EXISTENCE AND BIFURCATION OF SUB-HARMONIC SOLUTIONS IN REVERSIBLE NON-AUTONOMOUS DIFFERENTIAL EQUATIONS

by

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This dissertation is dedicated to my family

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We study the existence of subharmonic solutions in the system $\ddot{u}(t) = f(t, u(t))$ with $u(t) \in \mathbb{R}^k$, where $f(t, u)$ is a continuous map that is p-periodic and even with respect to t and odd and Γ-equivariant with respect to u (with the linear action of a finite group Γ). The problem of finding mp-periodic solutions is reformulated in an appropriate functional space, as a nonlinear $\Gamma \times \mathbb{Z}_2 \times D_m$-equivariant equation. Under certain conditions on the linearization of f at zero and Nagumo growth condition on f at infinity, we prove the existence of an infinite number of subharmonic solutions by means of the Brouwer equivariant degree. In addition, we discuss the bifurcation of subharmonic solutions for the system depending on an extra parameter α.

TABLE OF CONTENTS

LIST OF FIGURES

CHAPTER 1

INTRODUCTION

1.1 Outline of Context

Consider $p > 0$ and the system

$$\ddot{u}(t) = f(t, u(t)), \quad u(t) \in \mathbb{R}^k, \tag{1.1}$$

where $f(t, u)$ is a continuous map, p-periodic with respect to t-variable. It is clear that any p-periodic solution $u(t)$ to (1.1) is also pm-periodic for any $m \in \mathbb{N}$, i.e.

$$u(t + pm) = u(t) \quad \text{and} \quad \dot{u}(t + pm) = \dot{u}(t). \tag{1.2}$$

A pm-periodic solution to (1.1) which is not p-periodic and its minimal period is larger than p, will be called *subharmonic*.

The problem of existence and multiplicity of subharmonic solutions for second order nonlinear systems of the type (1.1) has been thoroughly investigated by many authors using various methods and techniques. The starting point was the ground-breaking work by D. Birkhoff and D.C. Lewis (cf. [4]), in which the authors established (for a Hamiltonian system) the existence of a sequence of subharmonic solutions with large minimal periods. The classical work by P.H. Rabinowitz (cf. [31]), where the mini-max theory was applied, opened the door to the systematic usage of variational methods (see also [29, 30, 32, 33]). Among multiple variational techniques used in the context relevant to subharmonic solutions, one should mention the work of C. Conley and E. Zehnder (see [8]), where the Morse theory based method was applied, I. Ekeland and H. Hofer (see [13]), where the duality principle was utilized. We refer to iconic monograph by J. Mawhin and M. Willem [27] for a systematic exposition of different variational methods used to study periodic solutions in Hamiltonian systems (including symmetric setting). A rich arsenal of symplectic geometry based was also successfully used to study subharmonic solutions for Hamiltonian systems (see for example [7]), where the classical result of J. Franks (cf. [16]), generalizing the Poincaré-Birkhoff theorem, was used (see also [11]).

Subharmonic solutions to second order systems without Hamiltonian structure were effectively studied by methods based on the coincidence degree and upper and lower solutions (cf. [18, 26, 14, 15, 5]. We should mention that a main difficulty in using degree to detect periodic solutions in second order systems lays in the fact that it does not guarantee that the detected periodic solutions are non-constant. It turns out that this issue is intimately related to symmetric properties of the considered system, which is a part of a larger classical problem: *how do the symmetries of a system determine symmetries of the actual dynamics?* As a matter of fact the recently developed equivariant degree theory allows to study effectively spatio-temporal patters of subharmonic solutions to second order non-Hamiltonian systems.

1.2 Statement of the Problem

In this dissertation we study the existence of subharmonic solutions of the system (1.1) for a continuous function $f : \mathbb{R} \times \mathbb{R}^k \to \mathbb{R}^k$ satisfying the following conditions:

(A_1) For all $t \in \mathbb{R}$ and $x \in \mathbb{R}^k$ we have $f(t + 2\pi, x) = f(t, x)$;

(A_2) For all $t \in \mathbb{R}$ and $x \in \mathbb{R}^k$ we have $f(-t, x) = f(t, x)$;

(A_3) For all $t \in \mathbb{R}$ and $x \in \mathbb{R}^k$ we have $f(t, -x) = -f(t, x)$.

Observe that in condition (A_1) one could assume that $f(t + p, x) = f(t, x)$ for some $p > 0$. However, by rescaling the time t, one can always obtain that $p = 2\pi$. That means this assumption doesn't lead to a loss of generality. Condition (A_1) implies that (on an appropriate functional space of $2\pi m$-periodic functions) (1.1) is equivalent to a $\mathbb{Z}_m$-equivariant operator equation. Condition (A_2) states the time-reversibility of the system (1.1), and as such, gives rise to the dihedral symmetries D_m. Condition (A_3) implies that the system (1.1) admits the antipodal $\mathbb{Z}_2$-symmetry. In other words, conditions (A_1)–(A_3) express the symmetric properties of (1.1), i.e. they allow to reformulate the problem of existence of subharmonic $2\pi m$-periodic solutions as a nonlinear operator equation with the $D_m \times \mathbb{Z}_2$-symmetries. Notice that the interplay between the D_m- and $\mathbb{Z}_2$-actions, in particular allows to distinguish non-constant solutions from the constant ones and is used to describe spatio-temporal patterns of subharmonic solutions.

Many problems of natural phenomena can be described as symmetric networks of coupled oscillators. Symmetries of couplings give rise to an extra spatial symmetry group Γ for the system (1.1) (see assumption (A_4) in section 2). Thus, the operator equation associated with (1.1) becomes an equivariant equation with respect to the group

$$G := \Gamma \times D_m \times \mathbb{Z}_2.$$

We point out that we do not require f to be of a gradient-type or to satisfy any differentiability properties, except for the existence of the linearization at 0 (see assumption (A_5) in subsection 4.2). The symmetric character of the problem together with mild differentiability requirements motivate the choice of the methodology.

1.3 Methodology

The Brouwer G-equivariant degree $G\text{-deg}(\boldsymbol{f}, \Omega)$ of a G-equivariant map $\boldsymbol{f} : V \to V$ on an open bounded G-invariant set $\Omega \subset V$ (here we assume that $\boldsymbol{f}$ is Ω-*admissible*, i.e. $\boldsymbol{f}(x) \neq 0$ for $x \in \partial\Omega$) is a finite collection of integers that are defined using the usual Brouwer degrees $d_H := \deg(\boldsymbol{f}^H, \Omega^H)$ of $\boldsymbol{f}$ on the H-fixed point set Ω^H (here (H) is an orbit type in Ω, cf. [2, 3]). To be more explicit, the equivariant degree is as an element of the free $\mathbb{Z}$-module $A(G)$ generated by the set $\Phi_0(G)$ consisting of the conjugacy classes (H) of subgroups H of

G with a finite Weyl group $W(H)$. The element $G\text{-deg}(\boldsymbol{f},\Omega)$ can be written as

$$G\text{-deg}(\boldsymbol{f},\Omega)=\sum_{(H)} n_H\,(H),\quad n_H\in\mathbb{Z}, \tag{1.3}$$

where the coefficients n_H are given by the following recurrence formula (with respect to the natural order relation in $\Phi_0(G)$)

$$n_H=\frac{1}{|W(H)|}\left(d_H-\sum_{(L)>(H)} n_L\, n(H,L)\,|W(L)|\right), \tag{1.4}$$

(see Preliminaries, Chapter 2, for more details). One can immediately recognize a connection between the two collections: $\{d_H\}$ and $\{n_H\}$, where $H\leq G$ and $W(H)$ is finite. Since d_H is just a classical Brouwer degree, it is obvious that $G\text{-deg}(f,\Omega)$ satisfies the standard degree-theoretical properties such as the *existence, additivity* and (equivariant) *homotopy* properties (see Preliminaries for more details). Furthermore, $G\text{-deg}(f,\Omega)$ admits an additional property, the so-called *product property* (see Chapter 3), which is important for the computation of the equivariant degree. To be more precise, the module $A(G)$ has a natural structure of a ring (which is called the *Burnside ring* of G) with the multiplication $\cdot : A(G)\times A(G)\to A(G)$, defined on generators by $(H)\cdot(K)=\sum_{(L)} m_L\,(L)$, where the integer m_L denotes the number of (L)-orbits in the G-space $G/H\times G/K$. The *product property* for two admissible G-pairs $(\boldsymbol{f}_1,\Omega_1)$ and $(\boldsymbol{f}_2,\Omega_2)$ means the following equality

$$G\text{-deg}(\boldsymbol{f}_1\times\boldsymbol{f}_2,\Omega_1\times\Omega_2)=G\text{-deg}(\boldsymbol{f}_1,\Omega_1)\cdot G\text{-deg}(\boldsymbol{f}_2,\Omega_2), \tag{1.5}$$

where the multiplication '$\cdot$' is taken in the Burnside ring $A(G)$. This property allows to effectively compute the full value of the G-equivariant Brouwer degree $G\text{-deg}(\boldsymbol{\mathcal{A}},B(V)$ (for G-equivariant isomorphisms $\boldsymbol{\mathcal{A}}:V\to V$) based on the information provided by the equivariant spectral decomposition of $\boldsymbol{\mathcal{A}}$ and the list of so-called *basic degrees* $\deg_{\mathcal{V}_k}:=G\text{-deg}(-\mathrm{Id}\,,B(\mathcal{V}_k)$ (here $\mathcal{V}_k$ stand for an irreducible G-representation and $B(\mathcal{V}_k)$ the unit ball in $\mathcal{V}_k$). Such a list can be 'prefabricated' in advance for any group G and, in many cases, these computations can be easily assisted by computer GAP programming, especially for large symmetry groups.

The idea used in this paper is the following: the problem (1.1)–(1.2) is set up as a nonlinear operator equation $\mathscr{F}(u) = 0$ in the functional space $\mathscr{E} := C^2_{2\pi m}(\mathbb{R};\mathbb{R}^k)$. The condition (A_5) assures the linearization $\mathscr{A} := D\mathscr{F}(0)$ at 0 exists and is non-degenerate. Thus, for sufficiently small $\varepsilon > 0$, the degree $G\text{-deg}(\mathscr{F}, \Omega_\varepsilon) = G\text{-deg}(\mathscr{A}, \Omega_\varepsilon)$ is well-defined in the open ball $\Omega_\varepsilon := B_\varepsilon(0)$. By the Nagumo growth condition (A_6), for a sufficiently large $R > 0$ we have $G\text{-deg}(\mathscr{F}, \Omega_R) = G\text{-deg}(\mathrm{Id}\,, \Omega_R) = (G)$ in the open ball $\Omega_R := B_R(0)$. Therefore, by the additivity property, we obtain

$$G\text{-deg}(\mathscr{F}, \Omega) = (G) - G\text{-deg}(\mathscr{A}, \Omega_\varepsilon), \quad \Omega := \Omega_R \setminus \overline{\Omega_\varepsilon}.$$

The degree $G\text{-deg}(\mathscr{F}, \Omega)$ (by the existence property) contains all the information needed to establish the existence of subharmonic $2\pi m$-periodic solutions. Symmetries of obtained solutions allows: (a) to provide multiplicity results, (b) to distinguish non-constant solutions from the constant ones and (c) to distinguish subharmonic solutions having different frequencies (see Theorems 4.5.1, 4.6.4). The main technical difficulty here lies in implementing the algorithms to compute the degree $G\text{-deg}(\mathscr{A}, \Omega)$. For this purpose, we use the equivariant degree package `EquiDeg` for the specific groups Γ considered in this dissertation. The package `EquiDeg` for GAP programming was created by Hao-Pin Wu and is available at `https://github.com/psistwu/equideg` (cf. [28]).

1.4 Overview of Dissertation

In Chapter 2 we present some preliminaries providing the theoretical background to the equivariant definitions and notations used in this dissertation. In particular we review some facts about Lie groups and the representation theory. In chapter 3, the Burnside ring, and the basic properties of Brouwer G-equivariant degree. Computational formulae for the

Brouwer G-equivariant degree of linear maps is also included. In Chapter 4, section 4.1, the system (1.1) is reformulated as a nonlinear G-equivariant equation $\mathscr{F}(u) = 0$ in the space $\mathscr{E} := C^2_{2\pi m}(\mathbb{R}; V)$. Then, in section 4.2, we study the linearization operator $\mathscr{A} := D\mathscr{F}(0)$, in section 4.3, we describe the G-isotypic decomposition of $\mathscr{E}$. The Nagumo growth condition is introduced in section 4.4 and the abstract existence result is obtained in section 4.5 (see Theorem 4.5.1). The existence of subharmonic solutions in the absence of additional symmetries (Theorem 4.6.4) is presented in section 4.6. In section 4.7, we show two examples of a Γ-symmetric system with $\Gamma = D_m$, for $m = 3$, 4, for which the exact value of the degree $G\text{-deg}(\mathscr{F}, \Omega)$ is obtained and the multiple subharmonic solutions with their symmetries are described. In section 4.8, we consider a bifurcation problem of subharmonic solutions for a parametrized system (4.25). The main result in the case of absence of additional symmetries is studied in Theorem 4.8.2 (section 4.8.1). An example of symmetric bifurcation for $\Gamma = D_3$ is worked out in section 4.8.2.

CHAPTER 2

PRELIMINARIES

2.1 Lie Groups and Their Properties

Recall a formal definition of a manifold: by a *smooth n-dimensional manifold* we mean a topological Hausdorff space M, satisfying the *second countability axiom* (i.e. a space admitting a countable basis of topology) together with a family $\{U_i, \varphi_i\}_{i \in I}$ (called an *atlas* on M) such that $\{U_i\}_{i \in I}$ is an open cover of M, $\varphi_i : U_i \to \mathbb{R}^n$, $i \in I$, is a homeomorphism onto an open set $\varphi_i(U_i)$ in $\mathbb{R}^n$, and for all i, $j \in I$, $\varphi_j \circ \varphi_i^{-1} : \varphi_i(U_i \cap U_j) \to \varphi_j(U_i \cap U_j)$ is a smooth map (i.e. C^∞-differentiable diffeomorphism). The pair (U_i, φ_i) is called a *chart* on M. For a point $x \in M$ and a chart (U, φ) such that $x \in U$, $\varphi^{-1}(U) \to M$ is called a *local coordinate system* near x.

A map $F : N \to M$ between two smooth manifolds is called *smooth* if for any $x \in N$, F is a smooth map with respect to some local coordinate system near x, i.e. there exist charts (U, φ) and (V, ψ) on N and M respectively such that $x \in U$, $F(U) \subset V$ and

$$\psi \circ F \circ \varphi^{-1} : \varphi(U) \to \psi(V) \tag{2.1}$$

is a smooth map. A point $x \in N$ is said to be *regular* if, in some charts (U, φ) and (V, ψ), the differential of (2.1) at x is surjective; otherwise x is called *critical.* A point $y \in M$ is called a *regular value* of F if its inverse image $F^{-1}(y)$ is either empty or consists of regular points only.

By combining the concepts of a group and a manifold one arrives at the notion of a *Lie group.* Let us briefly survey basic definitions and facts related to Lie groups.

Definition 2.1.1. Let G be simultaneously a group and (smooth) manifold. Then, G is called a *Lie group* if the group multiplication $p : G \times G \to G$ given by $p(u, v) = uv$ and inversion $i : G \to G$ given by $i(u) = u^{-1}$ are both smooth maps.

Several examples of Lie group will play an important role in this dissertation:

1. Any finite group is a zero-dimensional Lie group; $\mathbb{Z}$ considered with discrete topology is a Lie group, the cyclic group $\mathbb{Z}_n := \{z \in \mathbb{C} : z^n = 1\}$, $n \in \mathbb{N}$ is also a zero dimensional Lie group,

2. the set $O(n)$ consisting of all $(n \times n)$-matrices A satisfying $A^T = A^{-1}$, $n \in \mathbb{N}$, is $\frac{n(n-1)}{2}$-dimensional Lie group.

3. the set D_n, $n \in \mathbb{N}$, consisting of all matrices $A \in O(2)$, which are symmetries of a regular n-gon. It is convenient to identify D_n with the group $\mathbb{Z}_n \cup \mathbb{Z}_n\kappa$, where $e^{i\frac{2\pi k}{n}} \in \mathbb{Z}_n$, $k = 01, 2, \dots, n-1$, represents a rotation of plane by the angle $\frac{2\pi k}{n}$ and κ is the matrix of the complex conjugation.

4. For a real vector space V of dimnension n. Denote by $GL(V)$ the set of all linear operators such that $A^{-1} : V \to V$ exists. Then, $GL(V)$ is a n^2-dimensional Lie group.

It is important to mention that a closed subgroup of a Lie group is also a Lie group and a continuous homomorphism between two Lie groups is always smooth (see [19] for more details).

2.2 Weierstrass Theorem and Sard's Lemma

Weierstrass Theorem provides a passage from smooth maps to continuous ones by their small perturbation.

Proposition 2.2.1 (Weierstrass Theorem). *Let $\Omega \subset \mathbb{R}^n$ be an open set, $K \subset \Omega$ a compact subset and $f : \Omega \to \mathbb{R}^k$ a continuous map. Then, for any $\varepsilon > 0$, there exists a smooth map $g : \Omega \to \mathbb{R}^k$ such that*

$$\sup\{\|f(x) - g(x)\| : x \in K\} < \varepsilon. \tag{2.2}$$

The following result, commonly known as *Sard's Lemma* provides a passage from a non-regular value of a smooth map to the regular one for a slightly perturbed map.

Theorem 2.2.2 (Sard's Lemma)**.** *Let M be a manifold, K a compact subset of M and $f : M \to \mathbb{R}^k$ a smooth map. Then:*

(i) the set of all critical values of the map $f|_K$ has Lebesgue measure zero in $\mathbb{R}^k$, i.e. $\mu(f(C_K)) = 0$, where $C_K := \{x \in K : x \text{ is a critical point of } f\}$ and μ stands for the Lebesgue measure in $\mathbb{R}^k$;

(ii) the set of all regular values of $f|_K$ is open and dense in $\mathbb{R}^k$.

We refer to [12, 20] for the proofs and more details.

2.3 Brouwer Local Degree

Let V be a Euclidean space (i.e. $V = \mathbb{R}^n$ for some $n = 1, 2, 3, \dots$) and consider a continuous map $f : V \to V$. Assume that $\Omega \subset V$ is a bounded open subset such that $f(x) \neq 0$ for $x \in \partial\Omega$. Then, the pair (f, Ω) is called an *admissible pair* in V and f is called an Ω-*admissible map*. For a continuous map $F : [0, 1] \times V \to V$, we put $F_t(x) := F(t, x)$, $x \in V$, and we call F a *homotopy* between F_0 and F_1. If, in addition, F_t is Ω-admissible for every $t \in [0, 1]$, then F is called Ω-*admissible homotopy*. Then, we will also say that F_0 and F_1 are Ω-*admissibly homotopic*. In what follows, we will denote by $\mathcal{M}(V)$ the set of all admissible pairs in V, and put

$$\mathcal{M} := \bigcup_V \mathcal{M}(V), \tag{2.3}$$

where the union is taken over all finite-dimensional normed spaces V.

The following theorem provides us with the concept of the *Local Brouwer Degree*.

Theorem 2.3.1. *There exists a function* $\deg : \mathcal{M} \to \mathbb{Z}$*, called the (local Brouwer) degree, satisfying the following three conditions:*

(P1) (Additivity) For every $(f,\Omega) \in \mathcal{M}$ *and two open disjoint subsets* Ω_1 *and* Ω_2 *of* Ω *such that* $f^{-1}(0) \cap \Omega \subset \Omega_1 \cup \Omega_2$*, one has* $\deg(f,\Omega) = \deg(f,\Omega_1) + \deg(f,\Omega_2)$.

(P2) (Homotopy) For any open bounded set $\Omega \subset V$*, if* $F : [0,1] \times V \to V$ *is an* Ω*-admissible homotopy, then* $\deg(F_t,\Omega) =$ *constant for all* $t \in [0,1]$.

(P3) (Normalization) For any open bounded set $\Omega \subset V$ *and* $x_o \in V$ *such that* $x_o \notin \partial\Omega$*, one has* $\deg(\mathrm{Id} - x_o, \Omega) = \begin{cases} 1 & \text{if } x_o \in \Omega \\ 0 & \text{if } x_o \notin \Omega \end{cases}$.

The properties (P1)–(P3) allow easily to deduct additional properties of the Brouwer degree which is presented next. We focus on the standard properties following from conditions (P1)–(P3).

Proposition 2.3.2. *Suppose that* $\deg : \mathcal{M} \to \mathbb{Z}$ *is a degree. Then, one has the following additional properties*

(P4) (Existence) Suppose that for $(f,\Omega) \in \mathcal{M}$*, one has* $\deg(f,\Omega) \neq 0$*. Then, there exists* $x_o \in \Omega$ *such that* $f(x_o) = 0$.

(P5) (Excision) Suppose that $(f,\Omega) \in \mathcal{M}$ *and* Ω_o *is an open subset of* Ω *such that* $f^{-1}(0) \cap \Omega \subset \Omega_o$*. Then,* $\deg(f,\Omega) = \deg(f,\Omega_o)$.

(P6) (Rouché's Property) Suppose that (f,Ω) *is an admissible pair in* V *and let* $g : V \to V$ *be a continuous map such that*

$$\sup_{x\in\partial\Omega} |f(x) - g(x)| < \inf_{x\in\partial\Omega} |f(x)|. \tag{2.4}$$

Then, $(g,\Omega) \in \mathcal{M}$ *and* $\deg(f,\Omega) = \deg(g,\Omega)$.

(P7) (Boundary) If (f,Ω) *is an admissible pair in* V *and* $g : V \to V$ *is a continuous map such that* $f(x) = g(x)$ *for all* $x \in \partial\Omega$*, then* $\deg(f,\Omega) = \deg(g,\Omega)$.

Definition 2.3.3. Assume that (f, Ω) is an admissible pair in V. We say that f is a *regular* Ω-admissible map if f is smooth and zero is a regular value of $f|_\Omega$.

The following results provides a practical formula for the computation of the Local Brouwer degree

Theorem 2.3.4. *Assume that Ω is an open bounded set in $V := \mathbb{R}^n$ and $f : V \to V$ an Ω-admissible C^1-map such that 0 is a regular value of $f|_\Omega$. Then, $f^{-1}(0) \cap \Omega = \{x_1, x_2, \dots, x_m\}$ is a finite set and*

$$\deg(f, \Omega) = \sum_{j=1}^{m} \text{sign } \det Df(x_j). \tag{2.5}$$

In addition we have the following

Proposition 2.3.5. *The degree* $\deg : \mathcal{M} \to \mathbb{Z}$ *has the following property:*

(P8) (Product) *If (f, Ω), $(g, U) \in \mathcal{M}$, then*

$$\deg(f \times g, \Omega \times U) = \deg(f, \Omega) \cdot \deg(g, U).$$

For the existence and uniqueness of the Brouwer local degree we refer to [10] (where the reader can also find all that is related to the proofs in this material). We also refer to books [23, 24] for additional reading.

2.4 Elements of Equivariant Topology

Assume that G is a compact Lie group. Here, we assume all the considered subgroups of G to be closed.

For a subgroup H of G, i.e. $H \leq G$, we denote the normalizer of H in G by $N(H)$ and the Weyl group of H by $W(H) = N(H)/H$. We say K is *conjugate* to H if $\exists_{g\in G}$ with $K = g^{-1}Hg$, and we will write that $K \sim H$, i.e., $(H) := \{K : K \sim H\}$. The symbol (H) stands for the conjugacy class of H in G. We put $\Phi(G) := \{(H) : H \leq G\}$, i.e. $\Phi(G)$ is the set of conjugacy classes of subgroups in G. $\Phi(G)$ has a natural partial order defined by $(H) \leq (K)$ if and only if $\exists g \in G$ such that $gHg^{-1} \leq K$. We also put $\Phi_0(G) := \{(H) \in \Phi(G) : W(H) \text{ is finite}\}$. For two subgroups L, $H \in \Phi_0(G)$, we denote by $n(L, H)$ the number of different subgroups H' conjugate to H such that $L \leq H'$.

Definition 2.4.1. *Let X be a Hausdorff topological space, G a Lie group and $\varphi : G \times X \to X$ a continuous map such that:*

(i) $\varphi(g, \varphi(h, x)) = \varphi(gh, x)$ *for all* g, $h \in G$ *and* $x \in X$;

(ii) $\varphi(e, x) = x$ *for all* $x \in X$, *where* e *is the identity element of* G.

Then, φ is called a (left) G-action. In such a case, we will call X a G-space. We shall use the notation $g(x)$, $g \cdot x$ or simply gx, for $\varphi(g, x)$.

For $K \subset G$ and $A \subset X$, put $K(A) := \{gx : g \in K, \ \ x \in A\}$, and for $g \in G$, denote $gA := \{gx : x \in A\}$. A set $A \subset X$ is said to be *G-invariant*, if $G(A) = A$. Obviously, if A and G are compact, then so is $G(A)$.

For a G-space X and $x \in X$, we denote the *isotropy group* of x by $G_x := \{g \in G : gx = x\}$, the *orbit* of x by $G(x) := \{gx : g \in G\}$, and we call (G_x) the *orbit type* of $x \in X$. For a subgroup $H \leq G$ the subspace $X^H := \{x \in X : G_x \geq H\}$ is called the *H-fixed-point subspace* of X. We also put $X_H := \{x \in X : G_x = H\}$. Clearly, $W(H)$ acts on X^H and $W(H)$ acts freely on X_H.

In addition, for a G-space X and a *closed* subgroup H of G, we will use the following notations

$$X^{(H)} := \{x \in X : (G_x) \geq (H)\},$$
$$X_{(H)} := \{x \in X : (G_x) = (H)\}.$$

For two G-spaces X and Y, we say that a continuous map $f : X \to Y$ is G-equivariant if and only if $\forall_{g\in G}\forall_{x\in X}\ f(gx) = gf(x)$. We also say that $f : X \to \mathbb{R}$ is G-invariant if $\forall_{g\in G}\forall_{x\in X}$

$f(gx) = f(x)$. One can easily notice that for a G-equivariant map $f := X \to Y$, we have $\forall_{H \leq G}\ f(X^H) \subset Y^H$.

We refer to the book [22] for details, proofs and more reading about equivariant topological notions.

2.5 Elements of Representation Theory

Let G be a compact Lie group. We have the following definition.

Definition 2.5.1. Let V be a finite-dimensional (real) vector space. We say that V is a (real) *representation* of G (in short, *G-representation*), if V is a G-space such that the *translation map* $T_g : V \to V$ defined by $T_g(v) := gv$ for $v \in V$ is a linear operator for every $g \in G$.

It is easy to see that for a G-representation V, the map $T : G \to GL(V)$ given by $T(g) := T_g$ is a continuous homomorphism. If a basis in V is chosen, then one can associate with each T_g a matrix. In this case, a continuous homomorphism $T : G \to GL(n; \mathbb{R})$ is called a *matrix G-representation*. To make a connection to the space V, we may sometimes use the symbol T_V instead of T. Also, it is convenient to identify T_V with V. So, we will simply say that V is a G-representation. For two representations V_1 and V_2 of G, a homomorphism $A : V \to W$ is a linear map that is equivariant and they are represented by $L_G(V, W)$. The homomorphism is called an isomorphism if it is also a linear isomorphism. For two G-representations V_1 and V_2, we say that V_1 and V_2 are *equivalent* and write $V_1 \cong V_2$, if there is a G-equivariant isomorphism from $V_1 \to V_2$. From the viewpoint of representation theory, equivalent representations can be identified.

Let V be a G-representation. An inner product$\langle \cdot, \cdot \rangle : V \oplus V \to \mathbb{R}$ (resp. $\langle \cdot, \cdot \rangle : W \oplus W \to \mathbb{C}$) is called *$G$-invariant* if $\langle gu, gv \rangle = \langle u, v \rangle$ for all $g \in G$, u, $v \in V$. A G-representation together with a G-invariant inner product is called *orthogonal*. Obviously, orthogonal and unitary G-representations are easier to handle than arbitrary ones. It is well-known that for a compact Lie group G any G-representation is equivalent to an orthogonal one. An invariant linear subspace $\tilde{V} \subset V$ is called a *subrepresentation* of V, and V is an *irreducible* representation if it has no subrepresentation different from $\{0\}$ and V.

G.F. Frobenius suggested a concept of a character of representation which is both powerful and computable.

Definition 2.5.2. Let $T : G \to GL(V)$ be a finite-dimensional representation. The function $\chi_V : G \to \mathbb{K}$ given by

$$\chi_V(g) := \mathrm{Tr}(T(g)),$$

where Tr stands for the trace, is called a *character of the representation* T.

It follows immediately from the properties of the trace Tr that for two G-representations V and V', the following holds:

$$\forall_{g,h\in G} \quad \chi_{V\oplus V'}(g) = \chi_V(g) + \chi_{V'}(g) \quad \text{and} \quad \chi_W(hgh^{-1}) = \chi_W(g) \tag{2.6}$$

an important property of the character is formulated in the next proposition.

Proposition 2.5.3. *Let V_1 and V_2 be two arbitrary orthogonal G-representations. Then, V_1 and V_2 are equivalent if and only if*

$$\forall_{g\in G} \quad \chi_{V_1}(g) = \chi_{V_2}(g).$$

Remark 2.5.4. Conventions and Notations: (i) For a given compact Lie group G, denote by $\mathrm{Irr}(G)$ a complete collection of pairwise non-equivalent irreducible orthogonal G-representations. So, each irreducible G-representation is equivalent to exactly one element of $\mathrm{Irr}(G)$. We will assume that elements of $\mathrm{Irr}(G)$ are arranged into a list $\mathcal{V}_0$, $\mathcal{V}_1$, $\mathcal{V}_2$, ..., where the index zero is reserved for the trivial G-representations.

(ii) If G is finite, then the list $\mathrm{Irr}(G)$ of the characters of irreducible G-representations can be easily generated, for example, by such a programming platform as G.A.P.

Take a compact Lie group G and consider an orthogonal G-representation V. Given the list $\mathrm{Irr}(G) = \{\mathcal{V}_l\}_{l=0}^{\infty}$ (see Remark 2.5.4), then there exist the orthogonal projections $P_l : V \to V$ such that the subspaces $V_l := P_l(V)$, $l = 0, 1, 2, \ldots$, satisfy the properties

(i) V_l is G-invariant;

(ii) P_l is G-equivariant;

(iii) P_l is a projection onto V_l (i.e., $P_l \circ P_l = P_l$);

(iv) P_l is symmetric, i.e., $P_l^t = P_l$ (and therefore, orthogonal);

(v) if $x \in V_j$ and $j \neq l$, then $P_l(x) = 0$;

(vi) if $x \in V_l$, then $P_l(x) = x$.

Then the decomposition of V into a direct sum

$$V = V_0 \oplus V_1 \oplus \cdots \oplus V_r, \tag{2.7}$$

is called *G-isotypic decomposition of V* and the isotypic component V_i is said to be modeled on the irreducible G-representation $\mathcal{V}_l$.

Denote the $\mathbb{R}$-algebra (resp. group) of all G-equivariant linear (resp. invertible) operators on V by $\mathrm{L}^G(V)$ (resp. $\mathrm{GL}^G(V)$) . Clearly, the isotypic decomposition (2.7) induces the following direct sum decomposition of $\mathrm{GL}^G(V)$:

$$\mathrm{GL}^G(V) = \bigoplus_{i=0}^{r} \mathrm{GL}^G(V_i), \tag{2.8}$$

where for every

$$\mathrm{GL}^G(V_i) \simeq \mathrm{GL}(m_i, \mathbb{F}), \quad m_i = \dim V_i / \dim \mathcal{V}_i$$

and depending on the type of the irreducible representation $\mathcal{V}_i$, $\mathbb{F}$ ($= \mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$) is a finite-dimensional division algebra $\mathrm{L}^G(\mathcal{V}_i)$.

We refer to [6, 17, 35] for more theoretical background on the representation theory.

2.5.1 Describing subgroups of $\Gamma \times \mathbb{Z}_2$ (Twisted Notation)

Given a group Γ for example D_n, we consider the group $G := \Gamma \times \mathbb{Z}_2$. To describe (up to conjugacy classes) subgroups of the product $\Gamma \times \mathbb{Z}_2$. We notice there are two types of such subgroups

(a) The *product subgroups* $H \times \mathbb{Z}_2$. For $H \leq \Gamma$, we put $H^p := H \times \mathbb{Z}_2$

(b) The *twisted subgroups*; $H^\varphi \leq \Gamma \times \mathbb{Z}_2$ where $H \leq \Gamma$ and $\varphi : H \to \mathbb{Z}_2$ is a homomorphism,

$$H^\varphi : \{(h, z) \in \Gamma \times \mathbb{Z}_2 : \varphi(h) = z\}$$

We call the notation H^φ the twisted notation. For a trivial homomorphism $\varphi : H \to \{1\} \subset \mathbb{Z}_2$, the subgroup $H^\varphi = H \times \{1\}$ is simply denoted by H

Example 2.5.5. $G = D_3 \times \mathbb{Z}_2$, we classify the conjugacy classes of subgroups in G using twisted notation (and abbreviation)

(a) $\mathbb{Z}_1^p, \mathbb{Z}_3^p, D_1^p, D_3^p$

(b) $\mathbb{Z}_1, \mathbb{Z}_3, D_1, D_3$

(c) The homomorphism $z : D_3 \to \mathbb{Z}_2$ (resp. $z : D_1 \to \mathbb{Z}_2$) with $\operatorname{Ker} z = \mathbb{Z}_3$ (resp. $\operatorname{Ker} z = \mathbb{Z}_1$) we have D_3^z (resp. D_1^z)

2.5.2 Notation for subgroups in product group $G_1 \times G_2$ (Amalgamated Notation)

We consider the product group $G_1 \times G_2$ of two groups G_1 and G_2. The result of Goursat (see [9, 21]) provides a description of subgroups of the product group $G_1 \times G_2$. Namely, for any subgroup H of the product group $G_1 \times G_2$, there exists subgroups $H_1 \leq G_1$ and $H_2 \leq G_2$, a group L, and two epimorphisms $\psi_1 : H_1 \to L$ and $\psi_2 : H_2 \to L$ such that

$$H = \{(h_1, h_2) \in H_1 \times H_2 : \psi_1(h_1) = \psi_2(h_2)\}. \tag{2.9}$$

In this case, we use the notation

$$H =: H_1 {}^{\psi_1}\!\times_L^{\psi_2} H_2$$

and the group $H_1 {}^{\psi_1}\!\times_L^{\psi_2} H_2$ is called an *amalgamated* subgroup of $G_1 \times G_2$. As a particular case, we consider the group $\Gamma \times D_m \times \mathbb{Z}_2$ and its conjugacy classes of subgroups.

For any closed subgroup H of $\Gamma \times (D_m \times \mathbb{Z}_2)$, we will apply amalgamated notation $H^{\psi_1} \times_L^{\psi_2} H_2$, where $H_1 \leq \Gamma$ and $H_2 \leq D_m \times \mathbb{Z}_2$. To simplify the amalgamated subgroup notation and make it easy to understand, we put

$$L = H_2/\mathrm{Ker}\,(\psi_2).$$

With this condition, $\psi_2 : H_2 \to L$ is the natural projection, and there is no need to indicate it. On the other hand, the epimorphism ψ_2 can be identified by its kernel R. Since we are interested in describing conjugacy classes of H, we can identify the epimorphism $\psi_1 : H_1 \to L$ by indicating $Z = \mathrm{Ker}\,(\psi_1)$. Then, instead of using the notation $H_1^{\psi_1} \times_L^{\psi_2} H_2$, we write

$$H =: H_1 {}^{Z}\!\times_L^{R} H_2. \tag{2.10}$$

CHAPTER 3

BROUWER EQUIVARIANT DEGREE AND ITS PROPERTIES

3.1 Burnside Ring

We assume that G is a finite group. Denote the free abelian group generated by $(H) \in \Phi(G)$ by $A(G) := \mathbb{Z}[\Phi(G)]$, i.e., an element $a \in A(G)$ can be written as a sum

$$a = n_1(H_1) + \cdots + n_m(H_m),$$

where $n_i \in \mathbb{Z}$ and $(H_i) \in \Phi(G)$. There is a natural multiplication operation $\circ : A(G) \times A(G) \to A(G)$ which is defined on generators (H), $(K) \in \Phi(G)$ by

$$(H) \cdot (K) = \sum_{(L)\in\Phi(G)} m_L\,(L), \tag{3.1}$$

where the integer m_L represents the number of (L)-orbits contained in the space $G/H \times G/K$. The numbers m_L can be easily computed from the following *recurrence formula*

$$m_L = \frac{n(L,H)|W(H)|n(L,K)|W(K)| - \sum_{(\widetilde{L})>(L)} m_{\widetilde{L}}\, n(L,\widetilde{L})\,|W(\widetilde{L})|}{|W(L)|}. \tag{3.2}$$

Together with the multiplication '$\circ$', $A(G)$ becomes a ring with the unity (G), which is called the *Burnside ring* of G. For more details see [2, 3]

3.2 Axioms of Brouwer G-Equivariant Degree

Consider an orthogonal G-representation V, a continuous G-map $\boldsymbol{f} : V \to V$, and an open bounded G-invariant set $\Omega \subset V$ such that for all $x \in \partial\Omega$, we have $\boldsymbol{f}(x) \neq 0$. Then f is called Ω-*admissible* and $(\boldsymbol{f}, \Omega)$ is called a G-*admissible pair* (in V). The set of all possible G-pairs will be denoted by $\mathcal{M}^G$.

The following result (cf [2]) can be considered as an axiomatic definition of the G-*equivariant Brouwer degree*:

Theorem 3.2.1. *There exists a unique map* $G\text{-deg}: \mathcal{M}^G \to A(G)$*, which assigns to every admissible* G*-pair* (f, Ω) *an element* $G\text{-deg}(\boldsymbol{f}, \Omega) \in A(G)$

$$G\text{-deg}(\boldsymbol{f}, \Omega) = \sum_{(H)} n_H(H) = n_{H_1}(H_1) + \cdots + n_{H_m}(H_m), \tag{3.3}$$

satisfying the following properties:

- ***(Existence)*** *If* $G\text{-deg}(\boldsymbol{f}, \Omega) \neq 0$*, i.e.,* $n_{H_i} \neq 0$ *for some* i *in* (3.3)*, then there exists* $x \in \Omega$ *such that* $\boldsymbol{f}(x) = 0$ *and* $(G_x) \geq (H_i)$*.*

- ***(Additivity)*** *Let* Ω_1 *and* Ω_2 *be two disjoint open* G*-invariant subsets of* Ω *such that* $f^{-1}(0) \cap \Omega \subset \Omega_1 \cup \Omega_2$*. Then*

$$G\text{-deg}(\boldsymbol{f}, \Omega) = G\text{-deg}(\boldsymbol{f}, \Omega_1) + G\text{-deg}(\boldsymbol{f}, \Omega_2).$$

- ***(Homotopy)*** *If* $\boldsymbol{h} : [0,1] \times V \to V$ *is an* Ω*-admissible* G*-homotopy, then*

$$G\text{-deg}(\boldsymbol{h}_t, \Omega) = \text{constant}.$$

- ***(Normalization)*** *Let* Ω *be a* G*-invariant open bounded neighborhood of* 0 *in* V*. Then*

$$G\text{-deg}(\text{Id}, \Omega) = (G).$$

- ***(Product)*** *For any* $(\boldsymbol{f}_1, \Omega_1), (\boldsymbol{f}_2, \Omega_2) \in \mathcal{M}^G$*,*

$$G\text{-deg}(\boldsymbol{f}_1 \times \boldsymbol{f}_2, \Omega_1 \times \Omega_2) = G\text{-deg}(\boldsymbol{f}_1, \Omega_1) \cdot G\text{-deg}(\boldsymbol{f}_2, \Omega_2),$$

where the multiplication '$\cdot$' is taken in the Burnside ring $A(G)$*.*

- ***(Suspension)*** *If* W *is an orthogonal* G*-representation and* $\mathscr{B}$ *is an open bounded invariant neighborhood of* $0 \in W$*, then*

$$G\text{-deg}(\boldsymbol{f} \times \text{Id}_W, \Omega \times \mathscr{B}) = G\text{-deg}(\boldsymbol{f}, \Omega).$$

- ***(Recurrence Formula)*** *For an admissible G-pair (f,Ω), the G-degree (3.3) can be computed using the following recurrence formula*

$$n_H = \frac{\deg(\boldsymbol{f}^H, \Omega^H) - \sum_{(K)>(H)} n_K\, n(H,K)\ |W(K)|}{|W(H)|}, \tag{3.4}$$

where $|X|$ stands for the number of elements in the set X and $\deg(\boldsymbol{f}^H, \Omega^H)$ is the Brouwer degree of the map $\boldsymbol{f}^H := \boldsymbol{f}|_{V^H}$ on the set $\Omega^H \subset V^H$.

The G-$\deg(\boldsymbol{f}, \Omega)$ is called the *G-equivariant Brouwer degree* (or simply *G-degree*) of $\boldsymbol{f}$ in Ω.

3.2.1 Basic Degrees and Computations of G-Equivariant Degrees

Put $B(V) := \{x \in V : \|x\| < 1\}$. For each irreducible G-representation $\mathcal{V}_i$, $i = 0, 1, 2, \dots$, we define

$$\deg_{\mathcal{V}_i} := G\text{-}\deg(-\mathrm{Id}\,, B(\mathcal{V}_i)),$$

and will call $\deg_{\mathcal{V}_i}$ the *basic degree*.

Consider a G-equivariant linear isomorphism $T : V \to V$ and assume that V has a G-isotypic decomposition (2.7). Then by the Multiplicativity property, we have

$$G\text{-}\deg(T, B(V)) = \prod_{i=0}^{r} G\text{-}\deg(T_i, B(V_i)) = \prod_{i=0}^{r} \prod_{\mu \in \sigma_-(T)} \left(\deg_{\mathcal{V}_i}\right)^{m_i(\mu)} \tag{3.5}$$

where $T_i = T|_{V_i}$ and $\sigma_-(T) := \{\mu \in \sigma(T) : \mu < 0\}$ denotes the real negative spectrum of T.

Notice that the basic degrees can be effectively computed from (3.4), i.e. formula

$$\deg_{\mathcal{V}_i} = \sum_{(H)} n_H(H),$$

where

$$n_H = \frac{(-1)^{\dim \mathcal{V}_i^H} - \sum_{(H)>(K)} n_K\, n(H,K)\ |W(K)|}{|W(H)|}. \tag{3.6}$$

3.3 GAP Equivariant Degree Package

The **EquiDeg** is a GAP package which contains a collection of functions for computing equivariant degree and to analyze, for a given group G, its representations. The package **EquiDeg** was created by Hao-Pin Wu and it can be found on GitHub website at the location (`https://gitthub.com/psistwu/GAP-equideg/`) This package, depending on the operating system used, should be properly installed following the GAP installation procedures described in the GAP manual.

We recommend to use this package online through CoCalc (`http://cocal.com`). After registering as a user, one should download **EquiDeg** package from the GitHub site and install it following the procedures described in the GAP manual.

3.3.1 List of EquiDeg Functions

Group Theoretical Functions: ▷ `pCyclicGroup` — generates a cyclic group $\mathbb{Z}_n$ which consists of permutations.

▷ `mCyclicGroup` — generates a cyclic group $\mathbb{Z}_n$ which consists of 2-by-2 matrices.

▷ `pDihedralGroup` — generates a dihedral group D_n which consists of permutations.

▷ `mDihedralGroup` — generates a dihedral group D_n which consists of 2-by-2 matrices.

▷ `OrderOfWeylGroup` — provides the order of the Weyl group for a given H being a subgroup or a class from CCS: `OrderOfWeylGroup`(H).

▷ `nLHnumber` — this operation computes the numbers $n(L,H)$ for a given L and H being subgroups or classes from Conjugacy classes of subgroups (CCS): `nLHnumber`(L,H)

▷ `LatticeCCSs(`G`)` — this attribute contains lattice `lat` of conjugacy classes CCSs of the group G which can be later converted into `.eps` or `.pdf` files.

Functions for Representations: ▷ `DimensionOfFixedSet` — this operation computes dimension of the H-fixed point space V^H (here H is either a subgroup or a class $C=(H)$ from CCS) for the representation V associated with the character χ: `DimensionOfFixedSet`(χ,H) or `DimensionOfFixedSet`(χ,H).

▷ `OrbitTypes` — this attribute contains the list of the orbit types in the representation V associated with the character χ: `OrbitTypes` (χ).

▷ `LatticeOrbitTypes` — This attribute contains the lattice of the orbit types in the representation V associated with the character χ: `LatticeOrbitTypes` (χ).

▷ `MaximalOrbitTypes` — this attribute contains the list of maximal orbit types in the representation V associated with the character χ: `MaximalOrbitTypes` (χ).

Elementary Compact Lie Groups: These functions allow to describe finite subgroups and their conjugacy classes in the group $G := \Gamma \times O(2)$ (where Γ is a finite group).

▷ (CCSs, l, j) — this is the selector of the list of representatives in CCSs (with l being the mode in $O(2)$ and j being the index of a subgroup in Γ) for a compact Lie group of the type $\Gamma \times O(2)$ (Γ a finite group).

▷ `NumberOfNonzeroModeClasses` — this operation returns the number of non-zero mode CCSs: `NumberOfNonzeroModeClasses`(CCSs).

Utilties: ▷ `IsCompactLieGroupCharacterTable` — this operation verifies the category of the character table for the compact Lie group G.

▷ `IsCompactLieGroupIrrCollection` — this is the category of collection of irreducible representations of compact Lie group.

▷ `IsCompactLieGroupClassFunction` — this is the category of class function of compact Lie group.

▷ `IsCompactLieGroupCharacter` — this is the property of compact Lie group character.

▷ `IsCompactLieGroupVirtualCharacter` — this is the property of compact Lie group virtual character.

▷ `IsIrreducibleCharacter` — this is the property of compact Lie group irreducible character.

▷ `IdIrr` — this attribute contains the id of compact Lie group irreducible representation χ: `IdIrr(`χ`)`.

▷ `DimensionOfFixedSet` — This operation returns the dimension of H-fixed set of subgroup H or the conjugacy class C of subgroup with respect to character χ: tt DimensionOfFixedSet (χ, C) or tt DimensionOfFixedSet(χ, H)

▷ `OrthogonalGroupOverReal`(n) — this function creates the orthogonal group $O(n, \mathbb{R})$.

▷ `SpecialOrthogonalGroupOverReal`(n) — This function creates group $SO(n, \mathbb{R})$.

▷ `ElementaryCLGId` — this function creates an elementary compact Lie group of given id: `ElementaryCLGId`(id)

Direct Product of Two Finite Groups: ▷ `DirectProductDecomposition` — performs direct product decomposition for either a group G, a conjugacy class c or a group element a: `DirectProductDecomposition`(G), `DirectProductDecomposition`(c), `DirectProductDecomposition`(G, a). Here is an example.

```
gap> G1 := SymmetricGroup( 4 );;
gap> G2 := CyclicGroup( 2 );;
gap> G := DirectProduct( G1, G2 );;
gap> DirectProductDecomposition( G ); # for group
        [ Sym( [ 1 .. 4 ] ), Group([ (1,2) ]) ]
gap> c := ConjugayClasses( G )[ 10 ];
        (1,2,3,4)(5,6)^ G
gap> DirectProductDecomposition( c ); # for conjugacy class
        [ (1,2,3,4)^ G, (1,2)^ G ]
gap> a := List( G )[ 30 ];
        (1,3,2)(5,6)
gap> DirectProductDecomposition( G, a ); # for group element
     [ (1,3,2), (1,2) ]
```

BurnsideRing: ▷ `BurnsideRing`(G) — this is an attribute of a group G which contains the induced Burnside ring A (see subsection 3.1). Here is an example of creating a Burnside ring induced by a finite group and printing its additive and multiplicative identities.

```
gap> G := SymmetricGroup( 4 );
    Sym( [ 1 .. 4 ] )
gap> A := BurnsideRing( G );
    Brng( Sym( [ 1 .. 4 ] ) )
gap> Zero(A);
    <> in Brng( Sym( [ 1 .. 4 ] ) )
gap> One(A);
    < 1(11) > in Brng( Sym( [ 1 .. 4 ] ) )
```

▷ `UnderlyingGroup` — this attribute of the Burnside ring A contains the group G from which A is induced: `UnderlyingGroup`(A).

▷ `Basis` (for Burnside ring) — this is an attribute of a Burnside ring A which stores its basis: `Basis`(A). Here is an example of selecting certain element in the basis of A:

```
gap> G := SymmetricGroup( 4 );
gap> A := BurnsideRing( G );
gap> B := Basis( A );
gap> b := B[2];
```

Other Aspects: ▷ `BasicDegree` — this is an attribute of a character χ which stores the associated basic degree `BasicDegree`(χ) (see subsection 3.2.1 for exact definitions). Here is an example.

```
gap> G := SymmetricGroup(4);
gap> chi := Irr(G)[3];
```

Character(CharacterTable(Sym([1 .. 4])), [2, 0, 2, -1, 0])

```
gap> BasicDegree(chi);
```

< 1(5) − 2(9) + 1(11) > in Brng(Sym([1 .. 4]))

3.3.2 Examples of GAP Computations

In this subsection we consider an example of the the group $G := D_3 \times D_3 \times \mathbb{Z}_2$ acting on $\mathscr{H} = V \oplus V \oplus V$, with $V = \mathbb{R}^3$ in such a way that the first copy of D_3 permutes the V-coordinates and the second copy of D_3 permutes the coordinates in $V := \mathbb{R}^3$. To be more precise, we put

$$\boldsymbol{\gamma}_1 := (\gamma, 1, 1), \boldsymbol{\gamma}_2 := (1, \gamma, 1), \boldsymbol{\kappa}_1 := (\kappa, 1, 1),$$
$$\boldsymbol{\kappa}_2 := (1, \kappa, 1), -\mathbf{1} := (1, 1, -1),$$

where $D_3 = \{1, \gamma, \gamma^2, \kappa, \gamma\kappa, \gamma^2\kappa\}$, $\gamma := e^{i\frac{2\pi}{3}}$. Since D_3 can be represented as a group of permutations S_3, where $\gamma = (1, 2, 3)$ and $\kappa = (2, 3)$, we have the action of G on $\mathscr{H}$ given by

$$\boldsymbol{\gamma}_1(x_1, x_2, x_3, y_1, y_2, y_3, z_1, z_2, z_3) = (y_1, y_2, y_3, z_1, z_2, z_3, x_1, x_2, x_3),$$
$$\boldsymbol{\gamma}_2(x_1, x_2, x_3, y_1, y_2, y_3, z_1, z_2, z_3) = (x_2, x_3, x_1, y_2, y_3, y_1, z_2, z_3, z_1),$$
$$\boldsymbol{\kappa}_1(x_1, x_2, x_3, y_1, y_2, y_3, z_1, z_2, z_3) = (x_1, x_2, x_3, z_1, z_2, z_3, y_1, y_2, y_3),$$
$$\boldsymbol{\kappa}_2(x_1, x_2, x_3, y_1, y_2, y_3, z_1, z_2, z_3) = (x_1, x_3, x_2, y_1, y_3, y_2, z_1, z_3, z_2),$$
$$-\mathbf{1}(x_1, x_2, x_3, y_1, y_2, y_3, z_1, z_2, z_3) = (-x_1, -x_2, -x_3, -y_1, -y_2, -y_3, -z_1, -z_2, -z_3)$$

First, we start GAP and load the EquiDeg package (by using the command)

```
gap> LoadPackage( "equideg" );
```

Now we are ready to define the group G by the following commands:

```
gap> grp1 := SymmetricGroup( 3 );
gap> grp2 := SymmetricGroup( 2 );
gap> grp3 := DirectProduct(grp1,grp2);
gap> grp := DirectProduct(grp1,grp3);
```

Under the above identification we obtain that the product group $G < S_8$, which is denoted by the symbol `grp`, is generated by the permutations: $(1,2,3)$, $(1,2)$, $(4,5,6)$, $(4,5)$, $(7,8)$. One can easily notice that, according to the notation that was introduced earlier we have

$$\boldsymbol{\kappa}_1 = (\kappa, 1, 1) = (1,2), \quad \boldsymbol{\kappa}_2 = (1, \kappa, 1) = (4,5), \quad (-\mathbf{1}) = (1,1,-1) = (7,8),$$
$$\boldsymbol{\gamma}_1 = (\gamma, 1, 1) = (1,2,3), \quad \boldsymbol{\gamma}_2 = (1, \kappa, 1) = (4,5,6),$$

In the next step, we compute the lattice of the conjugacy classes of all subgroups of the group G

```
gap> latt := LatticeCCSs(grp);
```

One can list all these classes by issuing command

```
gap> Print( latt );
```

It is easy to check that there are exactly 69 conjugacy classes in the list `latt`. Indeed, we have

```
gap> latt;
<CCS lattice of Group([ (1,2,3), (1,2), (4,5,6), (4,5), (7,8) ]), 69 classes>
```

Next, we find the list of all irreducible representations of G

```
gap> irr_list := Irr( grp );
```

One can easily verify that we have exactly 18 irreducible G-representations:

```
gap> Length( irr_list );
18
```

Since GAP numbers these representations starting from one, and we would like to reserve the symbol $\mathcal{V}_0$ for the trivial representation, we adjust the GAP-numbering by shifting index by one unit, i.e. the k-th character in the list `irr_list[k]` corresponds to the irreducible G-representation $\mathcal{V}_{k-1}$. Since the character χ_{k-1}, corresponding to the G-representation $\mathcal{V}_{k-1}$ is a function on the conjugacy classes of the elements of G, we issue the following commands that will allow us to identify these classes:

```
gap> cc := ConjugacyClasses( grp );
gap> Print( cc );
 ()∧G,                  (7,8)∧G,            (5,6)∧G,
 (5,6)(7,8)∧G,          (4,5,6)∧G,          (4,5,6)(7,8)∧G,
 (2,3)∧G,               (2,3)(7,8)∧G,       (2,3)(5,6)∧G,
 (2,3)(5,6)(7,8)∧G,     (2,3)(4,5,6)∧G,     (2,3)(4,5,6)(7,8)∧G,
 (1,2,3)∧G,             (1,2,3)(7,8)∧G,     (1,2,3)(5,6)∧G,
 (1,2,3)(5,6)(7,8)∧G,  (1,2,3)(4,5,6)∧G,  (1,2,3)(4,5,6)(7,8)∧G .
```

One can easily recognize that these classes are (according to the order used for the character tables) are

$$(\pm\mathbf{1}),\ (\pm\boldsymbol{\kappa}_2),\ (\pm\boldsymbol{\gamma}_2),\ (\pm\boldsymbol{\kappa}_1),\ (\pm\boldsymbol{\kappa}_1\boldsymbol{\kappa}_2),\ (\pm\boldsymbol{\kappa}_1\boldsymbol{\gamma}_2),\ (\pm\boldsymbol{\gamma}_1),\ (\pm\boldsymbol{\gamma}_1\boldsymbol{\kappa}_2),\ (\pm\boldsymbol{\gamma}_1\boldsymbol{\gamma}_2),$$

where the symbol $(\pm a)$ stand for the pair (a), $(-a)$.

Then we have

```
gap> irr_list[5];
Character(CharacterTable(Group([ (1,2,3), (1,2), (4,5,6), (4,5), (7,8) ])),
```
$[1,\ -1,\ 1,\ -1,\ 1,\ -1,\ 1,\ -1,\ 1,\ -1,\ 1,\ -1,\ 1,\ -1,\ 1,\ -1,\ 1,\ -1])$

```
gap> irr_list[10];
Character(CharacterTable( Group([ (1,2,3), (1,2), (4,5,6), (4,5), (7,8) ])),
```
$[2,;-2,\ 2,\ -2,\ 2,\ -2,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ -1,\ 1,\ -1,\ 1,\ -1,\ 1])$

```
gap> irr_list[14];
Character(CharacterTable(Group([ (1,2,3), (1,2), (4,5,6), (4,5), (7,8) ])),
```
$[2,\ -2,\ 0,\ 0,\ -1,\ 1,2,\ -2,\ 0,\ 0,\ -1,\ 1,\ 2,\ -2,\ 0,\ 0,\ -1,\ 1])$

```
gap> irr_list[18];
Character(CharacterTable(Group([ (1,2,3), (1,2), (4,5,6), (4,5), (7,8) ])),
[4, −4, 0, 0, −2, 2, 0, 0, 0, 0, 0, 0, −2, 2, 0, 0, 1, −1])
```

In order to obtain the isotypic decomposition of the G-space $\mathscr{H}$, we need to solve the system of equations

$$\chi_{\mathscr{H}} = n_0\chi_{\mathcal{V}_0} + n_1\chi_{\mathcal{V}_1} + n_2\chi_{\mathcal{V}_2} + \cdots + n_{16}\chi_{\mathcal{V}_{16}} + n_{17}\chi_{\mathcal{V}_{17}},$$

By direct computation of the character $\chi_{\mathscr{H}}$ and the above GAP output, we obtain

	$(\pm\mathbf{1})$	$(\pm\boldsymbol{\kappa_2})$	$(\pm\boldsymbol{\gamma_2})$	$(\pm\boldsymbol{\kappa_1})$	$(\pm\boldsymbol{\kappa_1\kappa_2})$	$(\pm\boldsymbol{\kappa_1\gamma_2})$	$(\pm\boldsymbol{\gamma_1})$	$(\pm\boldsymbol{\gamma_1\kappa_2})$	$(\pm\boldsymbol{\gamma_1\gamma_2})$
$\chi_{\mathscr{H}}$	± 9	± 3	0	± 3	± 1	0	0	0	0
$\chi_{\mathcal{V}_4}$	± 1	± 1	± 1	± 1	± 1	± 1	± 1	± 1	± 1
$\chi_{\mathcal{V}_9}$	± 2	± 2	± 2	0	0	0	∓ 1	∓ 1	∓ 1
$\chi_{\mathcal{V}_{13}}$	± 2	0	∓ 1	± 2	0	∓ 1	± 2	0	∓ 1
$\chi_{\mathcal{V}_{17}}$	± 4	0	∓ 2	0	0	0	∓ 2	0	± 1

Consequently, we have the following G-isotypic decomposition of the G-representation $\mathscr{H}$:

$$\mathscr{H} := \mathscr{H}_4 \oplus \mathscr{H}_9 \oplus \mathscr{H}_{13} \oplus \mathscr{H}_{17}$$

where $\mathscr{H}_j \simeq \mathcal{V}_j$, $j = 4, 9, 13, 17$.

In order to determine the orbit types in the G-representation $\mathscr{H}$, we use the following GAP routine:

```
gap> repH := irr_list[5]+irr_list[10]+irr_list[14]+irr_list[18];
Character( CharacterTable( Group([ (1,2,3), (1,2), (4,5,6), (4,5), (7,8) ]) ),
[9, −9, 3, −3, 0, 0, 3, −3, 1, −1, 0, 0, 0, 0, 0, 0, 0, 0])
gap> latt1 := LatticeOrbitTypes( repH );
<object>
```

to create the lattice of all the orbit types in the representation $\mathscr{H}$. Next, we apply another routine to create the actual graphic representation of this lattice. In order to achieve it, first we do

```
gap> DotFileLattice (latt1,"dot/representation-H.dot");
```

where `"dot/representation-H.dot"` means that the output will be saved in the file *representation-H.dot* in the subfolder `dot`. Then, one needs to open another terminal and run program `dot`, outside GAP, in order to convert the dot-file into a pdf-file:

```
dot$ dot -Tpdf -o representation-H.pdf representation-H.dot
```

and as a result we obtain the following lattice:

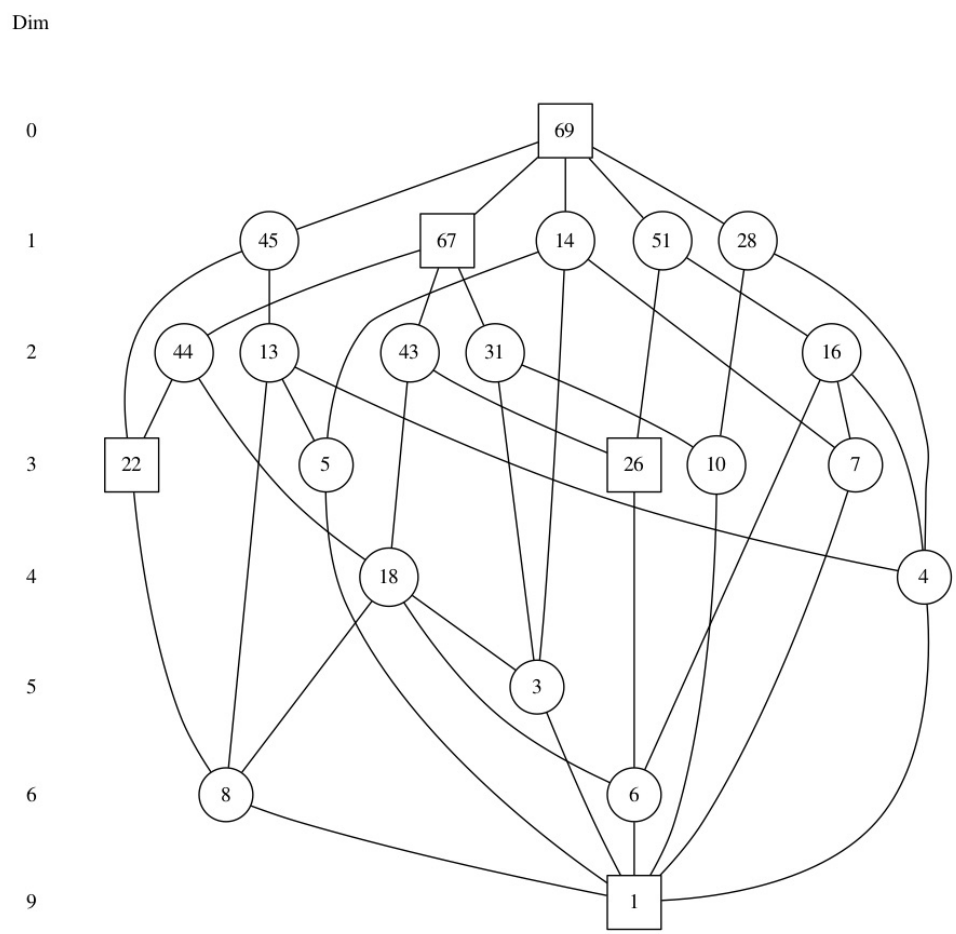

Figure 3.1. Lattice of orbit types in $D_3 \times D_3 \times \mathbb{Z}_2$-representation $\mathscr{H}$. The numbers k in the lattice correspond to the orbit types (H_k) and the numbers standing on the left margin denotes the dimensions of the corresponding H_k-fixed point spaces $\mathscr{H}^{H_k}$.

Our next task is to provide a notation, for the conjugacy classes in G, which is compatible with the amalgamated convention 2.5.2. We consider the group G as the product $D_3 \times (D_3 \times \mathbb{Z}_2)$, where in $D_3 \times \mathbb{Z}_2$ we have the following representatives of the conjugacy classes of subgroups

$$D_3 := D_3 \times \{1\}, \quad \mathbb{Z}_3 := \mathbb{Z}_3 \times \{1\}, \quad D_1 := D_1 \times \{1\}, \quad \mathbb{Z}_1 := \mathbb{Z}_1 \times \{1\}$$

$$D_3^p := D_3 \times \mathbb{Z}_2, \quad \mathbb{Z}_3^p := \mathbb{Z}_3 \times \mathbb{Z}_2, \quad D_1^p := D_1 \times \mathbb{Z}_2, \quad \mathbb{Z}_1^p := \mathbb{Z}_1 \times \mathbb{Z}_2$$

$$D_3^z := \mathbb{Z}_3 \times \{1\} \cup \mathbb{Z}_3\kappa \times \{-1\}, \quad D_1^z := \{(1,1),(\kappa,-1)\}.$$

Recall that the subgroups of the product $G_1 \times G_2$ can be represented in the form $H^\varphi \times_L^\psi K$, where $H \le G_1$, $K \le G_2$, and $\varphi : H \to L$, $\psi : K \to L$ are epimorphisms. Then

$$H^\varphi \times_L^\psi K := \{g,h) : \varphi(g) = \psi(h)\}.$$

For the group $D_3 \times (D_3 \times \mathbb{Z}_2)$, since we only want to list representatives of the conjugacy classes, we will denote the subgroup $H^\varphi \times_L^\psi K$ by

$$H^{Z_1} \times_L^{Z_2} K,$$

where $Z_1 := \operatorname{Ker}(\varphi)$ and $Z_2 := \operatorname{Ker}(\psi)$, $H \le D_3$, $K \le D_3^p$. In the case $L = \{e\}$ we will simply write $H \times K$ and if $\operatorname{Ker}(\varphi)$ or $\operatorname{Ker}(\psi)$ is trivial, the symbol Z_1 or Z_2 can be simply omitted.

Let us describe all the representatives H_k of the orbit types in $\mathscr{H}$. For this purpose we use the GAP command `ccs_list[[k]`. For example:

```
gap> ccs_list[44];
Group( [ (5,6), (4,5,6), (2,3) ] )^G
```

which means that there is a representative of (H_{44}) generated by $\boldsymbol{\kappa}_2$, $\boldsymbol{\gamma}_2$, $\boldsymbol{\kappa}_1$, and thus $H_{44} := D_1 \times D_3$ can be chosen as such a representative. In a similar way, we have that a representative of (H_{45}) is generated by $(2,3)(7,8)$, $(5,6)$ and $(4,5,6)$, which implies that we can take a subgroup H_{45} being generated by $(\kappa, 1, -1)$, $(1, \kappa, 1)$ and $1, \gamma, 1)$ which can be easily recognized the subgroup $H_{45} = D_1 \times_{\mathbb{Z}_2}^{D_3} D_3^p$. Similarly, for H_{28} we have the generators $(\kappa, \kappa, -1)$ and $(\gamma, \gamma, 1)$ which implies that $H_{28} = D_3 \times_{D_3} D_3^z$. Finally, for H_{51} we have generators $(1, \kappa, -1)$, $(\kappa, 1, 1)$ and $(\gamma, 1, 1)$, so $H_{51} = D_3 \times D_1^z$.

In summary we have the following orbit types in $\mathscr{H} \setminus \{0\}$:

$$
\begin{array}{lll}
(H_{67}) = (D_3 \times D_3), & (H_{45}) = (D_1 \times_{\mathbb{Z}_2}^{D_3} D_3^p), & (H_{14}) = (D_1 \times_{\mathbb{Z}_2}^{D_1^z} D_1^p), \\
(H_{51}) = (D_3 \times D_1^z), & (H_{28}) = (D_3 \times_{D_3} D_3^z), & (H_{44}) = (D_1 \times D_3), \\
(H_{13}) = (D_1 \times_{\mathbb{Z}_2}^{D_1} D_1^p), & (H_{43}) = (D_3 \times D_1), & (H_{31}) = (D_3 \times_{D_3} D_3), \\
(H_{16}) = (D_1 \times D_1^z), & (H_{22}) = (\mathbb{Z}_1 \times D_3), & (H_5) = (D_1 \times_{\mathbb{Z}_2} \mathbb{Z}_2), \\
(H_{26}) = (D_3 \times \mathbb{Z}_1), & (H_{10}) = (\mathbb{Z}_3 \times_{\mathbb{Z}_3} \mathbb{Z}_3), & (H_7) = (\mathbb{Z}_1 \times D_1^z), \\
(H_{18}) = (D_1 \times D_1), & (H_4) = (D_1 \times_{\mathbb{Z}_2} D_1^z), & (H_3) = (D_1 \times_{\mathbb{Z}_2} D_1) \\
(H_8) = (\mathbb{Z}_1 \times D_1), & (H_6) = (D_1 \times \mathbb{Z}_1), & (H_1) = (\mathbb{Z}_1).
\end{array}
$$

Moreover, clearly the maximal orbit types in $\mathscr{H} \setminus \{0\}$ are:

$$
(D_3 \times D_3), \quad (D_1 \times_{\mathbb{Z}_2}^{D_3} D_3^p), \quad (D_1 \times_{\mathbb{Z}_2}^{D_1^z} D_1^p), \quad (D_3 \times D_3^z), \quad (D_3 \times_{D_3} D_3^z).
$$

3.3.3 Example of GAP Computations of Basic Degrees

This example is a continuation of the subsection 3.3.2 where the group $G := D_3 \times D_3 \times \mathbb{Z}_2$ was consider with an action on the space $\mathscr{H} := V \oplus V \oplus V$, $V := \mathbb{R}^3$ (see subsection 3.3.2 for more details). Since $\mathscr{H}$ has the G-isotypical decomposition

$$
\mathscr{H} = \mathscr{H}_4 \oplus \mathscr{H}_9 \oplus \mathscr{H}_{13} \oplus \mathscr{H}_{17},
$$

in order to compute $G\text{-deg}(-\mathrm{Id}\,, B_1(\mathscr{H}))$ we need to apply the product formula

$$
G\text{-deg}(-\mathrm{Id}\,, B_1(\mathscr{H})) = \deg_{\mathcal{V}_4} \cdot \deg_{\mathcal{V}_9} \cdot \deg_{\mathcal{V}_{13}} \cdot \deg_{\mathcal{V}_{17}}.
$$

Next we apply the GAP routine to compute the basic degrees for the related to $\mathscr{H}$ irreducible G-representations:

```
gap> bdeg4 := BasicDegree(irr_list[5]);
<-1(67)+1(69)>
```

```
gap> bdeg9 := BasicDegree(irr_list[10]);
<1(22)-1(44)-1(45)+1(69)>
gap> bdeg13 := BasicDegree(irr_list[14]);
<1(26)-1(43)-1(51)+1(69)>
gap> bdeg17 := BasicDegree(irr_list[18]);
<-3(1)+2(3)+2(4)+1(5)+1(6)+1(7)+1(8)+1(10)-1(13)-1(14)
-1(16)-1(18)-1(28)-1(31)+1(69)>
```

Consequently, we have the following basic degrees

$$\begin{aligned}
\deg_{\mathcal{V}_4} &= (G) - ((D_3 \times D_3);\\
\deg_{\mathcal{V}_9} &= (G) - (D_1 \times^{D_3}_{\mathbb{Z}_2} D_3^p) - (D_1 \times D_3) + (\mathbb{Z}_1 \times D_3);\\
\deg_{\mathcal{V}_{13}} &= (G) - (D_3 \times D_1^z) - (D_3 \times D_1) + (D_3 \times \mathbb{Z}_1);\\
\deg_{\mathcal{V}_{17}} &= (G) - (D_3 \times_{D_3} D_3^z) - (D_1 \times^{D_1^z}_{\mathbb{Z}_2} D_1^p) - (D_3 \times_{D_3} D_3) - (D_1 \times D_1)\\
&\quad - (D_1 \times D_1^z) - (D_1 \times^{D_1}_{\mathbb{Z}_2} D_1^p) + (\mathbb{Z}_3 \times_{\mathbb{Z}_3} \mathbb{Z}_3) + (\mathbb{Z}_1 \times D_1) + (\mathbb{Z}_1 \times D_1^z)\\
&\quad + (D_1 \times \mathbb{Z}_1) + (D_1 \times_{\mathbb{Z}_2} \mathbb{Z}_2) + (D_1 \times_{\mathbb{Z}_2} D_1^z) + (D_1 \times_{\mathbb{Z}_2} D_1) - 3(\mathbb{Z}_1)
\end{aligned}$$

Finally, we compute $a := G\text{-deg}(-\mathrm{Id}\,, B_1(\mathscr{H}))$ using the following GAP routine;

```
gap> a := (bdeg4 * bdeg9 * bdeg13 * bdeg17);
<-1(3)+1(4)+1(13)-1(14)+1(16)-1(18)-1(28)+1(31)+1(43)+1(44)-1(45)-1(51)
-1(67)+1 (69)>
```

which implies that

$$\begin{aligned}
a &= -(H_3) + (H_4) + (H_{13}) - (H_{14}) + (H_{16}) - (H_{18}) - (H_{28}) + (H_{31}) + (H_{43}) + (H_{44})\\
&\quad - (H_{45}) - (H_{51}) - (H_{67}) + (H_{69})\\
&= -(D_1 \times_{\mathbb{Z}_2} D_1) + (D_1 \times_{\mathbb{Z}_2} D_1^z) + (D_1 \times^{D_1}_{\mathbb{Z}_2} D_1^p) - (D_1 \times^{D_1^z}_{\mathbb{Z}_2} D_1^p)\\
&\quad + (D_1 \times D_1^z) - (D_1 \times D_1) - (D_3 \times_{D_3} D_3^z) + (D_3 \times_{D_3} D_3)\\
&\quad + (D_3 \times D_1) - (D_1 \times^{D_3}_{\mathbb{Z}_2} D_3^p) - (D_3 \times D_1^z) - (D_3 \times D_3) + (G).
\end{aligned}$$

3.4 Local and Global Bifurcation Problems:

Assume that $a < b$, V is an orthogonal G-representation and $\Omega \subset V$ is an open bounded G-invariant subset. Let $\boldsymbol{F} : \mathbb{R} \oplus V \to V$ be a continuous G-equivariant map such that $(\boldsymbol{f}_a, \Omega)$, $(\boldsymbol{f}_b, \Omega) \in \mathcal{M}^G(V)$, where $\boldsymbol{f}_t(x) := \boldsymbol{F}(t, x)$, $t \in \mathbb{R}$, $x \in V$. Then a continuous G-invariant function $\varphi : \mathbb{R} \oplus V \to \mathbb{R}$ will be called Ω-*complementing function* for $\boldsymbol{f}_t$ at $t = a,\ b$, if

$$\begin{cases} \varphi(t,x) < 0 & \text{if } \ t = a,\ b,\ x \in \Omega \\ \varphi(t,x) > 0 & \text{if } \ t \in (a,b),\ x \in \partial\Omega. \end{cases} \tag{3.7}$$

In such a case we define the map $\boldsymbol{F}_\varphi : \mathbb{R} \oplus V \to \mathbb{R} \oplus V$ by

$$\boldsymbol{F}_\varphi(t,x) = (\varphi(t,x), f(t,x)), \quad t \in \mathbb{R},\ x \in V. \tag{3.8}$$

The following result is well-known in non-equivariant case.

Theorem 3.4.1. *Suppose that $\boldsymbol{F} : \mathbb{R} \oplus V \to V$ is a G-equivariant map such that $(\boldsymbol{f}_a, \Omega)$, $(\boldsymbol{f}_b, \Omega) \in \mathcal{M}^G(V)$, and $\varphi : \mathbb{R} \oplus V \to V$ is an Ω-complementing function for $\boldsymbol{f}_t$ at $t = a,\ b$. Then $(\boldsymbol{F}_\varphi, (a,b) \times \Omega) \in \mathcal{M}^G$, the G-equivariant degree $G\text{-deg}(\boldsymbol{F}_\varphi, (a,b) \times \Omega)$ doesn't depend on the choice of the Ω-complementing function φ and we have*

$$G\text{-deg}(\boldsymbol{F}_\varphi, (a,b) \times \Omega) = G\text{-deg}(\boldsymbol{f}_a, \Omega) - G\text{-deg}(\boldsymbol{f}_b, \Omega). \tag{3.9}$$

Let $\boldsymbol{F} : \mathbb{R} \times V \to V$ be a continuous G-equivariant map such that for all $t \in \mathbb{R}$, $\boldsymbol{F}(t, 0) = 0$. We are interested in solutions of the equation

$$\boldsymbol{F}(t,x) = 0. \tag{3.10}$$

Clearly any pair $(t, 0)$ satisfies (3.10), thus we will call them *trivial solutions* to (3.10). All other solutions to (3.10) will be called *nontrivial*. We denote the set of all nontrivial solutions to (3.10) by $\mathscr{S}$, i.e.

$$\mathscr{S} := \{(t,x) \in \mathbb{R} \times \mathbb{R}^n \oplus V : \boldsymbol{F}(t,x) = 0 \ \text{ and } x \neq 0\}.$$

Definition 3.4.2. Let $\mathcal{C} \subset \mathscr{S}$ and $\mathscr{U} \subset \mathbb{R} \times V \to V$ be a G-invariant open subset. The set $\mathcal{C} \subset \mathscr{S}$ is called a *branch* of nontrivial solutions to (3.10) in $\mathscr{U}$ if $\overline{\mathcal{C}}$ is a connected component of $\overline{\mathscr{S}} \cap \overline{\mathscr{U}}$ containing more than one point. Moreover, we say that the branch $\mathcal{C}$ bifurcates from a trivial solution $(t_o, 0)$ if $(t_o, 0) \in \overline{\mathcal{C}}$.

Theorem 3.4.3. *Let V be an orthogonal G-representation and $\Omega \subset V$ be a G-invariant bounded open set. Assume that $\boldsymbol{F} : [0,1] \times V \to V$ is a continuous G-equivariant map such that for every $t \in [0,1]$, $(\boldsymbol{f}_t, \Omega)$ (here $\boldsymbol{f}_t := \boldsymbol{F}(t, \cdot)$) is an admissible G-pair and $G\text{-deg}(\boldsymbol{f}_0, \Omega) \neq 0$. Then there exists a compact connected set $K_o \subset \boldsymbol{F}^{-1}(0) \cap [0,1] \times \Omega$ such that*

$$K_o \cap (\{0\} \times \Omega) \neq \emptyset \neq K_o \cap (\{1\} \times \Omega).$$

Let us discuss the *global bifurcation problem* for (3.10). Under the assumption that the derivative $D_x\boldsymbol{F}(t,0)$ exists for all $t \in \mathbb{R}$ and the map $t \mapsto D_x\boldsymbol{F}(t,0)$ is continuous, one can easily show that if $(t_o, 0)$ is a bifurcation point for (3.10), then $D_x\boldsymbol{F}(t_o,0)$ is not an isomorphism. Denote the set of all $t \in \mathbb{R}$ such that $(t,0)$ is a bifurcation point of (3.10) by $\mathscr{B}$ and put

$$\Lambda := \{t \in \mathbb{R} : \det D_x\boldsymbol{F}(t,0) = 0\}. \tag{3.11}$$

Λ is called the set of *critical points* for (3.10) and we clearly have $\mathscr{B} \subset \Lambda$. Suppose that $t_o \in \Lambda$ is an isolated point. We define the *local bifurcation invariant* $\omega_G(t_o)$ for $\boldsymbol{F}$ at $(t_o, 0)$ by

$$\omega_G(t_o) := G\text{-deg}(f_{t_-}, \beta_\epsilon(0)) - G\text{-deg}(f_{t_-}, \beta_\epsilon(0)) \tag{3.12}$$

Where $t_- < t_o < t_+$ intersects Λ at the point say $\{t_p\}$ and $\epsilon > 0$ is a sufficiently small number. By theorem 3.4.1, if $\omega_G(t_o) \neq 0$ then we have that $(t_o, 0) \in \mathscr{B}$

The following result is called *Rabinowitz's Alternative* (c.f [34]) for the proofs and more details:

Theorem 3.4.4. *Suppose that* $\boldsymbol{F} : \mathbb{R} \oplus V \to V$ *is a continuous* G*-equivariant map such that* $\boldsymbol{F}(t, 0) = 0$ *for all* $t \in \mathbb{R}$ *and* $D_x\boldsymbol{F}(t, 0)$ *exists and is continuous with respect to* $t \in \mathbb{R}$. *We also assume that the set of critical points* Λ *for* (3.10) *(given by* (3.11)*) is discrete, and consider an open bounded* G*-invariant set* $\mathcal{U} \subset \mathbb{R} \oplus V$ *such that* $(t_o, 0) \in \mathcal{U}$ *for some* $t_o \in \Lambda$. *For a connected component* $\mathscr{C}$ *of the set* $\overline{\mathcal{U}} \cap \overline{\mathscr{S}}$ *such that* $(t_o, 0) \in \mathscr{C}$ *we have the following alternative.*

(i) *either* $\mathscr{C} \cap \partial U \neq \emptyset$,

(ii) *or* $\mathscr{C} \cap (\Lambda \times \{0\}) = \{(t_1, 0), (t_2, 0), \dots, (t_n, 0)\}$ *for some* $n \in \mathbb{N}$ *(here* $t_j \neq t_k$ *for* $j \neq k$*) and*

$$\sum_{k=1}^{n} \omega_G(t_k) = 0. \tag{3.13}$$

CHAPTER 4

SUBHARMONIC SOLUTIONS IN REVERSIBLE NON-AUTONOMOUS DIFFERENTIAL EQUATIONS[1]

We are interested in studying the existence of the so-called *subharmonic* periodic solutions to (1.1), i.e. in finding non-constant solutions, which for some integer $m \geq 2$ satisfy

$$u(t) = u(t + 2\pi m), \quad \dot{u}(t) = \dot{u}(t + 2\pi m). \tag{4.1}$$

Consider a finite group Γ and assume that $V := \mathbb{R}^k$ is an orthogonal Γ-representation. In order to study the impact of additional symmetries on the existence of subharmonic solutions to (1.1), we introduce the following condition:

(A_4) For all $t \in \mathbb{R}$, $x \in \mathbb{R}^k$ and $\sigma \in \Gamma$, we have $f(t, \sigma x) = \sigma f(t, x)$.

The condition (A_4) means that f is Γ-equivariant.

4.1 Reformulation of (1.1) in Functional Spaces

Consider the Banach space $\mathbb{F} := C_{2\pi m}(\mathbb{R}, V)$ of all $2\pi m$-periodic continuous V-valued functions with the usual sup-norm

$$\|\varphi\|_\infty := \max_{t \in \mathbb{R}} |\varphi(t)|, \ \varphi \in \mathbb{F},$$

and denote by $\mathscr{E} := C^2_{2\pi m}(\mathbb{R}, V)$ the Banach space of all $2\pi m$-periodic C^2-differentiable V-valued functions with the norm $\| \cdot \| := \| \cdot \|_{2,\infty}$ given by

$$\|u\| = \|u\|_{2,\infty} := \max\{\|u\|_\infty, \|\dot{u}\|_\infty, \|\ddot{u}\|_\infty\}, \quad u \in \mathscr{E}. \tag{4.2}$$

[1]2020 Arxiv.2008.08132 Portions Adapted, with permission, from Izuchukwu Eze, Carlos Garcia-Azpeitia, W. Krawcewicz, and Yanli Lv, "Subharmonic Solutions in Reversible Non-Autonomous Differential Equations" Math.Ds, August 2020.

Define the natural injection operator $\boldsymbol{j} : \mathscr{E} \to \mathbb{F}$, $(\boldsymbol{j}(u))(t) := u(t)$, $t \in \mathbb{R}$ and notice that this operator is a compact linear operator. Next, define the operator $L : \mathscr{E} \to \mathbb{F}$ by $L(u)(t) := \ddot{u}(t) - u(t)$, $u \in \mathscr{E}$. Finally, define the map $N_f : \mathbb{F} \to \mathbb{F}$ given by $N_f(\varphi)(t) = f(t, \varphi(t))$, $\varphi \in \mathbb{F}$ and notice that N_f is continuous. Therefore, system (1.1) is equivalent to the following operator equation

$$Lu = N_f(\boldsymbol{j}(u)) - \boldsymbol{j}(u), \quad u \in \mathscr{E}. \tag{4.3}$$

The operator L is an isomorphism and therefore (4.3) can be rewritten as

$$u = L^{-1}\Big(N_f(\boldsymbol{j}(u)) - \boldsymbol{j}(u)\Big), \quad u \in \mathscr{E}.$$

To simplify the notation, define the map $\mathscr{F} : \mathscr{E} \to \mathscr{E}$, by

$$\mathscr{F}(u) := u - L^{-1}\Big(N_f(\boldsymbol{j}(u)) - \boldsymbol{j}(u)\Big), \quad u \in \mathscr{E}. \tag{4.4}$$

Then, clearly $u \in \mathscr{E}$ is a solution to (1.1) if and only if

$$\mathscr{F}(u) = 0. \tag{4.5}$$

Observe that by compactness of $\mathfrak{j}$, the map $\mathscr{F}$ is a completely continuous field on $\mathscr{E}$.

Obviously, by condition (A_3), $f(t, 0) = 0$ for all $t \in \mathbb{R}$, thus $\mathscr{F}(0) = 0$, i.e. the zero function is a solution to (1.1) (called the *trivial solution*). In what follows we are interested in finding non-constant $2\pi m$-periodic solutions to (1.1).

The group $G := \Gamma \times D_m \times \mathbb{Z}_2$ acts on the space $\mathscr{E}$ by

$$(\sigma, \gamma^j, \pm 1)u(t) := \pm\sigma u(t + 2\pi j), \quad j = 0, 1, \dots, m-1, \;\; \sigma \in \Gamma, \;\; \gamma = e^{\frac{i2\pi}{m}},$$
$$(\sigma, \kappa, \pm 1)u(t) := \pm\sigma u(-t), \quad t \in \mathbb{R}, \;\; u \in \mathscr{E},$$

thus $\mathscr{E}$ is an isometric Banach G-representation. One can easily verify that the properties (A_1)—(A_4) imply that $\mathscr{F}$ is G-equivariant.

4.2 Linearization of Equation (1.1) at 0

We make the following additional assumption

(A_5) There exists a **symmetric matrix** $A : \mathbb{R}^k \to \mathbb{R}^k$ such that

$$\lim_{u\to 0} \frac{f(t,u) - Au}{|u|} = 0 \tag{4.6}$$

uniformly with respect to $t \in \mathbb{R}$, and for all integers $j \geq 0$ and $\mu \in \sigma(A)$,

$$\frac{j^2}{m^2} + \mu \neq 0. \tag{4.7}$$

Condition (A_5), which means that $D_u(t,0) = A$, implies that $D\mathscr{F}(0)$ exists and is given as the following linear operator $\mathscr{A} : \mathscr{E} \to \mathscr{E}$ given by

$$\mathscr{A}u := u - L^{-1}\Big(N_A(\boldsymbol{j}(u) - \boldsymbol{j}(u)\Big), \quad u \in \mathscr{E}, \tag{4.8}$$

where $N_A(\varphi)(t) := A(\varphi(t))$, $t \in \mathbb{R}$, $\varphi \in C_{2\pi m}(\mathbb{R}; V)$. Under the assumption (A_5) the operator $\mathscr{A} : \mathscr{E} \to \mathscr{E}$ is an isomorphism.

Lemma 4.2.1. *Assume that $f : \mathbb{R} \times \mathbb{R}^k \to \mathbb{R}^k$ satisfies the conditions (A_1)—(A_5). Then there exists $\varepsilon > 0$ such that the G-map $\mathscr{F} : \mathscr{E} \to \mathscr{E}$ (given by (4.4)) and $\mathscr{A} : \mathscr{E} \to \mathscr{E}$ (given by (4.8)) are Ω_ε-admissibly G-homotopic (here $\Omega_\varepsilon := B_\varepsilon(0)$ in $\mathscr{E}$).*

Proof. Define the linear homotopy $\mathfrak{H} : [0,1] \times \mathscr{E} \to \mathscr{E}$ as $\mathfrak{H}(\lambda, u) := (1-\lambda)\mathscr{A}u + \lambda\mathscr{F}(u)$, $u \in \mathscr{E}$, and suppose for contradiction that there exists a sequence $\{\lambda_n, u_n\}$ such that $u_n \neq 0$, $\lambda_n \to \lambda_o$ and $u_n \to 0$ as $n \to \infty$. Then we have

$$\begin{aligned} 0 = \mathfrak{H}(\lambda_n, u_n) &= (1-\lambda_n)\mathscr{A}u_n + \lambda_n\mathscr{F}(u_n) \\ &= \mathscr{A}(u_n) + \lambda_n(\mathscr{F}(u_n) - \mathscr{A}u_n). \end{aligned}$$

Put $v_n := \frac{u_n}{\|u_n\|}$. Then

$$0 = \mathscr{A}v_n + \lambda_n \frac{\mathscr{F}(u_n) - \mathscr{A}u_n}{\|u_n\|}.$$

Since $\|u_n\| \to 0$ and λ_n is bounded, thus

$$\lim_{n\to\infty} \frac{\mathscr{F}(u_n) - \mathscr{A} u_n}{\|u_n\|} = 0,$$

which implies

$$0 = \lim_{n\to\infty} (\mathscr{A} v_n).$$

On the other hand, since $\mathscr{A} = \mathrm{Id} - \mathscr{K}$, where $\mathscr{K} := L^{-1}(A\boldsymbol{j} - \boldsymbol{j})$ is a compact operator, one can assume (by passing to a subsequence) that $\mathscr{K} v_n \to v_o$, which implies $v_n \to v_0$ and $\|v_o\| = 1$, so $v_o \in \operatorname{Ker} \mathscr{A}$, but this is a contradiction with (A_5). □

4.3 G-Isotypic Decomposition of $\mathscr{E}$

Isotypic decomposition of the G-representation $\mathscr{E}$ is an important technique providing a direct product decomposition of the space $\mathscr{E}$ into $\mathscr{A}$-invariant subspaces (see preliminaries for more details). In order to construct such a decomposition, first we notice that $\mathscr{E}$ is an isometric Banach $\Gamma \times O(2) \times \mathbb{Z}_2$-representation, with $O(2)$-action given by

$$e^{i\theta} u(t) = u\left(t + \theta m\right), \quad \kappa u(t) = u(-t), \quad u \in \mathscr{E},$$

and Γ-action given by $(\gamma u)(t) = \gamma u(t)$, $\gamma \in \Gamma$, $t \in \mathbb{R}$. Using the $\Gamma \times O(2) \times \mathbb{Z}_2$-action on $\mathscr{E}$, one can identify the $\Gamma \times D_m \times \mathbb{Z}_2$-isotypic decomposition of $\mathscr{E}$. Indeed, by applying the usual Fourier series expansions of functions $u \in \mathscr{E}$, we have the following $\Gamma \times O(2) \times \mathbb{Z}_2$-isotypic decomposition of $\mathscr{E}$

$$\mathscr{E} = \overline{\bigoplus_{j=0}^{\infty} \bigoplus_{l=0}^{\mathfrak{r}} \mathbb{V}_{j,l}}, \tag{4.9}$$

where

$$\mathbb{V}_{j,l} = \left\{ u \in \mathscr{E} : u(t) = \cos(jt/m)a + \sin(jt/m)b,\ a,\, b \in V_l \right\}.$$

and

$$V = V_0 \oplus V_1 \oplus \cdots \oplus V_{\mathfrak{r}},$$

is a Γ-isotypic decomposition of V, with the component V_l being modeled on the Γ-irreducible representation $\mathcal{U}_l$, $0 \le l \le \mathfrak{r}$. Then we have:

Lemma 4.3.1. *For $j > 0$, the $\Gamma \times O(2) \times \mathbb{Z}_2$-invariant subspace $\mathbb{V}_{j,l}$ can be identified with the complexification $V_l^c := V_l \oplus iV_l$ of V_l, on which $O(2)$ acts by*

$$e^{i\theta}(a+ib) := e^{-ij\theta} \cdot (a+ib), \quad \kappa(a+ib) = a-ib, \quad a, b \in V_l,$$

where '$\cdot$' stands for complex multiplication.

Proof. Although, the above statement is a well-known fact, for the sake of completeness, we give a short proof. Define the real isomorphism $\psi_j : V_l^c \to \mathbb{V}_{j,l}$ by $\psi(a+ib)(t) = \cos(jt/m)a + \sin(jt/m)b$, where $a, b \in V_l$. Then for $\mathbf{z} := a+ib$ we have

$$\begin{aligned}
\psi_j\big(e^{i\theta}(\mathbf{z})\big) &= \psi_j\big(e^{i\theta}(a+ib)\big) \\
&= \psi_j\Big(\cos(j\theta)a + \sin(j\theta)b + i(-\sin(j\theta)a + \cos(j\theta)b)\Big) \\
&= \cos(jt/m)(\cos(j\theta)a + \sin(j\theta)b) + \sin(jt/m)(-\sin(j\theta)a + \cos(j\theta)b) \\
&= \cos(\tfrac{j}{m}(t+m\theta))a + \sin(\tfrac{j}{m}(t+m\theta))b \\
&= e^{i\theta}(\cos(jt/m)a + \sin(jt/m)b) = e^{i\theta}\psi_j(a+ib) = e^{i\theta}\psi_j(\mathbf{z}).
\end{aligned}$$

□

The two-dimensional irreducible $O(2)$-representations are $\mathcal{W}_j \simeq \mathbb{C}$ for $j > 0$, where for $e^{i\theta} \in SO(2)$ one has that $e^{i\theta}z := e^{i\theta j} \cdot z$ and $\kappa z := \overline{z}$ with $z \in \mathbb{C}$. Since D_m is a subgroup of $O(2)$, $\mathcal{W}_j$ is a D_m-representation as well. Set $\mathfrak{s} := \left\lfloor \frac{m+1}{2} \right\rfloor$. Each $\mathcal{W}_j$ decompose in a direct sum of irreducible D_m-representations of the following list:

- if $i = 0$, the irreducible D_m-representation $\mathcal{V}_0 \simeq \mathbb{R}$ with the trivial D_m-action;
- if $0 < i < m/2$, the irreducible D_m-representation $\mathcal{V}_i \simeq \mathbb{R}^2 = \mathbb{C}$ with the D_m-action $\gamma z = \gamma^i \cdot z$ and $\kappa z = \overline{z}$ for $z \in \mathbb{C}$;

- if $i = \mathfrak{s}$, the irreducible D_m-representation $\mathcal{V}_{\mathfrak{s}} \simeq \mathbb{R}$ with the D_m-action $\gamma x = x$ and $\kappa x = -x$ for $x \in \mathbb{R}$;

- if $i = \mathfrak{s} + 1$, when m is even, the irreducible D_m-representation $\mathcal{V}_{\mathfrak{s}+1} \simeq \mathbb{R}$ with the D_m-action $\gamma x = -x$ and $\kappa x = x$ for $x \in \mathbb{R}$;

- if $i = \mathfrak{s} + 2$, when m is even, the irreducible D_m-representation $\mathcal{V}_{\mathfrak{s}+2} \simeq \mathbb{R}$ with the D_m-action $\gamma x = -x$ and $\kappa x = -x$ for $x \in \mathbb{R}$.

For the group $D_m \times \mathbb{Z}_2$, the corresponding irreducible representations (with non-trivial $\mathbb{Z}_2$-action) will be denoted by $\mathcal{V}_i^-$.

Proposition 4.3.2. *The space $\mathcal{W}_j$ considered as a D_m-representation has the following D_m-isotypic decomposition*

- $\mathcal{W}_{mj} \simeq \mathcal{V}_0 \oplus \mathcal{V}_{\mathfrak{s}}$,

- *for* $0 < i < \frac{m}{2}$, $\mathcal{W}_{mj+i} \simeq \mathcal{W}_{mj-i} \simeq \mathcal{V}_i$,

- *if* m *is even,* $\mathcal{W}_{mj-\frac{m}{2}} \simeq \mathcal{V}_{\mathfrak{s}+1} \oplus \mathcal{V}_{\mathfrak{s}+2}$.

For $j > 0$ and $0 \leq l \leq \mathfrak{r}$, we put

$$\begin{aligned}
\mathbb{V}^+_{jm,l} &= \{u \in \mathscr{E} : u(t) = \cos(jt)a,\ a \in V_l\}, \\
\mathbb{V}^-_{jm,l} &= \{u \in \mathscr{E} : u(t) = \sin(jt)b,\ b \in V_l\}, \\
\mathbb{V}^+_{jm-\frac{m}{2},l} &= \left\{u \in \mathscr{E} : u(t) = \cos((j - \tfrac{1}{2})t)a,\ a \in V_l\right\}, \\
\mathbb{V}^-_{jm-\frac{m}{2},l} &= \left\{u \in \mathscr{E} : u(t) = \sin((j - \tfrac{1}{2})t)b,\ b \in V_l\right\}.
\end{aligned}$$

Therefore, we have the following $\Gamma \times D_m \times \mathbb{Z}_2$-isotypic decomposition of the space $\mathscr{E}$:

$$\mathscr{E} = \bigoplus_{l=0}^{\mathfrak{r}} \bigoplus_{i=0}^{\mathfrak{s}^*} \mathcal{E}^-_{i,l}, \quad s^* := \begin{cases} \mathfrak{s} & \text{if } m \text{ is odd} \\ \mathfrak{s}+2 & \text{if } m \text{ is even} \end{cases}, \quad \mathfrak{s} = \left\lfloor \frac{m+1}{2} \right\rfloor,$$

where

$$\mathcal{E}_{0,l}^- = \mathbb{V}_{0,l} \oplus \overline{\bigoplus_{j=1}^{\infty} \mathbb{V}_{mj,l}^+}, \quad \mathcal{E}_{\mathfrak{s},l}^- = \overline{\bigoplus_{j=1}^{\infty} \mathbb{V}_{mj,l}^-},$$

for $0 < i < \frac{m}{2}$

$$\mathcal{E}_{i,l}^- = \overline{\bigoplus_{j=0}^{\infty} \mathbb{V}_{mj+i,l}} \oplus \overline{\bigoplus_{j=1}^{\infty} \mathbb{V}_{mj-i,l}},$$

and if m is even then

$$\mathcal{E}_{\mathfrak{s}+1,l}^- = \overline{\bigoplus_{j=1}^{\infty} \mathbb{V}_{mj-\frac{m}{2},l}^+}, \quad \mathcal{E}_{\mathfrak{s}+2,l}^- = \overline{\bigoplus_{j=1}^{\infty} \mathbb{V}_{mj-\frac{m}{2},l}^-}.$$

The component $\mathcal{E}_{i,l}^-$ ($0 \le i \le \mathfrak{s}^*$, $0 \le l \le \mathfrak{r}$) is modeled on the irreducible $\Gamma \times D_m \times \mathbb{Z}_2$-representation

$$\mathcal{V}_{i,l}^- := \mathcal{V}_i^- \otimes \mathcal{U}_l.$$

Since the operator L is $O(2) \times \mathbb{Z}_2$-equivariant isomorphism, thus $L(\mathbb{V}_j) = \mathbb{V}_j$ and $L|_{\mathbb{V}_j} = -(j^2/m^2 + 1)\mathrm{Id}_{\mathbb{V}_j}$.

4.4 Nagumo Growth Condition

To prove that for a sufficiently large R, $\mathscr{F}$ is an Ω_R-admissly G-homotopic to the identity, we introduce the following condition, which is often referred to as the *Nagumo growth condition*:

(A_6) There exists a constant $M > 0$ such that

$$\forall_{t\in\mathbb{R}} \ \forall_{x\in\mathbb{R}^k} \ |x| \ge M \quad \Rightarrow \quad f(t,x) \bullet x > 0.$$

Denote by Ω_R the ball $B_R(0)$ in $\mathscr{E}$. To establish a G-homotopy Ω_R-admissible (for sufficiently large R) of $\mathscr{F}$ with the identity map, we consider the following parametrized (by $\lambda \in [0,1]$) modification of system (1.1):

$$\begin{cases} \ddot{u}(t) = \lambda f(t, u(t)) + (1-\lambda)u(t), \quad t \in \mathbb{R}, \ u(t) \in V, \\ u(t) = u(t + 2\pi m), \ \dot{u}(t) = \dot{u}(t + 2\pi m). \end{cases} \tag{4.10}$$

Then:

Lemma 4.4.1. *Assume that $f:\mathbb{R}\oplus\mathbb{R}^k\to\mathbb{R}^k$ is a continuous function satisfying conditions (A_1)—(A_4) and (A_6). If $u(t)$ is a $2\pi m$-periodic function of class C^2 such that $\max_{t\in\mathbb{R}}|u(t)|\geq M$ (where M is given in (A_6)), then $u(t)$ cannot be a solution of* (4.10) *for* $\lambda\in[0,1]$.

Proof. Assume for the contradiction that $u(t)$ is a solution while $\max_{t\in\mathbb{R}}|u(t)|\geq M$. Consider the function $\phi(t):=\frac{1}{2}|u(t)|^2$. Suppose that $\phi(t_0)=\max_{t\in\mathbb{R}}\phi(t)$, then $\phi'(t_0)=u(t_0)\bullet\dot{u}(t_0)=0$ and $\phi''(t_0)=\dot{u}(t_0)\bullet\dot{u}(t_0)+\ddot{u}(t_0)\bullet u(t_0)\leq 0$. However, by condition (A_6), $\phi''(t_0)=\dot{u}(t_0)\bullet\dot{u}(t_0)+\ddot{u}(t_0)\bullet u(t_0)=(\lambda(f(u(t_0))-u(t_0))+u(t_0))\bullet u(t_0)+\dot{u}(t_0)\bullet\dot{u}(t_0)>(1-\lambda)u(t_0)\bullet u(t_0)+\lambda f(u(t_0))\bullet u(t_0)>0$, which leads to a contradiction with condition (A_6). □

Lemma 4.4.2. *Assume that $f:\mathbb{R}\oplus\mathbb{R}^k\to\mathbb{R}^k$ is a continuous function satisfying conditions (A_1)—(A_3) and (A_6). Then there exists $R>0$ such that for every solution $u\in\mathscr{E}$ to* (4.10), *$\lambda\in[0,1]$, we have $\|u\|<R$. In addition, for $\Omega_R:=B_R(0)$, the map $\mathscr{F}:\mathscr{E}\to\mathscr{E}$ is Ω_R-admissibly G-homotopic to* Id.

Proof. By Lemma 4.4.1, there exists a $M>0$ such that any $2\pi m$-periodic solution $u(t)$ to (4.10) satisfies $|u(t)|<M$. Take $A_R:=\{(t,x)\in[0,2\pi m]\times\mathbb{R}^k:|x|\leq M\}$. Observe that the function $F:[0,1]\times\mathbb{R}\oplus\mathbb{R}^k\to\mathbb{R}^k$ given by

$$F(\lambda,t,x)=\lambda f(t,x)+(1-\lambda)x,\quad x\in\mathbb{R}^k,\ \lambda\in[0,1],$$

is continuous, and the set $[0,1]\times A_R$ is compact. Hence for every solution $u(t)$ to (4.10) we have

$$|\ddot{u}(t)|=|F(\lambda,t,u)|\leq\sup\{|F(\lambda,t,x)|:(t,x)\in A_R,\ \lambda\in[0,1]\}=:M_2.$$

Put $\dot{u}(t)=(\dot{u}_1(t),\dot{u}_2(t),\dots,\dot{u}_k(t))^T$. Since for every $1\leq l\leq k$, the function $\dot{u}_l(t)$ is periodic, there exists $\tau_o\in[0,2\pi m]$ such that $\dot{u}_l(\tau_o)=0$. Thus the identity

$$\dot{u}_l(t)=\int_{\tau_o}^t\ddot{u}_l(s)ds,\quad t\in\mathbb{R}$$

implies $|\dot{u}_l(t)| \leq 2\pi m M_2$ for $t \in \mathbb{R}$ and consequently

$$\|\dot{u}\|_\infty = \max_{t\in\mathbb{R}} \sqrt{|\dot{u}_1(t)|^2 + |\dot{u}_2(t)|^2 + \cdots + |\dot{u}_k(t)|^2} \leq \sqrt{k} 2\pi m M_2 =: M_1.$$

Consequently,

$$\|\dot{u}\| = \max\{\|u\|_\infty, \|\dot{u}\|_\infty, \|\ddot{u}\|_\infty\} \leq \max\{M, M_1, M_2\} < \max\{M, M_1, M_2\} + 1 =: R, \quad (4.11)$$

and the conclusion follows. □

4.5 Abstract Existence Result

Assume that $f : \mathbb{R} \oplus V \to V$ satisfies the assumptions (A_1)—(A_6). We denote the set of negative eigenvalues of the operator $\mathscr{A}$ by $\sigma_-(\mathscr{A})$. Then by Lemma 4.2.1, there exists a sufficiently small $\varepsilon > 0$ such that $\mathscr{F}$ is Ω_ε-admissibly G-homotopic to $\mathscr{A}$ (given by (4.8)) and therefore

$$G\text{-deg}(\mathscr{F}, \Omega_\varepsilon) = G\text{-deg}(\mathscr{A}, B(\mathscr{E})) = \prod_{\lambda\in\sigma_-(\mathscr{A})} G\text{-deg}(-\mathrm{Id}\,|_{E(\lambda)}, B(E(\lambda))), \quad (4.12)$$

where $E(\lambda)$ denotes the eigenspace of $\mathscr{A}$ corresponding to λ, and $B(E(\lambda))$ stands for an open unit ball in $E(\lambda)$ (c.f formula 3.5).

In order to use the formula (4.12) we need to compute the negative spectrum $\sigma_-(\mathscr{A})$. Since $\mathscr{A}$ is $\Gamma \times O(2) \times \mathbb{Z}_2$-equivariant, one can use the isotypic decomposition (4.9) in order to determine eigenvalues (and eigenspaces) of $\mathscr{A}$:

$$\sigma(\mathscr{A}) = \left\{ \lambda_{j,\mu} := 1 + \frac{m^2(\mu - 1)}{j^2 + m^2} : j = 0, 1, 2, \ldots,\ \mu \in \sigma(A) \right\}. \quad (4.13)$$

Clearly,

$$\lambda_{j,\mu} = \frac{j^2 + m^2\mu}{j^2 + m^2} < 0$$

if and only if $\mu < -j^2/m^2$. Notice that, in such a case we also have

$$\lambda_{0,\mu} < \lambda_{1,\mu} < \cdots < \lambda_{j-1,\mu} < \lambda_{j,\mu} < \cdots < \lambda_{\mathrm{j}_\mu,\mu} < 0 < \lambda_{\boldsymbol{j}_\mu+1,\mu},$$

where j_μ is the integer number satisfying

$$-\frac{(\mathrm{j}_\mu+1)^2}{m^2} < \mu < -\frac{\mathrm{j}_\mu^2}{m^2}$$

By condition (A_5) such j_μ exists

On the other hand, by Lemma 4.4.2, there exists a sufficiently large $R > 0$ such that $\mathscr{F}$ is Ω_R-admissibly G-homotopic to Id. Therefore, $G\text{-deg}(\mathscr{F}, \Omega_R) = G\text{-deg}(\mathrm{Id}, \Omega_R) = (G)$. Put $\Omega := \Omega_R \setminus \overline{\Omega_\varepsilon}$. Then the $G\text{-deg}(\mathscr{F}, \Omega)$ is well defined and by additivity property we have

$$\begin{aligned} G\text{-deg}(\mathscr{F}, \Omega) &= G\text{-deg}(\mathscr{F}, \Omega_R) - G\text{-deg}(\mathscr{F}, \Omega_\varepsilon) \\ &= (G) - G\text{-deg}(\mathscr{A}, B(\mathscr{E})). \end{aligned}$$

In this way we can formulate the following abstract existence result:

Theorem 4.5.1. *Assume that $f : \mathbb{R} \oplus V \to V$ satisfies the assumptions (A_1)—(A_6), $R > 0$ is a sufficiently large number (given by Lemma 4.4.2), $\varepsilon > 0$ is a sufficiently small number (given by Lemma 4.2.1) and $\Omega := \Omega_R \setminus \overline{\Omega_\varepsilon}$. If the G-equivariant degree*

$$G\text{-deg}(\mathscr{F}, \Omega) = n_1(H_1) + n_2(H_2) + \cdots + n_s(H_s) \in A(G)$$

has a non-zero coefficient n_j, then there exists a $2\pi m$-periodic solution $u \in \Omega$ to (1.1) such that $G_u \geq H_j$. In addition, if (H_j) is a maximal orbit type in Ω such that $D_m \not\leq H_j$, then u is non-constant, and if for some $g \in D_m$, $g \neq 1$, we have $(g, -1) \in H_j$, then the solution u can not be 2π-periodic solution, i.e. its minimal period is not 2π.

Proof. The existence of a $2\pi m$-periodic solution x is a direct consequence of the existence property for G-equivariant degree. Moreover, if $u(t)$ is constant, then clearly $u(t+l2\pi) = u(t)$

and $u(-t) = u(t)$, for $t \in \mathbb{R}$ and $l \in \mathbb{Z}$, so $D_m \leq H_j$. Assume that there exists an element $(g, -1) \in H_j$ for some $1 \neq g \in D_m$, which implies that for some $1 \leq l \leq m-1$ we have $g = \gamma^l$ or $g = \gamma^l \kappa$. Then we also have

$$\forall_{t \in \mathbb{R}} \ \ ((g,-1)u)(t) = u(t) \ \ \Rightarrow u(t) = -u(t + 2\pi l).$$

Since $u \not\equiv 0$ it follows that for some t we have $u(t) \neq u(t + l2\pi)$ and consequently $u(t) \neq u(t + 2\pi)$. □

4.6 Subharmonic Solutions in Non-Equivariant Case

Theorem 4.5.1 can lead to more detailed information describing possible subharmonic solutions and their symmetries. First, we assume that $\Gamma = \{e\}$, i.e. we consider the case of (1.1) without additional symmetries. In this case we have $G \simeq D_m \times \mathbb{Z}_2$.

We begin with the following notation:

Definition 4.6.1. *Define*

$$\alpha(j) := j - \left\lfloor \frac{j}{m} \right\rfloor m \in \{0, 1, 2, \dots, m-1\}.$$

Next, define

$$i(j) := \begin{cases} \alpha(j) & \text{if } \alpha(j) \leq \lfloor \frac{m}{2} \rfloor, \\ m - \alpha(j) & \text{if } \alpha(j) > \lfloor \frac{m}{2} \rfloor, \end{cases} \tag{4.14}$$

We denote by $\mathfrak{m}(\mu)$ the algebraic multiplicity of μ belonging to the spectrum of A. The negative spectrum $\sigma_-(\mathscr{A})$ can be represented as

$$\sigma_-(\mathscr{A}) = \bigcup_{\mu \in \sigma_-(A)} \{\lambda_{0,\mu}, \lambda_{1,\mu}, \dots, \lambda_{\mathrm{j}_\mu - 1,\mu}, \lambda_{\mathrm{j}_\mu,\mu}\} \tag{4.15}$$

Denote by $E(\lambda_{j,\mu})$ the eigenspace of $\lambda_{j,\mu}$. The following notation that will be used to provide an exact formula for G-deg$(\mathscr{A}, B(\mathscr{E}))$. Namely we put

$$\beta_0(\mu) := \left(\left\lfloor \frac{\mathrm{j}_\mu}{m} \right\rfloor + 1 \right) \mathfrak{m}(\mu), \quad \beta_{\mathfrak{s}}(\mu) := \left\lfloor \frac{\mathrm{j}_\mu}{m} \right\rfloor \mathfrak{m}(\mu),$$

and for $i = \mathfrak{s}+1, \mathfrak{s}+2$ (in the case m is even),

$$\beta_i(\mu) := \begin{cases} \left\lfloor \frac{\mathfrak{j}_\mu}{m} \right\rfloor \mathfrak{m}(\mu), & \text{if } \alpha(\mathfrak{j}_\mu) < \frac{m}{2}, \\ \left(\left\lfloor \frac{\mathfrak{j}_\mu}{m} \right\rfloor + 1 \right) \mathfrak{m}(\mu), & \text{if } \alpha(\mathfrak{j}_\mu) \geq \frac{m}{2}, \end{cases}$$

and finally for $0 < i < \frac{m}{2}$,

$$\beta_i(\mu) := \begin{cases} 2 \left\lfloor \frac{\mathfrak{j}_\mu}{m} \right\rfloor \mathfrak{m}(\mu) & \text{if } \alpha(\mathfrak{j}_\mu) < i, \\ \left(2 \left\lfloor \frac{\mathfrak{j}_\mu}{m} \right\rfloor + 1 \right) \mathfrak{m}(\mu) & \text{if } i \leq \alpha(\mathfrak{j}_\mu) < m - i, \\ 2 \left(\left\lfloor \frac{\mathfrak{j}_\mu}{m} \right\rfloor + 1 \right) \mathfrak{m}(\mu) & \text{if } m - i \leq \alpha(\mathfrak{j}_\mu). \end{cases}$$

Then we define

$$\eta_i := \sum_{\mu \in \sigma_-(A)} \beta_i(\mu),$$

for $i = 0, 1, \dots, \mathfrak{s}, \mathfrak{s}+1, \mathfrak{s}+2$. The number η_i counts the "total number of times" the irreducible representation $\mathcal{V}_i^-$ appears in the eigenspaces of $\sigma_-(\mathscr{A})$.

We have the following list of basic degrees for the irreducible G-representations (see GAP section):

- for $0 \leq i \leq \lfloor \frac{m}{2} \rfloor$, $h := \gcd(m, i)$, m/h is odd then

$$\deg_{\mathcal{V}_i^-} = (D_m \times \mathbb{Z}_2) - (D_h) - (D_h^z) + (\mathbb{Z}_h);$$

- if $m/h \equiv 2 \pmod 4$ then

$$\deg_{\mathcal{V}_i^-} = (D_m \times \mathbb{Z}_2) - (D_{2h}^d) - (D_{2h}^{\hat{d}}) + (\mathbb{Z}_{2h}^d);$$

- if $m/h \equiv 0 \pmod 4$ then

$$\deg_{\mathcal{V}_i^-} = (D_m \times \mathbb{Z}_2) - (D_{2h}^d) - (\widetilde{D}_{2h}^d) + (\mathbb{Z}_{2h}^d);$$

- if $i = \mathfrak{s}$ then

$$\deg_{\mathcal{V}_\mathfrak{s}^-} = (D_m \times \mathbb{Z}_2) - (D_m^z);$$

- if $i = 0$ then

$$\deg_{\mathcal{V}_0^-} = (D_m \times \mathbb{Z}_2) - (D_m);$$

- if m is even and $i = \mathfrak{s} + 1$ then

$$\deg_{\mathcal{V}_{\mathfrak{s}+1}^-} = (D_m \times \mathbb{Z}_2) - (D_m^d);$$

- if m is even and $i = \mathfrak{s} + 2$ then

$$\deg_{\mathcal{V}_{\mathfrak{s}+2}^-} = (D_m \times \mathbb{Z}_2) - (D_m^{\hat{d}}).$$

Notice that $\deg_{\mathcal{V}_i^-} = \deg_{\mathcal{V}_{i'}^-}$ if and only if $\gcd(i, m) = \gcd(i', m)$. Therefore, we introduce the numbers ρ_i, $0 \leq i \leq \mathfrak{s} + 2$, that will allow us to determine how many times the basic degree $\deg_{\mathcal{V}_i^-}$ appears in the degree of G-$\deg(\mathscr{A}, B(\mathscr{E}))$.

Definition 4.6.2. We define $\rho_0 := \eta_0,\ \rho_{\mathfrak{s}} := \eta_{\mathfrak{s}},\ \rho_{\mathfrak{s}+1} := \eta_{\mathfrak{s}+1},\ \rho_{\mathfrak{s}+2} := \eta_{\mathfrak{s}+2}$, and

$$\rho_i := \sum_{\gcd(i',m)=\gcd(i,m)} \eta_{i'}, \qquad 0 < i < \frac{m}{2}. \tag{4.16}$$

Before proving our main theorem, we need to analyze the maximal G-orbit types in the space $\mathscr{E} \setminus \{0\}$.

Lemma 4.6.3. *Suppose that $m = 2^{\mathfrak{n}} m'$, where m' is an odd integer. Then the maximal orbit types in $\mathscr{E} \setminus \{0\}$ are:*

(a) (D_m^z), (D_m)

And if $\mathfrak{n} > 0$,

(b) (D_m^d), $(\widetilde{D}_m^d)$, $(D_{m/2}^d)$, $(\widetilde{D}_{m/2}^d)$,..., $(D_{m/2^{\mathfrak{n}-1}}^d)$, $(\widetilde{D}_{m/2^{\mathfrak{n}-1}}^d)$,

Proof. The maximal G-orbit types in $\mathscr{E} \setminus \{0\}$ are exactly the same as the maximal G-orbit types which occur in the space $\mathcal{V}^* \setminus \{0\}$, where

$$\mathcal{V}^* := \mathcal{V}_0^- \oplus \mathcal{V}_1^- \oplus \mathcal{V}_2^- \oplus \mathcal{V}_{\mathfrak{s}}^- \oplus \mathcal{V}_{\mathfrak{s}+1}^- \oplus \mathcal{V}_{\mathfrak{s}+2}^-.$$

First we identify the maximal orbit types in $\mathcal{V}_i^- \setminus \{0\}$, $i = 0, 1, \ldots, \mathfrak{s} + 2$:

- for $0 \leq i \leq \lfloor \frac{m}{2} \rfloor$, $h := \gcd(m, i)$, $p_h := \frac{m}{h}$ and if p_h is odd, then the orbit types are: (D_h), (D_h^z);
- if $p_h \equiv 2 \pmod 4$, then the maximal orbit types are: $((D_{2h}^d)$, $(D_{2h}^{\hat{d}})$;
- if $p_h \equiv 0 \pmod 4$, then the maximal orbit types are: (D_{2h}^d), $(\widetilde{D}_{2h}^d)$;
- if $i = \mathfrak{s}$, then the maximal orbit type is: (D_m^z);
- if $i = 0$, then the maximal orbit type is: (D_m);
- if m is even and $i = \mathfrak{s} + 1$, then the maximal orbit type is: (D_m^d);
- if m is even and $i = \mathfrak{s} + 2$, then the maximal orbit type is: $(D_m^{\hat{d}})$.

Notice that (D_m) and (D_m^z) are the maximal orbit types which occur in $\mathcal{V}_0^- \oplus \mathcal{V}_{\mathfrak{s}}^- \setminus \{0\}$. On the other hand for m being an even integer, we have (see Table 5.3 in [2]):

- $(D_{2n}^d \leq (D_m^d))$ if and only if $n|\frac{m}{2}$ and $\frac{m}{2n}$ is odd;
- $(\widetilde{D}_{2n}^d \leq (\widetilde{D}_m^d))$ if and only if $n|\frac{m}{2}$ and $\frac{m}{2n}$ is odd;
- $(D_{2n}^{\hat{d}} \leq (D_m^{\hat{d}}))$ if and only if $n|\frac{m}{2}$ and $\frac{m}{4n}$ is odd;

and the maximality of the orbit types (D_m^d), $(\widetilde{D}_m^d)$, $(D_{m/2}^d)$, $(\widetilde{D}_{m/2}^d)$,..., $(D_{m/2^{\mathfrak{n}-1}}^d)$, $(\widetilde{D}_{m/2^{\mathfrak{n}-1}}^d)$ follows from this result. □

We have the following main result:

Theorem 4.6.4. *Let m be a natural number and $f : \mathbb{R} \times \mathbb{R}^k \to \mathbb{R}^k$ be a continuous function satisfying the assumptions (A_1)—(A_3) and (A_5)—(A_6). Suppose $m = 2^{\varepsilon_0} p_1^{\varepsilon_1} p_2^{\varepsilon_2} \dots p_s^{\varepsilon_s}$, where $\varepsilon_0 \geq 0$, $\varepsilon_l > 0$, and p_l, $l = 1, 2, \dots, s$, are the prime numbers such that $2 < p_1 < p_2 < \dots < p_s$. For $l = 1, 2, \dots s$ put $m_l := \frac{m}{p_l}$. Then*

(i) *if ρ_0 is odd, then the system (1.1) admits a G-orbit of $2\pi m$-periodic solutions with symmetries (D_m) (these solutions are not subharmonic)*

(ii) *if $\rho_{\mathfrak{s}}$ is odd, then the system (1.1) admits a G-orbit of $2\pi m$-periodic solutions with symmetries (D_m^z) (these solutions are not subharmonic non-constant p-periodic)*

(iii) *if for some $l = 1, 2, \dots, s$, ρ_{m_l} is odd, then the system (1.1) admits a G-orbit of $2\pi m$-periodic solutions with symmetries either $(D_{m_i}^z)$ or (D_m^z) (solutions are the same as (iii)).*

(iv) *if $\varepsilon_0 > 0$, and $\rho_{\mathfrak{s}+1}$ is odd, then the system (1.1) admits a G-orbit of $2\pi m$-periodic solutions with symmetries exactly (D_m^d) (these solutions are subharmonic).*

(v) *if $\varepsilon_0 > 0$, and $\rho_{\mathfrak{s}+2}$ is odd, then the system (1.1) admits a G-orbit of $2\pi m$-periodic solutions with symmetries exactly $(D_m^{\hat{d}})$ (these solutions are subharmonic).*

(vi) *if $\varepsilon_0 = \mathfrak{k} > 0$, and for some $2 \leq k \leq \mathfrak{k}$, $\rho_{\frac{m}{2^k}}$ is odd, then the system (1.1) admits a G-orbit of $2\pi m$-periodic solutions with symmetries exactly $(D^d_{\frac{m}{2^{k-1}}})$ and $(D^{\hat{d}}_{\frac{m}{2^{k-1}}})$ (these solutions are subharmonic).*

Proof. Clearly for $\mu \in \sigma_-(A)$ we have the following formula for the $\mathcal{V}_i^-$-isotypic multiplicity of the eigenvalue $\lambda_{j,\mu}$

$$m_i^-(\lambda_{j,\mu}) = \begin{cases} \mathfrak{m}(\mu) & \text{if } \ i = i(j),\ 0 < i < \frac{m}{2}, \\ \mathfrak{m}(\mu) & \text{if } \ i = \mathfrak{s},\ j > 0,\ i(j) = 0, \\ \mathfrak{m}(\mu) & \text{if } \ i = \mathfrak{s}+1,\ i(j) = \frac{m}{2}, \\ \mathfrak{m}(\mu) & \text{if } \ i = \mathfrak{s}+2,\ i(j) = \frac{m}{2}, \\ 0 & \text{otherwise.} \end{cases} \tag{4.17}$$

Therefore (by (4.17)), we have

$$G\text{-deg}(\mathscr{A}, B(\mathscr{E})) = \prod_{\mu \in \sigma_-(A)} \prod_{j=0}^{\mathrm{j}_\mu} G\text{-deg}(-\mathrm{Id}\,, B(E(\lambda_{j,\mu})), \tag{4.18}$$

where

$$G\text{-deg}(-\mathrm{Id}\,, B(E(\lambda_{j,\mu})) = \begin{cases} (\deg_{\mathcal{V}_0^-})^{\mathfrak{m}(\mu)} & \text{if } \ j = 0, \\ (\deg_{\mathcal{V}_0^-})^{\mathfrak{m}(\mu)} \cdot (\deg_{\mathcal{V}_\mathfrak{s}^-})^{\mathfrak{m}(\mu)} & \text{if } \ j > 0,\ i(j) = 0, \\ (\deg_{\mathcal{V}_i^-})^{\mathfrak{m}(\mu)} & \text{if } \ 0 < i = i(j) < \frac{m}{2}, \\ (\deg_{\mathcal{V}_{\mathfrak{s}+1}^-})^{\mathfrak{m}(\mu)} \cdot (\deg_{\mathcal{V}_{\mathfrak{s}+2}^-})^{\mathfrak{m}(\mu)} & \text{if } \ i(j) = \frac{m}{2}. \end{cases} \tag{4.19}$$

Consequently, we obtain the following formula:

$$\begin{aligned} G\text{-deg}(\mathscr{F}, \Omega) &= (G) - G\text{-deg}(\mathscr{A}, B(\mathscr{E})) \\ &= (G) - \prod_{\mu \in \sigma_-(A)} \prod_{i=0}^{\mathfrak{s}+2} (\deg_{\mathcal{V}_i^-})^{\beta_i(\mu)}. \end{aligned}$$

Since η_i counts the "total number of times" that the irreducible representation $\mathcal{V}_i^-$ appears in $\sigma_-(\mathscr{A})$, we obtain that

$$G\text{-deg}(\mathscr{F}, \Omega) = (G) - \prod_{i=0}^{\mathfrak{s}+2} (\deg_{\mathcal{V}_i^-})^{\eta_i}. \tag{4.20}$$

Since

$$\deg_{\mathcal{V}_i^-} = \deg_{\mathcal{V}_{i'}^-} \qquad \Leftrightarrow \qquad \gcd(i,m) = \gcd(i',m),$$

and the numbers ρ_i indicate the number of occurrences of the basic degree $\deg_{\mathcal{V}_i^-}$ in the product (4.20), the conclusion follows from Lemma 4.6.3 and the fact that the square of any basic degree is the unit element $(G) \in A(G)$, i.e. $(\deg_{\mathcal{V}_i^-})^2 = (G)$. □

Remark 4.6.5. One could ask a question if the reversibility of the system (1.1) is really necessary in order to apply this method? Clearly, even without condition ((A_2) the map $\mathscr{F}$ is still $\mathbb{Z}_m \times \mathbb{Z}_2$-equivariant, so theoretically, the method described in this paper could be applied. However, one should observe that $G := \mathbb{Z}_m \times \mathbb{Z}_2$ is Abelian and thus all the irreducible two dimensional G-representations $\mathcal{V}_k^-$ are of complex type so $\deg_{\mathcal{V}_k^-} = (G)$, which makes formula (4.20) completely ineffective for $m \geq 3$.

In order to illustrate the applications of Theorem 4.6.4, we only consider a simple case when the operator A has one simple negative eigenvalue μ satisfying

$$-\frac{(\mathfrak{p}+1)^2}{m^2} < \mu < -\frac{\mathfrak{p}^2}{m^2}, \quad \mathfrak{p} := \left\lfloor \frac{m}{2} \right\rfloor. \tag{4.21}$$

Then we get the following result:

Corollary 4.6.6. *Let m be a natural number and $f : \mathbb{R} \times \mathbb{R}^k \to \mathbb{R}^k$ be a continuous function satisfying the assumptions (A_1)—(A_3), (A_5)—(A_6) and $\sigma_-(A)$ consists a single simple eigenvalue μ satisfying* (4.21). *Suppose $m = 2^{\varepsilon_0} p_1^{\varepsilon_1} p_2^{\varepsilon_2} \dots p_s^{\varepsilon_s}$, where $p_l > 2$ are distinct prime numbers, $\varepsilon_0 \geq 0$, $\varepsilon_l \geq 1$ for $l = 1, 2, \dots, s$. Then*

(i) the system (1.1) *admits G-orbit of non-zero $2\pi m$-periodic solutions with symmetries (D_m);*

(ii) if for some $l = 1, 2, \dots, s$, the number $\frac{p_l - 1}{2}$ is odd, the system (1.1) *admits G-orbit of $2\pi m$-periodic solutions with symmetries $(D^z_{m_l})$ or (D^z_m) (here $m_l = \frac{m}{p_l}$);*

(iii) *if* $\varepsilon_0 > 0$*, the system* (1.1) *admits* G*-orbit of* $2\pi m$*-periodic solutions with symmetries exactly* (D_m^d), $(D_m^{\hat{d}})$*;*

(iv) *if* $\varepsilon_0 > 1$*, the system* (1.1) *admits* G*-orbit of* $2\pi m$*-periodic solutions with symmetries exactly* $(D_{\frac{m}{2}}^d)$, $(\widetilde{D}_{\frac{m}{2}}^d)$*;*

Proof. Notice that we have

$$\beta_0(\mu) = 1, \;\; \beta_{\mathfrak{s}}(\mu) = 0, \;\; \beta_i(\mu) = 1 \;\text{ for }\; 0 < i < \frac{m}{2},$$

and in the case m is even

$$\beta_{\mathfrak{s}}(\mu + 1) = \beta_{\mathfrak{s}}(\mu + 2) = 1.$$

Consequently,

- Since $\rho_0 = \beta_0 = 1$, thus there exist a non-zero $2\pi m$-periodic solution to the system (1.1) with symmetries exactly (D_m);
- Since $\rho_{\mathfrak{s}} = \beta_{\mathfrak{s}} = 0$, we cannot conclude the existence of a $2\pi m$-periodic solution to the system (1.1) with symmetries exactly (D_m^z);
- Since $\rho_{m_l} = \beta_{m_l} \cdot \frac{p_l - 1}{2} = \frac{p_l - 1}{2}$, $l = 1, 2, \dots, s$, if $\frac{p_l - 1}{2}$ is odd there exists a $2\pi m$-periodic solution to the system (1.1) with symmetries either $(D_{m_l}^z)$ or (D_m^z) (notice that $(D_{m_l}^z)$ is not a maximal orbit type);
- In the case m is even, i.e. $\varepsilon_0 > 0$, since $\rho_{\mathfrak{s}+1} = \rho_{\mathfrak{s}+2} = 1$, it follows that the system (1.1) has an orbit of $2\pi m$-periodic solutions with symmetries either (D_m^d) or $(D_m^{\hat{d}})$;
- In addition, if $\varepsilon_0 > 1$, notice that $\rho_{\frac{m}{4}} = \beta_{\frac{m}{4}} = 1$, thus the system (1.1) admits G-orbit of $2\pi m$-periodic solutions with symmetries $(D_{\frac{m}{2}}^d)$, $(\widetilde{D}_{\frac{m}{2}}^d)$;

□

In order to illustrate that the other maximal orbit types in in $\mathscr{E} \setminus \{0\}$ can also appear as symmetries of $2\pi m$-periodic subharmonic solutions to (1.1), we assume that $m = 2^n p_1^{\varepsilon_1} p_2^{\varepsilon_2} \dots p_s^{\varepsilon_s}$, $n > 1$, and the operator A has one simple negative eigenvalue μ satisfying

$$-\frac{(\mathfrak{p}+1)^2}{m^2} < \mu < -\frac{\mathfrak{p}^2}{m^2}, \quad \mathfrak{p} := \left\lfloor \frac{m}{2^n} \right\rfloor. \tag{4.22}$$

Then we have:

Corollary 4.6.7. *Let m be a natural number and $f : \mathbb{R} \times \mathbb{R}^k \to \mathbb{R}^k$ be a continuous function satisfying the assumptions (A_1)—(A_3), (A_5)—(A_6) and $\sigma_-(A)$ consists a single simple eigenvalue μ satisfying* (4.22). *Suppose $m = 2^n p_1^{\varepsilon_1} p_2^{\varepsilon_2} \dots p_s^{\varepsilon_s}$, where $n > 2$, $p_l > 2$ are distinct prime numbers, $\varepsilon_0 \geq 0$, $\varepsilon_l \geq 1$ for $l = 1, 2, \dots, s$. Then*

(i) *The system* (1.1) *admits G-orbit of non-zero $2\pi m$-periodic solutions with symmetries (D_m),*

(ii) *The system* (1.1) *admits G-orbit of $2\pi m$-periodic solutions with symmetries exactly $(D^d_{\frac{m}{2^{n-1}}})$, $(\widetilde{D}^d_{\frac{m}{2^{n-1}}})$;*

4.7 Examples of Symmetric Systems

In this section we assume that $k = 3$, $f : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}^3$ satisfies assumptions (A_1)—(A_5) with $\Gamma = D_3$ and

$$A = \frac{1}{4} \begin{bmatrix} -4 & -2 & -2 \\ -2 & -4 & -2 \\ -2 & -2 & -4 \end{bmatrix}. \tag{4.23}$$

Then we have

$$\sigma(A) = \left\{ \mu_0 = -2, \mu_1 = -\frac{1}{2} \right\}.$$

Case $m = 3$: In this case $G = D_3 \times D_3 \times \mathbb{Z}_2$ and we have

$$\sigma(\mathscr{A}) = \left\{ \lambda_{j,l} := 1 + \frac{9(\mu_l - 1)}{j^2 + 9} : l = 0, 1,\ j = 0, 1, 2, \dots \right\}$$

Then

$$\begin{aligned}\sigma_-(\mathscr{A}) = &\Big\{ \lambda_{0,0} = -2,\ \lambda_{1,0} = -\frac{17}{10},\ \lambda_{2,0} = -\frac{14}{13},\ \lambda_{3,0} = -\frac{1}{2},\ \lambda_{4,0} = -\frac{2}{25}, \\ &\lambda_{0,1} = -\frac{1}{2},\ \lambda_{1,1} = -\frac{7}{20},\ \lambda_{2,1} = -\frac{1}{26} \Big\},\end{aligned}$$

and each of the eigenvalues $\lambda_{l,i}$ are isotypicaly simple, i.e. the eigenspaces $E(\lambda_{l,i})$ are irreducible G-representations. More precisely, we have

$$E(\lambda_{0,0}) \simeq \mathcal{V}_{0,0}^-, \quad E(\lambda_{3,0}) \simeq \mathcal{V}_{0,0}^- \oplus \mathcal{V}_{2,0}^-, \quad E(\lambda_{1,0}) \simeq E(\lambda_{2,0}) \simeq E(\lambda_{4,0}) \simeq \mathcal{V}_{1,0}^-,$$
$$E(\lambda_{0,1}) \simeq \mathcal{V}_{1,0}^-, \quad E(\lambda_{1,1}) \simeq E(\lambda_{2,1}) \simeq \mathcal{V}_{1,1}^-.$$

Therefore, we obtain

$$\begin{aligned} G\text{-deg}(\mathscr{F}, \Omega) &= (G) - G\text{-deg}(\mathscr{A}, B_1(0)) \\ &= (G) - (\deg_{\mathcal{V}_{0,0}^-})^2 \cdot \deg_{\mathcal{V}_{2,0}^-} \cdot (\deg_{\mathcal{V}_{1,0}^-})^3 \cdot \deg_{\mathcal{V}_{0,1}^-} \cdot (\deg_{\mathcal{V}_{1,1}^-})^2 \\ &= (G) - \deg_{\mathcal{V}_{0,1}^-} \cdot \deg_{\mathcal{V}_{1,0}^-} \cdot \deg_{\mathcal{V}_{2,0}^-} \end{aligned}$$

GAP Code: We use GAP package `EquiDeg` to compute G-deg$(\mathscr{F}, \Omega)$ for the groups $G := D_3 \times D_3 \times \mathbb{Z}_2$. The GAP code f is given below.

```
LoadPackage( "EquiDeg" );
gr1 := pDihedralGroup( 3 );
gr2 := SymmetricGroup( 2 );
# create the product of D_3 and Z_2
gr3:= DirectProduct( gr1, gr2 );
# create group G
G := DirectProduct( gr1, gr3 );
```

```
# create and name CCSs of gr1 and gr3
ccs_g1:= ConjugacyClassesSubgroups( gr1 );
ccs_g1_names := [ "Z1", "D1", "Z3", "D3" ];
ccs_gr3:=ConjugacyClassesSubgroups( gr3 );
ccs_gr3_names:=["Z1","Z1p","D1","D1z", "Z3",
"D1p","Z3p","D3","D3z","D3p"];
SetCCSsAbbrv(gr1, ccs_g1_names);
SetCCSsAbbrv(gr3, ccs_gr3_names);
ccs := ConjugacyClassesSubgroups( G );
cc := ConjugacyClasses( G );
# create characters of irreducible G-representations
irr := Irr( G );
# compute the corresponding to irr[k[] basic degree degk
```

For a subgroup $K \leq D_n$, we use the symbol $K^p := K \times \mathbb{Z}_2$, but in the code we simply write `Kp`. By using the list of conjugacy classes `cc`, one can easily recognize the irreducible G-representations $\mathcal{V}_{i,l}^{\pm} := \mathcal{V}_i^{\pm} \otimes \mathcal{U}_l$. For example, the character

	$(\pm\mathbf{1})$	$(\pm\kappa_2)$	$(\pm\boldsymbol{\gamma}_2)$	$(\pm\kappa_1)$	$(\pm\boldsymbol{\kappa}_1\kappa_2)$	$(\pm\kappa_1\gamma_2)$	$(\pm\gamma_1)$	$(\pm\gamma_1\kappa_2)$	$(\pm\gamma_1\gamma_2)$
`Irr(G)[9]`	± 2	∓ 2	± 2	0	0	0	∓ 1	± 1	∓ 1

is the character of the G-irreducible representation $\mathcal{V}_{2,1}^{-}$. To be more precise, we have the following correspondence of the characters.

`Irr(G)[1]`$= \chi_{\mathcal{V}_{0,0}^{+}}$	`Irr(G)[7]`$= \chi_{\mathcal{V}_{2,0}^{+}}$	`Irr(G)[13]`$= \chi_{\mathcal{V}_{1,2}^{-}}$
`Irr(G)[2]`$= \chi_{\mathcal{V}_{2,2}^{-}}$	`Irr(G)[8]`$= \chi_{\mathcal{V}_{0,2}^{+}}$	`Irr(G)[14]`$= \chi_{\mathcal{V}_{1,0}^{-}}$
`Irr(G)[3]`$= \chi_{\mathcal{V}_{2,0}^{-}}$	`Irr(G)[9]`$= \chi_{\mathcal{V}_{2,1}^{-}}$	`Irr(G)[15]`$= \chi_{\mathcal{V}_{1,2}^{+}}$
`Irr(G)[4]`$= \chi_{\mathcal{V}_{0,2}^{-}}$	`Irr(G)[10]`$= \chi_{\mathcal{V}_{0,1}^{-}}$	`Irr(G)[16]`$= \chi_{\mathcal{V}_{1,0}^{+}}$
`Irr(G)[5]`$= \chi_{\mathcal{V}_{0,0}^{-}}$	`Irr(G)[11]`$= \chi_{\mathcal{V}_{2,1}^{+}}$	`Irr(G)[17]`$= \chi_{\mathcal{V}_{1,1}^{+}}$
`Irr(G)[6]`$= \chi_{\mathcal{V}_{2,2}^{+}}$	`Irr(G)[12]`$= \chi_{\mathcal{V}_{0,1}^{+}}$	`Irr(G)[18]`$= \chi_{\mathcal{V}_{1,1}^{-}}$

To conclude the computations in GAP (we continue to use the package `EquiDeg`).

```
# unit element in A(G)
u := -BasicDegree( Irr( G )[1] );
# basic degrees
deg01 := BasicDegree( Irr( G )[10] );
deg10 := BasicDegree( Irr( G )[14] );
deg20 := BasicDegree( Irr( G )[3] );
deg := u-deg01*deg10*deg20;
```

The list of conjugacy classes of $G = D_3 \times D_3 \times \mathbb{Z}_2$ generated by GAP is $\{(H_k) : 1 \le k \le 69\}$, where $(G) = (H_{69})$. As a result we obtain

$$\begin{aligned} G\text{-deg}(\mathscr{F}, \Omega) = &-(H_1) + (H_4) + (H_6) + (H_8) + (H_{11}) - 2(H_{18}) - (H_{22}) - (H_{27}) \\ &+ (H_{35}) + (H_{43}) + (H_{44}) + (H_{45}) - (H_{51}) + (H_{63}) - (H_{67}), \end{aligned} \tag{4.24}$$

where the coefficient of G-deg$(\mathscr{F}, \Omega)$ can be easily described using amalgamated notation, for example

```
gap> Print( AmalgamationSymbol( ccs[45] ) );
```

In this way we get the following description of some orbit types represented in G-deg$(\mathscr{F}, \Omega)$:

$$\begin{aligned} H_4 &= -(D_1 \times_{\mathbb{Z}_2} D_1^z), \quad H_6 = (D_1 \times \mathbb{Z}_1), \\ H_8 &= (\mathbb{Z}_1 \times D_1), \quad H_{18} = (D_1 \times D_1), \\ H_{22} &= (\mathbb{Z}_1 \times D_3), \quad H_{21} = (D_1 \times \mathbb{Z}_3), \\ H_{43} &= (D_3 \times D_1), \quad H_{44} = (D_1 \times D_3), \\ H_{45} &= (D_1 \times_{\mathbb{Z}_2}^{D_3} D_3^p), \quad H_{51} = (D_3 \times D_1^z), \\ H_{63} &= (D_3 \times D_3^z), \quad H_{67} = (D_3 \times D_3). \end{aligned}$$

Next we find the maximal orbit types in the representation $\mathscr{E}$:

```
# characters appearing in E
chi :=  Irr(G)[3]+  Irr(G)[5]+ Irr(G)[9]+ Irr(G)[10]
+ Irr(G)[14]+ Irr(G)[18];
# find orbit types in E
orbtyps := ShallowCopy( OrbitTypes( chi ) );
Remove( orbtyps );
# find maximal orbit types in H-0
max_orbtyps := MaximalElements( orbtyps );
Print( List( max_orbtyps, IdCCS ) );
```

The maximal orbit types in $\mathscr{E} \setminus \{0\}$ are

$$(H_{45}) = (D_1 \times_{\mathbb{Z}_2}^{D_3} D_3^p), \quad (H_{52}) = (D_1 \times_{\mathbb{Z}_2}^{D_3^z} D_3^p),$$

$$(H_{63}) = (D_3 \times D_3^z), \quad (H_{67}) = (D_3 \times D_3).$$

Consequently, we obtain the following

Theorem 4.7.1. *Let* $m = 3$ *and* $\Gamma = D_3 = \langle (1,2,3), (2,3) \rangle$ *(acting on* $\mathbb{R}^3$ *by permuting the coordinates). Assume that* $f : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}^3$ *and* A *(given by* (4.31)*) satisfy the conditions* (A_1)*–*(A_6)*. Then there exist*

(i) at least one $(D_1 \times_{\mathbb{Z}_2}^{D_3} D_3^p)$*-orbit of (i.e. at least 6 different)* 6π*-periodic solutions to the system* (1.1) *(these solutions are subharmonic).*

(ii) at least one $(D_3 \times D_3^z)$*-orbit of (i.e. at least 2 different)* 6π*-periodic solutions to the system* (1.1) *(these solutions are not subharmonic).*

(iii) at least one $(D_3 \times D_3)$*-orbit of (i.e. at least 2 different)* 6π*-periodic solutions to the system* (1.1) *(these solutions are not subharmonic).*

Therefore, the system (1.1) *admits at least 10 different* 6π*-periodic solutions.*

Case $m = 4$**:** In this case $G = D_4 \times D_3 \times \mathbb{Z}_2$, we have

$$\sigma(\mathscr{A}) = \left\{ \lambda_{j,l} := 1 + \frac{16(\mu_l - 1)}{j^2 + 16} : l = 0, 1,\ j = 0, 1, 2, \dots \right\}$$

Then

$$\begin{aligned}\sigma_-(\mathscr{A}) = &\Big\{ \lambda_{0,0} = -2,\ \lambda_{1,0} = -\frac{31}{17},\ \lambda_{2,0} = -\frac{7}{5},\ \lambda_{3,0} = -\frac{23}{25},\ \lambda_{4,0} = -\frac{1}{2},\\ &\lambda_{5,0} = -\frac{7}{41},\ \lambda_{0,1} = -\frac{1}{2},\ \lambda_{1,1} = -\frac{7}{17},\ \lambda_{2,1} = -\frac{1}{5} \Big\},\end{aligned}$$

each of the eigenvalues $\lambda_{l,i}$ is isotypicly simple, i.e. the eigenspaces $E(\lambda_{l,i})$ are irreducible G-representations. More precisely, we have

$$E(\lambda_{0,0}) \simeq \mathcal{V}_{0,0}^-, \quad E(\lambda_{4,0}) \simeq \mathcal{V}_{0,0}^- \oplus \mathcal{V}_{2,0}^-, \quad E(\lambda_{1,0}) \simeq E(\lambda_{3,0}) \simeq E(\lambda_{5,0}) \simeq \mathcal{V}_{0,1}^-,$$

$$E(\lambda_{2,0}) \simeq \mathcal{V}_{3,0}^- \oplus \mathcal{V}_{4,0}^-,\ E(\lambda_{0,1}) \simeq \mathcal{V}_{0,1}^-, \quad E(\lambda_{1,1}) \simeq \mathcal{V}_{1,1}^-,\ E(\lambda_{2,1}) \simeq \mathcal{V}_{3,1}^- \oplus \mathcal{V}_{4,1}^-$$

Therefore, we obtain

$$\begin{aligned} G\text{-deg}(\mathscr{F}, \Omega) &= (G) - G\text{-deg}(\mathscr{A}, B_1(0)) \\ &= (G) - (\deg_{\mathcal{V}_{0,0}^-})^2 \cdot (\deg_{\mathcal{V}_{2,0}^-}) \cdot (\deg_{\mathcal{V}_{0,1}^-})^3 \cdot (\deg_{\mathcal{V}_{3,0}^-}) \cdot (\deg_{\mathcal{V}_{4,0}^-}) \\ &\quad \cdot \deg_{\mathcal{V}_{0,2}^-} \cdot \deg_{\mathcal{V}_{1,0}^-} \cdot \deg_{\mathcal{V}_{1,1}^-} \cdot \deg_{\mathcal{V}_{3,1}^-} \cdot (\deg_{\mathcal{V}_{4,1}^-}) \\ &= (G) - (\deg_{\mathcal{V}_{2,0}^-}) \cdot (\deg_{\mathcal{V}_{0,1}^-}) \cdot (\deg_{\mathcal{V}_{3,0}^-}) \cdot (\deg_{\mathcal{V}_{4,0}^-}) \cdot \deg_{\mathcal{V}_{0,2}^-} \\ &\quad \cdot \deg_{\mathcal{V}_{1,0}^-} \cdot \deg_{\mathcal{V}_{1,1}^-} \cdot \deg_{\mathcal{V}_{3,1}^-} \cdot (\deg_{\mathcal{V}_{4,1}^-}) \end{aligned}$$

GAP Code: $G := D_4 \times D_3 \times \mathbb{Z}_2$.

```
LoadPackage( "EquiDeg" );
gr1 := pDihedralGroup( 3 );
gr2 := SymmetricGroup( 2 );
# create the product of D_3 and Z_2
gr3:= DirectProduct( gr1, gr2 );
# create group G
```

```
gr4 := pDihedralGroup( 4 );
G := DirectProduct( gr4, gr3 );
# create and name CCSs of gr4 and gr3
ccs_g1:= ConjugacyClassesSubgroups( gr1 );
ccs_g4_names := [ "Z1", "Z2", "D1", "tD1", "D2",
"Z4", "tD2", "D4" ];
ccs_gr3:=ConjugacyClassesSubgroups( gr3 );
ccs_gr3_names:=["Z1","Z1p","D1","D1z", "Z3",
"D1p","Z3p","D3","D3z","D3p"];
SetCCSsAbbrv(gr4, ccs_g4_names);
SetCCSsAbbrv(gr3, ccs_gr3_names);
ccs := ConjugacyClassesSubgroups( G );
cc := ConjugacyClasses( G );
# create characters of irreducible G-representations
irr := Irr( G );
# compute the corresponding to irr[k[] basic degree degk
```

For a subgroup $K \leq D_n$, we use the symbol $K^p := K \times \mathbb{Z}_2$ and in the code we simply write `Kp`. The group $G = D_4 \times D_3 \times \mathbb{Z}_2$ has 236 conjugacy classes of subgroups. The conjugacy classes are denoted (H_k), $k = 1, 2, \ldots, 236$, and are according to the same order as it is generated by GAP. The group G has 30 irreducible representations which can be easily identified in GAP.

`Irr(G)[2]`$=\chi_{\mathcal{V}^-_{2,2}}$	`Irr(G)[7]`$=\chi_{\mathcal{V}^-_{3,0}}$	`Irr(G)[19]`$=\chi_{\mathcal{V}^-_{4,1}}$
`Irr(G)[3]`$=\chi_{\mathcal{V}^-_{3,2}}$	`Irr(G)[8]`$=\chi_{\mathcal{V}^-_{4,0}}$	`Irr(G)[20]`$=\chi_{\mathcal{V}^-_{0,1}}$
`Irr(G)[4]`$=\chi_{\mathcal{V}^-_{4,2}}$	`Irr(G)[9]`$=\chi_{\mathcal{V}^-_{0,0}}$	`Irr(G)[25]`$=\chi_{\mathcal{V}^-_{1,2}}$
`Irr(G)[5]`$=\chi_{\mathcal{V}^-_{0,2}}$	`Irr(G)[17]`$=\chi_{\mathcal{V}^-_{2,1}}$	`Irr(G)[26]`$=\chi_{\mathcal{V}^-_{1,0}}$
`Irr(G)[6]`$=\chi_{\mathcal{V}^-_{2,0}}$	`Irr(G)[18]`$=\chi_{\mathcal{V}^-_{3,1}}$	`Irr(G)[30]`$=\chi_{\mathcal{V}^-_{1,1}}$

Therefore, we are set up to compute the degree G-deg$(\mathscr{F},\Omega)$:

```
# unit element in A(G)
u := -BasicDegree( Irr( G )[1] );
# basic degrees
deg20 := BasicDegree( Irr( G )[6] );
deg01 := BasicDegree( Irr( G )[20] );
deg30 := BasicDegree( Irr( G )[7] );
deg40 := BasicDegree( Irr( G )[8] );
deg02 := BasicDegree( Irr( G )[5] );
deg10 := BasicDegree( Irr( G )[26] );
deg11 := BasicDegree( Irr( G )[30] );
deg31 := BasicDegree( Irr( G )[18] );
deg41 := BasicDegree( Irr( G )[19] );
deg := u-deg20*deg01*deg30* deg40* deg02* deg10
* deg11* deg31* deg41;
```

We also use the list of all irreducible G-representations generated by GAP. Using this list, the corresponding basic G-degrees are easily computed by the GAP program, so the exact value of G-deg$(\mathscr{F},\Omega)$ is given by

$$\begin{aligned} G\text{-deg}(\mathscr{F},\Omega) = & -3(H_1)+(H_2)+(H_3)+(H_4)-(H_6)+2(H_8)+(H_9)\\ & +(H_{10})+2(H_{11})-(H_{12})+(H_{13})+(H_{14})-(H_{17})+(H_{18})-(H_{19})\\ & -(H_{20})-(H_{23})-(H_{25})-(H_{26})+(H_{27})-(H_{28})-(H_{32})+(H_{33})\\ & -(H_{34})+(H_{35})-(H_{36})-(H_{39})+(H_{40})-(H_{43})-(H_{55})+2(H_{57})\\ & -(H_{60})+(H_{62})+(H_{68})+(H_{71})+(H_{72})-(H_{73})-(H_{76})+(H_{86})\\ & +(H_{87})-(H_{88})-(H_{90})-(H_{92})-(H_{95})+(H_{99})+(H_{100})+(H_{102})\\ & +(H_{103})-(H_{104})+(H_{105})+(H_{109})--(H_{116})+(H_{117})+(H_{119})-(H_{120})\\ & -(H_{122})+(H_{130})-(H_{135})-(H_{138})-(H_{144})+(H_{148})+(H_{151})+(H_{167})\\ & +(H_{169})+(H_{170})-m(H_{173})+(H_{174})+(H_{175})+(H_{177})+(H_{179})-(H_{192})\\ & -(H_{203})-(H_{208})-(H_{212})-(H_{214})+(H_{223})+(H_{225})+(H_{229})+(H_{233}). \end{aligned}$$

```
# characters appearing in E
chi := Irr(G)[2]+Irr(G)[3]+ Irr(G)[4]+ Irr(G)[5]
+ Irr(G)[6]+ Irr(G)[7]+Irr(G)[8]+Irr(G)[9]
+ Irr(G)[17]+ Irr(G)[18]+ Irr(G)[19]+ Irr(G)[20]
+ Irr(G)[25]+ Irr(G)[26]+ Irr(G)[30];
# find orbit types in E
orbtyps := ShallowCopy( OrbitTypes( chi ) );
Remove( orbtyps );
# find maximal orbit types in H-0
max_orbtyps := MaximalElements( orbtyps );
Print( List( max_orbtyps, IdCCS ) );
```

Since the G-isotypic components in $\mathscr{E}$ are easily identified, the GAP program also allows a quick computation of all maximal orbit types in $\mathscr{E} \setminus \{0\}$, namely

$$(H_{177}),\ (H_{178}),\ (H_{179}),\ (H_{180}),\ (H_{223}),\ (H_{224}),\ (H_{225}),$$
$$(H_{228}),\ (H_{229}),\ (H_{232}),\ (H_{233}).$$

One can notice that G-deg$(\mathscr{F}, \Omega)$ has non-zero coefficients for the following maximal orbit types:

$$\begin{aligned}
(H_{177}) &= (D_2^{D_1} \times_{\mathbb{Z}_2}^{D_3} D_3^p), \quad (H_{179}) &&= (\widetilde{D}_2^{\widetilde{D}_1} \times_{\mathbb{Z}_2}^{D_3} D_3^p) \\
(H_{223}) &= (D_4 \times D_3^z), \quad (H_{225}) &&= (D_4^{D_2} \times_{\mathbb{Z}_2}^{D_3} D_3^p), \\
(H_{229}) &= (D_4^{D_2} \times_{\mathbb{Z}_2}^{D_3} D_3^p), (H_{233}) &&= (D_4^{\mathbb{Z}_4} \times_{\mathbb{Z}_2}^{D_3} D_3^p)
\end{aligned}$$

Consequently, we obtain the following result.

Theorem 4.7.2. *Let $m = 4$, $k = 3$ and $\Gamma = D_3 = \langle (1, 2, 3), (2, 3) \rangle$ (acting on $\mathbb{R}^3$ by permuting the coordinates). Assume that $f : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}^3$ and A (given by* (4.31)*) satisfy the conditions (A_1)—(A_6). Then there exist*

(i) at least one $(D_2^{D_1} \times_{\mathbb{Z}_2}^{D_3} D_3^p)$-orbit of (i.e. at least 4 different) 8π-periodic solutions to the system (1.1) *(these solutions are subharmonic).*

(ii) at least one $(\widetilde{D}_2^{\widetilde{D}_1} \times_{\mathbb{Z}_2}^{D_3} D_3^p)$-orbit of (i.e. at least 4 different) 8π-periodic solutions to the system (1.1) *(these solutions are subharmonic).*

(iii) at least one $(D_4 \times D_3^z)$-orbit of (i.e. at least 2 different) 8π-periodic solutions to the system (1.1) *(solutions are not subharmonic).*

(iv) at least one $(D_4^{D_2} \times_{\mathbb{Z}_2}^{D_3} D_3^p)$-orbit of (i.e. at least 2 different) 8π-periodic solutions to the system (1.1) *(solutions are subharmonic).*

(v) at least one $(D_4^{D_2} \times_{\mathbb{Z}_2}^{D_3} D_3^p)$*-orbit of (i.e. at least 2 different)* 8π*-periodic solutions to the system* (1.1) *(these solutions are subharmonic).*

(vi) at least one $(D_4^{\mathbb{Z}_4} \times_{\mathbb{Z}_2}^{D_3} D_3^p)$*-orbit of (i.e. at least 2 different)* 8π*-periodic solutions to the system* (1.1) *(solutions are not subharmonic).*

Therefore, the system (1.1) *admits at least 16 different* 8π*-periodic solutions.*

4.8 Bifurcation in Reversible Non-Autonomous Second Order Differential Equations

Consider the following parametrized system

$$\ddot{u}(t) = \big(-\alpha \mathrm{Id} + A \big) u(t) + f(t, u(t)), \quad u(t) \in \mathbb{R}^k, \tag{4.25}$$

where A is a non-singular $k \times k$-matrix and $f : \mathbb{R} \times \mathbb{R}^k \to \mathbb{R}^k$ is a continuous function satisfying the conditions (A_1)—(A_3) and :

(B_4) $\quad \lim_{x \to 0} \dfrac{f(t,x)}{|x|} = 0$ uniformly with respect to t.

We are interested in studying the bifurcation of the *subharmonic* $2\pi m$-periodic solutions (for some integer m) to (4.25) from the trivial solution $(\alpha, 0)$, i.e. the solutions

$$u(t) = u(t + 2\pi m), \quad \dot{u}(t) = \dot{u}(t + 2\pi m). \tag{4.26}$$

We also consider a subgroup $\Gamma \le S_k$ which acts on $V := \mathbb{R}^k$ by permuting the coordinates of vectors $x = (x_1, x_2, \dots, x_k)^T$ in $\mathbb{R}^k$ given by

$$\sigma x = \sigma(x_1, x_2, \dots, x_k)^T := (x_{\sigma(1)}, x_{\sigma(2)}, \dots, x_{\sigma(k)})^T. \tag{4.27}$$

Clearly, the space $V := \mathbb{R}^k$ equipped with this Γ-action is an orthogonal Γ-representation. We also introduce the following conditions

(B_5) For all $t \in \mathbb{R}$, $x \in V$ and $\sigma \in \Gamma$, we have $f(t, \sigma x) = \sigma f(t, x)$ and $A\sigma x = \sigma A x$;

The condition (B_5) implies that the system (4.25) is Γ-symmetric. The bifurcation problem (4.25) with the periodic conditions (4.26) can be expressed as the following equation

$$\mathscr{F}(\alpha, u) = 0, \quad (\alpha, u) \in \mathbb{R} \oplus \mathscr{E}, \tag{4.28}$$

where

$$\mathscr{F}(\alpha, u) := u - L^{-1}\Big(N_{A+f}\big(\mathfrak{j}(u)\big) - (\alpha+1)\mathfrak{j}(u)\Big), \quad \alpha \in \mathbb{R},\ u \in \mathscr{E}. \tag{4.29}$$

Put $G := \Gamma \times D_m \times \mathbb{Z}_2$. Notice that under the assumptions (A_1)–(A_3) and (B_5), the map $\mathscr{F}$ is G-equivariant completely continuous field such that $\mathscr{F}(\alpha, 0) = 0$ for all $\alpha \in \mathbb{R}$. Moreover, the assumption (B_4) implies that $\mathscr{F}$ is differentiable at $(\alpha, 0)$ with respect to $u \in \mathscr{E}$, and

$$\mathscr{A}(\alpha) := D_u\mathscr{F}(\alpha, 0) = \mathrm{Id}\ - L^{-1}\Big(N_A \cdot \boldsymbol{j} - (\alpha+1)\boldsymbol{j}\Big) : \mathscr{E} \to \mathscr{E}.$$

The necessary condition for the point $(\alpha_o, 0)$ to be a bifurcation point for (4.28) is that $\mathscr{A}(\alpha_o) : \mathscr{E} \to \mathscr{E}$ is **not an isomorphism**, i.e. $0 \in \sigma(\mathscr{A}(\alpha_o))$.

The point α_o is called a *critical point* for (4.28) and the set of all such critical points α_o is denoted Λ. One can easily compute the spectrum of the operator $\mathscr{A}(\alpha_o)$:

$$\sigma(\mathscr{A}(\alpha_o)) := \left\{1 + \frac{m^2(\mu - \alpha_o - 1)}{j^2 + m^2} : j = 0, 1, 2, \dots,\ \mu \in \sigma(A)\right\},$$

which implies that

$$\Lambda = \left\{\alpha_{j,\mu} := \frac{j^2 + m^2\mu}{m^2} : j = 0, 1, 2, \dots,\ \mu \in \sigma(A)\right\}.$$

4.8.1 Bifurcation in System (4.25) without Symmetries

Suppose that $\Gamma = \{e\}$, i.e. $G := D_m \times \mathbb{Z}_2$, and that $\alpha_{j,\mu} \neq \alpha_{j',\mu'}$ for $(j, \mu) \neq (j', \mu')$. Let us put all the elements of Λ in increasing order, i.e. $\dots < \alpha_{j_k,\mu_k} < \alpha_{j_{k+1},\mu_{k+1}} < \dots$. Then for every $\alpha_{j_o,\mu_o} \in \Lambda$ we have

$$\omega_G(\alpha_{j_o,\mu_o}) = \prod_{\alpha_{j_k,\mu_k} < \alpha_{j_o,\mu_o}} (\deg_{\mathcal{W}^-_{i(j_k)}})^{m(\mu_k)}\Big((G) - (\deg_{\mathcal{W}^-_{i(j_o)}})^{m(\mu_o)}\Big). \tag{4.30}$$

Theorem 4.8.1. *Suppose that $A : V \to V$ and $f : \mathbb{R} \times V \to V$ satisfy the assumptions (A_1)—(A_3) and (B_4) and let Λ be the critical set for (4.28). Assume that for all $\alpha_{j,\mu}$, $\alpha_{j',\mu'} \in \Lambda$ we have $\alpha_{j,\mu} \neq \alpha_{j',\mu'}$ if $(j,\mu) \neq (j',\mu')$. Then for every $\alpha_{j_o,\mu_o} \in \Lambda$, suppose that $m(\mu_o)$ is odd we have $\omega(\alpha_{j_o,\mu_o}) \neq 0$, i.e. the point $(\alpha_{j_o,\mu_o}, 0)$ is a bifurcation point of non-trivial $2\pi m$-periodic solutions for (4.25).*

Proof. Notice that under the assumption that $m(\mu_o)$ is odd, we have

$$(G) - (\deg_{\mathcal{W}^-_{i(j_o)}})^{m(\mu_o)} = (G) - (\deg_{\mathcal{W}^-_{i(j_o)}}) \neq 0.$$

Since the product $\displaystyle\prod_{\alpha_{j_k,\mu_k} < \alpha_{j_o,\mu_o}} (\deg_{\mathcal{W}^-_{i(j_k)}})^{m(\mu_k)}$ is an invertible element in $A(G)$, it follows from formula (4.30) that $\omega_G(\alpha_{j_o,\mu_o})$ is non-zero. Therefore, by Theorem 3.4.1, the point $(\alpha_{j_o,\mu_o}, 0)$ is a bifurcation point of non-trivial $2\pi m$-periodic solutions for (4.25). □

Consequently, we obtain the following:

Theorem 4.8.2. *Suppose $A : V \to V$ and $f : \mathbb{R} \times V \to V$ satisfies the assumptions (A_1)—(A_3) and (B_4). Assume that $k > 0$ is odd and $\sigma(A)$ consists of exactly k different eigenvalues such that for all $(j,\mu) \neq (j',\mu')$ the critical points $\alpha_{j,\mu}$, $\alpha_{j',\mu'} \in \Lambda$ are also different. Suppose $m = 2^{\varepsilon_0} p_1^{\varepsilon_1} p_2^{\varepsilon_2} \dots p_s^{\varepsilon_s}$, where $\varepsilon_0 \geq 0$, $\varepsilon_l > 0$ and p_l are the prime numbers such that $2 \leq p_1 < p_2 < \dots < p_s$. For $l = 1, 2, \dots, s$ put $m_l := \frac{m}{p_l}$. Then*

- *(a) if $p_l > 2$ and ρ_l is odd, then the system (4.28) admits an unbounded branch of $2\pi m$-periodic solutions with symmetries at least $(D^z_{m_l})$, $m_l := \frac{m}{p_l}$*
- *(b) if $p_1 = 2$, $\varepsilon_1 = 1$, and ρ_1 is odd, then the system (4.28) admits an unbounded branch of $2\pi m$-periodic solutions with symmetries at least (D^d_m), $(D^{\hat{d}}_m)$*
- *(c) if $p_1 = 2$, $\varepsilon_1 > 1$, and ρ_l is odd, then the system (4.28) admits an unbounded branch of $2\pi m$-periodic solutions with symmetries at least $(D^d_{\frac{m}{2}})$, $(\widetilde{D}^d_{\frac{m}{2}})$.*

4.8.2 Bifurcation in System (4.25) with Additional Symmetries Γ

In this section we assume that $k = 3$, $f : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}^3$ satisfies assumptions (A_1)—(A_3) and (B_4)—(B_5) with $\Gamma = D_3$ and

$$A = \frac{1}{4}\begin{bmatrix} -4 & -2 & -2 \\ -2 & -4 & -2 \\ -2 & -2 & -4 \end{bmatrix}. \tag{4.31}$$

Then we have

$$\sigma(A) = \{\mu_0 = -2, \mu_1 = -\frac{1}{2}\}.$$

Case $m = 3$: In this case $G = D_3 \times D_3 \times \mathbb{Z}_2$ and we have

$$\sigma(\mathscr{A}(\alpha)) = \left\{\lambda_{j,l} := 1 + \frac{9(\mu_l - \alpha - 1)}{j^2 + 9} : l = 0, 1,\ j = 0, 1, 2, \dots\right\}$$

Then

$$\begin{aligned}\Lambda = \Big\{&\alpha_{0,0} = -2,\ \alpha_{0,1} = -1/2,\ \alpha_{1,0} = -\frac{17}{9},\ \alpha_{1,1} = -\frac{7}{18}, \dots, \\ &\alpha_{j,0} = \frac{j^2 - 18}{9},\ \alpha_{j,1} = \frac{j^2 - 9/2}{9},\ \dots\Big\},\end{aligned}$$

and each of the critical values $\alpha_{j,i}$ is isotypicly simple, i.e. the eigenspace $E(\lambda_{j,i})$ is an irreducible G-representation $\mathcal{V}^-_{i(j),l}$. Maximal orbit types in $\mathscr{E} \setminus \{0\}$ are listed in (4.24). Using the same GAP code as in Example (for D_3) in subsection 3.3.2, we can compute the exact bifurcation invariants. Indeed, we have the following critical points from Λ

$$\alpha_{0,0} < \alpha_{1,0} < \alpha_{2,0} < \alpha_{0,1} < \alpha_{1,1} < \alpha_{2,1} < \alpha_{3,0} < \alpha_{3,1} < \alpha_{4,0} < \alpha_{4,1} < \alpha_{5,1} < \dots$$

Thus

$$
\begin{aligned}
\omega(\alpha_{0,0}) &= (G) - \deg_{\mathcal{V}_{0,0}^-} = (H_{67}) \\
\omega(\alpha_{1,0}) &= \deg_{\mathcal{V}_{0,0}^-} \cdot ((G) - \deg_{\mathcal{V}_{1,0}^-}) = -(H_{43}) + (H_{51}) \\
\omega(\alpha_{2,0}) &= \deg_{\mathcal{V}_{0,0}^-} \cdot \deg_{\mathcal{V}_{1,0}^-} \cdot ((G) - \deg_{\mathcal{V}_{1,0}^-}) = (H_{43}) - (H_{51}) \\
\omega(\alpha_{0,1}) &= \deg_{\mathcal{V}_{0,0}^-} \cdot ((G) - \deg_{\mathcal{V}_{0,0}^-} \cdot \deg_{\mathcal{V}_{2,0}^-} = (H_{63}) + (H_{67})) \\
\omega(\alpha_{1,1}) &= \deg_{\mathcal{V}_{2,0}^-} \cdot ((G) - \deg_{\mathcal{V}_{1,1}^-}) = (H_1) - 2(H_3) + (H_4) - (H_5) - (H_8) + (H_{13}) \\
&\quad + (H_{14}) - (H_{16}) + (H_{18}) - (H_{28}) + (H_{31}) \\
\omega(\alpha_{4,0}) &= \deg_{\mathcal{V}_{2,0}^-} \cdot \deg_{\mathcal{V}_{1,1}^-} \cdot ((G) - \deg_{\mathcal{V}_{1,0}^-}) = (H_3) - (H_4) - (H_7) \\
&\quad + (H_8) + 2(H_{16}) - 2(H_{18}) + (H_{43}) - (H_{51}) \\
\omega(\alpha_{2,1}) &= \deg_{\mathcal{V}_{2,0}^-} \cdot \deg_{\mathcal{V}_{1,1}^-} \cdot \deg_{\mathcal{V}_{1,0}^-} \cdot ((G) - \deg_{\mathcal{V}_{1,1}^-}) = -(H_1) + (H_3) + (H_5) \\
&\quad + (H_7) - (H_{13}) - (H_{14}) - (H_{16}) + (H_{18}) + (H_{28}) - (H_{31})
\end{aligned}
$$

4.8.3 Conclusion and future Work

In this dissertation, using the Brouwer G-equivariant degree we were able to establish the existence of multiple non-constant subharmonic periodic solutions and to classify their spatio-temporal symmetries in non autonomous time-reversible systems. This research can be extended to similar systems involving multiple time delays and this is the focus of my current research which is near completion. Since differential systems $\ddot{u}(t) = f(t, u(t))$ with discontinuous right-hand side can be expressed as second order differential inclusions $\ddot{u}(t) \in F(t, u(t))$, the equivariant degree methods can be extended to such systems. This project will part of my future research.

Part of my future research plans also involve the problems related to the existence of non-constant periodic solution in symmetric higher order reversible differential systems with delays, global bifurcation of periodic solutions in symmetric differential system and symmetry breaking in elliptic systems on symmetric domains.

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Izuchukwu Eze obtained his bachelor's degree in mathematics at University of Nigeria in 2013. Izuchukwu continued his education at The University of Texas at Dallas while working as a Teaching Assistant. He went on to obtain his master's in mathematics in 2019 and began working on his doctorate under the supervision of Dr. Wieslaw Krawcewicz. He also obtained both the graduate and advanced graduate teaching certificates at The University of Texas at Dallas and has completed all the requirements for the data science certification which he will obtain in August 2021. Izuchukwu has accepted to join the faculty at Loyola University Chicago in the Department of Mathematics and Statistics from the fall of 2021.

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