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STOCHASTIC DIFFERENTIAL GAMES WITH A VARYING NUMBER OF PLAYERS

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Abstract. We consider a non zero sum stochastic differential game with a maximum n players, where the players control a diffusion in order to minimise a certain cost functional. During the game it is possible that present players may die or new players may appear. The death, respectively the birth time of a player is exponentially distributed with intensities that depend on the diffusion and the controls of the players who are alive. We show how the game is related to a system of partial differential equations with a special coupling in the zero order terms. We provide an existence result for solutions in appropriate spaces that allow to construct Nash optimal feedback controls. The paper is related to a previous result in a similar setting for two players leading to a parabolic system of Bellman equations [4]. Here, we study the elliptic case (infinite horizon) and present the generalisation to more than two players.

1. **Introduction.** We consider a discounted non zero sum stochastic differential game with infinite horizon and a variable number of players. During the game new players can appear and present players can get eliminated. We assume that the maximal number of players is fixed at $n \in \mathbb{N}$. The events of birth and death of a player are modelled by an n-dimensional counting process with intensities that are controlled by the players who are alive and depend on a controlled diffusion. The costs of any player develop as long as he is alive and is set zero as soon as he is dead. If no player is alive the game ends. Furthermore we assume that each player can observe the diffusion and is informed about the controls of the other player and about the set of players alive.

In this paper we investigate the game following the classical approach. We will work as in Bensoussan and Frehse [1] under quadratic growth conditions on the payoff. These conditions lead to the study of systems of PDE with quadratic growth in the gradient. Moreover we have a special coupling of the zero order terms due to the possible death and birth of a player. We note that in the n player case we have more states in between the state where only one player is alive and the state where

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all players are alive. So there are not only 3 sub games (P1 alive, P2 alive, P1 and P2 alive) as in Bensoussan, Frehse and Grün [4], but $2^n - 1$ with a strong coupling in the zero order term. This makes the structure much more involved. Under similar conditions as considered for the two player case in we show here L^∞ estimates, which are weaker but sufficient and generalize to the n player case. Moreover we consider the case $n = 2$ under weaker boundedness assumptions for the costs.

The study of differential games has its starting point in the pioneering work of Isaacs (see [14] and references given therein). Stochastic differential games have first been investigated by Friedman [10]. Later Bensoussan and Friedman [5] studied the related case where the players have the possibility to stop the game before a terminal time. For the classical investigation of differential games the existence of a sufficiently regular solution to an associated system of partial differential equations (PDE) respectively variational differential equations is used. With the help of the solution it is then possible to construct Nash optimal feedback strategies and thus determine the result of the game. The approach by Hamadène, Lepeltier and Peng [11] uses solutions to BSDE in the same manner allowing non Markovian cost functionals. For both approaches the non degeneracy of the diffusion matrix is crucial. Another approach is recently given by Buckdahn, Cardaliaguet and Rainer [7] who characterise a Nash optimal payoff without the assumption of non degeneracy of the diffusion matrix. This method however does not imply the construction of optimal strategies.

Our model can serve as a framework where market participants by too much exploiting the market can attract concurrents, that they have to share the market with. That means that the underlying diffusion and the costs will then be influenced by additional participants, which might lead to a fairly different behaviour of the market. On the other hand we can consider a situation where there is a shared market. The crucial question is then: when is it worth to accept some temporary losses in order to put a concurrent out of business and to exploit the market alone. Our model also naturally relates to switching games, respectively stopping games as in [5] and the recent work of Hamadène and Zhang [12], which constitute an extreme case of our approach. Indeed we will pose conditions on the model which will make immediate elimination, respectively apparition too costly in order to be optimal.

2. **Heuristic description of the game n=2.**

For the readers convenience we give here a short heuristic description of the game. All quantities used here will be defined in a concise manner in the next section. In order to simplify the notation we consider here only the case where the maximal number of players is $n = 2$.

For all $s \in [0, \infty)$ let $I(s)$ denote the set of players at time s. This is a random set with maximum number of 2, i.e. with the possible values

$$I(s) \in \{\emptyset, \{1\}, \{2\}, \{1, 2\}\}.$$

Let $\mathcal{O} \subset \mathbb{R}^d$ be an open, bounded domain with smooth boundary. The dynamics are given by a diffusion $(X_s^x)_{s \geq 0}$ in $\mathcal{O}$. They are controlled only by the players alive, hence the dynamics change whenever a player appears or disappears. More precisely we consider for $x \in \mathcal{O}$

$$\begin{aligned} X_s^x =& x + \int_0^s \Big(\beta_{\{1,2\}}(X_r^x, v_1(r), v_2(r)) 1_{I(r)=\{1,2\}} + \beta_{\{1\}}(X_r^x, v_1(r)) 1_{I(r)=\{1\}} \\ &+ \beta_{\{2\}}(X_r^{t,x}, v_2(r)) 1_{I(r)=\{2\}}\Big) dr + B_s, \end{aligned} \tag{1}$$

where B_s is a Brownian motion on some standard probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and v_i denotes the controls of the player i with values in some control space V.

The set of players alive $I(s)$ evolves according to birth respectively death processes with the following evolution rules for $\epsilon > 0$ small:

(i) If $I(s) = \{1\}$, then

$$I(s+\epsilon) = \begin{cases} \{1\} & \text{with probability } 1 - (\lambda_{1,\{1\}}(x, v_1(s)) + \mu_{2,\{1\}}(x, v_1(s)))\ \epsilon \\ \{1,2\} & \text{with probability } \mu_{2,\{1\}}(x, v_1(s))\ \epsilon \\ \emptyset & \text{with probability } \lambda_{1,\{1\}}(x, v_1(s))\ \epsilon. \end{cases}$$

(ii) If $I(s) = \{2\}$, then

$$I(s+\epsilon) = \begin{cases} \{2\} & \text{with probability } 1 - (\lambda_{2,\{2\}}(x, v_2(s)) + \mu_{1,\{2\}}(x, v_2(s)))\ \epsilon \\ \{1,2\} & \text{with probability } \mu_{1,\{2\}}(x, v_2(s))\ \epsilon \\ \emptyset & \text{with probability } \lambda_{2,\{2\}}(x, v_2(s))\ \epsilon. \end{cases}$$

(iii) If $I(s) = \{1,2\}$, then

$$I(s+\epsilon) = \begin{cases} \{1,2\} & \text{w.p. } 1 - (\lambda_{1,\{1,2\}}(x, v_1(s), v_2(s)) + \lambda_{2,\{1,2\}}(x, v_1(s), v_2(s)))\ \epsilon \\ \{1\} & \text{w.p. } \lambda_{2,\{1,2\}}(x, v_1(s), v_2(s))\ \epsilon \\ \{2\} & \text{w.p. } \lambda_{1,\{1,2\}}(x, v_1(s), v_2(s))\ \epsilon. \end{cases}$$

(iv) If $I(s) = \emptyset$, then $I(s+\epsilon) = \emptyset$.

Note that above evolution rules imply that only the death of one player can occur, when the two players are present.

The objective for each individual player Pi is to minimize the expected payoff

$$\mathbb{E}\Big[\int_0^{\tau^i} \big(f_{i,\{1,2\}}(X_s^{t,x}, v_1(s), v_2(s)) 1_{I(s)=\{1,2\}} + f_{i,\{i\}}(X_s^{t,x}, v_i(s)) 1_{I(s)=\{i\}}\big) ds\Big], \tag{2}$$

where τ^i is the death time of Pi himself, i.e. $\tau^i = \inf\{s \geq 0 : i \notin I(s)\}$ and $f_{i,\{1,2\}}$ $f_{i,\{i\}}$ are running costs which depend on the diffusion and are controlled by the players alive. The payoffs of dead players are set to zero.

In order to study Nash equilibria we considered in [4] the related system of four PDEs. The first two correspond to the evolution of the Nash optimal payoff in the game situation, i.e. when both players are alive. While the last two describe the payoff in the state where only one player is alive.

3. **General notation and standing assumption.** In case of n Players we use the notation v_i for the control of player $i \in \{1, \dots, n\}$. We assume that the controls take their values in $V = \mathbb{R}^{m_1} \times \mathbb{R}_+^{m_2}$ for $m_1, m_2 \in \mathbb{N}$ fixed with $m_2 > 0$. We set $m = m_1 + m_2$. We split the controls in order to have a separate (positive) control parameter for the possible birth and death rates. See Remark 3.1 below.

Let $\mathcal{I}$ be the set of ordered subsets of $\{1, \dots, n\}$. For any $I \in \mathcal{I}$ we set $I^c = \{1, \dots, n\} \setminus I$. Furthermore let $|I|$ denote the number of elements in $I \in \mathcal{I}$.

Note that for $n = 2$ we have $\mathcal{I} = \{\{1\}, \{2\}, \{1,2\}\}$.

Let $\mathcal{O} \subset \mathbb{R}^d$ be an open, bounded domain with smooth boundary. For $I \in \mathcal{I}$, $i \in I$ we denote:

- $f_{i,I} : \mathcal{O} \times V^{|I|} \to \mathbb{R}$ the running cost of Player i if the players $(Pj)_{j\in I}$ are alive
- $\lambda_{i,I} : \mathcal{O} \times V^{|I|} \to \mathbb{R}$ the death rate of Player i if the players $(Pj)_{j\in I}$ are alive
- $\mu_{i,I} : \mathcal{O} \times V^{|I|} \to \mathbb{R}$ the birth rate of Player i if the players $(Pj)_{j\in I}$ are alive
- $\beta_I : \mathcal{O} \times V^{|I|} \to \mathbb{R}$ the drift coefficient if the players $(Pj)_{j\in I}$ are alive.

We set for $i \in \{1,\dots,n\}$, $x \in \mathcal{O}$, $v \in V^{|I|}$

$$\nu_{i,I}(x,(v_j)_{j\in I}) = \lambda_{i,I}(x,(v_j)_{j\in I})1_{i\in I} + \mu_{i,I}(x,(v_j)_{j\in I})1_{i\notin I} \tag{3}$$

and define ν_I as the vector $(\nu_{i,I})_{i\in I}$.

Assumption (A)

(i) For all $I \in \mathcal{I}$, $i \in I$ the functions $f_{i,I}, \lambda_{i,I}, \mu_{i,I}, \beta_I$ are Carathéodory functions with Lipschitz continuous derivatives with respect to the arguments in $V^{|I|}$.

(ii) *Growth condition on the cost:* There exists $\delta > 0$ and K, K_δ, such that for all $I \in \mathcal{I}$ $i \in I$, $x \in \mathcal{O}$, $v \in V^{|I|}$

$$f_{i,I}(x,v) \le K|v_i|^2 + K|v|^{2-\delta} + K_\delta$$

Note that (ii) states, that the non market interaction is of lower order. The principle part however has quadratic growth.

(iii) *Convexity:* For all $I \in \mathcal{I}$, $i \in I$ the functions $f_{i,I} : \mathcal{O} \times V^{|I|} \to \mathbb{R}$ are convex with respect to the i-th control.

(iv) *Coercivity:* For all $I \in \mathcal{I}$, $i \in I$ there exist $c_0, \delta > 0$ and K, K_δ, such that for all $x \in \mathcal{O}$, $v \in V^{|I|}$

$$\frac{\partial}{\partial v_i} f_{i,I}(x,v) \cdot v_i \ge c_0|v_i|^2 - K|v|^{2-\delta} - K_\delta.$$

(v) *Linearity in the dynamics:* For all $I \in \mathcal{I}$ the functions β_I are of the form

$$\beta_I(x,v) = \sum_{i\in I} A_{i,I}(x)v_i + a_I(x),$$

where $A_{i,I}$ are $d\times m$ matrices with coefficients in $L^\infty(\mathcal{O})$ and $a_I \in L^\infty(\mathcal{O},\mathbb{R}^d)$.

(vi) *Linearity in the death/birth rates:* For all $I \in \mathcal{I}$, $i \in I$ the functions $\lambda_{i,I}, \mu_{i,I}$ are of the form

$$\lambda_{i,I}(x,v) = \sum_{i\in I} \Lambda_{i,I}(x)v_i'' + l_{i,I}(x) \tag{4}$$

$$\mu_{i,I}(x,v) = \sum_{i\in I} M_{i,I}(x)v_i'' + m_{i,I}(x), \tag{5}$$

where for any $v_i \in V$ we used the notation

$$v_i = \begin{pmatrix} {v_i}' \\ {v_i}'' \end{pmatrix} \text{ with } v_i' \in \mathbb{R}^{m_1}, v_i'' \in \mathbb{R}^{m_2}_+.$$

We assume that $\Lambda_{i,I}, M_{i,I}$ are vectors of length m_2 with non-negative coefficients in $L^\infty(\mathcal{O})$ and $l_{i,I}, m_{i,I} \in L^\infty(\mathcal{O})$ are non-negative functions.

Remark 1. We impose the linearity in order to have sufficient conditions for the existence of a Nash point for the Lagrangians. More general conditions (sub-quadratic growth) can be imposed as long as the non-negativity of the birth and death rates and the existence of a Nash point for the Lagrangians is guaranteed. (See Definition 5.1 below). The assumption (A)(iv) with the non-negativity of the controls ($v'' \in \mathbb{R}^{m_2}_+$) represents one typical case.

4. **Stochastic setup and description of the game.** Let $(\Omega^1, \mathcal{F}^1, \mathbb{P}^1)$ denote the Wiener space: i.e. $\Omega_1 = \mathcal{C}_0(\mathbb{R}_+, \mathbb{R}^d)$ the space of continuous functions from $\mathbb{R}_+$ to $\mathbb{R}^d$ with value zero at 0, endowed with the topology generated by the uniform convergence on compacts. $\mathcal{F}^1$ is the Borel σ algebra over Ω^1 completed by the Wiener measure $\mathbb{P}^1$. Under $\mathbb{P}^1$ the coordinate process $B_s(\omega^1) = \omega^1_s$, $s \in \mathbb{R}_+$, $\omega^1 \in \Omega^1$ is a d dimensional Brownian motion. We denote by $(\mathcal{F}^1_s)_{s\in\mathbb{R}_+}$ the natural filtration generated by B augmented by all $\mathbb{P}^1$ null sets.

Furthermore let $(\Omega^2, \mathcal{F}^2, \mathbb{P}^2)$ denote the n dimensional Poisson space

$$\Omega_2 = \left\{ \omega^2 = (\omega^2_i)_{i\in 1,\ldots,n}, \omega^2_i(s) = \sum_{j^i} 1_{s<t_{j^i}},\ j^1, \ldots, j^I \in \mathbb{N}, t_{j^i} \in \mathbb{R}_+ \right\}$$

and $\mathcal{F}^2$ the σ-algebra $\sigma(N_A : N_A(\omega^2) = \omega^2(A), A \in \mathcal{B}(A))$ completed by the probability measure $\mathbb{P}^2$, where $\mathbb{P}^2$ is such that the coordinate processes $N^i_s(\omega^2) = (\omega^2_i)(s)$ are independent Poisson processes with intensity 1. We denote by $(\mathcal{F}^2_s)_{s\in\mathbb{R}^+}$ the natural filtration generated by N augmented by all $\mathbb{P}^2$ null sets.

We will work on the space $(\Omega, \mathcal{F}, \mathbb{P})$, where $\Omega = (\Omega^1 \times \Omega^2)$, endowed with the product sigma algebra $\mathcal{F}^1 \otimes \mathcal{F}^2$ completed by $\mathbb{P} := \mathbb{P}^1 \otimes \mathbb{P}^2$. Furthermore we define $(\mathcal{F}_s)_{s\in\mathbb{R}_+}$ by $\mathcal{F}_s = \mathcal{F}^1_s \otimes \mathcal{F}^2_s$ augmented by all $\mathbb{P}$-null sets.

We define

$$X^x_s = x + B_s. \tag{6}$$

We set $\tau_x = \inf\{s \geq t : X^x_s \notin \mathcal{O}\}$ and stop the process X^x at τ_x without changing the notation.

We will describe the birth respectively death process of the players with the help of the canonical process N. For $I_0 \in \mathcal{I}$ we define the map $\phi^{I_0} : \mathbb{N}^n \to \mathcal{I}$ by setting

$$\phi^{I_0}(N_1, \ldots, N_n) := \{i : (1_{i\in I_0} + N_i) \bmod 2 = 1\}$$

and we define the set valued process

$$\mathbf{I}^{I_0}_s = \phi^{I_0}(N^1_s, \ldots, N^n_s). \tag{7}$$

We use $\mathbf{I}^{I_0}_s$ to describe the set of Players alive at time s. Clearly $\mathbf{I}^{I_0}_0 = I_0$ $\mathbb{P}$-a.s.

Definition 4.1. The set of all admissible control processes $\mathcal{V}$ is the set of all V valued $(\mathcal{F}_s)_{s\in\mathbb{R}_+}$ predictable processes which attain their values in a bounded set. (Not necessarily a common one!)

Definition 4.2. An admissible control process $v \in \mathcal{V}$ is called feedback control if it is of the form $v_s = v(s, \mathbf{I}^{I_0}_{s-}, X^x_s)$ for a measurable function $v : \mathbb{R}^+ \times \mathcal{I} \times \mathcal{O} \to V$.

In order to define the controlled dynamic of the game and hence the cost functionals we are considering the system under a Girsanov transformation. To that end we define for admissible controls $v^1, \ldots, v^n$ the density process $L^{v^1,\ldots,v^n} = L^{1;v^1,\ldots,v^n} L^{2;v^1,\ldots,v^n}$ given by

$$\begin{aligned} L^{1;v^1,\ldots,v^n}_t =& \exp\left(\int_0^t \left(\sum_{I\in\mathcal{I}} 1_{\mathbf{I}^{I_0}_s = I}\ \beta_I(X^x_s, (v^j_s)_{j\in I}) \right) dB_s \right. \\ & \left. - \frac{1}{2} \int_0^t \left(\sum_{I\in\mathcal{I}} 1_{\mathbf{I}^{I_0}_s = I}\ \left|\beta_I(X^x_s, (v^j_s)_{j\in I})\right|^2 \right) ds \right) \end{aligned} \tag{8}$$

and

$$L_t^{2;v^1,\dots,v^n} = \prod_{i=1}^{n} \Bigg(\exp\Bigg(\int_0^t \sum_{I\in\mathcal{I}} \Big(1 - \nu_{i,I}(X_s^x, (v_s^j)_{j\in I}) 1_{\mathbf{I}_{s-}^{I_0}=I} \Big)\ ds \Bigg)$$
$$\times \prod_{k\geq 1: T_k^i \leq t} \Bigg(\sum_{I\in\mathcal{I}} \nu_{i,I}(X_{T_k^i}^x, (v_{T_k^i}^j)_{j\in I}) 1_{\mathbf{I}_{T_k^i-}^{I_0}=I} \Bigg) \Bigg), \tag{9}$$

where T_k is a sequence of $\mathbb{R}_+$ valued random variables inductively defined by

$$T_0^i(\omega) = 0, \quad T_{k+1}^i(\omega) := \inf\{s > T_k^i : N_s^i(\omega) \neq N_{T_k^i(\omega)}^i(\omega)\} \tag{10}$$

with the convention that $T_{k+1}^i(\omega) = \infty$ if the set above is empty. (Note that by definition N and $\mathbf{I}^{I_0}$ have the same jump times.) Also we use the convention that the last product in (9) equals 1 for $T_1^i > s$. We note that assumptions (A) and the fact that the controls are admissible ensure that $L^{v^1,\dots,v^n}$ is a $\mathbb{P}$ martingale.

We define the equivalent measure $\mathbb{P}_{v^1,\dots,v^n}$ by a Girsanov transformation of $\mathbb{P}$ with the density process $L^{v^1,\dots,v^n}$. It is well known see e.g. [13] that under $\mathbb{P}_{v^1,\dots,v^n}$

$$dX_s^x = \Bigg(\sum_{I\in\mathcal{I}} 1_{\mathbf{I}_s^{I_0}=I}\ \beta_I(X_s^x, (v^j)_{j\in I}) \Bigg)\ ds + dB_s^{v^1,\dots,v^n}, \tag{11}$$

where $B_t^{v^1,\dots,v^n} = B_t - \int_0^t \Big(\sum_{I\in\mathcal{I}} 1_{\mathbf{I}_s^{I_0}=I}\ \beta_I(X_s^x, (v^j)_{j\in I}) \Big)\ ds$ is a $\mathbb{P}_{v^1,\dots,v^n}$ Brownian motion.

Furthermore the n-dimensional standard Poisson process $N = (N^i)_{i=1,\dots,n}$ has under $\mathbb{P}_{v^1,\dots,v^n}$ the same law as a n-dimensional counting process with the controlled stochastic intensity

$$\Bigg(\sum_{I\in\mathcal{I}} \nu_{i,I}(X_s^x, (v^j)_{j\in I}) 1_{\mathbf{I}_{s-}^{I_0}=I} \Bigg)_{i\in\{1,\dots,n\}} \tag{12}$$

(see also [6]).

Remark 2. Let $v^1,\dots,v^n \in \mathcal{V}$. By definition the probability that player $i \in I_0$ dies before time t is

$$\begin{aligned} \mathbb{P}_{v^1,\dots,v^n}\left[\inf\{s : i \notin \mathbf{I}_s^{I_0}\} \leq t\right] &= 1 - \mathbb{P}_{v^1,\dots,v^n}\left[\inf\{s : i \notin \mathbf{I}_s^{I_0}\} > t\right] \\ &= 1 - \mathbb{P}_{v^1,\dots,v^n}\left[N_t^i = 0\right] \\ &= 1 - \mathbb{E}_{v^1,\dots,v^n}\left[\exp\left(-\int_0^t \sum_{I\in\mathcal{I}} 1_{I=\mathbf{I}_s^{I_0}} \lambda_{i,I}(X_r^x, (v_j)_{j\in I}) dr \right)\right]. \end{aligned} \tag{13}$$

The probability that player $i' \notin I_0$ appears before time t is

$$\begin{aligned} &1 - \mathbb{P}_{v^1,\dots,v^n}\left[\inf\{s : i' \in \mathbf{I}_s^{I_0}\} > t\right] \\ &= 1 - \mathbb{E}_{v^1,\dots,v^n}\left[\exp\left(-\int_0^t \sum_{I\in\mathcal{I}} 1_{I=\mathbf{I}_s^{I_0}} \mu_{i',I}(X_r^x, (v_j)_{j\in I}) dr \right)\right]. \end{aligned} \tag{14}$$

We recover the death respectively birth probabilities for $t = \epsilon$ small as in section 2 by Taylor expansion.

For $i = 1, \dots, n$, $x \in \mathbb{R}^d$, $I_0 \in \mathcal{I}$ and admissible controls $v^1, \dots, v^n$ we define the payoff

$$J_i(x, I_0; v^1, \dots v^n) = \mathbb{E}_{v^1,\dots,v^n}\left[\int_0^{\tau^i} e^{-\alpha s}\left(\sum_{I\in\mathcal{I}} 1_{\mathbf{I}_s^{I_0}=I}\ f_{i,I}(X_s^x, (v_s^j)_{j\in I})\right)\ ds\right], \tag{15}$$

where the expectation $\mathbb{E}_{v^1,\dots,v^n}$ is taken under the measure $\mathbb{P}_{v^1,\dots,v^n}$ and τ^i is the death time of Player i himself (before X^x leaves the bounded domain $\mathcal{O}$), i.e.

$$\tau^i = \inf\{s \geq 0 : i \notin \mathbf{I}_s^{I_0}\} \wedge \tau_x. \tag{16}$$

Note for $i \notin I_0$ $J_i(x, I_0; v^1, \dots, v^n) = 0$.

Definition 4.3. The admissible feedback controls $(\hat{v}^1, \dots, \hat{v}^n)$ are a Nash equilibrium if and only if for all $i = 1, \dots, n$

$$J_i(x, I_0; \hat{v}^1, \dots, \hat{v}^n) \quad \leq \quad J_i(x, I_0; \hat{v}^1, \dots, v^i, \dots, \hat{v}^n) \tag{17}$$

for all controls v^i of player i. We call the functions

$$u_{i,I_0}(x) = J_i(x, I_0; \hat{v}^1, \dots, \hat{v}^n) \tag{18}$$

an equilibrium payoff or value of the game.

Remark 3. We note that it is crucial to have feedback rules in the definition. If Player i is not alive in the beginning, i.e. $i \notin I_0$, his payoff $J_i(x, I_0; v^1, \dots, v^n)$ is zero no matter what he and the others will play. So if he enters the game later he can play very far from the Nash equilibrium of that sub-game because anyway his payoff is zero and he can not be punished. However the fact that the controls are feedback rules that are optimal for any state "forces" him to stay in the equilibrium.

5. **Lagrangians and Hamiltonians.**

Let $n \in \mathbb{N}$ be the maximal number of players and $\mathcal{I}$ the set of ordered subsets of $\{1, \dots, n\}$. We define for all $I \in \mathcal{I}$ the Lagrangians for each sub-game played by players $i \in I$: for $1 < |I| < n$

$$\begin{aligned} &L_{i,I}(x, D_x u_{i,I}, u_{i,I}, (u_{i,I\setminus\{j\}})_{j\in I, j\neq i}, (u_{i,I\cup\{j\}})_{j\notin I}; (v_i)_{i\in I}) \\ &\quad = D_x u_{i,I}\beta_I(x, (v_i)_{i\in I}) + f_{i,I}(x, (v_i)_{i\in I}) \\ &\qquad - \left(\sum_{l\in I}\lambda_{l,I}(x, (v_i)_{i\in I}) + \sum_{m\in I^c}\mu_{m,I}(x, (v_i)_{i\in I})\right)\ u_{i,I} \\ &\qquad + \sum_{l\in I, l\neq i}\lambda_{l,I}(x, (v_i)_{i\in I})\ u_{i,I\setminus\{l\}} + \sum_{m\in I^c}\mu_{m,I}(x, (v_i)_{i\in I})\ u_{i,I\cup\{m\}}. \end{aligned} \tag{19}$$

The zero order terms $\mu_{m,I}(u_{i,I\cup\{m\}} - u_{i,I})$, respectively $\lambda_{l,I}(u_{i,I\setminus\{l\}} - u_{i,I})$ stem from the fact that new players appear, respectively present ones disappear.

In the case where the maximal number of players is present, i.e. $I = \{1, \dots, n\}$, we have

$$\begin{aligned} &L_{i,I}(x, D_x u_{i,I}, u_{i,I}, (u_{i,I\setminus\{j\}})_{j\in I, j\neq i}; (v_i)_{i\in I}) \\ &\quad = D_x u_{i,I}\beta_I(x, (v_i)_{i\in I}) + f_{i,I}(x, (v_i)_{i\in I}) \\ &\qquad - \left(\sum_{l\in I}\lambda_{l,I}(x, (v_i)_{i\in I}\right)\ u_{i,I} + \sum_{l\in I, l\neq i}\lambda_{l,I}(x, (v_i)_{i\in I})\ u_{i,I\setminus\{l\}}, \end{aligned} \tag{20}$$

since no additional players can join. On the other hand we have in the case where only one player is present, i.e. $I=\{i\}$, $i=1,\ldots,n$

$$\begin{aligned} &L_{i,I}(x, D_x u_{i,I}, u_{i,I}, (u_{i,I\cup\{j\}})_{j\notin I}; v_i)\\ &=D_x u_{i,I}\beta_I(x,v_i) + f_{i,I}(x,v_i) - \left(\lambda_{i,I}(x,v_i) + \sum_{m\in I^c}\mu_{m,I}(x,v_i)\right)\ u_{i,I}\\ &\quad + \sum_{m\in I^c}\mu_{m,I}(x,v_i)\ u_{i,I\cup\{m\}}, \end{aligned} \tag{21}$$

since the game ends with zero payoff if the last player dies.

Remark 4. If $n=2$ we only have the last two cases, i.e. for $i,j=1,2$, $i\neq j$

$$\begin{aligned} &L_{i,\{1,2\}}(x, D_x u_{i,\{1,2\}}, u_{i,\{1,2\}}, u_{i,\{i\}}; v_1, v_2)\\ &=D_x u_{i,\{1,2\}}\beta_{\{1,2\}}(x,v_1,v_2) + f_{i,\{1,2\}}(x,v_1,v_2)\\ &\quad - \lambda_{i,\{1,2\}}(x,v_1,v_2)\ u_{i,\{1,2\}} + \lambda_{j,\{1,2\}}(x,v_1,v_2)\ (u_{i,\{i\}} - u_{i,\{1,2\}}) \end{aligned} \tag{22}$$

and

$$\begin{aligned} &L_{i,\{i\}}(x, D_x u_{i,\{i\}}, u_{i,\{i\}}, u_{i,\{1,2\}}; v_i)\\ &=D_x u_{i,\{i\}}\beta_{\{i\}}(x,v_i) + f_{i,\{i\}}(x,v_i)\\ &\quad - \lambda_{i,\{i\}}(x,v_i)u_{i,\{i\}} + \mu_{j,\{i\}}(x,v_i)(u_{i,\{1,2\}} - u_{i,\{i\}}). \end{aligned} \tag{23}$$

We note that the discount like structure for $n\geq 3$ contains more terms and is hence much more involved than the case $n=2$. Compared to the standard games where the players have common dynamics the dynamics change as the number of player changes.

Definition 5.1. For all $I\in\mathcal{I}$, $i\in I$ let

$$\hat{v}^i(I,\cdot):\mathcal{O}\times(\mathbb{R}^d)^{|I|}\times\mathbb{R}^{|I|}\times(\mathbb{R}^{(|I|-1)})^{|I|}\times(\mathbb{R}^{(n-|I|)})^{|I|}\to V \tag{24}$$

be measurable functions. We say that $(\hat{v}^i(I,\cdot))_{I\in\mathcal{I},i\in I}$ is a Nash point for the Lagrangians $(L_{i,I})_{I\in\mathcal{I},i\in I}$ if for all $I\in\mathcal{I}$, $1<|I|<n$, $i\in I$, $x\in\mathcal{O}$, $p\in(\mathbb{R}^d)^{|I|}$, $r\in\mathbb{R}^{|I|}$, $s\in(\mathbb{R}^{(|I|-1)})^{|I|}$, $\bar{s}\in(\mathbb{R}^{(n-|I|)})^{|I|}$

$$\begin{aligned} &L_{i,I}(x,p_i,r_i,s_i,\bar{s}_i;(\hat{v}^j)_{j\in I})\\ &\quad\leq L_{i,I}(x,p_i,r_i,s_i,\bar{s}_i;(\hat{v}^j)_{j\in I,j<i},v,(\hat{v}^j)_{j\in I,j>i})\qquad\forall v\in V, \end{aligned} \tag{25}$$

for $I=\{1,\ldots,n\}$, $i\in I$, $x\in\mathcal{O}$, $p\in(\mathbb{R}^d)^{|I|}$, $r\in\mathbb{R}^{|I|}$, $s\in(\mathbb{R}^{(|I|-1)})^{|I|}$

$$L_{i,I}(x,p_i,r_i,s_i;(\hat{v}^j)_{j\in I})\leq L_{i,I}(x,p_i,r_i,s_i;(\hat{v}^j)_{j\in I,j<i},v,(\hat{v}^j)_{j\in I,j>i})\quad\forall v\in V \tag{26}$$

and for $I=\{i\}$, $x\in\mathcal{O}$, $p\in\mathbb{R}^d$, $r\in\mathbb{R}$, $\bar{s}\in\mathbb{R}^{(n-1)}$

$$L_{i,I}(x,p_i,r_i,\bar{s}_i;\hat{v}^i)\leq L_{i,I}(x,p_i,r_i,\bar{s}_i;v)\qquad\forall v\in V. \tag{27}$$

Under assumption (A) one can show:

Lemma 5.2. *There exist measurable functions $(\hat{v}^i(I,\cdot))_{I\in\mathcal{I},i\in I}$ that fulfill the Nash condition. Furthermore for all $I\in\mathcal{I}$, $i\in I$*

$$|\hat{v}^i(I,\cdot)|\quad\leq\quad K(|r|,|s|,|\bar{s}|)(|p_i|+|p|^{1-\delta}+1) \tag{28}$$

for all $x\in\mathcal{O}$, $p\in(\mathbb{R}^d)^{|I|}$, $r\in\mathbb{R}^{|I|}$, $s\in(\mathbb{R}^{(|I|-1)})^{|I|}$, $\bar{s}\in(\mathbb{R}^{(n-|I|)})^{|I|}$; $K(|r|,|s|,|\bar{s}|)$ denoting a constant depending on $(r,s,\bar{s})$ which remains bounded as long as r, s and $\bar{s}$ remain in a bounded set.

The proof is a straightforward generalisation of the corresponding result in the two player case [4]. We repeat it here for the reader's convenience.

Proof. We only prove the statement for the intermediate case (25). The result for the other two cases can then be shown in a similar way. Due to convexity the Nash conditions are equivalent to a system of variational inequalities:

$$\begin{aligned}
0 \geq & \langle \frac{\partial}{\partial v_i} L_{i,I}(x, p_i, r_i, s_i, \bar{s}_i; (v^j)_{j\in I})|_{(\hat{v}^j)_{j\in I}}, \hat{v}_i - v_i \rangle \\
= & \Big\langle (A_{i,I}(x))^T p_i + \frac{\partial}{\partial v_i} f_{i,I}(x, (\hat{v}^j)_{j\in I}) \\
& - \left(\sum_{l\in I} \frac{\partial}{\partial v_i} \lambda_{l,I}(x, (\hat{v}_i)_{i\in I}) + \sum_{m\in I^c} \frac{\partial}{\partial v_i} \mu_{m,I}(x, (\hat{v}_i)_{i\in I}) \right) \; r_i \\
& + \sum_{l\in I, l\neq i} \frac{\partial}{\partial v_i} \lambda_{l,I}(x, (\hat{v}_i)_{i\in I}) \; s_{i,l} \\
& + \sum_{m\in I^c} \frac{\partial}{\partial v_i} \mu_{m,I}(x, (\hat{v}_i)_{i\in I}) \; \bar{s}_{i,m}, \hat{v}_i - v_i \Big\rangle
\end{aligned} \tag{29}$$

for all $v \in V$. This is solvable due to the coercivity condition (note $\lambda_{l,I}, \mu_{m,I}$ are linear in the controls) see [15]. Setting $v_i = 0$ we have using the coercivity

$$c_0 |\hat{v}_i|^2 \leq K|\hat{v}|^{2-\delta} + K_\delta - \langle (A_{i,I}(x))^T p_i, \hat{v}_i \rangle + K(|r| + |s| + |\bar{s}|), \tag{30}$$

hence using Hölder, i.e. for all $a, b \in \mathbb{R}^d$ $\epsilon > 0$ $|ab| \leq \epsilon |a|^2 + \frac{1}{4\epsilon}|b|^2$, this gives

$$(c_0 - \epsilon)|\hat{v}_i|^2 \leq K|\hat{v}|^{2-\delta} + K_\delta + \frac{1}{4\epsilon} |(A_{i,I})^T|_\infty |p_i|^2 + K(|r| + |s| + |\bar{s}|). \tag{31}$$

So choosing $0 < \epsilon < c_0$ we get for $i = 1, 2$ an estimate for $|\hat{v}_i|^2$ in terms of $|\hat{v}|^{2-\delta} + |p_i|^2 + 1$ multiplied by a large enough constant $\tilde{K}(|r|, |s|, |\bar{s}|)$ depending on $|r|, |s|, |\bar{s}|$. To establish (28) remains to estimate $|\hat{v}|^{2-\delta}$ which is done by using the inequality above. $\square$

We define for all $I \in \mathcal{I}$ the Hamiltonians:

- for $1 < |I| < n$, $k \in I$

$$\begin{aligned}
& H_{k,I}(x, (D_x u_{i,I})_{i\in I}, (u_{i,I})_{i\in I}, (u_{i,I\setminus\{j\}})_{i,j\in I, j\neq i}, (u_{i,I\cup\{j\}})_{i\in I, j\notin I}) \\
& \quad = L_{k,I}(x, D_x u_{k,I}, u_{k,I}, (u_{k,I\setminus\{j\}})_{j\in I, j\neq k}, (u_{k,I\cup\{j\}})_{j\notin I}; (\hat{v}^i(I,x))_{i\in I}),
\end{aligned} \tag{32}$$

  where we used the notation

$$\hat{v}^k(I,x) = \hat{v}^k(I, x, (D_x u_{i,I})_{i\in I}, (u_{i,I})_{i\in I}, (u_{i,I\setminus\{j\}})_{i,j\in I, j\neq i}, (u_{i,I\cup\{j\}})_{i\in I, j\notin I}).$$

- for $I = \{1, \dots, n\}$, $k \in I$

$$\begin{aligned}
& H_{k,I}(x, (D_x u_{i,I})_{i\in I}, (u_{i,I})_{i\in I}, (u_{i,I\setminus\{j\}})_{i,j\in I, j\neq i}) \\
& \quad = L_{k,I}(x, D_x u_{k,I}, u_{k,I}, (u_{k,I\setminus\{j\}})_{j\in I, j\neq k}; (\hat{v}^i(I,x))_{i\in I})
\end{aligned} \tag{33}$$

  with

$$\hat{v}^k(I,x) = \hat{v}^k(I, x, (D_x u_{i,I})_{i\in I}, (u_{i,I})_{i\in I}, (u_{i,I\setminus\{j\}})_{i,j\in I, j\neq i}).$$

- for $I = \{k\}$

$$\begin{aligned}
& H_{k,\{k\}}(x, D_x u_{k,\{k\}}, u_{k,\{k\}}, (u_{k,\{k\}\cup\{j\}})_{j\neq k}) \\
& \quad = L_{k,I}(x, D_x u_{k,\{k\}}, u_{k,\{k\}}, (u_{k,\{k\}\cup\{j\}})_{j\neq k}; \hat{v}^k(\{k\}, x))
\end{aligned} \tag{34}$$

with

$$\hat{v}^k(\{k\},x) = \hat{v}^k(\{k\},x,D_x u_{k,\{k\}},u_{k,\{k\}},(u_{k,\{k\}\cup\{j\}})_{j=1,\dots,n,j\neq k}).$$

Furthermore we set $\hat{v}^k(I,x) = 0$ whenever $k \notin I$.

The previous Lemma implies under the growth and boundedness assumptions in (A) and the sub-optimality of the zero controls the following structure for the Hamiltonians:

Lemma 5.3. *For all $I \in \mathcal{I}$, $k \in I$ we can write the Hamiltonians $H_{k,I}$ in the following form*

$$\begin{aligned}
&H_{k,I}(x,D_x u_I,u_I,(u_{I\setminus\{j\}})_{j\in I},(u_{I\cup\{j\}})_{j\notin I})\\
&\quad=G_{k,I}(x,D_x u_I,u_I,(u_{I\setminus\{j\}})_{j\in I},(u_{I\cup\{j\}})_{j\notin I})\cdot D_x u_{k,I}\\
&\qquad+H^0_{k,I}(x,D_x u_I,u_I,(u_{I\setminus\{j\}})_{j\in I},(u_{I\cup\{j\}})_{j\notin I}),
\end{aligned} \tag{35}$$

where $G_{k,I}$ and $H^0_{k,I}$ satisfy

$$\begin{aligned}
&|G_{k,I}(x,D_x u_I,u_I,(u_{I\setminus\{j\}})_{j\in I},(u_{I\cup\{j\}})_{j\notin I})|\\
&\quad\le K(\|u_I\|_{L^\infty},(\|u_{I\setminus\{j\}}\|_{L^\infty})_{j\in I},(\|u_{I\cup\{j\}}\|_{L^\infty})_{j\notin I}))|D_x u_I|\\
&|H^0_{k,I}(x,D_x u_I,u_I,(u_{I\setminus\{j\}})_{j\in I},(u_{I\cup\{j\}})_{j\notin I})|\\
&\quad\le K(\|u_I\|_{L^\infty},(\|u_{I\setminus\{j\}}\|_{L^\infty})_{j\in I},(\|u_{I\cup\{j\}}\|_{L^\infty})_{j\notin I})\\
&\qquad(1+|D_x u_{k,I}|^2+|D_x u_I|^{2-\epsilon}).
\end{aligned} \tag{36}$$

6. Main result.

From the Lagrangians and their Nash points we defined the Hamiltonians. Those are now used to define the Bellman system (37). The solutions of the Bellman equations will allow to determine the values of the game following definition 4.3. The verification of the Nash property (17) relies on stochastic arguments and is given in theorem 6.2 below.

We consider the system of PDEs: for all $I \in \mathcal{I}$

$$\begin{aligned}
\alpha u_I - \frac{1}{2}\Delta u_I =& H_I(x,D_x u_I,u_I,(u_{I\setminus\{j\}})_{j\in I},(u_{I\cup\{j\}})_{j\notin I})\\
u_I =& 0, \quad \text{on } \partial\mathcal{O}
\end{aligned} \tag{37}$$

using the notation $u_I = (u_{k,I})_{k\in I}$ and with the convention to leave out the empty arguments $(u_{I\setminus\{j\}})_{j\in I}$ for $I = \{k\}$ and $(u_{I\cup\{j\}})_{j\notin I}$ for $I = \{1,\dots,n\}$.

In case of two players n=2 we have the following structure of (37): for $I = \{1,2\}$

$$\alpha u_{\{1,2\}} - \frac{1}{2}\Delta u_{\{1,2\}} = H_{\{1,2\}}(x,D_x u_{\{1,2\}},u_{\{1,2\}},u_{1,\{1\}},u_{2,\{2\}})$$

with $u_{\{1,2\}} = (u_{1,\{1,2\}},u_{2,\{1,2\}})$ and $(H_{\{1,2\}}(\cdot) = H_{1,\{1,2\}}(\cdot),H_{2,\{1,2\}}(\cdot))$.

Furthermore for $I = \{1\},\{2\}$

$$\begin{aligned}
\alpha u_{1,\{1\}} - \frac{1}{2}\Delta u_{1,\{1\}} =& H_{1,\{1\}}(x,D_x u_{1,\{1\}},u_{1,\{1\}},u_{1,\{1,2\}})\\
\alpha u_{2,\{2\}} - \frac{1}{2}\Delta u_{2,\{2\}} =& H_{2,\{2\}}(x,D_x u_{2,\{2\}},u_{2,\{2\}},u_{2,\{1,2\}}).
\end{aligned}$$

The following is the main theorem of this paper.

Theorem 6.1. *The system (37) has a solution $(u_{i,I})_{I\in\mathcal{I},i\in I}$ with $u_{i,I} \in W^{2,p}(\mathcal{O}) \cap L^\infty(\mathcal{O})$ for all $I \in \mathcal{I}$, $i \in I$.*

The proof for the existence of a solution $(u_{i,I})_{I\in\mathcal{I},i\in I}$ is done by an appropriate approximation of the system of equations (37). To that end we set

$$F_{\delta,I} = 1+\delta\Big(1+|u_I|^2+\sum_{j\in I}|u_{I\setminus\{j\}}|^2+\sum_{j\notin I}|u_{I\cup\{j\}}|^2\Big)\left(1+|D_x u_I|^2\right) \tag{38}$$

and consider for all $I\in\mathcal{I}$

$$\begin{aligned}\alpha u_I-\frac{1}{2}\Delta u_I =& F_{\delta,I}^{-1}H_I(x,D_x u_I,u_I,(u_{I\setminus\{j\}})_{j\in I},(u_{I\cup\{j\}})_{j\notin I})\\ u_I =& 0, \quad \text{ on } \partial\mathcal{O}.\end{aligned} \tag{39}$$

Since the right hand side of (39) is uniformly bounded the system (39) has for any $\delta>0$ a solution $(u_I^\delta)_{I\in\mathcal{I}}$ with $u_{i,I}^\delta\in W_0^{1,2}(\mathcal{O})$. We imply this regularity whenever we refer to a solution to (39), respectively (37). For a proof one can use the theory of monotone operators of Višik [20], Leray and Lions [18]. For further reference we refer to the book of Morrey [19] chapter 5.12 and references given therein. Using the standard elliptic theory for scalar equations one has the stronger regularity $u_{i,I}^\delta\in W^{2,p}(\mathcal{O})$ for all $\delta>0$, $I\in\mathcal{I}, i\in I$, since the right hand side of (39) is bounded.

The crucial point is to have uniform estimates in δ in order to go to the limit in the equation (39). We will provide those in the following sections. We note that while higher estimates can be achieved by combining known methods the crucial first step of establishing L^∞ estimates is more involved and requires additional conditions.

Using Theorem 6.1. we have

Theorem 6.2. *The feedback controls $\hat{v}^k:\mathcal{I}\times\mathcal{O}\to V$ (24) form a Nash equilibrium in the sense of Definition 4.3. moreover the payoff in this equilibrium is characterized by*

$$J_k(x,I_0;\hat{v}^1,\dots,\hat{v}^n)=u_{k,I_0}(x). \tag{40}$$

Proof. First we note that by the regularity of $u_{k,I}$, (28) and the boundedness of $\mathcal{O}$ the feedback controls $\hat{v}_k(I,x)$ are admissible. Furthermore we note that by the Itô Krylov formula (see e.g [13] and [16]) we have for any bounded $w:\mathcal{I}\times\mathbb{R}^d\to\mathbb{R}$ such that for all $I\in\mathcal{I}$ $w(I,\cdot)\in W^{2,p}(\mathcal{O})$

$$\begin{aligned}w(I_0,x)-e^{-\alpha\tau}w(\mathbf{I}_\tau^{I_0},X_\tau^x) =& \int_0^\tau e^{-\alpha s}(\alpha w(\mathbf{I}_{s-}^{I_0},X_s^x)-\frac{1}{2}\Delta_x w(\mathbf{I}_{s-}^{I_0},X_s^x))ds\\ &-\int_0^\tau e^{-\alpha s}D_x w(\mathbf{I}_{s-}^{I_0},X_s^x)dB_s\\ &-\sum_{m\ge 1,T_m\le\tau}e^{-\alpha T_m}(w(\mathbf{I}_{T_m}^{I_0},X_s^x)-w(\mathbf{I}_{T_m-}^{I_0},X_s^x))\end{aligned} \tag{41}$$

for stopping times $\tau\le\tau_x$. (τ_x was defined as the exit time of X^x of $\mathcal{O}$.) Here T_m is a sequence of $\mathbb{R}_+$ valued random variables inductively defined by

$$T_0(\omega)=0,\quad T_{m+1}(\omega):=\inf\{s>T_m:\mathbf{I}_s^{I_0}(\omega)\neq\mathbf{I}_{T_m}^{I_0}(\omega)\}.$$

Since by Theorem 6.1 $u_{k,I}\in W^{2,p}(\mathcal{O})\cap L^\infty(\mathcal{O})$ we can apply (41) to $w(I,x):=\sum_{J\in\mathcal{I}}1_{J=I}u_{k,J}(x)$ and $\tau=\tau^k$ (death time of player k). Taking expectation with respect to $\mathbb{P}_{\hat{v}^1,\dots,\hat{v}^n}$ yields

$$\begin{aligned}u_{k,I_0}(x) =&\mathbb{E}_{\hat{v}^1,\dots\hat{v}^n}\Bigg[\int_0^{\tau^k} e^{-\alpha s}\sum_{I\in\mathcal{I}} 1_{\mathbf{I}^{I_0}_{s-}=I}\Bigg(\alpha u_{k,I}(X^x_s) - \frac{1}{2}\Delta_x u_{k,I}(X^x_s)\\ &- \beta_I(X^x_s,(\hat{v}^j)_{j\in I})\ D_x u_{k,I}(X^x_s)\\ &- \sum_{i\in I}\ \lambda_{i,I}(X^x_s,(\hat{v}^j)_{j\in I})\ (u_{k,I\setminus\{i\}}(X^x_s) - u_{k,I}(X^x_s))\\ &- \sum_{i\notin I}\mu_{i,I}(X^x_s,(\hat{v}^j)_{j\in I})(u_{k,I\cup\{i\}}(X^x_s) - u_{k,I}(X^x_s))\Bigg)ds\Bigg].\end{aligned} \tag{42}$$

Note that $u_{k,I}=0$ whenever $k\notin I$. So by definition of the feedback controls and the fact that $(u_{k,I})_{I\in\mathcal{I},k\in I}$ is a solution to the system of PDEs (37) we have

$$\begin{aligned}u_{k,I_0}(x) =&\mathbb{E}_{\hat{v}^1,\dots,\hat{v}^n}\left[\int_0^{\tau^k} e^{-\alpha s}\left(\sum_{I\in\mathcal{I}}\ 1_{\mathbf{I}^{I_0}_{s-}=I}\ f_{k,I}(X^x_s,(\hat{v}^j_s)_{j\in I})\right)\ ds\right]\\ =&J_k(x,I_0;\hat{v}^1,\dots,\hat{v}^n).\end{aligned} \tag{43}$$

It remains to show that $(\hat{v}^i)_{i=1,\dots,n}$ form a Nash equilibrium. Because of the definition of a Nash point for the Lagrangian (24) we have (using again (41) for $w(I,x):=\sum_{J\in\mathcal{I}}1_{J=I}u_{k,J}(x)$ and the fact that $u_{k,I}$ is a solution) for any admissible control v^k_s

$$\begin{aligned}u_{k,I_0}(x) =& \int_0^{\tau^k} e^{-\alpha s}\ \left(\sum_{I\in\mathcal{I}}\ 1_{\mathbf{I}^{I_0}_{s-}=I}H_{k,I}(X^x_s,\cdot)\right)\ ds\\ &-\int_0^{\tau^k} e^{-\alpha s}\left(\sum_{I\in\mathcal{I}}\ 1_{\mathbf{I}^{I_0}_{s-}=I}D_x u_{k,I}(X^x_s)\right)dB_s\\ &-\sum_{m\geq 1,T_m\leq\tau^k} e^{-\alpha T_m}\left(\sum_{I\in\mathcal{I}}(1_{\mathbf{I}^{I_0}_{T_m}=I}u_{k,I}(X^x_s) - 1_{\mathbf{I}^{I_0}_{T_m-}=I}u_{k,I}(X^x_s))\right)\\ \leq& \int_0^{\tau^k}\left(\sum_{I\in\mathcal{I}}\ 1_{\mathbf{I}^{I_0}_{s-}=I}L_{k,I}\left(X^x_s,\cdot;(\hat{v}^i(I,X^x_s))_{i\in I,i\neq k},v^k_s\right)\right)\ ds\\ &-\int_0^{\tau^k} e^{-\alpha s}\left(\sum_{I\in\mathcal{I}}\ 1_{\mathbf{I}^{I_0}_{s-}=I}D_x u_{k,I}(X^x_s)\right)dB_s\\ &-\sum_{m\geq 1,T_m\leq\tau^k} e^{-\alpha T_m}\left(\sum_{I\in\mathcal{I}}(1_{\mathbf{I}^{I_0}_{T_m}=I}u_{k,I}(X^x_s) - 1_{\mathbf{I}^{I_0}_{T_m-}=I}u_{k,I}(X^x_s))\right).\end{aligned} \tag{44}$$

Taking expectation with respect to $\mathbb{P}_{\hat{v}^1,\dots,v^k,\dots,\hat{v}^n}$ yields

$$\begin{aligned}u_{k,I_0}(x) \leq&\mathbb{E}_{\hat{v}^1,\dots,v^k,\dots,\hat{v}^n}\left[\int_0^{\tau^k} e^{-\alpha s}\left(\sum_{I\in\mathcal{I}}\ 1_{\mathbf{I}^{I_0}_{s-}=I}\ f_{i,I}(X^x_s,(\hat{v}^i_s)_{i\in I,i\neq k},v^k_s\right)\ ds\right]\\ =&J_k(x,I_0;\hat{v}^1,\dots,v^k,\dots,\hat{v}^n).\end{aligned} \tag{45}$$

□

7. L^∞ **estimates.** The following assumption is crucial for our main result:

Comparability Assumption

We assume there is a $0 < \Theta < 1$ such that for all $x \in \mathcal{O}$, $I \in \mathcal{I}$, $i \in I$

$$(\sum_{j\in I, j\neq i} \lambda_{j,I}(x,\cdot) + \sum_{j\in I^c} \mu_{j,I}(x,\cdot))(\alpha + \sum_{j\in I} \lambda_{j,I}(x,\cdot) + \sum_{j\in I^c} \mu_{j,I}(x,\cdot))^{-1} \leq \Theta. \quad (46)$$

We remind that the the coefficients $\lambda_{j,I}$, $\mu_{j,I}$ depend on the controls $(v_i)_{i\in I}$ only via the components $v_i'' \in \mathbb{R}_+^{m_2}$.

Remark 5. For n=2 we have the Comparability Assumption of [4]: there exists a $0 < \Theta < 1$, such that for all $x \in \mathcal{O}$ $v_1, v_2 \in V$ $i, j = 1, 2$ $j \neq i$

$$\lambda_{j,\{1,2\}}(x, v_1, v_2) \leq \frac{\Theta}{(1-\Theta)}(\lambda_{i,\{1,2\}}(x, v_1, v_2) + \alpha) \quad (47)$$

and for all $x \in \mathbb{R}^d$, $v \in V$ $i, j = 1, 2$ $j \neq i$

$$\mu_{j,\{i\}}(x, v) \leq \frac{\Theta}{(1-\Theta)}(\lambda_{i,\{i\}}(x, v) + \alpha). \quad (48)$$

In the general case we get the equivalent condition

$$\sum_{j\in I} \lambda_{j,I}(x,\cdot) + \sum_{j\in I^c} \mu_{j,I}(x,\cdot) \leq \frac{\Theta}{1-\Theta}(\alpha + \lambda_{i,I}(x,\cdot)). \quad (49)$$

Remark 6. (46) is always satisfied with $\Theta = 1$ due to the non-negativity assumptions. The difficulty concerning $\Theta < 1$ arises if components of the controls pass to infinity.

In the ("typical") case that the control restriction for the birth-death controlling components consists in requiring $v_i'' \geq 0$. The comparability is fulfilled if $\lambda_{i,I}(., (v_i)_{i\in I})$ (which are affine linear functions in v_i'') is a linear combination of all (!) the components of v_i'' with positive coefficients, plus the inhomogeneity (Assumption (A) (vi)). Evidently there are other conditions under which we have a comparability. Indeed (46) holds we have estimates for the coefficients of each v_i'' in $\sum_{j\in I} \lambda_{j,I}(., (v_i)_{i\in I}) + \sum_{j\in I^c} \mu_{j,I}(., (v_i)_{i\in I})$ by the coefficients of each v_i'' in $\lambda_{i,I}$.

The following Lemmas are essential. We prove them under a smoothness assumption which is fulfilled by the solutions of (39) (for $\delta > 0$ fixed), in case the coefficients are Hölder continuous in x. In case that the dependence of the data with respect to x is only measurable, we may use an approximation by convolution with common mollifiers. Thereby the coefficients keep the conditions of the previous assumptions, notably the comparability condition above, as well as the additional conditions below.

7.1. L^∞ **estimates: general case.** For the L^∞ estimates we need also the following assumption.

Boundedness Assumption for zero control

Assume for all $I \in \mathcal{I}$ $i \in I$ there exists a $\gamma_{i,I} < \infty$ such that

$$f_{i,I}(x, (v_k)_{k\in I, k<i}, 0, (v_k)_{k\in I, k>i}) \leq \gamma_{i,I}$$

for all $x \in \mathcal{O}$, $(v_k)_{k\in I, k\neq i}$.

Lemma 7.1. *Let* (x^*) *be an inner positive maximum point of* $\max\{(u_{i,I})_{I\in\mathcal{I}, i\in I}\}$. *Let* i^*, I^* *be* $\operatorname{argmax}_{I\in\mathcal{I}, i\in I}\{u_{i,I}(x^*)\}$. *Then*

$$u_{i^*,I^*}(x^*) \leq K. \quad (50)$$

Proof. In the interior maximum x^* we have

$$\begin{aligned}\alpha u_{i^*,I^*}(x^*) \leq& \gamma_{i^*,I^*} - \left(\sum_{j\in I^*} \lambda_{j,I^*}(x^*,\cdot) + \sum_{j\in (I^*)^c} \mu_{j,I^*}(x^*,\cdot) \right) u_{i^*,I^*}(x^*) \\ &+ \left(\sum_{j\in I^*, j\neq i^*} \lambda_{j,I^*}(x^*,\cdot)\; u_{i^*,I^*\setminus\{j\}}(x^*) \right) \\ &+ \left(\sum_{j\in (I^*)^c} \mu_{j,I^*}(x^*,\cdot)\; u_{i^*,I^*\cup\{j\}}(x^*) \right) \end{aligned} \tag{51}$$

(using the convention that sums over empty sets are zero). Since x^* was chosen such that $u_{i^*,I^*}(x^*) \geq u_{i,I}(x^*)$ for all $I \in \mathcal{I}, i \in I$

$$\begin{aligned}&\left(\alpha + \sum_{j\in I^*} \lambda_{j,I^*}(x^*,\cdot) + \sum_{j\in (I^*)^c} \mu_{j,I^*}(x^*,\cdot) \right) u_{i^*,I^*}(x^*) \\ &\leq \gamma_{i^*,I^*} + \left(\sum_{j\in I^*, j\neq i^*} \lambda_{j,I^*}(x^*,\cdot) + \sum_{j\in (I^*)^c} \mu_{j,I^*}(x^*,\cdot) \right) u_{i^*,I^*}(x^*) \end{aligned} \tag{52}$$

thus by the comparability assumption

$$\frac{\sum_{j\in I^*, j\neq i^*} \lambda_{j,I^*}(x^*,\cdot) + \sum_{j\in (I^*)^c} \mu_{j,I^*}(x^*,\cdot)}{\alpha + \sum_{j\in I^*} \lambda_{j,I^*}(x^*,\cdot) + \sum_{j\in (I^*)^c} \mu_{j,I^*}(x^*,\cdot)} \leq \Theta < 1 \tag{53}$$

we have

$$u_{i^*,I^*}(x^*) \leq (1-\Theta)^{-1} \left(\alpha + \sum_{j\in I^*} \lambda_{j,I^*}(x^*,\cdot) + \sum_{j\in (I^*)^c} \mu_{j,I^*}(x^*,\cdot) \right)^{-1} \gamma_{i^*,I^*}. \tag{54}$$

□

The proof simplifies the one in [4]. In a similar way one shows the boundedness from above under the following assumption:

Semi-boundedness Assumption
Assume for all $I \in \mathcal{I}$, $i \in I$ there exist constants $F_i^{*,I}$, such that for all $x \in \mathcal{O}$, $v \in V^{|I|}$

$$f_{i,I}(x,v) \geq -F_i^{*,I}.$$

Lemma 7.2. *Let* (x^*) *be an inner negative minimum point of* $\min\{(u_{i,I})_{I\in\mathcal{I},i\in I}\}$. *Let* i^*, I^* *be* $\operatorname{argmin}_{I\in\mathcal{I},i\in I}\{u_{i,I}(x^*)\}$. *Then*

$$u_{i^*,I^*}(x^*) \geq -K. \tag{55}$$

7.2. **Improved L^∞ criterion via sum-coerciveness.** We note that the semi-boundedness condition excludes cost functions like $f_{i,I}(x,v) = \frac{1}{2}|v_i|^2 + v_i v_j + c$. Our condition below is more adapted for applications, since for example, this allows to put sub-quadratic growing powers of non market control interactions in $f_{i,I}$ in all control variables. Furthermore it also allows additional bilinear non market interactions. Since the proof is involved, although not difficult, we present the idea only in the case $n = 2$.

Sum Assumption
Assume for $I = \{1\}, \{2\}, \{1,2\}$ the following sum condition: there exists a $0 < F_I < \infty$, such that

$$\sum_{i \in I} f_{i,I}(x, v) \geq -F_I$$

for all $x \in \mathcal{O}$, $v \in V^{|I|}$.

Lemma 7.3. *For $I = \{1\}, \{2\}, \{1,2\}$ let x_I^* be an inner negative minimum point of the function $\sum_{i \in I} u_{i,I}(x)$. Then*

$$\sum_{i \in I} u_{i,I}(x_I^*) \geq -K. \tag{56}$$

Proof. Let x_I^* be an inner negative minimum point of the function $\sum_{i \in I} u_{i,I}(x)$. We set for any $l \in I$ $\lambda_I^* = \lambda_{l,I}(x_I^*, \cdot)$ and $m \in I^c$ $\mu_{m,I}^* = \mu_{m,I}(x_I^*, \cdot)$. We have at the inner negative minimum point $x_{\{1,2\}}^*$ of $u_{1,\{1,2\}} + u_{2,\{1,2\}}$

$$\begin{aligned}&\Big(\alpha + \lambda_{1,\{1,2\}}^* + \lambda_{2,\{1,2\}}^*\Big)\Big(u_{1,\{1,2\}}(x_{\{1,2\}}^*) + u_{2,\{1,2\}}(x_{\{1,2\}}^*)\Big)\\ &\geq -F_{\{1,2\}} + \lambda_{1,\{1,2\}}^* u_{2,\{2\}}(x_{\{1,2\}}^*) + \lambda_{2,\{1,2\}} u_{1,\{1\}}(x_{\{1,2\}}^*).\end{aligned} \tag{57}$$

Thus for any $x \in \mathbb{R}^d$

$$\begin{aligned}u_{1,\{1,2\}}(x) + u_{2,\{1,2\}}(x) \geq{}& \Big(\alpha + \lambda_{1,\{1,2\}}^* + \lambda_{2,\{1,2\}}^*\Big)^{-1}\\ &\Big(-F_{\{1,2\}} + \lambda_{1,\{1,2\}}^* u_{2,\{2\}}(x_{\{1,2\}}^*) + \lambda_{2,\{1,2\}}^* u_{1,\{1\}}(x_{\{1,2\}}^*)\Big)\end{aligned} \tag{58}$$

which yields with the upper bound K for $u_{1,\{1,2\}}$, respectively $u_{2,\{1,2\}}$

$$\begin{aligned}u_{i,\{1,2\}}(x) \geq{}& -K + \Big(\alpha + \lambda_{1,\{1,2\}}^* + \lambda_{2,\{1,2\}}^*\Big)^{-1}\\ &\Big(-F_{\{1,2\}} + \lambda_{1,\{1,2\}}^* u_{2,\{2\}}(x_{\{1,2\}}^*) + \lambda_{2,\{1,2\}}^* u_{1,\{1\}}(x_{\{1,2\}}^*)\Big)\end{aligned} \tag{59}$$

for $i = 1, 2$.

Furthermore at the inner negative minimum point $x_{\{1\}}^*$ of $u_{1,\{1\}}$ we have

$$u_{1,\{1\}}(x_{\{1\}}^*) \geq \Big(\alpha + \lambda_{1,\{1\}}^* + \mu_{2,\{1\}}^*\Big)^{-1} (-F_{\{1\}} + \mu_{2,\{1\}}^* \; u_{1,\{1,2\}}(x_{\{1\}}^*)). \tag{60}$$

Now we can use the estimate of $u_{1,\{1,2\}}$ (59) to get

$$\begin{aligned}u_{1,\{1\}}(x_{\{1\}}^*) \geq{}& -\frac{F_{\{1\}}}{\alpha + \lambda_{1,\{1\}}^* + \mu_{2,\{1\}}^*}\\ &+ \frac{\mu_{2,\{1\}}^*}{\alpha + \lambda_{1,\{1\}}^* + \mu_{2,\{1\}}^*} \; \Big(-K + \Big(\alpha + \lambda_{1,\{1,2\}}^* + \lambda_{2,\{1,2\}}^*\Big)^{-1}\\ &\Big(-F_{\{1,2\}} + \lambda_{1,\{1,2\}}^* u_{2,\{2\}}(x_{\{1,2\}}^*) + \lambda_{2,\{1,2\}}^* u_{1,\{1\}}(x_{\{1,2\}}^*)\Big).\end{aligned} \tag{61}$$

Remember $x^*_{\{1,2\}}$ is the inner minimum of $u_{1,\{1,2\}} + u_{2,\{1,2\}}$. We rewrite:

$$\begin{aligned}
&u_{1,\{1\}}(x^*_{\{1\}}) + \frac{F_{\{1\}}}{\alpha + \lambda^*_{1,\{1\}} + \mu^*_{2,\{1\}}} + \frac{\mu^*_{2,\{1\}} K}{\alpha + \lambda^*_{1,\{1\}} + \mu^*_{2,\{1\}}} \\
\geq &- \frac{\mu^*_{2,\{1\}}}{\alpha + \lambda^*_{1,\{1\}} + \mu^*_{2,\{1\}}} \frac{F_{\{1,2\}}}{\alpha + \lambda^*_{1,\{1,2\}} + \lambda^*_{2,\{1,2\}}} \\
&+ \left(\frac{\mu^*_{2,\{1\}}}{\alpha + \lambda^*_{1,\{1\}} + \mu^*_{2,\{1\}}} \right) \left(\frac{\lambda^*_{1,\{1,2\}}}{\alpha + \lambda^*_{1,\{1,2\}} + \lambda^*_{2,\{1,2\}}} \right) u_{2,\{2\}}(x^*_{\{1,2\}}) \\
&+ \left(\frac{\mu^*_{2,\{1\}}}{\alpha + \lambda^*_{1,\{1\}} + \mu^*_{2,\{1\}}} \right) \left(\frac{\lambda^*_{2,\{1,2\}}}{\alpha + \lambda^*_{1,\{1,2\}} + \lambda^*_{2,\{1,2\}}} \right) u_{1,\{1\}}(x^*_{\{1,2\}}). \qquad (62)
\end{aligned}$$

Since $x^*_{\{1\}}$ is an inner minimum point of $u_{1,\{1\}}$ we have $u_{1,\{1\}}(x^*_{\{1,2\}}) \geq u_{1,\{1\}}(x^*_{\{1\}})$ and similarly $u_{2,\{2\}}(x^*_{\{1,2\}}) \geq u_{2,\{2\}}(x^*_{\{2\}})$. This gives with the comparability

$$\begin{aligned}
(1 - \Theta^2)\; u_{1,\{1\}}(x^*_{\{1\}}) \geq &- \frac{F_{\{1\}}}{\alpha + \lambda^*_{1,\{1\}} + \mu^*_{2,\{1\}}} - \frac{\mu^*_{2,\{1\}} K}{\alpha + \lambda^*_{1,\{1\}} + \mu^*_{2,\{1\}}} \\
&- \frac{\mu^*_{2,\{1\}}}{\alpha + \lambda^*_{1,\{1\}} + \mu^*_{2,\{1\}}} \frac{F_{\{1,2\}}}{\alpha + \lambda^*_{1,\{1,2\}} + \lambda^*_{2,\{1,2\}}} \\
&+ \left(\frac{\mu^*_{2,\{1\}}}{\alpha + \lambda^*_{1,\{1\}} + \mu^*_{2,\{1\}}} \right) \left(\frac{\lambda^*_{1,\{1,2\}}}{\alpha + \lambda^*_{1,\{1,2\}} + \lambda^*_{2,\{1,2\}}} \right) u_{2,\{2\}}(x^*_{\{2\}}). \qquad (63)
\end{aligned}$$

Similarly for the inner negative minimum point $x^*_{\{2\}}$ of $u_{2,\{2\}}$ we get

$$u_{2,\{2\}}(x^*_{\{2\}}) \geq \left(\alpha + \lambda^*_{2,\{2\}} + \mu^*_{1,\{2\}} \right)^{-1} (-F_{\{2\}} + \mu^*_{1,\{2\}}\; u_{2,\{1,2\}}(x^*_{\{2\}})). \qquad (64)$$

By the very same arguments as above we get en estimate from below for $u_{2,\{2\}}(x^*_{\{2\}})$

$$\begin{aligned}
(1 - \Theta^2) u_{2,\{2\}}(x^*_{\{2\}}) \geq &- \frac{F_{\{2\}}}{\alpha + \lambda^*_{2,\{2\}} + \mu^*_{1,\{2\}}} - \frac{\mu^*_{1,\{2\}} K}{\alpha + \lambda^*_{2,\{2\}} + \mu^*_{1,\{2\}}} \\
&- \frac{\mu^*_{1,\{2\}}}{\alpha + \lambda^*_{2,\{2\}} + \mu^*_{1,\{2\}}} \frac{F_{\{1,2\}}}{\alpha + \lambda^*_{1,\{1,2\}} + \lambda^*_{2,\{1,2\}}} \\
&+ \left(\frac{\mu^*_{1,\{2\}}}{\alpha + \lambda^*_{2,\{2\}} + \mu^*_{1,\{2\}}} \right) \left(\frac{\lambda^*_{2,\{1,2\}}}{\alpha + \lambda^*_{2,\{1,2\}} + \lambda^*_{1,\{1,2\}}} \right) u_{1,\{1\}}(x^*_{\{1\}}). \qquad (65)
\end{aligned}$$

Plugging (63) in (65) yields under comparability assumptions an estimate from below of $u_{1,\{1\}}$ and $u_{2,\{2\}}$. These we can use in the estimate for $u_{1,\{1,2\}}$ and $u_{2,\{1,2\}}$ (59) in order to conclude. □

8. **Proof of theorem 6.1.** With Lemma 7.1 and 7.2, respectively 7.3 we have L^∞ a priori estimates for the solution to (37), respectively (39).

Proposition 1. *Let $(u_{i,I})_{I \in \mathcal{I}, i \in I}$ be a solution to (37). Then*

$$\|u_{i,I}\|_{L^\infty(\mathcal{O})} \leq K \qquad (66)$$

for all $I \in \mathcal{I}$, $i \in I$ where K is a constant independent of δ.

In order to establish uniform estimates for $D_x u_{i,I}$ the L^∞ estimates are crucial.

Lemma 8.1. *Let $(u_I)_{I\in\mathcal{I}}$ be a solution of (39) in L^∞ then*

$$\|u_I\|_{W^{1,2}(\mathcal{O})} \le K(\|u_I\|_{L^\infty}, (\|u_{I\setminus\{j\}}\|_{L^\infty})_{j\in I}, (\|u_{I\cup\{j\}}\|_{L^\infty})_{j\notin I}). \tag{67}$$

Here the structure of the Hamiltonians given in Lemma 5.3 (35) allows to apply known results. The estimate for $I = \{k\}$ is rather simple since it reduces to a scalar case which was treated in [17]. The main idea consists in using the test function

$$e^{\gamma u_{k,\{k\}}} - e^{-\gamma u_{k,\{k\}}} \tag{68}$$

for $\gamma > 0$ sufficiently large. Then the elliptic part yields

$$\gamma \int_{\mathcal{O}} |D_x u_{k,\{k\}}|^2 (e^{\gamma u_{k,\{k\}}} + e^{-\gamma u_{k,\{k\}}}) dx \tag{69}$$

which dominates the quadratic terms in $D_x u_{k,\{k\}}$ in the Hamiltonian.

The estimates for general $I \in \mathcal{I}$ is much more involved, however our structure allows to use the known results of [1] for $n = 2$, and [3], resp. [9] for general n . The proof relies on the method of "iterated exponentials", i.e. using

$$(e^{\gamma u_{k,I}} - e^{-\gamma u_{k,I}}) \exp\left(\frac{\nu}{\gamma}\sum_{i\in I}\left(e^{\gamma u_{i,I}} - e^{-\gamma u_{i,I}}\right)\right) \tag{70}$$

as test function. For a related case we refer to [2]. We note that the sub-quadratic term $|\nabla u_I|^{2-\epsilon}$ can be treated easily using Young's inequality.

The crucial step for our proof of existence of a solution to the system (39) are Hölder estimates. We recall that for $\alpha > 0$, the Hölder space $C^\alpha(\mathcal{O})$ denotes the set of functions f, such that

$$[f]_{C^\alpha(\bar{\mathcal{O}})} = \sup_{x\neq x'\in\bar{\mathcal{O}}} \frac{|f(x) - f(x')|}{|x - x'|^\alpha} < \infty. \tag{71}$$

In order to show the Hölder continuity of the solution $(u_I)_{I\in\mathcal{I}}$ one can show for all $x_0 \in \mathcal{O}$ such that $B_{2\rho}(x_0) \in \mathcal{O}$

$$\int_{B_\rho(x_0)} |D_x u_I|^2 |x - x_0|^{2-n} \le K \int_{B_{2\rho}(x_0)\setminus B_\rho(x_0)} |D_x u_I|^2 |x - x_0|^{2-n} + C\rho^\beta \tag{72}$$

in a similar way as the $W^{1,2}$ estimates using the above iterated exponential test functions in a localised setting. Then one applies the hole filling technique of Widman [21] to get:

Lemma 8.2. *Let $(u_I)_{I\in\mathcal{I}}$ be a solution of (39) in L^∞, then $(u_I)_{I\in\mathcal{I}}$ satisfies the following Morrey condition: for all $x \in \mathcal{O}$*

$$\int_{\mathcal{O}\cap B_\rho(x)} |D_x u_I|^2 \le K(\|u_I\|_{L^\infty}, (\|u_{I\setminus\{j\}}\|_{L^\infty})_{j\in I}, (\|u_{I\cup\{j\}}\|_{L^\infty})_{j\notin I}))\rho^{d-2+2\alpha}. \tag{73}$$

To establish Hölder estimates one can use the Poincaré inequality and the characterisation of Campanato [8].

The $W^{2,p}$ estimates for $u_{i,I}$

$$\|u_{i,I}\|_{W^{2,p}(\mathcal{O})} \le K(\|u_I\|_{L^\infty}, (\|u_{I\setminus\{j\}}\|_{L^\infty})_{j\in I}, (\|u_{I\cup\{j\}}\|_{L^\infty})_{j\notin I}) \tag{74}$$

can then be shown using standard theory.

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