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ON MATHEMATICAL CONTRIBUTIONS OF PETR PETROVICH ZABREĬKO

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1. **Introduction.** The present paper does not by any means pretend to serve as a complete survey of Petr Petrovich Zabreĭko's mathematical contributions. Moreover, his mathematical activities are so deep and diverse that it seems impossible to realize such a project in the framework of one paper of reasonable length. The goal of this paper is much more modest. Namely, to indicate the principal milestones in his works by pointing out the mathematical fields upon which his scientific activities had a great influence.

P.P. Zabreĭko has published about 450 mathematical works including ten monographs and surveys comparable with monographs (the greater part of them have been translated into several languages). His supply of scientific ideas seems to be endless: in the last ten years alone, he has published about a hundred papers and one monograph. The scientific community is greatly impressed not only by the amount of Zabreĭko's works, but also by the simplicity, lightness and elegance of his arguments. Everyone who has had the good luck to work with him, to attend his lectures or simply to talk to him, could not but admire his encyclopaedic knowledge (which goes far beyond the scope of mathematics). Many historians and philosophers regard him as their colleague.

In this paper, we will point out five principal directions in Zabreĭko's mathematical contributions:

(i) geometric methods of non-linear analysis;
(ii) function spaces, integral operators and integral equations;
(iii) fundamental problems of analysis;
(iv) differential equations;
(v) approximate methods.

2. **Geometric methods of non-linear analysis.** The first works of P. Zabreĭko related to geometric (sometimes called 'topological') methods of non-linear analysis have addressed a *winding number of a vector field* (resp. *a mapping degree*). The degree is an integer-valued characteristic assigned to a reasonable map and satisfying the standard *additivity, homotopy and normalization* properties, meaning that the degree is an algebraic 'count' of solutions to an operator equation that is not affected by small perturbations or even larger deformations. In the finite-dimensional case, the degree (known under the names "Brouwer degree", "Kronecker characteristics", "Poincaré index") is well–defined for continuous maps, while in the infinite dimensional spaces it is defined for special classes of maps (Leray-Schauder degree for compact fields, Caccioppoli-Smale degree for smooth maps, Nussbaum-Sadovskiĭ degree for the condensing fields, Browder-Petryshin-Skrypnik degree for monotone operators, to mention a few).

The application paradigm of the degree can be traced back to the intermediate value theorem in real analysis or global residue theorem in complex analysis resulting in the following basic principle (*existence property*): if the degree of a map is different from zero, then the corresponding operator equation has a solution. In addition, the degree theory allows to study the uniqueness of a solution, its stability, the structure of the solution set, and bifurcation phenomenon, as well as to look for approximate solutions, to estimate their errors, etc. Clearly, any application scheme of the degree theory can be effective only if there is a sufficiently rich class of maps with degrees that are evaluated/estimated and there is a way to deform a map in question to the one from this class.

The notion of degree allows to define an important topological characteristic of a singular point of a vector field (i.e. a point where the field either vanishes or is not defined). Namely, a degree of a vector field on a small sphere centred at an isolated singular point is called an index of the point. The first papers of P. Zabreĭko were addressed the computation of the index. In [17], he discovered a deep connection between the index of a singular point of the plane vector field with the classical Sturm Theorem on real roots of polynomials and the so-called Cauchy index of a pair of real polynomials. Based on this connection, an algorithm was suggested for the computation of the singular point index. This algorithm assigns to a singular point a tree, with the vertices being a pair of polynomials in such a way that their Cauchy indices completely determine the index in question. These results have constituted a special Appendix in the classical monograph [1] (see also [73] for the further formalizations). An elegant study of an essential singular point of a complex analytic function was given in a recent note [198].

J. Leray was the first to compute the index of a singular point of a completely continuous smooth vector field provided that the point is non-degenerate (in this case, the problem can be easily reduced to studying the spectrum of the linearization; see also [226] for the generalization of the Leray result to the vector fields leaving invariant a convex set). However, as it was observed by M. Krasnoselskiĭ, the case of degenerate singular point is incomparably much more important for the applications (in particular, for studying the bifurcation phenomenon). In [16, 18] (see also [2]), P. Zabreĭko and M. Krasnoselskiĭ suggested a general algorithm allowing a reduction of this problem to the analysis of an effectively constructed finite-dimensional vector field (the dimension of the field coincides with the geometric multiplicity of 1 for the linearization). Later, this algorithm was extended to other important classes of vector fields: (i) positive completely continuous [79] and pseudomonotone fields in Hilbert spaces [180]. Close results for the index at infinity of the pseudomonotone fields can be found in [182].

As is well-known, many non-linear problems may be given a fixed point formulation for the operators acting in different spaces (for the classical example, we refer to the oscillation theory problems which can be reduced to: (i) studying fixed points of the Poincaré shift operator in the phase space, as well as (ii) studying integral operators in appropriate functional spaces). The statements connecting the topological characteristics of the corresponding fields are called relatedness principles. It turned out that the abstract essence of these principles is the coincidence of the degrees of vector fields of the form $I - AB$ and $I - BA$. The corresponding degree coincidence theorems were established in [61, 72]. In addition, these papers contain the formulae connecting degrees of the fields $I - A$ and $I - A^n$ (a similar result was obtained independently by H. Steinlein). An abstract approach to the relatedness principles was suggested in [94]. This approach allowed P. Zabreĭko to give an elementary proof of the classical relatedness theorems for ODEs in [140].

Many vector fields appearing in the applications fail to satisfy the complete continuity. The problem of extending the degree methods to wider classes of the fields had attracted a lot of attention for a long time. For the fields with the so-called condensing operators (in particular, contracting + completely continuous) the degree theory was developed by several authors (F. Browder, C. Fenske, R. Nussbaum, W. Petryshin, B. Sadovskiĭ, Yu. Sapronov, G. Vainikko et al). An elementary approach to this concept based on the usage of a "fundamental" and "supporting" set was suggested in [9] (in particular, this scheme allows to define the degree with the so-called limit compact operators). In [36], the authors introduced a concept of the degree for the vector fields of the form $\Phi(x) = F(x, x)$, where F is locally invertible with respect to the first argument. By the way, this approach justified the Caccioppoli scheme of the degree theory of Fredholm maps.

In the early thirties, K. Borsuk established that the degree of an odd field on a finite-dimensional sphere is odd, i.e. K. Borsuk was the first to observe that symmetries (formally speaking, equivariance) of fields may lead to restrictions on possible values of the degree. Further generalizations of this result were due to M. Krasnoselskiĭ (geometric approach based on the usage of equivariant extensions) and P. Smith (homological approach based on the usage of the so-called Smith indices). It was P. Zabreĭko who observed for the first time that the degree of equivariant fields is intimately connected to the equivariant retract theory (see [67]; see also [87]). This observation was a starting point for Z. Balanov, S. Brodsky and A. Kushkuley in whose works the geometric approach acquired conceptual clarity and logical completeness. For a detailed historical survey on the ideas behind the geometric approach and its connections to the so-called equivariant degree and its applications to the symmetric Hopf bifurcation, we refer to the text [227] (as usual, both elegant and deep).

In [231], P. Zabreĭko returned to Krasnoselskiĭs old result on a connection between the degrees of a field on an $(n-1)$-dimensional sphere computed in the original (fixed) coordinate system and in a frame "moving continuously" along the sphere (known as the addition formula). It was established by M. Krasnoselskiĭ that this formula is non-trivial for $n = 2, 4$. Its non-triviality for $n = 8$ was proved in [231]. Moreover, a connection of the addition formula with the classical results of J. Milnor and M. Kervaire was indicated.

We would like to mention an interesting survey on the degree theory [14] with a special focus on the results obtained by Soviet mathematicians. These results are often unknown to Western readers. Last but not least, common features of the Conley index approach and the Krasnoselskiĭ-Perov guiding function approach to the existence of periodic solutions to autonomous dynamical systems are discussed in [216].

3. **Function spaces, integral operators and integral equations.** In 1964, P. Zabreiko began his investigations of integral operators in the Lebesgue spaces $L_p \quad (1 \leq p \leq \infty)$. In [20, 22, 23], in particular, he obtained several important theorems on the compactness in measure of linear integral operators (LIO). This property allows to formulate simple and effective criteria for the usual compactness of LIO. Additionally, in these papers, P. Zabreĭko and M. Krasnoselskiĭ introduced the notion of $\mathcal{L}$-characteristic of an operator (in general, non-linear) representing a "graph" of the action of the operator in the scale of Lebesgue spaces. This characteristic allows to determine important properties of fixed points of the operators (in particular, their regularity). In [19, 24, 27], it was established that if the Urysohn integral operator with a certain non-negative kernel is continuous, then this operator with a smaller (in modulo) kernel is continuous as well (majorant principle for continuity). At the same time, for the compactness property, the majorant principle is false. In addition, these papers contain new sufficient conditions for the complete continuity and differentiability (at points and domains) of the Urysohn operators. In the joint paper [21], P. Zabreĭko and E. Pystil'nik discovered a wide class of completely continuous Hammerstein integral operators for which the corresponding

linear integral operator is not compact. These investigations have been summarized in the monograph [3].

It was a pioneering observation of P. Zabreĭko that many arguments and constructions related to the integral operators in L_p are independent of the formula for the L_p-norm and, in fact, are based on certain properties connecting norms with the order relation on functions. In turn, the crucial role in these considerations belongs to geometric properties of cones of non-negative functions. It allowed to extend the results on integral operators in L_p to the operators in the so-called ideal functional spaces considered earlier in a different context by Yu. Gribanov and Luxemburg-Zaanen. In particular, the theory developed included integral operators in the Lebesque spaces, Orlicz spaces, Lorentz spaces, Marcinkiewicz spaces, to mention a few. Surprisingly, several results turned out to be new even in the Lebesque spaces. Observe also that ideal spaces are particular cases of the Riesz spaces, Kantorowich spaces, Nakano spaces, Banach function spaces. The main results on integral operators in ideal spaces were summarized in the monograph [4] (see also [5, 8]).

In later papers, P. Zabreĭko returned to the theory of ideal spaces many times. In [71], he focused on the geometry of these spaces as related to the additive properties of the norm on disjoint elements. The obtained results gave rise to a new scheme (based on the theorems on the products of kernels) of the proofs of Kantorovich's classical results on linear integral operators (see [89, 93, 98]). Perhaps more significantly, this scheme allowed to include important classes of non-linear integral operators. In [203], a complete description of quasipositive elements in important classes of ideal spaces was given, and as a result (see [204]), an analogue of the classical Perron-Frobenius Theorem for positive linear operators in ideal spaces was given.

A series of papers by P. Zabreĭko [95, 96, 102, 103, 104, 109, 114, 119, 128] is devoted to the superposition operator in different classes of function spaces. In particular, the classical theorem of Krasnoselskiĭ's providing the conditions for the action of the superposition operator in Lebesgue spaces is analysed in detail. The approach developed by the authors allowed to give explicit formulae for the growth of the superposition operator in the Lebesgue spaces. In addition, a complete study of analyticity, Lipschitz continuity, continuity (usual, weak, weakened, strengthened), boundedness, and differentiability of the superposition operator in ideal spaces was presented. It was established that the Lipschitz and Darbo conditions for the superposition operators in ideal spaces coincide. Also, it was discovered that the properties of the superposition operator in l_p, $(1 \leq p \leq \infty)$ are completely different from the ones in the case L_p, $(1 \leq p \leq \infty)$. It was M. Vainberg who observed for the first time that if the superposition operator in L_p is differentiable (resp. Gateaux differentiable), then it must be constant (resp. linear). The authors of the aforementioned papers gave a complete description of the pairs of ideal spaces for which these phenomena take place. The systematic exposition of the superposition operator theory can be found in the monograph [10] (see also [211, 215] for the further developments).

Similar investigations for the multivalued superposition operator were done in [133, 143, 144, 149] (see also the monograph [11], where these results were applied to studying the solubility of Hamerstein integral inclusions).

In a series of papers [105, 107, 115, 124, 125, 138, 139], a new variant of the theory of ideal spaces of vector-valued functions was developed; in these papers, the ideal space is understood to be the space of measurable functions closed with respect to the product by scalar bounded measurable functions. This viewpoint allowed to develop the duality theory as well as to establish the analogues of the majority of properties of usual scalar ideal spaces, among them the criteria of compactness of sets, separability, reflexivity. As a matter of fact, the ideal spaces of functions $\Omega \to \mathbb{R}^n$ are not ordered in the usual sense. Therefore, in this case, one needs to find a workable replacement of a cone of non-negative functions. A collection of the "tubes" (each of the tubes is a set of all selectors of a multivalued function defined on Ω and taking its values in the set of closed, convex, symmetric subsets of $\mathbb{R}^n$) serves as the required cone. This construction, being interesting in itself, allowed the authors to obtain analogues of the Banach (resp. Kantorovich) Theorem on the continuity (resp. regularity) of linear integral operators.

Let us also mention the work [39] where a general duality theory for Bochner-like ideal function spaces was constructed.

The main focus of Zabreĭko's investigations on non-linear equations is related to the Hammerstein integral equations. The standard scheme for studying these equations can be traced back to the classical works of A. Hammerstein (variational method), M. Krasnoselskiĭ (topological method) and Minty-Browder (monotone operator method). The main point of this scheme is an assumption on the adjustment of singularities of the kernel K of the corresponding linear operator and the growth of the non-linearity f with respect to the spacial variable. Usually, such an adjustment can be achieved by choosing a suitable function space X for which K acts from the dual space X' to X, while f takes X to X'. The original works of A. Hammerstein, M. Krasnoselskiĭ suggested L_p, $2 \leq p < \infty$, for X that allowed to consider the non-linearities of polynomial growth. The further development of the Orlicz space theory opened the door to considerations of non-polynomial non-linearities. In a series of works [33, 40, 52, 53, 58, 59, 63, 65, 66, 90], P. Zabreĭko and A. Povolockiĭ started to apply the theory of ideal spaces to the problem in question. In particular, this approach allowed them to formulate minimal requirements with respect to K and f, providing the validity of classical results for the solvability of Hammerstein equations.The same approach (see [78]) allowed the authors to explicitly construct the space X optimal for a priori given non-linearity.

It was H. Schaefer who established for the first time the existence of solutions to the Hammerstein equation using methods outside of functional analysis (special finite-dimensional approximations + a priori estimates). Actually, the non-linear operator considered by H. Schaefer was neither compact nor bounded. In [56], P. Zabreĭko obtained "Schaefer-like" results using functional analysis techniques and ideal function space theory. This approach, in compliance with a new method of M. Krasoselskiĭ, allowed (see [126, 135]) the authors to study the solubility of Hammerstein equation without the assumption that K (resp. f) takes X' to X (resp. X to X').

The existence of multiple solutions to variational Hammerstein equations was studied in [127, 131, 169, 172, 175], while the uniqueness theorems for the Hammerstein equations without symmetric kernels were established in [183].

At the end of the 19th/beginning of 20th century, the investigations of A. Lyapunov, E. Schmidt and L. Lichtenstein on the equilibria of rotating fluids resulted in the appearance of the so-called Lyapunov-Schmidt non-linear integral equations. Several partial results on these equations have been obtained by M. Krasnoselskiĭ and M. Vainberg. In [185, 212], these equations were given a systematic study in the framework of modern non-linear functional analysis.

A series of works of P. Zabreĭko is devoted to linear integral equations. It was established in [30, 35], that the classical result on the solubility of the Volterra integral equation of the second kind is a simple consequence of the compactness of the integral operator. Later on, a careful analysis of general theorems on integral Fredholm equations of the second kind (Fredholm alternative, Fredholm formula, integral presentation for the resolvent, etc.) was given in [141, 170] using the methods of functional analysis. In particular, it turned out that in the case of the space C of continuous functions (contrary to the case of L_2), the theorem on the integral presentation of the Fredholm resolvent (resp. Fredholm alternative) is a consequence of the boundedness (resp. compactness) of the integral operator. A systematic exposition of the theory of linear integral equations in the framework of the general theory of linear operators in Banach spaces was given in [6].

At the end of the 1990s, P. Zabreĭko began his investigations of the Riemann problem for PDEs. The solubility of this problem turned out to be connected to several properties of the so-called partial integral operators (almost always, noncompact), see [190, 199]. The obtained results turned out to be useful in the theory of the Barbashin integro-differential equations appearing in mathematical biology, astrophysics and radiation theory. A systematic exposition of the theory of partial integral operators and Barbashin equations can be found in [15].

4. **Foundations of analysis.** In [45], P. Zabreĭko suggested an elegant principle of continuity of semi-additive functionals, including as particular cases the following classical results: Gelfand's Lemma on the continuity of semi-additive functionals semi-continuous from below, Banach Theorem on the closed graph, a number of fundamental theorems in the theory of semi-ordered spaces, etc. In [97], a general uniform boundedness principle was established, allowing to unify the classical Banach-Steinhaus Theorem, Stechkin's theorem on the continuity of non-linear functionals, and the Hahn-Saks-Vitali Theorem on absolute equicontinuity of a sequence of measures. In [76], the authors suggested a new solution of A. Grothendieck's celebrated problem on extension of the closed graph theorem to the class of locally convex linear spaces containing Banach spaces and invariant with respect to taking inductive and projective limits of sequences of spaces. The method suggested by the authors was based on the analogue of Suslin's set-theoretic construction for locally convex spaces. In [70], a general compactness criterion of sets was formulated in terms of a given semigroup of operators (an abstract analogue of the classical criterion of F. Riesz). In [162], the authors gave a new characterization of Banach limits satisfying the multiplicativity property.

A number of works of P. Zabreĭko are devoted to positive linear operators (PLO), i.e. leaving invariant a cone in Banach spaces. In [60], there was described an abstract class of PLOs for which the analogues of the classical Ostrovskiĭ Theorem on positive matrices and E. Hopf Theorem on integral operators with positive kernels are valid (the so-called focusing operators). In particular, for these operators, the results on the convergence of Kellogg's approximations to the leading eigenvalue together with sharp estimates of the spectral clearance were established. In [196], the coincidence of the focusing constant and the constant estimating the spectral clearance was proved (see also [75] for further generalizations to the operators leaving invariant a closed convex set). In [80], a generalization of the classical Kreĭn-Rutman theorem on the existence of a positive eigenvector for compact PLOs with positive spectral radius was suggested. It turned out that the conclusion of this theorem is true for non-compact PLOs with the spectral radius greater than the Fredholm radius.

The paper [34] suggests an abstract scheme for the upper and lower estimates of the spectral radius of PLOs (concrete examples considered in [34] show that the obtained estimates are optimal). In [221, 234, 235, 240], the problem of the existence of the second positive eigenvalue was studied for the so-called bipositive operators (i.e. for PLOs for which the external square is positive on the external square of the space).

A series of papers by P. Zabreĭko is related to calculus in Banach spaces. In [157], a variant of the Stone-Weierstrass Theorem for Banach-valued functions was suggested. In [84], a delicate connection between the classical and Taylor higher derivatives for Banach-valued functions was considered. In [173], P. Zabreĭko suggested a surprising variant of the Lagrange mean value theorem for Banach-valued functions which does coincide with the classical Lagrange mean value theorem in the scalar case (many calculus textbooks state that it is impossible). In [188, 191, 195, 197], the authors obtained explicit (non-recurrent!) formulae for higher derivatives of inverse and implicit Banach-valued functions. In [200], a generalization of the well-known theorem on the global homeomorphism for Fréchet differentiable maps was extended to several classes of Gateaux differential maps. In [178, 217], the authors studied the Fréchet differentiability for multivalued maps.

P. Zabreĭko's contribution to Analysis concerning fixed point theory is extremely important. In [32], the authors generalized the well-known Minty fixed point theorem to the sums of monotone and strengthened continuous operators. In [57], it was observed that the assumption on the uniform convexity in Pokhozhaev's theorem on the solubility of nonlinear operator equations can be omitted (actually, the geometric lemma obtained in this paper turned out to be equivalent to a number of classical theorems in non-linear analysis: drop theorem, Bishop-Phelps theorem, petal theorem, etc.). In [222], P. Zabreĭko returned to the comparative analysis of the Banach-Cacciopolli contraction principle and the Kantorovich majorant principle. Although in many cases these principles lead to similar results, in general, they are not equivalent. Also, it was discovered that under the assumptions of the Kantorovich principle, the fixed point in question can be effectively localized in an explicit layer.

A number of papers by P. Zabreĭko were devoted to the implicit function theory (IFT). As is well-known, the classical implicit function theorems (for example, Hildenbrandt-Graves theorem) require continuous differentiability. However, in infinite dimensional spaces, this hypothesis is usually false (typical non-linear operators are differentiable at the points of dense subsets rather than on open subsets). This phenomenon requires the proofs of IFTs to use the fixed point principles with no a priori differentiability (the Schauder principle, Darbo-Sadovskiĭ principle, Minty-Browder principle give simple examples of this type). These arguments constituted the main background of the papers [26, 37, 46, 85, 166, 164]. The "classical" IFTs and a number of their modifications, together with different comments, have been presented in the monograph [7] while further developments have been reflected in the survey [12].

The theory of implicit functions is intimately connected to the theory of branching and bifurcation of solutions to operator equations with parameters. A series of papers [37, 38, 41, 42, 43, 62, 68, 237, 240] and monograph [7] are related to this direction. The main contributions are as follows. For a long time, it was a common belief that if the first coefficients of the Taylor decomposition of solutions to a given operator equation $f(\lambda, x) = 0$ are determined by the first coefficients for f, then the well-known Rücker-Lefschetz/Vainberg-Trenogin algorithms allow to compute these coefficients. This statement, as it was observed by M. Krasnoselskiĭ and P. Zabreĭko (see also [237, 240]), is not true starting from two-dimensional degeneracies of the linearizations. Furthermore, it was discovered that the Lyapunov branching equation does not coincide with the Schmidt branching equation (although, as is well-known, they are equivalent). Among other results, one should mention: (i) theorems on simple solutions (i.e., the ones on which $f'_x(\lambda, x(\lambda))$ is invertible for small λ) allowing to avoid the difficulties related to the Rücker-Lefschetz algorithm, (ii) iterative schemes for asymptotic approximations for the branching equations (it turned out that the "heuristic" van der Pol method of constructing amplitude curves well-known in the theory of non-linear oscillations is nothing else but the method of constructing asymptotic approximations), (iii) a satisfactory analysis of the behaviour of solutions around a bifurcation point in the case of one-dimensional degeneracy, (iv) description of the phenomenon of "linearization" in the method of undetermined coefficients: subsequent computation of the coefficients c_n related to λ^n leads to the decomposition of the coefficient space into a (finite) direct sum and, moreover, the components of c_n are evaluated by means of linear equations for sufficiently large n, (v) simple proofs of the M. Artin theorem on the convergence of series appearing in the method of undetermined coefficients.

In [83, 86, 91], a new definition of the cone of admissible directions for finite-dimensional extremal problems was suggested. In particular, it allowed to strengthen the classical results on extremum of functions in several variables. Recently (see [228, 229]), a general algorithm was suggested for testing a critical point of a real analytic function in two variables to be a minimum. This algorithm turned out to be intimately connected to the general Newton diagram based methods for studying singular points of plane curves.

5. **Differential equations.** P. Zabreĭko's great contribution to ordinary differential equations is deep and unquestionable. His first work in this area [31] was devoted to the uniqueness theorems for ODEs. A new approach based on the concept of ω-separated curve (being a refinement of upper and lower solutions to scalar DE) allowed P. Zabreĭko to formulate a general principle for the investigation of the conditions providing the uniqueness in ODEs. In [74, 82], by combining the concepts of measures of non-compactness and uniqueness theorems, the authors suggested new existence results for the Cauchy problems in Banach spaces.

A number of papers by P. Zabreĭko are related to the Bogolyubov-Krylov averaging principle. In [29, 44, 47],the authors suggested a new variant of the Bogolyubov-Krylov theorem on the solubility of the Cauchy problem and the estimates for higher order approximated solution on a finite but arbitrarily large interval. In [46, 48, 54, 77, 160], a series of fundamental results on the existence of solutions bounded on the whole axis were obtained. In particular, it was observed that the Bogolyubov theorem on the existence of bounded solutions: (i) is a simple corollary of the implicit function theorem suggested in [46], (ii) follows from the Banach-Cacciopolli theorem applied to the square of the integral operator with fixed points determined by the solution in question, (iii) is, to some extent, equivalent to the classical Bohl theorem. These considerations allowed P. Zabreĭko to essentially strengthen the Bogolyubov theorem as well as to extend this theorem to a wide class of differential equations (including delay equations). Moreover, these methods allowed (see [163]) to treat in a similar way the problem of the existence of solutions bounded on a semi-axis and to generalize the corresponding theorem of Yu. Mitropolsky.

In [55], the authors suggested a new scheme to justify the Galerkin method in the problem of constructing periodic solutions to non-autonomous ODEs. Also, this problem was given a lot of attention in [99, 100, 132], where the authors obtained optimal conditions for the applicability of the Samoĭlenko numeric-analytic method and studied the connections of this method to the classical methods of Poincaré and integral equations.

We would like to pay special attention to a series of papers [147, 148, 155] related to linear non-autonomous differential equations. The main idea of these works can be traced back to the famous method of N. Krasovsky (developed for FDEs), allowing to pass from the original equation to an evolutionary process in a suitable Banach space. In the considered case, the application of this method leads to analysis of a certain linear autonomous differential equation in a Banach space, i.e. to analysis of the corresponding semigroup generator. The construction suggested in these papers allowed the authors to connect spectral properties of the generator with Lyapunov exponents of the original system, stability and dichotomy of solutions, as well as to introduce the concept of monodromy operator for linear equations with (in general) non-periodic coefficients.

In [28, 110, 113, 122, 161], the aforementioned results on ODEs were applied to infinite systems of DEs and to integro-differential equations of Barbashin type (see also [16]).

Below, we review the contribution of P. Zabreĭko to PDEs. The non-local Cauchy problem was studied in [181, 194, 206, 213, 220, 224, 225] using the Banach-Cacciopolli contraction principle and different modifications of the Kantorovich majorant principle. The main difficulty here is related to constructing a Banach space invariant with respect to the integral operator corresponding to the Cauchy problem. This space was constructed for normal linear and quasi-linear equations, matrix systems of Fedorov-Riccati or Abel-Bernouli types, etc. A number of papers were devoted to elliptic problems: (i) in [25], $\mathcal{L}$-characteristics of fractional powers of elliptic operators in Sobolev spaces were evaluated; (ii) in [121], a series of solubility results for elliptic boundary value problems with non-monotone non-linearities was obtained; (iii) in [192, 193, 233], using the Browder-Petryshin-Skrypnik degree, the authors established new results on the solubility of the non-linear elliptic boundary value problem. Using the well-known fact that the semigroups related to linear parabolic equations take "large" function spaces to the "small" ones, in [167] the solubility of the Cauchy problem for parabolic equations with strong non-linearities was established. In [64, 69], analogues of averaging theorems of Bogolyubov-Krylov for hyperbolic equations were established using the van der Pol method. The existence of a periodic solution to the quasi-linear telegrapher equation was studied in [136] using the Kantorovich majorant principle. The Weyl decomposition for the Navier-Stokes equation was studied in the case of unbounded domains in [152, 158].

The first works of P. Zabreĭko on differential equations with unbounded operators (the so-called abstract equations) were related to the classical semigroup theory. In [49], a complete characterization of strongly continuous semigroups with integrable and power singularities at zero was given (for the prototypal examples, we refer to the classical Hille-Phillips-Miyadera theorem). In [106], a fixed point principle allowing to establish the solubility of the Cauchy problem for differential equations in discrete monotone scales of Banach spaces was suggested. This, in turn, led to new existence theorems for smooth and generalized solutions to PDEs. It was discovered in [111, 112] that the classical theorem of L. Ovsyanikov on the solubility of the Cauchy problem in scales of spaces is a consequence of a simple modification of the Banach-Cacciopolli principle for the so-called K-normed spaces (i.e. the spaces for which the "norm" takes its values in a suitable ordered linear space). These constructions gave rise to a series of new theorems on the solubility of the Cauchy problem in locally convex spaces, scales of Banach spaces, and more generally, K-normed spaces (see [118, 145, 146, 150, 179]). The novelty of this approach was based on the passage from a numeric Lipshitz constant to an operator valued "Lipschitz constant." This "innocent" observation lead to the usage of different solubility theorems for linear and non-linear operators in ordered linear spaces (see [201, 207, 208] for further developments).

In [165, 176], in terms of the Roumieu spaces and Burling spaces, a complete characterization of initial conditions ξ for which solutions to the Cauchy problem related to the equation $\dot{x} = Ax$ are determined by the formula $x(t) = e^{At}\xi$ was given. This led to a generalization of the well-known Gelfand theorem on the density of analytic and entire vectors. In [209, 223, 238], the existence and uniqueness theorems for the Cauchy problem for differential equations of fractional order with bounded and unbounded right-hand sides were established. We would like to conclude this section by emphasising the important role of P. Zabreĭko in the origin of the mathematical theory of hysteresis. P. Zabreĭko was a member of a group of Voronezh mathematicians and physicists led by M. Krasnoselskiĭ that suggested the first rigorous mathematical model of the hysteresis phenomenon (see [50, 51]). In particular, the hysterant operator was introduced, its continuity and Lipschitzian continuity were studied, and furthermore, applications to differential equations with this operator were considered.

6. **Approximate methods.** A number of P. Zabreĭkos papers are related to iterative methods for solving linear and non-linear operator equations. In [101, 123, 129, 159], the asymptotic behaviour of norms of iterations of a linear operator was analysed. In particular, (i) the connection between this behaviour and the spectral radius of the operator was studied, (ii) exact estimates of errors of approximate solutions to linear equations were established, and (iii) the case of linear operators taking one Banach space to another was considered. In [117, 168], successive approximations of solutions to non-linear equations with smooth operators were studied in the non-degenerate case (i.e. when the spectral radius of the linearization at the fixed point is less than one). The results obtained describe subspaces which are "approached" by the approximations. In [81], the degenerate situation (i.e. when the spectral radius of the linearization at a fixed point is equal to one) was studied. It was showed that in this case, the analysis of the convergence of successive approximations can be reduced to some smaller manifold invariant for the non-linear operator.

Another group of results obtained by P. Zabreĭko is related to the Newton-Kantorovich method (NKM) for approximate solving of non-linear operator equations. In [88, 92], a comparative analysis of different proofs of the convergence of the NKM (Kantorovich majorant method, geometric approach based on fixed point principles, etc.) was given. This led, on the one hand, to strengthening several theorems on the convergence of successive approximations, and, on the other hand, to elaborating a new version of the majorant method allowing to distinguish between the proofs of the existence, uniqueness and convergence of the Newton-Kantorovich approximations. As a result (see [108, 116, 134, 153]), (i) simple V. Pták's limit error estimates under the Kantorovich conditions were given (the original V. Pták's proof was based on the continuous induction principle obtained by means of the closed graph theorem; the new proof used only the Newton-Leibnitz formula), (ii) the obtained results were extended to the case when the derivative of the left-hand side of the equation satisfies only the local Lipschitz condition. Also, limit results for the Krasnoselskiĭ's version of the NKM for equations with non-smooth operators were obtained.

As a matter of fact, under the so-called Vertgeim conditions (which are more general than Kantorovichs), the above method did not lead to essentially new results. However, it was discovered in [156] that under the Vertgeim conditions, the results known at that moment described only "half" of the cases when the NKM converges. The further modification of the majorant method suggested in [134] (see also [174, 177]) allowed the authors: (i) to study the convergence of the NKM in the cases when, under the Vertgeim conditions, the hypotheses of the original Vertgeim theorems are not satisfied, and (ii) to obtain new results for Chebyshev and Newton's two-step methods. In [137, 154, 171, 174], the application of the NKM to non-linear integral equations was given. In particular, it was discovered that under the classical conditions suggested by L. Kantorovich, the usage of the NKM is impossible in L_p $(1 \leq p \leq 2)$ since these conditions imply that the kernel should be linear or even trivial! On the other hand, in this setting, the Vertgeim conditions turn out to be less restrictive: the degeneracy condition takes place only for L_p with $1 \leq p \leq 1+\theta$, where θ is the Hölder constant for the left-hand side of the equation. Among additional results in this direction, we should mention [184] (new local estimates of speed of convergence of the NKM), [202] (similar results for the chord method), and [218] (similar results for the Krasnoselskiĭ-Rutitskiĭ approximations).

In [210, 214, 232], under fairly general assumptions, the authors considered a class of iterative methods of constructing approximate solutions. This class includes classical minimal residual, minimal errors, steepest descent methods and the method of M. Altman, to mention a few. It turned out that the analysis of these methods can be reduced to studying the geometric properties of some scalar function describing the residual decrease. In addition, the same geometric properties determine convergence conditions of the methods in question and the speed of this convergence, as well as allowing to obtain a priori and a posteriori errors estimates.

In [142], there was suggested a generalization of the celebrated L. Kantorovich fixed point principle for non-smooth operators proved by means of successive approximations (see also [151, 154], where several analogues of this principle in K-normed spaces were considered). For the applications of the obtained results to differential equations, we refer to: [132] (convergence of approximate methods of constructing periodic solutions, [160] (averaging method), [179] (abstract Cauchy-Kovalevskaya theorems). For systematic expositions of the related results we refer to [13, 130] (see also [187, 201]). Recently (see [236]), applying the Kantorovich majorants principle to implicit successive approximation method, the authors obtained exact convergence estimates.

Finally, we would like to mention the paper [189], where a local Lipschitz constant of a map assigning to a positive definite symmetric matrix its lower-triangular factor in the LU-decomposition was effectively estimated in terms of the norm of the matrix, its dimension and principal minors.

7. **What has been left beyond the scope of this survey?** Unfortunately, being limited by the length of this section, we have only outlined several directions (not mentioned before) to which Petr Petrovich Zabreĭko has contributed greatly. Beyond the scope of the survey are the following: (i) dozens of papers related to impulse differential equations, (ii) important results on functional differential equations, (iii) investigations related to mathematical economics, (iv) educational textbooks related to different fields of linear and non-linear analysis. Finally, let us mention the papers [186, 205], where P. Zabreĭko wrote about his teacher – outstanding mathematician and pedagogue – Mark Alexandrovich Krasnoselskiĭ.

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