APPLICATIONS OF MEAN FIELD THEORY IN MANAGEMENT SCIENCE

by

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This dissertation is dedicated to my beloved wife, Yumi Kim.

APPLICATIONS OF MEAN FIELD THEORY IN MANAGEMENT SCIENCE

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The main objective of my PhD study is to understand an aggregate effect arising from a large number of agents who have a similar aspect of decision markings and objectives. The primary idea of mean field approach is that the individual agent makes a decision by considering the distribution of the other agents rather than assuming that all agents' detailed information on states is collectible. In the first essay of my dissertation, the primary objective is to study the optimal consumption and portfolio selection problem of risk-controlled investors who strive to maximize their utility of both consumption and terminal wealth. Risk is measured by the variance of terminal wealth, which introduces a nonlinear function of the expected value into the control problem, so a standard stochastic control theory is not properly applicable. This control problem is totally open until the discovery of mean field type control. The second essay explores the dynamic competition among a large number of interacting households who own local storage with a self-generated renewable energy system, and each can decide the amount of charging or discharging energy based on the market environment and the level of energy stored. Under the mean field setting, the optimal solution can be interpreted as an optimal policy suggestion by a central planner who is willing to increase the penetration of local storage to enhance the resilience of the grid system. The third essay investigates a new control

problem for dealer's optimal markup and inventory control regarding Over-The-Counter (OTC) trades. The explicit solutions obtained by the mean field approach can contribute to developing a decision support system for dealers willing to coordinate an inter-dealer and investor-dealer market simultaneously. The proposed decision-making rules may facilitate dealers' responses to imbalances in demand and supply to reduce the possibility of policy intervention about liquidity risk in OTC markets.

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CHAPTER 1

A RISK EXTENDED VERSION OF MERTON'S OPTIMAL CONSUMPTION AND PORTFOLIO SELECTION

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1.1 Introduction

Risk is no doubt essential in portfolio choices. The comprehensive review of portfolio choices with various risk measures can be found in Mitra and Ji (2010), Krokhmal, Zabarankin, and Uryasev (2011), Kolm, Tutuncu, and Fabozzi (2014). Also, utility theory is the foundation for the theory of choice under uncertainty. Hence, it is natural to combine the expected utility maximization framework with risk measures (utility-risk management framework hereafter) when considering a risk-controlled individual's optimal investment-consumption decision under uncertainty. The research along this line is scarce and growing (e.g., Basak and Shapiro 2001, Pfeiffer 2016, Wong, Yam, and Zheng 2017), and we aim to contribute to literature in this direction.

In this study, we combine the utility maximization framework of Merton (1969, 1971) with variance minimization idea from Markowitz (1952)'s mean-variance analysis to study the risk management on optimal portfolio and consumption decisions. Our primary interest is to observe how optimal consumption and portfolio rules are altered due to the introduction of the variance risk measurement. We consider an agent who strives to maximize total expected discounted utility of both consumption and terminal wealth while minimizing the variance of terminal wealth. The inclusion of the variance term introduces a nonlinear function of the expected value into the objective of the control problem. The problem is no longer a standard stochastic control problem, but rather a Mean field type control (MFTC hereafter) problem. We use the HJB-FP framework of Bensoussan, Frehse, and Yam (2013) to solve the MFTC problem and obtain a solution depending on the initial condition.

Our work makes four significant contributions. First, despite the growing attention to investigating portfolio selections under a utility-risk management framework, most work does not consider portfolio selections and intermediate consumption simultaneously. Our model fills the gap by integrating intermediate consumption, portfolio selections, and utility-risk management in a unified framework.

Second, our study makes a technical contribution to the literature. It turns out that solving a fixed point equation is the key difficulty of studying the MFTC problem by the HJB-FP framework in our study. We are not only able to rigorously prove the uniqueness and existence of the solution to the fixed point equation, but also able to obtain explicit formulas for the optimal consumption and portfolio choices. Furthermore, the fixed point has a crucial economic interpretation, the average terminal wealth. As a byproduct, we show that the optimal terminal wealth is deterministic for an individual whose penalty of the variance risk is infinite large. Moreover, we rigorously prove that this deterministic optimal terminal wealth is less than the expected optimal terminal wealth in classical Merton's model (i.e., zero penalty of the variance risk), showing that the conservative portfolio choice due to a zero tolerance of variance risk results in lower expected terminal wealth.

Third, we demonstrate the significance of embedding variance risk management criteria on optimal consumption and portfolio selections. Numerical analysis results show that a consumer-investor's investment in risky assets is inversely related to his perspective on the importance of the variance risk as well as the progress of time. More importantly, numerical results demonstrate the increasing-decreasing shape of optimal consumption rate with respect to a consumer-investor's perspective on the importance of the variance risk. We view this nonlinear relation a significant finding, which reveals that our model can not only allow a consumer-investor to control the variance risk, but also allow a consumer-investor to increase his consumption rate. This desirable feature is robustness regardless of values of a consumer-investor's risk aversion coefficient and the market price of risk.

Fourth, our model theoretically derives the expected terminal wealth depending on the different levels of variance risk. It allows a consumer-investor to choose the proper level of variance risk considering her risk aptitude by inputting a target expected terminal wealth. As a result, the extent of risk management for the investor is measurable and observable.

1.1.1 Literature Review: General Literature

Much of the current research on portfolio theory emanates from the path-breaking mean-variance portfolio model of Markowitz (1952), who refine the economic logic of diversification and offer a practical way to choose an "optimal portfolio" of assets by explicitly recognizing investment risk as measured by variance of return. Since then, there have been a considerable amount of studies devoted to the mean-variance framework including the extension from the single-period setting to the dynamic continuous-time formulation; see, for example, Li and Ng (2000), Zhou and Li (2000), Li, Zhou, and Lim (2002), Zhou and Yin (2003), Cesarone, Scozzari, and Tardella (2013), and Qin (2015), among others. In the extension to the multiperiod and continuous-time framework, before the works of Li and Ng (2000) and Zhou and Li (2000) who study the problem by the embedding technique and the stochastic linear-quadratic (LQ) control framework respectively, there is no analytical result. Moreover, to tackle the computational efficiency and to accommodate a broader class of risk measures for various considerations, many researchers have applied the concept of mean-variance analysis to the adoption of different risk measures, such as mean-absolute deviation risk measures, value-at-risk (VaR) risk measures, and coherent risk measures; see, for instance, Konno and Yamazaki (1991), Campbell, Huisman, and Koedijk (2013), He, Jin, and Zhou (2015), Gao, Xiong, and Li (2016), Gao, Zhou, and Li (2017), Rockafellar and Uryasev (2000, 2002), and Ahmadi-Javid (2012), among others.

In view that most analyses of portfolio, whether they are of Markowitz's mean-variance, maximized over one period, Samuelson (1969) formulate and solve a many-period generalization of portfolio selection as lifetime planning of consumption and investment decisions in a discrete time setup using expected utility maximization. Merton (1969, 1971) extend Samuelson (1969)'s work to a continuous-time setting. From that point on, dynamic portfolio optimization through expected utility maximization has been extensively studied; see, for example, Lehoczky, Sethi and Shreve (1983), Karatzas, Lehoczky, Sethi and Shreve (1987), Karatzas, Lehoczky, and Shreve (1987), Cox and Huang (1989), Shreve and Soner (1994), and Brown and Smith (2011), among others.

1.1.2 Literature Review: Mean Field Type Control

Due to time inconsistency, two different optimal strategies of MFTC are both studied. Bensoussan, Frehse, and Yam (2013) develop a coupled system of Hamilton-Jacobi-Bellman and Fokker-Planck equations (HJB-FP hereafter) to solve the MFTC problem and obtain a time inconsistent solution (or pre-commitment solution) which is depending on the initial condition. An alternative way is to find time-consistent strategies. Bjork, Murgoci, Zhou (2014) study the mean-variance problem within a game framework. Using the method introduced in Bjork, Khapko, Murgoci (2017), they derive time consistent equilibrium control by solving extended HJB equation. Pham and Wei (2017) adopt the dynamic programming for mean field type control and derive a solution which is independent of the initial condition. The current study uses the HJB-FP framework to obtain the optimal consumption and portfolio rules. The HJB-FP framework of Bensoussan, Frehse, and Yam (2013) has been applied to study a number of MFTC problems, for example, Bensoussan, Frehse, and Yam (2013) and Bensoussan, Hoe, and Yan (2019) apply this framework to study the continuous-time Markowitz portfolio with short-selling prohibition and capital investment problem respectively, in which closed-form solutions are obtained.

1.1.3 Literature Review: Risk Management and Utility Maximization Unified Framework

Our proposed problem follows the recent trend of embedding risk management criteria into the utility maximization framework. In this section, three studies closely related to our work are discussed. Basak and Shapiro (2001) present the first analytical research to embed the concept of risk management into a utility maximizing problem to analyze optimal dynamic portfolio and wealth/consumption policies. They first consider that a risk-managing investor, constrained to maintain the Value-at-Risk (VaR) of horizon wealth at a prespecified level for managing market-risk exposure, attempts to maximize the utility of terminal wealth. Extending the economic setting to a standard pure-exchange equilibrium model, the study then examines the problem that a VaR manager, who must comply with a VaR constraint imposed at some horizon (shorter than the agent's lifetime), strives to maximize the intertemporal utility of consumption over the lifetime. The dynamic optimization problems are solved using the martingale representation approach. Compared to Basak and Shapiro (2001)'s study, our work integrates intermediate consumption, portfolio selections, and utility-risk management in a unified framework to study the optimal consumption and portfolio choice problem using MFTC approach.

Wong, Yam, and Zheng (2017), henceforth WYZ, study the utility-risk portfolio selection problem by maximizing an investor's utility of terminal wealth with deviation risk. Although the work of WYZ is closely related to our study, they are different in several aspects. First, unlike our work, the work of WYZ does not consider intermediate consumption over the investment horizon in the utility-risk optimization framework.

Consequently, optimal portfolio rules are different, and our study is able to additionally provide the insights with respect to the optimal consumption policy, which is not intuitively guessable. Second, WYZ do not perform numerical analyses. We, on the other hand, quantitatively demonstrate the optimal consumption policy, the optimal investment policy and the wealth process through numerical studies, and obtain several important implications/insights. Third, WYZ convert the utility-risk problem into an equivalent nonlinear moment problem while we study the optimal investment-consumption problem using the HJB-FP framework of Bensoussan, Frehse, and Yam (2013).

Recently, Pfeiffer (2016) studies a continuous-time Merton's portfolio choice problem with cost functionals involving the probability distribution of the state variable. The problem takes the form of a mean-field type control problem. In this study, three cost functions, which are a cost involving the semi-deviation, the Conditional Value at Risk, and a cost with a penalization term with a target, are considered to allow for a risk averse consumer's risk management. Pfeiffer (2016) tackles the mean-field type control problem by solving a coupled system of HJB-FP equations numerically with an iterative method. The main idea of Pfeiffer (2016)'s work is close to the primary purpose of our study, and we both approach the problem by studying coupled system of HJB-FP equations. However, as in the work of WYZ, Pfeiffer (2016) does not consider intermediate consumption in the optimization problem. In addition, we do not only rigorously prove the existence and uniqueness of the solution by a fixed point argument, but also obtain analytical results. To the best of our knowledge, our study proposes the first analytical optimal investment-consumption policy for an extended Merton's model incorporating the idea of risk management proposed by Markowitz's portfolio selection problem.

1.1.4 Literature Review: Literature Related to Our Numerical Study

In our numerical example, we investigate the dynamic portfolio behavior of a consumer-investor who has a constant relative risk aversion (CRRA) utility function on consumption ($\frac{x^{1-\gamma}}{1-\gamma}$) and terminal wealth (log utility) with variance control (importance measured by ϵ). Note that the parameter γ in utility functions represents the consumer-investor's absolute risk aversion. Our MFTC model becomes the traditional Merton's portfolio selection problem when ϵ equals zero, which implies that the consumer-investor is not willing to manage the variation of terminal wealth. In the MFTC model, therefore, the consumer-investor chooses a nonmyopic dynamic portfolio regardless of γ by controlling the variance risk of final wealth.

Dai, Jin, Kou, and Xu (2020) set up a mean-variance model for log returns which is different from the standard mean-variance model for terminal wealth. They study the time-consistent portfolio investment in a complete and an incomplete market. They show that in a complete market the mean-variance optimization and the CRRA utility are equivalent. Therefore, the optimal investment strategy is to invest a constant fraction of wealth in the risky asset as Merton's classical result ($\epsilon = 0$ in our case). While in incomplete markets, they point out that the investment is decreasing as time progresses. We obtain the similar observation in our MFTC model with constant investment opportunity set. In general, the risk aversion coefficient is too difficult to measure in industry practice as well as academic research. Dai, Jin, Kou, and Xu (2020) suggest that the risk aversion can be inferred by inputting a target return, since they prove that there exists a one to one mapping between γ and annual target return in complete market. By analogy, our model demonstrate that expected terminal wealth ρ_ϵ depending on ϵ. Therefore, our MFTC model allows an investor to choose the proper ϵ considering her risk aptitude by inputting a target terminal wealth.

Sotomayor and Cadenillas (2009) study the optimal consumption-investment problem with regime switching, and obtain exact solutions for specific HARA utility functions. They observe a positive effect of consumption as in Merton (1969). That is, the investor increases his consumption as wealth increases. Interestingly, for all ϵ, the consumption in our model is increasing whereas the wealth is decreasing for some time t. In addition, Sotomayor and Cadenillas (2009) observe very high consumption wealth ratios (greater than 1) for investors, which is also observed in our model.

Liu (2007) introduce the analytical solutions for dynamic portfolio selection problem in a continuous-time model with CRRA-class utility functions in stochastic environments. His model indicates that the optimal terminal wealth for $\gamma = \infty$ becomes constant if the investment opportunity is constant, since an infinite risk averse investor constructs the optimal portfolio using only the riskless asset. Similarly, we investigate the extreme case for $\epsilon = \infty$ in the MFTC model. In this case, the final wealth (ρ_∞) becomes deterministic because of no variation in the optimal final wealth. The consumer-investor steadily reduces the weight of risk asset as the remaining time horizon goes to zero, and eventually allocates all her wealth to the risk-free asset at the terminal time horizon to attain the best deterministic final wealth.

In the remainder, Section 1.2 presents the model. Section 1.3 summarizes the theory and methodology of the mean field type control approach document in Bensoussan, Frehse, and Yam (2013) and Bensoussan, Hoe, Kim and Yan (2020), and lays out sufficient condition of optimality. Section 1.4 studies the existence and uniqueness solution of Merton's problem with variance control, and presents the optimal feedback and the optimal value of our model. Properties and solutions to two extreme cases are also studied. Section 1.5 reports the results of our numerical analysis, further highlighting the importance and benefits of our model. Section 1.6 lays out investment-consumption insights that can benefit investors. Section 1.7 presents some concluding remarks.

1.2 Merton's Problem with Variance Control

Merton (1969, 1971) extended Samuelson (1969)'s optimal investment-consumption model in a discrete-time setup to a continuous-time setting. A consumer-investor must choose his consumption and asset allocation strategy between risky assets (stocks) and a risk-free asset optimally so as to maximize expected utility. Merton used stochastic optimal control methodology to obtain the optimal portfolio strategy.

A potential risk presented to a consumer-investor in Merton's model is the deviation of his terminal wealth from the expected. In view of this, an extension of classical Merton's problem that incorporates the variance of a consumer-investor's terminal wealth to measure the risk is proposed. The inclusion of the variance term results in a mean field type control problem that cannot be solved by classical stochastic control methods. In the following, the financial market where a consumer-investor bases his investment-consumption decision is introduced first, followed by a brief review of classical Merton's problem. This section is concluded by presenting the model of Merton's problem with variance control.

1.2.1 Financial Market

A financial market consists of one non-risky asset with a constant interest rate r and n risky assets. Prices of risky assets $Y_i(t),\ i = 1, 2, ...n$, evolve as

$$\begin{aligned} dY_i(t) = Y_i(t)\Big[\alpha_i(t)dt + \sum_{j=1}^{n} \sigma_{ij}(t)dw_j\Big], \\ Y_i(0) = Y_i^0, \end{aligned} \tag{1.1}$$

where $w_j(t)$ are independent standard Wiener processes, constructed on a probability space $(\Omega, \mathcal{A}, P)$ and a filtration $\mathcal{F}^t$, and the coefficients $\alpha_i(t)$, $\sigma_{ij}(t)$ are deterministic functions.

The volatility matrix, $\sigma(t) = \{\sigma_{ij}(t)\}_{n\times n}$, is invertible. $\alpha(t) := (\alpha_1(t), \alpha_2(t), \cdots, \alpha_n(t))^T$, $\sigma(t)$, and $\sigma^{-1}(t)$ are assumed to be bounded. The Sharpe ratio is given by

$$\theta(t) = \sigma^{-1}(t)(\alpha(t) - r\mathbb{1}),$$

where $\mathbb{1} := (1, 1, ...)^T$ denotes a vector of R^n. Define the process $Z(t)$ by

$$\begin{aligned} dZ(t) &= -Z(t)\theta(t).dw(t), \\ Z(0) &= 1, \end{aligned} \tag{1.2}$$

which is called a martingale measure (market indicator). In addition, the process

$$Z(t)Y_i(t)e^{-rt},$$

is a $(P, \mathcal{F}^t)$ martingale.

1.2.2 Classical Merton's Problem

Consider a consumer-investor whose unique source of income comes from his portfolio investment on the market. The wealth at time s is

$$X(s) = \pi_0(s)e^{r(s-t)} + \sum_{i=1}^{n} \pi_i(s)Y_i(s), s > t, X(t) = x, \tag{1.3}$$

where $\pi_0(s)$ and $\pi_i(s)$ are respectively the amount of cash and the number of shares invested in the risky asset i. The portfolio is self-financed and the dynamics of controlled wealth process is given by

$$dX(s) = r\pi_0(s)e^{r(s-t)}ds + \sum_{i=1}^{n} \pi_i(s)dY_i(s) - C(s)ds, s > t, X(t) = x, \tag{1.4}$$

where $C(s)$, representing the consumption rate, $\pi_0(s)$ and $\pi(s) := (\pi_1(s), \pi_2(s), \cdots, \pi_n(s))^T$ are control variables.

Using (1.1) and introducing

$$\varpi_i(s) = \frac{\pi_i(s)Y_i(s)}{X(s)}, i = 1, 2, \cdots, n, \tag{1.5}$$

i.e, $\varpi_i(s)$ denotes the proportion of wealth invested in the risky asset i, it follows after rearrangements

$$\begin{aligned} dX(s) = rX(s)ds + X(s)\sigma^*(s)\varpi(s).(\theta(s)ds + dw(s)) - C(s)ds,\ s > t, \\ X(t) = x. \end{aligned} \tag{1.6}$$

Then $X(s)$ is the state of a dynamic system, with controls $\varpi(.) := (\varpi_1(.), \varpi_2(.), \cdots, \varpi_n(.))^T$ and $C(.)$.

The consumer-investor considers the intertemporal portfolio choice over a finite horizon T, where consumption and wealth allocation between risky assets and a risk-free asset must be made. The investment-consumption performance is measured by utility functions $U_1(c)$ for consumption and $U_2(x)$ for final wealth defined by

$$\begin{aligned} J(\varpi(.), C(.)) = E\int_0^T U_1(C(s))e^{-rs}ds \\ +EU_2(X(T))e^{-rT}, \end{aligned} \tag{1.7}$$

with

$$\begin{aligned} dX(s) = rX(s)ds + X(s)\sigma^*(s)\varpi(s).(\theta(s)ds + dw(s)) - C(s)ds, \\ X(0) = x_0. \end{aligned} \tag{1.8}$$

The consumer-investor's dynamic portfolio optimization problem is to maximize his expected utility over a finite horizon T through his choice of consumption and portfolio investments, that is

$$\Phi(x_0, 0) = \sup_{\varpi(.), C(.)} J(\varpi(.), C(.)). \tag{1.9}$$

The optimization problem of (1.9) can be solved applying dynamic programming and the value function, $\Phi(x_0, 0)$, is the solution of the Bellman equation.

1.2.3 Model of Merton's Problem with Variance Control

One potential problem associated with the classical Merton's model (cf. (1.7), (1.9)) is that a consumer-investor's terminal wealth may vary significantly. To control the variation risk, the penalty term, the variance of the terminal wealth, is added to the classical Merton's performance function (1.7). The consumer-investor's performance function becomes

$$\begin{aligned} J_\epsilon(\varpi(.), C(.)) = E \int_0^T U_1(C(s))e^{-rs}ds \\ +EU_2(X(T))e^{-rT} - \epsilon e^{-rT}\operatorname{var}(X(T)), \end{aligned} \tag{1.10}$$

subject to (1.8). In (1.10), $\epsilon \in [0, \infty)$ is a coefficient which weights the importance of variance.

The dynamic optimization problem is to maximize $J_\epsilon(\varpi(.), C(.))$, that is

$$u(x_0, 0) = \sup_{\varpi(.), C(.)} J_\epsilon(\varpi(.), C(.)). \tag{1.11}$$

The consumer-investor's optimization problem now deals not only with maximizing expected utility over a finite horizon T but also with minimizing the variance of the terminal wealth. Because of the presence of the variance term in (1.10), standard stochastic control cannot be applied to solve the optimization problem. The mean field type control theory is the right tool to study such a control problem.

Remark 1.1. *When $\epsilon = 0$, (1.10) reduces to the classical Merton's problem (cf. (1.7), (1.9)).*

Utility functions in (1.10) satisfy the following assumption:

Assumption 1.1. *$U_1(C), U_2(x) : R^+ \longrightarrow R^+$ is concave and twice differentiable in the interior, $U_i'(0) = +\infty, U_i'(+\infty) = 0, i = 1, 2$.*

1.3 General Mean Field Type Control

As stated in section 1.2.3, the proposed Merton's problem with variance control can be solved applying the mean field type control theory. In this section, the mean-field type control problem is briefly presented, followed by the sufficient conditions of optimality. Details can be found in Bensoussan, Frehse, and Yam (2013) and Bensoussan, Hoe, Kim and Yan (2020).

1.3.1 The Mean Field Type Control Problem

Let $(\Omega, \mathcal{A}, P)$ be a probability space and a filtration $\mathcal{F}^t$ generated by an $n-$dimensional standard Wiener process $w(t)$. Consider a diffusion process in R^n given by

$$\begin{aligned} dx = g(x, v(x,s))ds + \sigma(x, v(x,s))dw, \\ x(0) = x_0. \end{aligned} \tag{1.12}$$

where $x_0 \in R^n$ represents the initial state of the system, $v(x,s) \in R^m$ is the control obtained by feedback, and $\sigma(x)$ is an $n \times n$ matrix which is invertible.

Define payoff to be maximized as

$$\begin{aligned} J(v(.)) = \int_0^T e^{-rs}\, Ef(x(s), v(s))ds \\ +e^{-rT}\, Eh(x(T)) + e^{-rT}\, F(Ex(T)), \end{aligned} \tag{1.13}$$

where $v(s) = v(x(s), s)$ and $x(s)$ is the solution of (1.12) after inserting the feedback. In (1.13), $F(.)$ is a nonlinear function of the expected value of $x(T)$. Because of this term, this is not a standard control problem, but a mean filed type control problem.

Next, transform the stochastic problem (1.13) into a deterministic control problem for a P.D.E. by introducing the Fokker Planck equation

$$\begin{aligned} \frac{\partial m_{v(.)}}{\partial s} - \sum_{ij} \frac{\partial^2}{\partial x_i \partial x_j}(a_{ij}(x, v(s))m_{v(.)}) + \operatorname{div}(g(x, v(s))m_{v(.)}) = 0, \\ m_{v(.)}(x, 0) = \delta(x - x_0), \end{aligned} \tag{1.14}$$

and the solution is denoted by $m_{v(.)}(x,s)$, which is the probability distribution of $x_{v(.)}(s)$. Then the payoff function (1.13) can be rewritten as

$$\begin{aligned} J(v(.)) = \int_0^T e^{-rs} \int_{R^n} f(x,v(s))m_{v(.)}(x,s)dxds \\ +e^{-rT} \int_{R^n} h(x)m_{v(.)}(x,T)dx \\ +e^{-rT} F\left(\int_{R^n} xm_{v(.)}(x,T)dx\right) \int_{R^n} m_{v(.)}(x,T)dx. \end{aligned} \tag{1.15}$$

Remark 1.2. *The term $\int_{R^n} m_{v(.)}(x,T)dx$ equal to 1 is inserted in (1.15). So the functional (1.15) coincides with (1.13). The reason is that the problem (1.14), (1.15) is now considered with $m_{v(.)} \in L^2(R^n) \cap L^1(R^n)$ and not in the space of probability densities, in order to use standard variations of the control. A linear variation will not respect the normalization because the space of probability densities is not a vector space.*

1.3.2 Sufficient Conditions of Optimality

Following Bensoussan, Hoe, Kim and Yan (2020), the sufficient conditions of optimality are briefly presented. First, introduce the Lagrangian function

$$L(x,q,M,v) = f(x,v) + q \cdot g(x,v) + \text{tr } (a(x,v)M), \tag{1.16}$$

where $a(x,v) = \frac{1}{2}\sigma(x,v)\sigma^*(x,v)$, $q \in R^n$, $M \in \mathcal{L}(R^n;R^n)$ and the Hamiltonian function

$$H(x,q,M) = \sup_v L(x,q,M,v). \tag{1.17}$$

Let $\hat{v}(x,q,M)$ denote a measurable function, which attains the maximum in v in the Lagrangian, and write

$$H(x,q,M) = L(x,q,M,\hat{v}(x,q,M)), \tag{1.18}$$

$$G(x,q,M) = g(x,\hat{v}(x,q,M)), \tag{1.19}$$

$$P(x,q,M) = a(x,\hat{v}(x,q,M)). \tag{1.20}$$

Next, look for two functions $u(x,t) \in R$, $\Psi(x,t;T) \in R^n$ solutions of the coupled system of PDEs:

$$-\frac{\partial u}{\partial t} + ru = H(x, Du, D^2u),$$
$$u(x,T) = h(x) + x.DF(\rho) + F(\rho), \tag{1.21}$$

$$-\frac{\partial \Psi}{\partial t} = \mathrm{tr}P(x, Du, D^2u)D^2\Psi \;+ D\Psi.G(x, Du, D^2u),$$
$$\Psi(x,T;T) = x, \tag{1.22}$$

where

$$\Psi(x,t;T) = E_{x,t}[\hat{x}(T)], \tag{1.23}$$

and

$$\rho = \Psi(x_0, 0; T), \tag{1.24}$$

is the expected value of the optimal final state.

Solving this system, one obtains the optimal feedback

$$\hat{v}(x,t) = \hat{v}(x, Du, D^2u), \tag{1.25}$$

and the optimal value

$$J(\hat{v}(.)) = u(x_0, 0) - e^{-rT}\,\rho.DF(\rho). \tag{1.26}$$

Note that $u(x_0, 0)$ is not the optimal value.

1.4 The Existence and Uniqueness Solutions for Merton's Problem with Variance Control

In this section, solutions for Merton's problem with variance control are obtained. The correspondence of notation is stated first in order to apply the general theory presented in Section 1.3.

$$\begin{aligned}
v &= \begin{pmatrix} \varpi \\ C \end{pmatrix}, \\
f(x,v) = U_1(C), g(x,v) &= rx + x\varpi\sigma\theta - C, \\
h(x) &= U_2(x) - \epsilon x^2, \\
a(x,v) &= \frac{1}{2}x^2\varpi^*\sigma^*\sigma\varpi, \\
F(x) &= \epsilon x^2, \\
\beta_i &= (U_i')^{-1}, \\
L(x,q,M,v) &= U_1(C) + q(rx + x\varpi\sigma\theta - C) + tr(\frac{1}{2}x^2\varpi^*\sigma^*\sigma\varpi M), \\
U_1'(\hat{C}) &= q, \hat{C} = \beta_1(q), \\
\hat{\varpi} &= -\frac{q}{Mx}(\sigma^*)^{-1}\theta, M < 0, \\
H(x,q,M) &= U_1(\beta_1(q)) - q\beta_1(q) + qrx - \frac{1}{2}\frac{q^2|\theta|^2}{M}, \\
g(x,\hat{v}) &= rx - \frac{q|\theta|^2}{M} - \beta_1(q) = G(x,q,M), \\
a(x,\hat{v}) &= \frac{1}{2}\frac{q^2|\theta|^2}{M^2} = P(x,q,M).
\end{aligned} \tag{1.27}$$

In order for the Lagrangain, $L(x,q,M,v)$, to admit a maximum, we need to assume that $M < 0$.

Using the above notations, from Section 1.3.2, the system of coupled PDEs for the solutions of Merton's problem with variance control is

$$\begin{aligned}
-\frac{\partial u}{\partial t} + ru &= U_1(\beta_1(\frac{\partial u}{\partial x})) - \frac{\partial u}{\partial x}\beta_1(\frac{\partial u}{\partial x}) + rx\frac{\partial u}{\partial x} - \frac{1}{2}\frac{(\frac{\partial u}{\partial x})^2|\theta|^2}{\frac{\partial^2 u}{\partial x^2}}, \\
u(x,T) &= U_2(x) - \epsilon x^2 + \epsilon\rho_\epsilon^2 + 2\epsilon x\rho_\epsilon,
\end{aligned} \tag{1.28}$$

$$
-\frac{\partial \Psi}{\partial t} = \frac{\partial \Psi}{\partial x}\Big(rx - \frac{|\theta|^2 \frac{\partial u}{\partial x}}{\frac{\partial^2 u}{\partial x^2}} - \beta_1(\frac{\partial u}{\partial x})\Big) + \frac{1}{2}\frac{\partial^2 \Psi}{\partial x^2}\frac{|\theta|^2(\frac{\partial u}{\partial x})^2}{(\frac{\partial^2 u}{\partial x^2})^2}, \qquad (1.29)
$$

$$
\Psi(x, T; T) = x,
$$

and

$$
\rho_\epsilon = \Psi(x_0, 0; T). \qquad (1.30)
$$

In (1.28), (1.29) and (1.30), we have written ρ_ϵ instead of ρ to emphasize the dependence in ϵ. Then, the optimal feedback is given by

$$
\hat{C}_\epsilon(x,t) = \beta_1(\frac{\partial u}{\partial x}),
$$

$$
\hat{\varpi}_\epsilon(x,t) = -\frac{\frac{\partial u}{\partial x}}{x\frac{\partial^2 u}{\partial x^2}}(\sigma^*)^{-1}\theta, \qquad (1.31)
$$

and the optimal value is

$$
J_\epsilon(\hat{C}_\epsilon, \hat{\varpi}_\epsilon) = u(x_0, 0) - 2\epsilon\rho_\epsilon^2 e^{-rT}. \qquad (1.32)
$$

From (1.28), (1.29) and (1.30), ρ_ϵ is solution of a fixed point problem. As shown in the appendix 1.8.1, by introducing $\lambda(x,t) = \frac{\partial u}{\partial x}(x,t)$ and with some transformation, we can reduce the study to a linear P.D.E, and obtain ρ_ϵ as a solution of a fixed point equation given below:

$$
E\beta_{2\epsilon}(\lambda_\epsilon(\rho_\epsilon)\xi_0(T) - 2\epsilon\rho_\epsilon) = \rho_\epsilon, \qquad (1.33)
$$

where $\beta_{2\epsilon}(\mu)$ is the solution of $U_2'(x) - 2\epsilon x = \mu$, $\beta_{20}(\mu) = \beta_2(\mu)$, and μ must be greater than zero. The number ρ_ϵ has a crucial economic interpretation. It represents the average wealth at the horizon T. Namely, it is the target to which the final wealth must be close to. Solving this fixed point equation becomes the key difficulty of the mean field type control problem.

Assumption 1.2.

$$xU_2'(x) \text{ is concave },\tag{1.34}$$

$$xU_2'(x) \leq \gamma x + 1 - \gamma,\ 0 \leq \gamma \leq 1,\tag{1.35}$$

$$\gamma < \frac{1}{2}\bar{\lambda},\ \text{with } \bar{\lambda} \text{ the solution of } E\int_0^T \xi_0(s)\beta_1\big(\bar{\lambda}\xi_0(s)\big)e^{-rs}ds = x_0,\tag{1.36}$$

$$U_2'''(x) > 0.\tag{1.37}$$

Theorem 1.1. *Define* $\mathbb{T}_\epsilon(\rho) = E\beta_{2\epsilon}(\lambda_\epsilon(\rho)\xi_0(T) - 2\epsilon\rho)$. *Under Assumption 1.2, there exists a unique* $\rho_\varepsilon \in \mathbb{R}^+$ *such that* $T_\varepsilon(\rho_\varepsilon) = \rho_\varepsilon$. *Moreover,* $\rho_\varepsilon \in [0, \bar{\rho}]$, *where* $\bar{\rho} = \dfrac{8\epsilon(1-\gamma) + \gamma^2 + \bar{\lambda}^2 \exp|\theta|^2 T - (\bar{\lambda} - \gamma)^2}{4\epsilon(\bar{\lambda} - 2\gamma)}$.

Proof is given in the appendix 1.8.2.

By Theorem 1.1, the existence and uniqueness of solutions for Merton's problem with variance control can be stated explicitly in the following theorem. For notational convenience, we omit ρ_ϵ in the subscript.

Theorem 1.2. *The optimal consumption, investment, and wealth are respectively*

$$\hat{C}_\epsilon(s) = \beta_1(\lambda_\epsilon\xi_0(s)),\tag{1.38}$$

$$\hat{\varpi}_\epsilon(s) = -\frac{\lambda_\epsilon\xi_0(s)\dfrac{\partial G_\epsilon}{\partial\lambda}(\lambda_\epsilon\xi_0(s), s)}{G_\epsilon(\lambda_\epsilon\xi_0(s), s)}(\sigma^*)^{-1}\theta,\tag{1.39}$$

and

$$\hat{X}_\epsilon(s) = G_\epsilon(\lambda_\epsilon\xi_0(s), s).$$

The optimal value is

$$\begin{aligned} J_\epsilon(\hat{C}_\epsilon, \hat{\varpi}_\epsilon) &= E\int_0^T U_1(\beta_1(\lambda_\epsilon\xi_0(s)))e^{-rs}\,ds \\ &\quad + E[U_2(\beta_{2\epsilon}(\lambda_\epsilon\xi_0(T) - 2\epsilon\rho_\epsilon)) - \epsilon\beta_{2\epsilon}^2(\lambda_\epsilon\xi_0(T) - 2\epsilon\rho_\epsilon) - \epsilon\rho_\epsilon^2 \\ &\quad + 2\epsilon\rho_\epsilon\beta_{2\epsilon}(\lambda_\epsilon\xi_0(T) - 2\epsilon\rho_\epsilon)]e^{-rT}, \end{aligned}\tag{1.40}$$

where ρ_ϵ *represents the optimal expected wealth at time* T, *and* $\lambda_\epsilon = \lambda_\epsilon(x_0, 0)(\lambda_\epsilon > \bar{\lambda})$.

1.4.1 Extreme Cases

Since the proposed model of Merton's problem with variance control aims at managing the variance of a consumer-investor's wealth at the end of investment horizon T, a natural question to ask is how the expected optimal final wealth at the end of investment horizon is affected by a consumer-investor's perspective on the importance of variance risk measured by $\epsilon \in [0, \infty)$. The following subsections explore two extreme cases, that is, the cases when a consumer-investor displays no concern (i.e., $\epsilon = 0$) and extreme concern (i.e., $\epsilon = \infty$) of the variance risk. Finally, a proven relationship between these two expected optimal final wealth is presented.

When $\epsilon = 0$

Proposition 1.1. *In this case, the problem reduces to the classical Merton's problem, and*

$$\rho_0 = E\beta_2(\lambda_0 \xi_0(T)). \tag{1.41}$$

See the appendix 1.8.4 for proof.

Equation (2.31) gives an explicit formula for the optimal expected final wealth.

When $\epsilon \to +\infty$

Proposition 1.2.

$$\rho_\infty = \beta_2(\lambda_\infty), \tag{1.42}$$

with $\lambda_\infty = \lambda(\rho_\infty)$ *solution of the equation*

$$\beta_2(\lambda_\infty)e^{-rT} + E\int_0^T \xi_0(s)\beta_1(\lambda_\infty \xi_0(s))e^{-rs}\, ds = x_0. \tag{1.43}$$

See the appendix 1.8.5 for proof.

Equation (1.42) is the best deterministic final wealth guaranteed when $\epsilon \to +\infty$. It is no surprise because it costs too much for a consumer-investor to pay the penalty arising from the variance risk, leading to the case of zero variance.

Comparison between ρ_0 ($\epsilon = 0$) and ρ_∞ ($\epsilon \to +\infty$)

From Section 1.4.1, ρ_∞ turns out to be deterministic, that is, no variation in the optimal final wealth. It is thus natural to study the relation between ρ_0 when no control is made in the variance of the final wealth and ρ_∞ with zero variance.

Proposition 1.3. *Assume that $\beta_1, \beta_2 > 0$ decrease on $(0, \infty)$, and also that β_2 is strictly convex. Then $\rho_0 > \rho_\infty$.*

See the appendix 1.8.6 for proof.

Proposition 1.3 is intuitively explained since ρ_0 is the expected wealth at T without considering the variance risk of the final wealth, and ρ_∞ is the best deterministic wealth at T. To be of no surprise at the final wealth, that is, no deviation from the expected, a consumer-investor allocates all his wealth to risk-free assets. Consequently, ρ_∞ is smaller than ρ_0 as suggested by Proposition 1.3 because risk free assets yield less return than risky assets.

1.5 Numerical Analysis

A primary contribution of this research is to solve for optimal consumption−investment policies of Merton's problem with variance control. In this section, numerical studies are performed to illustrative the quantitative results. In the study, only one risky stock is considered, and utility functions take the following CRRA form which satisfies Assumption 1.1 and 1.2.

$$U_1(x) = \begin{cases} \dfrac{x^{1-\gamma_1}}{1-\gamma_1}, & 0 < \gamma_1 < 1, \\ \ln x, & \gamma_1 = 1, \end{cases} \tag{1.44}$$

$$U_2(x) = \ln x. \tag{1.45}$$

The base parameter values used for the numerical studies are:

$$x_0 = 1;\ T = 1;\ r = 0.04;\ \sigma = 0.2.$$

Remark 1.3. *The optimal controls for the CRRA utility function with variance control problem are given by*

$$\hat{C}_\epsilon(s) = (\lambda_\epsilon \xi_0(s))^{-\frac{1}{\gamma_1}}, \tag{1.46}$$

$$\hat{\varpi}_\epsilon(s) = -\frac{\lambda_\epsilon \xi_0(s) \dfrac{\partial G_\epsilon}{\partial \lambda}(\lambda_\epsilon \xi_0(s), s)}{G_\epsilon(\lambda_\epsilon \xi_0(s), s)} (\sigma^*)^{-1}\theta, \tag{1.47}$$

and

$$\hat{X}_\epsilon(s) = G_\epsilon(\lambda_\epsilon \xi_0(s), s).$$

Remark 1.4. *When $\gamma_1 = 1$ and $\epsilon = 0$, the optimal controls are given by*

$$\begin{aligned} \hat{C}_0(t) = \beta_1\big(\lambda_0 \xi_0(t)\big) &= \frac{r}{1 + (r-1)e^{-r(T-t)}} \hat{X}_0(t), \\ \hat{\varpi}_0(t) &= \frac{\theta}{\sigma}, \end{aligned} \tag{1.48}$$

which recover the results obtained by Merton (1969).

Remark 1.5. *When $\gamma_1 = 1$ and $\epsilon \to +\infty$, the optimal controls are given by*

$$\hat{C}_0(t) = \hat{C}_\infty(t), \tag{1.49}$$

$$\hat{\varpi}_\infty(t) = \frac{\frac{1}{r\lambda_\infty \xi_0(t)}\big(1 - e^{-r(T-t)}\big)\frac{\theta}{\sigma}}{\rho_\infty e^{-r(T-t)} + \frac{1}{r\lambda_\infty \xi_0(t)}\big(1 - e^{-r(T-t)}\big)} = \frac{\theta}{\sigma} - \Big(\frac{\rho_\infty e^{-r(T-t)}}{\hat{X}_\infty(t)}\Big)\Big(\frac{\theta}{\sigma}\Big). \tag{1.50}$$

Unlike Merton's classical result, the investment policy, $\hat{\varpi}_\infty(t)$, depends on both the current wealth and the terminal wealth, and decreases as time progresses.

Under log utility functions, Remark 1.5 shows a “surprising property”. A consumer-investor's optimal consumption behavior coincides in two extreme cases, that is, when he completely disregards the variation risk of terminal wealth and when he is excessively concerned about the variation risk of terminal wealth. In addition, when a consumer-investor extremely concerns about the variation of terminal wealth, the investment policy consists of myopic demand (the first term of equation 1.50, Merton's classical result) and hedging demand (the second term of equation 1.50).

1.5.1 Numerical Verification of Theorem 1.1

The fixed point equation (1.33) is numerically studied to verify Theorem 1.1. Figure **??**, which plots the solutions obtained by the intersection of ρ and T_ϵ, confirms a uniquely determined fixed point.

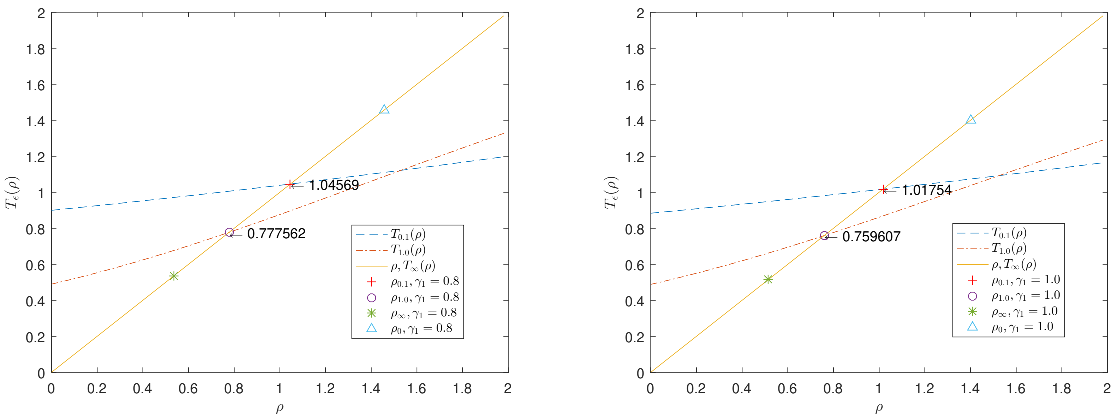

Figure 1.1. Fixed Point Problems depending on $\gamma_1 \in [0.8, 1.0]$ with $\theta = 1.0$

1.5.2 Impact of ϵ on Expected Value and Variance of Optimal Terminal Wealth, ρ_ϵ and $Var(\hat{X}_\epsilon(T))$

Numerical study exhibits that ρ_ϵ decreases with respect to ϵ. A proof of this relationship for ϵ small and $\gamma_1 = 1$ is given in the appendix 1.8.9. This inverse relation persists regardless of values of γ_1 and θ as shown in the left panels of Figure 1.4 and Figure 1.5. In other words, regardless of γ_1 and θ, the more a consumer-investor is concerned about the variance risk of his final wealth, the smaller his expected optimal final wealth will be. The result is expected since a consumer-investor decreases his portfolio holding in risky assets in an effort to reduce the variation of his final wealth as ϵ increases. The immediate effect of investing less wealth in risky assets and more wealth in risk-free assets is a decrease in expected investment returns because risky assets yield higher rates of return. This also explains why ρ_0 and ρ_∞ set the ceiling and the floor of the expected optimal terminal wealth shown in the left panels of Figure 1.2 and Figure 1.3.

Figure 1.9 (left panel) and Figure 1.8 (upper left panel) reveal the necessity of the MFTC model (Merton's problem with variance risk control) with preferable features. With ϵ small, the MFTC model allows a consumer-investor to enjoy a higher consumption rate with lower variation from his optimal terminal wealth. Furthermore, Figure 1.9 (left panel) shows that at terminal time T, $\hat{X}_0(T)$ (i.e. optimal terminal wealth from the traditional Merton's model) is extremely volatile. Obviously, the variance of terminal wealth in the traditional Merton's model is too substantial for a consumer-investor to ignore.

The optimal terminal wealth from MFTC model when ϵ goes to $+\infty$, $\hat{X}_\infty(T)$, is deterministic, a property proven in Section 1.4.1. In other words, a consumer-investor with excessive concerns about the variance risk in the terminal wealth can steer clear of variation in his terminal wealth. Given γ_1, ϵ can then be viewed as the risk aversion

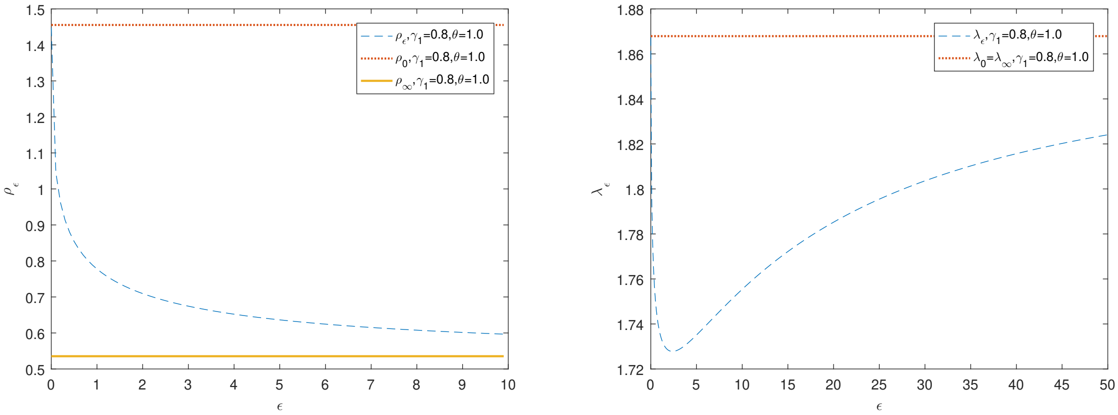

Figure 1.2. ρ_ϵ and λ_ϵ depending on $\epsilon \in (0, 50]$ with $\gamma_1 = 0.8$ and $\theta = 1.0$

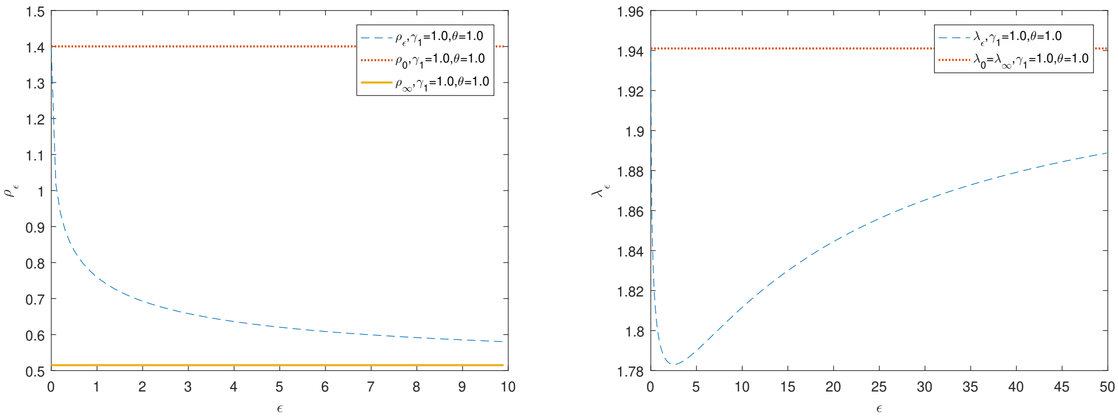

Figure 1.3. ρ_ϵ and λ_ϵ depending on $\epsilon \in (0, 50]$ with $\gamma_1 = 1.0$ and $\theta = 1.0$

of a consumer-investor. The result is then in line with Liu (2007)'s work, which studies dynamic portfolio choice with stochastic variation in investment opportunities, predicting the optimal terminal wealth for an investor with infinite risk aversion is a constant. More importantly, our model exhibits a distinct feature that a consumer-investor with constant terminal wealth can enjoy the same consumption rate as if the traditional Merton's model were implemented as shown in the upper right panels of Figure 1.8 and Remark 1.5.

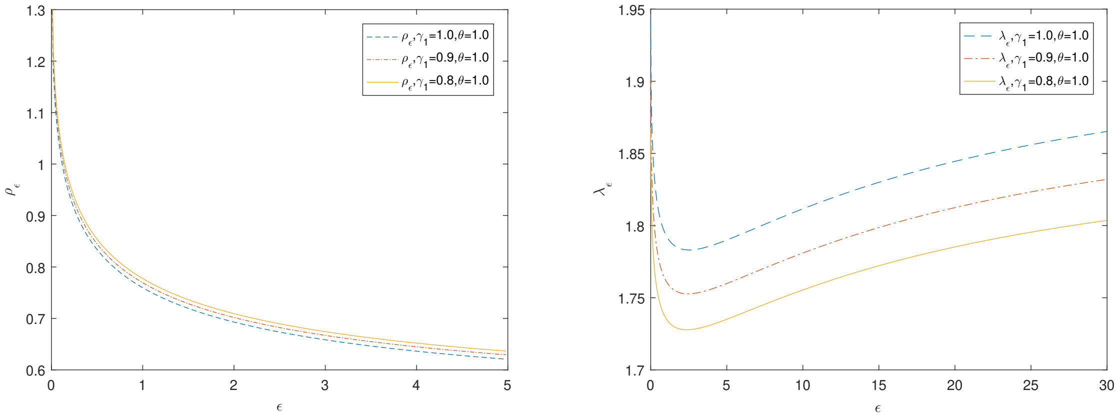

Figure 1.4. ρ_ϵ and λ_ϵ depending on $\epsilon \in (0, 30]$ with $\gamma_1 \in \{0.8, 0.9, 1.0\}$ and $\theta = 1.0$

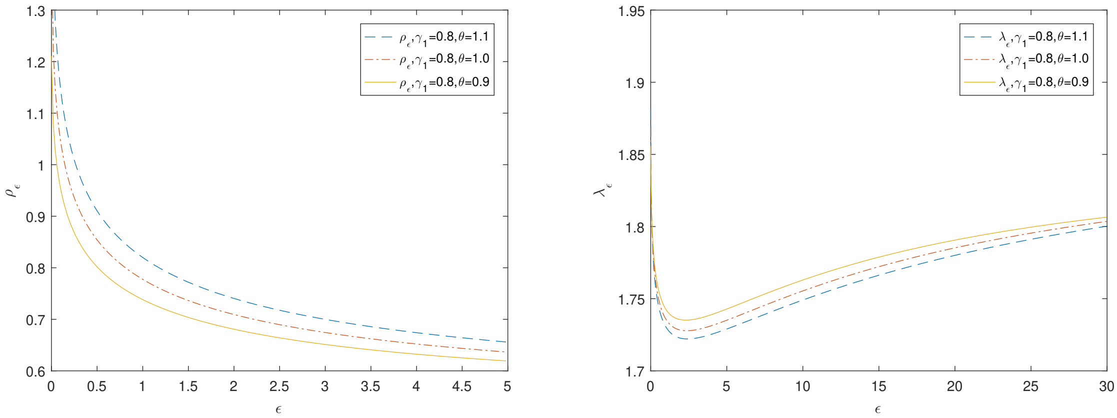

Figure 1.5. ρ_ϵ and λ_ϵ depending on $\epsilon \in (0, 30]$ with $\gamma_1 = 0.8$ and $\theta \in \{0.9, 1.0, 1.1\}$

1.5.3 Impact of ϵ on Optimal Consumption Rate $C_\epsilon(t)$

From equation (1.46), optimal consumption rate is inversely related to λ_ϵ; therefore, we will study λ_ϵ for the consumption rate behavior. The right panels of Figure 1.2 and Figure 1.3 depict a non-linear relation between ϵ and $1/\lambda_\epsilon$, and the relation holds regardless values of γ_1 and θ as shown in the right panels of Figure 1.4 and Figure 1.5.

As ϵ increases, the optimal consumption rate grows rapidly to a positive maximum and then decreases at a decreasing rate. This nonlinearity can be explained by risk-reward

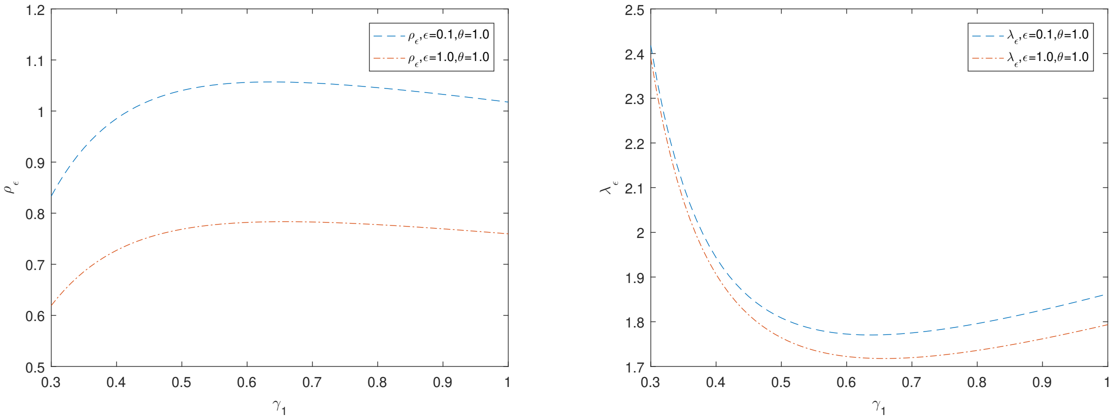

Figure 1.6. ρ_ϵ and λ_ϵ depending on $\gamma_1 \in [0.3, 1.0]$ with $\epsilon \in \{0.1, 1.0\}$ and $\theta = 1.0$

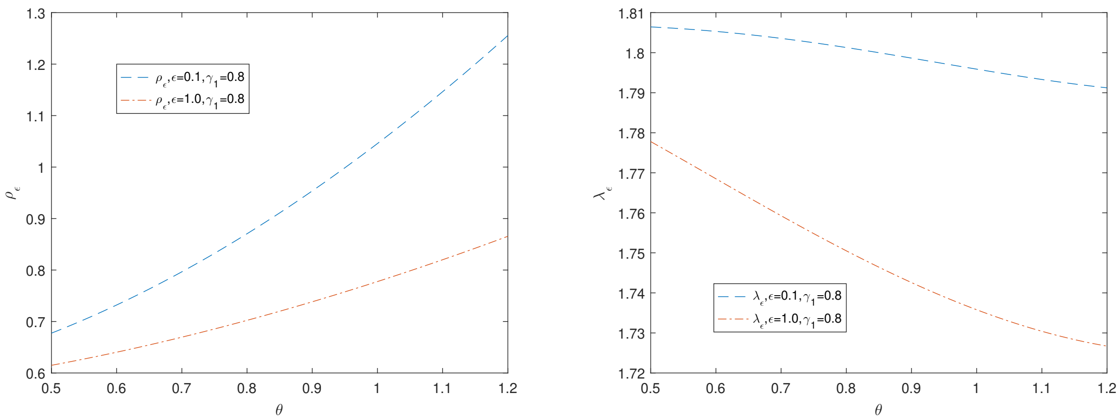

Figure 1.7. ρ_ϵ and λ_ϵ depending on $\theta \in [0.5, 1.2]$ with $\epsilon \in \{0.1, 1.0\}$ and $\gamma_1 = 0.8$

trade-off measured by the ratio between the expected optimal terminal wealth and the standard deviation of the terminal wealth (i.e., $\rho_\epsilon / SD(\hat{X}_\epsilon(T))$). As shown in the right panel of Figure 1.9, the reward (ρ_ϵ) per unit of risk (terminal standard deviation) for a consumer-investor is increasing as ϵ increases. Therefore, a consumer-investor is willing to rebalance his optimal investment and consumption policies to reduce the variance of the terminal wealth when ϵ starts kicking in. The consumer-investor can accomplish this goal by buying or selling assets at the market to change the asset allocation of his portfolio

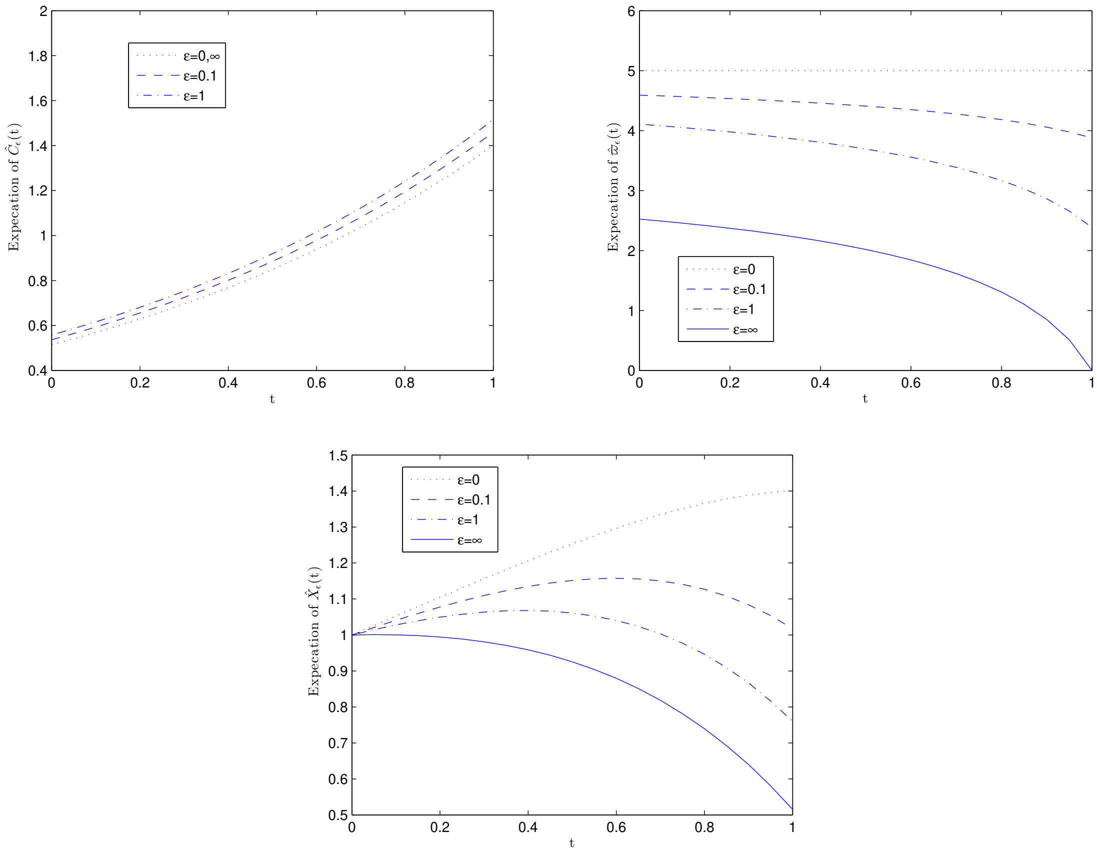

Figure 1.8. Expectation of $\hat{C}_\epsilon(t), \hat{\varpi}_\epsilon(t)$, and $\hat{X}_\epsilon(t)$ for $\epsilon \in \{0, 0.1, 1.0, +\infty\}$

from risky assets to risk free assets or to finance more consumption. As ϵ increases, he reduces his investment in risky assets and rebalances it between the risk-free asset and the consumption rate. When ϵ increases to the level where the variance of the terminal wealth is close to the level of zero, most of his investment is allocated toward the risk-free asset, and thus the amount of wealth that the consumer-investor can finance his consumption decreases. That is why the consumption rate starts decreasing after the point where the variance of the terminal wealth is close to the level of zero.

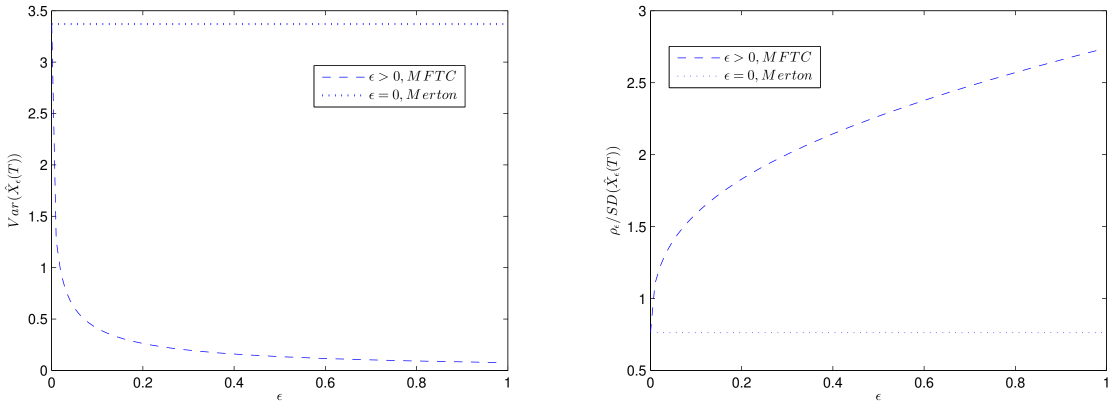

Figure 1.9. Variance of terminal wealth depending on $\epsilon \in [0, 1]$ and the ratio between the expected terminal wealth and standard deviation of terminal wealth depending on $\epsilon \in [0, 1]$

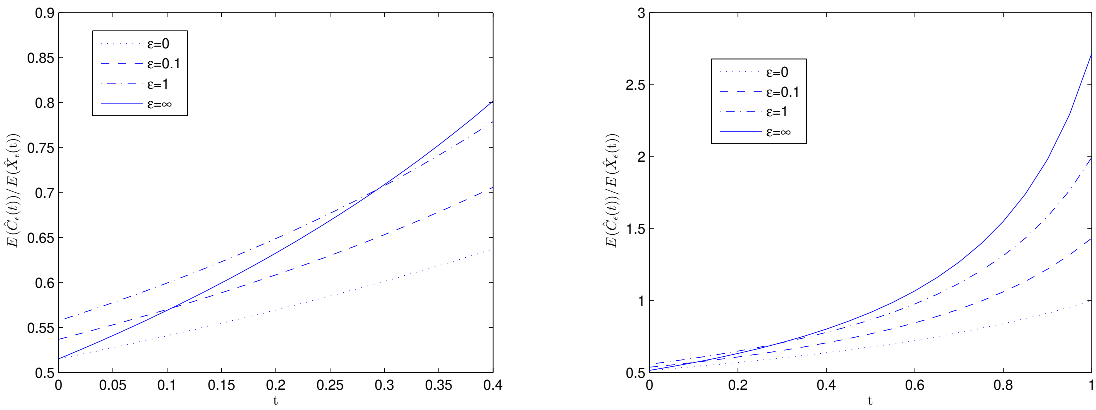

Figure 1.10. Ratio of the expectation of $\hat{C}_\epsilon(t)$), and the expectation of $\hat{X}_\epsilon(t)$

1.5.4 Expected Optimal Consumption Rate, Expected Optimal Percentage Allocations in Risky Assets and Expected Optimal Wealth, $E\hat{C}_\epsilon(t)$, $E\hat{\varpi}_\epsilon(t)$ and $E\hat{X}_\epsilon(t)$

Figure 1.8 graphically studies the expectation of optimal consumption, the expectation of optimal percentage allocations in risky assets, and the expectation of optimal wealth against time. The upper left panel confirms the increasing-decreasing pattern of optimal consumption process as ϵ increases, discussed in section 1.5.3. For a given ϵ, the expected optimal consumption is monotonically increasing against t similar to Merton (1969).

The upper right panel plots the expected optimal proportion of wealth allocated in risky assets, $E[\hat{\varpi}_\epsilon(t)]$, for different values of ϵ against time. It shows that percentage allocations in risky assets continuously decrease as the remaining investment horizon approaches zero and as ϵ increases. For a given γ_1, ϵ can be viewed as a consumer-investor's risk aversion toward terminal wealth given. As such, these observations are comparable to the empirical study by Barberis (2000). Barberis (2000) proposes the optimal portfolio choice for an investor who has a CRRA class utility over terminal wealth. This research shows that the allocation to stocks for the investor, optimally rebalancing the portfolio, steadily decreases as the remaining time horizon goes to zero, and the stock allocation falls as the risk aversion of investor's preferences over terminal wealth increases. In addition, Dai, Jin, Kou, and Xu (2020), who study a dynamic portfolio choice model with the mean-variance criterion for log-returns, also derive that $E[\hat{\varpi}_\epsilon(t)]$ decreases as time proceeds toward the end of the investment horizon, under the incomplete market setting. It is noted that, under a complete market setting, Dai, Jin, Kou, and Xu (2020) obtain the optimal fraction of the total wealth in risky assets as a constant independent of time and wealth, same as in Merton (1969).

Finally, the lower panel presents the expected optimal wealth process, a result of a consumer-investor's investment-consumption decision, for various ϵ. It shows that the expected optimal wealth process decreases as ϵ increases. Comparing to Merton (1969), our model leads to a non-monotonic expected optimal wealth process against time. As time t increases, the expected optimal wealth process increases first and then decreases. The non-monotonic shape holds true even for the extreme case, $\epsilon \to \infty$.

1.5.5 Expected Consumption-Wealth Ratio

In Figure 1.10, the expected consumption-wealth ratio is non-monotonic in ϵ at the beginning investment horizon. However, as t progresses, the expected consumption-wealth ratio becomes monotonically increasing in ϵ. The observation is expected. The differences in the expected wealth among ϵ are not significant at the beginning of investment horizon, and then increase at an increasing rate as time progresses, see Figure 1.8 (lower panel). On the contrary, the differences in expected consumption among ϵ stay quite constant over the entire investment horizon. Consequently, the consumption-wealth ratio becomes monotonically increasing in ϵ after the time when decreases in the expected wealth dominate.

It is not surprising that consumption-wealth ratio in our model is higher than Merton's since our model predicts higher consumption and lower wealth. Moreover, as shown in the figure, the ratios in our model can be greater than one, while in Merton's model the ratios are increasing to one at time T. Sotomayor and Cadenillas (2009) who study investment-consumption problems with regime switching under utility maximization framework, also observes ratios higher than 1 for the power utility x^α with $0 < \alpha < 1$ in every market regime (bull or bear).

1.6 Investment-Consumption Insights

Our MFTC model, combining Merton (1969)'s investment-consumption model and Markowitz (1952)'s mean-variance framework, investigates the optimal investment and consumption policies when the variance risk is explicitly incorporated into the investor-consumer's portfolio selection framework. We obtain the following investment-consumption insights that can benefit investors:

1. Ignoring variance risk of terminal wealth in a consumer-investor's portfolio selection framework is likely to end up yielding terminal wealth significantly lower than the expected. This message is important to both individual investors and professional investors because neither of them would want to be surprised at much lower accumulated terminal wealth than the expected at the end of the investment horizon.

2. The lower the variance risk of terminal wealth that a consumer-investor achieves, the lower the expected terminal wealth will be. This is consistent with the popular investment quote, *"In investing, what is comfortable is rarely profitable." by Robert Arnott*. There is no free lunch; it is a risk-reward tradeoff for investing comfort zone. Consequently, it is important to be able to quantify the variance risk in the portfolio selection instead of risk blindness. As said by Ben Graham, *"The individual investor should act consistently as an investor and not as a speculator."*.

3. Consumer-investors do not control variance risk at the expense of consumption. In fact, consumer-investors enjoy at least the same consumption rate as if he were not to control the variance risk. This again points out the necessity of incorporating the variance risk in the portfolio selection framework. The framework has the benefit of feeding two birds with one stone, that is, consumer-investors not only achieve the target terminal wealth at lower risk, but also enjoy higher consumption rates.

4. For a consumer-investor to achieve a guaranteed terminal wealth, it does not mean he will only invest in risk-free assets over the entire investment horizon. If he does so, he will not be able to afford the consumption rate as one who does not control

the variance risk at all because his gains will be minimal due to the extremely low interest rates from risk-free assets. Our MFTC model can actually help a consumer-investor achieve the goal through the investment policy which properly balances portfolios between risky assets and risk-free assets over the investment horizon.

1.7 Conclusion

A consumer-investor's investment-consumption problem is studied through integrating intermediate consumption, portfolio selections, and utility-risk management in a unified framework. Applying the mean field type control theory and overcoming the the key difficulty of solving a fixed point equation, explicit formulas for the optimal consumption and portfolio choices are obtained. When $\epsilon = 0$, closed form solutions for the traditional Merton's problem with logarithmic utilities ($U_1(x) = U_2(x) = \ln x$) are recovered. For comparison purposes, closed form solutions for our MFTC model are derived when $\epsilon \to \infty$. By inspecting the closed form solutions obtained, it reveals that, by implementing our MFTC model, a consumer-investor can obtain guaranteed terminal wealth and meanwhile enjoy the same consumption rate as the traditional Merton's model which bears high variation in the terminal wealth.

Numerical analysis results show that our MFTC model can not only effectively control the variance risk, but also allow a consumer-investor to increase his consumption rate. This desirable feature is illustrated by the increasing-decreasing shape of optimal consumption rate with respect to ϵ. Regardless of values of γ_1 and θ, the optimal consumption rate increases quickly to a positive maximum before starting to decrease at a decreasing rate as ϵ increases. By inspecting numerical results graphically, it reveals that the nonlinear shape is a consequence of a risk averse consumer-investor's risk-reward trade-off ($\rho_\epsilon / SD(\hat{X}_\epsilon(T))$) as well as the decreasing return from increasingly investing in risk-free assets. When ϵ increases to the level where the variance of the terminal wealth is close to the level of zero, most of his investment is allocated toward the risk-free asset, and thus the amount of wealth that the consumer-investor can finance his consumption decreases. That is why the consumption rate starts decreasing after the point where the variance of the terminal wealth is close to the level of zero.

1.8 Appendix. Proofs

1.8.1 Derivation of Merton's Solution

From (1.28), (1.29) and (1.30), ρ_ϵ is solution of a fixed point problem, we begin by setting $\rho > 0$ and look for $u_{\epsilon,\rho}$, the solution of

$$-\frac{\partial u}{\partial t} + ru = U_1(\beta_1(\frac{\partial u}{\partial x})) - \frac{\partial u}{\partial x}\beta_1(\frac{\partial u}{\partial x}) + rx\frac{\partial u}{\partial x} - \frac{1}{2}\frac{(\frac{\partial u}{\partial x})^2|\theta|^2}{\frac{\partial^2 u}{\partial x^2}}, \qquad u(x,T) = U_2(x) - \epsilon x^2 + \epsilon\rho^2 + 2\epsilon x\rho. \tag{1.51}$$

The Derivative Equation

In fact, $\lambda(x,t) = \frac{\partial u}{\partial x}(x,t)$ is the solution of the equation

$$-\frac{\partial \lambda}{\partial t} = \big(-\beta_1(\lambda) + rx\big)\frac{\partial \lambda}{\partial x} - |\theta|^2\lambda + \frac{1}{2}\lambda^2\frac{\frac{\partial^2\lambda}{\partial x^2}}{(\frac{\partial \lambda}{\partial x})^2}|\theta|^2, \tag{1.52}$$

with the boundary condition

$$\lambda(x,T) = U_2'(x) - 2\epsilon x + 2\epsilon\rho. \tag{1.53}$$

We postulate that $\lambda(x,t)$ can be obtained as the inverse of a function $G(\lambda,t)$ by solving $G(\lambda,t) = x$. We can write the derivatives

$$\frac{\partial \lambda}{\partial t} = -\frac{\frac{\partial G}{\partial t}}{\frac{\partial G}{\partial \lambda}}, \quad \frac{\partial \lambda}{\partial x} = \frac{1}{\frac{\partial G}{\partial \lambda}}, \quad \frac{\partial^2 \lambda}{\partial x^2} = -\frac{\frac{\partial^2 G}{\partial \lambda^2}}{(\frac{\partial G}{\partial \lambda})^3}. \tag{1.54}$$

Then from equation (1.52), $G(\lambda,t)$ is the solution of a linear P.D.E.

$$-\frac{\partial G}{\partial t} - |\theta|^2\lambda\Big(\frac{\partial G}{\partial \lambda} + \frac{1}{2}\lambda\frac{\partial^2 G}{\partial \lambda^2}\Big) + rG = \beta_1(\lambda). \tag{1.55}$$

At time T, we also have

$$G(\lambda, T) = x \Longleftrightarrow \lambda(x, T) = U_2'(x) - 2\epsilon x + 2\epsilon\rho.$$

For $\mu \in R$, there exists one and only one solution $x > 0$ of

$$U_2'(x) - 2\epsilon x = \mu. \tag{1.56}$$

We call it $\beta_{2\epsilon}(\mu)$. Of course $\beta_{20}(\mu) = \beta_2(\mu)$ and μ must be greater than zero. So the final condition is

$$G(\lambda, T) = \beta_{2\epsilon}(\lambda - 2\epsilon\rho). \tag{1.57}$$

Let $\xi_t(s)$ be the martingale

$$\begin{aligned} d\xi_t(s) &= -\xi_t(s)\theta.dw(s), \\ \xi_t(t) &= 1. \end{aligned} \tag{1.58}$$

Then from the Feynman-Kac formula, the solution of (1.55), (1.57) can be expressed by

$$\begin{aligned} G_{\epsilon,\rho}(\lambda, t) &= E\xi_t(T)\beta_{2\epsilon}(\lambda\xi_t(T) - 2\epsilon\rho)e^{-r(T-t)} \\ &+E\int_t^T \xi_t(s)\beta_1(\lambda\xi_t(s))e^{-r(s-t)}\,ds, \end{aligned} \tag{1.59}$$

and $\lambda_{\epsilon,\rho}(x,t) = \dfrac{\partial u_{\epsilon,\rho}}{\partial x}$ is obtained by solving $G_{\epsilon,\rho}(\lambda, t) = x$.

Finding $u_{\epsilon,\rho}(x,t)$

We can express

$$u_{\epsilon,\rho}(x,t) = \Phi_{\epsilon,\rho}(\lambda_{\epsilon,\rho}(x,t), t), \tag{1.60}$$

with $\Phi_{\epsilon,\rho}(\lambda, t)$ solution of

$$\begin{aligned} -\frac{\partial\Phi}{\partial t} + r\Phi - \frac{1}{2}\lambda^2|\theta|^2\frac{\partial^2\Phi}{\partial\lambda^2} &= U_1(\beta_1(\lambda)), \\ \Phi(\lambda, T) = U_2(\beta_{2\epsilon}(\lambda - 2\epsilon\rho)) - \epsilon\beta_{2\epsilon}^2(\lambda - 2\epsilon\rho) &+ \epsilon\rho^2 + 2\epsilon\rho\beta_{2\epsilon}(\lambda - 2\epsilon\rho), \end{aligned} \tag{1.61}$$

and we have

$$\Phi_{\epsilon,\rho}(\lambda,t) = E\int_t^T U_1(\beta_1(\lambda\xi_t(s)))e^{-r(s-t)}\,ds$$
$$+E\big[U_2(\beta_{2\epsilon}(\lambda\xi_t(T)-2\epsilon\rho)) - \epsilon\beta_{2\epsilon}^2(\lambda\xi_t(T)-2\epsilon\rho) + \epsilon\rho^2 + 2\epsilon\rho\beta_{2\epsilon}(\lambda\xi_t(T)-2\epsilon\rho)\big]e^{-r(T-t)}. \tag{1.62}$$

Finding $\Psi_{\epsilon,\rho}(x,t;T)$

Similarly, we have

$$\Psi_{\epsilon,\rho}(x,t;T) = \chi_{\epsilon,\rho}(\lambda_{\epsilon,\rho}(x,t),t), \tag{1.63}$$

with $\chi_{\epsilon,\rho}(\lambda,t)$ solution of

$$\begin{aligned}&\frac{\partial\chi}{\partial t} + \frac{1}{2}\lambda^2|\theta|^2\frac{\partial^2\chi}{\partial\lambda^2} = 0,\\ &\chi(\lambda,T) = \beta_{2\epsilon}(\lambda-2\epsilon\rho),\end{aligned} \tag{1.64}$$

We can then write

$$\chi_{\epsilon,\rho}(\lambda,t) = E\beta_{2\epsilon}(\lambda\xi_t(T)-2\epsilon\rho). \tag{1.65}$$

Therefore,

$$\chi_{\epsilon,\rho_\epsilon}(\lambda_\epsilon(\rho_\epsilon),0) = \Psi(G_{\epsilon,\rho_\epsilon}(\lambda_\epsilon(\rho_\epsilon)),0;T) = \Psi(x_0,0;T),$$

with

$$\lambda_\varepsilon(\rho_\varepsilon) = \lambda_{\epsilon,\rho}(x_0,0).$$

Equivalently,

$$E\beta_{2\epsilon}(\lambda_\epsilon(\rho_\epsilon)\xi_0(T)-2\epsilon\rho_\epsilon) = \rho_\epsilon. \tag{1.66}$$

Hence, ρ_ϵ is a solution of a fixed point equation.

1.8.2 Proof of Theorem 1.1

Before proving this theorem we need the following three lemmas. Lemma 1.1 is needed in the proof of Lemma 1.2, Lemma 1.2 establishes the property needed in the contraction mapping used in the proof a unique fixed point in Theorem 1.1, Lemma 1.3 establishes the property that ρ is bounded from above.

Lemma 1.1.

$$\lambda'_\epsilon(\rho) > 0. \tag{1.67}$$

Proof:

For fixed ρ, there exists a single $\lambda_\epsilon(\rho)$ solution of

$$G_{\epsilon,\rho}(\lambda, 0) = x_0. \tag{1.68}$$

Indeed

$$G_{\epsilon,\rho}(\lambda, 0) = E\xi_0(T)\beta_{2\epsilon}(\lambda\xi_0(T) - 2\epsilon\rho)e^{-rT} + E\int_0^T \xi_0(s)\beta_1\big(\lambda\xi_0(s)\big)e^{-rs}ds \tag{1.69}$$

As λ varies from 0 to $+\infty$, $G_{\epsilon,\rho}(\lambda, 0)$ decreases ∞ to 0. From (1.68) and (1.69), differentiate respect to ρ to deduce that

$$\begin{aligned} E\xi_0(T)\beta'_{2\epsilon}\big(\lambda_\epsilon(\rho)\xi_0(T) - 2\epsilon\rho\big)\big(\lambda'_\epsilon(\rho)\xi_0(T) - 2\epsilon\big)e^{-rT} \\ +E\int_0^T \xi_0^2(s)\beta'_1\big(\lambda_\epsilon(\rho)\xi_0(s)\big)\lambda'_\epsilon(\rho)e^{-rs}ds = 0. \end{aligned} \tag{1.70}$$

so that

$$\begin{aligned} \lambda'_\epsilon(\rho)\Big[E\xi_0^2(T)\beta'_{2\epsilon}\big(\lambda_\epsilon(\rho)\xi_0(T) - 2\epsilon\rho\big)e^{-rT} + E\int_0^T \xi_0^2(s)\beta'_1\big(\lambda_\epsilon(\rho)\xi_0(s)\big)e^{-rs}ds\Big] \\ = 2\epsilon e^{-rT}E\xi_0(T)\beta'_{2\epsilon}\big(\lambda_\epsilon(\rho)\xi_0(T) - 2\epsilon\rho\big). \end{aligned} \tag{1.71}$$

From Assumption 1.1, we have $\beta'_{2\epsilon} < 0$ and $\beta'_1 < 0$, therefore,

$$\lambda'_\epsilon(\rho) > 0. \tag{1.72}$$

$\square$

Lemma 1.2.

$$|\mathbb{T}'_\epsilon(\rho)| \leq -2\epsilon E\beta'_{2\epsilon}\big(\lambda_\epsilon(\rho)\xi_0(T) - 2\epsilon\rho\big). \tag{1.73}$$

Proof:

First, it is obvious that

$$\mathbb{T}'_\epsilon(\rho) \leq -2\epsilon E\beta'_{2\epsilon}\big(\lambda_\epsilon(\rho)\xi_0(T) - 2\epsilon\rho\big). \tag{1.74}$$

Second,

$$\begin{aligned} -\mathbb{T}'_\epsilon(\rho) \leq & \frac{2\epsilon e^{-rT}\Big(E\xi_0(T)\beta'_{2\epsilon}\big(\lambda_\epsilon(\rho)\xi_0(T) - 2\epsilon\rho\big)\Big)^2}{-E\xi_0^2(T)\beta'_{2\epsilon}\big(\lambda_\epsilon(\rho)\xi_0(T) - 2\epsilon\rho\big)e^{-rT} - E\int_0^T \xi_0^2(s)\beta'_1\big(\lambda_\epsilon(\rho)\xi_0(s)\big)e^{-rs}ds} \\ & \leq -2\epsilon E\beta'_{2\epsilon}\big(\lambda_\epsilon(\rho)\xi_0(T) - 2\epsilon\rho\big), \end{aligned} \tag{1.75}$$

and thus (1.73) is proved. $\square$

Lemma 1.3. *Assume (1.34),(1.35), and (1.36). Then a fixed point of the map $T_\epsilon(\rho)$ satisfies*

$$\rho \leq \bar{\rho} = \frac{8\epsilon(1-\gamma) + \gamma^2 + \bar{\lambda}^2 \exp|\theta|^2 T - (\bar{\lambda} - \gamma)^2}{4\epsilon(\bar{\lambda} - 2\gamma)}. \tag{1.76}$$

Proof:

From $G_{\epsilon,\rho}(\lambda_\epsilon(\rho), 0) = x_0$ and from (1.59), it follows that

$$E\int_0^T \xi_0(s)\beta_1\big(\lambda_\epsilon(\rho)\xi_0(s)\big)e^{-rs}ds \leq x_0. \tag{1.77}$$

Therefore, $\lambda_\epsilon(\rho) > \bar{\lambda} > 0$ with $\bar{\lambda}$ the solution of

$$E\int_0^T \xi_0(s)\beta_1\big(\bar{\lambda}\xi_0(s)\big)e^{-rs}ds = x_0. \tag{1.78}$$

Since $\lambda_\epsilon(\rho) > \bar{\lambda}$, we have

$$\mathbb{T}_\epsilon(\rho) \leq E\beta_{2\epsilon}(\bar{\lambda}\xi_0(T) - 2\epsilon\rho); \tag{1.79}$$

so ρ must satisfy

$$\rho \leq E\beta_{2\epsilon}(\bar{\lambda}\xi_0(T) - 2\epsilon\rho). \tag{1.80}$$

If we introduce $X_\epsilon(\rho) = \beta_{2\epsilon}(\bar{\lambda}\xi_0(T) - 2\epsilon\rho)$, then from (1.56), we obtain

$$U_2'(X_\epsilon(\rho)) - 2\epsilon X_\epsilon(\rho) = \bar{\lambda}\xi_0(T) - 2\epsilon\rho. \tag{1.81}$$

Multiplying both sides of the equation by $X_\epsilon(\rho)$ yields

$$2\epsilon X_\epsilon^2(\rho) + \big(\bar{\lambda}\xi_0(T) - 2\epsilon\rho\big)X_\epsilon(\rho) - X_\epsilon(\rho)U_2'(X_\epsilon(\rho)) = 0. \tag{1.82}$$

Therefore,

$$X_\epsilon(\rho) = \frac{-\big(\bar{\lambda}\xi_0(T) - 2\epsilon\rho\big) + \sqrt{\big(\bar{\lambda}\xi_0(T) - 2\epsilon\rho\big)^2 + 8\epsilon X_\epsilon(\rho)U_2'(X_\epsilon(\rho))}}{4\epsilon}. \tag{1.83}$$

Taking expectations, we obtain

$$EX_\epsilon(\rho) \leq \frac{-\big(\bar{\lambda} - 2\epsilon\rho\big) + \sqrt{\bar{\lambda}^2 \exp|\theta|^2T + 4\epsilon^2\rho^2 - 4\epsilon\rho\bar{\lambda} + 8\epsilon EX_\epsilon(\rho)U_2'(EX_\epsilon(\rho))}}{4\epsilon}, \tag{1.84}$$

which is equivalent to

$$\big(4\epsilon EX_\epsilon(\rho) + \bar{\lambda} - 2\epsilon\rho\big)^2 \leq \bar{\lambda}^2 \exp|\theta|^2T + 4\epsilon^2\rho^2 - 4\epsilon\rho\bar{\lambda} + 8\epsilon EX_\epsilon(\rho)U_2'(EX_\epsilon(\rho)). \tag{1.85}$$

From the assumption (1.35), we have

$$16\epsilon^2\big(EX_\epsilon(\rho)\big)^2 + 8\epsilon EX_\epsilon(\rho)\big(\bar{\lambda} - 2\epsilon\rho - \gamma\big) \leq 8\epsilon(1-\gamma) + \bar{\lambda}^2(\exp|\theta|^2T - 1); \tag{1.86}$$

then,

$$\rho \le EX_\epsilon(\rho) \le \frac{2\epsilon\rho + \gamma - \bar{\lambda} + \sqrt{8\epsilon(1-\gamma) + (2\epsilon\rho+\gamma)^2 + \bar{\lambda}^2 \exp|\theta|^2 T}}{4\epsilon}. \tag{1.87}$$

Therefore, we must have

$$2\epsilon\rho + \bar{\lambda} \le \gamma + \sqrt{8\epsilon(1-\gamma) + (2\epsilon\rho+\gamma)^2 + \bar{\lambda}^2 \exp|\theta|^2 T}. \tag{1.88}$$

By the assumption (1.36), we observe

$$\rho \le \bar{\rho} = \frac{8\epsilon(1-\gamma) + \gamma^2 + \bar{\lambda}^2 \exp|\theta|^2 T - (\bar{\lambda}-\gamma)^2}{4\epsilon(\bar{\lambda} - 2\gamma)}. \tag{1.89}$$

□

Now we prove Theorem 1.1. From Lemma 1.2,

$$|\mathbb{T}_\epsilon'(\rho)| \le -2\epsilon\beta_{2\epsilon}'(\lambda_\epsilon(\rho)\xi_0(T) - 2\epsilon\rho), \tag{1.90}$$

and

$$U_2^{'}(\beta_{2\epsilon}) - 2\epsilon\beta_{2\epsilon} = \mu. \tag{1.91}$$

Differentiate (1.91) twice with respect to μ to obtain

$$\beta_{2\epsilon}^{''}(\mu) = \frac{U_2^{'''}(\beta_{2\epsilon})}{\big(-U_2^{''}(\beta_{2\epsilon}) + 2\epsilon\big)^3}.$$

From the assumption (1.37), $\beta_{2\epsilon}^{'}(\mu)$ is increasing. So,

$$|\mathbb{T}_\epsilon'(\rho)| \le -2\epsilon\beta_{2\epsilon}'(-2\epsilon\rho) = \frac{2\epsilon}{2\epsilon - U_2^{''}\big(\beta_{2\epsilon}(-2\epsilon\rho)\big)}. \tag{1.92}$$

The function $-U_2^{''}\big(\beta_{2\epsilon}(-2\epsilon\rho)\big)$ is decreasing since

$$U_2^{'''}\big(\beta_{2\epsilon}(-2\epsilon\rho)\big)\beta_{2\epsilon}'(-2\epsilon\rho)2\epsilon < 0. \tag{1.93}$$

Therefore,

$$|\mathbb{T}_\epsilon'(\rho)| \le \frac{2\epsilon}{2\epsilon - U_2^{''}\big(\beta_{2\epsilon}(-2\epsilon\rho)\big)} \le \frac{2\epsilon}{2\epsilon - U_2^{''}\big(\beta_{2\epsilon}(-2\epsilon\bar{\rho})\big)} < 1, \tag{1.94}$$

we can conclude that $\mathbb{T}_\epsilon$ is a contraction map. Moreover, from Lemma 1.3, $\mathbb{T}_\epsilon(\rho) < \bar{\rho}$, Hence, we look for $\rho_\epsilon \in [0, \bar{\rho}]$, the desired result then follows. □

1.8.3 Proof of Theorem 1.2

From (2.18), the optimal feedbacks are

$$\hat{C}_\epsilon(x,t) = \beta_1(\lambda_\epsilon(x,t)), \tag{1.95}$$

$$x\hat{\varpi}_\epsilon(x,t) = -\lambda_\epsilon(x,t)\frac{\partial G_\epsilon}{\partial \lambda}(\lambda_\epsilon(x,t),t)(\sigma^*)^{-1}\theta. \tag{1.96}$$

In what follows, we first derive the optimal wealth $\hat{X}_\epsilon(s)$ and then use it to simplify the above optimal feedbacks. If we plug the optimal feedbacks in the state (wealth) equation (1.8), then the corresponding state denoted by $\hat{X}_\epsilon(s)$, the solution of (1.8), appears as the solution of

$$\begin{gathered} d\hat{X}_\epsilon = \Big(r\hat{X}_\epsilon - |\theta|^2\frac{\lambda_\epsilon}{\lambda'_\epsilon}(\hat{X}_\epsilon,s) - \beta_1\big(\lambda_\epsilon(\hat{X}_\epsilon,s)\big)\Big)ds - \frac{\lambda_\epsilon}{\lambda'_\epsilon}(\hat{X}_\epsilon,s)\theta.dw, \\ \hat{X}_\epsilon(0) = x_0. \end{gathered} \tag{1.97}$$

We already introduced the martingale (1.58), then we claim that

$$\hat{X}_\epsilon(s) = G_\epsilon(\lambda_\epsilon\xi_0(s),s), \tag{1.98}$$

where

$$\lambda_\epsilon = \lambda_\epsilon(x_0,0). \tag{1.99}$$

Let us check that the (1.98) satisfies (1.97). Fist we recall that

$$x_0 = G_\epsilon(\lambda_\epsilon,0) = \hat{X}_\epsilon(0).$$

Also we have

$$\lambda_\epsilon\xi_0(s) = \lambda_\epsilon(\hat{X}_\epsilon(s),s), \tag{1.100}$$

and

$$\frac{\partial G_\epsilon}{\partial \lambda}\big(\lambda_\epsilon(\rho)\xi_0(s),s\big) = \frac{\partial G_\epsilon}{\partial \lambda}\big(\lambda_\epsilon(\hat{X}_\epsilon(s),s),s\big) = \frac{1}{\dfrac{\partial \lambda_\epsilon}{\partial x}\big(\hat{X}_\epsilon(s),s\big)}. \tag{1.101}$$

We use Itô's formula for (1.98) and obtain

$$d\hat{X}_\epsilon = \frac{\partial G_\epsilon}{\partial s} - \frac{\partial G_\epsilon}{\partial \lambda}\lambda_\epsilon \xi_0(s)\theta dw + \frac{1}{2}\frac{\partial^2 G_\epsilon}{\partial \lambda^2}\lambda_\epsilon^2 \xi_0^2(s)|\theta|^2 ds, \tag{1.102}$$

so from (1.55) we have

$$\begin{aligned} d\hat{X}_\epsilon = \Big[-|\theta|^2 \lambda_\epsilon \xi_0(s) \frac{\partial G_\epsilon}{\partial \lambda}\big(\lambda_\epsilon \xi_0(s), s\big) + rG_\epsilon\big(\lambda_\epsilon \xi_0(s), s\big) - \beta_1\big(\lambda_\epsilon \xi_0(s)\big)\Big] ds \\ -\lambda_\epsilon \xi_0(s) \frac{\partial G_\epsilon}{\partial \lambda}\big(\lambda_\epsilon \xi_0(s), s\big)\theta dw \\ = \Big(r\hat{X}_\epsilon - |\theta|^2 \frac{\lambda_\epsilon}{\lambda'_\epsilon}(\hat{X}_\epsilon, s) - \beta_1\big(\lambda_\epsilon(\hat{X}_\epsilon, s)\big)\Big) ds - \frac{\lambda_\epsilon}{\lambda'_\epsilon}(\hat{X}_\epsilon, s)\theta dw. \end{aligned} \tag{1.103}$$

which is exactly (1.97). Therefore, we have the optimal feedbacks (2.28) and (2.29).

We now come to prove the optimal value. By (1.32), the optimal value is

$$J_\epsilon(\hat{C}_\epsilon, \hat{\varpi}_\epsilon) = u(x_0, 0) - 2\epsilon\rho_\epsilon^2 e^{-rT}. \tag{1.104}$$

We have

$$u(x_0, 0) = u\big(G_{\epsilon,\rho_\epsilon}(\lambda_\epsilon, 0), 0\big) = Z_{\epsilon,\rho_\epsilon}(\lambda_\epsilon, 0), \tag{1.105}$$

where

$$Z_{\epsilon,\rho}(\lambda, t) = u\big(G_{\epsilon,\rho}(\lambda, t), t\big), \tag{1.106}$$

a solution of

$$\begin{aligned} -\frac{\partial Z_{\epsilon,\rho}}{\partial t} - \frac{|\theta|^2}{2}\lambda^2 \frac{\partial^2 Z_{\epsilon,\rho}}{\partial \lambda^2} + rZ_{\epsilon,\rho} = U_1(\beta_1(\lambda)), \\ Z_{\epsilon,\rho}(\lambda, T) = U_2\big(G_{\epsilon,\rho}(\lambda, T)\big) - \epsilon G_{\epsilon,\rho}^2(\lambda, T) + \epsilon\rho_\epsilon^2 + 2\epsilon\rho_\epsilon G_{\epsilon,\rho}(\lambda, T), \\ G_{\epsilon,\rho}(\lambda, T) = \beta_{2\epsilon}(\lambda - 2\epsilon\rho). \end{aligned} \tag{1.107}$$

We obtain

$$Z_{\epsilon,\rho}(\lambda, 0) = e^{-rT} EZ(\lambda\xi_0(T), T) + E\int_0^T e^{-rs} U_1\Big(\beta_1\big(\lambda\xi_0(s)\big)\Big) ds, \tag{1.108}$$

and

$$Z_{\epsilon,\rho}(\lambda,0)=e^{-rT}\Big[EU_2\big(\beta_{2\epsilon}(\lambda\xi_0(T)-2\epsilon\rho)\big)-\epsilon E\beta_{2\epsilon}^2(\lambda\xi_0(T)-2\epsilon\rho)$$
$$+\epsilon\rho_\epsilon^2+2\epsilon\rho_\epsilon E\beta_{2\epsilon}(\lambda\xi_0(T)-2\epsilon\rho)\Big]+E\int_0^T e^{-rs}U_1\Big(\beta_1\big(\lambda\xi_0(s)\big)\Big)ds. \tag{1.109}$$

Therefore,

$$J_\epsilon\big(\hat{C}_\epsilon,\hat{\varpi}_\epsilon\big)=e^{-rT}\Big[EU_2\big(\beta_{2\epsilon}(\lambda_\epsilon\xi_0(T)-2\epsilon\rho_\epsilon)\big)-\epsilon E\beta_{2\epsilon}^2(\lambda_\epsilon\xi_0(T)-2\epsilon\rho_\epsilon)$$
$$-\epsilon\rho_\epsilon^2+2\epsilon\rho_\epsilon E\beta_{2\epsilon}(\lambda_\epsilon\xi_0(T)-2\epsilon\rho_\epsilon)\Big]+E\int_0^T e^{-rs}U_1\Big(\beta_1\big(\lambda_\epsilon\xi_0(s)\big)\Big)ds. \tag{1.110}$$

□

1.8.4 Proof of Proposition 1.1

In this case, the problem reduces to the classical Merton's problem with $\beta_{2\epsilon}(\mu)=\beta_2(\mu)$ and $G_{0,\rho}(\lambda,t)=G(\lambda,t)$ independent of ρ.

Then $G(\lambda_0,0)=x_0$; it follows

$$T_0(\rho)=E\beta_2(\lambda_0\xi_0(T)).$$

Therefore,

$$\rho_0=E\beta_2(\lambda_0\xi_0(T)). \tag{1.111}$$

□

1.8.5 Proof of Proposition 1.2

We first check that

$$\beta_{2\epsilon}(\lambda-2\epsilon\rho)\to\rho \text{ as } \epsilon\to+\infty;$$

hence $\lambda_\epsilon(\rho) \to \lambda_\infty(\rho) = \lambda(\rho)$ solution of

$$\rho e^{-rT} + E\int_0^T \xi_0(s)\beta_1(\lambda\xi_0(s))e^{-rs}\,ds = x_0. \tag{1.112}$$

We then see that $T_\epsilon(\rho) \to \rho$. To obtain the limit of ρ_ϵ, we need an asymptotic expansion.

In fact,

$$\beta_{2\epsilon}(\lambda - 2\epsilon\rho) \sim \rho + \frac{U_2'(\rho) - \lambda}{2\epsilon},$$

and

$$T_\epsilon(\rho) \sim \rho + \frac{U_2'(\rho) - \lambda(\rho)}{2\epsilon}.$$

It follows that ρ_∞ is the solution of

$$U_2'(\rho_\infty) - \lambda(\rho_\infty) = 0. \tag{1.113}$$

We need to find $\lambda_\infty = \lambda(\rho_\infty)$ solution of the equation

$$\beta_2(\lambda_\infty)e^{-rT} + E\int_0^T \xi_0(s)\beta_1(\lambda_\infty\xi_0(s))e^{-rs}\,ds = x_0, \tag{1.114}$$

which has a unique solution. Then $\rho_\infty = \beta_2(\lambda_\infty)$. □

1.8.6 Proof of Proposition 1.3

We have

$$\rho_0 = E\beta_0(\lambda_0\xi_0(T)), \tag{1.115}$$

$$\rho_\infty = \beta_2(\lambda_\infty), \tag{1.116}$$

$$\begin{aligned} d\xi_0(s) &= -\xi_0(s)\theta.dw, \\ \xi_0(0) &= 1. \end{aligned} \tag{1.117}$$

and

$$\rho_\infty e^{-rT} + E\int_0^T \xi_0(s)\beta_1(\lambda_\infty\xi_0(s))e^{-rs}ds = x_0, \tag{1.118}$$

$$E\xi_0(T)\beta_2(\lambda_0\xi_0(T))e^{-rT} + E\int_0^T \xi_0(s)\beta_1(\lambda_0\xi_0(s))e^{-rs}ds = x_0. \tag{1.119}$$

We can rewrite (1.118) as

$$\rho_\infty e^{-rT} + E\int_0^T \xi_0(s)\beta_1\big(\beta_2^{-1}(\rho_\infty)\xi_0(s)\big)e^{-rs}ds = x_0. \tag{1.120}$$

We are going to prove that

$$\rho_0 e^{-rT} + E\int_0^T \xi_0(s)\beta_1\big(\beta_2^{-1}(\rho_0)\xi_0(s)\big)e^{-rs}ds \geq x_0. \tag{1.121}$$

Since the function

$$\rho e^{-rT} + E\int_0^T \xi_0(s)\beta_1\big(\beta_2^{-1}(\rho)\xi_0(s)\big)e^{-rs}ds, \tag{1.122}$$

is monotone increasing in ρ. (1.120) and (1.121) implies $\rho_0 \geq \rho_\infty$.

To prove (1.121), we prove separately

$$\beta_2^{-1}(\rho_0) \leq \lambda_0, \tag{1.123}$$

$$E\xi_0(T)\beta_2(\lambda_0\xi_0(T)) \leq \rho_0. \tag{1.124}$$

To prove (1.123) we check that

$$\rho_0 \geq \beta_2(\lambda_0), \tag{1.125}$$

which means

$$E\beta_2(\lambda_0\xi_0(T)) \geq \beta_2(\lambda_0). \tag{1.126}$$

Since β_2 is strictly convex, Jensen's inequality implies

$$\beta_2(\lambda_0 E\xi_0(T)) < E\beta_2(\lambda_0\xi_0(T)), \tag{1.127}$$

and recall that $E\xi_0(T) = 1$, we obtain (1.126).

In the following, we shall prove that $\forall t \geq 0$, $\forall \lambda > 0$

$$E\lambda\xi_t(T)\beta_2(\lambda\xi_t(T)) \leq \lambda E\beta_2(\lambda\xi_t(T)), \tag{1.128}$$

then applying with $t = 0$ and $\lambda = \lambda_0$, we get (1.124).

We set

$$\begin{aligned} u(\lambda, t) &= E\lambda\xi_t(T)\beta_2(\lambda\xi_t(T)), \\ v(\lambda, t) &= E\beta_2(\lambda\xi_t(T)), \end{aligned} \tag{1.129}$$

where

$$\begin{aligned} d\xi_t(s) &= -\xi_t(s)\theta.dw, \\ \xi_t(t) &= 1. \end{aligned} \tag{1.130}$$

We also introduce the notation $\xi_{t\lambda}(s) = \lambda\xi_t(s)$, then

$$\begin{aligned} d\xi_{t\lambda}(s) &= -\xi_{t\lambda}(s)\theta.dw, \\ \xi_{t\lambda}(t) &= \lambda. \end{aligned} \tag{1.131}$$

Clearly, (1.128) means

$$u(\lambda, t) \leq \lambda v(\lambda, t). \tag{1.132}$$

Let $u(\lambda, t)$ is the solution of

$$\begin{aligned} \frac{\partial u}{\partial t} + \frac{1}{2}\lambda^2|\theta^2|\frac{\partial^2 u}{\partial \lambda^2} &= 0, \\ u(\lambda, T) &= \lambda\beta_2(\lambda). \end{aligned} \tag{1.133}$$

and $v(\lambda, t)$ is the solution of

$$\begin{aligned} \frac{\partial v}{\partial t} + \frac{1}{2}\lambda^2|\theta^2|\frac{\partial^2 v}{\partial \lambda^2} &= 0, \\ v(\lambda, T) &= \beta_2(\lambda). \end{aligned} \tag{1.134}$$

Set $\tilde{v}(\lambda, t) = \lambda v(\lambda, t)$, then we have

$$\begin{aligned} \frac{\partial \tilde{v}}{\partial t} + \frac{1}{2}\lambda^2|\theta^2|\frac{\partial^2 \tilde{v}}{\partial \lambda^2} &= \lambda^2|\theta|^2\frac{\partial v}{\partial t}, \\ \tilde{v}(\lambda, T) &= \lambda\beta_2(\lambda). \end{aligned} \tag{1.135}$$

From the definition of $v(\lambda, t)$ and the fact that β_2 is decreasing, we can check that

$$v'(\lambda, t) = \frac{\partial v}{\partial \lambda}(\lambda, t) \leq 0 \tag{1.136}$$

Set $\zeta(\lambda, t) = u(\lambda, t) - \tilde{v}(\lambda, t)$, then

$$\begin{aligned} \frac{\partial \zeta}{\partial t} + \frac{1}{2}\lambda^2|\theta^2|\frac{\partial^2 \zeta}{\partial \lambda^2} &= -\lambda^2|\theta|^2 v'(\lambda, t) > 0, \\ \zeta(\lambda, T) &= 0. \end{aligned} \tag{1.137}$$

Applying Itô's formula to $\zeta(\lambda, t)$, we obtain

$$d\zeta(\lambda, t) = (\frac{\partial \zeta}{\partial t} + \frac{1}{2}\lambda^2|\theta^2|\frac{\partial^2 \zeta}{\partial \lambda^2})dt + \theta\lambda\frac{\partial \zeta}{\partial \lambda}(\lambda, t)dw.$$

From (1.137), we have

$$d\zeta(\lambda, t) = -\lambda^2|\theta|^2 v'(\lambda, t) + \theta\lambda\frac{\partial \zeta}{\partial \lambda}(\lambda, t)dw,$$

integrating from t to T and taking expectations, it is easy to see that $\zeta(\lambda, t) \leq 0$, hence the result (1.128), which completes the proof. □

1.8.7 Proof of Remark 1.4

We note that

$$G_{\epsilon,\rho}(\lambda, t) = E\xi_t(T)\beta_{2\epsilon}(\lambda\xi_t(T) - 2\epsilon\rho)e^{-r(T-t)} + E\int_t^T \xi_t(s)\beta_1\big(\lambda\xi_t(s)\big)e^{-r(s-t)}ds, \tag{1.138}$$

$$\begin{aligned} G_{0,\rho}(\lambda, t) &= E\xi_t(T)\beta_2(\lambda\xi_t(T))e^{-r(T-t)} + E\int_t^T \xi_t(s)\beta_1\big(\lambda\xi_t(s)\big)e^{-r(s-t)}ds \\ &= \frac{1}{\lambda}e^{-r(T-t)} + \frac{1}{r\lambda}(1 - e^{-r(T-t)}), \end{aligned} \tag{1.139}$$

and

$$\frac{\partial G_{0,\rho}(\lambda, t)}{\partial \lambda} = -\frac{1}{\lambda^2}e^{-r(T-t)} - \frac{1}{r\lambda^2}(1 - e^{-r(T-t)}). \tag{1.140}$$

Therefore, we have

$$G_{0,\rho}(\lambda_0\xi_0(t),t) = \frac{1}{\lambda_0\xi_0(t)}e^{-r(T-t)} + \frac{1}{r\lambda_0\xi_0(t)}(1-e^{-r(T-t)}), \tag{1.141}$$

and

$$\frac{\partial G_{0,\rho}(\lambda_0\xi_0(t),t)}{\partial\lambda} = -\frac{1}{\lambda_0^2\xi_0^2(t)}e^{-r(T-t)} - \frac{1}{r\lambda_0^2\xi_0^2(t)}(1-e^{-r(T-t)}). \tag{1.142}$$

From

$$\hat{X}_0(t) = G_{0,\rho}(\lambda_0\xi_0(t),t), \tag{1.143}$$

and (1.141), we obtain that

$$\hat{C}_0(t) = \beta_1\big(\lambda_0\xi_0(t)\big) = \frac{r}{1+(r-1)e^{-r(T-t)}}\hat{X}_0(t). \tag{1.144}$$

Substitute (1.143) and (1.142) into

$$\hat{\varpi}_0(s)\hat{X}_0(t) = -\lambda_0\xi_0(t)\frac{\partial G_{0,\rho_0}}{\partial\lambda}\big(\lambda_0\xi_0(t),s\big)(\sigma^*)^{-1}\theta, \tag{1.145}$$

we have

$$\hat{\varpi}_0(t) = \frac{\alpha-r}{\sigma^2}. \tag{1.146}$$

□

1.8.8 Proof of Remark 1.5

At first, we have

$$\lim_{\epsilon\to\infty}\beta_{2\epsilon}(\lambda_\epsilon\xi_0(t)\xi_t(T)-2\epsilon\rho_\epsilon) = \rho_\infty, \tag{1.147}$$

and

$$\lim_{\epsilon\to\infty}\frac{\partial\beta_{2\epsilon}(\lambda_\epsilon\xi_0(t)\xi_t(T)-2\epsilon\rho_\epsilon)}{\partial\lambda} = \lim_{\epsilon\to\infty}\beta'_{2\epsilon}(\lambda_\epsilon\xi_0(t)\xi_t(T)-2\epsilon\rho_\epsilon)\xi_t(T) = 0. \tag{1.148}$$

Hence,

$$\hat{X}_\infty(t) = G_{\infty,\rho_\infty}(\lambda_\infty\xi_0(t), t) = \rho_\infty e^{-r(T-t)} + \frac{1}{r\lambda_\infty\xi_0(t)}(1 - e^{-r(T-t)}), \tag{1.149}$$

Therefore,

$$\frac{\partial G_{\infty,\rho_\infty}(\lambda_\infty\xi_0(t), t)}{\partial\lambda} = -\frac{1}{r\lambda_0^2\xi_0^2(t)}(1 - e^{-r(T-t)}). \tag{1.150}$$

which substituted in

$$\hat{\varpi}_\infty(t)\hat{X}_\infty(t) = -\lambda_\infty\xi_0(t)\frac{\partial G_{\infty,\rho_\infty}}{\partial\lambda}\big(\lambda_\infty\xi_0(t), t\big)(\sigma^*)^{-1}\theta,$$

gives

$$\hat{\varpi}_\infty(t) = \frac{\frac{1}{r\lambda_\infty\xi_0(t)}(1 - e^{-r(T-t)})\frac{\theta}{\sigma}}{\rho_\infty e^{-r(T-t)} + \frac{1}{r\lambda_\infty\xi_0(t)}(1 - e^{-r(T-t)})}. \tag{1.151}$$

In addition, (1.49) is obvious since

$$\hat{C}_\infty(t) = \beta_1\big(\lambda_\infty\xi_0(t)\big) = \beta_1\big(\lambda_0\xi_0(t)\big) = \hat{C}_0(t). \tag{1.152}$$

□

1.8.9 Proof of relationship between ρ_ϵ and ρ_0 for ϵ is small and $\gamma_1 = 1$.

Proposition 1.4. *When ϵ is small and $\gamma_1 = 1$, the relationship between ρ_ϵ and ρ_0 is given by*

$$\rho_\epsilon \approx \rho_0 + \alpha\rho_0^3\epsilon,$$

where

$$\alpha = e^{|\theta|^2T} - e^{3|\theta|^2T} + \frac{e^{-rT}(e^{|\theta|^2T-1})}{e^{-rT} + \frac{1-e^{-rt}}{r}}.$$

Since

$$\alpha < e^{|\theta|^2T} - e^{3|\theta|^2T} + e^{|\theta|^2T} - 1 < 0,$$

we have expected optimal terminal wealth ρ_ϵ is decreasing with ϵ.

Proof:

From Maclaurin series, we have

$$\beta_{2\epsilon}(\mu) = \frac{\mu}{4\epsilon}(-1+\sqrt{1+\frac{8\epsilon}{\mu^2}}) = \frac{\mu}{4\epsilon}(\frac{4\epsilon}{\mu^2} - \frac{8\epsilon^2}{\mu^4}) = \frac{1}{\mu} - \frac{2\epsilon}{\mu^3},$$

$$\frac{1}{\lambda\xi_0(T) - 2\epsilon\rho} = \frac{1}{\lambda\xi_0(T)(1-\frac{2\epsilon\rho}{\lambda\xi_0(T)})} = \frac{1}{\lambda\xi_0(T)}(1+\frac{2\epsilon\rho}{\lambda\xi_0(T)}).$$

Therefore,

$$\begin{aligned} G_{\epsilon,\rho}(\lambda,0) &= E\xi_0(T)\beta_{2\epsilon}(\lambda\xi_0(T)-2\epsilon\rho)e^{-r(T)} + E\int_0^T \xi_0(s)\beta_1\big(\lambda\xi_0(s)\big)e^{-r(s)}ds \\ &= \frac{e^{-rT}}{\lambda} + \frac{1-e^{-rT}}{r\lambda} + \frac{2\epsilon\rho}{\lambda^2}E\frac{1}{\xi_0(T)}e^{-rT} - \frac{2\epsilon}{\lambda^3}E\frac{1}{\xi_0^2(T)}e^{-rT}, \end{aligned} \tag{1.153}$$

From 1.117, and define $\eta_0(t) = \frac{1}{\xi_0(t)}$, we have

$$E\eta_0(T) = e^{|\theta|^2T},$$

$$E\eta_0^2(T) = e^{3|\theta|^2T}.$$

Plugging these into 1.153 yields

$$G_{\epsilon,\rho}(\lambda,0) = \frac{e^{-rT}}{\lambda} + \frac{1-e^{-rT}}{r\lambda} + \frac{2\epsilon\rho}{\lambda^2}e^{-(r-|\theta|^2)T} - \frac{2\epsilon}{\lambda^3}e^{-(r-3|\theta|^2)T}. \tag{1.154}$$

Since for $\epsilon = 0$, $\lambda_\epsilon(\rho) = \lambda_0(\rho)$, we assume

$$\lambda_\epsilon(\rho) = \lambda_0(\rho)(1+\epsilon a(\rho)),$$

where $a(\rho)$ is determined later.

Note that $G_{\epsilon,\rho}(\lambda,0) = x_0$, $\lambda_\epsilon^2(\rho) = \lambda_0^2(\rho)(1+2\epsilon a(\rho))$, and $\lambda_\epsilon^3(\rho) = \lambda_0(\rho)(1+3\epsilon a(\rho))$. Solving $\lambda^3 G_{\epsilon,\rho}(\lambda,0) = \lambda^3 x_0$ gives

$$a(\rho) = \frac{2\rho\lambda_0(\rho)e^{-(r-\theta|^2)T} - 2e^{-(r-3\theta|^2)T}}{x_0\lambda_0^3(\rho)}.$$

Therefore,

$$\lambda_\epsilon(\rho)\xi_0(T) - 2\epsilon\rho = \lambda_0(\rho)\xi_0(T)\{1 + 2\epsilon\frac{e^{-(r-|\theta|^2)T}}{x_0\lambda_0^2(\rho)}(\rho - \frac{e^{2|\theta|^2)T}}{\lambda_0(\rho)}) - \frac{2\epsilon\rho}{\lambda_0(\rho)\xi_0(T)}\}. \quad (1.155)$$

Note that $\rho_\epsilon = E\beta_{2\epsilon}\big(\lambda_\epsilon(\rho_\epsilon)\xi_0(T) - 2\epsilon\rho_\epsilon\big)$ and $T_\epsilon(\rho_\epsilon) = \rho_\epsilon$, a direct calculation gives

$$\rho_\epsilon = \rho_0 + \alpha\rho_0^3\epsilon,$$

where

$$\alpha = e^{|\theta|^2T} - e^{3|\theta|^2T} + \frac{e^{-rT}(e^{|\theta|^2T-1})}{e^{-rT} + \frac{1-e^{-rt}}{r}}.$$

$\square$

.

CHAPTER 2

OPTIMIZATION OF CONSUMER-PRODUCER STORAGE IN RENEWABLE ENERGY MARKETS: A MEAN FIELD APPROACH

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Note: A version of this chapter is a job market paper.

2.1 Introduction

2.1.1 Motivation and Some Related Literature

The growing concerns regarding an environmental issue, fossil fuel depletion, and energy independence have driven the introduction of renewable energy policy. In particular, the number of self-generated renewable energy facility (e.g., photovoltaic solar panel) for households has been considerably increased by an economic incentive from this energy policy (Paraschiv et al. 2014). In the past, the installation of a solar panel was an alternative energy generator to replace a traditional energy production plant. Increasing the penetration of self-generated renewable energy facility induces a new role in a smart grid market. For instance, by installing local storage with solar panels for individual households, various services such as spinning reserve, active power grid regulation, and dynamic load balancing are allowed. The main theme of these services is to cope with rapid changes in energy demand in the grid system by considering local storage as supplement energy storage. Since the short on energy in the grid would cause a blackout resulting in numerous cascading problems, it is indispensable for the smart grid operator to properly manage the surging demand. Therefore, it is evident that there are attractive advantages of the local storage strategy for the grid operator. The household who installs the local storage with the solar panel may reap a benefit from a bidirectional energy transaction by charging remnant energy during the off-peak times and by discharging stored energy during the on-peak time. This energy transaction would be profitable for the households because a dynamic market pricing scheme depends on the real-time demands in the grid system. Given how crucial the understanding of local storage strategy is for enhancing the resilience of the grid system, surprisingly little is explored analytically about it, as a lack of proper analytic models so far.

Qi et al. (2015) proposes the problem of economic planning for the energy storage system based on a linear programming model to decide the optimal size and location of energy storage for wind energy generation. This study shows how to choose the optimal size of energy storage by estimating a closed-form upper bound. However, this work is limited to understand a real-time operating decision to control the level of stored energy in energy storage. Cruise et al. (2019) studies for the large-scale energy storage optimization problem using a deterministic Lagrangian approach to derive the optimal level of inventory to minimize the cost structure. Lin et al. (2019) considers the approximate linear programming as a proper approach to manage high dimensional decision processes regarding inventory control in energy storage. The works of Cruise et al. (2019) and Lin et al. (2019) contribute the understanding of dynamic aspect of market system via efficient algorithms on calibrating the control rather than obtaining an explicit solution. In the viewpoint of a central planner, Zhou et al. (2016) suggests a finite Markov decision process to explain the comparison between the disposal strategy and the storage strategy by using the perishable electricity. This research work shows that a policymaker needs to establish economic incentives to promote the storage strategy because the disposal strategy is normally preferred by the merchants. Even though the market price is pivotal to draw a policy implication, this price impact is exogenous in their modelling. As discussed above, using only one of the techniques such as the linear programming, the Lagrangian approach, and the Markov decision process at a time can certainly illuminate one spectacular aspect of smart grid of energy storage though subject to a certain degree of limitation. In our opinion, the mean field approach can on one hand highlight the pros of these modellings, while on the other hand, remedy their cons too, henceforth a more comprehensive model for the energy-grid dynamic system can be obtained.

In our paper, we study a quantity competition model among the large number of interacting households who own local storage with a self-generated renewable energy system for the bidirectional energy transaction between the local storage and a grid in a dynamic setting, where the individual household can make a decision regarding an amount of charging or discharging energy based on the current spot price, the energy demand of the grid, and the level of stored energy. A game theory technique can be considered one of the tools to understand the optimal behavior of households and a grid operator. More specifically, a non-cooperative Stackelberg game allows us to capture their relationship. Stackelberg (1934) suggested the concept of an equilibrium to understand the optimal behavior for a leader and followers in 1934. In the grid system, the grid operator is considered as a leader and the group of households is considered as followers. The main objective of the group of households is to maximize their utility by strategically choosing the charging amount. Based on the group of households' optimal strategy, the grid operator is able to derive the optimal pricing strategy to maximize its profit. However, this approach is limited to describe an aspect of the dynamic market pricing scheme depending on the aggregated interaction among a large number of households. One of the pleasant ways to illustrate this aspect is the use of Stochastic Differential Games (SDGs) (see e.g., Friedman 1994). For example, Bensoussan and Frehse (2000) introduce the zero-sum SDGs with N players by using a dynamic programming approach, but this approach is limited to derive equilibrium outputs in numerous applications because of mathematical complexity.

In this paper, we propose an extended mean field type approach to study a dynamic integration model between a grid and a large number of households. Mean Field games have been introduced by Lasry and Lions (2007) and independently by Huang et al. (2006). See Bensoussan et al. (2013) for the comparison between Mean Field Games and Mean Field type Control Theory. To study the interacting particle system, mean field term was introduced as a medium. The novelty of MFGs is to describe the behavior of agents in social science as the interaction of particles in physics. The mean-field terms represent the probability distribution when the number of populations goes to infinity. In addition, this term can capture an aggregated effect arising from the status of populations.

The mean field type approach is a promising approach to describe a smart grid market mechanism (Couillet et al. 2012, Paola et al. 2016). However, the analysis of this approach requires several logical assumptions: (1) the sufficiently large number of players to be considered infinite; (2) homogeneity in strategic behavior of players; and (3) social interactions governed by mean field term. The first assumption is supported by the fact that there are a huge number of households who own local storage with a self-generated energy facility. The second assumption is justifiable since each player has similar constraints and objectives. This means that individual households do not vary and they have almost identical objectives such as minimizing costs and maximizing its utility. The last assumption explains the fact that each household can marginally contribute to the grid system, but the status of each household follows a probability distribution. In this research, the dynamic market pricing scheme is governed by the mean field term. We will explain this term more specifically in the Section 2.1.3.

A study for dynamic quantity competition with a large number of interacting players has been introduced by Chan and Sircar (2015). This study suggests a continuous time Cournot (1838) competition model based on the concept of MFGs. In this study, a market price is a function of aggregated quantity decisions from players, but this model does not take into account the concept of inventory management that should be considered to properly explain why the individual households strategically choose the amount of charging or discharging energy by considering the level of stored energy in local storage. In the existing literature, a mean field type control problem is concerned with controlling a

McKean-Vlasov type state process, one can refer to Carmona and Delarue (2017), or for a shorter presentation in Bensoussan et al. (2013) for discussion; in principle, on the top of the classical stochastic control theory, it adds the influence of the overall probability distribution of the state on both the state dynamics and objective functionals. The example, which motivates us, Alasseur et al. (2020) incorporates also the probability distribution of the control. The extension of mean field type control to situations where the equation and the pay off depend also on the evolution of the probability distribution of the control has been studied by Pham and Wei (2018), at the level of Bellman equation. We extend it at the level of the Master equation and system of Hamilton-Jacobi-Bellman-Fokker Planck equations. To best our knowledge, our work is the first theoretical work to derive a Master equation and a corresponding the system of HJB–FP equations for the extended mean field type control approach. Furthermore, we completely solve the application of an energy-grid dynamic optimization studied in the paper of Alasseur et al. (2020), who solve this problem using a stochastic maximum principle.

2.1.2 Contributions of our work

We explicitly derive the feedback control regarding the optimal charging or discharging rule for a household who owns a self-generated renewable energy system with a local storage facility based on the dynamic pricing scheme of a spot energy price. In our mean field model, the feedback control can help to modulate the behavior of the consumer by managing himself or herself to avoid over-charging or over-discharging the energy in the grid system. If the consumer charges the energy more than the average, then he or she will tend to use less energy or vice versa. Therefore, the movement of the spot energy price would be less volatile, so this is the main feature of having the feedback control that can reduce the furious fluctuation of the spot energy price. Our setting is especially relevant to the implementation of the dynamic pricing scheme in a grid as the popularity of a renewable energy facility increases. The dynamic pricing scheme triggers the economic incentive and ensures the energy bill stability because the consumers can adjust their consumption behavior in accordance with the spot energy price; despite of these advantages, there are only seven European countries (namely, Denmark, Estonia, Finland, Netherlands, Spain, Sweden, and the United Kingdom) which partially introduce the dynamic pricing scheme because there could be some risk inherent in the system from the high fluctuation of spot price energy (Dutta and Mitra 2017, European Commission 2019). Our mean field approach with a local storage strategy when combined with the dynamic pricing scheme can lead to a promising policy scheme for policymakers to reduce the volatility risk of the dynamic pricing scheme on one hand and also to ensure the mentioned advantages on the other hand.

The mean field approach allows us to propose a compelling model by overcoming the limitation of previous studies (see e.g., Qi et al. 2015, Zhou et al. 2016, Cruise et al. 2019, Lin et al. 2019) that have not yet considered an endogenous with feedback pricing scheme. Although their works remarkably illuminate one aspect of the control of local storage given an exogenous pricing scheme in a widespread energy-grid system, it is crucial to describe a tangible feedback mechanism between a local storage strategy and an endogenous pricing scheme in a future energy-grid system. In our mean field setting, we consider the endogenous pricing scheme based on the real-time amount of aggregate energy being consumed in the grid system. Our pricing scheme is modulated by the mean field term being the law of the control. This term can be regarded as the average of historical performance of one single player who tries to adhere to this pricing rule in alignment with his or her usual practice reflected by the mean field term. For instance, Alasseur et al. (2020) recently takes advantage of the novelty of mean field approach for a dynamic grid optimization problem; the analytical solution of their optimization problem is derived by a stochastic maximum principle via an adjoint process decoupled by the corresponding FBSDE. On the other hand, we consider the problem via the dynamic programming principle and so the solution can be immediately recovered. We also construct the corresponding system of stochastic HJB–FP equations into the Master equation (see Section 2.2). Our Master equation approach is more analytic in nature, while that of Alasseur et al. (2020) has a strong probabilistic favor. Unlike our approach, the stochastic maximum principle demands the existence of the FBSDE which is mostly not so immediate especially in the infinite dimensional setting.

With a plenty of numerical experiments, we interpret the role of feedback control obtained under the mean field setting for a local storage facility from the comparison of two energy pricing policies: (1) a dynamic pricing scheme; and (2) a static pricing scheme (see Section 2.7). Particularly, we show that our feedback control enables a proper management of the seasonality of energy flow in a grid system through the sensitivity analysis of the rate of mean reversion. It is our hope that our work can later be studied further by incorporating real energy transaction data between local storage facilities and a central grid system in the energy market. For example, this further study would capture an empirical relationship between the change of energy pricing scheme and the efficiency of dynamic energy load balancing service in the grid.

2.1.3 Application

In a recent interesting work of Alasseur et al. (2020), the authors considered the problem of an energy-grid dynamic optimization in which there are both local producers and consumers, and through out their work they addressed several possibilities of grouping the players. Without loss of generality, in the present study, we consider only one group of identical players. The interaction among the prosumers (who can generate and consume simultaneously) stems from the price of energy, which appears naturally in their individual payoffs. Note that this price is not exogenous and it depends among the decisions of all consumers. Particularly, none of the agents can individually influence this price but it depends on the overall average of decisions, and hence the price can be regarded as a mean field term. Here the community has a representative agent, whose state evolution is driven by a standard diffusion, while the mean field term influences the payoff functional of the representative. Since the representative agent represents the whole community, it is legitimate to use the mean field type control approach rather than via the mean field games.

The representative agent has two state variables $S(t)$ and $Q(t)$, here $S(t)$ is the amount of energy stored at any time t, and $Q(t)$ is the demand (or supply) rate of the agent by subtracting the rate of consumption from the rate of production. The evolution of $Q(t)$ is stochastic, described by a diffusion, however it is not controlled; instead the control is in the storage, which is defined by a feedback function of the state variable, $v(S, Q, t)$ (or $v(S, Q)$ in short). More precisely, the evolution dynamics is defined by the following equations

$$\begin{cases} \dfrac{dS}{dt} = v(S, Q), \ S(0) = 0; \\ dQ = b(Q)dt + \sigma(Q)dw(t), \ Q(0) = Q_0, \end{cases} \tag{2.1}$$

where $w(t)$ is the standard Wiener process and Q_0 is a random variable being independent of the Wiener process. The quantity $Q(t) - v(S(t), Q(t))$ is actual energy flow between a energy grid and a local storage per unit time, and its average is denoted by $\mathbb{E}(Q(t) - v(S(t), Q(t)))$. The price paid by the community declines with this average. Note that the positive sign of energy flow rate indicates the energy flow from the local storage to the energy grid and vice versa. Following the modelling in Alasseur et al. (2020), we denote this price by $p(-\mathbb{E}(Q(t) - v(S(t), Q(t))))$, here the price function $p(x)$ is a monotonic increasing function. The random revenue from the storage strategy is thus $p(-\mathbb{E}(Q(t)-v(S(t), Q(t))))\,(Q(t)-v(S(t), Q(t)))$. There are various costs to be incurred in front of this revenue. We choose to minimize, so the income enters in the payoff with a negative sign, and we take the costs as in Alasseur et al. (2020). The cost functional is now given by:

$$
\begin{aligned}
J(v) := \mathbb{E}\int_0^T & \Bigg[\underbrace{\frac{a}{2}S(t)^2 + lS(t) + \frac{c}{2}v^2(S(t), Q(t))}_{\text{current storage cost}} + \underbrace{\frac{K}{2}\Big|Q(t) - v(S(t), Q(t))\Big|^2}_{\text{demand charge}}\Bigg] dt \\
& - \mathbb{E}\int_0^T \underbrace{p(-\mathbb{E}(Q(t) - v(S(t), Q(t))))\mathbb{E}(Q(t) - v(S(t), Q(t)))}_{\text{volumetric charge}} dt + \underbrace{\mathbb{E}h(S(T)),}_{\text{terminal storage cost}}
\end{aligned}
\tag{2.2}
$$

where a, l, c, K are constant coefficient. The current storage cost represents the agent's effort to manage the load of demand. The demand charge depends on the maximum level of instantaneous power demand since an electricity system is designed to satisfy the peak level of demand, incurring an additional transmission cost and a loss of energy. The volumetric charge is a random profit or cost from a storage strategy based on a current spot price. The terminal storage cost requires the minimum level of stored energy at the end of the time horizon.

The rest of the paper is organized as follows: Section 2.2 describes a general extended mean field type control problem with a McKean-Vlasov stochastic differential equation which depends on not only the evolution of the probability distribution of state but also the probability distribution of control as in Pham and Wei (2018). In Section 2.3, we write down the master equation and the corresponding system of HJB-FP equations for extended mean field type control problems. In Section 2.4, we study the particular energy example proposed by the work of Alasseur et al. (2020). The explicit solutions can be obtained by the master equation and the system of HJB-FP equations we derive in Section 2.3. Section 2.5 and Section 2.6 introduce a theoretical model of extended mean field type control problem with common noise and its application, respectively. In section 2.7, we show the numerical results regarding the effectiveness of local storage strategy, the comparison of a dynamic pricing scheme and a fixed pricing scheme, and the grid resilience against perturbations of demand. Section 2.8 concludes with some managerial implications on operating the grid system.

2.2 General Extended Mean Field Type Control Problem

2.2.1 Formulation and Settings

The payoff (2.2) contains the expected values of both the state variables and some functions of the control variables, and due to the generic nature of p, the commonly found approach from mean field type control theory is insufficient. Pham and Wei (2018) discuss Bellman equation. We proceed further and also write down the master equation and the corresponding system of HJB-FP equations. This general mean field type control problem is stated as follows: we consider a probability space $(\Omega, \mathcal{A}, \mathbb{P})$, on which we construct the standard Wiener processes. We have a state process $x(t)$ or simply $x_t \in \mathbb{R}^n$, and its probability distribution is denoted by $\mathbb{P}_{x_t}$. The control take values in $\mathbb{R}^d$ and is defined by a feedback one, $v(x_t, \mathbb{P}_{x_t})$; it actually depends on time, but we omit the argument t for simplicity. The probability distribution of this control is denoted by $\mathbb{P}_{v(x_t,\mathbb{P}_{x_t})}$. The state evolution is defined by:

$$dx = g(x_t, \mathbb{P}_{x_t}, v(x_t, \mathbb{P}_{x_t}), \mathbb{P}_{v(x_t,\mathbb{P}_{x_t})})dt + \sigma(x_t)dw(t),\ x(0) = \xi, \tag{2.3}$$

where ξ is the initial random variable being independent of the Wiener process $w(t)$. In (2.3), the drift is a function $g(x, m, v, \mu)$ where the arguments $x \in \mathbb{R}^n$ and $v \in \mathbb{R}^d$, while the arguments m and μ are probability measures on $\mathbb{R}^n$ and $\mathbb{R}^d$, respectively. Note that the probability $\mathbb{P}_{v(x_t,\mathbb{P}_{x_t})}$ is the push forward image of the law of $\mathbb{P}_{x_t}$ by the map $x \mapsto v(x, \mathbb{P}_{x_t})$. In the sequel, we use the notation $v(\cdot, m)_{\#}m$ for the image or push forward measure of m by the map $x \mapsto v(x, m)$. Therefore, we can rewrite (2.3) as

$$dx = g(x_t, m_t, v(x_t, m_t), v(\cdot, m_t)_{\#}m_t)dt + \sigma(x_t)dw(t), \; x(0) = \xi, \tag{2.4}$$

where $m_t = m(t) = \mathbb{P}_{x_t}$. The functional we aim to minimize is:

$$J(v) = \mathbb{E}\left[\int_0^T f(x_t, m_t, v(x_t, m_t), v(\cdot, m_t)_{\#}m_t)dt + h(x_T, m_T)\right] \tag{2.5}$$

2.2.2 Reformulation

Assume that ξ has a probability distribution with a density $m_0(x)$. Denote by $A = A(t)$ the second order differential operator $A\varphi(x) := -\mathrm{tr}\Big(a(x)D^2\varphi(x)\Big)$, where $a(x) = \frac{1}{2}\sigma(x)\sigma^*(x)$. We call A^* the adjoint operator

$$A^*\varphi(x) = -\sum_{i,j=1}^{n} \frac{\partial^2}{\partial x_i \partial x_j}(a_{ij}(x)\varphi(x)).$$

Under some mild technical assumptions, the probability measure m_t should have a density $m(x, t)$ satisfying the Fokker-Planck equation:

$$\frac{\partial}{\partial t}m + A^*m + \mathrm{div}\Big(g(x, m, v(x, m), v(\cdot, m)_{\#}m)m(x, t)\Big) = 0, \; m(x, 0) = m_0(x), \tag{2.6}$$

and the objective functional (2.5) can be written as follows:

$$J(v) = \int_0^T \int_{\mathbb{R}^n} f(x, m_t, v(x, m_t), v(\cdot, m)_{\#}m_t)m(x, t)dxdt + \int_{\mathbb{R}^n} h(x, m_T)m(x, T)dx, \tag{2.7}$$

where $m_t = m(x,t)$. As usual we have reduced the original stochastic control problem to a deterministic control problem for a distributed parameter system, whose evolution is described by the FP equation (2.6). To apply Dynamic Programming principle, we embed problem (2.6) and (2.7) into the family, given $m(x)$,

$$\frac{\partial}{\partial s}m + A^*m + \operatorname{div}(g(x,m,v(x,m),v(\cdot,m)_{\#}m)m(x,s)) = 0,\ \ s > t,\ \ m(x,t) = m(x); \tag{2.8}$$

$$J_{m,t}(v) = \int_t^T \int_{\mathbb{R}^n} f(x,m_s,v(x,m_s),v(\cdot,m)_{\#}m_s)m(x,s)dxds + \int_{\mathbb{R}^n} h(x,m_T)m(x,T)dx.$$

The value function is given by

$$\Phi(m,t) := \inf_v J_{m,t}(v)$$

2.2.3 Bellman Equation

Subject to the existence of derivatives, the function $\Phi(m,t)$ is the solution of Bellman equation. In the following, we provide some formal derivation. The functional derivative $\frac{\partial}{\partial m}\Phi(m,t)(x)$ is the function (omitting the argument t) such that

$$\Phi(m') - \Phi(m) = \int_0^1 \frac{\partial}{\partial m}\Phi(\theta m' + (1-\theta)m)(x)(m'-m)(dx).$$

We introduce the Lagrangian

$$L(x,m,v,\mu,\rho) := f(x,m,v,\mu) + \rho g(x,m,v,\mu). \tag{2.9}$$

Now, the Bellman equation can be written as:

$$\begin{cases} -\dfrac{\partial \Phi}{\partial t} + \displaystyle\int_{\mathbb{R}^n} A_x \frac{\partial}{\partial m}\Phi(m,t)(x)m(x,t)dx \\ = \inf_v \int_{\mathbb{R}^n} L\left(x,m,v(x,m),v(\cdot,m)_{\#}m, D_x\dfrac{\partial}{\partial m}\Phi(m,t)(x)\right)m(x)dx, \\ \Phi(m,T) = \displaystyle\int_{\mathbb{R}^n} h(x,m)m(x)dx. \end{cases}$$

We shall use the notation

$$U(x,m,t) = \frac{\partial}{\partial m}\Phi(m,t)(x),$$

and then Bellman equation can be rewritten as

$$\begin{aligned} -\frac{\partial \Phi}{\partial t} + \int_{\mathbb{R}^n} A_x \frac{\partial}{\partial m}\Phi(m,t)(x) m(x,t)dx \\ = \inf_v \int_{\mathbb{R}^n} L(x,m,v(x,m),v(\cdot,m)_{\#}m, D_x U(x,m,t))m(x)dx \end{aligned} \tag{2.10}$$

and $\hat{v}(x,m) = \hat{v}(x,m,t)$ minimizes the functional, in v,

$$\mathcal{L}\left(v, v(\cdot,m)_{\#}m\right) := \int_{\mathbb{R}^n} L(x,m,v(x,m),v(\cdot,m)_{\#}m, D_x U(x,m,t))m(x)dx. \tag{2.11}$$

2.2.4 Rules of Derivation

To proceed, we need two important derivation rules. Details for the formal derivation rules can be found in Bensoussan et al. (2015, 2017). Consider a functional $\Psi(\mu)$ on probability measures on $\mathbb{R}^d$ and then $\Psi(v(\cdot,m)_{\#}m)$ is a functional of $v(x,m)$ and it is also a functional of m for Ψ having a functional derivative $\frac{\partial}{\partial \mu}\Psi(\mu)(w)$, with $w \in \mathbb{R}^d$, we then first claim that

$$\left.\frac{d}{d\theta}\Psi((v(\cdot,m)+\theta\tilde{v}(\cdot,m))_{\#}m)\right|_{\theta=0} = \int_{\mathbb{R}^n} D_w \frac{\partial}{\partial \mu}\Psi(v(\cdot,m)_{\#}m)(v(x,m)) \cdot \tilde{v}(x,m)\cdot m(x)dx. \tag{2.12}$$

For the second rule, we consider the map $m \mapsto \Psi(v(\cdot,m)_{\#}m)$ and establish its functional derivative as follows. We claim the formula

$$\begin{aligned} \frac{\partial}{\partial m}\Psi(v(\cdot,m)_{\#}m)(x) =& \frac{\partial}{\partial \mu}\Psi(v(\cdot,m)_{\#}m)(v(x,m)) \\ &+ \int_{\mathbb{R}^n} D_w \frac{\partial}{\partial \mu}\Psi(v(\cdot,m)_{\#}m)(v(\xi,m)) \cdot \frac{\partial}{\partial m}v(\xi,m)(x)m(\xi)d\xi. \end{aligned} \tag{2.13}$$

We want to make clear with the notations are used here, for the function $f(x,m,v(x,m),$ $v(\cdot,m)_{\#}m)$, when we write $\frac{\partial}{\partial m}f(x,m,v(x,m),v(\cdot,m)_{\#}m)(\xi)$, we mean the function

$\dfrac{\partial}{\partial m} f(x, m, v, \mu)(\xi)$ in which the arguments v and μ are replaced by $v(x, m)$ and $v(\cdot, m)_{\#}m$, respectively. We are not differentiating with respect to the argument m indexed in $v(x, m)$ or $v(\cdot, m)_{\#}m$.

Using the first rule of differentiation (2.12), we can write the Euler condition of optimality for the optimal feedback $\hat{v}(x, m)$ in the minimization of the objective functional (2.11); indeed, we first note that by using (2.12),

$$\frac{d}{d\theta}\mathcal{L}(v, v(\cdot, m)_{\#}m)\bigg|_{\theta=0} = 0 \quad \text{for any } \forall \tilde{v}(\cdot, m)$$

gives the following relation

$$\begin{aligned} &L_v(x, m, \hat{v}(x, m), \hat{v}(\cdot, m)_{\#}m, D_x U(x, m, t)) \\ &+ \int_{\mathbb{R}^n} D_w \frac{\partial}{\partial \mu} L(\xi, m, \hat{v}(\xi, m), \hat{v}(\cdot, m)_{\#}m, D_\xi U(\xi, m, t))(\hat{v}(x, m))m(\xi)d\xi = 0. \end{aligned} \tag{2.14}$$

It will be important in the sequel to compute the functional derivative of the objective functional

$$\mathcal{L}(m) = \int_{\mathbb{R}^n} L(x, m, \hat{v}(x, m), \hat{v}(\cdot, m)_{\#}m, D_x U(x, m, t))m(x)d\xi,$$

by taking into account that $\hat{v}(x, m)$ is already the optimal one, and thus that the necessary condition of optimality (2.14) is fufilled. We obtain the formula

$$\begin{aligned} \frac{\partial}{\partial m}\mathcal{L}(m)(x) =& L(x, m, \hat{v}(x, m), \hat{v}(\cdot, m)_{\#}m, D_x U(x, m, t)) \\ &+ \int_{\mathbb{R}^n} \frac{\partial}{\partial m} L(\xi, m, \hat{v}(\xi, m), \hat{v}(\cdot, m)_{\#}m, D_\xi U(\xi, m, t))(x)m(\xi)d\xi \\ &+ \int_{\mathbb{R}^n} \frac{\partial}{\partial \mu} L(\xi, m, \hat{v}(\xi, m), \hat{v}(\cdot, m)_{\#}m, D_\xi U(\xi, m, t))(\hat{v}(x, m))m(\xi)d\xi \\ &+ \int_{\mathbb{R}^n} D_\xi \frac{\partial}{\partial m} U(\xi, m, t)(x) \cdot g(\xi, m, \hat{v}(\xi, m), \hat{v}(\cdot, m)_{\#}m)m(\xi)d\xi, \end{aligned} \tag{2.15}$$

where to obtain the third term, we first apply (2.13) and then use Fubini's theorem, then the Euler-optimality condition (2.14) helps to simplify; to obtain the fourth term, just note that $\dfrac{\partial L}{\partial \rho} = g$ by using (2.9).

2.3 Master Equation and System of HJB-FP Equations

2.3.1 Master Equation

The master equation is obtained by differentiating Bellman equation (2.10) with respect to the probability measure m. Thanks to formula (2.15), we obtain the equation for $U(x, m, t)$ as follows:

$$\begin{cases} -\dfrac{\partial U}{\partial t} + A_x U(x, m, t) + \displaystyle\int_{\mathbb{R}^n} A_\xi \frac{\partial}{\partial m} U(\xi, m, t)(x) m(\xi) d\xi \\ = L(x, m, \hat{v}(x, m), \hat{v}(\cdot, m)_{\#} m, D_x U(x, m, t)) \\ + \displaystyle\int_{\mathbb{R}^n} \frac{\partial}{\partial m} L(\xi, m, \hat{v}(\xi, m), \hat{v}(\cdot, m)_{\#} m, D_\xi U(\xi, m, t))(x) m(\xi) d\xi \\ + \displaystyle\int_{\mathbb{R}^n} \frac{\partial}{\partial \mu} L(\xi, m, \hat{v}(\xi, m), \hat{v}(\cdot, m)_{\#} m, D_\xi U(\xi, m, t))(\hat{v}(x, m)) m(\xi) d\xi \\ + \displaystyle\int_{\mathbb{R}^n} D_\xi \frac{\partial}{\partial m} U(\xi, m, t)(x) \cdot g(\xi, m, \hat{v}(\xi, m), \hat{v}(\cdot, m)_{\#} m) m(\xi) d\xi, \\ U(x, m, T) = h(x, m) + \displaystyle\int_{\mathbb{R}^n} \frac{\partial}{\partial m} h(\xi, m)(x) m(\xi) d\xi; \end{cases}$$

and $\hat{v}(x, m)$ is related to $U(x, m, t)$ by the relation (2.14).

2.3.2 System of HJB-FP Equations

From the master equation, we can derive the system of HJB-FP equations. First of all, using the optimal feedback $\hat{v}(x, m)$ in the FP equation (2.8), we obtain a probability density that we here just simplify the notation $m(x, t)$ by adopting m_t; we also set $\hat{v}(x, t) = \hat{v}(x, m_t, t)$. Thus, the probability density $m(x, t)$ is the solution of

$$\frac{\partial m}{\partial t} + A^* m + \operatorname{div}(g(x, m_t, \hat{v}(x, t), \hat{v}(\cdot, t)_{\#} m_t) m(x, t)) = 0, \; m(x, 0) = m(x). \tag{2.16}$$

We also define $u(x,t) := U(x, m_t, t)$. We can state that the optimal control $\hat{v}(x,t)$ satisfies the Euler optimality condition:

$$\begin{aligned} & L_v(x, m_t, \hat{v}(x,t), \hat{v}(\cdot,t)_{\#}m_t, D_x u(x,t)) \\ + \int_{\mathbb{R}^n} D_w \frac{\partial}{\partial \mu} & L(\xi, m_t, \hat{v}(\xi,t), \hat{v}(\cdot,t)_{\#}m_t, D_\xi u(\xi,t))(\hat{v}(x,t))m(\xi,t)d\xi = 0. \end{aligned} \tag{2.17}$$

Finally, we obtain the HJB equation for $u(x,t)$ by using (2.16) and note that

$$\frac{\partial u}{\partial t} = \frac{\partial U}{\partial t} + \int_{\mathbb{R}^n} \frac{\partial U}{\partial m}(\xi, m_t, t) \cdot \frac{\partial m}{\partial t}(\xi) d\xi$$

$$\begin{cases} -\dfrac{\partial u}{\partial t} + A_x u = L(x, m_t, \hat{v}(x,t), \hat{v}(\cdot,t)_{\#}m_t, D_x u(x,t)) \\ + \displaystyle\int_{\mathbb{R}^n} \frac{\partial}{\partial m} L(\xi, m_t, \hat{v}(\xi,t), \hat{v}(\cdot,t)_{\#}m_t, D_\xi u(\xi,t))(x) m(\xi,t) d\xi \\ + \displaystyle\int_{\mathbb{R}^n} \frac{\partial}{\partial \mu} L(\xi, m_t, \hat{v}(\xi,t), \hat{v}(\cdot,t)_{\#}m_t, D_\xi u(\xi,t))(\hat{v}(x,t)) m(\xi,t) d\xi, \\ u(x,T) = h(x, m(T)) + \displaystyle\int_{\mathbb{R}^n} \frac{\partial}{\partial m} h(\xi, m(T)) m(\xi, T) d\xi. \end{cases} \tag{2.18}$$

Therefore, we have to solve jointly the system of HJB-FP equations (2.16) and (2.18) subject to the Euler optimality condition (2.17).

2.4 Application

2.4.1 Derivation of the Optimal Feedback

In this section, we derive the optimal feedback for a dynamic grid optimization problem described in Section 2.1.3. The household who owns local storage with a self-generated energy facility (e.g., a photovoltaic solar panel) makes a decision regarding how much charge or discharge energy by considering the current level of stored energy and the grid market environment. In our mean field type approach, we consider the fact that the individual household's decision marginally influences the market price, but the average of households' decision directly contribute to the formation of market price. For the derivation of the optimal feedback, we shall make the following assumption

$$h(S) = h_0 \frac{S^2}{2} + h_1 S + h_2, \tag{2.19}$$

where h_0, h_1, and h_2 are constant coefficient. Also, we use the notation

$$\overline{\hat{v}}(t) = \int \hat{v}(s,q,t) m_t(ds,dq); \ \overline{Q}(t) = \int q m_t(ds,dq).$$

Proposition 2.1. *We assume the terminal cost function is quadratic as in (2.19), then the feedback $\hat{v}(S,Q,t)$ is given by*

$$\hat{v}(S,Q,t) = \frac{KQ - \lambda(S,Q,t) - \zeta\left(\overline{\hat{v}}(t) - \overline{Q}(t)\right)}{c+K},$$

where $\zeta(x) = p(x) + xp'(x)$ and $\lambda(S,Q,t) = \frac{\partial u}{\partial S}(S,Q,t)$ is the solution of

$$\begin{cases} -\dfrac{\partial \lambda}{\partial t} - \dfrac{1}{2}\sigma^2(Q)\dfrac{\partial^2 \lambda}{\partial Q^2} - b(Q)\dfrac{\partial \lambda}{\partial Q} + \dfrac{1}{c+K}\lambda\dfrac{\partial \lambda}{\partial S} - \dfrac{1}{c+K}\left(KQ - \zeta\left(\overline{\hat{v}}(t) - \overline{Q}(t)\right)\right)\dfrac{\partial \lambda}{\partial S} \\ = aS + l, \\ \lambda(S,Q,T) = h_0 S + h_1. \end{cases}$$

Details in the proof of Proposition 2.1 can be found in the appendix. Once we know the $\lambda(S,Q,t)$, everything is explicit. We can obtain $\lambda(S,Q,t)$ as

$$\lambda(S,Q,t) = \lambda_0(Q,t)S + \lambda_1(Q,t);$$

and we have to solve the equation for $\lambda_0(Q,t)$ which is

$$-\frac{\partial \lambda_0}{\partial t} - \frac{1}{2}\sigma^2(Q)\frac{\partial^2 \lambda_0}{\partial Q^2} - b(Q)\frac{\partial \lambda_0}{\partial Q} + \frac{(\lambda_0)^2}{c+K} = a, \ \lambda_0(Q,T) = h_0;$$

and the equation for $\lambda_1(Q,t)$ which is

$$-\frac{\partial \lambda_1}{\partial t} - \frac{1}{2}\sigma^2(Q)\frac{\partial^2 \lambda_1}{\partial Q^2} - b(Q)\frac{\partial \lambda_1}{\partial Q} + \frac{\lambda_0 \lambda_1}{c+K} = \frac{1}{c+K}(KQ - \zeta(\overline{\hat{v}}(t) - \overline{Q}(t)))\lambda_0 + l,$$

$$\lambda_1(Q,T) = h_1.$$

Since a, c, K, h_0 are constants, $\lambda_0(Q,t) = \lambda_0(t)$ solution of

$$-\frac{d\lambda_0}{dt} + \frac{(\lambda_0)^2}{c+K} = a,\ \lambda_0(T) = h_0$$

whose solution is

$$\frac{\lambda_0(t)}{\sqrt{a(c+K)}} = \frac{(h_0 + \sqrt{a(c+K)}) \exp 2\sqrt{\frac{a}{c+K}}(T-t) + h_0 - \sqrt{a(c+K)}}{(h_0 + \sqrt{a(c+K)}) \exp 2\sqrt{\frac{a}{c+K}}(T-t) - (h_0 - \sqrt{a(c+K)})}$$

We can save notation in defining

$$\mu_0(t) = \frac{\lambda_0(t)}{c+K}, a_0 = \sqrt{\frac{a}{c+K}}, k_0 = \frac{h_0}{c+K},$$

then we have

$$\mu_0(t) = a_0 \frac{(k_0 + a_0) \exp 2a_0(T-t) + k_0 - a_0}{(k_0 + a_0) \exp 2a_0(T-t) - (k_0 - a_0)};$$

and $\lambda_1(Q,t)$ is the solution of

$$-\frac{\partial \lambda_1}{\partial t} - \frac{1}{2}\sigma^2(Q)\frac{\partial^2 \lambda_1}{\partial Q^2} - b(Q)\frac{\partial \lambda_1}{\partial Q} + \mu_0 \lambda_1 = \left(KQ - \zeta\left(\overline{\hat{v}}(t) - \overline{Q}(t)\right)\right)\mu_0 + l,\ \lambda_1(Q,T) = h_1, \tag{2.20}$$

which is a linear parabolic equation, the probabilistic interpretation of which, with respect to the diffusion (2.1) is immediate. The feedback $\hat{v}(S,Q,t)$ is given by

$$\hat{v}(S,Q,t) = \frac{KQ - \lambda_1(Q,t) - \zeta\left(\overline{\hat{v}}(t) - \overline{Q}(t)\right)}{c+K} - \mu_0(t)S. \tag{2.21}$$

Remark 2.1. *In the paper of Alasseur et al. (2020), the stochastic maximum principle is used. Our results concur naturally with the results of this work, and provide the dynamic programming version of the solution.*

Lemma 2.1. *Consider the situation in which the price function is linear*

$$p(x) = p_0 x + p_1$$

with $p_0 > 0$, then the feedback $\hat{v}(S,Q,t)$ is completely linear.

The proof of Lemma 3.1 is put in the appendix.

2.5 Extended Mean Field Type Control with Common Noise

2.5.1 Formulation of the Problem

We generalize the state equation (2.3) as follows. We introduce a new Brownian motion $b(t)$, independent of $w(t)$, and ξ, which we take scalar (to simplify) standard. Equation (2.4) becomes

$$dx = g(x, m_t, v(x, m_t), v(\cdot, m_t)_{\#}m_t)dt + \sigma(x_t)dw(t) + \beta db(t),\ x(0) = \xi, \tag{2.22}$$

where β is a constant vector in $\mathbb{R}^n$. This time, m_t is not the probability law of x_t, but the conditional probability law given the $\sigma-$algebra $\mathcal{B}^t = \sigma(b(s), 0 \leq s \leq t)$. Indeed, in the applications, $b(t)$ represents a common noise, which is observable. The probability measure will have a density, $m(x, t)$, which is a random field adapted to the filtration $\mathcal{B}^t$. Thanks to the fact that β is a constant vector, it is pretty standard to check that $m(x, t)$ is the solution of the stochastic Fokker-Planck equation

$$\begin{cases} d_t m(x,t) \\ +\Big(A^* m(x,t) - \dfrac{1}{2}\mathrm{tr} D^2 m(x,t)\beta\beta^* + \mathrm{div}\left(g(x, m_t, v(x, m_t), v(\cdot, m_t)_{\#}m_t)m(x,t)\right)\Big)dt \\ +\beta^* Dm(x,t)\, db(t), \\ m(x,0) = m_0(x), \end{cases} \tag{2.23}$$

where $m_0(x)$ is the density of the probability distribution of the variable ξ. Following invariant embedding framework, we shall consider arbitrary initial time t, a Brownian motion which vanishes at t, $b_t(s) = b(s) - b(t)$, $s \geq t$, and an initial random variable ξ, independent of $w(\cdot)$ and $b(\cdot)$. So we replace (2.23) by

$$\begin{cases} d_s m(x,s) \\ +\left(A^* m(x,s) - \dfrac{1}{2}\mathrm{tr} D^2 m(x,s)\beta\beta^* + \mathrm{div}\left(g(x, m_s, v(x,m_s), v(\cdot, m_s)_{\#} m_s) m(x,s)\right)\right) ds \\ +\beta^* D m(x,s)\, db(s), \\ m(x,0) = m_0(x). \end{cases}$$

We next define the cost functional

$$J_{m,t}(v) = \mathbb{E}\left[\int_t^T f(x, m_s, v(x,m_s), v(\cdot, m_s)_{\#} m_s) m(x,s) dx ds + \int_{\mathbb{R}^n} h(x, m_T) m(x,T) dx\right]$$

and the value function is given by

$$\Phi(m,t) = \inf_v J_{m,t}(v).$$

2.5.2 Bellman Equation

The functional $\Phi(m,t)$ is the solution of a second-order infinite dimensional P.D.E. To simplify notation, we introduce a Lagrangian as follows:

$$L(x, m, v, \mu, \rho) = f(x, m, v, \mu) + \rho \cdot g(x, m, v, \mu). \tag{2.24}$$

The arguments m and μ are probability measures, respectively on $\mathbb{R}^n$ and $\mathbb{R}^d$. The arguments x and v are in $\mathbb{R}^n$ and $\mathbb{R}^d$ respectively and $\rho \in \mathbb{R}^n$. By standard arguments, we check formally that $\Phi(m,t)$ is the solution of the equation

$$\begin{cases} -\dfrac{\partial \Phi}{\partial t} + \displaystyle\int_{\mathbb{R}^n} \left(A_x \dfrac{\partial \Phi}{\partial m}(m,t)(x) - \dfrac{1}{2}\mathrm{tr}\ D_x^2 \dfrac{\partial \Phi}{\partial m}(m,t)(x)\beta\beta^*\right) m(x) dx \\ -\dfrac{1}{2}\displaystyle\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \mathrm{tr} D_\xi D_\eta \dfrac{\partial^2 \Phi}{\partial m^2}(m,t)(\xi,\eta)\beta\beta^* m(\xi) m(\eta) d\xi d\eta \\ = \inf_v \displaystyle\int_{\mathbb{R}^n} L\left(x, m, v(x,m), v(\cdot,m)_{\#} m,\ D_x \dfrac{\partial \Phi}{\partial m}(m,t)(x)\right) m(x) dx, \\ \Phi(m,T) = \displaystyle\int_{\mathbb{R}^n} h(x,m) m(x) dx. \end{cases}$$

We note that the infimum is not achieved pointwise, because of the term $v(\cdot, m)_{\#}m$ which is not local. So, we cannot define an Hamiltonian as in the standard theory. In the sequel, we shall consider

$$U(x, m, t) = \frac{\partial \Phi}{\partial m}(m, t)(x).$$

We define the optimal feedback $\hat{v}(x, m) = \hat{v}(x, m, t)$ which achieves the infimum of the functional

$$\int_{\mathbb{R}^n} L\left(x, m, v(x, m), v(\cdot, m)_{\#}m,\ D_x U(x, m, t)\right) m(x) dx.$$

We write the Euler equation of optimality, taking into account the rules of differentiation, also see Section 2.2.4:

$$\begin{aligned} &L_v(x, m, \hat{v}(x, m), \hat{v}(\cdot, m)_{\#}m, D_x U(x, m, t)) \\ &\quad + \int_{\mathbb{R}^n} D_w \frac{\partial L}{\partial \mu}(\xi, m, \hat{v}(\xi, m), \hat{v}(\cdot, m)_{\#}m, D_\xi U(\xi, m, t))(\hat{v}(x, m)) m(\xi) d\xi = 0. \end{aligned}$$

2.5.3 Master Equation

The master equation is the equation giving the evolution of $U(x, m, t)$. It is obtained by differentiating Bellman equation, with respect to the argument m. After lengthy calculations, we can write formally the equation

$$\begin{cases}
-\dfrac{\partial U}{\partial t} + A_x U(x,m,t) - \dfrac{1}{2}\mathrm{tr} D_x^2 U(x,m,t)\beta\beta^* \\
+\displaystyle\int_{\mathbb{R}^n} \left(A_\xi \frac{\partial U}{\partial m}(\xi,m,t)(x) - \frac{1}{2}\mathrm{tr} D_\xi^2 \frac{\partial U}{\partial m}(\xi,m,t)(x)\beta\beta^* \right) m(\xi)d\xi \\
-\dfrac{1}{2}\displaystyle\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{\partial^2 U}{\partial m^2}(x,m,t)(\xi,\eta)\beta^* Dm(\xi)\beta^* Dm(\eta)d\xi d\eta \\
+\beta^* D_x \int_{\mathbb{R}^n} \dfrac{\partial U}{\partial m}(x,m,t)(\xi)\beta^* Dm(\xi)d\xi \\
= L(x,m,\hat{v}(x,m),\hat{v}(\cdot,m)_{\#}m, D_x U(x,m,t)) \\
+ \int_{\mathbb{R}^n} g(\xi,m,\hat{v}(\xi,m),\hat{v}(\cdot,m)_{\#}m)\cdot D_\xi \dfrac{\partial}{\partial m} U(\xi,m,t)(x)m(\xi)d\xi \\
+\displaystyle\int_{\mathbb{R}^n} \frac{\partial L}{\partial m}(\xi,m,\hat{v}(\xi,m),\hat{v}(\cdot,m)_{\#}m, D_\xi U(\xi,m,t))(x)m(\xi)d\xi \\
+\displaystyle\int_{\mathbb{R}^n} \frac{\partial L}{\partial \mu}(\xi,m,\hat{v}(\xi,m),\hat{v}(\cdot,m)_{\#}m, D_\xi U(\xi,m,t))(\hat{v}(x,m))m(\xi)d\xi, \\
U(x,m,T) = h(x,m) + \displaystyle\int_{\mathbb{R}^n} \frac{\partial h}{\partial m}(\xi,m)(x)m(\xi)d\xi.
\end{cases}$$

2.5.4 System of Stochastic HJB-FP Equations

Using the feedback $\hat{v}(x,m)$ in the stochastic Fokker-Planck equation (2.23), we obtain a solution, which we still call $m_t(x) = m(x,t)$ and we set

$$\hat{v}(x,t) = \hat{v}(x,m_t,t);\ u(x,t) = U(x,m_t,t);\ Z(x,t) = \int_{\mathbb{R}^n} \beta^* D_\xi \frac{\partial U}{\partial m}(\xi,m_t,t)(x)m_t(\xi)d\xi.$$

We obtain the following system

$$\begin{cases} -d_t u + \left(A_x u - \frac{1}{2}\mathrm{tr}D_x^2 u\beta\beta^* - \beta^* D_x Z(x,t)\right) dt \\ = \Bigg[L(x, m_t, \hat{v}(x,t), \hat{v}(\cdot,t)_{\#}m_t, D_x u(x,t)) \\ + \int_{\mathbb{R}^n} \frac{\partial L}{\partial m}(\xi, m_t, \hat{v}(\xi,t), \hat{v}(\cdot,t)_{\#}m_t, D_\xi u(\xi,t))(x) m_t(\xi) d\xi \\ + \displaystyle\int_{\mathbb{R}^n} \frac{\partial L}{\partial \mu}(\xi, m_t, \hat{v}(\xi,t), \hat{v}(\cdot,t)_{\#}m_t, D_\xi u(\xi,t))(\hat{v}(x,t)) m_t(\xi) d\xi \Bigg] dt - Z(x,t) db(t), \\ u(x,T) = h(x, m_T) + \displaystyle\int_{\mathbb{R}^n} \frac{\partial h}{\partial m}(\xi, m_T)(x) m_T(\xi) d\xi; \end{cases} \tag{2.25}$$

and

$$\begin{cases} d_t m_t(x) + \left(A^* m_t(x) - \frac{1}{2}\mathrm{tr}D^2 m_t(x)\beta\beta^* + \mathrm{div}\ (g(x, m_t, \hat{v}(x,t), \hat{v}(\cdot,t)_{\#}m_t) m_t(x))\right) dt \\ +\beta^* D m_t(x)\, db(t), \\ m(x,0) = m_0(x) \end{cases} \tag{2.26}$$

with the optimality condition

$$\begin{aligned} &L_v(x, m_t, \hat{v}(x,t), \hat{v}(\cdot,t)_{\#}m_t, D_x u(x,t)) \\ &\quad + \int_{\mathbb{R}^n} D_w \frac{\partial L}{\partial \mu}(\xi, m_t, \hat{v}(\xi,t), \hat{v}(\cdot,t)_{\#}m_t, D_\xi u(\xi,t))(\hat{v}(x,t)) m_t(\xi) d\xi = 0. \end{aligned} \tag{2.27}$$

Equation (2.25) is a backward stochastic P.D.E. and the random fields $u(x,t)$, $Z(x,t)$ are adapted to the filtration $\mathcal{B}^t$.

Remark 2.2. *In order to get the optimal control, we should use the Pontryagin Maximum Principle but the detail is put in the appendix.*

2.6 Application

2.6.1 Model

We extend the model described in the introduction, see Section 2.1.3, by considering that, besides the representative agent, there is an external producer with no storage facilities. Whereas the market price depends on only the prosumers who own the local storage in Section 2.1.3, this extended model can capture the more realistic environment, where the market price depends on the energy flow from prosumers who own the local storage and the energy flow from the external producer who have not installed the storage.

We denote by $Q_0(t)$ the energy delivered to the grid by this external producer, and by $Q_1(t)$ the energy delivered by the representative agent. The evolution of these energies is described by the equations

$$\begin{cases} dQ_0(t) = b_0(Q_0)dt + \beta_0 db(t),\ Q_0(0) = Q_{00}; \\ dQ_1(t) = b_1(Q_1)dt + \sigma(Q_1)dw(t) + \beta_1 db(t),\ Q_1(0) = Q_{10}. \end{cases} \tag{2.28}$$

Therefore, $b(t)$ is a common noise between the two productions. The initial values can be random. They are independent from the Wiener processes $w(t)$ and $b(t)$. We set $Q(t) = \begin{pmatrix} Q_0(t) \\ Q_1(t) \end{pmatrix}$. For the representative agent, there is the storage equation

$$\frac{dS}{dt} = v(S,Q),\ S(0) = 0,$$

in which $v(S,Q,t)$ is a feedback, which for S,Q fixed is a process adapted to the filtration $\mathcal{B}^t$. The global payoff is written as follow:

$$\begin{aligned} J(v) =& \mathbb{E}\int_0^T \left[\frac{a}{2}S^2(t) + lS(t) + \frac{c}{2}v^2(S(t),Q(t)) + \frac{K_0}{2}Q_0^2(t) + \frac{K_1}{2}\Big|Q_1(t) - v(S,Q)\Big|^2 \right. \\ &\left. + p\left(E^{\mathcal{B}^t}(v(S,Q) - Q_0(t) - Q_1(t))\right)(v(S,Q) - Q_0(t) - Q_1(t))\right]dt \\ &+ \mathbb{E}\left[\frac{h_0}{2}S^2(T) + h_1 S(T)\right]. \end{aligned}$$

2.6.2 Derivation of the Optimal Feedback

In Section 2.4.1, we consider the specific case where there is a continuum of households who operate local storage with a self-produced energy facility. Hence, the optimal feedback in Proposition 2.1 is obtained under an underlying assumption that the market price only depends on their decisions and states. For a further realistic approach to derive the optimal feedback, we take into account an external producer who does not operate a storage facility. In addition, a term of common noise is considered to capture aggregate shocks arising from an unpredictable change in the grid market.

The optimal feedback is denoted as $\hat{v}(S,Q,t)$. We also note $u(S,Q,t)$, $m_t(S,Q) = m(S,Q,t)$, and $Z(S,Q,t)$ the random fields solution of the stochastic HJB-FP equations (2.25) and (2.26). We also omit t as an argument, but not when we refer to m_t. The writing $m_t(S,Q)$ refers to a probability density. We also need $\mu_t = \hat{v}(\cdot,t)_{\#}m_t$ which is the conditional probability law of $\hat{v}(S,Q,t)$, given $\mathcal{B}^t$. We use the notation $\overline{\hat{v}}(t) = \displaystyle\int w\mu_t(dw)$; $\overline{Q}_0(t) = \displaystyle\int q_0 m_t(ds,dq)$; $\overline{Q}_1(t) = \displaystyle\int q_1 m_t(ds,dq)$ and set $\lambda(S,Q,t) = \dfrac{\partial u}{\partial S}(S,Q,t)$; $\Gamma(S,Q,t) = \dfrac{\partial Z}{\partial S}(S,Q,t)$.

Proposition 2.2. *The feedback $\hat{v}(S,Q,t)$ is given by*

$$\hat{v}(S,Q,t) = \frac{K_1 Q - \lambda(S,Q,t) - \zeta\left(\overline{\hat{v}}(t) - \overline{Q}_0(t) - \overline{Q}_1(t)\right)}{c+K_1},$$

where $\zeta(x) = p(x) + xp'(x)$ and $\lambda(S,Q,t)$ is the solution of

$$\begin{cases} -d_t\lambda + \Bigg(-\dfrac{1}{2}\beta_0^2\dfrac{\partial^2\lambda}{\partial Q_0^2} - \dfrac{1}{2}(\sigma^2(Q_1)+\beta_1^2)\dfrac{\partial^2\lambda}{\partial Q_1^2} - \beta_0\beta_1\dfrac{\partial^2\lambda}{\partial Q_0\partial Q_1} \\ -\dfrac{\partial\lambda}{\partial S}\left(\dfrac{K_1Q_1 - \lambda - \zeta(\overline{\hat{v}}(t) - \overline{Q}_0(t) - \overline{Q}_1(t))}{c+K_1}\right) \\ -\dfrac{\partial\lambda}{\partial Q_0}b_0(Q_0) - \dfrac{\partial\lambda}{\partial Q_1}b_1(Q_1) - \beta_0\dfrac{\partial\Gamma}{\partial Q_0} - \beta_1\dfrac{\partial\Gamma}{\partial Q_1}dt \\ = (aS+l)dt - \Gamma(S,Q)db(t), \\ \lambda(S,Q,T) = h_0S + h_1, \end{cases} \tag{2.29}$$

Let us consider next the process

$$\overline{\lambda}(t) = \int \lambda(s,q,t)m(ds,dq,t) \tag{2.30}$$

We can compute the Ito differential of $\overline{\lambda}(t)$ by combining (2.29) with the stochastic FP equation in Proposition 2.2 (see the proof of Proposition 2.2 in the appendix). After some calculations, recalling that

$$d\overline{\lambda}(t) = -(l + a\overline{S}(t))dt + \chi(t)db(t);\ \chi(t) = \int \left(\Gamma + \beta_0 \frac{\partial \lambda}{\partial Q_0} + \beta_1 \frac{\partial \lambda}{\partial Q_1}\right)(s,q,t)m(ds,dq,t) \tag{2.31}$$

with the final condition

$$\overline{\lambda}(T) = h_0 \overline{S}(T) + h_1$$

with the notation $\overline{S}(t) = \displaystyle\int sm(ds,dq,t)$. From the FP equation, we have also

$$\overline{S}(t) = \int_0^t \overline{\hat{v}}(\tau)d\tau \tag{2.32}$$

and from (2.30), we obtain the relation

$$(c + K_1)\overline{\hat{v}}(t) = K_1\overline{Q}_1(t) - \overline{\lambda}(t) - \zeta\left(\overline{\hat{v}}(t) - \overline{Q}_0(t) - \overline{Q}_1(t)\right). \tag{2.33}$$

From (2.31), (2.32), and (2.33) we can obtain the processes $\overline{\lambda}(t)$, $\overline{S}(t)$, and $\overline{\hat{v}}(t)$. From (2.31) and (2.32) we have

$$\overline{\lambda}(t) = l(T-t) + h_1 + \mathbb{E}^{\mathcal{B}^t}\left[\int_0^T \overline{\hat{v}}(s)(a(T - s \vee t) + h_0)ds\right]$$

and from (2.33) we obtain a functional equation for $\overline{\hat{v}}(t)$, namely

$$\begin{aligned}(c + K_1)\overline{\hat{v}}(t) + \zeta\left(\overline{\hat{v}}(t) - \overline{Q}_0(t) - \overline{Q}_1(t)\right) + \mathbb{E}^{\mathcal{B}^t}\left[\int_0^T \overline{\hat{v}}(s)(a(T - s \vee t) + h_0)ds\right] \\ = K_1\overline{Q}_1(t) - l(T-t) - h_1.\end{aligned} \tag{2.34}$$

This equation is the generalization of the integral equation for the function $\overline{\hat{v}}(t)$ studied in Proposition 2.1. Knowing $\overline{\hat{v}}(t)$, we can obtain $\lambda(S,Q,t)$ by solving (2.29). We postulate a solution of the form

$$\lambda(S,Q,t)=\lambda_0(t)S+\lambda_1(Q_1,t)+\nu(t),$$

where $\lambda_0(t)$ is a deterministic function of time, $\lambda_1(Q_1,t)$ is a deterministic function of Q_1 and time, and $\nu(t)$ is an adapted process. Using this expression in (2.29) yields

$$-\frac{d\lambda_0}{dt}+\frac{\lambda_0^2}{c+K_1}=a,\ \ \lambda_0(T)=h_0$$

then $\lambda_1(Q_1,t)$ is the solution of

$$-\frac{\partial\lambda_1}{\partial t}-\frac{1}{2}(\sigma^2(Q_1)+\beta_1^2)\frac{\partial^2\lambda_1}{\partial Q_1^2}-b_1(Q_1)\frac{\partial\lambda_1}{\partial Q_1}+\frac{\lambda_0(t)}{c+K_1}\lambda_1=\frac{K_1\lambda_0(t)}{c+K_1}Q_1,\ \ \lambda_1(Q_1,T)=0;$$

and $\nu(t)$ is solution of the BSDE

$$-d\nu(t)+\frac{\lambda_0(t)}{c+K_1}\nu(t)dt=\left(l-\frac{\lambda_0(t)}{c+K_1}\zeta\left(\overline{\hat{v}}(t)-\overline{Q}_0(t)-\overline{Q}_1(t)\right)\right)dt-\Gamma(t)db(t),\ \ \nu(T)=h_1.$$

Therefore, everything is explicit, provided we can solve the functional equation (2.34). We shall make the following monotonicity assumption

$$(\zeta(x_1)-\zeta(x_2))(x_1-x_2)\geq 2p_0|x_1-x_2|^2 \tag{2.35}$$

We then state

Proposition 2.3. *We assume (2.35) and*

$$\frac{1}{c+K_1+2p_0}\sqrt{\int_0^T\int_0^T(a(T-s\vee t)+h_0)^2dsdt}<1. \tag{2.36}$$

Then there exists a unique process $\overline{\hat{v}}(t)$, which is adapted to the filtration $\mathcal{B}^t$ and is square integrable, and solution of the functional equation (2.34).

2.6.3 Particular Price Function

We assume that the price function is linear as in Lemma 3.1. Then $\zeta(x) = 2p_0x + p_1$ and the equation (2.34) becomes (writing $\rho(t) = \overline{\hat{v}}(t)$ to simplify notation)

$$(c + K_1 + 2p_0)\rho(t) + \mathbb{E}^{\mathcal{B}^t}\int_0^T \rho(s)(a(T - s \vee t) + h_0)ds$$
$$= 2p_0\overline{Q}_0(t) + (K_1 + 2p_0)\overline{Q}_1(t) - l(T-t) - h_1 - p_1.$$

This equation is not a BSDE, and although linear does not seem to have an explicit solution. We can compute easily the expected value $\overline{\rho}(t) = \mathbb{E}\rho(t)$, solution of the deterministic functional equation,

$$(c + K_1 + 2p_0)\overline{\rho}(t) + \int_0^T \overline{\rho}(s)(a(T - s \vee t) + h_0)ds$$
$$= 2p_0\mathbb{E}\overline{Q}_0(t) + (K_1 + 2p_0)\mathbb{E}\overline{Q}_1(t) - l(T-t) - h_1 - p_1$$

which is the same problem in Lemma 3.1 (refer to the proof of Lemma 3.1 in the appendix). If we consider for equations (2.28), mean reverting models become

$$b_0(Q_0, t) = -\alpha_0(Q_0 - \gamma_0(t));\ b_1(Q_1, t) = -\alpha_1(Q_1 - \gamma_1(t)) \tag{2.37}$$

where $\gamma_0(t)$ and $\gamma_1(t)$ are deterministic functions, and take $\sigma(Q_1) = \sigma$, then we can check easily that

$$\begin{cases} \overline{Q}_0(t) & = \overline{Q}_{00}\exp(-\alpha_0 t) + \int_0^t \exp(-\alpha_0(t-s))\ (\alpha_0\gamma_0(s)ds + \beta_0 db(s))\,; \\ \overline{Q}_1(t) & = \overline{Q}_{10}\exp(-\alpha_1 t) + \int_0^t \exp(-\alpha_1(t-s))\,(\alpha_1\gamma_1(s)ds + \beta_1 db(s)). \end{cases} \tag{2.38}$$

By linearity, we can look for the solution $\rho(t)$ as a sum of a deterministic part (which gives the mean) and two stochastic parts corresponding to the stochastic integrals appearing at the right hand side of the two equations (2.38). We have to solve the stochastic integral equations

$$
\begin{aligned}
(c+K_1+2p_0)\rho_0(t) - a\int_0^t (t-s)\rho_0(s)ds + \mathbb{E}^{\mathcal{B}^t}\int_0^T \rho_0(s)(a(T-s)+h_0)ds \\
= 2p_0\beta_0 \exp(-\alpha_0 t)\int_0^t \exp\alpha_0 s\, db(s);
\end{aligned} \tag{2.39}
$$

$$
\begin{aligned}
(c+K_1+2p_0)\rho_1(t) - a\int_0^t (t-s)\rho_1(s)ds + \mathbb{E}^{\mathcal{B}^t}\int_0^T \rho_1(s)(a(T-s)+h_0)ds \\
= (K_1+2p_0)\beta_1 \exp(-\alpha_1 t)\int_0^t \exp\alpha_1 s\, db(s);
\end{aligned} \tag{2.40}
$$

and finally

$$\rho(t) = \overline{\rho}(t) + \rho_0(t) + \rho_1(t).$$

The new problem is (2.39) and (2.40) is equivalent to (2.39). We define

$$r_0 = -r_1 = \sqrt{\frac{a}{c+K_1+2p_0}}.$$

We shall prove the following:

Proposition 2.4. *The solution $\rho_0(t)$ of the functional stochastic equation (2.39) is given by*

$$\rho_0(t) = A(t)b(t) + \int_0^t B(t,s)b(s)ds, \tag{2.41}$$

where $A(t), B(t,s)$ are deterministic functions such that

$$B(t,s) = -\frac{2p_0\beta_0\alpha_0^3}{c+K_1+2p_0}\frac{1}{\alpha_0+r_0}\frac{1}{\alpha_0+r_1}\exp\big(-\alpha_0(t-s)\big) + B_1(s)\exp\big(r_0(t-s)\big) + B_2(s)\exp\big(r_1(t-s)\big) \tag{2.42}$$

and $A(t), B_1(t), B_2(t)$ are defined by the relations

$$
\begin{cases}
(c+K_1+2p_0)(A'(t)+B_1(t)+B_2(t)) = -\dfrac{2p_0\beta_0\alpha_0 r_0 r_1}{(\alpha_0+r_0)(\alpha_0+r_1)}, \\
A(T) = \dfrac{2p_0\beta_0}{c+K_1+2p_0};
\end{cases} \tag{2.43}
$$

$$A(t) - \frac{B_1(t)}{r_0} - \frac{B_2(t)}{r_1} = \frac{2p_0\beta_0\alpha_0^2}{(c+K_1+2p_0)(\alpha_0+r_0)(\alpha_0+r_1)}\left[1 - \frac{1-\exp\big(-\alpha_0(T-t)\big)}{\alpha_0(T-t)}\right], \tag{2.44}$$

and

$$\begin{aligned} h_0 A(t) + B_1(t)\left[(c+K_1+2p_0+\frac{h_0}{r_0})\exp\big(r_0(T-t)\big) - \frac{h_0}{r_0}\right] \\ +B_2(t)\left[(c+K_1+2p_0+\frac{h_0}{r_1})\exp\big(r_1(T-t)\big) - \frac{h_0}{r_1}\right] \\ = -\frac{2p_0\beta_0\alpha_0 r_0 r_1}{(\alpha_0+r_0)(\alpha_0+r_1)} + \frac{2p_0\beta_0\alpha_0^2}{c+K_1+2p_0}\frac{h_0}{(\alpha_0+r_0)(\alpha_0+r_1)}\Big(1-\exp\big(-\alpha_0(T-t)\big)\Big). \end{aligned} \tag{2.45}$$

Remark 2.3. *From formulas (2.44) and (2.45), we can express $B_1(t)$ and $B_2(t)$ as affine functions of $A(t)$. We can then solve the linear first order differential equation (2.43) and obtain the function $A(t)$ (see the proof of Proposition 2.4 in the appendix for details).*

2.7 Numerical Examples

2.7.1 The Effectiveness of Local Storage Strategy

We illustrate numerical results for the optimal storage strategy of a representative energy prosumer in a smart grid system considering the energy flow from an external producer with no storage facilities (see the description of numerical examples in the appendix). Figure 2.1 shows the net energy flow from the external producer Q_0 and the local prosumers Q_1, respectively. The bottom panel of Figure 2.1 presents the net energy flow deducting optimal storage decision. We can observe that the variation of prosumers' net energy flow decreases from the storage strategy.

As shown in the left panel of Figure 2.2, the local storage strategy would diminish the fluctuation of the spot energy price, especially that at the peak hours. The right panel of Figure 2.2 also shows that the overall variation of the net energy flow decreases. Both

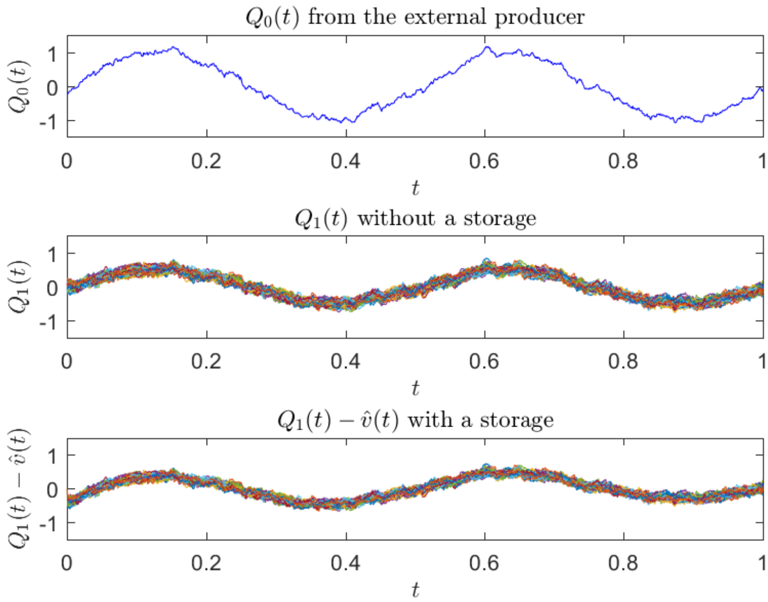
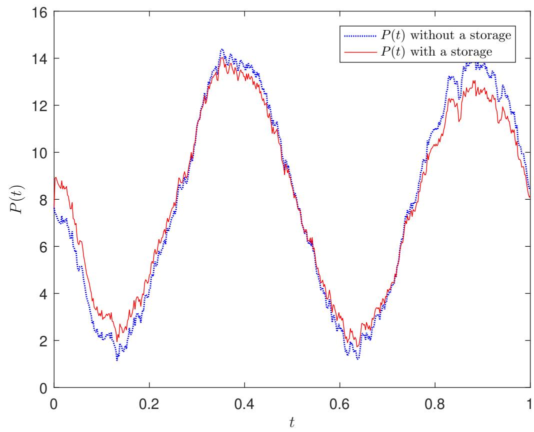


Figure 2.1. The net power flow from the external prosumer $Q_0(t)$ (upper graph), the net power flow from the local prosumers $Q_1(t)$ (middle graph), and the net delivered power flow $Q_1(t) - \hat{v}$ (lower graph).

of these results confirm that the local storage facility effectively rebalances the real-time demand or supply in the grid while it also reduces the furious fluctuation of the market price of energy. In our numerical experiments, we assume that the energy flow from the external producer to a grid is double compared to that of the representative local prosumer. This reflects the rationale that the households who installed the local storage used not to be dominant in the current grid market (Gur 2018). However, the investments on energy storage projects have been improved the global capacity of energy supply in the last few years; for instance, the installed storage capacity in 2016 increased by 28% compared to 2012 and globally 1,273 energy storage projects were operational as of 2018 (Gur 2018, Center for Sustainable Systems 2019). Considering the efforts on increasing the popularity of energy storage, we easily anticipate that the energy storage facility will play a crucial role in controlling the variation of the spot energy price and the real-time peak level of energy flow. The left panel on Figure 2.3 represents the relationship between



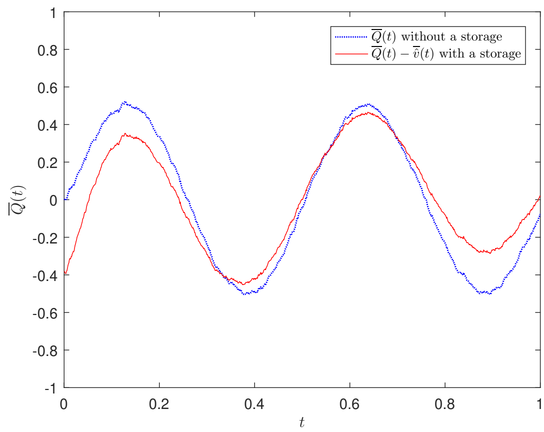


Figure 2.2. [1,left] The spot price $P(t)$ without a local storage (dot line) and with a local storage (straight line), [2, right] the trajectory of $\overline{Q}_1(t)$ without local a storage (dot line) and $\overline{Q}_1(t) - \bar{\tilde{v}}(t)$ with a local storage (straight line)

the trajectory of spot price and the amount of stored energy. Whereas the local prosumer is willing to save the produced energy when the spot price is relatively low, the prosumer discharges the stored energy to the grid system when the spot price is higher than the mean of the overall spot price.

As we observed the results of numerical analyses in Figure 2.1-2.3, a storage strategy

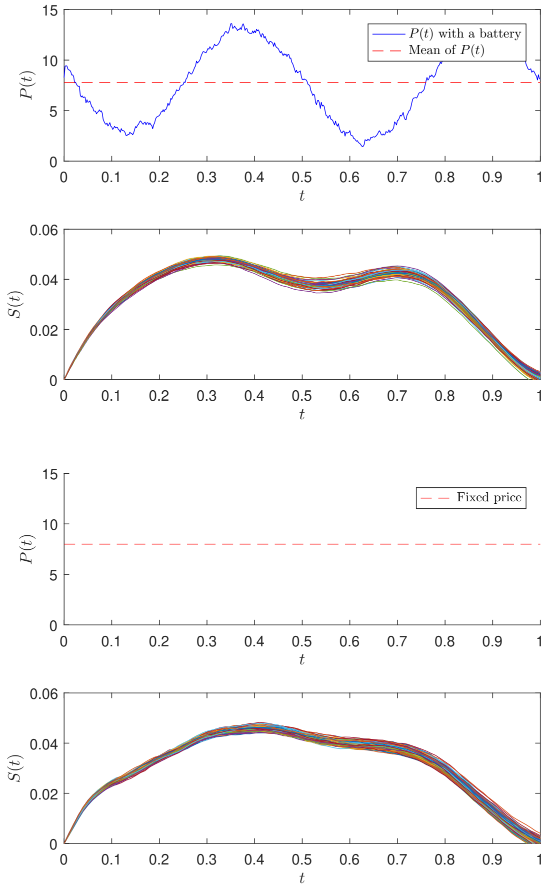


Figure 2.3. [1,left] The trajectory of $P(t)$ (straight line) with a mean field type pricing and mean of $P(t)$ (dashed line) with storage (upper graph) and $S(t)$ (lower graph), [2, right] the trajectory of $P(t)$ (dashed line) with a fixed pricing with storage (upper graph) and $S(t)$ (lower graph)

enables to reduce overall operating costs. Note that the summary of operational costs is shown in the appendix by considering two cases: the installation of a local storage system and no installation of local storage. The volumetric charges decrease in average by 28.46% after the installation of local storage. Figure 2.2 and Figure 2.3 support the local prosumer's strategic behavior to understand the reduction in the volumetric charges. The right of Figure 2.2 presents that the storage strategy reduces the maximum instantaneous power consumption, resulting in a decrease in demand charges on average by 35.78%. However, the storage strategy generates additional storage costs including a terminal cost. The Figure 2.3 also shows agents' storage level, incurring the current storage cost to manage a load of demand and the terminal cost to guarantee the minimum level of stored energy at the end of time horizon. Even considering the storage costs, the total operating costs decrease in average by 27.04% thanks to the storage strategy.

2.7.2 The Comparison of a Mean Field Type Pricing Scheme and a Fixed Pricing Scheme

We draw the policy implications on the structure of pricing scheme in the grid system by comparing the pricing scheme between the mean field type pricing and the fixed pricing. In our mean field type model, we consider a dynamic pricing scheme based on the real-time amount of aggregate energy being consumed in a grid system. While a fixed (or flat) rate is exogeneous, our dynamic pricing scheme is modulated by the mean field term being the law of the control. This term can be interpreted as the average of historical performance of one single player and be considered as the stringent restrictions on the behavior of this player by managing himself or herself to avoid over consumption or much saving. In other words, the single player has his or her own historical records over time, he or she needs to stick to the pricing rule in alignment with the mean field term by his or her feedback control. For instance, if the single player charges the energy more than the average, then he or she will tend to charge less energy. As described in the work of Alasseur et al. (2020), the single player can be regarded as a central planner who coordinates the local storage in the grid; thus, the solution of our approach is the optimal charging or discharging strategy of a central planner to allocate it to all local storage at once.

Since the European countries are eagerly willing to introduce a renewable energy facility, European Commission (2019) studies the profitability of a dynamic pricing scheme in a grid system. Even though there are only seven European countries (Denmark, Estonia, Finland, Netherlands, Spain, Sweden, and the United Kingdom) that partially allow consumers to choose the dynamic pricing scheme, European Commission (2019) sheds light on the benefit of the dynamic pricing scheme. According to this research, the annual saving with the dynamic pricing scheme can be in the range of $15-80$ Euros for small usage consumers whose annual consumption are less than 1000 kWh. Even though the dynamic pricing has an economic inventive for consumers, this pricing scheme is not prevalent because of the concern regarding a systemic risk from the fluctuation of a spot price of energy (Dutta and Mitra 2017, European Commission 2019). Our mean field approach with a local storage strategy is a proper way to lessen the system risk in a grid with dynamic pricing scheme as described in Section 2.7.1.

In Figure 2.4, we set a rate of fixed price as an overall mean of dynamic pricing rate in Figure 2.3. Then, the trajectory of optimal storage level with the fixed pricing scheme is close to an inverse U-shape while the mean field pricing scheme is similar to an M-shape. Whereas the M-shape storage level means that the prosumer optimally makes a decision how much charge or discharge energy in local storage against the variation of market spot price, the inverse U-shape storage level implies that the prosumer derives a decision based on the terminal time horizon. Numerically, the mean field pricing scheme reduces the average of costs for the storage strategy compared to the fixed pricing scheme by 18.05% (see the detailed comparison between mean field pricing scheme and fixed pricing scheme in the appendix). This implies that the dynamic pricing scheme modeled by a mean field type approach is implementable on the problem of an energy-grid dynamic optimization for prosumers with the storage strategy.

2.7.3 The Impact of the Rate of Mean Reversion for Local Storage Strategy

We assume that the energy flow, $Q_0(t)$ and $Q_1(t)$ are modeled by a mean reverting process as in (2.37). At here, α_0 and α_1 represent the rate of mean reversion which implies how strongly the system reacts to perturbations. The numerical analyses regarding the impact of the rate of mean reversion have been implemented for 1,000 times, and then we illustrate the average trajectory of storage level in Figure 2.4. In the left panel of Figure 2.4, the trajectory of storage level follows an M-shape with a relatively high mean revering property. Prosumers who operate the local storage tend to strategically react to the variation of the market price thanks to strong mean revering properties. In the right panel of Figure 2.4, however, the overall path of storage level is close to an inverse U-shape with weak mean revering property that may not appropriately cope with perturbations. This result is similar to the overall trajectory of storage level described in the case of a fixed pricing scheme as shown in the right panel of Figure 2.3.

As the rate of mean reversion increases, the peak level of storage decreases in Figure 2.4. A strong mean reverting-property guarantees that the system process reverts to its mean or average level without a significant perturbation. Therefore, prosumers properly cope with the variation of energy flow between a grid and local storage by efficiently controlling the level of stored energy. In addition, the rate of mean reversion is inversely related to the stationary variance of the Ornstein-Uhlenbeck process. The system of energy flow in the grid is under a certain level of uncertainty measured by the stationary variance of the stochastic process. The weaker the mean-reverting property is, the higher the uncertainty level of energy flow between the grid and local prosumers increases. Local prosumers are willing to bear a high storage cost to manage the uncertainty of energy flow by increasing the peak level of stored energy.

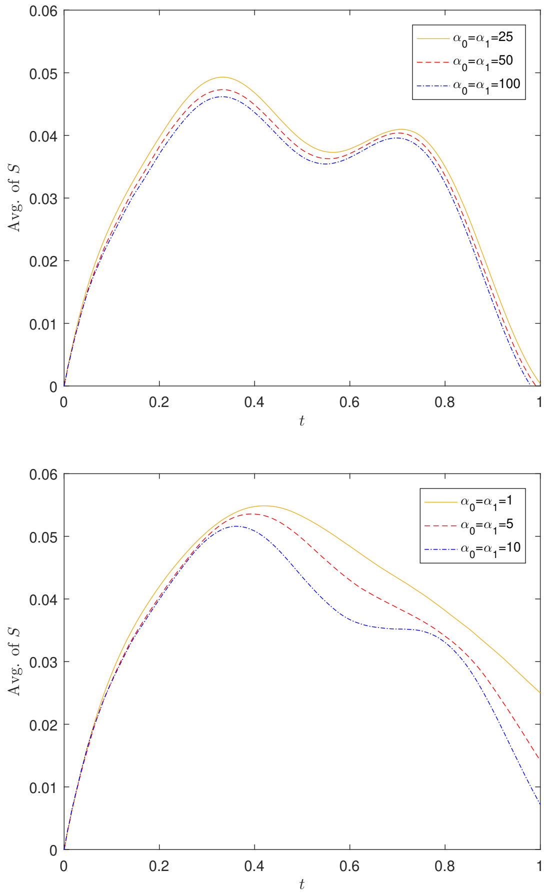


Figure 2.4. [1,left] The average trajectory of storage level with relatively high α_0 and α_1, [2,right] the average trajectory of storage level with relatively low α_0 and α_1

2.8 Managerial Implications and Concluding Remarks

In numerical experiments, we confirm that the prosumer's behavior to manage the level of energy stored in the grid system is highly close to the market maker and the trader's behavior to control the amount of holding assets in the security trading market (see e.g. Duffie et al. 2005, 2007, Weill 2007). In the viewpoint of prosumer's strategic decision, the stored energy can be considered as a tradable financial asset. The grid system allows buyers and sellers to quickly exchange their assets. Under the stable economic conditions, the traders are willing to stock up the inventory of securities when the market price goes down and to dispose of inventories when the market price goes up at profit (Weill 2007). Similarly, in the left panel of Figure 2.3, if the current spot price is relatively lower than the future price that the prosumer anticipates, she strategically reduces the actual energy flow sent to the grid by accumulating the energy in storage (Paola et al. 2016). During the financial disruption like the financial crisis 2007-2008, there would be strong selling pressure on traders. In the crisis, the main role of a market maker is to providing with liquidity by absorbing this selling pressure (Weill 2007). In a similar way, as shown in Figure 2.2, when the grid has excessive residual energy, households who own a local storage facility intentionally charge it by rebalancing a load of energy in the grid system (Paola et al. 2016).

In the viewpoint of a central planner who coordinates between a grid operator and a continuum of prosumers, increasing penetration of local storage may decrease the variation of spot price as well as the peak level of demand, resulting in the enhancement of the resilience of grid systems. In addition, the introduction of dynamic pricing scheme modeled by a mean field term gives prosumers an incentive to install local storage. Prosumers may reduce an electricity bill with a dynamic pricing scheme compared to a fixed pricing scheme even considering a storage cost. Our results are in line with the Dutta and Mitra (2017)'s survey study for dynamic pricing of electricity. This research shows that the dynamic pricing policy is preferred over the fixed pricing policy because the dynamic pricing scheme is more effective to provide customers with economic incentives and revenue stability. However, they pointed out that the dynamic pricing scheme should have inherent systemic risk engendered by a variation of real-time electricity price. In our research, we suggest a local storage strategy as a proper remedy to lessen this risk in a grid system. Prosumers who own local storage can charge or discharge supplement energy based on the the real-time spot price, resulting in the decrease of price variation. In addition, we show that the storage strategy is effective when a mean reverting property is even weak. The rate of mean reversion is directly related to uncertainty in energy flow between the grid and local storage. Prosumers may mitigate this uncertainty by increasing the peak level of stored energy. Eventually, a central planner's goal is achieved to assign optimal storage strategy so that a continuum of prosumers simultaneously choose controls for all of the prosumers to optimize the average pay off by enhancing the resilience of grid system.

The main task of our study has been the formulation of an extended mean field type control problem, applicable to an energy-grid dynamic optimization as studied in Alasseur et al. (2020), who solve this problem using a stochastic maximum principle. Our study is the first theoretical work to suggest a Master equation and the system of HJB-FP equations for the extended mean field type control approach. The model is extendable to a number of research questions for further studies. For example, we suggested a dynamic Cournot competition model among a large number of interacting players based on the concept of inventory control. It would be interesting to what quantity decision-making rules are proper under competition when a pricing rule is not external, but this is a function of the average decision of multiple players. In addition, this pricing rule influences the level of inventory related to the cost structure. This setting is especially relevant for the energy, online-retailing, and financial industry, where a large number of players compete the quantity decision by considering their current level of inventory.

2.9 Appendix. Proofs

2.9.1 Proof of Proposition 2.1

In the model (2.1) and (2.2) we have $x = (S, Q)$ and

$$g(S, Q, m, v, \mu) = \begin{vmatrix} v \\ b(Q) \end{vmatrix}, \ \sigma(x) = \begin{vmatrix} 0 \\ \sigma(Q) \end{vmatrix}$$

then

$$A\varphi(S, Q) = -\frac{1}{2}\sigma^2(Q)\frac{\partial^2\varphi}{\partial Q^2}$$

Also

$$f(S, Q, m, v, \mu) = \frac{a}{2}S^2 + lS + \frac{c}{2}v^2 + \frac{K}{2}|Q - v|^2 + p\left(\int w\mu(dw) - \int qm(ds, dq)\right)(v - Q)$$

$$h(x, m) = h(S)$$

We get the Lagrangian

$$L(S, Q, m, v, \mu, \rho) = \frac{a}{2}S^2 + lS + \frac{c}{2}v^2 + \frac{K}{2}|Q - v|^2$$
$$+p\left(\int w\mu(dw) - \int qm(ds, dq)\right)(v - Q) + \varrho v + \rho_1 b(Q)$$

with $\rho = \begin{pmatrix} \varrho \\ \rho_1 \end{pmatrix}$. We obtain the derivatives

$$L_v(S, Q, m, v, \mu, \rho) = (c + K)v - KQ + p\left(\int w\mu(dw) - \int qm(ds, dq)\right) + \varrho,$$
$$\frac{\partial}{\partial m}L(S, Q, m, v, \mu, \rho)(s, q) = -p'\left(\int w\mu(dw) - \int qm(ds, dq)\right)q(v - Q),$$
$$\frac{\partial}{\partial \mu}L(S, Q, m, v, \mu, \rho)(w) = p'\left(\int w\mu(dw) - \int qm(ds, dq)\right)w(v - Q).$$

We use the notation

$$\overline{\hat{v}}(t) = \int \hat{v}(s,q,t)m_t(ds,dq),$$
$$\overline{Q}(t) = \int q m_t(ds,dq).$$

We can then note that

$$\begin{aligned}
&L_v(S,Q,m_t,\hat{v}(S,Q,t),\hat{v}(\cdot,t)_{\#}m_t,\rho) \\
&= (c+K)\hat{v}(S,Q) - KQ + p\left(\overline{\hat{v}}(t) - \overline{Q}(t)\right) + \varrho, \\
&\frac{\partial}{\partial\mu}L(S,Q,m_t,\hat{v}(S,Q,t),\hat{v}(\cdot,t)_{\#}m_t,\rho)(w) \\
&= p'\left(\overline{\hat{v}}(t) - \overline{Q}(t)\right) w(\hat{v}(S,Q) - Q), \\
&\frac{\partial}{\partial m}L((S,Q,m_t,\hat{v}(S,Q,t),\hat{v}(\cdot,t)_{\#}m_t,\rho)(s,q) \\
&= -p'\left(\overline{\hat{v}}(t) - \overline{Q}(t)\right) q(\hat{v}(S,Q) - Q), \\
&\int D_w\frac{\partial}{\partial\mu}L(S,Q,m_t,\hat{v}(S,Q,t),\hat{v}(\cdot,t)_{\#}m_t,\rho)(\hat{v}(S,Q))m_t(ds,dq) \\
&= p'\left(\overline{\hat{v}}(t) - \overline{Q}(t)\right)\ (\overline{\hat{v}}(t) - \overline{Q}(t)).
\end{aligned}$$

Introduce the function

$$\zeta(x) = p(x) + xp'(x)$$

then the Euler condition (2.17) becomes

$$(c+K)\hat{v}(S,Q) - KQ + \frac{\partial u}{\partial S} + \zeta\left(\overline{\hat{v}}(t) - \overline{Q}(t)\right) = 0 \tag{2.46}$$

from which we derive

$$(c+K)\overline{\hat{v}}(t) - K\overline{Q}(t) + \int \frac{\partial u}{\partial S}(s,q,t)m_t(ds,dq) + \zeta\left(\overline{\hat{v}}(t) - \overline{Q}(t)\right) = 0. \tag{2.47}$$

The HJB equation (2.18) becomes

$$\begin{cases} -\dfrac{\partial u}{\partial t} - \dfrac{1}{2}\sigma^2(Q)\dfrac{\partial^2 u}{\partial Q^2} - \dfrac{\partial u}{\partial S}\hat{v}(S,Q) - \dfrac{\partial u}{\partial Q}b(Q) \\ = \dfrac{a}{2}S^2 + lS + \dfrac{c}{2}\hat{v}(S,Q)^2 + \dfrac{K}{2}|Q - \hat{v}(S,Q)|^2 + (\hat{v}(S,Q) - Q)\zeta\left(\overline{\hat{v}}(t) - \overline{Q}(t)\right), \\ u(S,Q,t) = h(S); \end{cases} \tag{2.48}$$

and the FP equation is

$$\begin{cases} \dfrac{\partial m}{\partial t} - \dfrac{1}{2}\dfrac{\partial^2}{\partial Q^2}(\sigma^2(Q)m) + \dfrac{\partial}{\partial S}(\hat{v}(S,Q)m) + \dfrac{\partial}{\partial Q}(b(Q)m), \\ m(S,Q,0) = \delta(S)\bigotimes m_0(Q), \end{cases} \tag{2.49}$$

where $m_0(Q)$ is the probability distribution of the initial value Q_0. Combining (2.48) with (2.46), we obtain the equation

$$\begin{cases} -\dfrac{\partial u}{\partial t} - \dfrac{1}{2}\sigma^2(Q)\dfrac{\partial^2 u}{\partial Q^2} - Q\dfrac{\partial u}{\partial S} - b(Q)\dfrac{\partial u}{\partial Q} + \dfrac{c+K}{2}\left|\hat{v}(S,Q) - Q\right|^2 = \dfrac{a}{2}S^2 + \dfrac{c}{2}Q^2 + lS, \\ u(S,Q,T) = h(S). \end{cases} \tag{2.50}$$

We then define $\lambda(S,Q,t) = \dfrac{\partial u}{\partial S}(S,Q,t)$. Differentiating (2.50) with respect to S we obtain

$$\begin{cases} -\dfrac{\partial \lambda}{\partial t} - \dfrac{1}{2}\sigma^2(Q)\dfrac{\partial^2 \lambda}{\partial Q^2} - Q\dfrac{\partial \lambda}{\partial S} - b(Q)\dfrac{\partial \lambda}{\partial Q} + (c+K)(\hat{v}(S,Q) - Q)\dfrac{\partial \hat{v}(S,Q)}{\partial S} = aS + l, \\ \lambda(S,Q,T) = h'(S). \end{cases}$$

On the other hand, from (2.46), we can write

$$(c+K)\frac{\partial \hat{v}}{\partial S} + \frac{\partial \lambda}{\partial S} = 0.$$

Then, we obtain the following equation for λ

$$\begin{cases} -\dfrac{\partial \lambda}{\partial t} - \dfrac{1}{2}\sigma^2(Q)\dfrac{\partial^2 \lambda}{\partial Q^2} - \hat{v}(S,Q)\dfrac{\partial \lambda}{\partial S} - b(Q)\dfrac{\partial \lambda}{\partial Q} = aS + l; \\ \lambda(S,Q,T) = h'(S). \end{cases} \tag{2.51}$$

From (2.49) and (2.51) , we can infer

$$\overline{\lambda}(t)=\int\lambda m_t(ds,dq)=\int h'(s)m_T(ds,dq)+l(T-t)+a\int_t^T\left(\int sm_\tau(ds,dq)\right)d\tau \quad (2.52)$$

On the other hand, from the FP equation we get also

$$\frac{d}{dt}\int sm_t(ds,dq)=\overline{\hat{v}}(t)$$

and since the initial condition in S is 0, we obtain

$$\int sm_t(ds,dq)=\int_0^t\overline{\hat{v}}(\tau)d\tau. \quad (2.53)$$

So (2.52) becomes

$$\overline{\lambda}(t)=\int h'(s)m_T(ds,dq)+l(T-t)+a\int_0^T(T-t\vee\tau)\overline{\hat{v}}(\tau)d\tau.$$

Finally, from (2.47) we obtain

$$\begin{aligned}(c+K)\overline{\hat{v}}(t)-K\overline{Q}(t)+\zeta\left(\overline{\hat{v}}(t)-\overline{Q}(t)\right)\\+\int h'(s)m_T(ds,dq)+l(T-t)+a\int_0^T(T-t\vee\tau)\overline{\hat{v}}(\tau)d\tau=0\end{aligned} \quad (2.54)$$

In this equation, the probability $m_T(ds,dq)$ stills intervenes and the full system HJB-FP and (2.54) remains coupled. We get a decoupling when the function $h(S)$ is quadratic. We then assume

$$h(S)=h_0\frac{S^2}{2}+h_1S+h_2.$$

Therefore, using (2.53), it follows that

$$\int h'(s)m_T(ds,dq)=h_0\int_0^T\overline{\hat{v}}(\tau)d\tau+h_1.$$

and we get from (2.54) an integral equation for the function $\overline{\hat{v}}(t)$, namely

$$(c+K)\overline{\hat{v}}(t)+\zeta\left(\overline{\hat{v}}(t)-\overline{Q}(t)\right)+\int_0^T\overline{\hat{v}}(\tau)(a(T-t\vee\tau)+h_0)d\tau=K\overline{Q}(t)-l(T-t)-h_1. \quad (2.55)$$

Knowing the function $\overline{\hat{v}}(t)$ we can obtain the function $\lambda(S,Q,t)$ by solving (2.51), taking account of (2.46). This concludes the proof. ■

2.9.2 Proof of Lemma 2.1

Consider the situation in which

$$p(x) = p_0 x + p_1,$$

with $p_0 > 0$. Then

$$\zeta(x) = 2p_0 x + p_1.$$

Equation (2.55) becomes linear. Setting $\rho(t) = \overline{\hat{v}}(t)$ to simplify notation, it reduces to, after rearrangements

$$(c + K + 2p_0)\rho(t) + \int_0^T \rho(\tau)(a(T - \tau \vee t) + h_0)d\tau = (K + 2p_0)\overline{Q}(t) - l(T - t) - h_1 - p_1$$

Differentiation twice, we obtain the second order differential equation

$$(c + K + 2p_0)\rho''(t) - a\rho(t) = (K + 2p_0)(\overline{Q}(t))'' \tag{2.56}$$

Setting $r_0 = \sqrt{\dfrac{a}{c + K + 2p_0}}$ and $r_1 = -r_0$, the general solution of (2.56) is given by

$$\begin{aligned} \rho(t) &= A_0 \exp r_0 t + A_1 \exp r_1 t \\ -\frac{K + 2p_0}{(c + K + 2p_0)(r_0 - r_1)} &\int_t^T (\overline{Q}(s))''(\exp\big(-r_0(s - t)\big) - \exp(-r_1(s - t))\,)ds \end{aligned} \tag{2.57}$$

We have two boundary conditions

$$(c + K + 2p_0)\rho(T) + h_0 \int_0^T \rho(\tau)d\tau = (K + 2p_0)\overline{Q}(T) - h_1 - p_1$$

$$(c + K + 2p_0)\rho(0) + \int_0^T (a(T - \tau) + h_0)\rho(\tau)d\tau = (K + 2p_0)\overline{Q}(0) - lT - h_1 - p_1$$

From (2.57) we obtain

$$\begin{aligned} \int_0^T \rho(\tau)d\tau =& A_0 \frac{\exp(r_0 T) - 1}{r_0} + A_1 \frac{\exp(r_1 T) - 1}{r_1} \\ &- \frac{K + 2p_0}{(c + K + 2p_0)(r_0 - r_1)} \int_0^T (\overline{Q}(s))'' \left(\frac{1 - \exp(-r_0 s)}{r_0} - \frac{1 - \exp(-r_1 s)}{r_1} \right) ds \end{aligned}$$

Similarly,

$$\int_0^T (T-\tau)\rho(\tau)d\tau = A_0 \frac{\exp(r_0 T) - 1 - r_0 T}{r_0^2} + A_1 \frac{\exp(r_1 T) - 1 - r_1 T}{r_1^2}$$
$$-\frac{K+2p_0}{c+K+2p_0}\frac{T}{(r_0-r_1)}\int_0^T (\overline{Q}(s))'' \left(\frac{1-\exp(-r_0 s)}{r_0} - \frac{1-\exp(-r_1 s)}{r_1}\right) ds$$
$$+\frac{K+2p_0}{(c+K+2p_0)(r_0-r_1)}\left[\int_0^T (\overline{Q}(s))'' \left(\frac{\exp(-r_0 s)+r_0 s}{r_0^2} - \frac{\exp(-r_1 s)+r_1 s}{r_1^2}\right) ds\right]$$

Combining results, we obtain the following linear algebraic system for the constants A_0, A_1

$$A_0\left[(c+K+2p_0)\exp(r_0 T) + h_0 \frac{\exp(r_0 T) - 1}{r_0}\right]$$
$$+A_1\left[(c+K+2p_0)\exp(r_1 T) + h_0 \frac{\exp(r_1 T) - 1}{r_1}\right]$$
$$= (K+2p_0)\overline{Q}(T) - h_1 - p_1$$
$$+h_0 \frac{K+2p_0}{(c+K+2p_0)(r_0-r_1)}\int_0^T (\overline{Q}(s))'' \left(\frac{1-\exp(-r_0 s)}{r_0} - \frac{1-\exp(-r_1 s)}{r_1}\right) ds$$

and

$$A_0\left[c+K+2p_0 + a\frac{\exp(r_0 T) - 1 - r_0 T}{r_0^2} + h_0 \frac{\exp(r_0 T) - 1}{r_0}\right]$$
$$+A_1\left[c+K+2p_0 + a\frac{\exp(r_1 T) - 1 - r_1 T}{r_1^2} + h_0 \frac{\exp(r_1 T) - 1}{r_1}\right]$$
$$=(K+2p_0)\overline{Q}(0) - lT - h_1 - p_1 + \frac{K+2p_0}{r_0-r_1}\int_0^T (\overline{Q}(s))''(\exp(-r_0 s) - \exp(-r_1 s))ds$$
$$+\frac{(K+2p_0)(aT+h_0)}{(c+K+2p_0)(r_0-r_1)}\int_0^T (\overline{Q}(s))'' \left(\frac{1-\exp(-r_0 s)}{r_0} - \frac{1-\exp(-r_1 s)}{r_1}\right) ds$$
$$-\frac{(K+2p_0)a}{(c+K+2p_0)(r_0-r_1)}\int_0^T (\overline{Q}(s))'' \left(\frac{\exp(-r_0 s)+r_0 s}{r_0^2} - \frac{\exp(-r_1 s)+r_1 s}{r_1^2}\right) ds$$

We obtain a completely explicit expression for the feedback (2.21). One still has to solve the linear PDE (2.20). As in the simulation developed in Alasseur et al. (2020), one has $\sigma(Q) = \sigma$ and $b(Q) = -\alpha(Q-\beta)$. Equation (2.20) becomes

$$\begin{cases} -\dfrac{\partial \lambda_1}{\partial t} - \dfrac{1}{2}\sigma^2 \dfrac{\partial^2 \lambda_1}{\partial Q^2} + \alpha(Q-\beta)\dfrac{\partial \lambda_1}{\partial Q} + \mu_0 \lambda_1 = \mu_0\Big[KQ - p_1 - 2p_0(\rho(t) - \overline{Q}(t))\Big] + l, \\ \lambda_1(Q,T) = h_1; \end{cases}$$

and the solution is a linear function in Q, namely

$$\lambda_1(Q,t) = \nu(t)Q + \theta(t)$$

with

$$\begin{cases} -\dfrac{d\nu}{dt} + (\alpha + \mu_0)\nu = \mu_0 K, \ \nu(T) = 0; \\ -\dfrac{d\theta}{dt} + \mu_0\theta = \alpha\beta\nu - \mu_0\Big[p_1 + 2p_0(\rho(t) - \overline{Q}(t))\Big] + l, \ \theta(T) = h_1. \end{cases}$$

The feedback $\hat{v}(S,Q,t)$ is completely linear. ■

2.9.3 Proof of Proposition 2.2

This problem enters into the general theory developed in section 2.5.1. with the state $x = (S,Q)$ and

$$g(S,Q,m,v,\mu) = \begin{vmatrix} v \\ b_0(Q_0) \\ b_1(Q_1) \end{vmatrix}, \ \sigma(S,Q) = \begin{vmatrix} 0 \\ 0 \\ \sigma(Q_1) \end{vmatrix}, \ \beta = \begin{vmatrix} 0 \\ \beta_0 \\ \beta_1 \end{vmatrix}$$

$$f(S,Q,m,v,\mu) = \frac{a}{2}S^2 + lS + \frac{c}{2}v^2 + \frac{K_0}{2}Q_0^2 + \frac{K_1}{2}\Big|Q_1 - v\Big|^2 + (v - Q_0 - Q_1)p\left(\int w\mu(dw) - \int (q_0 + q_1)m(ds,dq)\right),$$

in which μ is a probability on $\mathbb{R}$ (the control v) and m is a probability on $\mathbb{R}^3$ (the state S and Q). We write the system (2.25), (2.26), and (2.27). We first write the Lagrangian (2.24). We get

$$L(S,Q,m,v,\mu,\rho) = \frac{a}{2}S^2 + lS + \frac{c}{2}v^2 + \frac{K_0}{2}Q_0^2 + \frac{K_1}{2}\Big|Q_1 - v\Big|^2 + (v - Q_0 - Q_1)p\left(\int w\mu(dw) - \int (q_0 + q_1)m(ds,dq)\right) + \varrho v + \rho_0 b_0(Q_0) + \rho_1 b_1(Q_1),$$

where $\rho = \begin{pmatrix} \varrho \\ \rho_0 \\ \rho_1 \end{pmatrix}$. We need its derivatives

$$\begin{aligned}
L_v(S,Q,m,v,\mu,\rho) &= (c+K_1)v - K_1Q_1 + p\left(\int w\mu(dw) - \int (q_0+q_1)m(ds,dq)\right) \\
&\frac{\partial L}{\partial m}(S,Q,m,v,\mu,\rho)(s,q) \\
&= -(v-Q_0-Q_1)p'\left(\int w\mu(dw) - \int (q_0+q_1)m(ds,dq)\right)(q_0+q_1) \\
&\frac{\partial L}{\partial \mu}(S,Q,m,v,\mu,\rho)(w) \\
&= (v-Q_0-Q_1)p'\left(\int w\mu(dw) - \int (q_0+q_1)m(ds,dq)\right)w
\end{aligned} \tag{2.58}$$

We use the notation

$$\overline{\hat{v}}(t) = \int w\mu_t(dw);\ \overline{Q}_0(t) = \int q_0 m_t(ds,dq);\ \overline{Q}_1(t) = \int q_1 m_t(ds,dq)$$

We begin with the Euler condition (2.27). We have, using (2.58)

$$\begin{aligned}
&\int_{\mathbb{R}^n} D_w \frac{\partial L}{\partial \mu}(\xi, m_t, \hat{v}(\xi,t), \hat{v}(\cdot,t)_{\#}m_t, D_\xi u(\xi,t))(\hat{v}(x,t))m_t(\xi)d\xi \\
&\qquad = (\overline{\hat{v}}(t) - \overline{Q}_0(t) - \overline{Q}_1(t))p'(\overline{\hat{v}}(t) - \overline{Q}_0(t) - \overline{Q}_1(t))
\end{aligned}$$

and

$$\begin{aligned}
&L_v(x, m_t, \hat{v}(x,t), \hat{v}(\cdot,t)_{\#}m_t, D_x u(x,t)) \\
&= (c+K_1)\hat{v}(S,Q,t) - K_1Q_1 + p\left(\overline{\hat{v}}(t) - \overline{Q}_0(t) - \overline{Q}_1(t)\right) + \frac{\partial u}{\partial S}(S,Q,t).
\end{aligned}$$

If we set

$$\zeta(x) = p(x) + xp'(x),$$

we can write the Euler condition as (omitting arguments)

$$(c+K_1)\hat{v} - K_1Q_1 + \zeta\left(\overline{\hat{v}}(t) - \overline{Q}_0(t) - \overline{Q}_1(t)\right) + \frac{\partial u}{\partial S} = 0. \tag{2.59}$$

We turn to HJB equation (2.25). We first have

$$L(x, m_t, \hat{v}(x,t), \hat{v}(\cdot,t)_{\#}m_t, D_x u(x,t)) = \frac{a}{2}S^2 + lS + \frac{c}{2}\hat{v}^2 + \frac{K_0}{2}Q_0^2 + \frac{K_1}{2}\Big|Q_1 - \hat{v}\Big|^2$$
$$+ (\hat{v} - Q_0 - Q_1)p\left(\overline{\hat{v}}(t) - \overline{Q}_0(t) - \overline{Q}_1(t)\right) + \frac{\partial u}{\partial S}\hat{v} + \frac{\partial u}{\partial Q_0}b_0(Q_0) + \frac{\partial u}{\partial Q_1}b_1(Q_1)$$

then

$$\int_{\mathbb{R}^n} \frac{\partial L}{\partial m}(\xi, m_t, \hat{v}(\xi,t), \hat{v}(\cdot,t)_{\#}m_t, D_\xi u(\xi,t))(x) m_t(\xi) d\xi$$
$$= -(\overline{\hat{v}}(t) - \overline{Q}_0(t) - \overline{Q}_1(t))p'\left(\overline{\hat{v}}(t) - \overline{Q}_0(t) - \overline{Q}_1(t)\right)(Q_0 + Q_1)$$
$$\int_{\mathbb{R}^n} \frac{\partial L}{\partial \mu}(\xi, m_t, \hat{v}(\xi,t), \hat{v}(\cdot,t)_{\#}m_t, D_\xi u(\xi,t))(\hat{v}(x,t)) m_t(\xi) d\xi$$
$$= (\overline{\hat{v}}(t) - \overline{Q}_0(t) - \overline{Q}_1(t))p'\left(\overline{\hat{v}}(t) - \overline{Q}_0(t) - \overline{Q}_1(t)\right)\hat{v}$$

Adding up, we obtain

$$\begin{aligned}
X =& L\left(x, m_t, \hat{v}(x,t), \hat{v}(\cdot,t)_{\#}m_t, D_x u(x,t)\right) \\
&+ \int_{\mathbb{R}^n} \frac{\partial L}{\partial m}(\xi, m_t, \hat{v}(\xi,t), \hat{v}(\cdot,t)_{\#}m_t, D_\xi u(\xi,t))(x) m_t(\xi) d\xi \\
&+ \int_{\mathbb{R}^n} \frac{\partial L}{\partial \mu}(\xi, m_t, \hat{v}(\xi,t), \hat{v}(\cdot,t)_{\#}m_t, D_\xi u(\xi,t))(\hat{v}(x,t)) m_t(\xi) d\xi \\
=& \frac{a}{2}S^2 + lS + \frac{c}{2}\hat{v}^2 + \frac{K_0}{2}Q_0^2 + \frac{K_1}{2}\Big|Q_1 - \hat{v}\Big|^2 \\
&+ (\hat{v} - Q_0 - Q_1)\zeta\left(\overline{\hat{v}}(t) - \overline{Q}_0(t) - \overline{Q}_1(t)\right) + \frac{\partial u}{\partial S}\hat{v} + \frac{\partial u}{\partial Q_0}b_0(Q_0) + \frac{\partial u}{\partial Q_1}b_1(Q_1) \\
=& \frac{a}{2}S^2 + lS + \frac{c}{2}\hat{v}^2 + \frac{K_0}{2}Q_0^2 + \frac{K_1}{2}\Big|Q_1 - \hat{v}\Big|^2 \\
&+ (\hat{v} - Q_0 - Q_1)\left(\frac{\partial u}{\partial S} + \zeta\left(\overline{\hat{v}}(t) - \overline{Q}_0(t) - \overline{Q}_1(t)\right)\right) \\
&+ \frac{\partial u}{\partial S}(Q_0 + Q_1) + \frac{\partial u}{\partial Q_0}b_0(Q_0) + \frac{\partial u}{\partial Q_1}b_1(Q_1).
\end{aligned}$$

Using the Euler condition (2.59), we have to compute

$$\frac{a}{2}S^2 + lS + \frac{c}{2}\hat{v}^2 + \frac{K_0}{2}Q_0^2 + \frac{K_1}{2}\Big|Q_1 - \hat{v}\Big|^2 + (\hat{v} - Q_0 - Q_1)(K_1 Q_1 - (c + K_1)\hat{v})$$
$$= \frac{a}{2}S^2 + lS - \frac{K_1 + c}{2}\Big|\hat{v} - Q_0 - Q_1\Big|^2 + \frac{K_0 + K_1}{2}Q_0^2 + \frac{c}{2}(Q_0 + Q_1)^2.$$

Combining results we obtain the HJB equation

$$
\begin{cases}
-d_t u + \Big(-\dfrac{1}{2}\beta_0^2 \dfrac{\partial^2 u}{\partial Q_0^2} - \dfrac{1}{2}(\sigma^2(Q_1)+\beta_1^2)\dfrac{\partial^2 u}{\partial Q_1^2} - \beta_0\beta_1 \dfrac{\partial^2 u}{\partial Q_0 \partial Q_1} \\
-\dfrac{\partial u}{\partial S}(Q_0+Q_1) - \dfrac{\partial u}{\partial Q_0} b_0(Q_0) - \dfrac{\partial u}{\partial Q_1} b_1(Q_1) - \beta_0 \dfrac{\partial Z}{\partial Q_0} - \beta_1 \dfrac{\partial Z}{\partial Q_1} \\
+\dfrac{K_1+c}{2}\Big| \hat{v} - Q_0 - Q_1 \Big|^2 \Big) dt \\
= \left(\dfrac{a}{2}S^2 + lS + \dfrac{K_0+K_1}{2}Q_0^2 + \dfrac{c}{2}(Q_0+Q_1)^2 \right) dt - Z(S,Q)db(t), \\
u(S,Q,T) = \dfrac{h_0}{2}S^2 + h_1 S.
\end{cases}
\tag{2.60}
$$

We finally write the FP equation

$$
\begin{cases}
d_t m + \left(-\dfrac{1}{2}\beta_0^2 \dfrac{\partial^2 m}{\partial Q_0^2} - \dfrac{1}{2}\dfrac{\partial^2}{\partial Q_1^2}((\sigma^2(Q_1)+\beta_1^2)m) - \beta_0\beta_1 \dfrac{\partial^2 m}{\partial Q_0 \partial Q_1} \right. \\
\left. +\dfrac{\partial}{\partial S}(\hat{v}m) + \dfrac{\partial}{\partial Q_0}(b_0(Q_0)m) + \dfrac{\partial}{\partial Q_1}(b_1(Q_1)m) \right) dt + \left(\beta_0 \dfrac{\partial m}{\partial Q_0} + \beta_1 \dfrac{\partial m}{\partial Q_1} \right) db(t) = 0, \\
m(S,Q,0) = \delta(S) \bigotimes m_0(Q).
\end{cases}
$$

In the general theory, we have introduced the gradient $\lambda(x,t) = D_x u(x,t)$. Here, only $\lambda(S,Q,t) = \dfrac{\partial u}{\partial S}(S,Q,t)$ will play a role. We obtain the equation for λ, by differentiating (2.60) with respect to S. We use the expression of $\hat{v}$ in terms of λ given by (2.59), and its derivative

$$
\frac{\partial \hat{v}}{\partial S}(S,Q,t) = -\frac{1}{c+K_1}\frac{\partial \lambda}{\partial S}(S,Q,t)
$$

Rearranging and setting $\Gamma(S,Q,t) = \dfrac{\partial Z}{\partial S}(S,Q,t)$, we obtain the results. ■

2.9.4 Proof of Proposition 2.3

We shall use a fixed point argument, in the Hilbert space $L^2_{\mathcal{B}^t}(0,T)$, closed subspace of $L^2(0,T;L^2(\Omega,\mathcal{A},P))$ of processes which are adapted to the filtration $\mathcal{B}^t$. For any $z \in$

$L^2_{\mathcal{B}^t}(0,T)$, there exists a unique $\rho \in L^2_{\mathcal{B}^t}(0,T)$, solution of

$$
\begin{aligned}
&(c+K_1)\rho(t) + \zeta(\rho(t) - \overline{Q}_0(t) - \overline{Q}_1(t)) \\
&= K_1\overline{Q}_1(t) - l(T-t) - h_1 - \mathbb{E}^{\mathcal{B}^t}\left[\int_0^T z(s)(a(T - s\vee t) + h_0)ds\right]
\end{aligned}
$$

This follows from the fact that, by assumption (2.35), the function $\zeta(x)$ is monotone. Considering two processes z_1 and z_2 and the corresponding images ρ_1 and ρ_2, we have

$$
\begin{aligned}
&(c+K_1)\mathbb{E}\left(\rho_1(t)-\rho_2(t)\right)^2 \\
&+\mathbb{E}\left[\zeta\left(\rho_1(t) - \overline{Q}_0(t) - \overline{Q}_1(t)\right) - \zeta\left(\rho_2(t) - \overline{Q}_0(t) - \overline{Q}_1(t)\right)\right]\left(\rho_1(t)-\rho_2(t)\right) \\
&= -\mathbb{E}\left(\rho_1(t)-\rho_2(t)\right)\int_0^T (z_1(s)-z_2(s))(a(T - s\vee t) + h_0)ds
\end{aligned}
$$

By easy majorations, using the assumption (2.35), we obtain

$$
(c+K_1+2p_0)^2\mathbb{E}\Big|\rho_1(t)-\rho_2(t)\Big|^2 \leq \mathbb{E}\int_0^T \Big|z_1(s)-z_2(s)\Big|^2 ds \int_0^T (a(T - s\vee t) + h_0)^2 ds
$$

Integrating in t, we obtain immediately

$$
||z_1 - z_2||_{L^2_{\mathcal{B}^t}(0,T)} \leq \frac{1}{c+K_1+2p_0}\sqrt{\int_0^T\int_0^T (a(T - s\vee t) + h_0)^2 dsdt}\, ||\rho_1-\rho_2||_{L^2_{\mathcal{B}^t}(0,T)}
$$

Thanks to the assumption (2.36), the map $z \to \rho$ is a contraction, which leads to the result. ■

2.9.5 Proof of Proposition 2.4

We write the right hand side of (2.39) as

$$
2p_0\beta_0 b(t) - 2p_0\beta_0\alpha_0\exp(-\alpha_0 t)\int_0^t b(s)\exp(\alpha_0 s)ds
$$

We then insert (2.41) into equation (2.39) and equate terms in $b(t)$ and in $\int_0^t b(s)...ds$. We obtain the relations

$$
\begin{aligned}
&(c+K_1+2p_0)A(t)+\int_t^T A(s)(a(T-s)+h_0)ds \\
&+\int_t^T\Big(\int_s^T B(\tau,s)(a(T-\tau)+h_0)d\tau\Big)\,ds = 2p_0\beta_0,
\end{aligned}
\tag{2.61}
$$

and

$$
\begin{aligned}
(a(T-t)+h_0)A(s)&+(c+K_1+2p_0)B(t,s)+(a(T-t)+h_0)\int_s^t B(\tau,s)d\tau \\
&+\int_t^T (a(T-\tau)+h_0)B(\tau,s)d\tau = -2p_0\beta_0\alpha_0\exp\big(-\alpha_0(t-s)\big).
\end{aligned}
\tag{2.62}
$$

If we differentiate the relation (2.62) twice in t, we see that the function $t\longmapsto B(t,s)$ satisfies the second order differential equation

$$
(c+K_1+2p_0)B_{tt}(t,s)-aB(t,s)=-2p_0\beta_0\alpha_0^3\exp\big(-\alpha_0(t-s)\big)
\tag{2.63}
$$

from which it is easy to obtain the result (2.42), the first function being a particular solution of the second order differential equation (2.63). Next, differentiating (2.61) in t yields

$$
(c+K_1+2p_0)A'(t)-A(t)(a(T-t)+h_0)-\int_t^T B(\tau,t)(a(T-\tau)+h_0)d\tau=0.
$$

But applying (2.62) with $s=t$ yields

$$
(a(T-t)+h_0)A(t)+(c+K_1+2p_0)B(t,t)+\int_t^T (a(T-\tau)+h_0)B(\tau,t)d\tau=-2p_0\beta_0\alpha_0,
$$

and from (2.42) with $s=t$ we obtain

$$
B(t,t)=-\frac{2p_0\beta_0\alpha_0^3}{c+K_1+2p_0}\frac{1}{\alpha_0+r_0}\frac{1}{\alpha_0+r_1}+B_1(t)+B_2(t).
$$

Combining we get (2.43). The value of $A(T)$ is obtained by taking $t = T$ in (2.61). Combining the last two relations, it follows that

$$\begin{aligned} &(a(T-t)+h_0)A(t)+(c+K_1+2p_0)(B_1(t)+B_2(t)) \\ &+\int_t^T (a(T-\tau)+h_0)B(\tau,t)d\tau = -\frac{2p_0\beta_0\alpha_0 r_0 r_1}{(\alpha_0+r_0)(\alpha_0+r_1)}. \end{aligned} \tag{2.64}$$

Also applying (2.62) with $t = T$, $s = t$ we obtain

$$h_0 A(t)+(c+K_1+2p_0)B(T,t)+h_0\int_t^T B(\tau,t)d\tau = -2p_0\beta_0\alpha_0 \exp\big(-\alpha_0(T-t)\big). \tag{2.65}$$

We use (2.42) to compute

$$\begin{aligned} \int_t^T (a(T-\tau)+h_0)B(\tau,t)d\tau &= B_1(t)\left[-\frac{a}{r_0}(T-t)+\frac{1}{r_0}(h_0+\frac{a}{r_0})\big(\exp\big(r_0(T-t)\big)-1\big)\right] \\ &+B_2(t)\left[-\frac{a}{r_1}(T-t)+\frac{1}{r_1}(h_0+\frac{a}{r_1})\big(\exp\big(r_1(T-t)\big)-1\big)\right] \\ -\frac{2p_0\beta_0\alpha_0^2}{c+K_1+2p_0}\frac{1}{\alpha_0+r_0}\frac{1}{\alpha_0+r_1}&\left[(h_0-\frac{a}{\alpha_0})\big(1-\exp\big(-\alpha_0(T-t)\big)\big)+a(T-t)\right], \\ \int_t^T B(\tau,t)d\tau &= B_1(t)\frac{\exp r_0(T-t)-1}{r_0}+B_2(t)\frac{\exp(r_1(T-t))-1}{r_1} \\ &-\frac{2p_0\beta_0\alpha_0^2}{c+K_1+2p_0}\frac{1}{\alpha_0+r_0}\frac{1}{\alpha_0+r_1}\big(1-\exp\big(-\alpha_0(T-t)\big)\big). \end{aligned}$$

Then (2.64) yields

$$\begin{aligned} (a(T-t)+h_0)A(t)+B_1(t)&\left[-\frac{a}{r_0}(T-t)+\frac{1}{r_0}(h_0+\frac{a}{r_0})\exp\big(r_0(T-t)\big)-\frac{h_0}{r_0}\right] \\ +B_2(t)&\left[-\frac{a}{r_1}(T-t)+\frac{1}{r_1}(h_0+\frac{a}{r_1})\exp\big(r_1(T-t)\big)-\frac{h_0}{r_1}\right] \\ =\frac{2p_0\beta_0\alpha_0^2}{c+K_1+2p_0}\frac{1}{\alpha_0+r_0}\frac{1}{\alpha_0+r_1}&\left[a(T-t)-(h_0-\frac{a}{\alpha_0})\exp\big(-\alpha_0(T-t)\big)+h_0\right] \end{aligned} \tag{2.66}$$

and (2.65) yields

$$
\begin{aligned}
h_0 A(t) + B_1(t)\left[(c+K_1+2p_0+\frac{h_0}{r_0})\exp\big(r_0(T-t)\big) - \frac{h_0}{r_0}\right]\\
+B_2(t)\left[(c+K_1+2p_0+\frac{h_0}{r_1})\exp\big(r_1(T-t)\big) - \frac{h_0}{r_1}\right]\\
= -\frac{2p_0\beta_0\alpha_0 r_0 r_1}{(\alpha_0+r_0)(\alpha_0+r_1)} + \frac{2p_0\beta_0\alpha_0^2}{c+K_1+2p_0}\frac{h_0}{(\alpha_0+r_0)(\alpha_0+r_1)}\big(1-\exp\big(-\alpha_0(T-t)\big)\big),
\end{aligned}
\tag{2.67}
$$

Substracting (2.67) from (2.66) and then dividing the difference by $a(T-t)$ yields (2.44). This concludes the proof. ■

2.10 Appendix. Pontryagin Maximum Principle

We begin by considering the gradient $\lambda(x,t) = D_x u(x,t)$. We differentiate (2.25), and take account of the optimality condition, as well as the property that $D_x\lambda(x,t)$ is a symmetric matrix. We obtain

$$
\begin{cases}
-d_t\lambda + \left(-\mathrm{tr}(D_x\sigma^* D_x\lambda\sigma) + A_x\lambda - \frac{1}{2}\mathrm{tr}(D_x^2\lambda\beta\beta^*) - \beta^* D_x^2 Z(x,t)\right)dt\\
= \Bigg[\ D_x L(x,m_t,\hat{v}(x,t),\hat{v}(\cdot,t)_{\#}m_t,\lambda(x,t)) + D_x\lambda(x,t)g(x,m_t,\hat{v}(x,t),\hat{v}(\cdot,t)_{\#}m_t)\\
+D_x\displaystyle\int_{\mathbb{R}^n}\frac{\partial L}{\partial m}(\xi,m_t,\hat{v}(\xi,t),\hat{v}(\cdot,t)_{\#}m_t,\lambda(\xi,t))(x)m_t(\xi)d\xi\Bigg]dt - D_x Z(x,t)db(t),\\
\lambda(x,T) = D_x h(x,m_T) + D_x\displaystyle\int_{\mathbb{R}^n}\frac{\partial h}{\partial m}(\xi,m_T)(x)m_T(\xi)d\xi.
\end{cases}
$$

We recall that $\lambda(x,t)$ is a vector, so $D_x\lambda$ is a symmetric matrix, hence $\mathrm{tr}(D_x\sigma^* D_x\lambda\sigma)$ is the vector

$$
(\mathrm{tr}(D_x\sigma^* D_x\lambda\sigma))_i = \sum_{jkl}\frac{\partial\sigma_{jl}}{\partial x_i}\sigma_{kl}\frac{\partial\lambda_j}{\partial x_k}.
$$

We consider next the optimal trajectory, corresponding to (2.22) when we use the optimal feedback $\hat{v}(x,t)$. We obtain a process, which we denote by $y(t)$. To be precise with the meaning of m_t, we denote it $\mathbb{P}^{\mathcal{B}^t}_{y(t)}$, which means the conditional probability of $y(t)$ given the σ-algebra $\mathcal{B}^t$. We next call $\hat{v}(t) = \hat{v}(y(t), t)$. Clearly $\hat{v}(\cdot, t)_{\#} m_t$ is the conditional probability of $\hat{v}(t)$ given $\mathcal{B}^t$, that we denote by $\mathbb{P}^{\mathcal{B}^t}_{\hat{v}(t)}$. With this notation the optimal state $y(t)$ is the solution of the SDE

$$dy = g\left(y(t), \mathbb{P}^{\mathcal{B}^t}_{y(t)}, \hat{v}(t), \mathbb{P}^{\mathcal{B}^t}_{\hat{v}(t)}\right) dt + \sigma(y(t))dw(t) + \beta db(t), \ y(0) = \xi.$$

We then introduce the adjoint state $p(t)$ by the formula

$$p(t) = \lambda(y(t), t). \tag{2.68}$$

We turn to the necessary condition of optimality (2.27) in which we take $x = y(t)$. The first term can be interpreted easily. To avoid confusion, we consider an independent copy of $(y(t),\ \hat{v}(t),\ p(t))$ that we call $(\tilde{y}(t),\ \widetilde{\hat{v}(t)},\ \tilde{p}(t))$, and we can interpret condition (2.27) as follows:

$$\begin{aligned} &L_v\left(y(t), \mathbb{P}^{\mathcal{B}^t}_{y(t)}, \hat{v}(t), \mathbb{P}^{\mathcal{B}^t}_{\hat{v}(t)}, p(t)\right) \\ +\mathbb{E}^{\mathcal{B}^t}&\left[D_w \frac{\partial L}{\partial \mu}\left(\tilde{y}(t), \mathbb{P}^{\mathcal{B}^t}_{y(t)}, \widetilde{\hat{v}(t)}, \mathbb{P}^{\mathcal{B}^t}_{\hat{v}(t)}, \tilde{y}(t), \widetilde{\hat{v}(t)}, \tilde{p}(t)\right)\right](\hat{v}(t)) = 0, \end{aligned}$$

in which the conditional expectation with respect to $\mathcal{B}^t$ refers to the random variables $(\tilde{y}(t),\ \widetilde{\hat{v}(t)},\ \tilde{p}(t))$. It remains to find the equation of the adjoint state $p(t)$. It is obtained by taking the Ito differential of the right hand side of (2.68). However, because the function $\lambda(x, t)$ is not deterministic and there is a correlation at the level of the Wiener process $b(t)$ between its Ito differential for fixed x and the Ito differential of $y(t)$, we cannot use the standard Ito's formula. We need to use a generalization due to Kunita, see Kunita (1982). We obtain the following backward SDE

$$\begin{cases} -dp(t) = \Bigg(D_x L\left(y(t), \mathbb{P}^{\mathcal{B}^t}_{y(t)}, \hat{v}(t), \mathbb{P}^{\mathcal{B}^t}_{\hat{v}(t)}, p(t)\right) + \mathrm{tr}\big(D_x \sigma^*(y(t))\big) r(t) \\ +D_x \mathbb{E}^{\mathcal{B}^t}\left[\dfrac{\partial L}{\partial m}(\tilde{y}(t), \mathbb{P}^{\mathcal{B}^t}_{y(t)}, \widetilde{\hat{v}(t)}, \mathbb{P}^{\mathcal{B}^t}_{\hat{v}(t)}, \tilde{y}(t), \widetilde{\hat{v}(t)}, \tilde{p}(t))\right](y(t))\Bigg) dt - r(t)dw(t) - \theta(t)db(t), \\ p(T) = D_x h\left(y(T), P^{\mathcal{B}^T}_{y(T)}\right) + D_x \mathbb{E}^{\mathcal{B}^T}\left[\dfrac{\partial h}{\partial m}(\tilde{y}(T), P^{\mathcal{B}^T}_{y(T)})\right](y(T)), \end{cases}$$

where $r(t)$ and $\theta(t)$ are stochastic processes with values in $\mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$ and $\mathbb{R}^n$ respectively. Denoting $\mathcal{F}^t = \sigma(w(s), b(s), s \le t)$, all processes $y(t)$, $\hat{v}(t)$, $p(t)$, $r(t)$, and $\theta(t)$ are adapted to the filtration $\mathcal{F}^t$. We can express the processes $r(t)$ and $\theta(t)$ in terms of the gradients of the random fields $\lambda(x,t)$ and $Z(x,t)$ as follows:

$$r(t) = D_x\lambda(y(t),t)\sigma(y(t));\ \theta(t) = D_x Z(y(t),t) + D_x\lambda(y(t),t)\beta.$$

2.11 Appendix. Numerical Results

Table 2.1. The average of costs without storage and with storage.

	without storage	with storage	percentage decrease
volumetric charges	1.4751	1.0553	28.46%
demand charges	0.6780	0.4354	35.78%
storage costs (inc., terminal cost)	0	0.0802	N/A
total costs	2.1531	1.5709	27.04 %

Table 2.2. The average of costs with fixed pricing scheme and with mean field pricing scheme.

	fixed pricing	mean field pricing	percentage decrease
volumetric charges	1.2291	1.0553	16.47%
demand charges	0.5540	0.4354	27.24%
storage costs (inc., terminal cost)	0.0713	0.0802	-11.09%
total costs	1.8544	1.5709	18.05 %

We present numerical examples based on the theoretical model. The main objective of each prosumer is to minimize her electricity bill and a storage cost by optimizing the control $v(t)$ in the storage $S(t)$. The underlying assumption of energy flow system is seasonal. The net energy sent to the grid after deducting consumption of energy, $Q_0(t)$ and $Q_1(t)$ are random, but $Q_0(t)$ and $Q_1(t)$ are mean reverting stochastic processes by modeled an Ornstein-Uhlenbeck (OU) process as described in (2.37). The seasonality components of net energy sent to grid are defined by

$$\gamma_1(t) = 0.5cos(4\pi t - \pi/2) \text{ and } \gamma_0(t) = cos(4\pi t - \pi/2).$$

In addition, the other parameters are given as follows: $\alpha_0 = \alpha_1 = 100$, $\sigma_1 = \beta_0 = 1.0$, $\beta_1 = 0.5$, $p_0 = 4$, $p_1 = 8$, $K = 11$, $c = 55$, $a = 125$, $l = -0.15a$, $h_0 = 25$, $h_1 = -0.11h_0$, and $h_2 = h_1^2/2h_0$. In the simulations, we observe the short-term behavior of the agent's operational decision, so we set the terminal time as 1 day (i.e., $T = 1$). We draw the numerical results to check whether a storage strategy and a mean field pricing scheme are effective by implementing 1,000 times. The details are shown in Table 3.4.1 and Table 2.2. From these results, we conclude that the local storage is profitable and the mean field pricing is superior than the fixed pricing for individual households.

Our model does not enforce constraints on the capacity of local storage, but the level of storage nearly maintain positive value by adjusting numerical parameters for reasonable interpretations. In the numerical analysis of Alasseur et al.(2020), the negative value of storage level is allowed, so we may not directly compare how much the storage strategy reduce the local prosumer's electricity bill. In addition, the setting on parameters is different in both models. Even though the specific figures in numerical results are not identical, both models show that the installation of local storage for prosumers is effective in diminishing the volatility of spot price and in reducing the electricity bill. This implies that a central planner has an incentive to increase the penetration of local storage for prosumers in the grid system.

2.12 Miscellaneous Appendix

2.12.1 Assumptions

Assumption 2.1. (Lipschitz Condition) *g and σ are globally Lipschitz continuous in all arguments, i.e., $\exists L > 0$, such that*

$$|g(x,m,v,\mu) - g(x',m',v',\mu')| \leq L\left(|x-x'| + ||m-m'|| + |v-v'| + ||\mu-\mu'||\right);$$
$$|\sigma(x) - \sigma(x')| \leq L\left(|x-x'|\right).$$

Assumption 2.2. (Linear Growth) *g and σ are of linear growth in all arguments, i.e., $\exists L > 0$, such that*

$$|g(x,m,v,\mu)| \leq L\left(1 + |x| + ||m|| + |v| + ||\mu||\right);$$
$$|\sigma(x)| \leq L\left(1 + |x|\right).$$

Assumption 2.3. (Quadratic Condition on the Cost Functional (See (A.5) in Carmona and Delarue (20)) *$\exists L > 0$, such that*

$$\begin{aligned} &|f(x,m,v,\mu) - f(x',m',v',\mu')| \\ &\leq L\Big[1 + |x| + ||m|| + |v| + ||\mu|| + |x'| + ||m'|| + |v'| + ||\mu'||\Big] \\ &\quad \cdot \Big[|x-x'| + ||m-m'|| + |v-v'| + ||\mu-\mu'||\Big]; \\ |h(x,m) - h(x',m')| &\leq L\Big[1 + |x| + ||m|| + |x'| + ||m'||\Big] \cdot \Big[|x-x'| + ||m-m'||\Big]. \end{aligned}$$

2.12.2 Bellman Equation

We assume that the optimal control is approximately constant for $s \in [t, t+\epsilon]$. The dynamic programming principle tells us that

$$\Phi(m,t) = \inf_v \left[\int_t^{t+\epsilon} \int_{\mathbb{R}^n} f(x, m_s, v(x,m_s), v(\cdot,m)_{\#}m_s) m(x,s) dx ds + \Phi(m(t+\epsilon), t+\epsilon)\right]. \tag{2.69}$$

At here,

$$\Phi(m(t+\epsilon), t+\epsilon) = \Phi(m,t) + \epsilon\frac{\partial \Phi}{\partial t} + \epsilon\int_{\mathbb{R}^n} \frac{\partial \Phi}{\partial m}\frac{\partial m}{\partial t} dx + o(\epsilon).$$

Now, plug this into (2.69)

$$\begin{aligned}\Phi(m,t) = \inf_v \Big[\int_t^{t+\epsilon} \int_{\mathbb{R}^n} f(x, m_s, v(x, m_s), v(\cdot, m)_{\#} m_s) m(x,s) dx ds + \Phi(m,t) \\ + \epsilon\frac{\partial \Phi}{\partial t} + \epsilon\int_{\mathbb{R}^n} \frac{\partial \Phi}{\partial m}\frac{\partial m}{\partial t} dx + o(\epsilon)\Big].\end{aligned}$$

The term $\Phi(m,t)$ can be pulled out of the infimum, then

$$\begin{aligned}0 = \inf_v \Bigg[\int_t^{t+\epsilon} \int_{\mathbb{R}^n} f(x, m_s, v(x, m_s), v(\cdot, m)_{\#} m_s) m(x,s) dx ds + \epsilon\frac{\partial \Phi}{\partial t} \\ + \epsilon\int_{\mathbb{R}^n} \frac{\partial \Phi}{\partial m}\frac{\partial m}{\partial t} dx + o(\epsilon)\Bigg].\end{aligned}$$

After dividing by ϵ and let $\epsilon \to 0$

$$0 = \inf_v \left[\int_{\mathbb{R}^n} f(x, m, v(x, m), v(\cdot, m)_{\#} m) m(x) dx + \frac{\partial \Phi}{\partial t} + \int_{\mathbb{R}^n} \frac{\partial \Phi}{\partial m}\frac{\partial m}{\partial t} dx \right].$$

Using the Fokker-Plank equation (2.6), we obtain that

$$\begin{aligned}-\frac{\partial \Phi}{\partial t} + \int_{\mathbb{R}^n} A_x \frac{\partial}{\partial m}\Phi(m,t)(x) m(x,t) dx \\ = \inf_v \Big[\int_{\mathbb{R}^n} f(x, m, v(x, m), v(\cdot, m)_{\#} m) m(x) dx \\ - \int_{\mathbb{R}^n} \frac{\partial}{\partial m}\Phi(m,t)(x) \operatorname{div}\big(g(x, m, v(x,m), v(\cdot, m)_{\#} m) m(x)\big) dx\Big].\end{aligned}$$

This equation can be rewritten with the Lagrangian (2.9), then we get the Bellman equation (2.10).

Rules of Derivation

The first derivative rule is obtained by

$$\begin{aligned}
\frac{d}{d\theta}\Psi((v(\cdot,m)+\theta\tilde{v}(\cdot,m))_{\#}m)\Big|_{\theta=0} &= \lim_{\theta\to 0}\frac{\Psi((v(\cdot,m)+\theta\tilde{v}(\cdot,m))_{\#}m)-\Psi(v(\cdot,m))}{\theta} \\
&= \lim_{\theta\to 0}\frac{\Psi(\overbrace{(v(x_0,m)+\theta\tilde{v}(x_0,m))}^{\text{random variable}})-\Psi(v(x_0,m))}{\theta},\ x_0\sim m \\
&= \mathbb{E}\Big(D_x\Psi(v(x_0,m))\cdot\tilde{v}(x_0,m)\Big) \\
&= \int_{\mathbb{R}^n} D_x\Psi(v(x,m))\cdot\tilde{v}(x,m)\cdot m(dx) \\
&= \int_{\mathbb{R}^n} D_w\frac{\partial}{\partial\mu}\Psi(v(\cdot,m)_{\#}m)(v(x,m))\cdot\tilde{v}(x,m)\cdot m(x)dx.
\end{aligned} \tag{2.70}$$

We derive the second derivative rule based on the result of the first derivative rule (2.70) as follow:

$$\begin{aligned}
\frac{\partial}{\partial m}\Psi(v(\cdot,m)_{\#}m)(x) &= \underbrace{\lim_{\theta\to 0}\frac{\Psi(v(\cdot,m)_{\#}(m+\theta\tilde{m}))-\Psi(v(\cdot,m)_{\#}m)}{\theta}}_{\text{(i)}} \\
&\quad+\underbrace{\lim_{\theta\to 0}\frac{\Psi(v(\cdot,m+\theta\tilde{m})_{\#}m)-\Psi(v(\cdot,m)_{\#}m)}{\theta}}_{\text{(ii)}}.
\end{aligned}$$

$$\begin{aligned}
\text{(i)} &= \lim_{\theta\to 0}\frac{\Psi(v(\cdot,m)_{\#}(m+\theta\tilde{m})-v(\cdot,m)_{\#}m+v(\cdot,m)_{\#}m)-\Psi(v(\cdot,m)_{\#}m)}{\theta} \\
&= \lim_{\theta\to 0}\frac{1}{\theta}\left[\int_{\mathbb{R}^n}\frac{\partial\Psi}{\partial\mu}(v(\cdot,m)_{\#}m)(x)\Big(v(\cdot,m)_{\#}(m+\theta\tilde{m})-v(\cdot,m)_{\#}m\Big)(dx)\right] \\
&= \lim_{\theta\to 0}\frac{1}{\theta}\left[\int_{\mathbb{R}^n}\frac{\partial\Psi}{\partial\mu}(v(\cdot,m)_{\#}m)(v(x,m))d(m+\theta\tilde{m})-\int_{\mathbb{R}^n}\frac{\partial\Psi}{\partial\mu}(v(\cdot,m)_{\#}m)(v(x,m))dm\right] \\
&= \int_{\mathbb{R}^n}\frac{\partial\Psi}{\partial\mu}(v(\cdot,m)_{\#}m)(v(x,m))d\tilde{m} = \frac{\partial}{\partial\mu}\Psi(v(\cdot,m)_{\#}m)(v(x,m)).
\end{aligned}$$

$$\begin{aligned}
\text{(ii)} &= \lim_{\theta\to 0}\frac{\Psi(v(x_0,m+\theta\tilde{m}))-\Psi(v(x_0,m))}{\theta},\ x_0\sim m \\
&= \int_{\mathbb{R}^n} D_w\frac{\partial}{\partial\mu}\Psi(v(\cdot,m)_{\#}m)(v(\xi,m))\cdot\lim_{\theta\to 0}\frac{v(\xi,m+\theta\tilde{m})-v(\xi,m)}{\theta}m(d\xi) \\
&= \int_{\mathbb{R}^n} D_w\frac{\partial}{\partial\mu}\Psi(v(\cdot,m)_{\#}m)(v(\xi,m))\cdot\frac{\partial}{\partial m}v(\xi,m)(x)m(\xi)d\xi.
\end{aligned}$$

Combining the results (i) and (ii), we obtain the second derivative rule (2.13).

CHAPTER 3

INVENTORY MANAGEMENT IN OVER-THE-COUNTER MARKETS

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3.1 Introduction

An Over-The-Counter (OTC) market is a decentralized market that can be split into two levels: (i) a primary market and (ii) a secondary market. Whereas a primary market refers to a market, where an issuer creates a new asset, a secondary market is where investors trade the asset that they already possess. A huge number of OTC trades is negotiated in the secondary market that mainly consists of a competitive inter-dealer market and an OTC trade market between investors and dealers. An investor who is willing to sell or buy an asset traded in the OTC market makes an effort to search for dealers who take the role of market makers like intermediaries in bilateral trades. There are search and bargaining processes between two parties, but it is not necessary to publicly disclose the settled price. OTC tradings frequently occur with trade bonds, derivatives, mortgage-backed securities, and commodities. In addition, the volume of OTC tradings cannot be overlooked. For example, the average daily trending volume of U.S. corporate bonds publicly traded in 2018 is nearly 24.8 USD billion.

In a traditional OTC market, market participants such as dealers, banks, insurance companies, and funds managers mainly negotiated OTC trades through a simple electronic chatting system or telephone. Investors needed to individually contact one or multiple dealers to get quotes regarding the current available price and tradable amounts of assets. There were also voice brokers who may help a matching process between dealers by facilitating the exchange of information about dealers' transaction preferences. Considering a market environment, past theoretical works (see e.g. Duffie et al. 2005, Duffie et al. 2007, Weill 2007, Lagos et al. 2011) focused on an understanding of searching behaviors within market participants. After the late 1990s, Electronic Trading Platforms (ETPs) have been developed for inter-dealers trades about Treasury bonds in the United States. Dealer-investors OTC trades were also popularly negotiated via ETPs from the early 2000s. There are two types of trending platforms in OTC markets. Single-Dealer Platforms (SDPs) are trading systems to assist a bilateral asset trade between a single dealer and its customer. Multi-Dealer Platforms (MDPs) allow investors to obtain multiple quotes simultaneously from many dealers, so this system enables dealers' competition for asset trades.

The major shift from voice-based tradings to electronic-based tradings induces a variety of interactions between market participants (Bech et al., 2016). The majority of protocols in the inter-dealer market is a central limit order book (CLOB). The CLOB is a trading protocol that has a virtual queue, where the limit orders are stored with a specified trading rule. Under the CLOB setting, the inter-dealer market is relatively transparent because dealers may monitor the real-time bid-ask spreads and obtain information about historical settled price data and the volume of transactions. Thus, this market is somewhat similar to a central exchange. In this research, we assume that the inter-dealer market is competitive, and dealers are available to obtain a fair price of trading assets. By contrast, a dealer-investor market is mainly traded by the request for quote (RFQ) trading protocol, an asymmetric execution model where an investor requests quotes from dealers who respond to a bid or offer. The RFQ protocol not only allows the investor to obtain multiple quotes simultaneously from dealers but also dealer have advantages of markup pricing using information from the inter-dealer market.

The primary incentive of electronic trading has been the potential to reduce the cost of price discovery and improve market liquidity (Bech et al., 2016). This trading system reduces the need for human intervention, lowering operational costs and risk. Our mean field setting helps develop a decision support system for dealers willing to coordinate inter-dealer and dealer-investor markets simultaneously. Via an inter-dealer platform, the dealers tend to control the position of inventory in a competitive market with a large number of dealers. Also, dealers promptly respond to an investor's request for a price of tradable securities. The demand of investors depends on the collective behavior of dealers' pricing strategy, so an individual dealer needs to predict other dealers' price markup decisions. Under the mean field setting, the dealer derives the decision considering the distribution of other players' decisions rather than assuming that the dealer can have all information about market participants. The mean field approach can give a theoretical insight to capture dynamic inter-dependencies between dealers via ETPs.

3.2 Related Literature and Contributions

The theoretical framework of control problems in a dealership market has been widely studied (see e.g. Stoll 1978, Amihud and Mendelson 1980, Ho and Stoll 2017), but the theoretical models for the OTC market has recently suggested as the popularity of OTC trade increases. Whereas the seminal works of Duffie et al. (2005, 2007) only consider the role of match makers who cannot hold inventory to understand the concept of long term liquidity provision, Weill (2007) extends their theoretical models by introducing the market makers who can adjust inventory positions to provide market liquidity by absorbing selling pressure during financial disruptions. In addition, the work of Lagos et al. (2011) demonstrates the conditions of policy intervention by a regulator when market makers cannot properly perform the role of liquidity provider during financial crises.

Previous theoretical studies of Weill (2007) and Lagos et al. (2011) focus on the role of market makers during the financial crisis as liquidity providers rather than describing the underlying mechanism of OTC trades to explain why market makers hold inventory even in the stable market environment. Past works (see e.g. Duffie et al. 2005, Duffie et al. 2007, Weill 2007, Lagos et al. 2011) assume that dealers act as matchmakers who never hold inventory because they buy and resell assets immediately under a normal market condition. However, a recent empirical work of Randall (2015) show that dealers naturally have a finite amount of inventory regardless of market status. This is because previous works have overlooked the rationale that dealers can be considered as investors who exploit the asymmetric information advantage between dealers and customers in the opaque OTC market. The solution obtained by the mean field approach represents market makers' optimal price markup strategy to explain why dealers are willing to adjust the position of inventory in stable market conditions.

Eventually, our model can explicitly obtain the feedback control policies to increase the efficiency of market makers' liquidity provision and to reduce the desirability of policy intervention in OTC markets. In our model, the market demand is modulated by the mean field term being the law of the dealer's price markup control. This term can be regarded as the stringent restrictions on the dealer's behavior to avoid over price markup or under price markup. In other words, the dealer has historical performance over time, she or he needs to adhere to the endogenous demand rule described by her feedback control. For example, if the market price is higher than the average, then she or he is willing to set less price markup, and vice versa. Therefore, both the movement of the market demand and the spot price would be less volatile, so these are the main feature of having the feedback control that can guarantee a stable investment environment. Consequently, the stable movement of market price encourages greater market participation by latent investors as the market transparency increases in the opaque OTC market.

Our theoretical model can contribute to developing a more affordable ETP for dealers who are willing to simultaneously integrate both the inter-dealer and dealer-investor trades. In particular, the closed form solution obtained by the extended version of mean field type control may reduce a computational load to derive the optimal markup decision and inventory control in electronic OTC tradings. For dealers, ETPs with optimal markup decisions obtained by the analytical approach can contribute to ensuring economic incentives as well as reducing the average costs of asset trades. Understanding the strategic behavior of dealers' inventory control may increase the efficiency of dealers' liquidity provision and reduce the desirability of policy intervention. Consequently, this research can contribute to lowering the entry barriers for dealers by introducing a more favorable trading system. We are expected that our theoretical model can be a driver of electronic trading so that ETPs lessen a liquidity provision in OTC markets by giving dealers economic incentives to readily respond to inventors' quotes.

To the best of our knowledge, our study is the first mean field type approach for the understanding of a secondary OTC market, where the competitive interaction among a large number of players and the corresponding inventory management are implemented. We hope that our theoretical model can later be investigated further by using real transaction data between dealers and customers in the secondary OTC market. For instance, this further study would illustrate an empirical relationship between the price markup decision and the change of market demand in the OTC market. Our model can be extendable to understand the relationship between market transparency and prevailing markets in a secondary search OTC market (Duffie et al. 2017) and the inventory management between a core and peripheral dealers in an inter-dealer network (Colliard et al. 2018).

3.3 A Theoretical Model for Mean Field Type Control

3.3.1 Theoretical Motivation

The main contribution of this article is to explain the role of a market maker and her inventory control considering a prevailing market price and investors' demand in the OTC market. We consider a secondary OTC market with a continuum of market makers (e.g. dealers). The market maker trades a single risky asset in a finite time horizon, T. We assume that the market maker is risk-averse to hold a risky asset. This means that there is a penalty to hold the assets for market makers. Therefore, the market maker needs to derive the optimal level of inventory accumulation by considering the trends of the market price. There is an incentive for the market maker to reap a profit when she buys assets at a relatively low price and then she resells them at a relatively high price even considering the holding penalty.

The secondary OTC market consists of a competitive inter-dealer market and a dealer-investor trade market in which dealers and investors bargain over the price. We assume that the inter-dealer market is a perfect competition market, where the asset is traded at a prevailing market price. This is because the majority of trades in the inter-dealer market are executed under the protocol of CLOB. The prevailing market price is close to the concept of a true value of the asset. In the dealer-investor market, we assume that a dealer has an information advantage, so the bargaining price is set based on the prevailing market price and the dealer's price markup. This reflects the rationale that the dealer is relatively accessible to the true value of an asset compared to the investor. In addition, the assumption is supported by the fact that a dealer-investor market is mostly traded by the RFQ protocol.

In our trade model, we consider two types of trades: (i) a paired trade and (ii) an unpaired trade. The paired trade is a dealer-investor trade, where the dealer immediately unwinds the imbalance of selling demand and buying demand into the inter-dealer market. The role of a market maker becomes a broker rather than a dealer in the paired trade. Therefore, the broker is not willing to stock up the inventory by implementing the inter-dealer trades with the exact different volume of the selling demand and buying demand. In the unpaired trade, on the contrary, the dealer strategically manages the level of inventory. This means that the inter-dealer trading volume is not identical to the imbalance of selling demand and buying demand. It is crucial for the dealer to manage the level of inventory in the unpaired trade.

We introduce a new control problem for dealers' optimal markup and inventory control regarding OTC market trades based on a mean field type approach. The investor's aggregate demand on OTC trades is modeled by a mean field term, a medium to describe interactions among a continuum of players. Using the mean field type approach, we can explicitly obtain the market maker's optimal feedback regarding price markup and inventory management. Under the legitimate of mean field type control, this feedback can be considered as the policy suggestion of a central planner like a market regulator in the OTC market.

Since a seminal work by Aumann (1964) introduced static games with a continuum of players, there is a huge number of works regarding related topics on non-atomic games (e.g. Aumann and Shapley 1964, Mas-Colell 1984, Schmeidler 1973). In the viewpoint of dynamic games, Lasry and Lions (2007) introduced the mean field approach based on the concept of differential games from the mean field theory in the field of physics. Unlike a zero-intelligence model in a particle interaction system, a mean-field model in a social interaction system considers a large number of players who are willing to optimize their objective. The mean field model consists of two approaches, namely, mean field games and mean field type control. The key difference between the two approaches is that the former is close to the concept of a non-cooperative game with a continuum of players to find a mean field Nash equilibrium. However, the mean field type control is to simultaneously assign a decision for all players at once so that the average payoff of players is optimized. Therefore, the optimal feedback in mean field type control models can be interpreted as the policy suggestion of a central planner. The details in a theoretical comparison of mean field games and mean field type control can be found in the work of

Huang et al. (2006) and Bensoussan et al. (2013).

We believe that the mean field approach is a promising methodology to understand market makers' behavior in OTC markets, where there is a competitive competition among a continuum of players. The mean field system consists of a Hamilton-Jacobi-Bellman (HJB) equation for a value function and a Fokker-Planck (FP) equation for a density of mean field. The evolution of an individual player's value function is written by the HJB equation and the decision of each player is coupled to the density of the rest of players governed by the FP equation. In mean field approach, this FP equation depends on not only the feedback but also the solution of the HJB equation. This system allows us to analyze an optimization problem with a large number of interacting players. Intuitively, the individual player makes a decision by considering the distribution of the other players rather than assuming that all players' detailed information on states is collectible. This reflects the fact that it is unrealistic to fully gather market information about a huge number of players. Therefore, the mean field setting enables us to more realistically interpret the role of market maker in the OTC market.

3.3.2 General Formulation

A mean field type control model considers a Mckean-Vlasov type process, where both the state dynamics and objective functional depend on the overall probability distribution of the state (Bensoussan et al. 2013, Carmona and Delarue 2017). Alasseur et al. (2020) extended the concept of mean field type control model by incorporating the probability distribution of the control into the evolution of states and the payoff using stochastic maximum principle. Pham and Wei (2018) studied this theoretical model at the level of Bellman equation, then Bensoussan et al. (2021) suggested the Master equation with a corresponding system consist of HJB-FP equations. In our study, we use the theoretical result of Bensoussan et al. (2021)'s extended mean field type control model to understand the dealers' inventory management in a secondary OTC market. The main difference between Bensoussan et al. (2021)'s model and our model is that the former study is close to a dynamic Cournot competition, but our model is related to a dynamic Bertrand competition with a continuum of players.

In this section, we briefly introduce the theoretical model for an extended mean field type approach. The details on the notation and the derivation of this model can be found in Bensoussan et al. (2021). Consider a probability space $(\Omega, A, \mathbb{P})$ on which Wiener processes are defined. We set $\mathcal{F}^t = \sigma(x_0, w(s); s \leq t)$. The control $v(x,t)$ at time t is a feedback. We consider function $f(x,m,v)$, $g(x,m,v)$, $h(x,m)$ and $\sigma(x)$ where $x \in \mathbb{R}^n$; m is a probability measure on $\mathbb{R}^n$. v is a control in $\mathbb{R}^d$. The function f and h are scalar, but g is a vector in $\mathbb{R}^n$ and $\sigma(x)$ is $n \times n$ matrix. We have a state $x(t) = x_t \in \mathbb{R}^n$. Its probability distribution is denoted by $\mathbb{P}_{x_t}$. The control belongs to $\mathbb{R}^d$ and is defined by a feedback $v(x_t, \mathbb{P}_{x_t})$. The probability distribution at the control is denoted by $\mathbb{P}_{v(x_t, \mathbb{P}_{x_t})}$. We assume that ξ has a probability distribution, obtained from a density $m_0(x)$. We define the state equation as

$$dx = g\big(x_t, \mathbb{P}_{x_t}, v(x_t, \mathbb{P}_{x_t}), \mathbb{P}_{v(x_t, \mathbb{P}_{x_t})}\big)dt + \sigma(x_t)dw(t),\ x(0) = \xi.$$

The drift term is a function $g(x, m, v, \mu)$ where the arguments x, v are in $\mathbb{R}^n$, $\mathbb{R}^d$ respectively. In addition, the argument m, μ are probability measures on $\mathbb{R}^n$, $\mathbb{R}^d$ respectively. Note that the probability $\mathbb{P}_{v(x_t, \mathbb{P}_{x_t})}$ is the image of $\mathbb{P}_{x_t}$. In the sequel, we use the notation $v(\cdot, m) * m$ for image measure of m by the map $x \to v(x, m)$. Therefore, we rewrite the state equation as

$$dx = g\big(x_t, m_t, v(x_t, m_t), v(\cdot, m_t) * m_t\big)dt + \sigma(x_t)dw(t),\ x(0) = \xi,$$

where $m_t = m(t) = \mathbb{P}_{x_t}$ represents the probability density of $x(t)$, given the $\sigma-$algebra $\mathcal{F}^t$. We want to maximize the objective functional

$$J\big(v(\cdot)\big) = \mathbb{E}\Big[\int_0^T f(x_t, m_t, v(x_t, m_t), v(\cdot, m_t) * m_t)dt + h(x_T, m_T)\Big].$$

We denote $A = A(t)$ as the second order differential operator

$$A\phi(x) = -\text{tr } a(x)D^2\phi(x),$$

where $a(x) = \frac{1}{2}\sigma(x)\sigma^*(x)$. We call A^* the operator

$$A^*\phi(x) = -\sum_{i,j=1}^{n} \frac{\partial^2}{\partial x_i \partial x_j}\big(a_{ij}(x)\phi(x)\big).$$

The probability measure m_t has a density $m(x,t)$, solution of Fokker-Planck (FP) equation

$$\frac{\partial m}{\partial t} + A^*m + \text{div}\Big(g\big(x, m, v(x,m), v(\cdot, m) * m\big)m(x,t)\Big) = 0, \;\; m(x,0) = m_0(x).$$

The objective functional is

$$J\big(v(\cdot)\big) = \int_0^T \int_{\mathbb{R}^n} f\big(x, m(t), v(x, m(t)), v(\cdot, m(t)) * m(t)\big)m(x,t)dxdt \\ + \int_{\mathbb{R}^n} h(x, m(T))m(x,T)dx.$$

Using the invariant embedding, we rewrite this system indexed by m, t

$$\frac{\partial m}{\partial s} + A^*m + \text{div}\Big(g\big(x, m, v(x,m), v(\cdot, m) * m\big)m(x,s)\Big) = 0, \;\; s > t, \;\; m(x,t) = m_0(x)$$

and

$$J_{m,t}\big(v(\cdot)\big) = \int_t^T \int_{\mathbb{R}^n} f\big(x, m(s), v(x, m(s)), v(\cdot, m(s)) * m(s)\big)m(x,s)dxds \\ + \int_{\mathbb{R}^n} h(x, m(T))m(x,T)dx.$$

We define the value function by

$$\Phi(m,t) = \sup_{v(\cdot)} J_{m,t}\big(v(\cdot)\big).$$

3.3.3 Rules of Derivation

To proceed, we introduce two important derivation rules. Details for the formal derivation rules can be found in Bensoussan et al. (2021). Consider a functional $\Psi(\mu)$ on probability measures on $\mathbb{R}^d$ and then $\Psi(v(\cdot,m)*m)$ is a functional of $v(x,m)$ and it is also a functional of m for Ψ having a functional derivative $\frac{\partial}{\partial\mu}\Psi(\mu)(w)$, with $w \in \mathbb{R}^d$, we then first claim that

$$\frac{d}{d\theta}\Psi((v(\cdot,m)+\theta\tilde{v}(\cdot,m))*m)\bigg|_{\theta=0} = \int_{\mathbb{R}^n} D_w \frac{\partial}{\partial\mu}\Psi(v(\cdot,m)*m)(v(x,m))\cdot\tilde{v}(x,m)\cdot m(x)dx. \tag{3.1}$$

For the second rule, we consider the map $m \mapsto \Psi(v(\cdot,m)*m)$ and establish its functional derivative as follows. We claim the formula

$$\begin{aligned}\frac{\partial}{\partial m}\Psi(v(\cdot,m)*m)(x) =& \frac{\partial}{\partial\mu}\Psi(v(\cdot,m)*m)(v(x,m))\\ &+\int_{\mathbb{R}^n} D_w\frac{\partial}{\partial\mu}\Psi(v(\cdot,m)*m)(v(\xi,m))\cdot\frac{\partial}{\partial m}v(\xi,m)(x)m(\xi)d\xi.\end{aligned} \tag{3.2}$$

Using two derivative rules (3.1) and (3.2), we derive the master equation and the corresponding HJB-FP equations, but we do not describe the detail derivation here. The details can be found in Bensoussan et al. (2021).

3.3.4 Bellman Equation

$\Phi(m,t)$ satisfies the Dynamic Programming equation

$$\begin{cases} -\dfrac{\partial\Phi}{\partial t}+\displaystyle\int_{\mathbb{R}^n} A_x\frac{\partial\Phi(m,t)}{\partial m}(x)m(x,t)dx \\ =\displaystyle\sup_{v(\cdot)}\Bigg(\int_{\mathbb{R}^n}\Big[f(x,m,v(x,m),v(\cdot,m)*m) \\ +D_x\frac{\partial\Phi(m,t)}{\partial m}(x)\cdot g(x,m,v(x,m),v(\cdot,m)*m)\Big]m(x)dx\Bigg), \\ \Phi(m,T)=\displaystyle\int_{\mathbb{R}^n} h(x,m)m(x)dx. \end{cases} \tag{3.3}$$

To obtain the master equation, we shall use the notation

$$U(x,m,t) = \frac{\partial \Phi(m,t)}{\partial m}(x)$$

and from the Bellman equation (3.3) we obtain

$$\begin{cases} -\dfrac{\partial \Phi}{\partial t} + \displaystyle\int_{\mathbb{R}^n} A_x \frac{\partial \Phi(m,t)}{\partial m}(x) m(x,t) dx \\ = \sup\limits_{v(\cdot)} \Bigg(\displaystyle\int_{\mathbb{R}^n} \Big[f(x,m,v(x,m),v(\cdot,m)*m) \\ +D_x U(x,m,t) \cdot g(x,m,v(x,m),v(\cdot,m)*m) \Big] m(x) dx \Bigg), \\ \Phi(m,T) = \displaystyle\int_{\mathbb{R}^n} h(x,m) m(x) dx. \end{cases} \tag{3.4}$$

and $\hat{v}(x,m) = \hat{v}(x,m,t)$ maximize the functional

$$\int_{\mathbb{R}^n} \Big[f(x,m,v(x,m),v(\cdot,m)*m) + D_x U(x,m,t).g(x,m,v(x,m),v(\cdot,m)*m) \Big] m(x) dx. \tag{3.5}$$

We write the Euler equation of optimality by following derivation rules suggested by Bensoussan et al., (15) to maximize the functional (3.5).

$$\begin{aligned} &\frac{\partial f}{\partial v}(x,m,\hat{v}(x,m),\hat{v}(\cdot,m)*m) + D_x U(x,m,t) \cdot \frac{\partial g}{\partial v}(x,m,\hat{v}(x,m),\hat{v}(\cdot,m)*m) \\ &\quad + \int_{\mathbb{R}^n} D_w \frac{\partial f}{\partial \mu}(\xi,m,\hat{v}(\xi,m),\hat{v}(\cdot,m)*m) \Big(\hat{v}(x,m) \Big) m(\xi) d\xi \\ &\quad + \int_{\mathbb{R}^n} D_\xi U(\xi,m,t) \cdot D_w \frac{\partial g}{\partial \mu}(\xi,m,\hat{v}(\xi,m),\hat{v}(\cdot,m)*m) \Big(\hat{v}(x,m) \Big) m(\xi) d\xi = 0. \end{aligned} \tag{3.6}$$

3.3.5 Master Equation and HJB-FP Equations

We then differentiate the Bellman equation with respect to m to obtain the master equation.

$$\begin{cases} -\dfrac{\partial U}{\partial t} + A_x U(x,m,t) + \displaystyle\int_{\mathbb{R}^n} A_\xi \frac{\partial}{\partial m} U(\xi,m,t)(x) m(\xi) d\xi \\ = f(x,m,\hat{v}(x,m),\hat{v}(\cdot,m)*m) + D_x U(x,m,t)\cdot g(x,m,\hat{v}(x,m),\hat{v}(\cdot,m)*m) \\ + \displaystyle\int_{\mathbb{R}^n} \Bigg[\frac{\partial}{\partial m} f(\xi,m,\hat{v}(\xi,m),\hat{v}(\cdot,m)*m)(x) \\ + D_\xi U(\xi,m,t)\cdot \frac{\partial}{\partial m} g(\xi,m,\hat{v}(\xi,m),\hat{v}(\cdot,m)*m)(x) \Bigg] m(\xi) d\xi \\ + \displaystyle\int_{\mathbb{R}^n} \Bigg[\frac{\partial}{\partial \mu} f(\xi,m,\hat{v}(\xi,m),\hat{v}(\cdot,m)*m)(\hat{v}(x,m)) \\ + D_\xi U(\xi,m,t)\cdot \frac{\partial}{\partial \mu} g(\xi,m,\hat{v}(\xi,m),\hat{v}(\cdot,m)*m)(\hat{v}(x,m)) \Bigg] m(\xi) d\xi \\ + \displaystyle\int_{\mathbb{R}^n} D_\xi \frac{\partial}{\partial m} U(\xi,m,t)(x)\cdot g(\xi,m,\hat{v}(\xi,m),\hat{v}(\cdot,m)*m) m(\xi) d\xi, \\ U(x,m,T) = h(x,m) + \displaystyle\int_{\mathbb{R}^n} \frac{\partial h(\xi,m)}{\partial m}(x) m(\xi) d\xi. \end{cases} \tag{3.7}$$

Consider the probability density process corresponding to the optimal feedback to derive a system of coupled HJB-FP (Hamilton-Jacobi-Bellman-Fokker-Planck) equations. We set $\hat{v}(x,t) = \hat{v}(x,m(t),t)$ and thus the probability density $m(x,t)$ is the solution of

$$\begin{cases} \dfrac{\partial m}{\partial t} + A^* m + \operatorname{div}\Big(g\big(x,m,v(x,m),v(\cdot,m)*m\big) m(x,t) \Big) = 0, \\ m(x,0) = m_0(x). \end{cases} \tag{3.8}$$

Set $u(x,t) = U(x,m,t)$. We can state that the function $\hat{v}(x,t)$ satisfies the Euler condition

$$\begin{aligned} &\frac{\partial f}{\partial v}\Big(x,m(t),\hat{v}(x,t),\hat{v}(\cdot,t)*m(t)\Big) + D_x u(x,t)\cdot \frac{\partial g}{\partial v}\Big(x,m(t),\hat{v}(x,t),\hat{v}(\cdot,t)*m(t)\Big) \\ &\quad + \int_{\mathbb{R}^n} D_w \frac{\partial f}{\partial \mu}\Big(\xi,m(t),\hat{v}(\xi,t),\hat{v}(\cdot,t)*m(t)\Big)\Big(\hat{v}(x,t)\Big) m(\xi,t) d\xi \\ &\quad + \int_{\mathbb{R}^n} D_\xi u(\xi,t)\cdot D_w \frac{\partial g}{\partial \mu}\Big(\xi,m(t),\hat{v}(\xi,t),\hat{v}(\cdot,t)*m(t)\Big)\Big(\hat{v}(x,t)\Big) m(\xi,t) d\xi = 0. \end{aligned} \tag{3.9}$$

Finally, we obtain the HJB equation for $u(x,t)$

$$\begin{cases} -\dfrac{\partial u}{\partial t} + A_x u = f(x, m(t), \hat{v}(x,t), \hat{v}(\cdot, m) * m(t)) \\ +D_x u(x,t) \cdot g(x, m(t), \hat{v}(x,t), \hat{v}(\cdot,t) * m(t)) \\ + \displaystyle\int_{\mathbb{R}^n} \left[\frac{\partial f}{\partial m}(\xi, m(t), \hat{v}(\xi,t), \hat{v}(\cdot,t) * m(t))(x) \right. \\ \left. +D_\xi u(\xi,t) \cdot \frac{\partial g}{\partial m}(\xi, m(t), \hat{v}(\xi,t), \hat{v}(\cdot,t) * m(t))(x) \right] m(\xi,t) d\xi \\ + \displaystyle\int_{\mathbb{R}^n} \left[\frac{\partial f}{\partial \mu}(\xi, m(t), \hat{v}(\xi,t), \hat{v}(\cdot,t) * m(t))(\hat{v}(x,t)) \right. \\ \left. +D_\xi u(\xi,t) \cdot \frac{\partial g}{\partial \mu}(\xi, m(t), \hat{v}(\xi,t), \hat{v}(\cdot,t) * m(t))(\hat{v}(x,t)) \right] m(\xi,t) d\xi, \\ u(x,T) = h(x, m(T)) + \displaystyle\int_{\mathbb{R}^n} \frac{\partial h(\xi, m(T))}{\partial m}(x) m(\xi,T) d\xi. \end{cases} \tag{3.10}$$

3.4 Model

3.4.1 Setting of the Problem

We propose a new dynamic trading model of an OTC market by introducing the concept of mean field approach. Under the legitimate of mean field type control, we assume that there is a representative agent to represent a continuum of dealers. The representative agent has three state variables $S(t)$, $Y(t)$, and $Z(t)$. $S(t)$ is the cash position of the dealer at any time t. The evolution of $S(t)$ depends on the demand of investors with the settled price and cash earning with the constant interest rate r. $Y(t)$ is the rate of the prevailing market price and this is modeled by a continuous-time stochastic differential equation. The evolution of $Y(t)$ is random, described by diffusion but is not controlled. Lastly, $Z(t)$ represents the rate of change of a dealer's inventory. The evolution of $Z(t)$ is modeled by the rate of the difference of the inventor's selling demand and buying demand and inter-dealer trades.

The dealer-investor trades have two types of trading prices: (i) the price of selling trade and (ii) the price of buying trade. Each price is settled by the sum of the prevailing market price and markup, a dealer's decision. At each time t, the dealer derives the markup for selling trade, $v_S(Y, S, Z, t)$ and the markup for buying trade, $v_B(Y, S, Z, t)$, where the letters S and B designate the selling trade and the buying trade, respectively. We note that $Y(t) + v_S(Y(t), S(t), Z(t))$ is the rate of unit price in selling trades considering the prevailing market price and the markup. Its expected value is denoted by $\mathbb{E}\Big(Y(t) + v_S(Y(t), S(t), Z(t)))\Big)$. Also, $Y(t) - v_B(Y(t), S(t), Z(t))$ is the rate of unit price in buying trades based on the prevailing market price and the markup. $\mathbb{E}\Big(Y(t) - v_B(Y(t), S(t), Z(t))\Big)$ denotes the expected value of price in buying trades. The rate of demand is denoted by $\mathbf{D}_S\Big(\mathbb{E}\big(- Y(t) - v_S(Y(t), S(t), Z(t))\big)$ and $\mathbf{D}_B\Big(\mathbb{E}\big(Y(t) - v_B(Y(t), S(t), Z(t))\big)$. We assume that the demand function in selling and buying trades are monotone increasing functions. The dealer consumes cash at the rate $v_C(Y(t), S(t), Z(t))$. $v_I(Y(t), S(t), Z(t))$ denotes the rate of inter-dealer trades settled at the prevailing market price $Y(t)$. The sign of $v_I(Y(t), S(t), Z(t))$ represents the direction of trades. The evolution of states are defined by

$$
\begin{aligned}
dY =& \alpha(Y)dt + \beta(Y)dw(t),\ Y(0) = Y_0, \\
\frac{dS}{dt} =& \underbrace{rS}_{\text{cash interest}} + \underbrace{\mathbf{D}_S\Big(- \mathbb{E}\big(Y(t) + v_S(Y(t), S(t), Z(t))\big)\Big)\Big(Y(t) + v_S(Y(t), S(t), Z(t))\Big)}_{\text{dealer-customer selling trades}} \\
& - \underbrace{\mathbf{D}_B\Big(+ \mathbb{E}\big(Y(t) - v_B(Y(t), S(t), Z(t))\big)\Big)\Big(Y(t) - v_B(Y(t), S(t), Z(t))\Big)}_{\text{dealer-customer buying trades}} \\
& - \underbrace{v_I(Y(t), S(t), Z(t))Y(t)}_{\text{inter-dealer trades}} - \underbrace{v_C(Y(t), S(t), Z(t))}_{\text{consumption}},\ S(0) = S_0, \\
\frac{dZ}{dt} =& - \underbrace{\mathbf{D}_S\Big(- \mathbb{E}\big(Y(t) + v_S(Y(t), S(t), Z(t))\big)\Big)}_{\text{dealer-customer selling demand}} + \underbrace{\mathbf{D}_B\Big(+ \mathbb{E}\big(Y(t) - v_B(Y(t), S(t), Z(t))\big)\Big)}_{\text{dealer-customer buying demand}} \\
& + \underbrace{v_I(Y(t), S(t), Z(t))}_{\text{inter-dealer trade volume}},\ Z(0) = Z_0,
\end{aligned} \tag{3.11}
$$

where Y_0, S_0, and Z_0 are random variables. The dealer's objective is to maximize the present value of the utility of consumption stream and to minimize the present value of the utility of efforts on information advantage and the present value of the utility of holding inventory. In addition, the dealer's objective is to minimize the penalty utility of short cash and short inventory by considering the utility of transaction efforts on inter-dealer trades.

The dealer takes information advantage and charges price markups in an opaque OTC market. $U_I(\cdot)$ is a utility function for efforts on information advantage. Based on this information advantage, dealers make a price markup decision but the penalty of the utility of price markup guarantee that the dealer cannot make an excessive price markup for dealer-customer trades. $U_Z(\cdot)$ is a utility function for holding inventory, similar to the concept of carrying (holding) costs in the traditional inventory management problem.

$U_C(\cdot)$ is a utility function for the dealer's consumption. This is the dealer's incentive for providing a secondary OTC market with liquidity. $U_{SC}(\cdot)$ and $U_{SI}(\cdot)$ denote the penalty utility of dealers of short cash and short inventory, respectively. Because of these utilities, dealers are willing to maintain the positive value of cash position and inventory position. Whereas dealers stock up inventory from the inter-dealer trades to cope with short inventory, dealers unwind accumulated inventory to the inter-dealer market to secure a sufficient cash position. $U_{IT}(\cdot)$ represents the utility flow of transaction efforts on inter-dealer trades. In our model, we assume that an investor needs to pay the transaction costs of dealer-customer trades instead of a dealer. Lastly, $U_H(\cdot)$ is the utility of salvage value depending on the inventory position at the terminal time T. In addition, we set $v(Y,S,Z) = \Big(v_S(Y,S,Z), v_B(Y,S,Z), v_I(Y,S,Z), v_C(Y,S,Z)\Big)^\intercal$. The objective functional is

$$
\begin{aligned}
J(v(\cdot)) =& \mathbb{E}\int_0^T e^{-rt}\Bigg[\underbrace{-U_I\Big(v_S(Y(t),S(t),Z(t))\Big) - U_I\Big(v_B(Y(t),S(t),Z(t))\Big)}_{\text{the utility flow of efforts on information advantage}} \\
&- \underbrace{U_Z\Big(Z(t)^+\Big)}_{\text{the utility flow of holding inventory}} \\
&- \underbrace{U_{IT}\Big(v_I(Y(t),S(t),Z(t))\Big)}_{\text{the utility flow of transaction efforts on inter-dealer trades}} \\
&- \underbrace{U_{SC}\Big(S(t)^-\Big)}_{\text{the utility flow of short cash}} - \underbrace{U_{SI}\Big(Z(t)^-\Big)}_{\text{the utility flow of short inventory}} \\
&+ \underbrace{U_C\Big(v_C(Y(t),S(t),Z(t))\Big)}_{\text{the utility flow of consumption}}\Bigg]dt + \underbrace{\mathbb{E}U_{H_0}(S,T)e^{-rT}}_{\text{the utility of terminal cash position}} \\
&+ \underbrace{\mathbb{E}U_{H_1}(Z,T)e^{-rT}}_{\text{the utility of salvage value}}
\end{aligned}
\tag{3.12}
$$

where r is the discount rate the same as an interest rate. In the OTC market, there is information asymmetry between a dealer and an investor, resulting in a dominance of bargaining power. Dealers would reap a benefit from the asymmetry of bargaining power by setting a price markup, but this also generates utility efforts that incorporate into the objective functional.

The comprehensive model (3.11) and (3.12) capture simultaneously the paired and unpaired trades. In the following sections, we study a simple model at first, then we attempt to build up the complicated model like an induction process.

Table 3.1. The summary of studies regarding the OTC trades model

	price markup	consumption	inventory management
Paired trades model (Sec. 3.4.2)	✓	✓	
Unpaired trades model (Sec. 3.4.3)	✓		✓
Comprehensive model (Sec. 3.4.4)	✓	✓	✓

3.4.2 A Paired Trade Model

At first, we study for a paired trade which is a dealer-customer trade immediately unwound in the inter-dealer market. The main feature of the paired trade is to reap a profit from the price markup, so the dealer's role becomes the broker's role. In this model, we do not consider the level of inventory $Z(t)$. The primary objective of the broker is to maximize the terminal cash position and the utility of consumption by minimizing the penalty utility of efforts on information advantage and short on the cash position. Then, the evolution of states are defined by

$$
\begin{aligned}
dY =& \alpha(Y)dt + \beta(Y)dw(t), \; Y(0) = Y_0, \\
\frac{dS}{dt} =& \underbrace{rS}_{\text{the cash interest flow}} + \underbrace{\mathbf{D}_S\Big(\mathbb{E}\big(-Y(t) - v_S(Y(t),S(t))\big)\Big)v_S(Y(t),S(t))}_{\text{the profit flow of trades}} \\
& + \underbrace{\mathbf{D}_B\Big(\mathbb{E}\big(Y(t) - v_B(Y(t),S(t))\big)\Big)v_B(Y(t),S(t))}_{\text{the profit flow of trades}} - \underbrace{v_C(Y(t),S(t))}_{\text{the consumption flow}}, \; S(0) = 0,
\end{aligned}
\tag{3.13}
$$

and the objective functional of a broker is

$$
\begin{aligned}
J(v(\cdot)) =& \mathbb{E}\int_0^T e^{-rt}\Bigg[- \underbrace{U_I\Big(v_S(Y(t),S(t))\Big) - U_I\Big(v_B(Y(t),S(t))\Big)}_{\text{the utility flow of efforts on information advantage}} \\
& + \underbrace{U_C\Big(v_C(Y(t),S(t))\Big)}_{\text{the utility flow of consumption}} \\
& - \underbrace{U_{SC}\Big(S(t)^-\Big)}_{\text{the utility flow of short on cash}} \Bigg]dt + \underbrace{\mathbb{E}U_{H_0}(S,T)e^{-rT}}_{\text{the utility of terminal cash position}}
\end{aligned}
\tag{3.14}
$$

In the model (3.13) and (3.14) we have $v(x,m) = \Big(v_S(x,m), v_B(x,m), v_C(x,m)\Big)$ and $x = (Y(t), S(t))^\intercal$. We denote the component of state variables: $Y(t) = x_1$, $S(t) = x_2$ and the component of controls: $v_S(x,m) = v_1(x,m)$, $v_B(x,m) = v_2(x,m)$, and $v_C(x,m) = v_3(x,m)$.

Proposition 3.1. *Once we assume that $\boldsymbol{D}_S(\cdot)$, $\boldsymbol{D}_B(\cdot)$, $U_{SC}(\cdot)$, and $U_H(\cdot)$ are linear and $U_I(\cdot)$ and $U_C(\cdot)$ are quadratic, the optimal feedback $v(x,m)$ is completely explicit.*

Proof. The proof of Proposition 3.1 can be found in the appendix.

Remark 3.1. *When we assume that* $\boldsymbol{D}_S(\cdot)$*,* $\boldsymbol{D}_B(\cdot)$*,* $U_{SC}(\cdot)$*, and* $U_{H_0}(\cdot)$ *are linear and* $U_I(\cdot)$ *and* $U_C(\cdot)$ *are quadratic,* $\lambda_{x_2}(x,t) = \overline{\lambda}_{x_2}(t)$*.*

Proof. The proof of Remark 3.1 can be found in the appendix

3.4.3 An unpaired trade model

An unpaired trade is a dealer-customer trade, where the dealer strategically accumulates the assets rather than immediately unwinding them in the inter-dealer market. The dealer's objective is to maximize the revenue of dealer-customer trades and inter-dealer trades by optimizing the level of inventory. In this unpaired trade model, a dealer is willing to focus on the inventory control $v_I(\cdot,t)$ rather than managing the cash position $S(t)$ and the consumption rate $v_C(\cdot,t)$. Then, the evolution of states are defined by

$$
\begin{aligned}
dY =& \alpha(Y)dt + \beta(Y)dw(t),\ Y(0) = Y_0,\\
\frac{dZ}{dt} =& \underbrace{-\mathbf{D}_S\Big(\mathbb{E}\big(-Y(t) - v_S(Y(t),Z(t))\big)\Big) + \mathbf{D}_B\Big(\mathbb{E}\big(Y(t) - v_B(Y(t),Z(t))\big)\Big)}_{\text{the change rate of inventory position from dealer-customer trades}}\\
&+ \underbrace{v_I(Y(t),Z(t))}_{\text{the change rate of inventory position from inter-dealer trades}},\ Z(0) = 0,
\end{aligned}
\tag{3.15}
$$

and the objective functional of a dealer is

$$
\begin{aligned}
J(v(\cdot)) = \mathbb{E}\int_0^T \Bigg[& \underbrace{\mathbf{D}_S\Big(-\mathbb{E}\big(Y(t)+v_S(Y(t),Z(t))\big)\Big)\Big(Y(t)+v_S(Y(t),Z(t))\Big)}_{\text{the profit flow of dealer-customer trades}} \\
& \underbrace{-\mathbf{D}_B\Big(+\mathbb{E}\big(Y(t)-v_B(Y(t),Z(t))\big)\Big)\Big(Y(t)-v_B(Y(t),Z(t))\Big)}_{\text{the profit flow of dealer-customer trades}} \\
& - \underbrace{v_I(Y(t),Z(t))Y(t)}_{\text{the profit flow of inter-dealer trades}} \\
& \underbrace{-U_I\Big(v_S(Y(t),Z(t))\Big) - U_I\Big(v_B(Y(t),Z(t))\Big)}_{\text{the cost flow of efforts on information advantage}} \\
& - \underbrace{U_{IT}\Big(v_I(Y(t),Z(t))\Big)}_{\text{the cost flow of transaction efforts on inter-dealer trades}} \\
& - \underbrace{U_{SI}\Big(Z(t)^-\Big)}_{\text{the cost flow of short inventory}} - \underbrace{U_Z\Big(Z(t)^+\Big)}_{\text{the cost flow of holding inventory}} \Bigg] dt \\
& + \underbrace{\mathbb{E}U_{H_1}(Z,T)}_{\text{the profit flow of salvage value}}
\end{aligned}
\tag{3.16}
$$

In the model (3.15) and (3.16) we have $v(x,m) = \Big(v_S(x,m), v_B(x,m), v_I(x,m)\Big)$ and $x = (Y(t), Z(t))^\intercal$. We denote the component of state variables: $Y(t) = x_1$, $Z(t) = x_2$ and the component of controls: $v_S(x,m) = v_1(x,m)$, $v_B(x,m) = v_2(x,m)$, and $v_I(x,m) = v_3(x,m)$.

Proposition 3.2. *Once we assume that $\boldsymbol{D}_S(\cdot)$, $\boldsymbol{D}_B(\cdot)$, $U_{SI}(\cdot)$, $U_Z(\cdot)$ and $U_{H_1}(\cdot)$ are linear and $U_I(\cdot)$ and $U_{IT}(\cdot)$ are quadratic, the optimal feedback $v(x,m)$ is completely explicit.*

Proof. The proof of Proposition 3.2 can be found in the appendix.

Remark 3.2. *When we assume that $\boldsymbol{D}_S(\cdot)$, $\boldsymbol{D}_B(\cdot)$, $U_{SI}(\cdot)$, $U_Z(\cdot)$ and $U_{H_1}(\cdot)$ are linear and $U_I(\cdot)$ and $U_{IT}(\cdot)$ are quadratic, $\lambda_{x_2}(x,t) = \overline{\lambda}_{x_2}(t)$.*

Proof. The proof of Remark 3.2 can be found in the appendix.

3.4.4 The Comprehensive Model

In this section, we attempt to solve the proposed model (3.11) and (3.12). This model reflects the fact that there are paired trades and unpaired trades simultaneously in the OTC market. Therefore, a market maker is willing to maximize the utility of inter-dealer trades and the utility of dealer-customer trades by optimizing the cash position and inventory position and by maximizing the utility of consumption. In the model (3.11) and (3.12), we have $x = (Y(t), S(t), Z(t))^{\intercal}$ and $v(x,m) = \Big(v_S(x,m), v_B(x,m), v_I(x,m), v_C(x,m)\Big)$. We denote the component of state variables: $Y(t) = x_1$, $S(t) = x_2$, and $Z(t) = x_3$ and the component of controls: $v_S(x,m) = v_1(x,m)$, $v_B(x,m) = v_2(x,m)$, $v_I(x,m) = v_3(x,m)$, and $v_C(x,m) = v_4(x,m)$.

Proposition 3.3. *Once we assume that $\boldsymbol{D}_S(\cdot)$, $\boldsymbol{D}_B(\cdot)$, $U_{SC}(\cdot)$, $U_{SI}(\cdot)$, $U_Z(\cdot)$, $U_{H_0}(\cdot)$, and $U_{H_1}(\cdot)$ are linear and $U_I(\cdot)$, $U_{IT}(\cdot)$, and $U_C(\cdot)$ are quadratic, the optimal feedback $v(x,m)$ is completely explicit.*

Proof. The proof of Proposition 3.3 can be found in the appendix.

3.5 Concluding Remarks and Further Studies

We introduce a new control problem for dealers' optimal markup and inventory control regarding OTC market trades based on a mean field type approach. An OTC market is a decentralized market in which dealers and investors bilaterally trade securities not listed in exchanges. Previous theoretical studies focus on the role of market makers during the financial crisis as liquidity providers rather than describing the underlying mechanism of OTC trades to explain why market makers hold inventory even in a stable market environment. This is because previous works have overlooked the rationale that dealers can be considered as investors who exploit the asymmetric information advantage between dealers and customers in the opaque OTC market. On the contrary, the solution obtained by the mean field approach represents market makers' optimal inventory control strategy to explain why dealers are willing to adjust the position of inventory in stable market conditions. Furthermore, our theoretical model also proposes a dealer's pricing strategy when investors are able to simultaneously access many dealer's quotes via MDPs. In a traditional voice trading platform, the investors are limited to obtain multiple quotes simultaneously because the opportunity to accept the quote lapses quickly. Recently, The development of electronic platforms makes it easier to obtain multiple quotes. Therefore, it is crucial to understand the dynamic inter-dependencies in supporting dealers and ETPs. The mean field approach enables to capture a tangible interaction among a huge number of dealers who have similar aspects of decision makings and objectives in OTC markets. I believe that the mean field approach is a promising modeling technique that supports the development of decision making processes for dealers willing to derive the optimal control of inventory and the price markup decision.

This research contributes to understanding how to overcome transparency and liquidity issues in OTC markets. Our theoretical model explores the dynamic inventory control problem using mean field approach to develop a decision support system. We obtained explicit solutions about the control of inventory and the price mark-up decision. These solutions can reduce the computational load for ETPs that helps to enhance price efficiency and market liquidity. In addition, our theoretical model may suggest more affordable electronic trading platforms for dealers who are willing to coordinate the inter-dealer and investor-dealer markets simultaneously. Considering the position of inventory and the market information, the dealer can respond to the request of quotes from an investor via multi dealer platform. It ensures an economic incentive for dealers to make a profit from mark-up decision and also reduce the average costs of asset trades. A more affordable trading environment leads to lowering the entry barriers of dealers. Consequently, the main outcome of our research may answer the question about how to reduce the policy intervention to manage liquidity issues. Our decision-making rules may facilitate dealers' responses to imbalances in demand and supply to ensure the market transparency in matching processes between especially dealers and investors.

Lastly, We hope that our theoretical model can later be investigated further by using real transaction data between dealers and customers in the secondary OTC market. For instance, this further study would illustrate an empirical relationship between the price markup decision and the change of market demand in the OTC market. Our model can be extendable to understand the relationship between market transparency and prevailing market prices in a secondary search OTC market and the inventory management between core and peripheral dealers in an inter-dealer network.

3.6 Appendix. Proof

3.6.1 Proof of Proposition 3.1

$$g(x, m, v, \mu)$$

$$= \left| \begin{array}{c} \alpha(x_1) \\ rx_2 + v_1 \mathbf{D}_S \left(-\int \xi_1 m(\xi) d\xi - \int \eta_1 d\mu(\eta) \right) + v_2 \mathbf{D}_B \left(\int \xi_1 m(\xi) d\xi - \int \eta_2 d\mu(\eta) \right) - v_3 \end{array} \right|,$$

$$\sigma(x) = \left| \begin{array}{c} \beta(x_1) \\ 0 \end{array} \right|,$$

then

$$A\varphi(x) = -\frac{1}{2}\beta^2(x_1)\frac{\partial^2 \varphi}{\partial x_1^2}.$$

Also,

$$f(x, m, v, \mu) = -U_I(v_1) - U_I(v_2) + U_C(v_3) - U_{SC}(x_2^-)$$

$$h(x,m) = U_{H_0}(x_2).$$

We obtain the derivatives

$$\frac{\partial f}{\partial v}(x,m,v,\mu) = \begin{vmatrix} -U_I'(v_1) & -U_I'(v_2) & U_C'(v_3) \end{vmatrix}$$

$$\frac{\partial f}{\partial m}\Big((x,m,v,\mu)\Big)(\xi) = \frac{\partial f}{\partial \mu}\Big((x,m,v,\mu)\Big)(\eta) = 0,$$

$$\frac{\partial g}{\partial v}(x,m,v,\mu)$$

$$= \begin{vmatrix} 0 & 0 & 0 \\ \mathbf{D}_S\left(-\int \xi_1 m(\xi)d\xi - \int \eta_1 d\mu(\eta)\right) & \mathbf{D}_B\left(\int \xi_1 m(\xi)d\xi - \int \eta_2 d\mu(\eta)\right) & -1 \end{vmatrix},$$

$$\frac{\partial g}{\partial m}\big((x,m,v,\mu)\big)(\xi)$$

$$= \begin{vmatrix} 0 \\ -v_1\xi_1\mathbf{D}_S'\left(-\int \xi_1 m(\xi)d\xi - \int \eta_1 d\mu(\eta)\right) + v_2\xi_1\mathbf{D}_B'\left(\int \xi_1 m(\xi)d\xi - \int \eta_2 d\mu(\eta)\right) \end{vmatrix},$$

$$\frac{\partial g}{\partial \mu}\big((x,m,v,\mu)\big)(\eta)$$

$$= \begin{vmatrix} 0 & 0 & 0 \\ -v_1\eta_1\mathbf{D}_S'\left(-\int \xi_1 m(\xi)d\xi - \int \eta_1 d\mu(\eta)\right) & -v_2\eta_2\mathbf{D}_B'\left(\int \xi_1 m(\xi)d\xi - \int \eta_2 d\mu(\eta)\right) & 0 \end{vmatrix},$$

We use the notation

$$\overline{x}_1(t) = \int \xi_1 m(\xi,t)d\xi;\ \overline{\hat{v}}_1(t) = \int \eta_1 d\mu(\eta,t);\ \overline{\hat{v}}_2(t) = \int \eta_2 d\mu(\eta,t).$$

$$\frac{\partial f}{\partial v}(x,m(t),\hat{v}(x,t),\hat{v}(\cdot,t)*m(t)) = \begin{vmatrix} -U_I'(\hat{v}_1(x,t)) & -U_I'(\hat{v}_2(x,t)) & U_C'(\hat{v}_3(x,t)) \end{vmatrix},$$

$$\frac{\partial f}{\partial m}\Big((x,m(t),\hat{v}(x,t),\hat{v}(\cdot,t)*m(t))\Big)(\xi) = \frac{\partial f}{\partial \mu}\Big((x,m(t),\hat{v}(x,t),\hat{v}(\cdot,t)*m(t))\Big)(\eta) = 0.$$

$$\frac{\partial g}{\partial v}(x,m(t),\hat{v}(x,t),\hat{v}(\cdot,t)*m(t)) = \begin{vmatrix} 0 & 0 & 0 \\ \mathbf{D}_S\left(-\overline{x}_1(t) - \overline{\hat{v}}_1(t)\right) & \mathbf{D}_B\left(\overline{x}_1(t) - \overline{\hat{v}}_2(t)\right) & -1 \end{vmatrix},$$

$$\frac{\partial g}{\partial m}\big((x,m(t),\hat{v}(x,t),\hat{v}(\cdot,t)*m(t))\big)(\xi)$$

$$= \begin{vmatrix} 0 \\ -\hat{v}_1(x,t)\xi_1\mathbf{D}_S'\left(-\overline{x}_1(t) - \overline{\hat{v}}_1(t)\right) + \hat{v}_2(x,t)\xi_1\mathbf{D}_B'\left(\overline{x}_1(t) - \overline{\hat{v}}_2(t)\right) \end{vmatrix},$$

$$\frac{\partial g}{\partial \mu}\big((x, m(t), \hat{v}(x,t), \hat{v}(\cdot,t) * m(t))\big)(\eta)$$

$$= \left| \begin{array}{ccc} 0 & 0 & 0 \\ -\hat{v}_1(x,t)\eta_1 \mathbf{D}'_S\left(-\overline{x}_1(t) - \overline{\hat{v}}_1(t)\right) & -\hat{v}_2(x,t)\eta_2 \mathbf{D}'_B\left(\overline{x}_1(t) - \overline{\hat{v}}_2(t)\right) & 0 \end{array} \right|,$$

$$\int D_w \frac{\partial f}{\partial \mu}\Big((\xi, m(t), \hat{v}(\xi,t), \hat{v}(\cdot,t) * m(t))\Big)\big(\hat{v}(x,t)\big) m(\xi,t) d\xi = 0$$

$$\int D_\xi u(\xi,t) \cdot D_w \frac{\partial g}{\partial \mu}\Big((\xi, m(t), \hat{v}(\xi,t), \hat{v}(\cdot,t) * m(t))\Big)\big(\hat{v}(x,t)\big) m(\xi,t) d\xi$$
$$= \left| \begin{array}{c} -\mathbf{D}'_S\left(-\overline{x}_1(t) - \overline{\hat{v}}_1(t)\right)\overline{\hat{v}}_1(t) \displaystyle\int \frac{\partial u}{\partial x_2}(\xi,t) m(\xi,t) d\xi \\ -\mathbf{D}'_B\left(\overline{x}_1(t) - \overline{\hat{v}}_2(t)\right)\overline{\hat{v}}_2(t) \displaystyle\int \frac{\partial u}{\partial x_2}(\xi,t) m(\xi,t) d\xi \\ 0 \end{array} \right|^{\intercal}.$$

The Euler condition (3.9) becomes

$$\begin{aligned} &-U'_I(\hat{v}_1(x,t)) + \frac{\partial u(x,t)}{\partial x_2}\mathbf{D}_S\left(-\overline{x}_1(t) - \overline{\hat{v}}_1(t)\right) - \mathbf{D}'_S\left(-\overline{x}_1(t) - \overline{\hat{v}}_1(t)\right)\overline{\hat{v}}_1(t) \int \frac{\partial u}{\partial x_2}(\xi,t) m(\xi,t) d\xi = 0, \\ &-U'_I(\hat{v}_2(x,t)) + \frac{\partial u(x,t)}{\partial x_2}\mathbf{D}_B\left(\overline{x}_1(t) - \overline{\hat{v}}_2(t)\right) - \mathbf{D}'_B\left(\overline{x}_1(t) - \overline{\hat{v}}_2(t)\right)\overline{\hat{v}}_2(t) \int \frac{\partial u}{\partial x_2}(\xi,t) m(\xi,t) d\xi = 0, \\ &U'_C(\hat{v}_3(x,t)) - \frac{\partial u(x,t)}{\partial x_2} = 0. \end{aligned} \tag{3.17}$$

Next,

$$f(x, m(t), \hat{v}(x,t), \hat{v}(\cdot,t) * m(t)) = -U_I(\hat{v}_1(x,t)) - U_I(\hat{v}_2(x,t)) + U_C(\hat{v}_3(x,t)) - U_{SC}(x_2^-)$$

$$\int \frac{\partial f}{\partial m}\Big((\xi, m(t), \hat{v}(\xi,t), \hat{v}(\cdot,t) * m(t))\Big)(x) m(\xi,t) d\xi = 0,$$

$$\int \frac{\partial f}{\partial \mu}\Big((\xi, m(t), \hat{v}(\xi,t), \hat{v}(\cdot,t) * m(t))\Big)(\hat{v}(x,t)) m(\xi,t) d\xi = 0,$$

$$\begin{aligned} &\int D_\xi u(\xi,t) \cdot \frac{\partial g}{\partial m}\Big((\xi, m(t), \hat{v}(\xi,t), \hat{v}(\cdot,t) * m(t))\Big)(x) m(\xi,t) d\xi \\ &\quad = -x_1 \mathbf{D}'_S\left(-\overline{x}_1(t) - \overline{\hat{v}}_1(t)\right)\overline{\hat{v}}_1(t) \int \frac{\partial u}{\partial x_2}(\xi,t) m(\xi,t) d\xi \\ &\quad + x_1 \mathbf{D}'_B\left(\overline{x}_1(t) - \overline{\hat{v}}_2(t)\right)\overline{\hat{v}}_2(t) \int \frac{\partial u}{\partial x_2}(\xi,t) m(\xi,t) d\xi \end{aligned}$$

$$
\begin{aligned}
&\int D_\xi u(\xi,t)\cdot\frac{\partial g}{\partial \mu}\Big((\xi,m(t),\hat v(\xi,t),\hat v(\cdot,t)*m(t))\Big)(\hat v(x,t))m(\xi,t)d\xi\\
&\quad=-\hat v_1(x,t)\mathbf{D}_S'\left(-\overline{x}_1(t)-\overline{\hat v}_1(t)\right)\overline{\hat v}_1(t)\int\frac{\partial u}{\partial x_2}(\xi,t)m(\xi,t)d\xi\\
&\quad-\hat v_2(x,t)\mathbf{D}_B'\left(\overline{x}_1(t)-\overline{\hat v}_2(t)\right)\overline{\hat v}_2(t)\int\frac{\partial u}{\partial x_2}(\xi,t)m(\xi,t)d\xi,
\end{aligned}
$$

The HJB equation (3.10) becomes

$$
\begin{cases}
-\dfrac{\partial u}{\partial t}-\dfrac{1}{2}\beta^2(x_1)\dfrac{\partial^2 u}{\partial x_1^2}-\dfrac{\partial u}{\partial x_1}\alpha(x_1)\\
-\dfrac{\partial u}{\partial x_2}\Big(rx_2+\hat v_1(x,t)\mathbf{D}_S\big(-\overline{x}_1(t)-\overline{\hat v}_1(t)\big)+\hat v_2(x,t)\mathbf{D}_B\big(\overline{x}_1(t)-\overline{\hat v}_2(t)\big)-\hat v_3(x,t)\Big)+ru\\
=-U_I(\hat v_1(x,t))-U_I(\hat v_2(x,t))+U_C(\hat v_3(x,t))-U_{SC}(x_2^-)\\
+(-x_1-\hat v_1(x,t))\mathbf{D}_S'\left(-\overline{x}_1(t)-\overline{\hat v}_1(t)\right)\overline{\hat v}_1(t)\displaystyle\int\frac{\partial u}{\partial x_2}(\xi,t)m(\xi,t)d\xi\\
+(x_1-\hat v_2(x,t))\mathbf{D}_B'\left(\overline{x}_1(t)-\overline{\hat v}_2(t)\right)\overline{\hat v}_2(t)\displaystyle\int\frac{\partial u}{\partial x_2}(\xi,t)m(\xi,t)d\xi\\
u(x,T)=U_{H_0}(x_2).
\end{cases}
$$

Using the Euler conditions (3.17), we rewrite the HJB equation:

$$
\begin{cases}
-\dfrac{\partial u}{\partial t}-\dfrac{1}{2}\beta^2(x_1)\dfrac{\partial^2 u}{\partial x_1^2}-\dfrac{\partial u}{\partial x_1}\alpha(x_1)\\
-\dfrac{\partial u}{\partial x_2}\Big(rx_2-x_1\mathbf{D}_S\big(-\overline{x}_1(t)-\overline{\hat v}_1(t)\big)+x_1\mathbf{D}_B\big(\overline{x}_1(t)-\overline{\hat v}_2(t)\big)\Big)+ru\\
=\hat v_1(x,t)U_I'(\hat v_1(x,t))-U_I(\hat v_1(x,t))+\hat v_2(x,t)U_I'(\hat v_2(x,t))-U_I(\hat v_2(x,t))\\
-\hat v_3(x,t)U_C'(\hat v_3(x,t))+U_C(\hat v_3(x,t))-U_{SC}(x_2^-)+x_1U_I'(\hat v_1(x,t))-x_1U_I'(\hat v_2(x,t))\\
u(x,T)=U_{H_0}(x_2).
\end{cases}
$$

For the derivation of the optimal solution, we shall make the following assumption for the utility function:

$$
U_I(v(x,t))=\frac{a_0}{2}v^2(x,t);\ U_C(v(x,t))=c_1v(x,t)-\frac{c_0}{2}v^2(x,t)
$$

and

$$
U_{SC}(x(t))=p_0x(t);\ U_{H_0}(x(t))=h_0x(t),
$$

where a_0, c_0, c_1, h_0, and p_0 are positive constant coefficient. Then,

$$\begin{cases} -\dfrac{\partial u}{\partial t} - \dfrac{1}{2}\beta^2(x_1)\dfrac{\partial^2 u}{\partial x_1^2} - \dfrac{\partial u}{\partial x_1}\alpha(x_1) \\ -\dfrac{\partial u}{\partial x_2}\Big(rx_2 - x_1\mathbf{D}_S\big(-\overline{x}_1(t) - \overline{v}_1(t)\big) + x_1\mathbf{D}_B\big(\overline{x}_1(t) - \overline{v}_2(t)\big)\Big) + ru \\ = \dfrac{a_0}{2}\Big|x_1 + \hat{v}_1(x,t)\Big|^2 + \dfrac{a_0}{2}\Big|x_1 - \hat{v}_2(x,t)\Big|^2 + \dfrac{c_0}{2}\hat{v}_3^2(x,t) - p_0x_2^- - a_0x_1^2 \\ u(x,T) = h_0x_2. \end{cases} \tag{3.18}$$

and the FP equation is

$$\begin{cases} \dfrac{\partial m}{\partial t} - \dfrac{1}{2}\dfrac{\partial^2}{\partial x_1^2}\Big((\beta^2(x_1))m\Big) + \dfrac{\partial}{\partial x_1}\Big((\alpha(x_1))m\Big) \\ +\dfrac{\partial}{\partial x_2}\Big(\big(rx_2 + \hat{v}_1(x,t)\mathbf{D}_S\big(-\overline{x}_1(t) - \overline{v}_1(t)\big) + \hat{v}_2(x,t)\mathbf{D}_B\big(\overline{x}_1(t) - \overline{v}_2(t)\big) - \hat{v}_3(x,t)\big)m\Big) = 0, \\ m(x,0) = \delta(x)\bigotimes m_0(x), \end{cases} \tag{3.19}$$

In addition, the Euler conditions become

$$\begin{aligned} -a_0\hat{v}_1(x,t) + &\frac{\partial u(x,t)}{\partial x_2}\mathbf{D}_S\left(-\overline{x}_1(t) - \overline{v}_1(t)\right) - \mathbf{D}_S'\left(-\overline{x}_1(t) - \overline{v}_1(t)\right)\overline{v}_1(t)\int \frac{\partial u}{\partial x_2}(\xi,t)m(\xi,t)d\xi = 0, \\ -a_0\hat{v}_2(x,t) + &\frac{\partial u(x,t)}{\partial x_2}\mathbf{D}_B\left(\overline{x}_1(t) - \overline{v}_2(t)\right) - \mathbf{D}_B'\left(\overline{x}_1(t) - \overline{v}_2(t)\right)\overline{v}_2(t)\int \frac{\partial u}{\partial x_2}(\xi,t)m(\xi,t)d\xi = 0, \\ c_1 - &c_0\hat{v}_3(x,t) - \frac{\partial u(x,t)}{\partial x_2} = 0. \end{aligned} \tag{3.20}$$

We define

$$\lambda_{x_2}(x,t) = \frac{\partial u}{\partial x_2}(x,t)$$

Differentiating (3.18) with respect to x_2, we obtain

$$\begin{cases} -\dfrac{\partial \lambda_{x_2}}{\partial t} - \dfrac{1}{2}\beta^2(x_1)\dfrac{\partial^2 \lambda_{x_2}}{\partial x_1^2} - \dfrac{\partial \lambda_{x_2}}{\partial x_1}\alpha(x_1) - \dfrac{\partial \lambda_{x_2}}{\partial x_2}\left(rx_2 - x_1\mathbf{D}_S\big(-\overline{x}_1(t) - \overline{v}_1(t)\big) + x_1\mathbf{D}_B\big(\overline{x}_1(t) - \overline{v}_2(t)\big)\right) \\ = a_0(x_1 + \hat{v}_1(x,t))\dfrac{\partial \hat{v}_1(x,t)}{\partial x_2} - a_0(x_1 - \hat{v}_2(x,t))\dfrac{\partial \hat{v}_2(x,t)}{\partial x_2} + c_0\hat{v}_3(x,t)\dfrac{\partial \hat{v}_3(x,t)}{\partial x_2} - p_0\mathbf{1}_{x_2<0} \\ \lambda_{x_2}(x,T) = h_0. \end{cases}$$

From the Euler condition, we have

$$\begin{aligned} -a_0\frac{\partial \hat{v}_1(x,t)}{\partial x_2} + \frac{\partial \lambda_{x_2}(x,t)}{\partial x_2}\mathbf{D}_S\left(-\overline{x}_1(t) - \overline{v}_1(t)\right) &= 0, \\ -a_0\frac{\partial \hat{v}_2(x,t)}{\partial x_2} + \frac{\partial \lambda_{x_2}(x,t)}{\partial x_2}\mathbf{D}_B\left(\overline{x}_1(t) - \overline{v}_2(t)\right) &= 0, \\ -c_0\frac{\partial \hat{v}_3(x,t)}{\partial x_2} - \frac{\lambda_{x_2}(x,t)}{\partial x_2} &= 0. \end{aligned}$$

Eventually, we get

$$\begin{cases} -\dfrac{\partial \lambda_{x_2}}{\partial t} - \dfrac{1}{2}\beta^2(x_1)\dfrac{\partial^2 \lambda_{x_2}}{\partial x_1^2} - \dfrac{\partial \lambda_{x_2}}{\partial x_1}\alpha(x_1) \\ -\dfrac{\partial \lambda_{x_2}}{\partial x_2}\Big(rx_2 + \hat{v}_1(x,t)\mathbf{D}_S\big(-\overline{x}_1(t) - \overline{\hat{v}}_1(t)\big) + \hat{v}_2(x,t)\mathbf{D}_B\big(\overline{x}_1(t) - \overline{\hat{v}}_2(t)\big) - \hat{v}_3(x,t)\Big) = -p_0\mathbf{1}_{x_2<0} \\ \lambda_{x_2}(x,T) = h_0. \end{cases} \tag{3.21}$$

From (3.19) with (3.21), we can infer

$$\overline{\lambda}_{x_2}(t) = \int \lambda_{x_2}(\xi,t)m(\xi,t)d\xi = \int \lambda_{x_2}(\xi,T)m(\xi,T)d\xi - p_0\mathbf{1}_{x_2<0}(T-t) = h_0 - p_0\mathbf{1}_{x_2<0}(T-t). \tag{3.22}$$

We assume that the evolution of a prevailing price of an asset is described by the mean revering model as below:

$$\alpha(x_1,t) = -\alpha_1(x_1 - \gamma_1(t))$$

, where $\gamma_1(t)$ is deterministic function and we take $\beta(x_1) = \beta$. Then, we can easily obtain

$$\overline{x}_1(t) = \overline{x}_{10}\exp -\alpha_1 t + \int_0^t \exp -\alpha_1(t-s)(\alpha_1\gamma_1(s)ds + \beta dw(s)).$$

From the Euler conditions (3.20), we have

$$-a_0\overline{\hat{v}}_1(t) + \mathbf{D}_S\left(-\overline{x}_1(t) - \overline{\hat{v}}_1(t)\right)\overline{\lambda}_{x_2}(t) - \mathbf{D}'_S\left(-\overline{x}_1(t) - \overline{\hat{v}}_1(t)\right)\overline{\hat{v}}_1(t)\overline{\lambda}_{x_2}(t) = 0,$$
$$-a_0\overline{\hat{v}}_2(t) + \mathbf{D}_B\left(\overline{x}_1(t) - \overline{\hat{v}}_2(t)\right)\overline{\lambda}_{x_2}(t) - \mathbf{D}'_B\left(\overline{x}_1(t) - \overline{\hat{v}}_2(t)\right)\overline{\hat{v}}_2(t)\overline{\lambda}_{x_2}(t) = 0.$$

Consider the linear demand in which

$$\mathbf{D}_S(x) = d_{S_0}x + d_{S_1} \text{ and } \mathbf{D}_B(x) = d_{B_0}x + d_{B_1},$$

where d_{S_0}, d_{S_1}, d_{B_0}, and d_{B_1} are positive constant coefficients. Then, we obtain

$$\overline{\hat{v}}_1(t) = \frac{-d_{S_0}\overline{x}_1(t)\overline{\lambda}_{x_2}(t) + d_{S_1}\overline{\lambda}_{x_2}(t)}{a_0 + 2d_{S_0}\overline{\lambda}_{x_2}(t)};$$
$$\overline{\hat{v}}_2(t) = \frac{d_{B_0}\overline{x}_1(t)\overline{\lambda}_{x_2}(t) + d_{B_1}\overline{\lambda}_{x_2}(t)}{a_0 + 2d_{B_0}\overline{\lambda}_{x_2}(t)}.$$

In addition, the optimal feedback is defined:

$$\hat{v}_1(x,t) = \frac{1}{a_0}\left[\left(d_{S_0}\left(-\overline{x}_1(t) - \overline{\hat{v}}_1(t)\right) + d_{S_1}\right)\lambda_{x_2}(x,t) - d_{S_0}\overline{\hat{v}}_1(t)\overline{\lambda}_{x_2}(t)\right]$$
$$\hat{v}_2(x,t) = \frac{1}{a_0}\left[\left(d_{B_0}\left(\overline{x}_1(t) - \overline{\hat{v}}_2(t)\right) + d_{B_1}\right)\lambda_{x_2}(x,t) - d_{B_0}\overline{\hat{v}}_2(t)\overline{\lambda}_{x_2}(t)\right]$$
$$\hat{v}_3(x,t) = \frac{1}{c_0}\Big(c_1 - \lambda_{x_2}(x,t)\Big)$$

In addition, we set

$$\varphi_{S_0}(t) = \frac{d_{S_0}\left(-\overline{x}_1(t) - \overline{\hat{v}}_1(t)\right) + d_{S_1}}{a_0} \text{ and } \varphi_{B_0}(t) = \frac{d_{B_0}\left(\overline{x}_1(t) - \overline{\hat{v}}_2(t)\right) + d_{B_1}}{a_0};$$
$$\varphi_{S_1}(t) = \frac{d_{S_0}\overline{\hat{v}}_1(t)\overline{\lambda}_{x_2}(t)}{a_0} \text{ and } \varphi_{B_1}(t) = \frac{d_{B_0}\overline{\hat{v}}_2(t)\overline{\lambda}_{x_2}(t)}{a_0}$$

Then, the optimal feedback is

$$\begin{aligned}
\hat{v}_1(x,t) &= \varphi_{S_0}(t)\lambda_{x_2}(x,t) - \varphi_{S_1}(t) \\
\hat{v}_2(x,t) &= \varphi_{B_0}(t)\lambda_{x_2}(x,t) - \varphi_{B_1}(t) \\
\hat{v}_3(x,t) &= c_0^{-1}c_1 - c_0^{-1}\lambda_{x_2}(x,t)
\end{aligned} \tag{3.23}$$

Once we know $\lambda_{x_2}(x,t)$, everything is explicit. We rewrite (3.21) using the optimal feedback (3.23) as below:

$$\begin{aligned}
&-\frac{\partial \lambda_{x_2}}{\partial t} - \frac{1}{2}\beta^2\frac{\partial^2 \lambda_{x_2}}{\partial x_1^2} + \frac{\partial \lambda_{x_2}}{\partial x_1}\alpha_1\big(x_1 - \gamma_1(t)\big) \\
&-\frac{\partial \lambda_{x_2}}{\partial x_2}\Big(rx_2 + \Big(\varphi_{S_0}(t)\lambda_{x_2}(x,t) - \varphi_{S_1}(t)\Big)a_0\varphi_{S_0}(t) + \Big(\varphi_{B_0}(t)\lambda_{x_2}(x,t) - \varphi_{B_1}(t)\Big)a_0\varphi_{B_0}(t) \\
&\qquad - c_0^{-1}c_1 + c_0^{-1}\lambda_{x_2}(x,t)\Big) = -p_0\mathbf{1}_{x_2<0} \\
&\quad \lambda_{x_2}(x,T) = h_0.
\end{aligned} \tag{3.24}$$

We can obtain $\lambda_{x_2}(x,t)$ as

$$\lambda_{x_2}(x,t) = \lambda_{20}(t)x_2 + \lambda_{21}(x_1,t)$$

Then, we have to solve the equation for $\lambda_{20}(t)$ which is

$$-\frac{\partial \lambda_{20}}{\partial t} - (a_0\varphi_{S_0}(t)^2 + a_0\varphi_{B_0}(t)^2 + c_0^{-1})\lambda_{20}^2 - r\lambda_{20} = 0,\ \lambda_{20}(T) = 0. \tag{3.25}$$

and the equation for $\lambda_{21}(x_1,t)$ which is

$$-\frac{\partial \lambda_{21}}{\partial t} - \frac{1}{2}\beta^2\frac{\partial^2 \lambda_{21}}{\partial x_1^2} + \frac{\partial \lambda_{21}}{\partial x_1}\alpha_1\big(x_1 - \gamma_1(t)\big) - \theta_0\lambda_{21} + \theta_1 = -p_0\mathbf{1}_{x_2<0},\ \lambda_{21}(x_1,T) = h_0, \tag{3.26}$$

where

$$\begin{aligned}
\theta_0(t) &= (a_0\varphi_{S_0}(t)^2 + a_0\varphi_{B_0}(t)^2 + c_0^{-1})\lambda_{20}(t) \\
\theta_1(t) &= (a_0\varphi_{S_0}(t)\varphi_{S_1}(t) + a_0\varphi_{B_0}(t)\varphi_{B_1}(t) + c_0^{-1}c_1)\lambda_{20}(t).
\end{aligned}$$

From (3.25), we can infer $\lambda_{20}(t) = 0$. Then we rewrite (3.26):

$$-\frac{\partial \lambda_{21}}{\partial t} - \frac{1}{2}\beta^2\frac{\partial^2 \lambda_{21}}{\partial x_1^2} + \frac{\partial \lambda_{21}}{\partial x_1}\alpha_1\big(x_1 - \gamma_1(t)\big) = -p_0\mathbf{1}_{x_2<0},\ \lambda_{21}(x_1,T) = h_0, \tag{3.27}$$

The solution of (3.27) is a linear function of x_1, namely

$$\lambda_{21}(x_1,t) = \psi_0(t)x_1 + \psi_1(t)$$

with

$$\begin{cases} -\dfrac{\partial \psi_0}{\partial t} + \alpha_1\psi_0 = 0, \ \psi_0(T) = 0. \\ -\dfrac{\partial \psi_1}{\partial t} - \gamma_1\psi_0 = -p_0\mathbf{1}_{x_2<0}, \ \psi_1(T) = h_0. \end{cases}$$

whose solutions are

$$\psi_0(t) = 0 \text{ and } \psi_1(t) = -p_0\mathbf{1}_{x_2<0}(T-t) + h_0$$

Therefore, we conclude that

$$\lambda_{x_2}(x,t) = -p_0\mathbf{1}_{x_2<0}(T-t) + h_0. \tag{3.28}$$

Eventually, we obtain the optimal feedback

$$\begin{aligned}
\hat{v}_1(x,t) &= \varphi_{S_0}(t)(-p_0\mathbf{1}_{x_2<0}(T-t)+h_0) - \varphi_{S_1}(t) \\
\hat{v}_2(x,t) &= \varphi_{B_0}(t)(-p_0\mathbf{1}_{x_2<0}(T-t)+h_0) - \varphi_{B_1}(t) \\
\hat{v}_3(x,t) &= c_0^{-1}c_1 - c_0^{-1}(-p_0\mathbf{1}_{x_2<0}(T-t)+h_0). \ \blacksquare
\end{aligned}$$

3.6.2 Proof of Remark 3.1

Proof. Refer to (3.22) and (3.28). ■

3.6.3 Proof of Proposition 3.2

Proof.

$$g(x,m,v,\mu) = \left| \begin{array}{c} \alpha(x_1) \\ -\mathbf{D}_S\left(-\int \xi_1 m(\xi)d\xi - \int \eta_1 d\mu(\eta)\right) + \mathbf{D}_B\left(\int \xi_1 m(\xi)d\xi - \int \eta_2 d\mu(\eta)\right) + v_3 \end{array} \right|,$$

$$\sigma(x) = \left| \begin{array}{c} \beta(x_1) \\ 0 \end{array} \right|,$$

then

$$A\varphi(x) = -\frac{1}{2}\beta^2(x_1)\frac{\partial^2 \varphi}{\partial x_1^2}.$$

Also,

$$f(x,m,v,\mu) =(x_1+v_1)\mathbf{D}_S\left(-\int \xi_1 m(\xi)d\xi - \int \eta_1 d\mu(\eta)\right) - (x_1-v_2)\mathbf{D}_B\left(\int \xi_1 m(\xi)d\xi - \int \eta_2 d\mu(\eta)\right)$$
$$- v_3x_1 - U_I(v_1) - U_I(v_2) - U_{IT}(v_3) - U_{SI}(x_2^-) - U_Z(x_2^+)$$
$$h(x,m) = U_{H_1}(x_2).$$

We obtain the derivatives

$$\frac{\partial f}{\partial v}(x,m,v,\mu)$$
$$= \left| \begin{array}{ccc} \mathbf{D}_S\left(-\int \xi_1 m(\xi)d\xi - \int \eta_1 d\mu(\eta)\right) - U_I'(v_1) & \mathbf{D}_B\left(\int \xi_1 m(\xi)d\xi - \int \eta_2 d\mu(\eta)\right) - U_I'(v_2) & -x_1 - U_{IT}'(v_3) \end{array} \right|,$$

$$\frac{\partial f}{\partial m}\Big((x,m,v,\mu)\Big)(\xi)$$
$$= -(x_1+v_1)\xi_1\mathbf{D}_S'\left(-\int \xi_1 m(\xi)d\xi - \int \eta_1 d\mu(\eta)\right) - (x_1-v_2)\xi_1\mathbf{D}_B'\left(\int \xi_1 m(\xi)d\xi - \int \eta_2 d\mu(\eta)\right)$$

$$\frac{\partial f}{\partial \mu}\Big((x,m,v,\mu)\Big)(\eta)$$
$$= \left| \begin{array}{ccc} -(x_1+v_1)\eta_1\mathbf{D}_S'\left(-\int \xi_1 m(\xi)d\xi - \int \eta_1 d\mu(\eta)\right) & (x_1-v_2)\eta_2\mathbf{D}_B'\left(\int \xi_1 m(\xi)d\xi - \int \eta_2 d\mu(\eta)\right) & 0 \end{array} \right|$$

$$\frac{\partial g}{\partial v}(x,m,v,\mu) = \left| \begin{array}{ccc} 0 & 0 & 0 \\ 0 & 0 & 1 \end{array} \right|,$$

$$\frac{\partial g}{\partial m}((x,m,v,\mu))(\xi) = \left| \begin{array}{c} 0 \\ \xi_1\mathbf{D}_S'\left(-\int \xi_1 m(\xi)d\xi - \int \eta_1 d\mu(\eta)\right) + \xi_1\mathbf{D}_B'\left(\int \xi_1 m(\xi)d\xi - \int \eta_2 d\mu(\eta)\right) \end{array} \right|,$$

$$\frac{\partial g}{\partial \mu}((x,m,v,\mu))(\eta) = \left| \begin{array}{ccc} 0 & 0 & 0 \\ \eta_1\mathbf{D}_S'\left(-\int \xi_1 m(\xi)d\xi - \int \eta_1 d\mu(\eta)\right) & -\eta_2\mathbf{D}_B'\left(\int \xi_1 m(\xi)d\xi - \int \eta_2 d\mu(\eta)\right) & 0 \end{array} \right|,$$

We use the notation

$$\overline{x}_1(t) = \int \xi_1 m(\xi,t)d\xi;\ \overline{\hat{v}}_1(t) = \int \eta_1 d\mu(\eta,t);\ \overline{\hat{v}}_2(t) = \int \eta_2 d\mu(\eta,t).$$

$$\frac{\partial f}{\partial v}(x,m(t),\hat{v}(x,t),\hat{v}(\cdot,t)*m(t))$$
$$= \left| \begin{array}{ccc} \mathbf{D}_S\left(-\overline{x}_1(t) - \overline{\hat{v}}_1(t)\right) - U_I'(\hat{v}_1(x,t)) & \mathbf{D}_B\left(\overline{x}_1(t) - \overline{\hat{v}}_2(t)\right) - U_I'(\hat{v}_2(x,t)) & -x_1 - U_{IT}'(\hat{v}_3(x,t)) \end{array} \right|,$$

$$\frac{\partial f}{\partial m}\Big((x,m(t),\hat{v}(x,t),\hat{v}(\cdot,t)*m(t))\Big)(\xi)$$
$$= -(x_1+\hat{v}_1(x,t))\xi_1\mathbf{D}_S'\left(-\overline{x}_1(t) - \overline{\hat{v}}_1(t)\right) - (x_1-\hat{v}_2(x,t))\xi_1\mathbf{D}_B'\left(\overline{x}_1(t) - \overline{\hat{v}}_2(t)\right),$$

$$\frac{\partial f}{\partial \mu}\Big((x,m(t),\hat{v}(x,t),\hat{v}(\cdot,t)*m(t))\Big)(\eta)$$
$$= \left| \begin{array}{ccc} -(x_1+\hat{v}_1(x,t))\eta_1\mathbf{D}_S'\left(-\overline{x}_1(t) - \overline{\hat{v}}_1(t)\right) & (x_1-\hat{v}_2(x,t))\eta_2\mathbf{D}_B'\left(\overline{x}_1(t) - \overline{\hat{v}}_2(t)\right) & 0 \end{array} \right|.$$

$$\frac{\partial g}{\partial v}(x, m(t), \hat{v}(x,t), \hat{v}(\cdot,t) * m(t)) = \begin{vmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{vmatrix},$$

$$\frac{\partial g}{\partial m}\big((x, m(t), \hat{v}(x,t), \hat{v}(\cdot,t) * m(t))\big)(\xi) = \begin{vmatrix} 0 \\ \xi_1 \mathbf{D}'_S\left(-\overline{x}_1(t) - \overline{\hat{v}}_1(t)\right) + \xi_1 \mathbf{D}'_B\left(\overline{x}_1(t) - \overline{\hat{v}}_2(t)\right) \end{vmatrix},$$

$$\frac{\partial g}{\partial \mu}\big((x, m(t), \hat{v}(x,t), \hat{v}(\cdot,t) * m(t))\big)(\eta) = \begin{vmatrix} 0 & 0 & 0 \\ \eta_1 \mathbf{D}'_S\left(-\overline{x}_1(t) - \overline{\hat{v}}_1(t)\right) & -\eta_2 \mathbf{D}'_B\left(\overline{x}_1(t) - \overline{\hat{v}}_2(t)\right) & 0 \end{vmatrix},$$

$$\begin{aligned} &\int D_w \frac{\partial f}{\partial \mu}\Big((\xi, m(t), \hat{v}(\xi,t), \hat{v}(\cdot,t) * m(t))\Big)\big(\hat{v}(x,t)\big) m(\xi,t) d\xi \\ &= \begin{vmatrix} -\mathbf{D}'_S\left(-\overline{x}_1(t) - \overline{\hat{v}}_1(t)\right)\left(\overline{x}_1(t) + \overline{\hat{v}}_1(t)\right) & \mathbf{D}'_B\left(\overline{x}_1(t) - \overline{\hat{v}}_2(t)\right)\left(\overline{x}_1(t) - \overline{\hat{v}}_1(t)\right) & 0 \end{vmatrix} \end{aligned}$$

$$\begin{aligned} &\int D_\xi u(\xi,t) \cdot D_w \frac{\partial g}{\partial \mu}\Big((\xi, m(t), \hat{v}(\xi,t), \hat{v}(\cdot,t) * m(t))\Big)\big(\hat{v}(x,t)\big) m(\xi,t) d\xi \\ &= \begin{vmatrix} \mathbf{D}'_S\left(-\overline{x}_1(t) - \overline{\hat{v}}_1(t)\right) \displaystyle\int \frac{\partial u}{\partial x_2}(\xi,t) m(\xi,t) d\xi & -\mathbf{D}'_B\left(\overline{x}_1(t) - \overline{\hat{v}}_2(t)\right) \displaystyle\int \frac{\partial u}{\partial x_2}(\xi,t) m(\xi,t) d\xi & 0 \end{vmatrix}. \end{aligned}$$

The Euler condition (3.9) becomes

$$\begin{aligned} &\mathbf{D}_S\left(-\overline{x}_1(t) - \overline{\hat{v}}_1(t)\right) - U'_I(\hat{v}_1(x,t)) + \mathbf{D}'_S\left(-\overline{x}_1(t) - \overline{\hat{v}}_1(t)\right)\big(-\overline{x}_1(t) - \overline{\hat{v}}_1(t)\big) \\ &\qquad + \mathbf{D}'_S\left(-\overline{x}_1(t) - \overline{\hat{v}}_1(t)\right) \int \frac{\partial u}{\partial x_2}(\xi,t) m(\xi,t) d\xi = 0, \\ &\mathbf{D}_B\left(\overline{x}_1(t) - \overline{\hat{v}}_2(t)\right) - U'_I(\hat{v}_2(x,t)) + \mathbf{D}'_B\left(\overline{x}_1(t) - \overline{\hat{v}}_2(t)\right)\left(\overline{x}_1(t) - \overline{\hat{v}}_2(t)\right) \\ &\qquad - \mathbf{D}'_B\left(\overline{x}_2(t) - \overline{\hat{v}}_2(t)\right) \int \frac{\partial u}{\partial x_2}(\xi,t) m(\xi,t) d\xi = 0, \\ &-x_1 - U'_{IT}(v_3) + \frac{\partial u(x,t)}{\partial x_2} = 0. \end{aligned} \tag{3.29}$$

Next,

$$\begin{aligned} &f(x, m(t), \hat{v}(x,t), \hat{v}(\cdot,t) * m(t)) \\ &= (x_1 + \hat{v}_1(x,t)) \mathbf{D}_S\left(-\overline{x}_1(t) - \overline{\hat{v}}_1(t)\right) - (x_1 - \hat{v}_2(x,t)) \mathbf{D}_B\left(\overline{x}_1(t) - \overline{\hat{v}}_2(t)\right) - \hat{v}_3(x,t) x_1 \\ &\quad - U_I(\hat{v}_1(x,t)) - U_I(\hat{v}_2(x,t)) - U_{IT}(\hat{v}_3(x,t)) - U_{SI}(x_2^-) - U_Z(x_2^+) \end{aligned}$$

$$\begin{aligned} &\int \frac{\partial f}{\partial m}\Big((\xi, m(t), \hat{v}(\xi,t), \hat{v}(\cdot,t) * m(t))\Big)(x) m(\xi,t) d\xi \\ &= -\mathbf{D}'_S\left(-\overline{x}_1(t) - \overline{\hat{v}}_1(t)\right) x_1\big(\overline{x}_1(t) + \overline{\hat{v}}_1(t)\big) - \mathbf{D}'_B\left(\overline{x}_1(t) - \overline{\hat{v}}_2(t)\right) x_1\big(\overline{x}_1(t) - \overline{\hat{v}}_2(t)\big), \end{aligned}$$

$$
\begin{aligned}
&\int \frac{\partial f}{\partial \mu}\Big((\xi, m(t), \hat{v}(\xi,t), \hat{v}(\cdot,t) * m(t))\Big)(\hat{v}(x,t))m(\xi,t)d\xi \\
&= -\,\mathbf{D}_S'\left(-\overline{x}_1(t) - \overline{\hat{v}}_1(t)\right)\hat{v}_1(x,t)\big(\overline{x}_1(t) + \overline{\hat{v}}_1(t)\big) + \mathbf{D}_B'\left(\overline{x}_1(t) - \overline{\hat{v}}_2(t)\right)\hat{v}_2(x,t)\big(\overline{x}_1(t) - \overline{\hat{v}}_2(t)\big) \\
&\int D_\xi u(\xi,t)\cdot \frac{\partial g}{\partial m}\Big((\xi, m(t), \hat{v}(\xi,t), \hat{v}(\cdot,t) * m(t))\Big)(x)m(\xi,t)d\xi \\
&= x_1\mathbf{D}_S'\left(-\overline{x}_1(t) - \overline{\hat{v}}_1(t)\right)\int \frac{\partial u}{\partial x_2}(\xi,t)m(\xi,t)d\xi + x_1\mathbf{D}_B'\left(\overline{x}_1(t) - \overline{\hat{v}}_2(t)\right)\int \frac{\partial u}{\partial x_2}(\xi,t)m(\xi,t)d\xi
\end{aligned}
$$

$$
\begin{aligned}
&\int D_\xi u(\xi,t)\cdot \frac{\partial g}{\partial \mu}\Big((\xi, m(t), \hat{v}(\xi,t), \hat{v}(\cdot,t) * m(t))\Big)(\hat{v}(x,t))m(\xi,t)d\xi \\
&= \hat{v}_1(x,t)\mathbf{D}_S'\left(-\overline{x}_1(t) - \overline{\hat{v}}_1(t)\right)\int \frac{\partial u}{\partial x_2}(\xi,t)m(\xi,t)d\xi - \hat{v}_2(x,t)\mathbf{D}_B'\left(\overline{x}_1(t)\right. \\
&\left. - \overline{\hat{v}}_2(t)\right)\int \frac{\partial u}{\partial x_2}(\xi,t)m(\xi,t)d\xi,
\end{aligned}
$$

The HJB equation (3.10) becomes

$$
\begin{cases}
-\dfrac{\partial u}{\partial t} - \dfrac{1}{2}\beta^2(x_1)\dfrac{\partial^2 u}{\partial x_1^2} - \dfrac{\partial u}{\partial x_1}\alpha(x_1) - \dfrac{\partial u}{\partial x_2}\Big(-\mathbf{D}_S\big(-\overline{x}_1(t) - \overline{\hat{v}}_1(t)\big) + \mathbf{D}_B\big(\overline{x}_1(t) - \overline{\hat{v}}_2(t)\big) + \hat{v}_3(x,t)\Big) \\
= -(-x_1 - \hat{v}_1(x,t))\mathbf{D}_S\left(-\overline{x}_1(t) - \overline{\hat{v}}_1(t)\right) - (x_1 - \hat{v}_2(x,t))\mathbf{D}_B\left(\overline{x}_1(t) - \overline{\hat{v}}_2(t)\right) - \hat{v}_3(x,t)x_1 \\
-U_I(\hat{v}_1(x,t)) - U_I(\hat{v}_2(x,t)) - U_{IT}(\hat{v}_3(x,t)) - U_{SI}(x_2^-) - U_Z(x_2^+) \\
-\mathbf{D}_S'\left(-\overline{x}_1(t) - \overline{\hat{v}}_1(t)\right)\left(-x_1 - \hat{v}_1(x,t)\big(-\overline{x}_1(t) - \overline{\hat{v}}_1(t)\big)\right. \\
-\mathbf{D}_B'\left(\overline{x}_1(t) - \overline{\hat{v}}_2(t)\right)(x_1 - \hat{v}_2(x,t))\big(\overline{x}_1(t) - \overline{\hat{v}}_2(t)\big) \\
-\mathbf{D}_S'\left(-\overline{x}_1(t) - \overline{\hat{v}}_1(t)\right)(-x_1 - \hat{v}_1(x,t))\displaystyle\int \frac{\partial u}{\partial x_2}(\xi,t)m(\xi,t)d\xi \\
+\mathbf{D}_B'\left(\overline{x}_1(t) - \overline{\hat{v}}_2(t)\right)(x_1 - \hat{v}_2(x,t))\displaystyle\int \frac{\partial u}{\partial x_2}(\xi,t)m(\xi,t)d\xi \\
u(x,T) = U_{H_1}(x_2).
\end{cases}
$$

Using the Euler conditions (3.29), we rewrite the HJB equation:

$$
\begin{cases}
-\dfrac{\partial u}{\partial t} - \dfrac{1}{2}\beta^2(x_1)\dfrac{\partial^2 u}{\partial x_1^2} - \dfrac{\partial u}{\partial x_1}\alpha(x_1) - \dfrac{\partial u}{\partial x_2}\Big(-\mathbf{D}_S\big(-\overline{x}_1(t) - \overline{\hat{v}}_1(t)\big) + \mathbf{D}_B\big(\overline{x}_1(t) - \overline{\hat{v}}_2(t)\big)\Big) \\
= \hat{v}_1(x,t)U_I'(\hat{v}_1(x,t)) - U_I(\hat{v}_1(x,t)) + \hat{v}_2(x,t)U_I'(\hat{v}_2(x,t)) - U_I(\hat{v}_2(x,t)) \\
+\hat{v}_3(x,t)U_{IT}'(\hat{v}_3(x,t)) - U_{IT}(\hat{v}_3(x,t)) - U_{SI}(x_2^-) - U_Z(x_2^+) + x_1U_I'(\hat{v}_1(x,t)) - x_1U_I'(\hat{v}_2(x,t)) \\
u(x,T) = U_{H_1}(x_2).
\end{cases}
$$

For the derivation of the optimal solution, we shall make the following assumption for the cost function:

$$
U_I(v(x,t)) = \frac{a_0}{2}v^2(x,t);\ U_{IT}(v(x,t)) = \frac{k_0}{2}v^2(x,t)
$$

and

$$U_{SI}(x(t)) = q_0 x(t);\ U_Z(x(t)) = q_1 x(t);\ U_{H_1}(x(t)) = h_1 x(t),$$

where a_0, k_0, h_1, q_0, and q_1 are positive constant coefficient. Then,

$$\begin{cases} -\dfrac{\partial u}{\partial t} - \dfrac{1}{2}\beta^2(x_1)\dfrac{\partial^2 u}{\partial x_1^2} - \dfrac{\partial u}{\partial x_1}\alpha(x_1) - \dfrac{\partial u}{\partial x_2}\Big(-\mathbf{D}_S\big(-\overline{x}_1(t) - \overline{\hat{v}}_1(t)\big) + \mathbf{D}_B\big(\overline{x}_1(t) - \overline{\hat{v}}_2(t)\big)\Big) \\ = \dfrac{a_0}{2}\Big|x_1 + \hat{v}_1(x,t)\Big|^2 + \dfrac{a_0}{2}\Big|x_1 - \hat{v}_2(x,t)\Big|^2 + \dfrac{k_0}{2}\hat{v}_3^2(x,t) - q_0 x_2^- - q_1 x_2^+ - a_0 x_1^2 \\ u(x,T) = h_1 x_2. \end{cases} \tag{3.30}$$

and the FP equation is

$$\begin{cases} \dfrac{\partial m}{\partial t} - \dfrac{1}{2}\dfrac{\partial^2}{\partial x_1^2}\Big((\beta^2(x_1))m\Big) + \dfrac{\partial}{\partial x_1}\Big((\alpha(x_1))m\Big) \\ +\dfrac{\partial}{\partial x_2}\Big(\big(-\mathbf{D}_S\big(-\overline{x}_1(t) - \overline{\hat{v}}_1(t)\big) + \mathbf{D}_B\big(\overline{x}_1(t) - \overline{\hat{v}}_2(t)\big) + \hat{v}_3(x,t)\big)m\Big) = 0, \\ m(x,0) = \delta(x)\bigotimes m_0(x), \end{cases} \tag{3.31}$$

In addition, the Euler conditions become

$$\begin{aligned} -a_0\hat{v}_1(x,t) &+ \mathbf{D}_S\left(-\overline{x}_1(t) - \overline{\hat{v}}_1(t)\right) + \mathbf{D}'_S\left(-\overline{x}_1(t) - \overline{\hat{v}}_1(t)\right)\big(-\overline{x}_1(t) - \overline{\hat{v}}_1(t)\big) \\ &+ \mathbf{D}'_S\left(-\overline{x}_1(t) - \overline{\hat{v}}_1(t)\right)\int \frac{\partial u}{\partial x_2}(\xi,t)m(\xi,t)d\xi = 0, \\ -a_0\hat{v}_2(x,t) &+ \mathbf{D}_B\left(\overline{x}_1(t) - \overline{\hat{v}}_2(t)\right) + \mathbf{D}'_B\left(\overline{x}_1(t) - \overline{\hat{v}}_2(t)\right)\left(\overline{x}_1(t) - \overline{\hat{v}}_2(t)\right) \\ &- \mathbf{D}'_B\left(\overline{x}_1(t) - \overline{\hat{v}}_2(t)\right)\int \frac{\partial u}{\partial x_2}(\xi,t)m(\xi,t)d\xi = 0, \\ -k_0\hat{v}_3(x,t) - x_1 &+ \frac{\partial u(x,t)}{\partial x_2} = 0. \end{aligned} \tag{3.32}$$

We define

$$\lambda_{x_2}(x,t) = \frac{\partial u}{\partial x_2}(x,t)$$

Differentiating (3.30) with respect to x_2, we obtain

$$\begin{cases} -\dfrac{\partial \lambda_{x_2}}{\partial t} - \dfrac{1}{2}\beta^2(x_1)\dfrac{\partial^2 \lambda_{x_2}}{\partial x_1^2} - \dfrac{\partial \lambda_{x_2}}{\partial x_1}\alpha(x_1) - \dfrac{\partial \lambda_{x_2}}{\partial x_2}\Big(-\mathbf{D}_S\big(-\overline{x}_1(t) - \overline{\hat{v}}_1(t)\big) + \mathbf{D}_B\big(\overline{x}_1(t) - \overline{\hat{v}}_2(t)\big)\Big) \\ = a_0(x_1 + \hat{v}_1(x,t))\dfrac{\partial \hat{v}_1(x,t)}{\partial x_2} - a_0(x_1 - \hat{v}_2(x,t))\dfrac{\partial \hat{v}_2(x,t)}{\partial x_2} + k_0\hat{v}_3(x,t)\dfrac{\partial \hat{v}_3(x,t)}{\partial x_2} - q_0\mathbf{1}_{x_2<0} - q_1\mathbf{1}_{x_2>0} \\ \lambda_{x_2}(x,T) = h_1. \end{cases}$$

From the Euler condition, we have

$$\begin{aligned}\frac{\partial \hat{v}_1(x,t)}{\partial x_2} &= 0,\\ \frac{\partial \hat{v}_2(x,t)}{\partial x_2} &= 0,\\ -k_0\frac{\partial \hat{v}_3(x,t)}{\partial x_2} + \frac{\lambda_{x_2}(x,t)}{\partial x_2} &= 0.\end{aligned}$$

Eventually, we get

$$\begin{cases} -\dfrac{\partial \lambda_{x_2}}{\partial t} - \dfrac{1}{2}\beta^2(x_1)\dfrac{\partial^2 \lambda_{x_2}}{\partial x_1^2} - \dfrac{\partial \lambda_{x_2}}{\partial x_1}\alpha(x_1) \\ -\frac{\partial \lambda_{x_2}}{\partial x_2}\Big(- \mathbf{D}_S\big(- \overline{x}_1(t) - \overline{\hat{v}}_1(t)\big) + \mathbf{D}_B\big(\overline{x}_1(t) - \overline{\hat{v}}_2(t)\big) + \hat{v}_3(x,t)\Big) \\ = -q_0\mathbf{1}_{x_2<0} - q_1\mathbf{1}_{x_2>0} \\ \lambda_{x_2}(x,T) = h_1. \end{cases} \tag{3.33}$$

From (3.31) with (3.33), we can infer

$$\begin{aligned}\overline{\lambda}_{x_2}(t) &= \int \lambda_{x_2}(\xi,t)m(\xi,t)d\xi \\ &= \int \lambda_{x_2}(\xi,T)m(\xi,T)d\xi - q_0\mathbf{1}_{x_2<0}(T-t) - q_1\mathbf{1}_{x_2>0}(T-t) \\ &= h_1 - q_0\mathbf{1}_{x_2<0}(T-t) - q_1\mathbf{1}_{x_2>0}(T-t).\end{aligned} \tag{3.34}$$

We assume that the evolution of a prevailing price of an asset is described by the mean revering model as below:

$$\alpha(x_1,t) = -\alpha_1(x_1 - \gamma_1(t))$$

, where $\gamma_1(t)$ is deterministic function and we take $\beta(x_1) = \beta$. Then, we can easily obtain

$$\overline{x}_1(t) = \overline{x}_{10}\exp -\alpha_1 t + \int_0^t \exp -\alpha_1(t-s)(\alpha_1\gamma_1(s)ds + \beta dw(s)).$$

From the Euler conditions (3.32), we have

$$\begin{aligned} -a_0\overline{\hat{v}}_1(t) + \mathbf{D}_S\left(-\overline{x}_1(t) - \overline{\hat{v}}_1(t)\right) + \mathbf{D}'_S\left(-\overline{x}_1(t) - \overline{\hat{v}}_1(t)\right)\big(- \overline{x}_1(t) - \overline{\hat{v}}_1(t)\big) \\ +\mathbf{D}'_S\left(-\overline{x}_1(t) - \overline{\hat{v}}_1(t)\right)\overline{\lambda}_{x_2}(t) = 0, \\ -a_0\overline{\hat{v}}_2(t) + \mathbf{D}_B\left(\overline{x}_1(t) - \overline{\hat{v}}_2(t)\right) + \mathbf{D}'_B\left(\overline{x}_1(t) - \overline{\hat{v}}_2(t)\right)\left(\overline{x}_1(t) - \overline{\hat{v}}_2(t)\right) - \mathbf{D}'_B\left(\overline{x}_1(t) - \overline{\hat{v}}_2(t)\right)\overline{\lambda}_{x_2}(t) = 0.\end{aligned}$$

Consider the linear demand in which

$$\mathbf{D}_S(x) = d_{S_0}x + d_{S_1} \text{ and } \mathbf{D}_B(x) = d_{B_0}x + d_{B_1},$$

where d_{S_0}, d_{S_1}, d_{B_0}, and d_{B_1} are positive constant coefficients. Then, we obtain

$$\begin{aligned}\bar{\hat{v}}_1(t) &= \frac{-2d_{S_0}\bar{x}_1(t) + d_{S_0}\bar{\lambda}_{x_2}(t) + d_{S_1}}{a_0 + 2d_{S_0}};\\ \bar{\hat{v}}_2(t) &= \frac{2d_{B_0}\bar{x}_1(t) - d_{B_0}\bar{\lambda}_{x_2}(t) + d_{B_1}}{a_0 + 2d_{B_0}}.\end{aligned}$$

In addition, the optimal feedback is defined:

$$\begin{aligned}\hat{v}_1(x,t) &= \frac{1}{a_0}\left[2d_{S_0}\big(-\bar{x}_1(t) - \bar{\hat{v}}_1(t)\big) + d_{S_0}\bar{\lambda}_{x_2}(t) + d_{S_1}\right],\\ \hat{v}_2(x,t) &= \frac{1}{a_0}\left[2d_{B_0}\big(\bar{x}_1(t) - \bar{\hat{v}}_2(t)\big) - d_{B_0}\bar{\lambda}_{x_2}(t) + d_{B_1}\right],\\ \hat{v}_3(x,t) &= \frac{1}{k_0}\Big(-x_1 + \lambda_{x_2}(x,t)\Big).\end{aligned} \tag{3.35}$$

Once we know $\lambda_{x_2}(x,t)$, everything is explicit. We rewrite (3.33) using the optimal feedback (3.35) as below:

$$\begin{cases} -\dfrac{\partial \lambda_{x_2}}{\partial t} - \dfrac{1}{2}\beta^2\dfrac{\partial^2 \lambda_{x_2}}{\partial x_1^2} + \dfrac{\partial \lambda_{x_2}}{\partial x_1}\alpha_1\big(x_1 - \gamma_1(t)\big)\\ -\dfrac{\partial \lambda_{x_2}}{\partial x_2}\Big((d_{S_0} + d_{B_0})\bar{x}_1(t) + d_{S_0}\bar{\hat{v}}_1(t) - d_{B_0}\bar{\hat{v}}_2(t) - d_{S_1} + d_{B_1} - k_0^{-1}x_1 + k_0^{-1}\lambda_{x_2}\Big)\\ = -q_0\mathbf{1}_{x_2<0} - q_1\mathbf{1}_{x_2>0}\\ \lambda_{x_2}(x,T) = h_1. \end{cases}$$

We can obtain $\lambda_{x_2}(x,t)$ as

$$\lambda_{x_2}(x,t) = \lambda_{20}(t)x_2 + \lambda_{21}(x_1,t)$$

Then, we have to solve the equation for $\lambda_{20}(t)$ which is

$$-\frac{\partial \lambda_{20}}{\partial t} - k_0^{-1}\lambda_{20}^2 = 0,\ \lambda_{20}(T) = 0. \tag{3.36}$$

and the equation for $\lambda_{21}(x_1,t)$ which is

$$\begin{aligned}-\frac{\partial \lambda_{21}}{\partial t} - \frac{1}{2}\beta^2\frac{\partial^2 \lambda_{21}}{\partial x_1^2} + \frac{\partial \lambda_{21}}{\partial x_1}\alpha_1\big(x_1 - \gamma_1(t)\big) - k_0^{-1}\lambda_{20}\lambda_{21}\\ -\lambda_{20}\big((d_{S_0} + d_{B_0})\bar{x}_1(t) + d_{S_0}\bar{\hat{v}}_1(t) - d_{B_0}\bar{\hat{v}}_2(t) - d_{S_1} + d_{B_1}\big) - k_0^{-1}\lambda_{20}x_1 = -q_0\mathbf{1}_{x_2<0} - q_1\mathbf{1}_{x_2>0},\\ \lambda_{21}(x_1,T) = h_0,\end{aligned} \tag{3.37}$$

From (3.36), we can infer $\lambda_{20}(t) = 0$. Then we rewrite (3.37):

$$-\frac{\partial \lambda_{21}}{\partial t} - \frac{1}{2}\beta^2\frac{\partial^2 \lambda_{21}}{\partial x_1^2} + \frac{\partial \lambda_{21}}{\partial x_1}\alpha_1\big(x_1 - \gamma_1(t)\big) = -q_0\mathbf{1}_{x_2<0} - q_1\mathbf{1}_{x_2>0},\ \lambda_{21}(x_1,T) = h_1, \tag{3.38}$$

The solution of (3.38) is a linear function of x_1, namely

$$\lambda_{21}(x_1,t) = \varrho_0(t)x_1 + \varrho_1(t)$$

with

$$\begin{cases} -\dfrac{\partial \varrho_0}{\partial t} + \alpha_1 \varrho_0 = 0, \ \varrho_0(T) = 0. \\ -\dfrac{\partial \varrho_1}{\partial t} - \gamma_1 \varrho_0 = -q_0 \mathbf{1}_{x_2<0} - q_1 \mathbf{1}_{x_2>0}, \ \varrho_1(T) = h_1. \end{cases}$$

whose solutions are

$$\varrho_0(t) = 0 \text{ and } \varrho_1(t) = -q_0 \mathbf{1}_{x_2<0}(T-t) - q_1 \mathbf{1}_{x_2>0}(T-t) + h_1$$

Therefore, we conclude that

$$\lambda_{x_2}(x,t) = -q_0 \mathbf{1}_{x_2<0}(T-t) - q_1 \mathbf{1}_{x_2>0}(T-t) + h_1. \tag{3.39}$$

Eventually, we obtain the optimal feedback

$$\begin{aligned} \hat{v}_1(x,t) &= \frac{1}{a_0}\left[2d_{S_0}\left(-\overline{x}_1(t) - \overline{\hat{v}}_1(t)\right) + d_{S_0}\overline{\lambda}_{x_2}(t) + d_{S_1}\right], \\ \hat{v}_2(x,t) &= \frac{1}{a_0}\left[2d_{B_0}\left(\overline{x}_1(t) - \overline{\hat{v}}_2(t)\right) - d_{B_0}\overline{\lambda}_{x_2}(t) + d_{B_1}\right], \\ \hat{v}_3(x,t) &= \frac{1}{k_0}\Big(-x_1 - q_0 \mathbf{1}_{x_2<0}(T-t) - q_1 \mathbf{1}_{x_2>0}(T-t) + h_1\Big). \ \blacksquare \end{aligned}$$

3.6.4 Proof of Remark 3.2

Proof. Refer to (3.34) and (3.39). ■

3.6.5 Proof of Proposition 3.3

Proof.

$$g(x,m,v,\mu)$$

$$= \left| \begin{array}{c} \alpha(x_1) \\ rx_2 + (x_1+v_1)\mathbf{D}_S\left(-\int \xi_1 m(\xi)d\xi - \int \eta_1 d\mu(\eta)\right) - (x_1 - v_2)\mathbf{D}_B\left(\int \xi_1 m(\xi)d\xi - \int \eta_2 d\mu(\eta)\right) - v_3 x_1 - v_4 \\ -\mathbf{D}_S\left(-\int \xi_1 m(\xi)d\xi - \int \eta_1 d\mu(\eta)\right) + \mathbf{D}_B\left(\int \xi_1 m(\xi)d\xi - \int \eta_2 d\mu(\eta)\right) + v_3 \end{array} \right|,$$

$$\sigma(x) = \left| \begin{array}{c} \beta(x_1) \\ 0 \\ 0 \end{array} \right|,$$

then

$$A\varphi(x) = -\frac{1}{2}\sigma^2(x_1)\frac{\partial^2\varphi}{\partial x_1^2}.$$

Also,

$$f(x,m,v,\mu) = -U_I(v_1) - U_I(v_2) - U_{IT}(v_3) + U_C(v_4) - U_Z(x_3^+) - U_{SI}(x_3^-) - U_{SC}(x_2^-)$$

$$h(x,m) = U_{H_0}(x_2) + U_{H_1}(x_3).$$

We obtain the derivatives

$$\frac{\partial f}{\partial v}(x,m,v,\mu) = \begin{vmatrix} -U_I'(v_1) & -U_I'(v_2) & -U_{IT}'(v_3) & U_C'(v_4) \end{vmatrix}$$

$$\frac{\partial f}{\partial m}\Big((x,m,v,\mu)\Big)(\xi) = \frac{\partial f}{\partial \mu}\Big((x,m,v,\mu)\Big)(\eta) = 0,$$

$$\frac{\partial g}{\partial v}(x,m,v,\mu) = \begin{vmatrix} 0 & 0 & 0 & 0 \\ \mathbf{D}_S\left(-\int \xi_1 m(\xi)d\xi - \int \eta_1 d\mu(\eta)\right) & \mathbf{D}_B\left(\int \xi_1 m(\xi)d\xi - \int \eta_2 d\mu(\eta)\right) & -x_1 & -1 \\ 0 & 0 & 1 & 0 \end{vmatrix},$$

$$\frac{\partial g}{\partial m}((x,m,v,\mu))(\xi)$$

$$= \begin{vmatrix} 0 \\ -(x_1+v_1)\xi_1\mathbf{D}_S'\left(-\int \xi_1 m(\xi)d\xi - \int \eta_1 d\mu(\eta)\right) - (x_1-v_2)\xi_1\mathbf{D}_B'\left(\int \xi_1 m(\xi)d\xi - \int \eta_2 d\mu(\eta)\right) \\ \xi_1\mathbf{D}_S'\left(-\int \xi_1 m(\xi)d\xi - \int \eta_1 d\mu(\eta)\right) + \xi_1\mathbf{D}_B'\left(\int \xi_1 m(\xi)d\xi - \int \eta_2 d\mu(\eta)\right) \end{vmatrix},$$

$$\frac{\partial g}{\partial \mu}((x,m,v,\mu))(\eta)$$

$$= \begin{vmatrix} 0 & 0 & 0 & 0 \\ -(x_1+v_1)\eta_1\mathbf{D}_S'\left(-\int \xi_1 m(\xi)d\xi - \int \eta_1 d\mu(\eta)\right) & (x_1-v_2)\eta_2\mathbf{D}_B'\left(\int \xi_1 m(\xi)d\xi - \int \eta_2 d\mu(\eta)\right) & 0 & 0 \\ \eta_1\mathbf{D}_S'\left(-\int \xi_1 m(\xi)d\xi - \int \eta_1 d\mu(\eta)\right) & -\eta_2\mathbf{D}_B'\left(\int \xi_1 m(\xi)d\xi - \int \eta_2 d\mu(\eta)\right) & 0 & 0 \end{vmatrix},$$

We use the notation

$$\overline{x}_1(t) = \int \xi_1 m(\xi,t)d\xi;\ \overline{\hat{v}}_1(t) = \int \eta_1 d\mu(\eta,t);\ \overline{\hat{v}}_2(t) = \int \eta_2 d\mu(\eta,t).$$

$$\frac{\partial f}{\partial v}(x, m(t), \hat{v}(x,t), \hat{v}(\cdot,t) * m(t))$$

$$= \begin{vmatrix} -U_I'(\hat{v}_1(x,t)) & -U_I'(\hat{v}_2(x,t)) & -U_{IT}'(\hat{v}_3(x,t)) & U_C'(\hat{v}_4(x,t)) \end{vmatrix},$$

$$\frac{\partial f}{\partial m}\Big((x, m(t), \hat{v}(x,t), \hat{v}(\cdot,t) * m(t))\Big)(\xi) = \frac{\partial f}{\partial \mu}\Big((x, m(t), \hat{v}(x,t), \hat{v}(\cdot,t) * m(t))\Big)(\eta) = 0.$$

$$\frac{\partial g}{\partial v}(x, m(t), \hat{v}(x,t), \hat{v}(\cdot,t) * m(t)) = \left|\begin{array}{cccc} 0 & 0 & 0 & 0 \\ \mathbf{D}_S\left(-\overline{x}_1(t) - \overline{\hat{v}}_1(t)\right) & \mathbf{D}_B\left(\overline{x}_1(t) - \overline{\hat{v}}_2(t)\right) & -x_1 & -1 \\ 0 & 0 & 1 & 0 \end{array}\right|,$$

$$\begin{aligned} &\frac{\partial g}{\partial m}\big((x, m(t), \hat{v}(x,t), \hat{v}(\cdot,t) * m(t))\big)(\xi) \\ = &\left|\begin{array}{c} 0 \\ -(x_1 + \hat{v}_1(x,t))\xi_1 \mathbf{D}'_S\left(-\overline{x}_1(t) - \overline{\hat{v}}_1(t)\right) - (x_1 - \hat{v}_2(x,t))\xi_1 \mathbf{D}'_B\left(\overline{x}_1(t) - \overline{\hat{v}}_2(t)\right) \\ \xi_1 \mathbf{D}'_S\left(-\overline{x}_1(t) - \overline{\hat{v}}_1(t)\right) + \xi_1 \mathbf{D}'_B\left(\overline{x}_1(t) - \overline{\hat{v}}_2(t)\right) \end{array}\right|, \end{aligned}$$

$$\begin{aligned} &\frac{\partial g}{\partial \mu}\big((x, m(t), \hat{v}(x,t), \hat{v}(\cdot,t) * m(t))\big)(\eta) \\ = &\left|\begin{array}{cccc} 0 & 0 & 0 & 0 \\ -(x_1 + \hat{v}_1(x,t))\eta_1 \mathbf{D}'_S\left(-\overline{x}_1(t) - \overline{\hat{v}}_1(t)\right) & (x_1 - \hat{v}_2(x,t))\eta_2 \mathbf{D}'_B\left(\overline{x}_1(t) - \overline{\hat{v}}_2(t)\right) & 0 & 0 \\ \eta_1 \mathbf{D}'_S\left(-\overline{x}_1(t) - \overline{\hat{v}}_1(t)\right) & -\eta_2 \mathbf{D}'_B\left(\overline{x}_1(t) - \overline{\hat{v}}_2(t)\right) & 0 & 0 \end{array}\right|, \end{aligned}$$

$$\int D_w \frac{\partial f}{\partial \mu}\Big((\xi, m(t), \hat{v}(\xi,t), \hat{v}(\cdot,t) * m(t))\Big)\big(\hat{v}(x,t)\big) m(\xi,t) d\xi = 0$$

$$\begin{aligned} &\int D_\xi u(\xi,t) \cdot D_w \frac{\partial g}{\partial \mu}\Big((\xi, m(t), \hat{v}(\xi,t), \hat{v}(\cdot,t) * m(t))\Big)\big(\hat{v}(x,t)\big) m(\xi,t) d\xi \\ = &\left|\begin{array}{c} -\mathbf{D}'_S\Big(-\overline{x}_1(t) - \overline{\hat{v}}_1(t)\Big)\Big(\overline{x}_1(t) + \overline{\hat{v}}_1(t)\Big)\int \frac{\partial u}{\partial x_2}(\xi,t) m(\xi,t) d\xi + \mathbf{D}'_S\Big(-\overline{x}_1(t) - \overline{\hat{v}}_1(t)\Big)\int \frac{\partial u}{\partial x_3}(\xi,t) m(\xi,t) d\xi \\ \mathbf{D}'_B\Big(\overline{x}_1(t) - \overline{\hat{v}}_2(t)\Big)\Big(\overline{x}_1(t) - \overline{\hat{v}}_2(t)\Big)\int \frac{\partial u}{\partial x_2}(\xi,t) m(\xi,t) d\xi - \mathbf{D}'_B\Big(\overline{x}_1(t) - \overline{\hat{v}}_2(t)\Big)\int \frac{\partial u}{\partial x_3}(\xi,t) m(\xi,t) d\xi \\ 0 \\ 0 \end{array}\right|^{\intercal}. \end{aligned}$$

The Euler condition (3.9) becomes

$$
\begin{aligned}
-U_I'(\hat{v}_1(x,t)) &+ \frac{\partial u(x,t)}{\partial x_2}\mathbf{D}_S\left(-\overline{x}_1(t)-\overline{\hat{v}}_1(t)\right)\\
&+\mathbf{D}_S'\left(-\overline{x}_1(t)-\overline{\hat{v}}_1(t)\right)\left(-\overline{x}_1(t)-\overline{\hat{v}}_1(t)\right)\int\frac{\partial u}{\partial x_2}(\xi,t)m(\xi,t)d\xi\\
&+\mathbf{D}_S'\left(-\overline{x}_1(t)-\overline{\hat{v}}_1(t)\right)\int\frac{\partial u}{\partial x_3}(\xi,t)m(\xi,t)d\xi=0,\\
-U_I'(\hat{v}_2(x,t)) &+ \frac{\partial u(x,t)}{\partial x_2}\mathbf{D}_B\left(\overline{x}_1(t)-\overline{\hat{v}}_2(t)\right)+\mathbf{D}_B'\left(\overline{x}_1(t)-\overline{\hat{v}}_2(t)\right)\left(\overline{x}_1(t)-\overline{\hat{v}}_2(t)\right)\int\frac{\partial u}{\partial x_2}(\xi,t)m(\xi,t)d\xi\\
&-\mathbf{D}_B'\left(\overline{x}_1(t)-\overline{\hat{v}}_2(t)\right)\int\frac{\partial u}{\partial x_3}(\xi,t)m(\xi,t)d\xi=0,\\
-U_{IT}'(\hat{v}_3(x,t)) &- \frac{\partial u(x,t)}{\partial x_2}x_1+\frac{\partial u(x,t)}{\partial x_3}=0,\\
U_C'(\hat{v}_4(x,t)) &- \frac{\partial u(x,t)}{\partial x_2}=0.
\end{aligned}
\tag{3.40}
$$

Next,

$$
\begin{aligned}
&f(x,m(t),\hat{v}(x,t),\hat{v}(\cdot,t)*m(t))\\
&=-U_I(\hat{v}_1(x,t))-U_I(\hat{v}_2(x,t))-U_{IT}(\hat{v}_3(x,t))+U_C(\hat{v}_4(x,t))-U_Z(x_3^+)-U_{SI}(x_3^-)-U_{SC}(x_2^-)
\end{aligned}
$$

$$
\int\frac{\partial f}{\partial m}\Big((\xi,m(t),\hat{v}(\xi,t),\hat{v}(\cdot,t)*m(t))\Big)(x)m(\xi,t)d\xi=0,
$$

$$
\int\frac{\partial f}{\partial \mu}\Big((\xi,m(t),\hat{v}(\xi,t),\hat{v}(\cdot,t)*m(t))\Big)(\hat{v}(x,t))m(\xi,t)d\xi=0,
$$

$$
\begin{aligned}
&\int D_\xi u(\xi,t)\cdot\frac{\partial g}{\partial m}\Big((\xi,m(t),\hat{v}(\xi,t),\hat{v}(\cdot,t)*m(t))\Big)(x)m(\xi,t)d\xi\\
&=-x_1\mathbf{D}_S'\left(-\overline{x}_1(t)-\overline{\hat{v}}_1(t)\right)\left(\overline{x}_1(t)+\overline{\hat{v}}_1(t)\right)\int\frac{\partial u}{\partial x_2}(\xi,t)m(\xi,t)d\xi\\
&-x_1\mathbf{D}_B'\left(\overline{x}_1(t)-\overline{\hat{v}}_2(t)\right)\left(\overline{x}_1(t)-\overline{\hat{v}}_2(t)\right)\int\frac{\partial u}{\partial x_2}(\xi,t)m(\xi,t)d\xi\\
&+x_1\mathbf{D}_S'\left(-\overline{x}_1(t)-\overline{\hat{v}}_1(t)\right)\int\frac{\partial u}{\partial x_3}(\xi,t)m(\xi,t)d\xi+x_1\mathbf{D}_B'\left(\overline{x}_1(t)-\overline{\hat{v}}_2(t)\right)\int\frac{\partial u}{\partial x_3}(\xi,t)m(\xi,t)d\xi,
\end{aligned}
$$

$$
\begin{aligned}
&\int D_\xi u(\xi,t)\cdot\frac{\partial g}{\partial \mu}\Big((\xi,m(t),\hat{v}(\xi,t),\hat{v}(\cdot,t)*m(t))\Big)(\hat{v}(x,t))m(\xi,t)d\xi\\
&=-\hat{v}_1(x,t)\mathbf{D}_S'\left(-\overline{x}_1(t)-\overline{\hat{v}}_1(t)\right)\left(\overline{x}_1(t)+\overline{\hat{v}}_1(t)\right)\int\frac{\partial u}{\partial x_2}(\xi,t)m(\xi,t)d\xi\\
&+\hat{v}_2(x,t)\mathbf{D}_B'\left(\overline{x}_1(t)-\overline{\hat{v}}_2(t)\right)\left(\overline{x}_1(t)-\overline{\hat{v}}_2(t)\right)\int\frac{\partial u}{\partial x_2}(\xi,t)m(\xi,t)d\xi\\
&+\hat{v}_1(x,t)\mathbf{D}_S'\left(-\overline{x}_1(t)-\overline{\hat{v}}_1(t)\right)\int\frac{\partial u}{\partial x_3}(\xi,t)m(\xi,t)d\xi\\
&-\hat{v}_2(x,t)\mathbf{D}_B'\left(\overline{x}_1(t)-\overline{\hat{v}}_2(t)\right)\int\frac{\partial u}{\partial x_3}(\xi,t)m(\xi,t)d\xi,
\end{aligned}
$$

The HJB equation (3.10) becomes

$$
\begin{cases}
-\dfrac{\partial u}{\partial t} - \dfrac{1}{2}\beta^2(x_1)\dfrac{\partial^2 u}{\partial x_1^2} - \dfrac{\partial u}{\partial x_1}\alpha(x_1) \\
-\dfrac{\partial u}{\partial x_2}\Big(rx_2 + (x_1 + \hat{v}_1(x,t))\mathbf{D}_S\big(-\overline{x}_1(t) - \overline{\hat{v}}_1(t)\big) \\
-(x_1 - \hat{v}_2(x,t))\mathbf{D}_B\big(\overline{x}_1(t) - \overline{\hat{v}}_2(t)\big) - \hat{v}_3(x,t)x_1 - \hat{v}_4(x,t)\Big) \\
-\dfrac{\partial u}{\partial x_3}\Big(-\mathbf{D}_S\big(-\overline{x}_1(t) - \overline{\hat{v}}_1(t)\big) + \mathbf{D}_B\big(\overline{x}_1(t) - \overline{\hat{v}}_2(t)\big) + \hat{v}_3(x,t)\Big) + ru \\
= -U_I(\hat{v}_1(x,t)) - U_I(\hat{v}_2(x,t)) - U_{IT}(\hat{v}_3(x,t)) + U_C(\hat{v}_4(x,t)) - U_Z(x_3^+) - U_{SI}(x_3^-) - U_{SC}(x_2^-) \\
-(-x_1 - \hat{v}_1(x,t))\mathbf{D}'_S\left(-\overline{x}_1(t) - \overline{\hat{v}}_1(t)\right)\left(-\overline{x}_1(t) - \overline{\hat{v}}_1(t)\right)\displaystyle\int \dfrac{\partial u}{\partial x_2}(\xi,t)m(\xi,t)d\xi \\
-(x_1 - \hat{v}_2(x,t))\mathbf{D}'_B\left(\overline{x}_1(t) - \overline{\hat{v}}_2(t)\right)\left(\overline{x}_1(t) - \overline{\hat{v}}_2(t)\right)\displaystyle\int \dfrac{\partial u}{\partial x_2}(\xi,t)m(\xi,t)d\xi \\
-(-x_1 - \hat{v}_1(x,t))\mathbf{D}'_S\left(-\overline{x}_1(t) - \overline{\hat{v}}_1(t)\right)\displaystyle\int \dfrac{\partial u}{\partial x_3}(\xi,t)m(\xi,t)d\xi \\
+(x_1 - \hat{v}_2(x,t))\mathbf{D}'_B\left(\overline{x}_1(t) - \overline{\hat{v}}_2(t)\right)\displaystyle\int \dfrac{\partial u}{\partial x_3}(\xi,t)m(\xi,t)d\xi \\
u(x,T) = U_{H_0}(x_2) + U_{H_1}(x_3).
\end{cases}
$$

Using the Euler conditions (3.40), we rewrite the HJB equation:

$$
\begin{cases}
-\dfrac{\partial u}{\partial t} - \dfrac{1}{2}\beta^2(x_1)\dfrac{\partial^2 u}{\partial x_1^2} - \dfrac{\partial u}{\partial x_1}\alpha(x_1) - \dfrac{\partial u}{\partial x_2}(rx_2) \\
-\dfrac{\partial u}{\partial x_3}\Big(-\mathbf{D}_S\big(-\overline{x}_1(t) - \overline{\hat{v}}_1(t)\big) + \mathbf{D}_B\big(\overline{x}_1(t) - \overline{\hat{v}}_2(t)\big)\Big) + ru \\
= \hat{v}_1(x,t)U'_I(\hat{v}_1(x,t)) - U_I(\hat{v}_1(x,t)) + \hat{v}_2(x,t)U'_I(\hat{v}_2(x,t)) \\
-U_I(\hat{v}_2(x,t)) + \hat{v}_3(x,t)U'_{IT}(\hat{v}_3(x,t)) - U_{IT}(\hat{v}_3(x,t)) \\
-\hat{v}_4(x,t)U'_C(\hat{v}_4(x,t)) + U_C(\hat{v}_4(x,t)) - U_Z(x_3^+) - U_{SI}(x_3^-) - U_{SC}(x_2^-) + x_1U'_I(\hat{v}_1(x,t)) - x_1U'_I(\hat{v}_2(x,t)) \\
u(x,T) = U_{H_0}(x_2) + U_{H_1}(x_3).
\end{cases}
$$

For the derivation of the optimal solution, we shall make the following assumption for the utility function:

$$
U_I(v(x,t)) = \frac{a_0}{2}v^2(x,t);\ U_{IT}(v(x,t)) = \frac{k_0}{2}v^2(x,t);\ U_C(v(x,t)) = c_1v(x,t) - \frac{c_0}{2}v^2(x,t)
$$

and

$$
U_Z(x(t)) = q_1x(t);\ U_{SI}(x(t)) = q_0x(t);\ U_{SC}(x(t)) = p_0x(t);\ U_{H_0}(x(t)) = h_0x(t);\ U_{H_1}(x(t)) = h_1x(t),
$$

where a_0, c_0, c_1, h_0, h_1, k_0, p_0, q_0, and q_1 are positive constant coefficient. Then,

$$\begin{cases} -\dfrac{\partial u}{\partial t} - \dfrac{1}{2}\beta^2(x_1)\dfrac{\partial^2 u}{\partial x_1^2} - \dfrac{\partial u}{\partial x_1}\alpha(x_1) - \dfrac{\partial u}{\partial x_2}(rx_2) \\ -\dfrac{\partial u}{\partial x_3}\Big(- \mathbf{D}_S\big(- \overline{x}_1(t) - \overline{v}_1(t)\big) + \mathbf{D}_B\big(\overline{x}_1(t) - \overline{v}_2(t)\big)\Big) + ru \\ = \dfrac{a_0}{2}\Big|x_1 + \hat{v}_1(x,t)\Big|^2 + \dfrac{a_0}{2}\Big|x_1 - \hat{v}_2(x,t)\Big|^2 + \dfrac{k_0}{2}\hat{v}_3^2(x,t) + \dfrac{c_0}{2}\hat{v}_4^2(x,t) - q_1 x_3^+ - q_0 x_3^- - p_0 x_2^- - a_0 x_1^2 \\ u(x,T) = h_0 x_2 + h_1 x_3. \end{cases} \tag{3.41}$$

and the FP equation is

$$\begin{cases} \dfrac{\partial m}{\partial t} - \dfrac{1}{2}\dfrac{\partial^2}{\partial x_1^2}(\beta^2(x_1)m) + \dfrac{\partial}{\partial x_1}\Big(\big(\alpha(x_1)\big)m\Big) \\ +\dfrac{\partial}{\partial x_2}\Big(\big(rx_2 + (x_1 + \hat{v}_1(x,t))\mathbf{D}_S\big(- \overline{x}_1(t) - \overline{v}_1(t)\big) \\ -(x_1 - \hat{v}_2(x,t))\mathbf{D}_B\big(\overline{x}_1(t) - \overline{v}_2(t)\big) - \hat{v}_3(x,t)x_1 - \hat{v}_4(x,t)\big)m\Big) \\ +\dfrac{\partial}{\partial x_3}\Big(\big(- \mathbf{D}_S\big(- \overline{x}_1(t) - \overline{v}_1(t)\big) + \mathbf{D}_B\big(\overline{x}_1(t) - \overline{v}_2(t)\big) + \hat{v}_3(x,t)\big)m\Big) = 0, \\ m(x,0) = \delta(x)\bigotimes m_0(x), \end{cases} \tag{3.42}$$

In addition, the Euler conditions become

$$\begin{aligned} -a_0\hat{v}_1(x,t) + &\frac{\partial u(x,t)}{\partial x_2}\mathbf{D}_S\left(-\overline{x}_1(t) - \overline{v}_1(t)\right) + \mathbf{D}'_S\left(-\overline{x}_1(t) - \overline{v}_1(t)\right)\left(-\overline{x}_1(t) - \overline{v}_1(t)\right)\int \frac{\partial u}{\partial x_2}(\xi,t)m(\xi,t)d\xi \\ &+ \mathbf{D}'_S\left(-\overline{x}_1(t) - \overline{v}_1(t)\right)\int \frac{\partial u}{\partial x_3}(\xi,t)m(\xi,t)d\xi = 0, \\ -a_0\hat{v}_2(x,t) + &\frac{\partial u(x,t)}{\partial x_2}\mathbf{D}_B\left(\overline{x}_1(t) - \overline{v}_2(t)\right) + \mathbf{D}'_B\left(\overline{x}_1(t) - \overline{v}_2(t)\right)\left(\overline{x}_1(t) - \overline{v}_2(t)\right)\int \frac{\partial u}{\partial x_2}(\xi,t)m(\xi,t)d\xi \\ &- \mathbf{D}'_B\left(\overline{x}_1(t) - \overline{v}_2(t)\right)\int \frac{\partial u}{\partial x_3}(\xi,t)m(\xi,t)d\xi = 0, \\ -k_0\hat{v}_3(x,t) - &\frac{\partial u(x,t)}{\partial x_2}x_1 + \frac{\partial u(x,t)}{\partial x_3} = 0, \\ c_1 - c_0\hat{v}_4(x,t) - &\frac{\partial u(x,t)}{\partial x_2} = 0. \end{aligned} \tag{3.43}$$

We define

$$\lambda_{x_2}(x,t) = \frac{\partial u}{\partial x_2}(x,t)$$

Differentiating (3.41) with respect to x_2, we obtain

$$
\begin{cases}
-\dfrac{\partial \lambda_{x_2}}{\partial t} - \dfrac{1}{2}\beta^2(x_1)\dfrac{\partial^2 \lambda_{x_2}}{\partial x_1^2} - \dfrac{\partial \lambda_{x_2}}{\partial x_1}\alpha(x_1) - \dfrac{\partial \lambda_{x_2}}{\partial x_2}(rx_2) \\
-\dfrac{\partial \lambda_{x_2}}{\partial x_3}\Big(- \mathbf{D}_S\big(- \overline{x}_1(t) - \overline{\hat{v}}_1(t)\big) + \mathbf{D}_B\big(\overline{x}_1(t) - \overline{\hat{v}}_2(t)\big)\Big) \\
= a_0(x_1 + \hat{v}_1(x,t))\dfrac{\partial \hat{v}_1(x,t)}{\partial x_2} - a_0(x_1 - \hat{v}_2(x,t))\dfrac{\partial \hat{v}_2(x,t)}{\partial x_2} \\
+k_0\hat{v}_3(x,t)\frac{\partial \hat{v}_3(x,t)}{\partial x_2} + c_0\hat{v}_4(x,t)\frac{\partial \hat{v}_4(x,t)}{\partial x_2} - p_0\mathbf{1}_{x_2<0} \\
\lambda_{x_2}(x,T) = h_0.
\end{cases}
$$

From the Euler condition, we have

$$
\begin{aligned}
&-a_0\frac{\partial \hat{v}_1(x,t)}{\partial x_2} + \frac{\partial \lambda_{x_2}(x,t)}{\partial x_2}\mathbf{D}_S\left(-\overline{x}_1(t) - \overline{\hat{v}}_1(t)\right) = 0, \\
&-a_0\frac{\partial \hat{v}_2(x,t)}{\partial x_2} + \frac{\partial \lambda_{x_2}(x,t)}{\partial x_2}\mathbf{D}_B\left(\overline{x}_1(t) - \overline{\hat{v}}_2(t)\right) = 0, \\
&-k_0\frac{\partial \hat{v}_3(x,t)}{\partial x_2} - \frac{\lambda_{x_2}(x,t)}{\partial x_2}x_1 + \frac{\lambda_{x_2}(x,t)}{\partial x_3} = 0, \\
&-c_0\frac{\partial \hat{v}_4(x,t)}{\partial x_2} - \frac{\lambda_{x_2}(x,t)}{\partial x_2} = 0.
\end{aligned}
$$

Eventually, we get

$$
\begin{cases}
-\dfrac{\partial \lambda_{x_2}}{\partial t} - \dfrac{1}{2}\beta^2(x_1)\dfrac{\partial^2 \lambda_{x_2}}{\partial x_1^2} - \dfrac{\partial \lambda_{x_2}}{\partial x_1}\alpha(x_1) \\
-\dfrac{\partial \lambda_{x_2}}{\partial x_2}\Big(rx_2 + (x_1 + \hat{v}_1(x,t))\mathbf{D}_S\big(- \overline{x}_1(t) - \overline{\hat{v}}_1(t)\big) \\
-(x_1 - \hat{v}_2(x,t))\mathbf{D}_B\big(\overline{x}_1(t) - \overline{\hat{v}}_2(t)\big) - \hat{v}_3(x,t)x_1 - \hat{v}_4(x,t)\Big) \\
-\dfrac{\partial \lambda_{x_2}}{\partial x_3}\Big(- \mathbf{D}_S\big(- \overline{x}_1(t) - \overline{\hat{v}}_1(t)\big) + \mathbf{D}_B\big(\overline{x}_1(t) - \overline{\hat{v}}_2(t)\big) + \hat{v}_3(x,t)\Big) = -p_0\mathbf{1}_{x_2<0} \\
\lambda_{x_2}(x,T) = h_0.
\end{cases}
\tag{3.44}
$$

In addition, we define

$$
\lambda_{x_3}(x,t) = \frac{\partial u}{\partial x_3}(x,t)
$$

Then,

$$\begin{cases} -\dfrac{\partial \lambda_{x_3}}{\partial t} - \dfrac{1}{2}\beta^2(x_1)\dfrac{\partial^2 \lambda_{x_3}}{\partial x_1^2} - \dfrac{\partial \lambda_{x_3}}{\partial x_1}\alpha(x_1) \\ -\dfrac{\partial \lambda_{x_3}}{\partial x_3}\Big(rx_2 + (x_1 + \hat{v}_1(x,t))\mathbf{D}_S\big(- \overline{x}_1(t) - \overline{\hat{v}}_1(t)\big) \\ -(x_1 - \hat{v}_2(x,t))\mathbf{D}_B\big(\overline{x}_1(t) - \overline{\hat{v}}_2(t)\big) - \hat{v}_3(x,t)x_1 - \hat{v}_4(x,t)\Big) \\ -\dfrac{\partial \lambda_{x_3}}{\partial x_3}\Big(- \mathbf{D}_S\big(- \overline{x}_1(t) - \overline{\hat{v}}_1(t)\big) + \mathbf{D}_B\big(\overline{x}_1(t) - \overline{\hat{v}}_2(t)\big) + \hat{v}_3(x,t)\Big) = -q_0\mathbf{1}_{x_3<0} - q_1\mathbf{1}_{x_3>0} \\ \lambda_{x_3}(x,T) = h_1. \end{cases} \tag{3.45}$$

From (3.42) with (3.44) and (3.45), we can infer

$$\begin{aligned} \overline{\lambda}_{x_2}(t) &= \int \lambda_{x_2}(\xi,t)m(\xi,t)d\xi = \int \lambda_{x_2}(\xi,T)m(\xi,T)d\xi - p_0\mathbf{1}_{x_2<0}(T-t) = h_0 - p_0\mathbf{1}_{x_2<0}(T-t) \\ \overline{\lambda}_{x_3}(t) &= \int \lambda_{x_3}(\xi,t)m(\xi,t)d\xi = \int \lambda_{x_3}(\xi,T)m(\xi,T)d\xi - q_0\mathbf{1}_{x_3<0}(T-t) - q_1\mathbf{1}_{x_3>0}(T-t) \\ &= h_1 - q_0\mathbf{1}_{x_3<0}(T-t) - q_1\mathbf{1}_{x_3>0}(T-t) \end{aligned}$$

We assume that the evolution of a prevailing price of an asset is described by the mean revering model as below:

$$\alpha(x_1,t) = -\alpha_1(x_1 - \gamma_1(t))$$

, where $\gamma_1(t)$ is deterministic function and we take $\beta(x_1) = \beta$. Then, we can easily obtain

$$\overline{x}_1(t) = \overline{x}_{10}\exp -\alpha_1 t + \int_0^t \exp -\alpha_1(t-s)(\alpha_1\gamma_1(s)ds + \beta dw(s)).$$

From the Euler conditions (3.43), we have

$$\begin{aligned} -a_0\overline{\hat{v}}_1(t) + \mathbf{D}_S\left(-\overline{x}_1(t) - \overline{\hat{v}}_1(t)\right)\overline{\lambda}_{x_2}(t) + \mathbf{D}_S'\left(-\overline{x}_1(t) - \overline{\hat{v}}_1(t)\right)\left(-\overline{x}_1(t) - \overline{\hat{v}}_1(t)\right)\overline{\lambda}_{x_2}(t) \\ +\mathbf{D}_S'\left(-\overline{x}_1(t) - \overline{\hat{v}}_1(t)\right)\overline{\lambda}_{x_3}(t) = 0, \\ -a_0\overline{\hat{v}}_2(t) + \mathbf{D}_B\left(\overline{x}_1(t) - \overline{\hat{v}}_2(t)\right)\overline{\lambda}_{x_2}(t) + \mathbf{D}_B'\left(\overline{x}_1(t) - \overline{\hat{v}}_2(t)\right)\left(\overline{x}_1(t) - \overline{\hat{v}}_2(t)\right)\overline{\lambda}_{x_2}(t) \\ -\mathbf{D}_B'\left(\overline{x}_1(t) - \overline{\hat{v}}_2(t)\right)\overline{\lambda}_{x_3}(t) = 0. \end{aligned}$$

Consider the linear demand in which

$$\mathbf{D}_S(x) = d_{S_0}x + d_{S_1} \text{ and } \mathbf{D}_B(x) = d_{B_0}x + d_{B_1},$$

where d_{S_0}, d_{S_1}, d_{B_0}, and d_{B_1} are positive constant coefficients. Then, we obtain

$$\bar{\hat{v}}_1(t) = \frac{-2d_{S_0}\overline{x}_1(t)\overline{\lambda}_{x_2}(t) + d_{S_0}\overline{\lambda}_{x_3}(t) + d_{S_1}\overline{\lambda}_{x_2}(t)}{a_0 + 2d_{S_0}\overline{\lambda}_{x_2}(t)};$$
$$\bar{\hat{v}}_2(t) = \frac{2d_{B_0}\overline{x}_1(t)\overline{\lambda}_{x_2}(t) - d_{B_0}\overline{\lambda}_{x_3}(t) + d_{B_1}\overline{\lambda}_{x_2}(t)}{a_0 + 2d_{B_0}\overline{\lambda}_{x_2}(t)}.$$

In addition, the optimal feedback is defined:

$$\begin{aligned}
\hat{v}_1(x,t) &= \frac{1}{a_0}\Big(d_{S_0}\big(-\overline{x}_1(t) - \bar{\hat{v}}_1(t)\big)\big(\lambda_{x_2}(x,t) + \overline{\lambda}_{x_2}(t)\big) + d_{S_0}\overline{\lambda}_{x_3}(t) + d_{S_1}\lambda_{x_2}(x,t)\Big)\\
\hat{v}_2(x,t) &= \frac{1}{a_0}\Big(d_{B_0}\big(\overline{x}_1(t) - \bar{\hat{v}}_2(t)\big)\big(\lambda_{x_2}(x,t) + \overline{\lambda}_{x_2}(t)\big) - d_{B_0}\overline{\lambda}_{x_3}(t) + d_{B_1}\lambda_{x_2}(x,t)\Big)\\
\hat{v}_3(x,t) &= \frac{1}{k_0}\Big(-\lambda_{x_2}(x,t)x_1 + \lambda_{x_3}(x,t)\Big)\\
\hat{v}_4(x,t) &= \frac{1}{c_0}\Big(c_1 - \lambda_{x_2}(x,t)\Big)
\end{aligned} \tag{3.46}$$

In addition, we set

$$\varphi_{S_0}(t) = \frac{d_{S_0}\big(-\overline{x}_1(t) - \bar{\hat{v}}_1(t)\big) + d_{S_1}}{a_0} \text{ and } \varphi_{B_0}(t) = \frac{d_{B_0}\big(\overline{x}_1(t) - \bar{\hat{v}}_2(t)\big) + d_{B_1}}{a_0};$$
$$\varphi_{S_1}(t) = \frac{d_{S_0}\big(-\overline{x}_1(t) - \bar{\hat{v}}_1(t)\big)}{a_0}\overline{\lambda}_{x_2}(t) + \frac{d_{S_0}}{a_0}\overline{\lambda}_{x_3}(t) \text{ and } \varphi_{B_1}(t) = \frac{d_{B_0}\big(\overline{x}_1(t) - \bar{\hat{v}}_2(t)\big)}{a_0}\overline{\lambda}_{x_2}(t) - \frac{d_{B_0}}{a_0}\overline{\lambda}_{x_3}(t)$$

Then, the optimal feedback is

$$\begin{aligned}
\hat{v}_1(x,t) &= \varphi_{S_0}(t)\lambda_{x_2}(x,t) + \varphi_{S_1}(t)\\
\hat{v}_2(x,t) &= \varphi_{B_0}(t)\lambda_{x_2}(x,t) + \varphi_{B_1}(t)\\
\hat{v}_3(x,t) &= -k_0^{-1}\lambda_{x_2}(x,t)x_1 + k_0^{-1}\lambda_{x_3}(x,t)\\
\hat{v}_4(x,t) &= c_0^{-1}c_1 - c_0^{-1}\lambda_{x_2}(x,t)
\end{aligned} \tag{3.47}$$

Once we know $\lambda_{x_2}(x,t)$ and $\lambda_{x_3}(x,t)$, everything is explicit. From the Lemma 3.1 and Lemma 3.2, we can infer that

$$\lambda_{x_2}(x,t) = \overline{\lambda}_{x_2}(t) \text{ and } \lambda_{x_3}(x,t) = \overline{\lambda}_{x_3}(t).$$

Then, we can rewrite the optimal feedback (3.47) as below:

$$\begin{aligned}
\hat{v}_1(x,t) &= \varphi_{S_0}(t)\overline{\lambda}_{x_2}(t) + \varphi_{S_1}(t)\\
\hat{v}_2(x,t) &= \varphi_{B_0}(t)\overline{\lambda}_{x_2}(t) + \varphi_{B_1}(t)\\
\hat{v}_3(x,t) &= -k_0^{-1}\overline{\lambda}_{x_2}(t)x_1 + k_0^{-1}\overline{\lambda}_{x_3}(t)\\
\hat{v}_4(x,t) &= c_0^{-1}c_1 - c_0^{-1}\overline{\lambda}_{x_2}(t),
\end{aligned}$$

where

$$\overline{\lambda}_{x_2}(t) = h_0 - p_0 \mathbf{1}_{x_2<0}(T-t) \text{ and } \overline{\lambda}_{x_3}(t) = h_1 - q_0 \mathbf{1}_{x_3<0}(T-t) - q_1 \mathbf{1}_{x_3>0}(T-t). \blacksquare$$

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Research Projects

"A Risk Extended Version of Merton's Optimal Consumption and Portfolio Selection." With Alain Bensoussan, Celine Hoe, and Zhongfeng Yan; *under 2nd round review (major revision) at Operations Research.*
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Publications

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Operations Management — Fall 2019; Spring 2020 and 2021, UT Dallas

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