, sum rate capacity

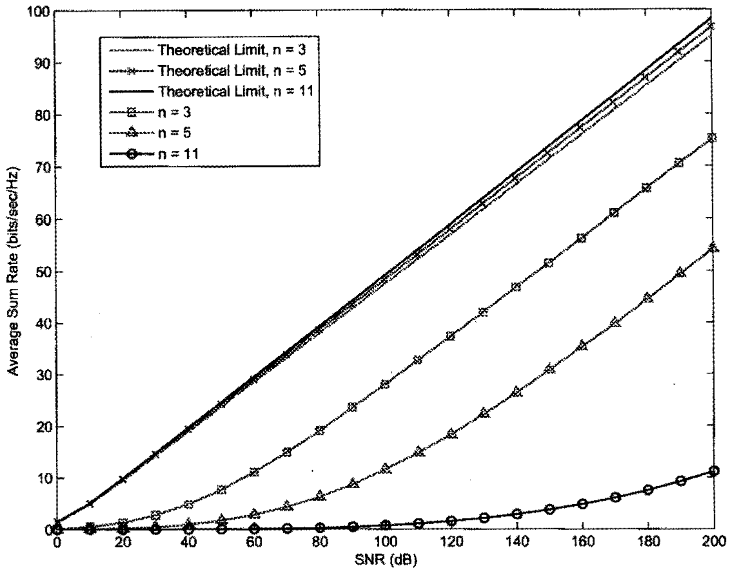

Fig. 1. Average sum rate of the original IA scheme over various n. $K = 3$.

I. Introduction

Recent studies have shown that the interference alignment (IA) scheme proposed by Cadambe and Jafar (CJ) achieves the maximum degrees of freedom in the K user, single antenna interference channel [1]. By aligning the interference at each receiver to within a fraction of the overall receive signal space, the remaining portion of the observed signal space is left to contain and preserve the desired signal free of interference. In this way, CJ have proven that at high SNR and with a large number of dimensions in which to align the interference, the theoretical outerbound of $K/2$ degrees of freedom may be achieved. Thus, no matter the size of the network, it is theoretically possible that each user may be able to achieve half the degrees of freedom were there no interference at all.

While studies have been conducted on the feasibility of IA for arbitrary numbers of users, antennas (in the case of MIMO) or other signaling dimensions [2], [3], [4], few have been made on reducing the necessary transmit power required in achieving the maximum number of degrees of freedom. Only achievable at high SNR and for large n, IA incurs very large penalties in SNR when approaching the theoretical outerbound. To illustrate this point, refer to Fig. 1.

Here we plot the average sum rate for the $K = 3$ users, SISO network where the number of frequency slots, $n = 3, 5, 11$. Channel coefficients are assumed to be drawn independently identically distributed (i.i.d.) from a continuous distribution. The absolute value of the channel coefficients are also assumed to be bounded between a non-zero minimum value and a finite maximum value. Also included are plots of the theoretical rate limits, $\mathcal{D}\log(1 + \text{SNR})$, where $\mathcal{D} = \frac{3n+1}{2n+1}$ for $K = 3$. As shown, for $n = 3$, a minimum of 70 dB is required for only $\mathcal{D} = 10/7$, approximately 95% of the 3/2 outerbound. And for $n = 5$, a minimum of 150 dB is required to achieve $\mathcal{D} = 16/11$, approximately 97% of the outerbound. And in order to reach roughly 99% of the outerbound, n must equal 11, for which the minimum required SNR exceeds **200 dB**. Therefore, improvements to the IA scheme must be made in order to reduce the minimum required SNR needed to more closely approach the maximum degrees of freedom limit, $\mathcal{D}$.

Shen, Host-Madsen, and Vidal (SHV) have made such improvements by maximizing the high-SNR offset of the sum rate capacity (20), thereby minimizing the difference between the average sum rate and theoretical rate limit curves in Fig. 1 [5]. By using two additional precoders at transmitters 2 and 3, SHV have achieved significant savings in SNR of the overall sum rate.

Our proposed scheme also maximizes the high-SNR offset in a way similar to SHV, however, rather than amending the transmit schemes of only users 2 and 3, we design a new set of transmit beamforming vectors for user 1 that effectively modifies the beamforming vectors for user 2 and 3 as well. By no longer using a vector of all ones as originally proposed by CJ, we design a weighted vector $\mathbf{w}$ that distributes the power among the $n + 1$ beamforming vectors for all three users more evenly. In doing so, we show that a greater portion of the total allocated $3n + 1$ degrees of freedom may be utilized even at low to mid SNR levels. We show that this seemingly simple modification to $\mathbf{w}$ produces very large savings in SNR in order to attain the maximum degrees of freedom for a given n.

Contributions made by the update to user 1, in addition to those of users 2 and 3, yield even greater savings in SNR, bringing the average sum rate curve to within just 13 dB of the theoretical limit for $n = 5$, a tremendous difference to the original scheme which is 88 dB away from the limit, for an enormous SNR savings of **74 dB**. Furthermore, by combining our new scheme with SHV precoding, an additional 1 dB of SNR savings is gained, for a total difference of **75 dB** from that of the original IA scheme. Thus, with the use of our new design, the achieveability of a given degrees of freedom is far more feasible at lower SNR levels that are more likely to occur in actual applications.

II. System Model

Although the CJ interference alignment scheme achieves unprecedented multiplexing gains for the interference channel, there are no closed-form solutions for the transmit beamforming vectors needed to implement such a system for networks with four or more users and a limited number of dimension over which to align interference. Attempts to solve this problem have been made by Peters and Heath who have created a numerical, iterative technique for calculating beamforming vectors for an arbitrary number of users, antennas, or spatial stream [3]. However, their results, like those of [2], conclude that interference alignment may not always be feasible for some arbitrary set of channel matrices and a particular degree of freedom allocation. Therefore, the scope of our analysis shall be limited to the $K = 3$ interference channel for which closed-formed expressions for the transmit beamforming vectors do exist.

For the $K = 3$ user interference channel, user 1 is allocated $n + 1$ streams of data while users 2 and 3 are both allocated n; and all three users transmit over a $2n + 1$ symbol extension of the channel by means of beamforming. Here, $n \in \mathbb{N}$. The channel output at the kth receiver is then defined as

$$\mathbf{Y}_k = \mathbf{H}_{k1}\mathbf{V}_1\mathbf{X}_1 + \mathbf{H}_{k2}\mathbf{V}_2\mathbf{X}_2 + \mathbf{H}_{k3}\mathbf{V}_3\mathbf{X}_3 + \mathbf{Z}_k \tag{1}$$

where, $k \in \{1, 2, 3\}$ is the user index, $\mathbf{Y}_k$ is a $(2n + 1) \times 1$ vector representing the output signal of the kth receiver. $\mathbf{H}_{kj}$ is a diagonal $(2n+1) \times (2n+1)$ matrix of channel fading coefficients from transmitter j to receiver k. The elements of $\mathbf{H}_{kj}$ are assumed to be drawn i.i.d. from a continuous distribution with absolute values assumed to be bounded between a non-zero minimum value and a finite maximum value. Perfect channel knowledge is also assumed to be globally available at all transmitters and receivers.

$\mathbf{X}_1$ is a $(n + 1) \times 1$ vector representing the input signal of the first transmitter and $\mathbf{X}_k$ for $k = 2, 3$ is a $n \times 1$ vector representing the input signal of the kth transmitter. The elements of $\mathbf{X}_k$ for all k are assumed to be i.i.d. zero-mean complex Gaussian with variance ρ, i.e. $\mathbf{X}_1 \sim \mathcal{N}(0, \rho\mathbf{I}_{(n+1)\times(n+1)})$, and $\mathbf{X}_2$ and $\mathbf{X}_3$ are both $\mathcal{N}(0, \rho\mathbf{I}_{n\times n})$.

$\mathbf{V}_1$ is a $(2n + 1) \times (n + 1)$ matrix, the columns of which are the transmit beamforming vectors used to project each of the $n+1$ independent input streams at transmitter 1 through the extended channel and into the desired receive signal subspace at receiver 1 while simultaneously aligning the streams along the interference subspaces at both receivers 2 and 3. $\mathbf{V}_2$ and $\mathbf{V}_3$ are $(2n + 1) \times n$ matrices that serve similar roles for transmitters 2 and 3, respectively.

Lastly, $\mathbf{Z}_k$ is a $(2n + 1) \times 1$ vector representing the additive white Gaussian noise (AWGN). All noise terms are also assumed to be i.i.d. zero-mean complex Gaussian with unit variance, $\mathbf{Z}_k \sim \mathcal{N}(0, \mathbf{I}_{(2n+1)\times(2n+1)})$.

Now, in order to guarantee the alignment of interference at each of the three receivers to achieve $\frac{3n+1}{2n+1}$ degrees of freedom, CJ propose the following three equations.

$$\mathbf{H}_{12}\mathbf{V}_2 = \mathbf{H}_{13}\mathbf{V}_3 \tag{2}$$

stating that the extended channel projection matrices of the interference at RX_1 must be equal, thus confining the interference to within n dimensions of the overall $2n + 1$ dimension of the receive signal space. Similarly, in order to ensure the interference at RX_2 is simultaneously aligned to within $n+1$ dimensions,

$$\mathbf{H}_{23}\mathbf{V}_3 \prec \mathbf{H}_{21}\mathbf{V}_1 \tag{3}$$

where $\mathbf{P} \prec \mathbf{Q}$ means the set of column vectors of matrix $\mathbf{P}$ is a subset of the columns of matrix $\mathbf{Q}$. Likewise for RX_3,

$$\mathbf{H}_{32}\mathbf{V}_2 \prec \mathbf{H}_{31}\mathbf{V}_1. \tag{4}$$

Thus, the set of beamforming matrices $\mathbf{V}_1$, $\mathbf{V}_2$, and $\mathbf{V}_3$ must be chosen such that (2), (3), and (4) are satisfied. Equivalently, CJ express (2) through (4) as

$$\mathbf{B} = \mathbf{T}\mathbf{C} \tag{5}$$

$$\mathbf{B} \prec \mathbf{A} \tag{6}$$

$$\mathbf{C} \prec \mathbf{A} \tag{7}$$

where

$$\mathbf{A} = \mathbf{V}_1 \tag{8}$$

$$\mathbf{B} = (\mathbf{H}_{21})^{-1}\mathbf{H}_{23}\mathbf{V}_3 \tag{9}$$

$$\mathbf{C} = (\mathbf{H}_{31})^{-1}\mathbf{H}_{32}\mathbf{V}_2 \tag{10}$$

$$\mathbf{T} = \mathbf{H}_{12}(\mathbf{H}_{21})^{-1}\mathbf{H}_{23}(\mathbf{H}_{32})^{-1}\mathbf{H}_{31}(\mathbf{H}_{13})^{-1}. \tag{11}$$

In order to solve (5) through (7) for $\mathbf{V}_1$, $\mathbf{V}_2$, and $\mathbf{V}_3$, CJ propose the following structures.

$$\mathbf{A} = \begin{bmatrix} \mathbf{w}_1 & \mathbf{T}\mathbf{w}_1 & \mathbf{T}^2\mathbf{w}_1 & \dots & \mathbf{T}^n\mathbf{w}_1 \end{bmatrix} \tag{12}$$

$$\mathbf{B} = \begin{bmatrix} \mathbf{T}\mathbf{w}_1 & \mathbf{T}^2\mathbf{w}_1 & \dots & \mathbf{T}^n\mathbf{w}_1 \end{bmatrix} \tag{13}$$

$$\mathbf{C} = \begin{bmatrix} \mathbf{w}_1 & \mathbf{T}\mathbf{w}_1 & \dots & \mathbf{T}^{n-1}\mathbf{w}_1 \end{bmatrix} \tag{14}$$

where $\mathbf{w}_1$ is a $(2n+1)\times 1$ vector of all ones. Thus establishing all the necessary conditions in order to achieve $\frac{3n+1}{2n+1}$ degrees of freedom for any n.

III. Zero-Forcing Decoding and Sum Rate Capacity

When combined with the CJ interference alignment scheme, zero-forcing decoding proves to be an efficient means of eliminating the interference confined to within a reduced-dimension of the overall receive signal space, while also preserving the degrees of freedom outerbound of $K/2$ as shown in [6].

Zero-forcing decoding begins with the creation of an orthogonal projection matrix for receiver k, $\mathbf{P}_k^{\perp}$, in the following way. Let the desired receive signal subspace for receiver k be defined as the span of the columns of matrix $\mathbf{H}_{kk}\mathbf{V}_k$. The interference subspace of the received signal is then the span of the columns of matrices $\mathbf{H}_{kj}\mathbf{V}_j$ for $j \neq k$. As guaranteed by the interference alignment scheme, the multiple interference subspaces related to transmitters $j \neq k$ are perfectly aligned at receiver k. Therefore, the $(2\mathrm{n}+1) \times (2\mathrm{n}+1)$ orthogonal projection matrix $\mathbf{P}_k^{\perp}$ may be created based upon $\mathbf{H}_{kj}\mathbf{V}_j$ for any single $j \neq k$, rather than all $j \neq k$. Then $\mathbf{P}_k^{\perp} = \mathbf{I}_{\mathrm{N}\times\mathrm{N}} - \mathbf{P}_k$, for $\mathrm{N} = 2n+1$, where $\mathbf{P}_k$ is the projection matrix of the interference subspace defined as

$$\mathbf{P}_k = \mathbf{H}_{kj}\mathbf{V}_j[(\mathbf{H}_{kj}\mathbf{V}_j)^H\mathbf{H}_{kj}\mathbf{V}_j]^{-1}(\mathbf{H}_{kj}\mathbf{V}_j)^H \tag{15}$$

for any single $j \neq k$ and $(\ )^H$ represents the Hermitian transpose of a matrix. When multiplied by the received signal $\mathbf{Y}_k$,

$$\mathbf{P}_k^{\perp}\mathbf{Y}_k = \mathbf{P}_k^{\perp}\mathbf{H}_{kk}\mathbf{V}_k\mathbf{X}_k + \mathbf{P}_k^{\perp}\mathbf{Z}_k \tag{16}$$

the interference terms are eliminated and the desired signal $\mathbf{X}_k$ can be decoded by inverting $\mathbf{P}_k^{\perp}\mathbf{H}_{kk}\bar{\mathbf{V}}_k$.

Using the definition of the zero-forcing receiver from above, [5] defines the rate of the kth user as

$$R_k = \log(|\mathbf{I}_{d_k \times d_k} + \rho(\mathbf{H}_{kk}\mathbf{V}_k - \mathbf{P}_k(\mathbf{H}_{kk}\mathbf{V}_k))^H(\mathbf{H}_{kk}\mathbf{V}_k - \mathbf{P}_k(\mathbf{H}_{kk}\mathbf{V}_k)|) \tag{17}$$

where d_k is the degrees of freedom for user k, i.e.

$$d_k = \begin{cases} n+1 & k=1 \\ n & k=2,3 \end{cases}. \tag{18}$$

The sum rate per symbol of the CJ interference alignment scheme is then

$$R = \frac{1}{N}(R_1 + R_2 + R_3). \tag{19}$$

Lastly, the high-SNR offset of the sum rate capacity based upon the finding in [7], [8], and [9] is defined by [5] as

$$\lim_{\mathrm{SNR}\to\infty} R - D\log(\mathrm{SNR}) = \frac{\log(\prod_{k=1}^{K} |(\mathbf{H}_{kk}\mathbf{V}_k - \mathbf{P}_k(\mathbf{H}_{kk}\mathbf{V}_k))^H(\mathbf{H}_{kk}\mathbf{V}_k - \mathbf{P}_k(\mathbf{H}_{kk}\mathbf{V}_k))|)}{N} \tag{20}$$

where D is the total degrees of freedom per symbol of the system, i.e. $\frac{3n+1}{2n+1}$, and shall serve as the object function to the optimization problem presented in the following section.

IV. Beamforming Vector Optimization

The high-SNR offest performance measure, first used in studies of CDMA [8], and in recent years, MIMO technologies [7], [9], is adopted here in the context of interference alignment as a means of measuring the performance of a systems's sum rate capacity with respect to high SNR theoretical limits, namely $D\log(1+\text{SNR})$. Noting the dependency of the high-SNR offset on the matrices of beamforming vectors, $\mathbf{V}_k$, we propose the design of a new set of beamforming vectors that approaches the sum rate capacity limit to a far greater extent than the original.

SHV in [5], showed that when using ZF at the receiver, maximizing the high-SNR offset (20) was equivalent to

$$\max \prod_{k=1}^{K} p_k |(\mathbf{H}_{kk}\mathbf{V}_k)^H(\mathbf{H}_{kk}\mathbf{V}_k)| \tag{21}$$

$$\text{s.t.} \frac{1}{N}\text{tr}(\mathbf{V}_k\mathbf{V}_k^H) = 1 \tag{22}$$

where p_k is a positive projection constant at receiver k. They then reduced the problem further to the maximization of $|\mathbf{V}_k^H\mathbf{V}_k|$, dropping the dependency on p_k and $\mathbf{H}_{kk}$, while maintaining the same power constraint. Using two new precoding matrices at receivers 2 and 3, $\mathbf{F}$ and $\mathbf{E}$ respectively, they maximize the overall sum rate by orthonormalizing the columns of $\mathbf{V}_2$ and $\mathbf{V}_3$, thereby diagonalizing $\mathbf{V}_k^H\mathbf{V}_k$ for $k = 2, 3$. Using $\mathbf{F}$ and $\mathbf{E}$, SHV define the improved beamforming vectors for transmitters 2 and 3, $\tilde{\mathbf{V}}_\mathbf{2}$ and $\tilde{\mathbf{V}}_\mathbf{3}$, respectively, as

$$\tilde{\mathbf{V}}_2 = \mathbf{V}_2\mathbf{F} \tag{23}$$

$$\tilde{\mathbf{V}}_3 = \mathbf{V}_3\mathbf{E} \tag{24}$$

In contrast, our approach to solving the maximization problem does not utilize another pair of precoding matrices at only transmitters 2 and 3, but instead uses an entirely new $\mathbf{V}_1$ matrix and $\mathbf{w}$ vector that maximizes the rates of *all* three users as well as the overall sum rate. Our new solution to the maximization problem, what we refer to as *weighted vector normalization* (WVN), is as follows.

A. Weighted Vector Normalization

The main results of this subsection are summarized in the following theorem.

Theorem 1: For the $K = 3$ SISO interference channel with $\mathcal{D} = \frac{3n+1}{2n+1}$, the high SNR offset of the sum rate capacity is maximized when the elements of vector $\mathbf{w}$ are equal to the following

$$w_i = \left(\sum_{m=0}^{n} \left(|t_i|^2 \right)^m \right)^{-\frac{1}{2}} \quad \text{for } i = 1, 2, ..., N \tag{25}$$

where $w_i \in \mathbb{R}^+$, excluding zero, and t_i are the diagonal elements of matrix $\mathbf{T}$.

Proof: First observing that $\mathbf{V}_2$ and $\mathbf{V}_3$ are both dependent upon $\mathbf{V}_1$, we begin by maximizing $|\mathbf{V}_1^H \mathbf{V}_1|$. We rewrite $\mathbf{V}_1$ as

$$\mathbf{V}_1 = \mathbf{W}\mathbf{\Gamma} \tag{26}$$

where $\mathbf{W}$ is an $N \times N$ diagonal matrix with elements equal to those of vector $\mathbf{w}$, i.e. $\mathbf{W} = \text{diag}([w_1 \, w_2 \ldots w_N])$. Note that the elements in $\mathbf{w}$ are no longer all ones as originally defined by CJ in (12), but are now variable and constrained to be within the set of all positive and real numbers excluding zero, i.e. $w_i \in \mathbb{R}^+$ for $i = 1, 2, ..., N$. The terms w_i, will be used as optimization variables in the maximization of the object function, $|\mathbf{V}_1^H \mathbf{V}_1|$.

$\mathbf{\Gamma}$ is an $N \times (n+1)$ Vandermonde matrix whose columns consist of the diagonal elements of $\mathbf{T}^0, \mathbf{T}^1, ..., \mathbf{T}^n$, i.e.

$$\mathbf{\Gamma} = \begin{bmatrix} t_1^0 & t_1^1 & \ldots & t_1^n \\ t_2^0 & t_2^1 & \ldots & t_2^n \\ \vdots & \vdots & \ddots & \vdots \\ t_N^0 & t_N^1 & \ldots & t_N^n \end{bmatrix}. \tag{27}$$

We also define the N rows of $\mathbf{\Gamma}$ as vectors $\mathbf{t}_i$ for $i = 1, 2, ..., N$. Next, we normalize the rows of $\mathbf{\Gamma}$ by factoring the norm of each row into the $N \times N$ diagonal matrix $\mathbf{\Lambda}$ such that

$$\mathbf{\Gamma} = \mathbf{\Lambda}\bar{\mathbf{\Gamma}} \tag{28}$$

where $\mathbf{\Lambda} = \text{diag}([\|\mathbf{t}_1\| \; \|\mathbf{t}_2\| ... \; \|\mathbf{t}_N\|])$. Then matrix $\bar{\mathbf{\Gamma}}$ consists of the unit norm row vectors of $\mathbf{\Gamma}$, i.e. $\bar{\mathbf{\Gamma}} = [\bar{\mathbf{t}}_1 \; \bar{\mathbf{t}}_2 \; ... \; \bar{\mathbf{t}}_N]^T$, where $\bar{\mathbf{t}}_i = \frac{\mathbf{t}_i}{\|\mathbf{t}_i\|}$. Thus,

$$\mathbf{V}_1 = \mathbf{W}\mathbf{\Lambda}\bar{\mathbf{\Gamma}} \tag{29}$$

and

$$\mathbf{V}_1^H \mathbf{V}_1 = \bar{\mathbf{\Gamma}}^H \mathbf{\Lambda}^H \mathbf{W}^H \mathbf{W} \mathbf{\Lambda} \bar{\mathbf{\Gamma}}. \tag{30}$$

Letting $\tilde{\mathbf{W}} = \mathbf{\Lambda}^H\mathbf{W}^H\mathbf{W}\mathbf{\Lambda}$, i.e.

$$\tilde{\mathbf{W}} = \begin{bmatrix} |w_1|^2\|\mathbf{t}_1\|^2 & 0 & 0 & 0 \\ 0 & |w_1|^2\|\mathbf{t}_2\|^2 & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & |w_1|^2\|\mathbf{t}_N\|^2 \end{bmatrix}, \tag{31}$$

we express the inner product $\mathbf{V}_1^H\mathbf{V}_1$ as

$$\mathbf{V}_1^H\mathbf{V}_1 = \bar{\mathbf{\Gamma}}^H\tilde{\mathbf{W}}\bar{\mathbf{\Gamma}} = \sum_{i=1}^{N} |w_i|^2\|\mathbf{t}_i\|^2\bar{\mathbf{t}}_i^H\bar{\mathbf{t}}_i. \tag{32}$$

Here, we define the rank one projection matrix of the unit norm vector $\bar{\mathbf{t}}_i$ as $\mathbf{P}_i = \bar{\mathbf{t}}_i^H\bar{\mathbf{t}}_i$, where it can easily be shown that $\mathbf{P}_i = \mathbf{P}_i^2 = \mathbf{P}_i^H$, then $\mathbf{V}_1^H\mathbf{V}_1$ can be expressed as the weighted sum of N such rank one projections with weights $|w_i|^2\|\mathbf{t}_i\|^2$. Note the resemblance of (32) to Mercer's theorem [10] except for the fact that the basis vectors $\bar{\mathbf{t}}_i$ are not orthogonal to one another. Were that the case, $|\mathbf{V}_1^H\mathbf{V}_1|$ would be maximum and equal to the product of the N diagonal elements of $\tilde{\mathbf{W}}$. With this in mind, we upper bound the determinant as

$$|\mathbf{V}_1^H\mathbf{V}_1| < \prod_{i=1}^{N} |w_i|^2\|\mathbf{t}_i\|^2. \tag{33}$$

Noting that the closed-form expression of the determinant above becomes exceedingly complex to derive for $n > 2$, as is the case in our analysis, we choose instead to maximize its upper bound under the power constraint in (22). Using an alternate expression for the power constraint, $\mathrm{tr}(\mathbf{V}_1\mathbf{V}_1^H) = \sum_{i=1}^{N} |w_i|^2\|\mathbf{t}_i\|^2$, we reformulate the maximization problem as

$$\max_{\mathbf{w}} \prod_{i=1}^{N} |w_i|^2\|\mathbf{t}_i\|^2 \tag{34}$$

$$\text{s.t. } \frac{1}{N}\sum_{i=1}^{N} |w_i|^2\|\mathbf{t}_i\|^2 = 1. \tag{35}$$

As a solution to this problem, we choose the following

$$w_i = \left(\frac{1}{\|\mathbf{t}_i\|^2}\right)^{\frac{1}{2}} \text{ for } i = 1, 2, ..., N. \tag{36}$$

In this way, vector $\mathbf{w}$ maximizes $|\mathbf{V}_1^H\mathbf{V}_1|$ by more evenly distributing the available transmit power across the $n+1$ columns of the beamforming matrix $\mathbf{V}_1$ as opposed to the original CJ scheme which allocates the vast majority of the power onto the columns with the largest exponents, the largest being n.

To show this, let vector $\mathbf{t}$ be defined as an $N \times 1$ column vector containing the diagonal elements of matrix $\mathbf{T}$, i.e. $\mathbf{t} = \text{diag}(\mathbf{T})$. Then it can easily be shown that

$$E\left[\|\mathbf{t}^n\|^2\right] \gg E\left[\|\mathbf{t}^0\|^2\right]. \tag{37}$$

As a result, the columns possessing the smallest exponents will be allocated the least amount of power, effectively turning them off relative to the columns with the largest exponents after normalization; and it is for this reason that $|\mathbf{V}_1^H \mathbf{V}_1|$ is so ill conditioned. Furthermore, it is this imbalance of power among the transmit beamforming vectors that we attribute the severe penalty in SNR that the original CJ scheme incurs in achieving the maximum degrees of freedom which grows exceedingly worse as n increases.

At very high SNR, the imbalance of power becomes negligible and the maximum achievable degrees of freedom may still be attained. However, at low to mid SNRs, the imbalance of power effectively reduces the number of dimensions over which user 1 may align its interference and the number of data streams user 1 may transmit. By allocating the vast majority of power onto those columns with the the largest exponents, the remaining columns are effectively turned off. Therefore, at low to mid SNR levels, user 1 is left with only a fraction of the intended $n+1$ degree of freedom. Similarly, because the transmit beamforming matrices of users 2 and 3 are scaled subsets of $\mathbf{V}_1$, they too will experience a reduction in their degrees of freedom allocation - no longer transmitting over the full n data streams as intended.

In contrast, our design distributes power over the $n+1$ columns of $\mathbf{V}_1$ more evenly, so whether at low, mid, or high SNR, each user is more fully utilizing its allocated degrees of freedom. Therefore, our new scheme achieves the maximum degrees of freedom of the sum rate capacity for some n at a far lower SNR than the original CJ scheme.

In doing so, we note that the singluar value spread of $\mathbf{V}_1$ is reduced, thereby improving the condition number of $|\mathbf{V}_1^H \mathbf{V}_1|$. As will be seen in Section V, our new transmit beamforming design is far better conditioned than the original CJ design.

B. Reduced Order Optimization

Using a similar strategy whereby the power across the columns of $\mathbf{V}_1$ is more evenly distributed, we show that an increased savings in SNR of the sum rate capacity may also be achieved without redesigning vector $\mathbf{w}$, a vector of all ones. This is done so by using the following beamforming matrix.

$$\mathbf{V}_1 = \begin{bmatrix} \mathbf{T}^{-\frac{n}{2}}\mathbf{w} & \mathbf{T}^{-\frac{n}{2}+1}\mathbf{w} & \ldots & \mathbf{T}^{\frac{n}{2}-1}\mathbf{w} & \mathbf{T}^{\frac{n}{2}}\mathbf{w} \end{bmatrix}. \tag{38}$$

Here, the greatest exponent among the $n+1$ columns of $\mathbf{V}_1$ is $n/2$ rather than n as proposed by CJ. Likewise, the smallest exponent is now $-n/2$ rather than 0. By using this new structure, we show that the greatest portion of the available transmit power may be allocated more evenly among a greater number of data streams than in the CJ scheme. To show this, we first write $\mathbf{V}_1$ as

$$\mathbf{V}_1 = \left[\begin{array}{cccccc} \mathbf{t}^{[-\frac{n}{2}]} & \mathbf{t}^{[-\frac{n}{2}+1]} & \dots & \mathbf{t}^{[\frac{n}{2}-1]} & \mathbf{t}^{[\frac{n}{2}]} \end{array} \right], \tag{39}$$

where the column vectors $\mathbf{t}^{[m]}$ contain the N diagonal elements of matrix $\mathbf{T}^m$ for $m = -\frac{n}{2}, ..., \frac{n}{2}$, i.e. $\mathbf{t}^{[m]} = \text{diag}(\mathbf{T}^m)$. Note that the m term in the superscript of $\mathbf{t}^{[m]}$ is used for notational convenience, only representing an exponent, and should not be mistaken as an exponent. Next, we examine the power of each column in $\mathbf{V}_1$ by examining the diagonal elements of $\mathbf{V}_1^H\mathbf{V}_1$, i.e.

$$\text{diag}(\mathbf{V}_1^H\mathbf{V}_1)^T = \left[\begin{array}{ccccc} \|\mathbf{t}^{[-\frac{n}{2}]}\|^2 & \|\mathbf{t}^{[-\frac{n}{2}+1]}\|^2 & \dots & \|\mathbf{t}^{[\frac{n}{2}-1]}\|^2 & \|\mathbf{t}^{[\frac{n}{2}]}\|^2 \end{array} \right]. \tag{40}$$

From here, two distinct cases arise: 1) for n is even 2) for n is odd.

For the first case, we make use of the following theorem taken from [11].

Theorem 2: For a pair of i.i.d. non-negative and real random variables, X and Y, whose joint probability density function (PDF), $f_{X,Y}(x, y)$, is symmetric about the line $x = y$ for all x and y, the cumulative distribution function (CDF) of $Z = X/Y$, $F_Z(z)$, is equal to the CDF of $1/Z$, i.e.

$$F_Z(z) = F_{1/Z}(z). \tag{41}$$

Proof provided in [11]. With the use of this result, it can easily be shown that

$$E\left[\|\mathbf{t}^{[-\frac{n}{2}+m]}\|^2\right] = E\left[\|\mathbf{t}^{[\frac{n}{2}-m]}\|^2\right] \text{ for } m = 0, ..., \frac{n}{2} - 1 \tag{42}$$

and

$$E\left[\|\mathbf{t}^{[\frac{n}{2}]}\|^2\right] \gg E\left[\|\mathbf{t}^{[1]}\|^2\right]. \tag{43}$$

These results show that $\text{diag}(\mathbf{V}_1^H\mathbf{V}_1)^T$ is symmetric about the center of the vector which also implies that power, on average, is more equally allocated across pairs of columns centered about the central column of $\mathbf{V}_1$. That is, columns 1 and $n+1$ share the greatest portion of the available power more equally while the two columns closest to the center of $\mathbf{V}_1$ share a far smaller portion of the available power, though more equally amongst the two. The center column, is allocated the least amount of power and does not share it equally with any other column.

Because power is now shared more equally, on average, amongst pairs of columns, in contrast to the original CJ scheme which does not, the new scheme effectively makes use of twice as many degrees of freedom at low SNR than the CJ scheme, though not the full $n + 1$. Nonetheless, the extra degrees of freedom at low SNR result in a greater savings in SNR in achieving the maximum degrees of freedom of the sum rate capacity for a given n as will be shown in Section V.

Similarly, for the second case where n is odd,

$$E\left[\|\mathbf{t}^{\left[-\frac{n}{2}+m\right]}\|^2\right] = E\left[\|\mathbf{t}^{\left[\frac{n}{2}-m\right]}\|^2\right] \text{ for } m = 0, ..., \left\lfloor\frac{n}{2}\right\rfloor \tag{44}$$

where $\lfloor\ \rfloor$ represents the floor function. Thus, power is allocated more equally among pairs of columns centered about $\mathbf{V}_1$ as in the first case; however, unlike the first case, there is no longer a central column that does not share its power with another column. Rather, every column now shares its power in pairs. The result, however, is the same. Using the new structure at low to mid SNR, more degrees of freedom may be attained over the original CJ scheme, though not the full $n+1$, thereby reducing the minimum SNR required to achieve the maximum degrees of freedom for a given n. And as in the case of the optimized vector $\mathbf{w}$, the new $\mathbf{V}_1$ structure is far better conditioned than the original as will be shown in Section V.

Note that in order to double the *effective* number of degrees of freedom at low SNR over that of the CJ scheme, the absolute value of the maximum and minimum exponents in $\mathbf{V}_1$ must be equal such that *two* columns share exponents whose absolute values are equal and maximum. Otherwise, the majority of the available power would still be allocated to a single column as in the CJ scheme. Thus, in this sense, the new maximum exponent of $n/2$ is optimum. In reference to the reduced order of the maximum exponent in $\mathbf{V}_1$, we refer to this scheme as reduced order optimization (ROO).

Now, while the ROO technique achieves a greater savings in SNR by reducing the order of the columns in $\mathbf{V}_1$, the weighted vector normalization (WVN) scheme achieves far greater savings because it more evenly distributes the available transmit power over *all* of the $n+1$ beamforming vectors in $\mathbf{V}_1$. Therefore, it suffices to just apply the WVN scheme in practice. And although the ROO scheme may not be needed in practice, it has been included in this analysis as insight into the problem.

C. Joint Weighted Vector Normalization and SHV Orthonormalization Precoding

As a third and final means of increasing the SNR savings of the sum rate capacity, we combine our WVN scheme with that of the SHV orthonormalization precoding, referring to the combined scheme as WVN-SHVOP.

SHV maximize the high-SNR offset by creating two new precoding matrices of dimension $n \times n$ for transmitters 2 and 3, $\mathbf{F}$ and $\mathbf{E}$, respectively. These new matrices are designed to orthonormalize the columns of matrices $\mathbf{V}_2$ and $\mathbf{V}_3$, such that the inner products in the determinants $|\mathbf{V}_2^H\mathbf{V}_2|$ and $|\mathbf{V}_3^H\mathbf{V}_3|$ are diagonal, thus maximizing them and the resultant high-SNR offset of the sum rate (20). Matrices $\mathbf{F}$ and $\mathbf{E}$ are defined as

$$\mathbf{F} = \hat{\mathbf{V}}_2\hat{\mathbf{\Sigma}}_2^{-1} \tag{45}$$

and

$$\mathbf{E} = \hat{\mathbf{V}}_3 \hat{\boldsymbol{\Sigma}}_3^{-1} \tag{46}$$

where $\hat{\mathbf{V}}_2$ and $\hat{\mathbf{V}}_3$ are matrices of the left most singular vectors of the singular value decomposition (SVD) of $\mathbf{V}_2$ and $\mathbf{V}_3$, respectively; and $\hat{\boldsymbol{\Sigma}}_2^{-1}$ and $\hat{\boldsymbol{\Sigma}}_3^{-1}$ are diagonal matrices containing the singular values of the two SVDs, i.e.

$$\mathbf{V}_2 = \hat{\mathbf{U}}_2 \hat{\boldsymbol{\Sigma}}_2 \hat{\mathbf{V}}_2^H \tag{47}$$

and

$$\mathbf{V}_3 = \hat{\mathbf{U}}_3 \hat{\boldsymbol{\Sigma}}_3 \hat{\mathbf{V}}_3^H. \tag{48}$$

As a result of the orthonomalization of the columns in $\mathbf{V}_2$ and $\mathbf{V}_3$, the power across all n columns in both cases is distributed equally at N/n per column. Thus, even at low SNR, users 2 and 3 make use of all n degrees of freedom allocated to them, normally not in full use in either the original CJ or the new WVN schemes. Thus, further reducing the minimum SNR required to achieve the maximum degrees of freedom of the sum rate capacity.

V. Simulation Results

As a demonstration of the savings in SNR that can be achieved using the proposed new scheme, we present the following two examples under the assumptions given in Section II.[1] We begin with the $n = 11$ case, corresponding to a maximum of $34/23$ degrees of freedom or approximately 99% of the theoretical outerbound, $3/2$. Shown in Fig. 2 are the average sum rate curves using the original scheme proposed by CJ and the new scheme using 1) the reduced order optimization (ROO), 2) weighted vector normalization (WVN) without ROO, and 3) the joint weighted vector normalization technique and SHV orthonormalization precoding (WVN-SHVOP). Also shown is the theoretical sum rate capacity limit which serves as the benchmark to measure the performance of the various schemes.

As shown, an enormous reduction in the minimum SNR required to achieve the maximum degrees of freedom is made with use of the joint WVN-SHVOP scheme over the original CJ. Not even visible on the plot, the point at which the CJ scheme begins increasing linearly is well beyond 250 dB, whereas the WNV-SHVOP is only 21 dB away from the theoretical rate limit - miniscule in comparison. And only

[1]Channel coefficients are drawn i.i.d. from a complex Gaussian distribution. However, the channel coefficients were further bounded to be within a non-zero minimum value and a finite maximum value in order to avoid degenerate channel conditions. These conditions also apply to the simulation results in Fig. 1.

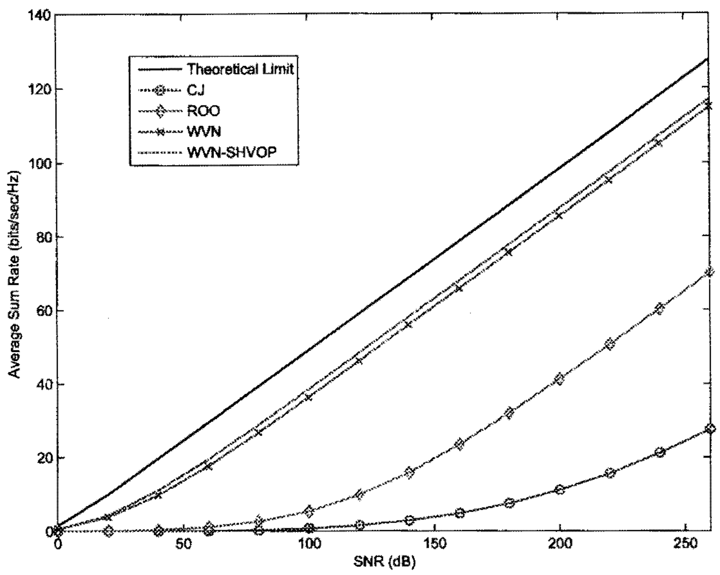

Fig. 2. Average sum rate comparison under various schemes for n = 11.

5 dB greater than the WNV-SHVOP scheme, the stand-alone WNV is 26 dB away from the theoretical limit. The ROO scheme, however, at 115 dB away from the limit, is still considerably closer than the original, proving that reducing the order of the columns of $\mathbf{V_1}$ can indeed produce significant savings in SNR. Thus, 99% of the theoretical degrees of freedom outerbound can be achieved at an SNR of approximately 60 dB as opposed to the original CJ scheme of well over 260 dB by simply using a weighted vector $\mathbf{w}$ rather than of all ones and still less if combined with SHVOP at transmitters 2 and 3.

If one is willing to trade-off degrees of freedom for an increase in SNR savings, $n = 4$ may be chosen for a maximum degrees of freedom of $13/9$ or approximately 96% of the theoretical outerbound. As shown in Fig. 3, using the joint WNV-SHVOP scheme, the maximum degrees of freedom may be achieved at an SNR of approximately 25 dB and slightly higher if using just WVN. The original CJ scheme, however, becomes linear at a far greater SNR of approximately 90 dB, and better still is the ROO scheme at approximately 50 dB. Thus, for $n = 4$, 96% of the theoretical outerbound may be achieved at an SNR that is 65 dB less than that required of the original CJ scheme.

Furthermore, for the $n = 4$ case, Fig. 4 shows that considerable gains in the sum rate can be achieved even at mid SNR levels. For instance, the WVN-SHVOP scheme reaches the maximum degrees of freedom at approximately 25 dB for a sum rate of about 8.4 bits/sec/Hz, whereas the original CJ scheme

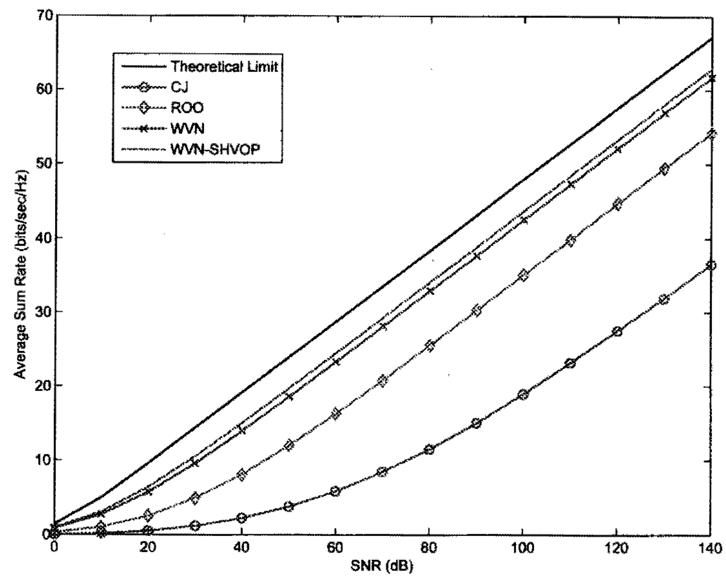

Fig. 3. Average sum rate comparison under various schemes for n = 4.

only achieves an approximate sum rate of 0.8 bits/sec/Hz (not even a full bit/sec/Hz) for the same SNR, and still very far from reaching the maximum achievable degrees of freedom. Approximately 10.5 times (over an order of magnitude) greater in sum rate, the WVN-SHVOP clearly produces significant gains over the original CJ scheme. Additionally, the stand-alone WVN scheme also produces considerable gains over the original CJ scheme with a sum rate of approximately 7.6 bits/sec/Hz, approximately 9.5 times greater. Not quite a full order of magnitude, but still many times greater. Even the ROO scheme, with a sum rate of approximately 3.6 bits/sec/Hz still out-performs the original CJ scheme by approximately 4.5 times. Thus, by more evenly distributing the available transmit power over each of the $n+1$ beamforming vectors for user 1 and the n beamforming vectors for users 2 and 3, a tremendous gain in SNR savings can be achieved in order to attain the maximum degrees of freedom for a given n.

We also show that by distributing each transmitter's total available power across the beamforming vectors $\mathbf{V}_1$, the condition number of $\mathbf{V}_1^H\mathbf{V}_1$ reduces considerably in comparison to that of the original CJ scheme. Letting $\kappa = \text{cond}\left(\mathbf{V}_1^H\mathbf{V}_1\right)$, we plot the CDF of κ (in units of dB) for the WVN-SHVOP, ROO, and CJ schemes in Fig. 5. Here, $n = 4$ at an SNR of 25 dB, the point at which WVN-SHVOP begins increasing linearly.

In comparing the three schemes, we note that for the WNV-SHVOP scheme, $P(\kappa \leq 31dB) = 0.99$. That is, the condition number of the WVN scheme is less than or equal to 31 dB for 99% of the time,

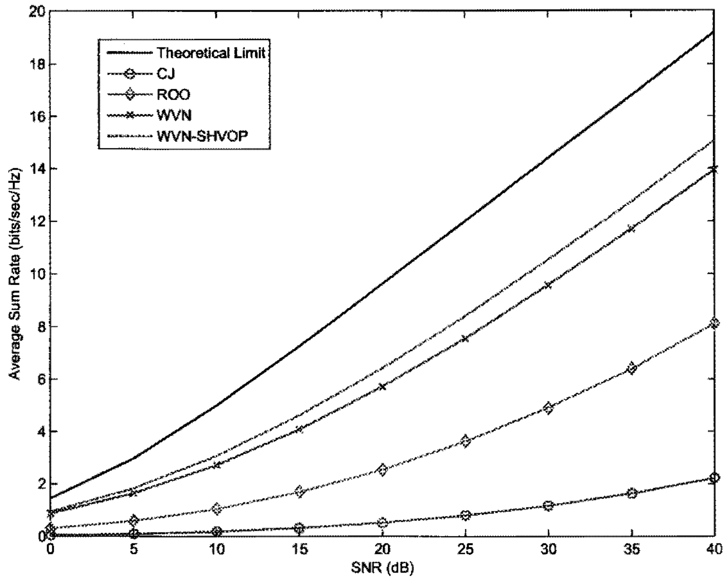

Fig. 4. Average sum rate comparison under various schemes for n = 4 at low to mid SNR.

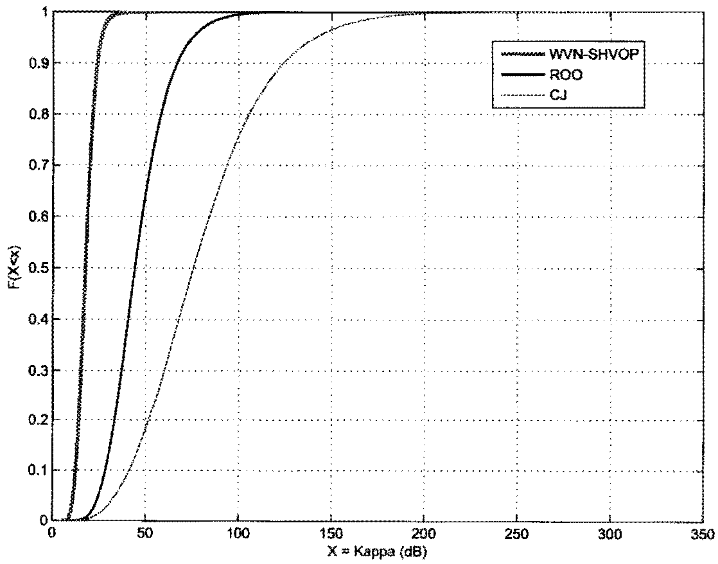

Fig. 5. CDF of CND of schemes WVN-SHVOP and CJ at 25 dB for $n = 4$.

whereas the original CJ scheme attains a condition number of 176 dB for 99% of the time, for a difference of **148 dB**. Though not as large of an improvement, the ROO scheme is still 83 dB better than the CJ scheme, with a condition number that is less than or equal to 93 dB for 99% of the time. Clearly, the two new scheme, WVN-SHVOP and ROO, produces a far better conditioned set of beamforming matrices for user 1 as compared to the original scheme which implies that the new scheme is more fully utilizing the total $3n+1$ degrees of freedom for all three users.

VI. CONCLUSION

Interference alignment, while optimum in its theoretical outerbound of $K/2$ degrees of freedom for the K user interference channel, is severely limited in practice by very large minimum SNR requirements needed to achieve the maximum degrees of freedom for a given resource allocation, n. We have shown that even for moderate values of n such as 3 and 5, a minimum SNR of 70 dB and 150 dB is required to reach the maximum degrees of freedom, respectively. As a solution to this problem for the $K=3$, SISO network, we have presented three improved transmit beamforming vector designs that significantly reduces the minimum SNR needed to fully exploit the advantages of IA.

We have shown that the original IA scheme is largely limited to the imbalance of power distributed over the $n+1$ beamforming vectors in $\mathbf{V}_1$. Because the vast majority of the available transmit power is allocated to those beamforming vectors possessing the largest exponents, the data streams associated with the beamforming vectors of lowest power are not fully utilized at low and mid SNR levels. Under these circumstances, user 1 effectively does not transmit over the full $n+1$ degrees of freedom as intended; likewise, users 2 and 3 do not transmit over the full n degrees of freedom per user allocated to them. Thus, a very large SNR is required to achieve the maximum degrees of freedom for a given n.

In response to this problem, we have presented three new beamforming vector designs that drastically reduce the required minimum SNR in achieving the maximum degrees of freedom, $\frac{3n+1}{2n+1}$ by maximizing the high-SNR offset of the sum rate capacity.

The first design, WVN, does so by no longer using a vector of all ones as originally proposed by CJ, and instead uses a new weighted vector that when multiplied by the $n+1$ $\mathbf{T}$ matrices in $\mathbf{V}_1$, distributes the available transmit power more evenly over the $n+1$ beamforming vectors. By doing so, user 1 is able to more fully utilize the full $n+1$ degrees of freedom allocated to it even at low and mid SNR levels. Similarly, because of the relationship of $\mathbf{V}_2$ and $\mathbf{V}_3$ to $\mathbf{V}_1$, users 2 and 3 are able to more fully utilize the n degrees of freedom allocated to them. As a result, a far smaller SNR is required to achieve the maximum degrees of freedom. Thus, by simply using a weighted vector rather than of all ones, significant savings in SNR are achievable.

The second design, ROO, also achieves greater savings in SNR over the original scheme by distributing the total available transmit power more evenly over the $n+1$ beamforming vectors of user 1, done so by reducing the $n+1$ exponents in $\mathbf{V}_1$ by $-\frac{n}{2}$. We show that by doing so, the new $\mathbf{V}_1$ structure allocates the majority of the available transmit power more equally among the two columns of $\mathbf{V}_1$ whose exponents have the largest absolute value, $n/2$, whereas the original scheme allocates the majority of the power to just a single column whose exponent is maximum, n. By increasing the number of columns which share the available transmit power more equally, more of the $n+1$ degrees of freedom allocated to user 1 may be used at low and mid SNRs. Though not all $n+1$ may be used, the improvement in the utilization of the degrees of freedom at low to mid SNRs produces significant savings in SNR.

Finally, the third design, WVN-SHVOP, makes use of SHV orthonormalization precoding by combining the optimization of transmitters 2 and 3 with our new WVN scheme. By far the best technique among the three, WVN-SHVOP not only ensures that power is distributed more evenly across the transmit beamforming vectors for user 1, the added SHV procoding guarantees that the power across the beamforming vectors at transmitters 2 and 3 is perfectly divided into N/n per column. This joint technique produces the greatest savings in SNR among the three designs and may easily be implemented in an actual system. Thus, in addition to the unprecedented savings in SNR achieved by the joint WVN-SHVOP scheme which makes IA that much more practical for $K=3$ SISO networks, it may be implemented by making minor modifications to the vector $\mathbf{w}$ and with the addition of one new precoder at both transmitters 2 and 3.

As an example of the potential savings in SNR afforded by the three new designs, the case where $n=4$ is examined. Here, a maximum degrees of freedom of $13/9$ or approximately 96% of the theoretical outerbound is achievable. As shown in Fig. 3, using the joint WNV-SHVOP scheme, the maximum degrees of freedom may be achieved at an SNR of approximately 25 dB and slightly higher if using just WVN. The original CJ scheme, however, becomes linear at a far greater SNR of approximately 90 dB, for an enormous savings in SNR of **65 dB**. Though not as great of a savings, the ROO scheme reaches the maximum degrees of freedom at approximately 50 dB. Nonetheless, for a significant savings in SNR of 40 dB.

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