INVERSION OF THE COVERING MAP FOR THE INDEFINITE SPIN GROUPS

by

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To my amazing mother, Alice Osei and my beloved son, Bernard Osei Adjei.

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INVERSION OF THE COVERING MAP FOR THE INDEFINITE SPIN GROUPS

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The University of Texas at Dallas, 2017

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The double covering of Orthogonal groups by the Spin groups is vital for many applications. In this dissertation, we address the important question of inverting the covering map $\phi_{p,q} : Spin^{+}(p, q) \rightarrow SO^{+}(p, q)$ in the indefinite case. We also develop explicit formulae for the forward map $\phi_{p,q} : Spin^{+}(p, q) \rightarrow SO^{+}(p, q)$, for $(p, q) \in \{(1, 2), (1, 3), (2, 3)\}$. We do not work with the even subalgebra of $Cl(p, q)$ and thus our formulae are quite explicit. Our method relies significantly on standard Givens and hyperbolic Givens decompositions.

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CHAPTER 1

INTRODUCTION

Clifford Algebra has shown to be a highly efficient and powerful language available for solving problems in mathematical physics, control engineering, group representation theory, signal and image processing, numerical analysis, integral transforms as well as Lie algebras, geographic information systems, cosmology, medical imaging, and neural computation. Most often, it is easy to work in some matrix algebra that is isomorphic to a Clifford algebra, and there are known explicit matrix realizations of $Cl(p,q)$ for any (p,q) [9] [14]. Therefore, the spin groups also have standard representations as matrix groups. For $q \geq 1$ and $p \geq 1$ the even subalgebra of $Cl(p,q)$ is isomorphic to $Cl(p,q-1)$ and $Cl(q,p-1)$ respectively. $Spin^+(p,q)$ being a subset of the even subalgebra can be viewed as living in the matrix algebra corresponding to the even subalgebra. However this representation of $Spin^+(p,q)$ of the matrix algebra does not lead to an identification of the Spin group as a subset of $Cl(p,q)$. This is inconvenient for many applications. Therefore we avoid usage of this expression.

In [1] by Emily Herzig and Viswanath Ramakrishna, the spin homomorphism $\phi_{p,q} : Spin^+(p,q) \to SO^+(p,q)$ of classical matrix Lie groups was explicitly formulated for several (p,q) with $3 \leq p+q \leq 6$. Further the double covering map from the definite $Spin(0,n)$ to $SO(n,\mathbb{R})$ was inverted for $n = 5, 6$ [2] which, among other applications, makes possible a parametrization of $SO(5,\mathbb{R})$ and $SO(6,\mathbb{R})$. In this dissertation, we extend the explicit construction of indefinite spin groups to Clifford algebras with $(p,q) \in \{(1,2),(1,3),(2,3)\}$. We also invert the covering map for several important cases of (p,q). The applications of the indefinite spin group, the double covering map,and its inversion are manifold in the field of mathematical physics. For instance, the Lorentz group of special relativity is $SO(3,1,\mathbb{R})$. The study of hydrogen atom employs the $SO^+(4,2)$ and $SO^+(3,2)$.

Let us briefly describe how we are going to proceed of the task of inverting the forward map:

(i) We use the work by Emily Herzig and Viswanath Ramakrishna [1] to explicitly construct the forward map $\phi_{p,q}$ as a matrix of quadratic maps taking values in the indefinite orthogonal groups. This requires several intricate calculations.

(ii) Therefore, the question of inverting $\phi_{p,q}$ is reduce to solving a large dimensional system of quadratic equations. Further, we want the inversion formulae to be explicitly parameterized by the target matrix in $SO^+(p,q)$.

(iii) To ameliorate this, we make use of the fact that $\phi_{p,q}$ is a group homomorphism and therefore it suffices to solve the systems of equations in (ii) when the target is a factor in some decomposition of elements of the indefinite orthogonal groups. For us the most useful such factors are the hyperbolic Givens or standard Givens.

For the last chapter in the dissertation, we begin with a low dimensional $Cl(p,q)$ and use an iterative construction to extend $Cl(p+1,q+1)$.

The organization of the dissertation is as follows. In chapter 2, we present preliminary background on Clifford algebras, spin groups, hyperbolic Givens and standard Givens, quaternions, and other facts needed for the study. In chapters 3, we develop explicit formulations of the forward map for $(p,q) \in \{(2,1),(2,2),(3,1),(3,2),(3,3)\}$. Chapter 4, fully addresses the important question of inverting the forward maps for each component of $Spin^+(p,q)$ for $(p,q) \in \{(2,1),(2,2),(3,1),(3,2),(3,3)\}$. Finally, in chapter 5, we provide details of a direct approach to the indefinite spin groups for $(p,q) \in \{(1,2),(0,2),(1,3),(2,3)\}$.

CHAPTER 2

PRELIMINARIES

2.1 Clifford Algebras

Associated to any vector space V with symmetric bilinear form B is a Clifford algebra *Cl(V ; B)*. In the special case $B = 0$, the Clifford algebra is just the exterior algebra $\wedge(V)$. Let V be a finite-dimensional vector space over a field $\mathbb{K}$, with a symmetric bilinear form $B : V \times V \to \mathbb{K}$. The Clifford algebra *Cl(V; B)* is the quotient map $Cl(V;B) = \frac{T(V)}{I(V;B)}$, where *T(V)* $\subset I(V;B)$ is the two-sided ideal generated by all elements of the form

$v \otimes w + w \otimes v = 2B(v,w)1\ \forall v, w \in V$. So, given an associative algebras $\mathcal{A}$ and $\mathcal{B}$ over $\mathbb{C}$ with unit and a linear maps $\tau : V \to \mathcal{A}$ and $\rho : V \to \mathcal{B}$ such that $\rho(v)\rho(w)+\rho(w)\rho(v) = 2B(v,w)1$ $\forall v, w \in V$,there exists an associative algebra homomorphism $\lambda : \mathcal{A} \to B$ such that $\rho = \lambda \circ \tau$.

Thus, $\tau(\mathcal{A})$ generates $\mathcal{A}$ and $\mathcal{A}$ is called a Clifford algebra for *Cl(V; B)*. The existence of Clifford algebra for any pair *(V, B)* is guaranteed and unique up to isomorphism (see [9] for details). Besides, if $\{v_i\}$ for $i = 1, 2..., n$ form a basis for V, then $\mathcal{A}$ is the algebra generated by $\{1, \tau(v_1), \tau(v_2), ..., \tau(v_n)\}$ due to the fact that B is bilinear. There are several alternative definitions for such algebra depending on application (see Clifford's original definition [5]). As this work seeks to formulate explicit construction of the indefinite spin group as $Spin^+(p,q)$ as a matrix subalgebra of a matrix algebra that *Cl(p,q)* is isomorphic to. So, our Clifford algebras will be realized as real,complex or quaternion matrix with explicitly given generating set. Therefore, our approach will rely on constructive formulations and definitions of a Clifford algebra.

Definition 1. Let p, q with $n = p + q$ be two nonnegative integers and suppose we find a collection of matrices $e_1, e_2, e_3, ..., e_p$ and $e_{p+1}, e_{p+2}, e_{p+3}, ..., e_n$ within some suitable ambient space $M(m, \mathbb{F})$ (a family of $m \times m$ matrices with entries in a field $\mathbb{F}$), and the algebra $\mathcal{A}$ generated by $\{I_m, e_1, e_2, e_3, ..., e_n\}$, such that:

(i) $e_i^2 = I_m \; \forall i = 1, 2, 3..., p$ and $e_i^2 = -I_m \; \forall i = p+1, p+2, p+3, ...n$,

(ii) $e_i e_j = -e_j e_i \; \forall i \neq j$, and

(iii) *Cl(p,q)* cannot be generated by any proper subset of $\{I_m, e_1, e_2, e_3, ..., e_n\}$.

Then $\mathcal{A}$ is the Clifford algebra and is denoted by *Cl(p,q)*. The algebra is expressible as a span over a field $\mathbb{F}$ of the following elements:

- the identity element I_m, called the 0-vector or scalar;
- n elements e_i, called the 1-vectors;
- $\binom{n}{k}$ elements $e_i e_j$, for distinct i, j, called the 2-vectors or bivectors;
- $\binom{n}{k}$ elements $e_i e_j e_k$, for distinct i, j,and k, called 3-vectors or trivectors;

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- the n-vectors or the volume element $e_1 e_2 ... e_n$. It follows from (iii) that, the basis for the algebra contains $\sum_{k=1}^{n} \binom{n}{k} = 2^n$ vectors. We refer to 0-vectors, 2-vectors, 4-vectors,etc as the even vectors, and 1-vectors, 3-vectors, 5-vectors,etc as the odd vectors. So the algebra decomposed into the direct sum : *Cl(p,q)* $= Cl^+(p,q) \oplus Cl^-(p,q)$, where the even(resp. odd) subalgebra of $Cl(p,q)$, denoted by $Cl^+(p,q)$ (resp $Cl^-(p,q)$) is the span of the even (resp. odd) vectors of $Cl(p,q)$. For any pair (p,q), a Clifford algebra $Cl(p,q)$ can be realized as a particular matrix algebra $M(m,\mathbb{F})$, where $\mathbb{F} = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$ or as a double ring of one of these algebras. Nevertheless, whereas the algebra containing $Cl(p,q)$ is known, the explicit constructions of the spin groups within that algebra relies intricately on the particular choice of 1-vectors. As such, the choice of 1-vector used will be specified whenever a particular construction is pertinent to our results.

Definition 2.

(i) Define a map ϕ^{cc} on the basis elements with $\phi^{cc}(I) = I$, and $\phi^{cc}(v) = -v$ for all 1-vectors v, and $\phi^{cc}(vw) = \phi^{cc}(w)\phi^{cc}(v)$ for any basis vectors v and w. Then, by linearity, extend ϕ^{cc} to all of $Cl(p,q)$. The function ϕ^{cc} is known as the **Clifford conjugation** antiautotomorphism.

(ii) Define a function ϕ^{rev} on the basis elements with $\phi^{rev}(I) = I$, $\phi^{rev}(v) = v, \quad \forall$ 1-vectors v, and $\phi^{rev}(vw) = \phi^{rev}(w)\phi^{rev}(v)$ for any basis vectors v and w. Then ,by linearity, extend ϕ^{rev} to all of $Cl(p,q)$. The map ϕ^{rev} is called the **reversion** antiautotomorphism.

(iii) Define a map $\phi^{gr} = \phi^{rev} \circ \phi^{cc}$ on $Cl(p,q)$. We refer to this map as the **grade** anti-autotomorphism. It satisfies $\phi^{gr}(I) = I$, $\phi^{gr}(v) = -v$ forall 1-vectors v, $\phi^{gr}(vw) = \phi^{gr}(v)\phi^{gr}(w)$ for any basis vectors v and w and extends linearly over $Cl(p,q)$.

For brevity, we denote $x^{cc} := \phi^{cc}(x)$, $x^{rev} := \phi^{rev}(x)$, and $x^{gr} := \phi^{gr}(x)$. Thus, $x \in Cl(p,q)$ is an even vector if and only if $x^{gr} = x$ and x is an odd vector if and only if $x^{gr} = -x$.

It is easy to formulate examples of low-dimensional Clifford algebras as isomorphic to known matrix algebras. For instance, $Cl(3,0)$ is $M(2,\mathbb{C})$, and [15] gives conditions for identifying a suitable matrix algebra for any $Cl(p,q)$. Nevertheless, as the size of the pair (p,q) increases, the task of finding practical family of 1- vectors in order to define the Clifford conjugation, reversion and the grade map as explicit matrix automorphisms becomes a hurdle. However, our task will be simplified by the following iteration denoted **IC**:

IC: Assume $Cl(p,q)$ has a known matrix representation generated by 1-vectors $\{e_1 \cdots e_p, f_1 \cdots f_q\}$, where $e_i^2 = 1$, $f_j^2 = -1 \; \forall i = 1, \cdots p$ and $j = 1, \cdots q$. Let $\widetilde{e_i}$ and $\widetilde{f_j}$ denote the set of 1-vectors for the new Clifford algebra whose squares are I and $-I$ respectively. Then $Cl(p+1, q+1)$ can be represented as $M(2, Cl(p,q))$ with 1-vectors defined as follows:

$\widetilde{e_i} = \sigma_z \otimes e_i, \; i = 1 \cdots p, \;\; \widetilde{e}_{p+1} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$; the identity $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, where 0 and 1 denote the zero and identity elements of $Cl(p,q)$.

The following result also tells us how to extend reversion and Clifford conjugation from that for (p, q) to $(p+1, q+1)$:

For $A, B, C, D \in Cl(p, q)$, let $X = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Cl(p+1, q+1)$. Then Clifford conjugation and reversion on $Cl(p+1, q+1)$ is given by:

$X^{cc} = \begin{pmatrix} D^{rev} & -B^{rev} \\ -C^{rev} & A^{rev} \end{pmatrix}$ and $X^{rev} = \begin{pmatrix} D^{cc} & B^{cc} \\ C^{cc} & A^{cc} \end{pmatrix}$ respectively.

Now our path to inversion of the covering map follows the steps below:

1. Identify a set of 1-vectors for Clifford algebra $Cl(p, q)$ and formulate the Clifford conjugation ϕ^{cc}, reversion ϕ^{rev}, and grade ϕ^{gr} map with respect to the 1-vectors.

2. Identify a representation of $Spin^{+}(p, q)$ consistent with ϕ^{cc}, ϕ^{rev} and ϕ^{gr}. Thus $Spin^{+}(p, q)$ should lie in the same algebra as $Cl(p, q)$.

   Steps 1 and 2 were achieved in reference [1].

3. Compute the explicit form, as a matrix of quadratic functions, of the forward map, a two-to-one, surjective homomorphism $\phi_{p,q} : Spin^{+}(p, q) \rightarrow SO^{+}(p, q)$

4. Finally, calculate the inverse map by solving an intricate system quadratic equations when the target matrix is standard Givens or hyperbolic Givens.

2.2 Quaternions

To define the quaternions, we first introduce the symbols i, j, k. These symbols satisfy the following properties: $i^2 = j^2 = k^2 = ijk = -1$, $ij = k$, $jk = i$ and $ki = j$. Any *quaternion* $q \in \mathbb{H}$ is an object of the form $a + bi + cj + dk$, where $a, b, c, d \in \mathbb{R}$. The real part of q is a and its imaginary part is a vector $(b, c, d) \in \mathbb{R}^3$. So, we define *pure quaternion* as $\mathcal{P} = \{q \in \mathbb{H} : a = 0\}$

The conjugate of a quaternion q is $\bar{q} = a - bi - cj - dk$ and its norm squared is $q\bar{q} = a^2 + b^2 + c^2 + d^2$. So the norm of q is defined to be $|q| = \sqrt{\bar{q}q}$. q is called a unit quaternion if $|q| = 1$.

It is also known that $\mathbb{H} \otimes \mathbb{H}$ is isomorphic to $M(4, \mathbb{R})$ [9]. We make significant use of the following matrices stemming from this isomorphism, $\mathbb{H} \otimes \mathbb{H} \cong M(4, \mathbb{R})$:

$$M_{1\otimes i} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} ; M_{1\otimes j} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} ; M_{1\otimes k} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} ,$$

2.3 Indefinite Orthogonal groups and Spin Groups

- Denote by $O(p,q)$ the set of real $n \times n$ matrices X satisfying $X^T I_{p,q} X = I_{p,q}$. Then $SO(p,q) = \{X \in O(p,q) : det(X) = 1\}$. This set is not connected for $pq \neq 0$. Therefore, we define $SO^+(p,q)$ as the connected component of the identity I of the set $\{X \in M(p+q, \mathbb{R}) : X^T I_{p,q} X = I_{p,q}, det(X) = 1\}$. Here $n = p+q$ and $I_{p,q} = \begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix}$. Unlike the case, $pq = 0$, the set is not connected. Therefore, there is no simple algebraic formulation of $SO^+(p,q)$.

- $Spin^+(p,q)$ is the set of all $X \in Cl(p,q)$ satisfying: i) X is even; ii) $X^{cc}X = I$; iii) The map $v \to XvX^{cc}$ leaves the space of 1-vectors invariant.

- The forward map $\phi_{p,q}(X)$ is the matrix of the linear map that sends $v \to xvx^{cc}$ for any 1-vector v.

2.4 Given-like Actions

Define $R = \begin{pmatrix} c & -s \\ s & c \end{pmatrix}$ where $c^2 + s^2 = 1$ and $H = \begin{pmatrix} a & b \\ b & a \end{pmatrix}$, for $a^2 - b^2 = 1$ respectively. It is well known that

- Given a vector $(x,y)^T$ there is an $R = \begin{pmatrix} c & -s \\ s & c \end{pmatrix}$ where $c^2+s^2 = 1$ such that $R(x,y)^T = (\sqrt{x^2+y^2}, 0)$

- Similarly given a vector $(x,y)^T$, with $\mid x \mid \geq \mid y \mid$, there is an $H = \begin{pmatrix} a & b \\ b & a \end{pmatrix}$, for $a^2 - b^2 = 1$ such that $H(x,y)^T = (\sqrt{x^2-y^2}, 0)$

R, H are called plane standard Givens and hyperbolic Givens respectively. Embedding R, resp. H as a principal submatrix of the identity matrix I_n, yields matrices known as standard Givens (respectively, Hyperbolic Givens)[19].

$R_{i,j}$($H_{i,j}$ respectively) stands for standard Givens (respectively, Hyperbolic Givens) obtained by embedding a plane standard Givens and hyperbolic Givens respectively in rows and columns (i, j). We now present an example which shows how elements of $SO^+(p, q)$ can be factored into products of standard Givens and hyperbolic Givens.

Let $X \in SO^+(2, 2)$. Consider the first column of X,

$$v_1 = \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$$

Since $X \in SO^+(2, 2)$, $a^2 + b^2 - c^2 - d^2 = 1$. Therefore there are $R_{1,2}, R_{3,4}$ such that the first column of $R_{1,2}, R_{3,4}X = \begin{pmatrix} \alpha \\ 0 \\ \beta \\ 0 \end{pmatrix}$, where $\alpha = \sqrt{a^2 + b^2}$ and $\beta = \sqrt{c^2 + d^2}$. Since $a^2 + b^2 - c^2 - d^2 = 1 = \alpha^2 - \beta^2$, it follows that $|\alpha| > |\beta|$. Hence there is an $H_{1,3}$ such that the first column of $H_{1,3}R_{1,2}R_{3,4}X$

$$v_1 = \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Since $H_{1,3}R_{1,2}R_{3,4}X \in SO^+(2, 2)$ also, it follows that the first row $H_{1,3}R_{1,2}R_{3,4}X$ is also $\begin{pmatrix} 1 & 0 & 0 & 0 \end{pmatrix}$. Therefore, the second column of the product $H_{1,3}R_{1,2}R_{3,4}X$ is of the form $\begin{pmatrix} 0 \\ b \\ c \\ d \end{pmatrix}$ since $b^2 - c^2 - d^2 = 1$. So there is an $R_{3,4}$ such that $R_{3,4}(c,d)^T = (\gamma, 0)^T$, where $\gamma^2 = c^2 + d^2$. As before $b^2 - \gamma^2 = 1$, so there is an $H_{2,3}$ such that $H_{2,3}$ with $H_{2,3}(b, \gamma)^T = (1, 0)$. So it follows that the first and the second column equal the first two standard unit vectors.

Since $H_{2,3}R_{3,4}H_{1,3}R_{1,2}R_{3,4}X \in S0^+(2,2)$, it follows that it equals

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & y_{33} & y_{34} \\ 0 & 0 & y_{43} & y_{44} \end{pmatrix}$$

Again the condition $H_{2,3}R_{3,4}H_{1,3}R_{1,2}R_{3,4}X \in S0^+(2,2)$, ensures that

$$\begin{pmatrix} y_{33} & y_{34} \\ y_{43} & y_{44} \end{pmatrix}$$

must itself be a plane standard Givens, whose inverse is, of course, is also a plane standard Givens. Therefore, pre-multiplying be the corresponding $R_{3,4}$ we get that

$$R_{3,4}H_{2,3}R_{3,4}H_{1,3}R_{1,2}R_{3,4}X = I_4$$

Since the inverse of each $R_{i,j}$ (respectively $H_{k,l}$) is itself an $R_{i,j}$ (respectively $H_{k,l}$), it follows that X can be expressed **constructively** as a product of $R_{3,4}, H_{2,3}, H_{1,3}, R_{1,2}$. The above factorization is the only way to factor an element of $SO^+(2,2)$ into a product of standard and hyperbolic Givens

CHAPTER 3

EXPLICIT FORMULATIONS OF THE MAP

In the work [1], explicit choice of 1-vectors were used to identify Clifford conjugation, reversion and grade map for several $Spin^+(p,q)$. In this chapter, we build on that work to explicitly describe $\phi_{p,q}(X)$ as a matrix of quadratic maps. Recall that the forward map $\phi_{p,q}(X)$ is the matrix of the linear map $v \to xvx^{cc}$, for any 1-vector v.

3.1 The double cover of $SO^+(2,1)$ by $Spin^+(2,1)$

Let $\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $\mathrm{Id}_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

Then $\{Y_1, Y_2, Y_3\}$, the set of 1-vectors for $Cl(2,1)$ is defined as follows:

$$Y_1 = \sigma_z \otimes \sigma_z = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \; Y_2 = \sigma_x \otimes \mathrm{Id}_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix},$$

$$Y_3 = i\sigma_x \otimes \mathrm{Id}_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}.$$

Let $\widetilde{Z} = \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix} \in SL(2,\mathbb{R})$ which is abstractly isomorphic to $Spin^+(2,1)$.

We embed $\widetilde{Z}$ in $Z \in Cl(2,1)$ as follows: $Z = \begin{pmatrix} x_1 & 0 & x_2 & 0 \\ 0 & -x_1 & 0 & x_2 \\ x_3 & 0 & x_4 & 0 \\ 0 & x_3 & 0 & -x_4 \end{pmatrix}$.

Note $g_1 = x_1x_4 - x_2x_3 - 1 = 0$. Then $Z \in Cl(2,1)$ and also in $Spin^+(2,1)$.

Denote the Gröbner basis by GB $= \{g_1\}$

Clearly

$$Z^{-1} = \begin{pmatrix} x_4 & 0 & -x_2 & 0 \\ 0 & -x_4 & 0 & -x_2 \\ -x_3 & 0 & x_1 & 0 \\ 0 & -x_3 & 0 & -x_1 \end{pmatrix}$$

We now compute modulo Gröbner basis GB $= \{g_1\}$:

$$U_1 = ZY_1Z^{-1} mod \mathrm{GB} = \begin{pmatrix} 1+2x_2x_3 & 0 & -2x_1x_2 & 0 \\ 0 & -(1+2x_2x_3) & 0 & -2x_1x_2 \\ 2x_3x_4 & 0 & -(1+2x_2x_3) & 0 \\ 0 & 2x_3x_4 & 0 & 1+2x_2x_3 \end{pmatrix}$$

,

$$U_2 = ZY_2Z^{-1} mod \mathrm{GB} = \begin{pmatrix} x_2x_4 - x_1x_3 & 0 & x_1^2 - x_2^2 & 0 \\ 0 & x_1x_3 - x_2x_4 & 0 & x_1^2 - x_2^2 \\ x_4^2 - x_3^2 & 0 & x_1x_3 - x_2x_4 & 0 \\ 0 & x_4^2 - x_3^2 & 0 & x_2x_4 - x_1x_3 \end{pmatrix}$$

$$U_3 = ZY_3Z^{-1} mod \mathrm{GB} = \begin{pmatrix} -(x_1x_3 + x_2x_4) & 0 & x_1^2 + x_2^2 & 0 \\ 0 & x_1x_3 + x_2x_4 & 0 & x_1^2 + x_2^2 \\ -(x_3^2 + x_4^2) & 0 & x_1x_3 + x_2x_4 & 0 \\ 0 & -(x_3^2 + x_4^2) & 0 & -(x_1x_3 + x_2x_4) \end{pmatrix}$$

,

Define the 3×3 matrix D:

$$D = \begin{pmatrix} d_{11} & d_{12} & d_{13} \\ d_{21} & d_{22} & d_{23} \\ d_{31} & d_{32} & d_{33} \end{pmatrix}$$

where

$$U_j = d_{1j}Y_1 + d_{2j}Y_2 + d_{3j}Y_3, \;\; j = 1, 2, 3,$$

an the entries of matrix D are given by

$$\begin{aligned} d_{ij} &= \frac{\langle U_j, Y_i \rangle}{\langle Y_i, Y_i \rangle} mod \text{GB}, \; i, j = 1, 2, 3 \text{ and} \\ \\ \langle A, B \rangle &= \operatorname{tr}\left(A^T B\right). \end{aligned}$$

Then, $D = \phi_{2,1}\left[\begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix}\right]$. We obtain the following matrix:

$$D = \begin{pmatrix} 1 + 2x_2x_3 & x_2x_4 - x_1x_3 & -\left(x_1x_3 + x_2x_4\right) \\ x_3x_4 - x_1x_2 & \frac{1}{2}\left(x_1^2 - x_2^2 - x_3^2 + x_4^2\right) & \frac{1}{2}\left(x_1^2 + x_2^2 - x_3^2 - x_4^2\right) \\ -\left(x_1x_2 + x_3x_4\right) & \frac{1}{2}\left(x_1^2 - x_2^2 + x_3^2 - x_4^2\right) & \frac{1}{2}\left(x_1^2 + x_2^2 + x_3^2 + x_4^2\right) \end{pmatrix}. \tag{3.1}$$

Theorem 3.1.

The forward map $\phi_{2,1} : Spin^+(2,1) \to SO^+(2,1)$ is the $\phi_{2,1}\left[\begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix}\right] = D$, where D is as in Equation (3.1)

3.2 The double cover of $SO^+(2,2)$ by $Spin^+(2,2)$

Let $\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $\mathrm{Id} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

Let $X_1 = \sigma_z \otimes \sigma_x = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$, $X_2 = \sigma_x \otimes \mathrm{Id} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$,

$X_3 = \sigma_z \otimes (i\sigma_y) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$, and $X_4 = (i\sigma_y) \otimes \mathrm{Id}$.

Then $\{X_1, X_2, X_3, X_4\}$ is the set of 1-vectors for $Cl(2,2)$.

Now let $\left[\widetilde{A} = \begin{pmatrix} x_1 & x_2 \\ x_7 & x_8 \end{pmatrix}, \widetilde{B} = \begin{pmatrix} x_3 & x_4 \\ x_5 & x_6 \end{pmatrix}\right] \in SL(2,\mathbb{R}) \times SL(2,\mathbb{R}) \cong Spin^+(2,2)$.

Let us embed $\widetilde{A}$,$\widetilde{B}$ in $Cl(2,2)$ as follows: $A = \begin{pmatrix} x_1 & 0 & 0 & x_2 \\ 0 & x_3 & x_4 & 0 \\ 0 & x_5 & x_6 & 0 \\ x_7 & 0 & 0 & x_8 \end{pmatrix}$;

Then the entries of matrix A satisfy the following identities:

Define: $h_1 = \det\left[\begin{pmatrix} x_1 & x_2 \\ x_7 & x_8 \end{pmatrix}\right] - 1 = x_1x_8 - x_2x_7 - 1$ and

$h_2 = \det\left[\begin{pmatrix} x_3 & x_4 \\ x_5 & x_6 \end{pmatrix}\right] - 1 = x_3x_6 - x_4x_5 - 1$.

Clearly, the Gröbner Basis for the ideal generated by these two polynomials is:

GB $= \{h_1,\ h_2\}$. Obviously

$$A^{-1} = \begin{pmatrix} x_8 & 0 & 0 & -x_2 \\ 0 & x_6 & -x_4 & 0 \\ 0 & -x_5 & x_3 & 0 \\ -x_7 & 0 & 0 & x_1 \end{pmatrix}$$

Next, we compute matrices

$$C_i = AX_iA^{-1} mod \mathrm{GB},\ i = 1, 2, 3, 4.$$

We have

$$\begin{aligned}
C_1 &= \begin{pmatrix} 0 & x_2x_5 + x_1x_6 & -x_2x_3 - x_1x_4 & 0 \\ x_4x_7 + x_3x_8 & 0 & 0 & -(x_2x_3 + x_1x_4) \\ x_6x_7 + x_5x_8 & 0 & 0 & -(x_2x_5 + x_1x_6) \\ 0 & x_6x_7 + x_5x_8 & -(x_4x_7 + x_3x_8) & 0 \end{pmatrix} \\
C_2 &= \begin{pmatrix} 0 & x_2x_6 - x_1x_5 & x_1x_3 - x_2x_4 & 0 \\ x_4x_8 - x_3x_7 & 0 & 0 & x_1x_3 - x_2x_4 \\ x_6x_8 - x_5x_7 & 0 & 0 & x_1x_5 - x_2x_6 \\ 0 & x_6x_8 - x_5x_7 & x_3x_7 - x_4x_8 & 0 \end{pmatrix} \\
C_3 &= \begin{pmatrix} 0 & x_1x_6 - x_2x_5 & x_2x_3 - x_1x_4 & 0 \\ x_4x_7 - x_3x_8 & 0 & 0 & x_2x_3 - x_1x_4 \\ x_6x_7 - x_5x_8 & 0 & 0 & x_2x_5 - x_1x_6 \\ 0 & x_6x_7 - x_5x_8 & x_3x_8 - x_4x_7 & 0 \end{pmatrix} \\
C_4 &= \begin{pmatrix} 0 & -(x_1x_5 + x_2x_6) & x_1x_3 + x_2x_4 & 0 \\ -(x_3x_7 + x_4x_8) & 0 & 0 & x_1x_3 + x_2x_4 \\ -(x_5x_7 + x_6x_8) & 0 & 0 & x_1x_5 + x_2x_6 \\ 0 & -(x_5x_7 + x_6x_8) & x_3x_7 + x_4x_8 & 0 \end{pmatrix}
\end{aligned}$$

Define $D = \phi^{+}_{2,2} : Spin^{+}(2,2) \rightarrow SO^{+}(2,2)$

$$D = \begin{pmatrix} d_{11} & d_{12} & d_{13} & d_{14} \\ d_{21} & d_{22} & d_{23} & d_{24} \\ d_{31} & d_{32} & d_{33} & d_{34} \\ d_{41} & d_{42} & d_{43} & d_{44} \end{pmatrix} \tag{3.2}$$

where $C_j = d_{1,j}X_1 + d_{2,j}X_2 + d_{3,j}X_3 + d_{4,j},\ j = 1,2,3,4$, Thus, one obtains:

$$\begin{aligned} d_{ij} &= \frac{\langle C_j, X_i\rangle}{\langle X_i, X_i\rangle} mod\mathrm{GB},\ i,j = 1,2,3,4 \text{ and} \\ \langle A, B\rangle &= \mathrm{tr}\left(A^T B\right). \end{aligned}$$

Therefore

$$\begin{aligned}
d_{1,1} &= \tfrac{1}{2}\left(x_2x_5 + x_1x_6 + x_4x_7 + x_3x_8\right) \\
d_{2,1} &= \tfrac{1}{2}\left(x_6x_7 + x_5x_8 - x_2x_3 - x_1x_4\right) \\
d_{3,1} &= \tfrac{1}{2}\left(x_2x_5 + x_1x_6 - x_4x_7 - x_3x_8\right) \\
d_{4,1} &= \tfrac{1}{2}\left(x_2x_6 - x_1x_5 - x_3x_7 + x_4x_8\right) \\
d_{1,2} &= \tfrac{1}{2}\left(x_2x_6 - x_1x_5 - x_3x_7 + x_4x_8\right) \\
d_{2,2} &= \tfrac{1}{2}\left(x_1x_3 - x_2x_4 - x_5x_7 + x_6x_8\right) \\
d_{3,2} &= \tfrac{1}{2}\left(x_2x_6 - x_1x_5 + x_3x_7 - x_4x_8\right) \\
d_{4,2} &= \tfrac{1}{2}\left(x_1x_3 - x_2x_4 + x_5x_7 - x_6x_8\right) \\
d_{1,3} &= \tfrac{1}{2}\left(x_1x_6 - x_2x_5 + x_4x_7 - x_3x_8\right) \\
d_{2,3} &= \tfrac{1}{2}\left(x_2x_3 - x_1x_4 + x_6x_7 - x_5x_8\right) \\
d_{3,3} &= \tfrac{1}{2}\left(x_1x_6 - x_2x_5 - x_4x_7 + x_3x_8\right) \\
d_{4,3} &= \tfrac{1}{2}\left(x_2x_3 - x_1x_4 - x_6x_7 + x_5x_8\right) \\
d_{1,4} &= \tfrac{1}{2}\left(-x_1x_5 - x_2x_6 - x_3x_7 - x_4x_8\right) \\
d_{2,4} &= \tfrac{1}{2}\left(x_1x_3 + x_2x_4 - x_5x_7 - x_6x_8\right) \\
d_{3,4} &= \tfrac{1}{2}\left(x_3x_7 + x_4x_8 - x_1x_5 - x_2x_6\right) \\
d_{4,4} &= \tfrac{1}{2}\left(x_1x_3 + x_2x_4 + x_5x_7 + x_6x_8\right)
\end{aligned}$$

Theorem 3.2.

The forward map for $\phi_{2,2} : Spin^{+}(2,2) \to SO^{+}(2,2)$ is
$\phi_{2,2}\left[\begin{pmatrix} x_1 & x_2 \\ x_7 & x_8 \end{pmatrix}, \begin{pmatrix} x_3 & x_4 \\ x_5 & x_6 \end{pmatrix}\right] = D$ *with D given by Equation (3.2).*

3.3 The double cover of $SO^{+}(3,1)$ by $Spin^{+}(3,1)$

Let

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \mathrm{Id}_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \mathrm{Id}_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Define $X_1 = \sigma_z \otimes \sigma_z,\ X_2 = \sigma_z \otimes \sigma_x,\ X_3 = \sigma_x \otimes \mathrm{Id}_2,\ X_4 = (i\sigma_x) \otimes \mathrm{Id}_2$, and $\mathrm{q} = \frac{1}{\sqrt{2}}\left(1 + \mathrm{k}\right)$, then

$$M_{1\otimes \mathrm{q}} = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{2}} \end{pmatrix}$$

and using $M_{1\otimes \mathrm{q}}$, we change the basis $\{X_1, X_2, X_3, X_4\}$ of $Cl(3,1)$ into the basis of 1-vectors for $Cl(3,1)$: $\{Z_1, Z_2, Z_3, Z_4\}$, where

$$Z_i = M_{1\otimes \mathrm{q}}^T X_i M_{1\otimes \mathrm{q}},\ i = 1,2,3,4.$$

We thus have

$$Z_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},\ Z_2 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix} Z_3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix},\ Z_4 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

Let $A = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}$, $B = \begin{pmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{pmatrix}$, $J_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, and let us consider matrix $Y = \begin{pmatrix} A & -B \\ B & A \end{pmatrix}$. One shows that

$$Y \in Spin^{+}(3,1)$$

if and only if

$$\begin{aligned} -J_2A^TJ_2A + J_2B^TJ_2B &= \mathrm{Id}_2 \text{ and} \\ J_2A^TJ_2B + J_2B^TJ_2A &= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \end{aligned}$$

The above identities imply that $Y \in Spin^{+}(3,1)$

iff $p = -1 - x_{12}x_{21} + x_{11}x_{22} + y_{12}y_{21} - y_{11}y_{22} = 0$ and $q = -x_{22}y_{11} + x_{21}y_{12} + x_{12}y_{21} - x_{11}y_{22}$.

One verifies that Gröebner Basis of $I = \langle p, q \rangle$ is $\mathrm{GB} = \{f_1, f_2, f_3\}$, where

$$\begin{aligned} f_1 &= x_{22}^2y_{11} - x_{21}x_{22}y_{12} - x_{12}x_{22}y_{21} + y_{22} + x_{12}x_{21}y_{22} - y_{12}y_{21}y_{22} + y_{11}y_{22}^2; \\ f_2 &= x_{22}y_{11} - x_{21}y_{12} - x_{12}y_{21} + x_{11}y_{22}; \\ f_3 &= -1 - x_{12}x_{21} + x_{11}x_{22} + y_{12}y_{21} - y_{11}y_{22}. \end{aligned}$$

One sees that $\det(Y)\, mod\mathrm{GB} = 1$. Thus, $Y^{-1} mod\mathrm{GB} = Y^{-1}$ is computed using the matrix of cofactors of Y

$$Y^{-1} = \begin{pmatrix} t_{11} & t_{12} & t_{13} & t_{14} \\ t_{21} & t_{22} & t_{23} & t_{24} \\ t_{31} & t_{32} & t_{33} & t_{34} \\ t_{41} & t_{42} & t_{43} & t_{44} \end{pmatrix}^T$$

where

$$t_{ij} = (-1)^{i+j} M_{ij} mod\mathrm{GB},$$

and M_{ij} is the (i,j) minor of Y. We have

$$Y^{-1} = \begin{pmatrix} x_{22} & -x_{12} & -y_{22} & y_{12} \\ -x_{21} & x_{11} & y_{21} & -y_{11} \\ y_{22} & -y_{12} & x_{22} & -x_{12} \\ -y_{21} & y_{11} & -x_{21} & x_{11} \end{pmatrix}$$

Next, we compute matrices $C_1 mod \text{GB}, C_2 mod \text{GB}, C_3 mod \text{GB}, C_4 mod \text{GB}$, as follows:

$$C_i = Y Z_i Y^{-1} mod \text{GB} \ , i = 1, 2, 3, 4.$$

We have,if $C_k = \left(c_{ij}^k\right)$,then

$$\begin{array}{lllllllll}
c_{11}^1 & = & 1 + 2x_{12}x_{21} + 2y_{11}y_{22} & c_{12}^1 & = & -2x_{11}x_{12} - 2y_{11}y_{12} & c_{13}^1 & = & 2x_{22}y_{11} - 2x_{12}y_{21} \\
c_{21}^1 & = & 2x_{21}x_{22} + 2y_{21}y_{22} & c_{22}^1 & = & -1 - 2x_{12}x_{21} - 2y_{11}y_{22} & c_{23}^1 & = & 0 \\
c_{31}^1 & = & 2x_{22}y_{11} - 2x_{12}y_{21} & c_{32}^1 & = & 0 & c_{33}^1 & = & -1 - 2x_{12}x_{21} - 2y_{11}y_{22} \\
c_{41}^1 & = & 0 & c_{42}^1 & = & 2x_{22}y_{11} - 2x_{12}y_{21} & c_{43}^1 & = & -2x_{21}x_{22} - 2y_{21}y_{22}
\end{array}$$

$$\begin{array}{llllll}
c_{11}^2 & = & -x_{11}x_{21} + x_{12}x_{22} - y_{11}y_{21} + y_{12}y_{22} & c_{12}^2 & = & x_{11}^2 - x_{12}^2 + y_{11}^2 - y_{12}^2 \\
c_{21}^2 & = & -x_{21}^2 + x_{22}^2 - y_{21}^2 + y_{22}^2 & c_{22}^2 & = & x_{11}x_{21} - x_{12}x_{22} + y_{11}y_{21} - y_{12}y_{22} \\
c_{31}^2 & = & -x_{21}y_{11} + x_{22}y_{12} + x_{11}y_{21} - x_{12}y_{22} & c_{32}^2 & = & 0 \\
c_{41}^2 & = & 0 & c_{42}^2 & = & -x_{21}y_{11} + x_{22}y_{12} + x_{11}y_{21} - x_{12}y_{22}
\end{array}$$

$$\begin{array}{llllll}
c_{13}^2 & = & -x_{21}y_{11} + x_{22}y_{12} + x_{11}y_{21} - x_{12}y_{22} & c_{12}^2 & = & 0 \\
c_{23}^2 & = & 0 & c_{22}^2 & = & -x_{21}y_{11} + x_{22}y_{12} + x_{11}y_{21} - x_{12}y_{22} \\
c_{33}^2 & = & x_{11}x_{21} - x_{12}x_{22} + y_{11}y_{21} - y_{12}y_{22} & c_{32}^2 & = & -x_{11}^2 + x_{12}^2 - y_{11}^2 + y_{12}^2 \\
c_{43}^2 & = & x_{21}^2 - x_{22}^2 + y_{21}^2 - y_{22}^2 & c_{42}^2 & = & -x_{11}x_{21} + x_{12}x_{22} - y_{11}y_{21} + y_{12}y_{22}
\end{array}$$

$$\begin{array}{llllll}
c_{11}^3 & = & -2x_{22}y_{11} + 2x_{21}y_{12} & c_{12}^3 & = & 2x_{12}y_{11} - 2x_{11}y_{12} \\
c_{21}^3 & = & -2x_{22}y_{21} + 2x_{21}y_{22} & c_{22}^3 & = & 2x_{22}y_{11} - 2x_{21}y_{12} \\
c_{31}^3 & = & 1 - 2y_{12}y_{21} + 2y_{11}y_{22} & c_{32}^3 & = & 0 \\
c_{41}^3 & = & 0 & c_{42}^3 & = & 1 - 2y_{12}y_{21} + 2y_{11}y_{22}
\end{array}$$

$$\begin{array}{llllll}
c_{13}^3 & = & 1 - 2y_{12}y_{21} + 2y_{11}y_{22} & c_{12}^3 & = & 0 \\
c_{23}^3 & = & 0 & c_{22}^3 & = & 1 - 2y_{12}y_{21} + 2y_{11}y_{22} \\
c_{33}^3 & = & 2x_{22}y_{11} - 2x_{21}y_{12} & c_{32}^3 & = & -2x_{12}y_{11} + 2x_{11}y_{12} \\
c_{43}^3 & = & 2x_{22}y_{21} - 2x_{21}y_{22} & c_{42}^3 & = & -2x_{22}y_{11} + 2x_{21}y_{12}
\end{array}$$

$$
\begin{array}{rclrcl}
c^4_{11} & = & -x_{11}x_{21} - x_{12}x_{22} - y_{11}y_{21} - y_{12}y_{22} & c^4_{12} & = & x^2_{11} + x^2_{12} + y^2_{11} + y^2_{12} \\
c^4_{21} & = & -x^2_{21} - x^2_{22} - y^2_{21} - y^2_{22} & c^4_{22} & = & x_{11}x_{21} + x_{12}x_{22} + y_{11}y_{21} + y_{12}y_{22} \\
c^4_{31} & = & -x_{21}y_{11} - x_{22}y_{12} + x_{11}y_{21} + x_{12}y_{22} & c^4_{32} & = & 0 \\
c^4_{41} & = & 0 & c^4_{42} & = & -x_{21}y_{11} - x_{22}y_{12} + x_{11}y_{21} + x_{12}y_{22}
\end{array}
$$

$$
\begin{array}{rclrcl}
c^4_{13} & = & -x_{21}y_{11} - x_{22}y_{12} + x_{11}y_{21} + x_{12}y_{22} & c^4_{12} & = & 0 \\
c^4_{23} & = & 0 & c^4_{22} & = & -x_{21}y_{11} - x_{22}y_{12} + x_{11}y_{21} + x_{12}y_{22} \\
c^4_{33} & = & x_{11}x_{21} + x_{12}x_{22} + y_{11}y_{21} + y_{12}y_{22} & c^4_{32} & = & -x^2_{11} - x^2_{12} - y^2_{11} - y^2_{12} \\
c^4_{43} & = & x^2_{21} + x^2_{22} + y^2_{21} + y^2_{22} & c^4_{42} & = & -x_{11}x_{21} - x_{12}x_{22} - y_{11}y_{21} - y_{12}y_{22}
\end{array}
$$

Using these matrices, we compute the matrix

$$
M = \begin{pmatrix} m_{11} & m_{12} & m_{13} & m_{14} \\ m_{21} & m_{22} & m_{23} & m_{24} \\ m_{31} & m_{32} & m_{33} & m_{34} \\ m_{41} & m_{42} & m_{43} & m_{44} \end{pmatrix}, \tag{3.3}
$$

where $C_j = \sum_{i=1}^{4} m_{i,j} Z_i,\ m_{i,j} = \frac{1}{\langle Z_i, Z_i \rangle} \langle C_j, Z_i \rangle\, mod\mathrm{GB},\ j = 1,2,3,4$ and $\langle A, B \rangle = \mathrm{Tr}\left(B^T A\right).$

We obtain the following:

$$m_{11} = 1 + 2x_{12}x_{21} + 2y_{11}y_{22}$$

$$m_{21} = -x_{11}x_{12} + x_{21}x_{22} - y_{11}y_{12} + y_{21}y_{22}$$

$$m_{31} = 2x_{22}y_{11} - 2x_{12}y_{21}$$

$$m_{41} = -x_{11}x_{12} - x_{21}x_{22} - y_{11}y_{12} - y_{21}y_{22} m_{12} = -x_{11}x_{21} + x_{12}x_{22} - y_{11}y_{21} + y_{12}y_{22}$$

$$m_{22} = \frac{1}{2}\left(x_{11}^2 - x_{12}^2 - x_{21}^2 + x_{22}^2 + y_{11}^2 - y_{12}^2 - y_{21}^2 + y_{22}^2\right)$$

$$m_{32} = -x_{21}y_{11} + x_{22}y_{12} + x_{11}y_{21} - x_{12}y_{22}$$

$$m_{42} = \frac{1}{2}\left(x_{11}^2 - x_{12}^2 + x_{21}^2 - x_{22}^2 + y_{11}^2 - y_{12}^2 + y_{21}^2 - y_{22}^2\right) m_{13} = -2x_{22}y_{11} + 2x_{21}y_{12}$$

$$m_{23} = x_{12}y_{11} - x_{11}y_{12} - x_{22}y_{21} + x_{21}y_{22}$$

$$m_{33} = 1 - 2y_{12}y_{21} + 2y_{11}y_{22}$$

$$m_{43} = x_{12}y_{11} - x_{11}y_{12} + x_{22}y_{21} - x_{21}y_{22} m_{14} = -x_{11}x_{21} - x_{12}x_{22} - y_{11}y_{21} - y_{12}y_{22}$$

$$m_{24} = \frac{1}{2}\left(x_{11}^2 + x_{12}^2 - x_{21}^2 - x_{22}^2 + y_{11}^2 + y_{12}^2 - y_{21}^2 - y_{22}^2\right)$$

$$m_{34} = -x_{21}y_{11} - x_{22}y_{12} + x_{11}y_{21} + x_{12}y_{22}$$

$$m_{44} = \frac{1}{2}\left(x_{11}^2 + x_{12}^2 + x_{21}^2 + x_{22}^2 + y_{11}^2 + y_{12}^2 + y_{21}^2 + y_{22}^2\right)$$

Hence we obtain:

Theorem 3.3.

The forward map $\Phi_{3,1}$ sends the matrix $A+iB$, with $A = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}$, $B = \begin{pmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{pmatrix}$ to the matrix M in Equation (3.3).

3.4 The double cover of $SO^+(3,2)$ by $Spin^+(3,2)$

Let $\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $\mathrm{Id}_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $\mathrm{Id}_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

We choose the basis $\mathcal{B} = \{X_1,\ X_2,\ X_3,\ X_4,\ X_5\}$, as our basis of 1-vectors for $Cl(3,2)$,

$$\text{where } X_1 = \sigma_x \otimes \mathrm{Id}_4 = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$X_2 = \sigma_z \otimes \sigma_x \otimes \mathrm{Id}_2 = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \end{pmatrix}$$

$$X_3 = \sigma_z \otimes \sigma_z \otimes \sigma_z = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix},$$

$$X_4 = (i\sigma_y) \otimes \mathrm{Id}_4 = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$X_5 = \sigma_z \otimes (i\sigma_y) \otimes \mathrm{Id}_2 = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

This choice of 1-vectors produces $\mathrm{S}pin^+(3,2)$ as the following nonstandard copy of the real symplectic group

$$\widehat{Sp}(4, \mathbf{R}) = \{\widehat{A} \in M(4, \mathbf{R}) : \widehat{A}^T M_{1\otimes k} \widehat{A} = M_{1\otimes k}\}$$

Suppose

$$\widehat{A} = \begin{pmatrix} x_1 & x_2 & x_3 & x_4 \\ x_5 & x_6 & x_7 & x_8 \\ x_9 & x_{10} & x_{11} & x_{12} \\ x_{13} & x_{14} & x_{15} & x_{16} \end{pmatrix} \tag{3.4}$$

We embed $\widehat{A}$ into an element of $Cl(3,2)$ as follows:

$$A = \begin{pmatrix} x_1 & 0 & x_2 & 0 & x_3 & 0 & x_4 & 0 \\ 0 & x_1 & 0 & -x_2 & 0 & -x_3 & 0 & x_4 \\ x_5 & 0 & x_6 & 0 & x_7 & 0 & x_8 & 0 \\ 0 & -x_5 & 0 & x_6 & 0 & x_7 & 0 & -x_8 \\ x_9 & 0 & x_{10} & 0 & x_{11} & 0 & x_{12} & 0 \\ 0 & -x_9 & 0 & x_{10} & 0 & x_{11} & 0 & -x_{12} \\ x_{13} & 0 & x_{14} & 0 & x_{15} & 0 & x_{16} & 0 \\ 0 & x_{13} & 0 & -x_{14} & 0 & -x_{15} & 0 & x_{16} \end{pmatrix}$$

Then A lives in $Cl(3,2)$ and belongs to $\mathrm{S}pin^+(3,2)$. The conditions that $\widehat{A}$ belongs to $\widehat{Sp}(4,\mathbf{R})$ are the following:

$f_1 = x_1x_{16} - x_4x_{13} - x_5x_{12} + x_8x_9 - 1 = 0$

$f_2 = x_2x_{16} - x_4x_{14} - x_6x_{12} + x_8x_{10} = 0$

$f_3 = x_3x_{16} - x_4x_{15} - x_7x_{12} + x_8x_{11} = 0$

$f_4 = x_1x_{15} - x_3x_{13} - x_5x_{11} + x_7 * x_9 = 0$

$f_5 = x_2x_{15} - x_3x_{14} - x_6x_{11} + x_7x_{10} + 1 = 0$

$f_6 = x_1x_{14} - x_2x_{13} - x_5x_{10} + x_6x_9 = 0$

Let $K = \{f_1, f_2, f_3, f_4, f_5, f_6\}$ and a Gröbner Basis for the ideal $(K) \leq \mathbb{R}[x_1, x_2, ..., x_{16}]$ with respect to the *Lex Order* is given by $\mathrm{GB} = \{g_1, g_2, ..., g_{22}\}$, where

$g_1 = -x_{12}x_{13} + x_{11}x_{14} - x_{10}x_{15} + x_{16}x_9,$

$g_2 = -x_{16} - x_{12}x_{15}x_6 + x_{11}x_{16}x_6 + x_{12}x_{14}x_7 - x_{10}x_{16}x_7 - x_{11}x_{14}x_8 + x_{10}x_{15}x_8,$

$g_3 = x_{12}x_{13} - x_{11}x_{14} + x_{10}x_{15} - x_{11}x_{12}x_{13}x_6 + x_{11}^2x_{14}x_6 - x_{10}x_{11}x_{15}x_6 + x_{10}x_{12}x_{13}x_7 - x_{10}x_{11}x_{14}x_7$
$+ x_{10}^2x_{15}x_7 + x_{12}x_{15}x_6x_9 - x_{12}x_{14}x_7x_9 + x_{11}x_{14}x_8x_9 - x_{10}x_{15}x_8x_9,$

$g_4 = x_{16}x_5 - x_{15}x_6 + x_{14}x_7 - x_{13}x_8,$

$g_5 = 1 + x_{12}x_5 - x_{11}x_6 + x_{10}x_7 - x_8x_9,$

$g_6 = -x_{13} - x_{11}x_{14}x_5 + x_{10}x_{15}x_5 + x_{11}x_{13}x_6 - x_{10}x_{13}x_7 - x_{15}x_6x_9 + x_{14}x_7x_9,$

$g_7 = x_{16}x_3 - x_{15}x_4 - x_{12}x_7 + x_{11}x_8,$

$g_8 = x_{12}x_{13}x_3 - x_{11}x_{14}x_3 + x_{10}x_{15}x_3 - x_{15}x_4x_9 - x_{12}x_7x_9 + x_{11}x_8x_9,$

$g_9 = -x_{15}x_4x_5 + x_{15}x_3x_6 + x_7 - x_{14}x_3x_7 - x_{11}x_6x_7 + x_{10}x_7^2 + x_{13}x_3x_8 + x_{11}x_5x_8 - x_7x_8x_9,$

$g_{10} = -x_{13}x_4 - x_{11}x_{14}x_4x_5 - x_{12}x_{13}x_3x_6 + x_{11}x_{14}x_3x_6 + x_{11}x_{13}x_4x_6 + x_{10}x_7 - x_{10}x_{14}x_3x_7 - x_{10}x_{13}x_4x_7 - x_{10}x_{11}x_6x_7 + x_{10}^2x_7^2 + x_{10}x_{13}x_3x_8 + x_{10}x_{11}x_5x_8 + x_{14}x_4x_7x_9 + x_{12}x_6x_7x_9 - x_{11}x_6x_8x_9 - x_{10}x_7x_8x_9,$

$g_{11} = x_{16}x_2 - x_{14}x_4 - x_{12}x_6 + x_{10}x_8,$

$g_{12} = 1 + x_{15}x_2 - x_{14}x_3 - x_{11}x_6 + x_{10}x_7,$

$g_{13} = x_{10} - x_{12}x_{13}x_2 + x_{11}x_{14}x_2 - x_{10}x_{14}x_3 - x_{10}x_{11}x_6 + x_{10}^2x_7 + x_{14}x_4x_9 + x_{12}x_6x_9 - x_{10}x_8x_9,$

$g_{14} = x_{14}x_4x_5 - x_{14}x_3x_6 + x_{14}x_2x_7 - x_{13}x_2x_8 - x_{10}x_5x_8 + x_6x_8x_9,$

$g_{15} = -x_4 - x_{12}x_3x_6 + x_{11}x_4x_6 + x_{12}x_2x_7 - x_{10}x_4x_7 - x_{11}x_2x_8 + x_{10}x_3x_8,$

$g_{16} = x_4x_5 - x_3x_6 - x_{11}x_4x_5x_6 + x_{11}x_3x_6^2 + x_2x_7 + x_{10}x_4x_5x_7 - x_{11}x_2x_6x_7 - x_{10}x_3x_6x_7 + x_{10}x_2x_7^2 + x_{11}x_2x_5x_8 - x_{10}x_3x_5x_8 + x_3x_6x_8x_9 - x_2x_7x_8x_9,$

$g_{17} = x_1x_{16} - x_{13}x_4 - x_{11}x_6 + x_{10}x_7,$

$g_{18} = x_1x_{15} - x_{13}x_3 - x_{11}x_5 + x_7x_9,$

$g_{19} = x_1x_{14} - x_{13}x_2 - x_{10}x_5 + x_6x_9,$

$g_{20} = x_1x_{12} - x_{11}x_2 + x_{10}x_3 - x_4x_9,$

$g_{21} = -x_4x_5 + x_3x_6 - x_2x_7 + x_1x_8,$

$g_{22} = -x_1 - x_{11}x_2x_5 + x_{10}x_3x_5 + x_1x_{11}x_6 - x_1x_{10}x_7 - x_3x_6x_9 + x_2x_7x_9.$

Note that $\det(A)\, mod \mathrm{GB} = 1$, and hence

$$A^{-1}modGB = \begin{pmatrix} x_{16} & 0 & -x_{12} & 0 & x_8 & 0 & -x_4 & 0 \\ 0 & x_{16} & 0 & x_{12} & 0 & -x_8 & 0 & -x_4 \\ -x_{15} & 0 & x_{11} & 0 & -x_7 & 0 & x_3 & 0 \\ 0 & x_{15} & 0 & x_{11} & 0 & -x_7 & 0 & -x_3 \\ x_{14} & 0 & -x_{10} & 0 & x_6 & 0 & -x_2 & 0 \\ 0 & -x_{14} & 0 & -x_{10} & 0 & x_6 & 0 & x_2 \\ -x_{13} & 0 & x_9 & 0 & -x_5 & 0 & x_1 & 0 \\ 0 & -x_{13} & 0 & -x_9 & 0 & x_5 & 0 & x_1 \end{pmatrix}$$

Next, we compute matrices C_1modGB, C_2modGB, C_3modGB, C_4modGB, C_5modGB as follows: $C_i = AX_iA^{-1}modGB, i = 1, 2, 3, 4, 5$. We then have $C_1 = \left(c^1_{i,j}\right)$, where the $c^1_{i,j}$ are

$$\begin{aligned}
c^1_{11} &= x_{10}x_5 + x_{12}x_7 - x_{11}x_8 - x_6x_9 & c^1_{37} &= -x_2x_5 + x_1x_6 - x_4x_7 + x_3x_8 \\
c^1_{13} &= -x_1x_{10} - x_{12}x_3 + x_{11}x_4 + x_2x_9 & c^1_{42} &= x_{14}x_5 - x_{13}x_6 + x_{16}x_7 - x_{15}x_8 \\
c^1_{15} &= -x_2x_5 + x_1x_6 - x_4x_7 + x_3x_8 & c^1_{44} &= x_{10}x_5 + x_{12}x_7 - x_{11}x_8 - x_6x_9 \\
c^1_{22} &= -x_{10}x_5 - x_{12}x_7 + x_{11}x_8 + x_6x_9 & c^1_{48} &= -x_2x_5 + x_1x_6 - x_4x_7 + x_3x_8 \\
c^1_{24} &= -x_1x_{10} - x_{12}x_3 + x_{11}x_4 + x_2x_9 & c^1_{51} &= -x_{10}x_{13} - x_{12}x_{15} + x_{11}x_{16} + x_{14}x_9 \\
c^1_{26} &= -x_2x_5 + x_1x_6 - x_4x_7 + x_3x_8 & c^1_{55} &= -x_{10}x_5 - x_{12}x_7 + x_{11}x_8 + x_6x_9 \\
c^1_{31} &= x_{14}x_5 - x_{13}x_6 + x_{16}x_7 - x_{15}x_8 & c^1_{57} &= x_1x_{10} + x_{12}x_3 - x_{11}x_4 - x_2x_9 \\
c^1_{33} &= -x_{10}x_5 - x_{12}x_7 + x_{11}x_8 + x_6x_9 & c^1_{62} &= -x_{10}x_{13} - x_{12}x_{15} + x_{11}x_{16} + x_{14}x_9
\end{aligned}$$

$$\begin{aligned}
c^1_{66} &= x_{10}x_5 + x_{12}x_7 - x_{11}x_8 - x_6x_9 \\
c^1_{68} &= x_1x_{10} + x_{12}x_3 - x_{11}x_4 - x_2x_9 \\
c^1_{73} &= -x_{10}x_{13} - x_{12}x_{15} + x_{11}x_{16} + x_{14}x_9 \\
c^1_{75} &= -x_{14}x_5 + x_{13}x_6 - x_{16}x_7 + x_{15}x_8 \\
c^1_{77} &= x_{10}x_5 + x_{12}x_7 - x_{11}x_8 - x_6x_9 \\
c^1_{74} &= -x_{10}x_{13} - x_{12}x_{15} + x_{11}x_{16} + x_{14}x_9 \\
c^1_{86} &= -x_{14}x_5 + x_{13}x_6 - x_{16}x_7 + x_{15}x_8 \\
c^1_{88} &= -x_{10}x_5 - x_{12}x_7 + x_{11}x_8 + x_6x_9
\end{aligned}$$

$C_2 = \left(c^2_{i,j}\right)$, where $c^2_{i,j} \neq 0$ are:

$$
\begin{aligned}
c^2_{11} &= -x_{11}x_5 + x_{12}x_6 - x_{10}x_8 + x_7x_9 & c^2_{37} &= x_3x_5 - x_4x_6 - x_1x_7 + x_2x_8 \\
c^2_{13} &= x_1x_{11} - x_{12}x_2 + x_{10}x_4 - x_3x_9 & c^2_{42} &= -x_{15}x_5 + x_{16}x_6 + x_{13}x_7 - x_{14}x_8 \\
c^2_{15} &= x_3x_5 - x_4x_6 - x_1x_7 + x_2x_8 & c^2_{44} &= -x_{11}x_5 + x_{12}x_6 - x_{10}x_8 + x_7x_9 \\
c^2_{22} &= x_{11}x_5 - x_{12}x_6 + x_{10}x_8 - x_7x_9 & c^2_{48} &= x_3x_5 - x_4x_6 - x_1x_7 + x_2x_8 \\
c^2_{24} &= x_1x_{11} - x_{12}x_2 + x_{10}x_4 - x_3x_9 & c^2_{51} &= x_{11}x_{13} - x_{12}x_{14} + x_{10}x_{16} - x_{15}x_9 \\
c^2_{26} &= x_3x_5 - x_4x_6 - x_1x_7 + x_2x_8 & c^2_{55} &= x_{11}x_5 - x_{12}x_6 + x_{10}x_8 - x_7x_9 \\
c^2_{31} &= -x_{15}x_5 + x_{16}x_6 + x_{13}x_7 - x_{14}x_8 & c^2_{57} &= -x_1x_{11} + x_{12}x_2 - x_{10}x_4 + x_3x_9 \\
c^2_{33} &= x_{11}x_5 - x_{12}x_6 + x_{10}x_8 - x_7x_9 & c^2_{62} &= x_{11}x_{13} - x_{12}x_{14} + x_{10}x_{16} - x_{15}x_9
\end{aligned}
$$

$$
\begin{aligned}
c^2_{66} &= -x_{11}x_5 + x_{12}x_6 - x_{10}x_8 + x_7x_9 \\
c^2_{68} &= -x_1x_{11} + x_{12}x_2 - x_{10}x_4 + x_3x_9 \\
c^2_{73} &= x_{11}x_{13} - x_{12}x_{14} + x_{10}x_{16} - x_{15}x_9 \\
c^2_{75} &= x_{15}x_5 - x_{16}x_6 - x_{13}x_7 + x_{14}x_8 \\
c^2_{77} &= -x_{11}x_5 + x_{12}x_6 - x_{10}x_8 + x_7x_9 \\
c^2_{84} &= x_{11}x_{13} - x_{12}x_{14} + x_{10}x_{16} - x_{15}x_9 \\
c^2_{86} &= x_{15}x_5 - x_{16}x_6 - x_{13}x_7 + x_{14}x_8 \\
c^2_{88} &= x_{11}x_5 - x_{12}x_6 + x_{10}x_8 - x_7x_9
\end{aligned}
$$

$C_3 = \left(c^3_{i,j}\right)$, where $c^3_{i,j} \neq 0$ So, we have :

$$
\begin{aligned}
c^3_{11} &= -1 + 2x_{11}x_6 - 2x_{10}x_7 & c^3_{37} &= -2x_3x_6 + 2x_2x_7 & c^3_{66} &= -1 + 2x_{11}x_6 - 2x_{10}x_7 \\
c^3_{13} &= -2x_{11}x_2 + 2x_{10}x_3 & c^3_{42} &= 2x_{15}x_6 - 2x_{14}x_7 & c^3_{68} &= 2x_{11}x_2 - 2x_{10}x_3 \\
c^3_{15} &= -2x_3x_6 + 2x_2x_7 & c^3_{44} &= -1 + 2x_{11}x_6 - 2x_{10}x_7 & c^3_{73} &= -2x_{11}x_{14} + 2x_{10}x_{15} \\
c^3_{22} &= 1 - 2x_{11}x_6 + 2x_{10}x_7 & c^3_{48} &= -2x_3x_6 + 2x_2x_7 & c^3_{75} &= -2x_{15}x_6 + 2x_{14}x_7 \\
c^3_{24} &= -2x_{11}x_2 + 2x_{10}x_3 & c^3_{51} &= -2x_{11}x_{14} + 2x_{10}x_{15} & c^3_{77} &= -1 + 2x_{11}x_6 - 2x_{10}x_7 \\
c^3_{26} &= -2x_3x_6 + 2x_2x_7 & c^3_{55} &= 1 - 2x_{11}x_6 + 2x_{10}x_7 & c^3_{84} &= -2x_{11}x_{14} + 2x_{10}x_{15} \\
c^3_{31} &= 2x_{15}x_6 - 2x_{14}x_7 & c^3_{57} &= 2x_{11}x_2 - 2x_{10}x_3 & c^3_{86} &= -2x_{15}x_6 + 2x_{14}x_7 \\
c^3_{33} &= 1 - 2x_{11}x_6 + 2x_{10}x_7 & c^3_{62} &= -2x_{11}x_{14} + 2x_{10}x_{15} & c^3_{88} &= 1 - 2x_{11}x_6 + 2x_{10}x_7
\end{aligned}
$$

$C_4 = \left(c^4_{i,j}\right)$, where $c^4_{i,j} \neq 0$ are:

$$
\begin{array}{lcl}
c^4_{11} & = & x_{10}x_5 - x_{12}x_7 + x_{11}x_8 - x_6x_9 \\
c^4_{13} & = & -x_1x_{10} + x_{12}x_3 - x_{11}x_4 + x_2x_9 \\
c^4_{15} & = & -x_2x_5 + x_1x_6 + x_4x_7 - x_3x_8 \\
c^4_{22} & = & -x_{10}x_5 + x_{12}x_7 - x_{11}x_8 + x_6x_9 \\
c^4_{24} & = & -x_1x_{10} + x_{12}x_3 - x_{11}x_4 + x_2x_9 \\
c^4_{26} & = & -x_2x_5 + x_1x_6 + x_4x_7 - x_3x_8 \\
c^4_{31} & = & x_{14}x_5 - x_{13}x_6 - x_{16}x_7 + x_{15}x_8 \\
c^4_{33} & = & -x_{10}x_5 + x_{12}x_7 - x_{11}x_8 + x_6x_9
\end{array}
$$

$$
\begin{array}{lcl}
c^4_{37} & = & -x_2x_5 + x_1x_6 + x_4x_7 - x_3x_8 \\
c^4_{42} & = & x_{14}x_5 - x_{13}x_6 - x_{16}x_7 + x_{15}x_8 \\
c^4_{44} & = & x_{10}x_5 - x_{12}x_7 + x_{11}x_8 - x_6x_9 \\
c^4_{48} & = & -x_2x_5 + x_1x_6 + x_4x_7 - x_3x_8 \\
c^4_{51} & = & -x_{10}x_{13} + x_{12}x_{15} - x_{11}x_{16} + x_{14}x_9 \\
c^4_{55} & = & -x_{10}x_5 + x_{12}x_7 - x_{11}x_8 + x_6x_9 \\
c^4_{57} & = & x_1x_{10} - x_{12}x_3 + x_{11}x_4 - x_2x_9 \\
c^4_{62} & = & -x_{10}x_{13} + x_{12}x_{15} - x_{11}x_{16} + x_{14}x_9 \\
c^4_{66} & = & x_{10}x_5 - x_{12}x_7 + x_{11}x_8 - x_6x_9 \\
c^4_{68} & = & x_1x_{10} - x_{12}x_3 + x_{11}x_4 - x_2x_9 \\
c^4_{73} & = & -x_{10}x_{13} + x_{12}x_{15} - x_{11}x_{16} + x_{14}x_9 \\
c^4_{75} & = & -x_{14}x_5 + x_{13}x_6 + x_{16}x_7 - x_{15}x_8 \\
c^4_{77} & = & x_{10}x_5 - x_{12}x_7 + x_{11}x_8 - x_6x_9 \\
c^4_{84} & = & -x_{10}x_{13} + x_{12}x_{15} - x_{11}x_{16} + x_{14}x_9 \\
c^4_{86} & = & -x_{14}x_5 + x_{13}x_6 + x_{16}x_7 - x_{15}x_8 \\
c^4_{88} & = & -x_{10}x_5 + x_{12}x_7 - x_{11}x_8 + x_6x_9
\end{array}
$$

$C_5 = \left(c^5_{i,j}\right)$, where $c^5_{i,j} \neq 0$ are:

$$\begin{array}{rclrcl}
c^5_{11} &=& -x_{11}x_5 - x_{12}x_6 + x_{10}x_8 + x_7x_9 & c^5_{37} &=& x_3x_5 + x_4x_6 - x_1x_7 - x_2x_8 \\
c^5_{13} &=& x_1x_{11} + x_{12}x_2 - x_{10}x_4 - x_3x_9 & c^5_{42} &=& -x_{15}x_5 - x_{16}x_6 + x_{13}x_7 + x_{14}x_8 \\
c^5_{15} &=& x_3x_5 + x_4x_6 - x_1x_7 - x_2x_8 & c^5_{44} &=& -x_{11}x_5 - x_{12}x_6 + x_{10}x_8 + x_7x_9 \\
c^5_{22} &=& x_{11}x_5 + x_{12}x_6 - x_{10}x_8 - x_7x_9 & c^5_{48} &=& x_3x_5 + x_4x_6 - x_1x_7 - x_2x_8 \\
c^5_{24} &=& x_1x_{11} + x_{12}x_2 - x_{10}x_4 - x_3x_9 & c^5_{51} &=& x_{11}x_{13} + x_{12}x_{14} - x_{10}x_{16} - x_{15}x_9 \\
c^5_{26} &=& x_3x_5 + x_4x_6 - x_1x_7 - x_2x_8 & c^5_{55} &=& x_{11}x_5 + x_{12}x_6 - x_{10}x_8 - x_7x_9 \\
c^5_{31} &=& -x_{15}x_5 - x_{16}x_6 + x_{13}x_7 + x_{14}x_8 & c^5_{57} &=& -x_1x_{11} - x_{12}x_2 + x_{10}x_4 + x_3x_9 \\
c^5_{33} &=& x_{11}x_5 + x_{12}x_6 - x_{10}x_8 - x_7x_9 & c^5_{62} &=& x_{11}x_{13} + x_{12}x_{14} - x_{10}x_{16} - x_{15}x_9
\end{array}$$

$$\begin{array}{rcl}
c^5_{66} &=& -x_{11}x_5 - x_{12}x_6 + x_{10}x_8 + x_7x_9 \\
c^5_{68} &=& -x_1x_{11} - x_{12}x_2 + x_{10}x_4 + x_3x_9 \\
c^5_{73} &=& x_{11}x_{13} + x_{12}x_{14} - x_{10}x_{16} - x_{15}x_9 \\
c^5_{75} &=& x_{15}x_5 + x_{16}x_6 - x_{13}x_7 - x_{14}x_8 \\
c^5_{77} &=& -x_{11}x_5 - x_{12}x_6 + x_{10}x_8 + x_7x_9 \\
c^5_{84} &=& x_{11}x_{13} + x_{12}x_{14} - x_{10}x_{16} - x_{15}x_9 \\
c^5_{86} &=& x_{15}x_5 + x_{16}x_6 - x_{13}x_7 - x_{14}x_8 \\
c^5_{88} &=& x_{11}x_5 + x_{12}x_6 - x_{10}x_8 - x_7x_9
\end{array}$$

Now we express the matrices C_i in terms of the basis $\mathcal{B} = \{X_1,\ X_2,\ X_3,\ X_4,\ X_5\}$ as follows:

$$C_j = \sum_{i=1}^{5} m_{i,j} X_i, \text{ where } m_{i,j} = \frac{1}{\langle X_i, X_i\rangle} \langle C_j, X_i\rangle\, modGB \text{ and } \langle Y, Z\rangle = \mathrm{Tr}\left(Y^T Z\right)$$

Let M be the matrix of coefficients:

$$M = \begin{pmatrix} m_{1,1} & m_{1,2} & m_{1,3} & m_{1,4} & m_{1,5} \\ m_{2,1} & m_{2,2} & m_{2,3} & m_{2,4} & m_{2,5} \\ m_{3,1} & m_{3,2} & m_{3,3} & m_{3,4} & m_{3,5} \\ m_{4,1} & m_{4,2} & m_{4,3} & m_{4,4} & m_{4,5} \\ m_{5,1} & m_{5,2} & m_{5,3} & m_{5,4} & m_{5,5} \end{pmatrix} \tag{3.5}$$

The $m_{i,j}$ are given as follows:

$$
\begin{aligned}
m_{1,1} &= \tfrac{1}{2}(-x_{10}x_{13} - x_{12}x_{15} + x_{11}x_{16} - x_2x_5 + x_1x_6 - x_4x_7 + x_3x_8 + x_{14}x_9) \\
m_{2,1} &= \tfrac{1}{2}(-x_1x_{10} - x_{12}x_3 + x_{11}x_4 + x_{14}x_5 - x_{13}x_6 + x_{16}x_7 - x_{15}x_8 + x_2x_9) \\
m_{3,1} &= x_{10}x_5 + x_{12}x_7 - x_{11}x_8 - x_6x_9 \\
m_{4,1} &= \tfrac{1}{2}(x_{10}x_{13} + x_{12}x_{15} - x_{11}x_{16} - x_2x_5 + x_1x_6 - x_4x_7 + x_3x_8 - x_{14}x_9) \\
m_{5,1} &= \tfrac{1}{2}(-x_1x_{10} - x_{12}x_3 + x_{11}x_4 - x_{14}x_5 + x_{13}x_6 - x_{16}x_7 + x_{15}x_8 + x_2x_9) \\
m_{1,2} &= \tfrac{1}{2}(x_{11}x_{13} - x_{12}x_{14} + x_{10}x_{16} + x_3x_5 - x_4x_6 - x_1x_7 + x_2x_8 - x_{15}x_9) \\
m_{2,2} &= \tfrac{1}{2}(x_1x_{11} - x_{12}x_2 + x_{10}x_4 - x_{15}x_5 + x_{16}x_6 + x_{13}x_7 - x_{14}x_8 - x_3x_9) \\
m_{3,2} &= -x_{11}x_5 + x_{12}x_6 - x_{10}x_8 + x_7x_9 \\
m_{4,2} &= \tfrac{1}{2}(-x_{11}x_{13} + x_{12}x_{14} - x_{10}x_{16} + x_3x_5 - x_4x_6 - x_1x_7 + x_2x_8 + x_{15}x_9) \\
m_{5,2} &= \tfrac{1}{2}(x_1x_{11} - x_{12}x_2 + x_{10}x_4 + x_{15}x_5 - x_{16}x_6 - x_{13}x_7 + x_{14}x_8 - x_3x_9) \\
m_{1,3} &= -x_{11}x_{14} + x_{10}x_{15} - x_3x_6 + x_2x_7 \\
m_{2,3} &= -x_{11}x_2 + x_{10}x_3 + x_{15}x_6 - x_{14}x_7 \\
m_{3,3} &= -1 + 2x_{11}x_6 - 2x_{10}x_7 \\
m_{4,3} &= x_{11}x_{14} - x_{10}x_{15} - x_3x_6 + x_2x_7 \\
m_{5,3} &= -x_{11}x_2 + x_{10}x_3 - x_{15}x_6 + x_{14}x_7 \\
m_{1,4} &= \tfrac{1}{2}(-x_{10}x_{13} + x_{12}x_{15} - x_{11}x_{16} - x_2x_5 + x_1x_6 + x_4x_7 - x_3x_8 + x_{14}x_9) \\
m_{2,4} &= \tfrac{1}{2}(-x_1x_{10} + x_{12}x_3 - x_{11}x_4 + x_{14}x_5 - x_{13}x_6 - x_{16}x_7 + x_{15}x_8 + x_2x_9) \\
m_{3,4} &= x_{10}x_5 - x_{12}x_7 + x_{11}x_8 - x_6x_9 \\
m_{4,4} &= \tfrac{1}{2}(x_{10}x_{13} - x_{12}x_{15} + x_{11}x_{16} - x_2x_5 + x_1x_6 + x_4x_7 - x_3x_8 - x_{14}x_9) \\
m_{5,4} &= \tfrac{1}{2}(-x_1x_{10} + x_{12}x_3 - x_{11}x_4 - x_{14}x_5 + x_{13}x_6 + x_{16}x_7 - x_{15}x_8 + x_2x_9) \\
m_{1,5} &= \tfrac{1}{2}(x_{11}x_{13} + x_{12}x_{14} - x_{10}x_{16} + x_3x_5 + x_4x_6 - x_1x_7 - x_2x_8 - x_{15}x_9) \\
m_{2,5} &= \tfrac{1}{2}(x_1x_{11} + x_{12}x_2 - x_{10}x_4 - x_{15}x_5 - x_{16}x_6 + x_{13}x_7 + x_{14}x_8 - x_3x_9) \\
m_{3,5} &= -x_{11}x_5 - x_{12}x_6 + x_{10}x_8 + x_7x_9 \\
m_{4,5} &= \tfrac{1}{2}(-x_{11}x_{13} - x_{12}x_{14} + x_{10}x_{16} + x_3x_5 + x_4x_6 - x_1x_7 - x_2x_8 + x_{15}x_9) \\
m_{5,5} &= \tfrac{1}{2}(x_1x_{11} + x_{12}x_2 - x_{10}x_4 + x_{15}x_5 + x_{16}x_6 - x_{13}x_7 - x_{14}x_8 - x_3x_9
\end{aligned}
$$

Theorem 3.4. *The map $\Phi_{3,2}$ sends $\widehat{A} \in Spin^+(3,2) = \widehat{Sp}(4,\mathbf{R})$ to M in $SO^+(3,2)$, where $\widehat{A}$ is as in Equation (3.4) and M is as in Equation (3.5).*

3.5 The double cover of $SO^+(3,3)$ by $Spin^+(3,3)$

Let $\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $\mathrm{Id}_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $\mathrm{Id}_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

Define $Y_1 = \sigma_z \otimes \sigma_z \otimes \sigma_x$, $Y_2 = \sigma_z \otimes \sigma_x \otimes \mathrm{Id}_2$, $Y_3 = \sigma_x \otimes \mathrm{Id}_2 \otimes \mathrm{Id}_2$, $Y_4 = \sigma_z \otimes \sigma_z \otimes i\sigma_y$, $Y_5 = \sigma_z \otimes i\sigma_y \otimes \mathrm{Id}_2$, $Y_6 = i\sigma_y \otimes \mathrm{Id}_2 \otimes \mathrm{Id}_2$;

Let $Q = [e_1|e_4|e_6|e_7|e_2|e_3|e_5|e_8] = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$ be a permutation matrix.

Define

$$Z_1 = Q^T Y_1 Q = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}, Z_2 = Q^T Y_2 Q = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$Z_3 = Q^T Y_3 Q = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, Z_4 = Q^T Y_4 Q = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$Z_5 = Q^T Y_5 Q = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, Z_6 = Q^T Y_6 Q = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

We use the set $\{Z_i, i = 1, \ldots, 6\}$ as our basis of 1-vectors for $Cl(3,3)$.

Let

$$T = \begin{pmatrix} x_1 & x_2 & x_3 & x_4 \\ x_5 & x_6 & x_7 & x_8 \\ x_9 & x_{10} & x_{11} & x_{12} \\ x_{13} & x_{14} & x_{15} & x_{16} \end{pmatrix} \tag{3.6}$$

We suppose $T \in \mathrm{SL}(4,\mathbb{R}) = \mathrm{S}pin^{+}(3,3)$. Denote by $p = \det(T) - 1$; $p \in \mathbb{R}[x_1, x_2, ..., x_{16}]$;
Since the Gröbner Basis GB of the ideal $\langle p \rangle$ is $\mathrm{GB}(\langle p \rangle) = \{p\}$, we compute

$$T^{-1} = \begin{pmatrix} x_1 & x_2 & x_3 & x_4 \\ x_5 & x_6 & x_7 & x_8 \\ x_9 & x_{10} & x_{11} & x_{12} \\ x_{13} & x_{14} & x_{15} & x_{16} \end{pmatrix}^{-1} \quad modp$$

which yields the following matrix

$$T^{-1} = \begin{pmatrix} t_{11} & t_{12} & t_{13} & t_{14} \\ t_{21} & t_{22} & t_{23} & t_{24} \\ t_{31} & t_{32} & t_{33} & t_{34} \\ t_{41} & t_{42} & t_{43} & t_{44} \end{pmatrix},$$

where

$t_{11} = -x_{12}x_{15}x_6 + x_{11}x_{16}x_6 + x_{12}x_{14}x_7 - x_{10}x_{16}x_7 - x_{11}x_{14}x_8 + x_{10}x_{15}x_8;$

$t_{21} = x_{12}x_{15}x_5 - x_{11}x_{16}x_5 - x_{12}x_{13}x_7 + x_{11}x_{13}x_8 + x_{16}x_7x_9 - x_{15}x_8x_9;$

$t_{31} = -x_{12}x_{14}x_5 + x_{10}x_{16}x_5 + x_{12}x_{13}x_6 - x_{10}x_{13}x_8 - x_{16}x_6x_9 + x_{14}x_8x_9;$

$t_{41} = x_{11}x_{14}x_5 - x_{10}x_{15}x_5 - x_{11}x_{13}x_6 + x_{10}x_{13}x_7 + x_{15}x_6x_9 - x_{14}x_7x_9;$

$t_{12} = x_{12}x_{15}x_2 - x_{11}x_{16}x_2 - x_{12}x_{14}x_3 + x_{10}x_{16}x_3 + x_{11}x_{14}x_4 - x_{10}x_{15}x_4;$

$t_{22} = -x_1x_{12}x_{15} + x_1x_{11}x_{16} + x_{12}x_{13}x_3 - x_{11}x_{13}x_4 - x_{16}x_3x_9 + x_{15}x_4x_9;$

$t_{32} = x_1x_{12}x_{14} - x_1x_{10}x_{16} - x_{12}x_{13}x_2 + x_{10}x_{13}x_4 + x_{16}x_2x_9 - x_{14}x_4x_{\neg 9};$

$t_{42} = -x_1x_{11}x_{14} + x_1x_{10}x_{15} + x_{11}x_{13}x_2 - x_{10}x_{13}x_3 - x_{15}x_2x_9 + x_{14}x_3x_9;$

$t_{13} = -x_{16}x_3x_6 + x_{15}x_4x_6 + x_{16}x_2x_7 - x_{14}x_4x_7 - x_{15}x_2x_8 + x_{14}x_3x_{8;}$

$t_{23} = x_{16}x_3x_5 - x_{15}x_4x_5 - x_1x_{16}x_7 + x_{13}x_4x_7 + x_1x_{15}x_8 - x_{13}x_3x_8;$

$t_{33} = -x_{16}x_2x_5 + x_{14}x_4x_5 + x_1x_{16}x_6 - x_{13}x_4x_6 - x_1x_{14}x_8 + x_{13}x_2x_8;$

$t_{43} = x_{15}x_2x_5 - x_{14}x_3x_5 - x_1x_{15}x_6 + x_{13}x_3x_6 + x_1x_{14}x_7 - x_{13}x_2x_7;$

$t_{14} = x_{12}x_3x_6 - x_{11}x_4x_6 - x_{12}x_2x_7 + x_{10}x_4x_7 + x_{11}x_2x_8 - x_{10}x_3x_8;$

$t_{24} = -x_{12}x_3x_5 + x_{11}x_4x_5 + x_1x_{12}x_7 - x_1x_{11}x_8 - x_4x_7x_9 + x_3x_8x_9;$

$t_{34} = x_{12}x_2x_5 - x_{10}x_4x_5 - x_1x_{12}x_6 + x_1x_{10}x_8 + x_4x_6x_9 - x_2x_8x_9;$

$t_{44} = -x_{11}x_2x_5 + x_{10}x_3x_5 + x_1x_{11}x_6 - x_1x_{10}x_7 - x_3x_6x_9 + x_2x_7x_9$

Next let $G = -M_{1\otimes \mathrm{k}} \left(T^{-1}\right)^T M_{1\otimes \mathrm{k}} mod p$ where we recall that

$$M_{1\otimes \mathrm{k}} = \sigma_x \otimes \mathrm{i}\sigma_y = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$$

Thus,

$$G = \begin{pmatrix} g_{11} & g_{12} & g_{13} & g_{14} \\ g_{21} & g_{22} & g_{23} & g_{24} \\ g_{31} & g_{32} & g_{33} & g_{34} \\ g_{41} & g_{42} & g_{43} & g_{44} \end{pmatrix},$$

where

$g_{11} = -x_{11}x_2x_5 + x_{10}x_3x_5 + x_1x_{11}x_6 - x_1x_{10}x_7 - x_3x_6x_9 + x_2x_7x_9;$

$g_{12} = -x_{12}x_2x_5 + x_{10}x_4x_5 + x_1x_{12}x_6 - x_1x_{10}x_8 - x_4x_6x_9 + x_2x_8x_9;$

$g_{13} = -x_{12}x_3x_5 + x_{11}x_4x_5 + x_1x_{12}x_7 - x_1x_{11}x_8 - x_4x_7x_9 + x_3x_8x_9;$

$g_{14} = -x_{12}x_3x_6 + x_{11}x_4x_6 + x_{12}x_2x_7 - x_{10}x_4x_7 - x_{11}x_2x_8 + x_{10}x_3x_8;$

$g_{21} = -x_{15}x_2x_5 + x_{14}x_3x_5 + x_1x_{15}x_6 - x_{13}x_3x_6 - x_1x_{14}x_7 + x_{13}x_2x_7;$

$g_{22} = -x_{16}x_2x_5 + x_{14}x_4x_5 + x_1x_{16}x_6 - x_{13}x_4x_6 - x_1x_{14}x_8 + x_{13}x_2x_8;$

$g_{23} = -x_{16}x_3x_5 + x_{15}x_4x_5 + x_1x_{16}x_7 - x_{13}x_4x_7 - x_1x_{15}x_8 + x_{13}x_3x_8;$

$g_{24} = -x_{16}x_3x_6 + x_{15}x_4x_6 + x_{16}x_2x_7 - x_{14}x_4x_7 - x_{15}x_2x_8 + x_{14}x_3x_8;$

$g_{31} = -x_1x_{11}x_{14} + x_1x_{10}x_{15} + x_{11}x_{13}x_2 - x_{10}x_{13}x_3 - x_{15}x_2x_9 + x_{14}x_3x_9;$

$g_{32} = -x_1x_{12}x_{14} + x_1x_{10}x_{16} + x_{12}x_{13}x_2 - x_{10}x_{13}x_4 - x_{16}x_2x_9 + x_{14}x_4x_9;$

$g_{33} = -x_1x_{12}x_{15} + x_1x_{11}x_{16} + x_{12}x_{13}x_3 - x_{11}x_{13}x_4 - x_{16}x_3x_9 + x_{15}x_4x_9;$

$g_{34} = -x_{12}x_{15}x_2 + x_{11}x_{16}x_2 + x_{12}x_{14}x_3 - x_{10}x_{16}x_3 - x_{11}x_{14}x_4 + x_{10}x_{15}x_4;$

$g_{41} = -x_{11}x_{14}x_5 + x_{10}x_{15}x_5 + x_{11}x_{13}x_6 - x_{10}x_{13}x_7 - x_{15}x_6x_9 + x_{14}x_7x_9;$

$g_{42} = -x_{12}x_{14}x_5 + x_{10}x_{16}x_5 + x_{12}x_{13}x_6 - x_{10}x_{13}x_8 - x_{16}x_6x_9 + x_{14}x_8x_9;$

$g_{43} = -x_{12}x_{15}x_5 + x_{11}x_{16}x_5 + x_{12}x_{13}x_7 - x_{11}x_{13}x_8 - x_{16}x_7x_9 + x_{15}x_8x_9;$

$g_{44} = -x_{12}x_{15}x_6 + x_{11}x_{16}x_6 + x_{12}x_{14}x_7 - x_{10}x_{16}x_7 - x_{11}x_{14}x_8 + x_{10}x_{15}x_8;$

We then embed T into a matrix U living in $Cl(3,3)$ via

$$U = \begin{pmatrix} x_1 & x_2 & x_3 & x_4 & 0 & 0 & 0 & 0 \\ x_5 & x_6 & x_7 & x_8 & 0 & 0 & 0 & 0 \\ x_9 & x_{10} & x_{11} & x_{12} & 0 & 0 & 0 & 0 \\ x_{13} & x_{14} & x_{15} & x_{16} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & g_{11} & g_{12} & g_{13} & g_{14} \\ 0 & 0 & 0 & 0 & g_{21} & g_{22} & g_{23} & g_{24} \\ 0 & 0 & 0 & 0 & g_{31} & g_{32} & g_{33} & g_{34} \\ 0 & 0 & 0 & 0 & g_{41} & g_{42} & g_{43} & g_{44} \end{pmatrix};$$

As one verifies, $\det(U) \equiv 1 mod \langle p \rangle$. Therefore, we can compute the inverse of $U\ mod \langle p \rangle$ by taking the transpose of the matrix of all cofactors $mod \langle p \rangle$. In particular, we have, if

$$U^{-1} = \begin{pmatrix} u_{11} & u_{12} & u_{13} & u_{14} & u_{15} & u_{16} & u_{17} & u_{18} \\ u_{21} & u_{22} & u_{23} & u_{24} & u_{25} & u_{26} & u_{27} & u_{28} \\ u_{31} & u_{32} & u_{33} & u_{34} & u_{35} & u_{36} & u_{37} & u_{38} \\ u_{41} & u_{42} & u_{43} & u_{44} & u_{45} & u_{46} & u_{47} & u_{48} \\ u_{51} & u_{52} & u_{53} & u_{54} & u_{55} & u_{56} & u_{57} & u_{58} \\ u_{61} & u_{62} & u_{63} & u_{64} & u_{65} & u_{66} & u_{67} & u_{68} \\ u_{71} & u_{72} & u_{73} & u_{74} & u_{75} & u_{76} & u_{77} & u_{78} \\ u_{81} & u_{82} & u_{83} & u_{84} & u_{85} & u_{86} & u_{87} & u_{88} \end{pmatrix}^T mod \langle p \rangle$$

then $u_{ij} = (-1)^{i+j} M_{ij} mod \langle p \rangle$, where M_{ij} is the $(i,j)$the minor of matrix U, $i, j = 1, 2, ..., 8$.

In particular, we have:

$u_{11} = -x_{12}x_{15}x_6 + x_{11}x_{16}x_6 + x_{12}x_{14}x_7 - x_{10}x_{16}x_7 - x_{11}x_{14}x_8 + x_{10}x_{15}x_8;$

$u_{21} = x_{12}x_{15}x_2 - x_{11}x_{16}x_2 - x_{12}x_{14}x_3 + x_{10}x_{16}x_3 + x_{11}x_{14}x_4 - x_{10}x_{15}x_4;$

$u_{31} = -x_{16}x_3x_6 + x_{15}x_4x_6 + x_{16}x_2x_7 - x_{14}x_4x_7 - x_{15}x_2x_8 + x_{14}x_3x_8;$

$u_{41} = x_{12}x_3x_6 - x_{11}x_4x_6 - x_{12}x_2x_7 + x_{10}x_4x_7 + x_{11}x_2x_8 - x_{10}x_3x_8;$

$u_{j1} = 0,\ j = 5, 6, 7, 8.$

$u_{12} = x_{12}x_{15}x_5 - x_{11}x_{16}x_5 - x_{12}x_{13}x_7 + x_{11}x_{13}x_8 + x_{16}x_7x_9 - x_{15}x_8x_9;$

$u_{22} = -x_1x_{12}x_{15} + x_1x_{11}x_{16} + x_{12}x_{13}x_3 - x_{11}x_{13}x_4 - x_{16}x_3x_9 + x_{15}x_4x_9;$

$u_{32} = x_{16}x_3x_5 - x_{15}x_4x_5 - x_1x_{16}x_7 + x_{13}x_4x_7 + x_1x_{15}x_8 - x_{13}x_3x_8;$

$u_{42} = -x_{12}x_3x_5 + x_{11}x_4x_5 + x_1x_{12}x_7 - x_1x_{11}x_8 - x_4x_7x_9 + x_3x_8x_9;$

$u_{j2} = 0,\ j = 5, 5, 7, 8;$

$u_{13} = -x_{12}x_{14}x_5 + x_{10}x_{16}x_5 + x_{12}x_{13}x_6 - x_{10}x_{13}x_8 - x_{16}x_6x_9 + x_{14}x_8x_9;$

$u_{23} = x_1x_{12}x_{14} - x_1x_{10}x_{16} - x_{12}x_{13}x_2 + x_{10}x_{13}x_4 + x_{16}x_2x_9 - x_{14}x_4x_9;$

$u_{33} = -x_{16}x_2x_5 + x_{14}x_4x_5 + x_1x_{16}x_6 - x_{13}x_4x_6 - x_1x_{14}x_8 + x_{13}x_2x_8;$

$u_{43} = x_{12}x_2x_5 - x_{10}x_4x_5 - x_1x_{12}x_6 + x_1x_{10}x_8 + x_4x_6x_9 - x_2x_8x_9;$

$u_{j3} = 0,\ j = 5, 6, 7, 8.$

$u_{14} = x_{11}x_{14}x_5 - x_{10}x_{15}x_5 - x_{11}x_{13}x_6 + x_{10}x_{13}x_7 + x_{15}x_6x_9 - x_{14}x_7x_9;$

$u_{24} = -x_1x_{11}x_{14} + x_1x_{10}x_{15} + x_{11}x_{13}x_2 - x_{10}x_{13}x_3 - x_{15}x_2x_9 + x_{14}x_3x_9;$

$u_{34} = x_{15}x_2x_5 - x_{14}x_3x_5 - x_1x_{15}x_6 + x_{13}x_3x_6 + x_1x_{14}x_7 - x_{13}x_2x_7;$

$u_{44} = -x_{11}x_2x_5 + x_{10}x_3x_5 + x_1x_{11}x_6 - x_1x_{10}x_7 - x_3x_6x_9 + x_2x_7x_9;$

$u_{j4} = 0,\ j = 5, 5, 7, 8;$

$u_{j5} = 0,\ j = 1, 2, 3, 4;$

$u_{55} = x_{16},\ u_{65} = -x_{12},\ u_{75} = x_8,\ u_{85} = -x_4;$

$u_{j6} = 0,\ j = 1, 2, 3, 4;$

$u_{56} = -x_{15},\ u_{66} = x_{11},\ u_{76} = -x_7,\ u_{87} = x_3;$

$u_{j7} = 0,\ j = 1, 2, 3, 4;$

$u_{57} = x_{14},\ u_{67} = -x_{10},\ u_{77} = x_6,\ u_{87} = -x_2;$

$u_{j8} = 0,\ j = 1, 2, 3, 4;$

$u_{58} = -x_{13},\ u_{68} = x_9,\ u_{78} = -x_5,\ u_{88} = x_1;$

One verifies that $UU^{-1} mod\, \langle p \rangle = \mathrm{Id}_8$

Now, we compute matrices $C_1 mod \langle p \rangle$, $C_2 mod \langle p \rangle$, $C_3 mod \langle p \rangle$, $C_4 mod \langle p \rangle$, $C_5 mod \langle p \rangle$, $C_6 mod \langle p \rangle$ as follows:

$C_i = UZ_iU^{-1} mod \langle p \rangle$, $i = 1, 2, 3, 4, 5, 6$. We have:

$C_1 = \left(c^1_{i,j}\right)$, where $c^1_{i,j} \neq 0$ are given as follows:

$$\begin{array}{lllllll}
c^1_{51} & = & x_{12}x_5 + x_{11}x_6 - x_{10}x_7 - x_8x_9 & c^1_{52} & = & -x_1x_{12} - x_{11}x_2 + x_{10}x_3 + x_4x_9 \\
c^1_{61} & = & x_{16}x_5 + x_{15}x_6 - x_{14}x_7 - x_{13}x_8 & c^1_{62} & = & -x_1x_{16} - x_{15}x_2 + x_{14}x_3 + x_{13}x_4 \\
c^1_{71} & = & -x_{12}x_{13} - x_{11}x_{14} + x_{10}x_{15} + x_{16}x_9 & c^1_{82} & = & -x_{12}x_{13} - x_{11}x_{14} + x_{10}x_{15} + x_{16}x_9 \\
c^1_{53} & = & -x_4x_5 - x_3x_6 + x_2x_7 + x_1x_8 & c^1_{64} & = & -x_4x_5 - x_3x_6 + x_2x_7 + x_1x_8 \\
c^1_{73} & = & -x_1x_{16} - x_{15}x_2 + x_{14}x_3 + x_{13}x_4 & c^1_{74} & = & x_1x_{12} + x_{11}x_2 - x_{10}x_3 - x_4x_9 \\
c^1_{83} & = & -x_{16}x_5 - x_{15}x_6 + x_{14}x_7 + x_{13}x_8 & c^1_{84} & = & x_{12}x_5 + x_{11}x_6 - x_{10}x_7 - x_8x_9 \\
c^1_{15} & = & x_1x_{16} + x_{15}x_2 - x_{14}x_3 - x_{13}x_4 & c^1_{16} & = & -x_1x_{12} - x_{11}x_2 + x_{10}x_3 + x_4x_9 \\
c^1_{25} & = & x_{16}x_5 + x_{15}x_6 - x_{14}x_7 - x_{13}x_8 & c^1_{26} & = & -x_{12}x_5 - x_{11}x_6 + x_{10}x_7 + x_8x_9 \\
c^1_{35} & = & -x_{12}x_{13} - x_{11}x_{14} + x_{10}x_{15} + x_{16}x_9 & c^1_{46} & = & -x_{12}x_{13} - x_{11}x_{14} + x_{10}x_{15} + x_{16}x_9 \\
c^1_{17} & = & -x_4x_5 - x_3x_6 + x_2x_7 + x_1x_8 & c^1_{28} & = & -x_4x_5 - x_3x_6 + x_2x_7 + x_1x_8 \\
c^1_{37} & = & -x_{12}x_5 - x_{11}x_6 + x_{10}x_7 + x_8x_9 & c^1_{38} & = & x_1x_{12} + x_{11}x_2 - x_{10}x_3 - x_4x_9 \\
c^1_{47} & = & -x_{16}x_5 - x_{15}x_6 + x_{14}x_7 + x_{13}x_8 & c^1_{48} & = & x_1x_{16} + x_{15}x_2 - x_{14}x_3 - x_{13}x_4
\end{array}$$

$C_2 = \left(c^2_{i,j}\right)$, where $c^2_{i,j} \neq 0$ are given in the table:

$$\begin{array}{lllllll}
c^2_{51} & = & -x_{11}x_5 + x_{12}x_6 - x_{10}x_8 + x_7x_9 & c^2_{52} & = & x_1x_{11} - x_{12}x_2 + x_{10}x_4 - x_3x_9 \\
c^2_{61} & = & -x_{15}x_5 + x_{16}x_6 + x_{13}x_7 - x_{14}x_8 & c^2_{62} & = & x_1x_{15} - x_{16}x_2 - x_{13}x_3 + x_{14}x_4 \\
c^2_{71} & = & x_{11}x_{13} - x_{12}x_{14} + x_{10}x_{16} - x_{15}x_9 & c^2_{82} & = & x_{11}x_{13} - x_{12}x_{14} + x_{10}x_{16} - x_{15}x_9 \\
c^2_{53} & = & x_3x_5 - x_4x_6 - x_1x_7 + x_2x_8 & c^2_{64} & = & x_3x_5 - x_4x_6 - x_1x_7 + x_2x_8 \\
c^2_{73} & = & x_1x_{15} - x_{16}x_2 - x_{13}x_3 + x_{14}x_4 & c^2_{74} & = & -x_1x_{11} + x_{12}x_2 - x_{10}x_4 + x_3x_9 \\
c^2_{83} & = & x_{15}x_5 - x_{16}x_6 - x_{13}x_7 + x_{14}x_8 & c^2_{84} & = & -x_{11}x_5 + x_{12}x_6 - x_{10}x_8 + x_7x_9 \\
c^2_{15} & = & -x_1x_{15} + x_{16}x_2 + x_{13}x_3 - x_{14}x_4 & c^2_{16} & = & x_1x_{11} - x_{12}x_2 + x_{10}x_4 - x_3x_9 \\
c^2_{25} & = & -x_{15}x_5 + x_{16}x_6 + x_{13}x_7 - x_{14}x_8 & c^2_{26} & = & x_{11}x_5 - x_{12}x_6 + x_{10}x_8 - x_7x_9 \\
c^2_{35} & = & x_{11}x_{13} - x_{12}x_{14} + x_{10}x_{16} - x_{15}x_9 & c^2_{46} & = & x_{11}x_{13} - x_{12}x_{14} + x_{10}x_{16} - x_{15}x_9
\end{array}$$

$$\begin{aligned}
c^2_{17} &= x_3x_5 - x_4x_6 - x_1x_7 + x_2x_8 & c^2_{28} &= x_3x_5 - x_4x_6 - x_1x_7 + x_2x_8 \\
c^2_{37} &= x_{11}x_5 - x_{12}x_6 + x_{10}x_8 - x_7x_9 & c^2_{38} &= -x_1x_{11} + x_{12}x_2 - x_{10}x_4 + x_3x_9 \\
c^2_{47} &= x_{15}x_5 - x_{16}x_6 - x_{13}x_7 + x_{14}x_8 & c^2_{48} &= -x_1x_{15} + x_{16}x_2 + x_{13}x_3 - x_{14}x_4
\end{aligned}$$

$C_3 = \left(c^3_{i,j}\right)$, where $c^3_{i,j} \neq 0$ are:

$$\begin{aligned}
c^3_{51} &= x_{10}x_5 + x_{12}x_7 - x_{11}x_8 - x_6x_9 & c^3_{52} &= -x_1x_{10} - x_{12}x_3 + x_{11}x_4 + x_2x_9 \\
c^3_{61} &= x_{14}x_5 - x_{13}x_6 + x_{16}x_7 - x_{15}x_8 & c^3_{62} &= -x_1x_{14} + x_{13}x_2 - x_{16}x_3 + x_{15}x_4 \\
c^3_{71} &= -x_{10}x_{13} - x_{12}x_{15} + x_{11}x_{16} + x_{14}x_9 & c^3_{82} &= -x_{10}x_{13} - x_{12}x_{15} + x_{11}x_{16} + x_{14}x_9 \\
c^3_{53} &= -x_2x_5 + x_1x_6 - x_4x_7 + x_3x_8 & c^3_{64} &= -x_2x_5 + x_1x_6 - x_4x_7 + x_3x_8 \\
c^3_{73} &= -x_1x_{14} + x_{13}x_2 - x_{16}x_3 + x_{15}x_4 & c^3_{74} &= x_1x_{10} + x_{12}x_3 - x_{11}x_4 - x_2x_9 \\
c^3_{83} &= -x_{14}x_5 + x_{13}x_6 - x_{16}x_7 + x_{15}x_8 & c^3_{84} &= x_{10}x_5 + x_{12}x_7 - x_{11}x_8 - x_6x_9 \\
c^3_{15} &= x_1x_{14} - x_{13}x_2 + x_{16}x_3 - x_{15}x_4 & c^3_{16} &= -x_1x_{10} - x_{12}x_3 + x_{11}x_4 + x_2x_9 \\
c^3_{25} &= x_{14}x_5 - x_{13}x_6 + x_{16}x_7 - x_{15}x_8 & c^3_{26} &= -x_{10}x_5 - x_{12}x_7 + x_{11}x_8 + x_6x_9 \\
c^3_{35} &= -x_{10}x_{13} - x_{12}x_{15} + x_{11}x_{16} + x_{14}x_9 & c^3_{46} &= -x_{10}x_{13} - x_{12}x_{15} + x_{11}x_{16} + x_{14}x_9 \\
c^3_{17} &= -x_2x_5 + x_1x_6 - x_4x_7 + x_3x_8 & c^3_{28} &= -x_2x_5 + x_1x_6 - x_4x_7 + x_3x_8 \\
c^3_{37} &= -x_{10}x_5 - x_{12}x_7 + x_{11}x_8 + x_6x_9 & c^3_{38} &= x_1x_{10} + x_{12}x_3 - x_{11}x_4 - x_2x_9 \\
c^3_{47} &= -x_{14}x_5 + x_{13}x_6 - x_{16}x_7 + x_{15}x_8 & c^3_{48} &= x_1x_{14} - x_{13}x_2 + x_{16}x_3 - x_{15}x_4
\end{aligned}$$

$C_4 = \left(c^4_{i,j}\right)$, where $c^4_{i,j} \neq 0$ are given in the table:

$$\begin{aligned}
c^4_{51} &= x_{12}x_5 - x_{11}x_6 + x_{10}x_7 - x_8x_9 & c^4_{52} &= -x_1x_{12} + x_{11}x_2 - x_{10}x_3 + x_4x_9 \\
c^4_{61} &= x_{16}x_5 - x_{15}x_6 + x_{14}x_7 - x_{13}x_8 & c^4_{62} &= -x_1x_{16} + x_{15}x_2 - x_{14}x_3 + x_{13}x_4 \\
c^4_{71} &= -x_{12}x_{13} + x_{11}x_{14} - x_{10}x_{15} + x_{16}x_9 & c^4_{82} &= -x_{12}x_{13} + x_{11}x_{14} - x_{10}x_{15} + x_{16}x_9 \\
c^4_{53} &= -x_4x_5 + x_3x_6 - x_2x_7 + x_1x_8 & c^4_{64} &= -x_4x_5 + x_3x_6 - x_2x_7 + x_1x_8 \\
c^4_{73} &= -x_1x_{16} + x_{15}x_2 - x_{14}x_3 + x_{13}x_4 & c^4_{74} &= x_1x_{12} - x_{11}x_2 + x_{10}x_3 - x_4x_9 \\
c^4_{83} &= -x_{16}x_5 + x_{15}x_6 - x_{14}x_7 + x_{13}x_8 & c^4_{84} &= x_{12}x_5 - x_{11}x_6 + x_{10}x_7 - x_8x_9 \\
c^4_{15} &= x_1x_{16} - x_{15}x_2 + x_{14}x_3 - x_{13}x_4 & c^4_{16} &= -x_1x_{12} + x_{11}x_2 - x_{10}x_3 + x_4x_9 \\
c^4_{25} &= x_{16}x_5 - x_{15}x_6 + x_{14}x_7 - x_{13}x_8 & c^4_{26} &= -x_{12}x_5 + x_{11}x_6 - x_{10}x_7 + x_8x_9 \\
c^4_{35} &= -x_{12}x_{13} + x_{11}x_{14} - x_{10}x_{15} + x_{16}x_9 & c^4_{46} &= -x_{12}x_{13} + x_{11}x_{14} - x_{10}x_{15} + x_{16}x_9
\end{aligned}$$

$$
\begin{aligned}
c^4_{17} &= -x_4x_5 + x_3x_6 - x_2x_7 + x_1x_8 & c^4_{28} &= -x_4x_5 + x_3x_6 - x_2x_7 + x_1x_8 \\
c^4_{37} &= -x_{12}x_5 + x_{11}x_6 - x_{10}x_7 + x_8x_9 & c^4_{38} &= x_1x_{12} - x_{11}x_2 + x_{10}x_3 - x_4x_9 \\
c^4_{47} &= -x_{16}x_5 + x_{15}x_6 - x_{14}x_7 + x_{13}x_8 & c^4_{48} &= x_1x_{16} - x_{15}x_2 + x_{14}x_3 - x_{13}x_4
\end{aligned}
$$

$C_5 = \left(c^5_{i,j}\right)$, where $c^5_{i,j} \neq 0$:

$$
\begin{aligned}
c^5_{51} &= -x_{11}x_5 - x_{12}x_6 + x_{10}x_8 + x_7x_9 & c^5_{52} &= x_1x_{11} + x_{12}x_2 - x_{10}x_4 - x_3x_9 \\
c^5_{61} &= -x_{15}x_5 - x_{16}x_6 + x_{13}x_7 + x_{14}x_8 & c^5_{62} &= x_1x_{15} + x_{16}x_2 - x_{13}x_3 - x_{14}x_4 \\
c^5_{71} &= x_{11}x_{13} + x_{12}x_{14} - x_{10}x_{16} - x_{15}x_9 & c^5_{82} &= x_{11}x_{13} + x_{12}x_{14} - x_{10}x_{16} - x_{15}x_9 \\
c^5_{53} &= x_3x_5 + x_4x_6 - x_1x_7 - x_2x_8 & c^5_{64} &= x_3x_5 + x_4x_6 - x_1x_7 - x_2x_8 \\
c^5_{73} &= x_1x_{15} + x_{16}x_2 - x_{13}x_3 - x_{14}x_4 & c^5_{74} &= -x_1x_{11} - x_{12}x_2 + x_{10}x_4 + x_3x_9 \\
c^5_{83} &= x_{15}x_5 + x_{16}x_6 - x_{13}x_7 - x_{14}x_8 & c^5_{84} &= -x_{11}x_5 - x_{12}x_6 + x_{10}x_8 + x_7x_9 \\
c^5_{15} &= -x_1x_{15} - x_{16}x_2 + x_{13}x_3 + x_{14}x_4 & c^5_{16} &= x_1x_{11} + x_{12}x_2 - x_{10}x_4 - x_3x_9 \\
c^5_{25} &= -x_{15}x_5 - x_{16}x_6 + x_{13}x_7 + x_{14}x_8 & c^5_{26} &= x_{11}x_5 + x_{12}x_6 - x_{10}x_8 - x_7x_9 \\
c^5_{35} &= x_{11}x_{13} + x_{12}x_{14} - x_{10}x_{16} - x_{15}x_9 & c^5_{46} &= x_{11}x_{13} + x_{12}x_{14} - x_{10}x_{16} - x_{15}x_9 \\
c^5_{17} &= x_3x_5 + x_4x_6 - x_1x_7 - x_2x_8 & c^5_{28} &= x_3x_5 + x_4x_6 - x_1x_7 - x_2x_8 \\
c^5_{37} &= x_{11}x_5 + x_{12}x_6 - x_{10}x_8 - x_7x_9 & c^5_{38} &= -x_1x_{11} - x_{12}x_2 + x_{10}x_4 + x_3x_9 \\
c^5_{47} &= x_{15}x_5 + x_{16}x_6 - x_{13}x_7 - x_{14}x_8 & c^5_{48} &= -x_1x_{15} - x_{16}x_2 + x_{13}x_3 + x_{14}x_4
\end{aligned}
$$

$C_6 = \left(c^6_{i,j}\right)$, where $c^6_{i,j} \neq 0$:

$$
\begin{aligned}
c^6_{51} &= x_{10}x_5 - x_{12}x_7 + x_{11}x_8 - x_6x_9 & c^6_{52} &= -x_1x_{10} + x_{12}x_3 - x_{11}x_4 + x_2x_9 \\
c^6_{61} &= x_{14}x_5 - x_{13}x_6 - x_{16}x_7 + x_{15}x_8 & c^6_{62} &= -x_1x_{14} + x_{13}x_2 + x_{16}x_3 - x_{15}x_4 \\
c^6_{71} &= -x_{10}x_{13} + x_{12}x_{15} - x_{11}x_{16} + x_{14}x_9 & c^6_{82} &= -x_{10}x_{13} + x_{12}x_{15} - x_{11}x_{16} + x_{14}x_9 \\
c^6_{53} &= -x_2x_5 + x_1x_6 + x_4x_7 - x_3x_8 & c^6_{64} &= -x_2x_5 + x_1x_6 + x_4x_7 - x_3x_8 \\
c^6_{73} &= -x_1x_{14} + x_{13}x_2 + x_{16}x_3 - x_{15}x_4 & c^6_{74} &= x_1x_{10} - x_{12}x_3 + x_{11}x_4 - x_2x_9 \\
c^6_{83} &= -x_{14}x_5 + x_{13}x_6 + x_{16}x_7 - x_{15}x_8 & c^6_{84} &= x_{10}x_5 - x_{12}x_7 + x_{11}x_8 - x_6x_9 \\
c^6_{15} &= x_1x_{14} - x_{13}x_2 - x_{16}x_3 + x_{15}x_4 & c^6_{16} &= -x_1x_{10} + x_{12}x_3 - x_{11}x_4 + x_2x_9 \\
c^6_{25} &= x_{14}x_5 - x_{13}x_6 - x_{16}x_7 + x_{15}x_8 & c^6_{26} &= -x_{10}x_5 + x_{12}x_7 - x_{11}x_8 + x_6x_9 \\
c^6_{35} &= -x_{10}x_{13} + x_{12}x_{15} - x_{11}x_{16} + x_{14}x_9 & c^6_{46} &= -x_{10}x_{13} + x_{12}x_{15} - x_{11}x_{16} + x_{14}x_9
\end{aligned}
$$

$$
\begin{array}{rclrcl}
c^6_{17} & = & -x_2x_5 + x_1x_6 + x_4x_7 - x_3x_8 & c^6_{28} & = & -x_2x_5 + x_1x_6 + x_4x_7 - x_3x_8 \\
c^6_{37} & = & -x_{10}x_5 + x_{12}x_7 - x_{11}x_8 + x_6x_9 & c^6_{38} & = & x_1x_{10} - x_{12}x_3 + x_{11}x_4 - x_2x_9 \\
c^6_{47} & = & -x_{14}x_5 + x_{13}x_6 + x_{16}x_7 - x_{15}x_8 & c^6_{48} & = & x_1x_{14} - x_{13}x_2 - x_{16}x_3 + x_{15}x_4
\end{array}
$$

Now we express the matrices C_i in terms of the basis $\mathcal{B} = \{Z_1,\ Z_2,\ Z_3,\ Z_4,\ Z_5,\ Z_6\}$ as follows:

$$C_j = \sum_{i=1}^{6} m_{i,j} Z_i, \text{ where } m_{i,j} = \frac{1}{\langle Z_i, Z_i\rangle} \langle C_j, Z_i\rangle \, mod \, \langle p\rangle \text{ and } \langle A, B\rangle = \mathrm{Tr}\left(B^T A\right)$$

This produces a matrix M:

$$M = \begin{pmatrix} m_{1,1} & m_{1,2} & m_{1,3} & m_{1,4} & m_{1,5} & m_{1,6} \\ m_{2,1} & m_{2,2} & m_{2,3} & m_{2,4} & m_{2,5} & m_{2,6} \\ m_{3,1} & m_{3,2} & m_{3,3} & m_{3,4} & m_{3,5} & m_{3,6} \\ m_{4,1} & m_{4,2} & m_{4,3} & m_{4,4} & m_{4,5} & m_{4,6} \\ m_{5,1} & m_{5,2} & m_{5,3} & m_{5,4} & m_{5,5} & m_{5,6} \\ m_{6,1} & m_{6,2} & m_{6,3} & m_{6,4} & m_{6,5} & m_{6,6} \end{pmatrix} \tag{3.7}$$

The entries $m_{i,j}$ are given by

$$
\begin{array}{rcl}
m_{1,1} & = & \frac{1}{2}(x_1x_{16} + x_{15}x_2 - x_{14}x_3 - x_{13}x_4 + x_{12}x_5 + x_{11}x_6 - x_{10}x_7 - x_8x_9) \\
m_{2,1} & = & \frac{1}{2}(-x_1x_{12} - x_{11}x_2 + x_{10}x_3 + x_{16}x_5 + x_{15}x_6 - x_{14}x_7 - x_{13}x_8 + x_4x_9) \\
m_{3,1} & = & \frac{1}{2}(-x_{12}x_{13} - x_{11}x_{14} + x_{10}x_{15} - x_4x_5 - x_3x_6 + x_2x_7 + x_1x_8 + x_{16}x_9) \\
m_{4,1} & = & \frac{1}{2}(x_1x_{16} + x_{15}x_2 - x_{14}x_3 - x_{13}x_4 - x_{12}x_5 - x_{11}x_6 + x_{10}x_7 + x_8x_9) \\
m_{5,1} & = & \frac{1}{2}(-x_1x_{12} - x_{11}x_2 + x_{10}x_3 - x_{16}x_5 - x_{15}x_6 + x_{14}x_7 + x_{13}x_8 + x_4x_9) \\
m_{6,1} & = & \frac{1}{2}(x_{12}x_{13} + x_{11}x_{14} - x_{10}x_{15} - x_4x_5 - x_3x_6 + x_2x_7 + x_1x_8 - x_{16}x_9) \\
m_{1,2} & = & \frac{1}{2}(-x_1x_{15} + x_{16}x_2 + x_{13}x_3 - x_{14}x_4 - x_{11}x_5 + x_{12}x_6 - x_{10}x_8 + x_7x_9) \\
m_{2,2} & = & \frac{1}{2}(x_1x_{11} - x_{12}x_2 + x_{10}x_4 - x_{15}x_5 + x_{16}x_6 + x_{13}x_7 - x_{14}x_8 - x_3x_9) \\
m_{3,2} & = & \frac{1}{2}(x_{11}x_{13} - x_{12}x_{14} + x_{10}x_{16} + x_3x_5 - x_4x_6 - x_1x_7 + x_2x_8 - x_{15}x_9) \\
m_{4,2} & = & \frac{1}{2}(-x_1x_{15} + x_{16}x_2 + x_{13}x_3 - x_{14}x_4 + x_{11}x_5 - x_{12}x_6 + x_{10}x_8 - x_7x_9) \\
m_{5,2} & = & \frac{1}{2}(x_1x_{11} - x_{12}x_2 + x_{10}x_4 + x_{15}x_5 - x_{16}x_6 - x_{13}x_7 + x_{14}x_8 - x_3x_9) \\
m_{6,2} & = & \frac{1}{2}(-x_{11}x_{13} + x_{12}x_{14} - x_{10}x_{16} + x_3x_5 - x_4x_6 - x_1x_7 + x_2x_8 + x_{15}x_9)
\end{array}
$$

$$
\begin{aligned}
m_{1,3} &= \tfrac{1}{2}(x_1x_{14} - x_{13}x_2 + x_{16}x_3 - x_{15}x_4 + x_{10}x_5 + x_{12}x_7 - x_{11}x_8 - x_6x_9) \\
m_{2,3} &= \tfrac{1}{2}(-x_1x_{10} - x_{12}x_3 + x_{11}x_4 + x_{14}x_5 - x_{13}x_6 + x_{16}x_7 - x_{15}x_8 + x_2x_9) \\
m_{3,3} &= \tfrac{1}{2}(-x_{10}x_{13} - x_{12}x_{15} + x_{11}x_{16} - x_2x_5 + x_1x_6 - x_4x_7 + x_3x_8 + x_{14}x_9) \\
m_{4,3} &= \tfrac{1}{2}(x_1x_{14} - x_{13}x_2 + x_{16}x_3 - x_{15}x_4 - x_{10}x_5 - x_{12}x_7 + x_{11}x_8 + x_6x_9) \\
m_{5,3} &= \tfrac{1}{2}(-x_1x_{10} - x_{12}x_3 + x_{11}x_4 - x_{14}x_5 + x_{13}x_6 - x_{16}x_7 + x_{15}x_8 + x_2x_9) \\
m_{6,3} &= \tfrac{1}{2}(x_{10}x_{13} + x_{12}x_{15} - x_{11}x_{16} - x_2x_5 + x_1x_6 - x_4x_7 + x_3x_8 - x_{14}x_9) \\
m_{1,4} &= \tfrac{1}{2}(x_1x_{16} - x_{15}x_2 + x_{14}x_3 - x_{13}x_4 + x_{12}x_5 - x_{11}x_6 + x_{10}x_7 - x_8x_9) \\
m_{2,4} &= \tfrac{1}{2}(-x_1x_{12} + x_{11}x_2 - x_{10}x_3 + x_{16}x_5 - x_{15}x_6 + x_{14}x_7 - x_{13}x_8 + x_4x_9) \\
m_{3,4} &= \tfrac{1}{2}(-x_{12}x_{13} + x_{11}x_{14} - x_{10}x_{15} - x_4x_5 + x_3x_6 - x_2x_7 + x_1x_8 + x_{16}x_9) \\
m_{4,4} &= \tfrac{1}{2}(x_1x_{16} - x_{15}x_2 + x_{14}x_3 - x_{13}x_4 - x_{12}x_5 + x_{11}x_6 - x_{10}x_7 + x_8x_9) \\
m_{5,4} &= \tfrac{1}{2}(-x_1x_{12} + x_{11}x_2 - x_{10}x_3 - x_{16}x_5 + x_{15}x_6 - x_{14}x_7 + x_{13}x_8 + x_4x_9) \\
m_{6,4} &= \tfrac{1}{2}(x_{12}x_{13} - x_{11}x_{14} + x_{10}x_{15} - x_4x_5 + x_3x_6 - x_2x_7 + x_1x_8 - x_{16}x_9) \\
m_{1,5} &= \tfrac{1}{2}(-x_1x_{15} - x_{16}x_2 + x_{13}x_3 + x_{14}x_4 - x_{11}x_5 - x_{12}x_6 + x_{10}x_8 + x_7x_9) \\
m_{2,5} &= \tfrac{1}{2}(x_1x_{11} + x_{12}x_2 - x_{10}x_4 - x_{15}x_5 - x_{16}x_6 + x_{13}x_7 + x_{14}x_8 - x_3x_9) \\
m_{3,5} &= \tfrac{1}{2}(x_{11}x_{13} + x_{12}x_{14} - x_{10}x_{16} + x_3x_5 + x_4x_6 - x_1x_7 - x_2x_8 - x_{15}x_9) \\
m_{4,5} &= \tfrac{1}{2}(-x_1x_{15} - x_{16}x_2 + x_{13}x_3 + x_{14}x_4 + x_{11}x_5 + x_{12}x_6 - x_{10}x_8 - x_7x_9) \\
m_{5,5} &= \tfrac{1}{2}(x_1x_{11} + x_{12}x_2 - x_{10}x_4 + x_{15}x_5 + x_{16}x_6 - x_{13}x_7 - x_{14}x_8 - x_3x_9) \\
m_{6,5} &= \tfrac{1}{2}(-x_{11}x_{13} - x_{12}x_{14} + x_{10}x_{16} + x_3x_5 + x_4x_6 - x_1x_7 - x_2x_8 + x_{15}x_9) \\
m_{1,6} &= \tfrac{1}{2}(x_1x_{14} - x_{13}x_2 - x_{16}x_3 + x_{15}x_4 + x_{10}x_5 - x_{12}x_7 + x_{11}x_8 - x_6x_9) \\
m_{2,6} &= \tfrac{1}{2}(-x_1x_{10} + x_{12}x_3 - x_{11}x_4 + x_{14}x_5 - x_{13}x_6 - x_{16}x_7 + x_{15}x_8 + x_2x_9) \\
m_{3,6} &= \tfrac{1}{2}(-x_{10}x_{13} + x_{12}x_{15} - x_{11}x_{16} - x_2x_5 + x_1x_6 + x_4x_7 - x_3x_8 + x_{14}x_9) \\
m_{4,6} &= \tfrac{1}{2}(x_1x_{14} - x_{13}x_2 - x_{16}x_3 + x_{15}x_4 - x_{10}x_5 + x_{12}x_7 - x_{11}x_8 + x_6x_9) \\
m_{5,6} &= \tfrac{1}{2}(-x_1x_{10} + x_{12}x_3 - x_{11}x_4 - x_{14}x_5 + x_{13}x_6 + x_{16}x_7 - x_{15}x_8 + x_2x_9) \\
m_{6,6} &= \tfrac{1}{2}(x_{10}x_{13} - x_{12}x_{15} + x_{11}x_{16} - x_2x_5 + x_1x_6 + x_4x_7 - x_3x_8 - x_{14}x_9)
\end{aligned}
$$

Theorem 3.5. *The forward map* $\phi_{3,3} : Spin^{+}(3,3) : SO^{+}(3,3)$ *sends the matrix* T *as in Equation (3.6) to the matrix* M *as in Equation (3.7).*

CHAPTER 4

INVERSION OF THE DOUBLE-COVERING MAP

Since the statements of theorems will get extremely long, we follow the following format. Each target in $SO^+(p,q)$ can be expressed constructively as a product of standard and hyperbolic Givens. One just has to mimic Example (2.4). Therefore, in each section we will just develop the output of the inversion when the target is one of these factors.

4.1 From $SO^+(2,1)$ to $Spin^+(2,1)$

Following Example (2.4) every matrix in $SO^+(2,1)$ is a product of $R_{1,2}H_{1,3}H_{2,3}$.

Let $c = \cos\theta$, $s = \sin\theta$, $a = \cosh(\theta)$ and $b = \sinh(\theta)$ and

$$R_{1,2} = \begin{pmatrix} c & -s & 0 \\ s & c & 0 \\ 0 & 0 & 1 \end{pmatrix}, H_{1,3} = \begin{pmatrix} a & 0 & b \\ 0 & 1 & 0 \\ b & 0 & a \end{pmatrix}, H_{2,3} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & a & b \\ 0 & b & a \end{pmatrix}.$$

- $R_{1,2}$ case

  We solve the system of quadratic equations which arise when the target is standard Givens $R_{1,2}$.

$$D = R_{1,2},$$

If $\theta \in (0, 2\pi)$, then $\sin\left(\frac{\theta}{2}\right) > 0$. Let $\widehat{\mathrm{c}} = \cos\left(\frac{\theta}{2}\right)$, $\widehat{\mathrm{s}} = \sin\left(\frac{\theta}{2}\right)$, $\widehat{\mathrm{ch}} = \cosh\left(\frac{\beta}{2}\right)$ and $\widehat{\mathrm{sh}} = \sinh\left(\frac{\beta}{2}\right)$ then

$$\mathrm{S}\left(R_{1,2}\right) = \pm \begin{pmatrix} \widehat{\mathrm{c}} & 0 & -\widehat{\mathrm{s}} & 0 \\ 0 & -\widehat{\mathrm{c}} & 0 & -\widehat{\mathrm{s}} \\ \widehat{\mathrm{s}} & 0 & \widehat{\mathrm{c}} & 0 \\ 0 & \widehat{\mathrm{s}} & 0 & -\widehat{\mathrm{c}} \end{pmatrix}$$

So $\phi_{2,1}^{-1}(R_{1,2}) = \pm \begin{pmatrix} \widehat{\mathrm{c}} & -\widehat{\mathrm{s}} \\ \widehat{\mathrm{s}} & \widehat{\mathrm{c}} \end{pmatrix}$. For $\theta = 0, 2\pi$ we get

$$\pm \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

For $\theta = 0, 2\pi$ the inverse image is $\phi_{2,1}^{-1}(R_{1,2}) = \mp \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

- $H_{1,3}$ case

  We solve the system of quadratic equations which arise when the target is hyperbolic Givens $H_{1,3}$.

$$D = H_{1,3}$$

If $\beta > 0$ then we have

$$\mathrm{S}\,(H_{1,3}) = \pm \begin{pmatrix} \widehat{\mathrm{ch}} & 0 & -\widehat{\mathrm{sh}} & 0 \\ 0 & -\widehat{\mathrm{ch}} & 0 & -\widehat{\mathrm{sh}} \\ -\widehat{\mathrm{sh}} & 0 & \widehat{\mathrm{ch}} & 0 \\ 0 & -\widehat{\mathrm{sh}} & 0 & -\widehat{\mathrm{ch}} \end{pmatrix}$$

So the inverse image is $\phi_{2,1}^{-1}(H_{1,3}) = \pm \begin{pmatrix} \widehat{\mathrm{ch}} & -\widehat{\mathrm{sh}} \\ -\widehat{\mathrm{sh}} & \widehat{\mathrm{ch}} \end{pmatrix}$.

Now, if $\beta < 0$, then we have

$$\mathrm{S}\,(H_{1,3}) = \mp \begin{pmatrix} \widehat{\mathrm{ch}} & 0 & -\widehat{\mathrm{sh}} & 0 \\ 0 & -\widehat{\mathrm{ch}} & 0 & -\widehat{\mathrm{sh}} \\ -\widehat{\mathrm{sh}} & 0 & \widehat{\mathrm{ch}} & 0 \\ 0 & -\widehat{\mathrm{sh}} & 0 & -\widehat{\mathrm{ch}} \end{pmatrix}$$

Thus the inverse image is $\phi_{2,1}^{-1}(H_{1,3}) = \pm \begin{pmatrix} \widehat{\mathrm{ch}} & -\widehat{\mathrm{sh}} \\ -\widehat{\mathrm{sh}} & \widehat{\mathrm{ch}} \end{pmatrix}$. The remaining case is $\beta = 0$, we have in such a case

$$\pm \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

So the inverse image is $\phi_{2,1}^{-1}(H_{1,3}) = \mp \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

- $H_{2,3}$ case

  We solve the system of quadratic equations which arise when the target is hyperbolic Givens $H_{2,3}$.

$$D = H_{2,3}$$

and we have the following solutions:

$$\mathrm{S}\left(H_{2,3}\right) = \pm \begin{pmatrix} -e^{\beta/2} & 0 & 0 & 0 \\ 0 & e^{\beta/2} & 0 & 0 \\ 0 & 0 & -e^{-\beta/2} & 0 \\ 0 & 0 & 0 & e^{-\beta/2} \end{pmatrix}$$

So the inverse image is $\phi_{2,1}^{-1}(H_{2,3}) = \pm \begin{pmatrix} -e^{\frac{\beta}{2}} & 0 \\ 0 & e^{-\frac{\beta}{2}} \end{pmatrix}$.

4.2 From $SO^{+}(2,2)$ to $Spin^{+}(2,2)$

Following Example (2.4) every matrix in $SO^{+}(2,2)$ is a product of $R_{1,2}, R_{3,4}, H_{1,3}, H_{1,4}, H_{2,4}$, where

$$R_{1,2} = \begin{pmatrix} c & -s & 0 & 0 \\ s & c & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, R_{3,4} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & c & -s \\ 0 & 0 & s & c \end{pmatrix}, H_{1,3} = \begin{pmatrix} a & 0 & b & 0 \\ 0 & 1 & 0 & 0 \\ b & 0 & a & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$H_{1,4} = \begin{pmatrix} a & 0 & 0 & b \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ b & 0 & 0 & a \end{pmatrix}, H_{2,4} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & a & 0 & b \\ 0 & 0 & 1 & 0 \\ 0 & b & 0 & a \end{pmatrix}.$$

Let $c = \cos\theta$, $s = \sin\theta$, $a = \cosh(\theta)$ and $b = \sinh(\theta)$

- $R_{1,2}$ case

  We solve the system of quadratic equations which arise when the target is stndard Givens $R_{1,2}$.

$$D = R_{1,2}$$

If $\theta \in (0,\pi)$, then

$$\mathrm{S}(R_{1,2}) = \mp \begin{pmatrix} \widehat{\mathrm{c}} & 0 & 0 & -\widehat{\mathrm{s}} \\ 0 & \widehat{\mathrm{c}} & -\widehat{\mathrm{s}} & 0 \\ 0 & \widehat{\mathrm{s}} & \widehat{\mathrm{c}} & 0 \\ \widehat{\mathrm{s}} & 0 & 0 & \widehat{\mathrm{c}} \end{pmatrix}$$

So the inverse image is $\phi_{2,2}^{-1}(R_{1,2}) = \mp \left[\begin{pmatrix} \widehat{\mathrm{c}} & -\widehat{\mathrm{s}} \\ \widehat{\mathrm{s}} & \widehat{\mathrm{c}} \end{pmatrix}, \begin{pmatrix} \widehat{\mathrm{c}} & -\widehat{\mathrm{s}} \\ \widehat{\mathrm{s}} & \widehat{\mathrm{c}} \end{pmatrix} \right]$.

If $\theta \in (\pi, 2\pi)$, then

$$\mathrm{S}\left(R_{1,2}\right) = \pm \begin{pmatrix} \widehat{\mathrm{c}} & 0 & 0 & -\widehat{\mathrm{s}} \\ 0 & \widehat{\mathrm{c}} & -\widehat{\mathrm{s}} & 0 \\ 0 & \widehat{\mathrm{s}} & \widehat{\mathrm{c}} & 0 \\ \widehat{\mathrm{s}} & 0 & 0 & \widehat{\mathrm{c}} \end{pmatrix}$$

So the inverse image is $\phi_{2,2}^{-1}(R_{1,2}) = \pm \left[\begin{pmatrix} \widehat{\mathrm{c}} & -\widehat{\mathrm{s}} \\ \widehat{\mathrm{s}} & \widehat{\mathrm{c}} \end{pmatrix}, \begin{pmatrix} \widehat{\mathrm{c}} & -\widehat{\mathrm{s}} \\ \widehat{\mathrm{s}} & \widehat{\mathrm{c}} \end{pmatrix} \right]$.

- $R_{3,4}$ case

  We solve the system of quadratic equations which arise when the target is standard Givens $R_{3,4}$.

$$D = R_{3,4}$$

If $\theta \in (0, \pi)$, then

$$\mathrm{S}\left(R_{3,4}\right) = \mp \begin{pmatrix} \widehat{\mathrm{c}} & 0 & 0 & \widehat{\mathrm{s}} \\ 0 & \widehat{\mathrm{c}} & -\widehat{\mathrm{s}} & 0 \\ 0 & \widehat{\mathrm{s}} & \widehat{\mathrm{c}} & 0 \\ -\widehat{\mathrm{s}} & 0 & 0 & \widehat{\mathrm{c}} \end{pmatrix}$$

So the inverse image is $\phi_{2,2}^{-1}(R_{3,4}) = \mp \left[\begin{pmatrix} \widehat{\mathrm{c}} & \widehat{\mathrm{s}} \\ -\widehat{\mathrm{s}} & \widehat{\mathrm{c}} \end{pmatrix}, \begin{pmatrix} \widehat{\mathrm{c}} & -\widehat{\mathrm{s}} \\ \widehat{\mathrm{s}} & \widehat{\mathrm{c}} \end{pmatrix} \right]$.

If $\theta \in (\pi, 2\pi)$, then

$$\mathrm{S}\left(R_{3,4}\right) = \pm \begin{pmatrix} \widehat{\mathrm{c}} & 0 & 0 & \widehat{\mathrm{s}} \\ 0 & \widehat{\mathrm{c}} & -\widehat{\mathrm{s}} & 0 \\ 0 & \widehat{\mathrm{s}} & \widehat{\mathrm{c}} & 0 \\ -\widehat{\mathrm{s}} & 0 & 0 & \widehat{\mathrm{c}} \end{pmatrix}.$$

So the inverse image is $\phi_{2,2}^{-1}(R_{3,4}) = \pm\left[\begin{pmatrix} \widehat{\mathrm{c}} & \widehat{\mathrm{s}} \\ -\widehat{\mathrm{s}} & \widehat{\mathrm{c}} \end{pmatrix}, \begin{pmatrix} \widehat{\mathrm{c}} & -\widehat{\mathrm{s}} \\ \widehat{\mathrm{s}} & \widehat{\mathrm{c}} \end{pmatrix}\right]$.

- $H_{1,3}$ case

  We solve the system of quadratic equations which arise when the target is hyperbolic Givens $H_{1,3}$.

$$D = H_{1,3}$$

We have

$$\mathrm{S}\left(H_{1,3}\right) = \mp\begin{pmatrix} e^{\beta/2} & 0 & 0 & 0 \\ 0 & e^{-\beta/2} & 0 & 0 \\ 0 & 0 & e^{\beta/2} & 0 \\ 0 & 0 & 0 & e^{-\beta/2} \end{pmatrix},$$

So the inverse image is $\phi_{2,2}^{-1}(H_{1,3}) = \mp\left[\begin{pmatrix} e^{\beta/2} & 0 \\ 0 & e^{-\beta/2} \end{pmatrix}, \begin{pmatrix} e^{-\beta/2} & 0 \\ 0 & e^{\beta/2} \end{pmatrix}\right]$.

- $H_{1,4}$ case

  We solve the system of quadratic equations which arise when the target is hyperbolic Givens $H_{1,4}$

$$D = H_{1,4}$$

We have

$$\mathrm{S}\left(H_{1,4}\right) = \pm\begin{pmatrix} \widehat{\mathrm{ch}} & 0 & 0 & -\widehat{\mathrm{sh}} \\ 0 & \widehat{\mathrm{ch}} & -\widehat{\mathrm{sh}} & 0 \\ 0 & -\widehat{\mathrm{sh}} & \widehat{\mathrm{ch}} & 0 \\ -\widehat{\mathrm{sh}} & 0 & 0 & \widehat{\mathrm{ch}} \end{pmatrix}$$

So the inverse image is $\phi_{2,2}^{-1}(H_{1,4}) = \pm\left[\begin{pmatrix} \widehat{\mathrm{ch}} & -\widehat{\mathrm{sh}} \\ -\widehat{\mathrm{sh}} & \widehat{\mathrm{ch}} \end{pmatrix}, \begin{pmatrix} \widehat{\mathrm{ch}} & -\widehat{\mathrm{sh}} \\ -\widehat{\mathrm{sh}} & \widehat{\mathrm{ch}} \end{pmatrix}\right]$.

- $H_{2,4}$ case

  We solve the system of quadratic equations which arise when the target is hyperbolic Givens $H_{2,4}$.

$$D = H_{2,4}$$

Finally, we have

$$\mathrm{S}\left(H_{2,4}\right) = \mp \begin{pmatrix} e^{\beta/2} & 0 & 0 & 0 \\ 0 & e^{\beta/2} & 0 & 0 \\ 0 & 0 & e^{-\beta/2} & 0 \\ 0 & 0 & 0 & e^{-\beta/2} \end{pmatrix}$$

So the inverse image is $\phi_{2,2}^{-1}(H_{2,4}) = \mp \left[\begin{pmatrix} e^{\beta/2} & 0 \\ 0 & e^{-\beta/2} \end{pmatrix}, \begin{pmatrix} e^{\beta/2} & 0 \\ 0 & e^{-\beta/2} \end{pmatrix} \right]$

4.3 From $SO^+(3,1)$ to $Spin^+(3,1)$

Following Example (2.4) every matrix in $SO^+(3,1)$ is a product of

$R_{1,2}$, $R_{2,3}$, $R_{1,3}$, $H_{1,4}$, $H_{2,4}$, $H_{3,4}$, where

$$R_{1,2} = \begin{pmatrix} c & -s & 0 & 0 \\ s & c & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, R_{2,3} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & c & -s & 0 \\ 0 & s & c & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, R_{1,3} = \begin{pmatrix} c & 0 & -s & 0 \\ 0 & 1 & 0 & 0 \\ s & 0 & c & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$H_{1,4} = \begin{pmatrix} a & 0 & 0 & b \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ b & 0 & 0 & a \end{pmatrix}, \ H_{2,4} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & a & 0 & b \\ 0 & 0 & 1 & 0 \\ 0 & b & 0 & a \end{pmatrix}, \ H_{3,4} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & a & b \\ 0 & 0 & b & a \end{pmatrix};$$

Let $c = \cos(\theta)$, $s = \sin(\theta)$, $a = \cosh(\beta)$, and $b = \sinh(\beta)$.

- $R_{1,2}$ case

  We solve the system of quadratic equations which arise when the target is standard Givens $R_{1,2}$.

$$M = R_{1,2}$$

a) If $\theta \neq 0, \pi$ and $\theta \in (0, \pi)$, then

$$\mathrm{S}(R_{1,2}) = \mp \begin{pmatrix} \widehat{\mathrm{c}} & -\widehat{\mathrm{s}} & 0 & 0 \\ \widehat{\mathrm{s}} & \widehat{\mathrm{c}} & 0 & 0 \\ 0 & 0 & \widehat{\mathrm{c}} & -\widehat{\mathrm{s}} \\ 0 & 0 & \widehat{\mathrm{s}} & \widehat{\mathrm{c}} \end{pmatrix}$$

So the inverse image is $\phi_{3,1}^{-1}(R_{1,2}) = \mp \begin{pmatrix} \widehat{\mathrm{c}} & -\widehat{\mathrm{s}} \\ \widehat{\mathrm{s}} & \widehat{\mathrm{c}} \end{pmatrix}$

b) If $\theta \neq \pi, 2\pi$ and $\theta \in (\pi, 2\pi)$, then

$$\mathrm{S}(R_{1,2}) = \pm \begin{pmatrix} \widehat{\mathrm{c}} & -\widehat{\mathrm{s}} & 0 & 0 \\ \widehat{\mathrm{s}} & \widehat{\mathrm{c}} & 0 & 0 \\ 0 & 0 & \widehat{\mathrm{c}} & -\widehat{\mathrm{s}} \\ 0 & 0 & \widehat{\mathrm{s}} & \widehat{\mathrm{c}} \end{pmatrix},$$

So the inverse image is $\phi_{3,1}^{-1}(R_{1,2}) = \pm \begin{pmatrix} \widehat{\mathrm{c}} & -\widehat{\mathrm{s}} \\ \widehat{\mathrm{s}} & \widehat{\mathrm{c}} \end{pmatrix}$

- The equation:

$$M = R_{2,3}$$

gives the following solutions.

a) If $\theta \neq 0, \pi$ and $\theta \in (0, \pi)$, then

$$\mathrm{S}(R_{2,3}) = \mp \begin{pmatrix} \widehat{\mathrm{c}} & 0 & 0 & -\widehat{\mathrm{s}} \\ 0 & \widehat{\mathrm{c}} & -\widehat{\mathrm{s}} & 0 \\ 0 & \widehat{\mathrm{s}} & \widehat{\mathrm{c}} & 0 \\ \widehat{\mathrm{s}} & 0 & 0 & \widehat{\mathrm{c}} \end{pmatrix}$$

Therefore, the inverse image is $\phi_{3,1}^{-1}(R_{2,3}) = \mp \begin{pmatrix} \widehat{\mathrm{c}} & \imath\widehat{\mathrm{s}} \\ \imath\widehat{\mathrm{s}} & \widehat{\mathrm{c}} \end{pmatrix}$

b) If $\theta \neq \pi, 2\pi$ and $\theta \in (\pi, 2\pi)$, then

$$\mathrm{S}(R_{2,3}) = \pm \begin{pmatrix} \widehat{\mathrm{c}} & 0 & 0 & -\widehat{\mathrm{s}} \\ 0 & \widehat{\mathrm{c}} & -\widehat{\mathrm{s}} & 0 \\ 0 & \widehat{\mathrm{s}} & \widehat{\mathrm{c}} & 0 \\ \widehat{\mathrm{s}} & 0 & 0 & \widehat{\mathrm{c}} \end{pmatrix}.$$

So the inverse image is $\phi_{3,1}^{-1}(R_{2,3}) = \pm \begin{pmatrix} \widehat{\mathrm{c}} & \imath\widehat{\mathrm{s}} \\ \imath\widehat{\mathrm{s}} & \widehat{\mathrm{c}} \end{pmatrix}$

- $R_{1,3}$ case

  We solve the system of quadratic equations which arise when the target is standard Givens $R_{1,3}$.

$$M = R_{1,3}$$

a) If $\theta \neq 0, \pi$ and $\theta \in (0, \pi)$, then

$$\mathrm{S}(R_{1,3}) = \pm \begin{pmatrix} \widehat{\mathrm{c}} & 0 & -\widehat{\mathrm{s}} & 0 \\ 0 & \widehat{\mathrm{c}} & 0 & \widehat{\mathrm{s}} \\ \widehat{\mathrm{s}} & 0 & \widehat{\mathrm{c}} & 0 \\ 0 & -\widehat{\mathrm{s}} & 0 & \widehat{\mathrm{c}} \end{pmatrix}$$

So the inverse image is $\phi_{3,1}^{-1}(R_{1,3}) = \pm \begin{pmatrix} \widehat{\mathrm{c}} + \imath\widehat{\mathrm{s}} & 0 \\ 0 & \widehat{\mathrm{c}} - \imath\widehat{\mathrm{s}} \end{pmatrix}$

b) If $\theta \neq \pi, 2\pi$ and $\theta \in (\pi, 2\pi)$, then

$$\mathrm{S}(R_{1,3}) = \mp \begin{pmatrix} \widehat{\mathrm{c}} & 0 & -\widehat{\mathrm{s}} & 0 \\ 0 & \widehat{\mathrm{c}} & 0 & \widehat{\mathrm{s}} \\ \widehat{\mathrm{s}} & 0 & \widehat{\mathrm{c}} & 0 \\ 0 & -\widehat{\mathrm{s}} & 0 & \widehat{\mathrm{c}} \end{pmatrix}$$

So the inverse image is $\phi_{3,1}^{-1}(R_{1,3}) = \mp \begin{pmatrix} \widehat{\mathrm{c}} + \imath\widehat{\mathrm{s}} & 0 \\ 0 & \widehat{\mathrm{c}} - \imath\widehat{\mathrm{s}} \end{pmatrix}$

- $H_{1,4}$ case

  We solve the system of quadratic equations which arise when the target is hyperbolic Givens $H_{1,4}$.

$$M = H_{1,4}$$

$$S(H_{1,4}) = \pm \begin{pmatrix} -\widehat{\mathrm{ch}} & \widehat{\mathrm{sh}} & 0 & 0 \\ \widehat{\mathrm{sh}} & -\widehat{\mathrm{ch}} & 0 & 0 \\ 0 & 0 & -\widehat{\mathrm{ch}} & \widehat{\mathrm{sh}} \\ 0 & 0 & \widehat{\mathrm{sh}} & -\widehat{\mathrm{ch}} \end{pmatrix}$$

So the inverse image is $\phi_{3,1}^{-1}(H_{1,4}) = \pm \begin{pmatrix} -\widehat{\mathrm{ch}} & \widehat{\mathrm{sh}} \\ \widehat{\mathrm{sh}} & -\widehat{\mathrm{ch}} \end{pmatrix}$

- $H_{2,4}$ case

  We solve the system of quadratic equations which arise when the target is hyperbolic Givens $H_{2,4}$.

$$M = H_{2,4}$$

$$S(H_{2,4}) = \mp \begin{pmatrix} \mathrm{e}^{\beta/2} & 0 & 0 & 0 \\ 0 & \mathrm{e}^{-\beta/2} & 0 & 0 \\ 0 & 0 & \mathrm{e}^{\beta/2} & 0 \\ 0 & 0 & 0 & \mathrm{e}^{-\beta/2} \end{pmatrix},$$

So the inverse image is $\phi_{3,1}^{-1}(H_{2,4}) = \mp \begin{pmatrix} e^{\beta/2} & 0 \\ 0 & e^{-\beta/2} \end{pmatrix}$

- $H_{3,4}$ case

  We solve the system of quadratic equations which arise when the target is hyperbolic Givens $H_{3,4}$.

$$M = H_{3,4}$$

$S(H_{3,4}) = \mp \begin{pmatrix} \widehat{\mathrm{ch}} & 0 & 0 & \widehat{\mathrm{sh}} \\ 0 & \widehat{\mathrm{ch}} & -\widehat{\mathrm{sh}} & 0 \\ 0 & -\widehat{\mathrm{sh}} & \widehat{\mathrm{ch}} & 0 \\ \widehat{\mathrm{sh}} & 0 & 0 & \widehat{\mathrm{ch}} \end{pmatrix}$. So the inverse image is $\phi_{3,1}^{-1}(H_{3,4}) = \mp \begin{pmatrix} \widehat{\mathrm{ch}} & -i\widehat{\mathrm{sh}} \\ -i\widehat{\mathrm{sh}} & \widehat{\mathrm{ch}} \end{pmatrix}$

4.4 From $SO^+(3,2)$ to $Spin^+(3,2)$

Following Example (2.4) every matrix in $SO^+(3,2)$ is a product of

$R_{1,2}, R_{1,3},\ R_{2,3},\ R_{4,5}, H_{1,4}, H_{2,5}$, where

$$R_{1,2} = \begin{pmatrix} c & -s & 0 & 0 & 0 \\ s & c & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, R_{1,3} = \begin{pmatrix} c & 0 & -s & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ s & 0 & c & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, R_{2,3} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & c & -s & 0 & 0 \\ 0 & s & c & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$R_{4,5} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & c & -s \\ 0 & 0 & 0 & s & c \end{pmatrix},\ H_{1,4} = \begin{pmatrix} a & 0 & 0 & b & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ b & 0 & 0 & a & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, H_{2,5} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & a & 0 & 0 & b \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & b & 0 & 0 & a \end{pmatrix}.$$

Solving the following equations for variables x_1, x_2, ..., x_{16} in terms of s, c, a, b

- $R_{1,2}$ case

  We solve the system of quadratic equations which arise when the target is standard Givens $R_{1,2}$.

$$M = R_{1,2}$$

If $\theta \neq 0, \pi, 2\pi$, and $\theta \in (0, \pi)$, then

$$
\mathrm{S}\left(R_{1,2}\right)=\mp\begin{pmatrix}
\widehat{\mathrm{c}} & 0 & 0 & 0 & 0 & 0 & \widehat{\mathrm{s}} & 0\\
0 & \widehat{\mathrm{c}} & 0 & 0 & 0 & 0 & 0 & \widehat{\mathrm{s}}\\
0 & 0 & \widehat{\mathrm{c}} & 0 & \widehat{\mathrm{s}} & 0 & 0 & 0\\
0 & 0 & 0 & \widehat{\mathrm{c}} & 0 & \widehat{\mathrm{s}} & 0 & 0\\
0 & 0 & -\widehat{\mathrm{s}} & 0 & \widehat{\mathrm{c}} & 0 & 0 & 0\\
0 & 0 & 0 & -\widehat{\mathrm{s}} & 0 & \widehat{\mathrm{c}} & 0 & 0\\
-\widehat{\mathrm{s}} & 0 & 0 & 0 & 0 & 0 & \widehat{\mathrm{c}} & 0\\
0 & -\widehat{\mathrm{s}} & 0 & 0 & 0 & 0 & 0 & \widehat{\mathrm{c}}
\end{pmatrix}
$$

So the inverse image is $\phi_{3,2}^{-1}(R_{1,2})=\mp\begin{pmatrix}
\widehat{\mathrm{c}} & 0 & 0 & \widehat{\mathrm{s}}\\
0 & \widehat{\mathrm{c}} & \widehat{\mathrm{s}} & 0\\
0 & -\widehat{s} & \widehat{c} & 0\\
-\widehat{s} & 0 & 0 & \widehat{c}
\end{pmatrix}$ For $\theta\neq 0,\pi,2\pi,$ and $\theta\in(\pi,2\pi)$,

then

$$
\mathrm{S}\left(R_{1,2}\right)=\pm\begin{pmatrix}
\widehat{\mathrm{c}} & 0 & 0 & 0 & 0 & 0 & \widehat{\mathrm{s}} & 0\\
0 & \widehat{\mathrm{c}} & 0 & 0 & 0 & 0 & 0 & \widehat{\mathrm{s}}\\
0 & 0 & \widehat{\mathrm{c}} & 0 & \widehat{\mathrm{s}} & 0 & 0 & 0\\
0 & 0 & 0 & \widehat{\mathrm{c}} & 0 & \widehat{\mathrm{s}} & 0 & 0\\
0 & 0 & -\widehat{\mathrm{s}} & 0 & \widehat{\mathrm{c}} & 0 & 0 & 0\\
0 & 0 & 0 & -\widehat{\mathrm{s}} & 0 & \widehat{\mathrm{c}} & 0 & 0\\
-\widehat{\mathrm{s}} & 0 & 0 & 0 & 0 & 0 & \widehat{\mathrm{c}} & 0\\
0 & -\widehat{\mathrm{s}} & 0 & 0 & 0 & 0 & 0 & \widehat{\mathrm{c}}
\end{pmatrix}
$$

So the inverse image is $\phi_{3,2}^{-1}(R_{1,2})=\pm\begin{pmatrix}
\widehat{\mathrm{c}} & 0 & 0 & \widehat{\mathrm{s}}\\
0 & \widehat{\mathrm{c}} & \widehat{\mathrm{s}} & 0\\
0 & -\widehat{s} & \widehat{c} & 0\\
-\widehat{s} & 0 & 0 & \widehat{c}
\end{pmatrix}$

- $R_{1,3}$ case

We solve the system of quadratic equations which arise when the target is standard Givens $R_{1,3}$.

$$M = R_{1,3}$$

If $\theta \neq 0, \pi, 2\pi$, and $\theta \in (0, \pi)$, then

$$\mathrm{S}\left(R_{1,3}\right) = \mp \begin{pmatrix} \widehat{\mathrm{c}} & 0 & 0 & 0 & \widehat{\mathrm{s}} & 0 & 0 & 0 \\ 0 & \widehat{\mathrm{c}} & 0 & 0 & 0 & -\widehat{\mathrm{s}} & 0 & 0 \\ 0 & 0 & \widehat{\mathrm{c}} & 0 & 0 & 0 & -\widehat{\mathrm{s}} & 0 \\ 0 & 0 & 0 & \widehat{\mathrm{c}} & 0 & 0 & 0 & \widehat{\mathrm{s}} \\ -\widehat{\mathrm{s}} & 0 & 0 & 0 & \widehat{\mathrm{c}} & 0 & 0 & 0 \\ 0 & \widehat{\mathrm{s}} & 0 & 0 & 0 & \widehat{\mathrm{c}} & 0 & 0 \\ 0 & 0 & \widehat{\mathrm{s}} & 0 & 0 & 0 & \widehat{\mathrm{c}} & 0 \\ 0 & 0 & 0 & -\widehat{\mathrm{s}} & 0 & 0 & 0 & \widehat{\mathrm{c}} \end{pmatrix}$$

So the inverse image is $\phi_{3,2}^{-1}(R_{1,3}) = \mp \begin{pmatrix} \widehat{\mathrm{c}} & 0 & \widehat{\mathrm{s}} & 0 \\ 0 & \widehat{\mathrm{c}} & 0 & -\widehat{\mathrm{s}} \\ -\widehat{s} & 0 & \widehat{c} & 0 \\ 0 & \widehat{s} & 0 & \widehat{c} \end{pmatrix}$

If $\theta \neq 0, \pi, 2\pi$, and $\theta \in (\pi, 2\pi)$, then

$$
\mathrm{S}\left(R_{1,3}\right)=\pm\begin{pmatrix}
\widehat{\mathrm{c}} & 0 & 0 & 0 & \widehat{\mathrm{s}} & 0 & 0 & 0\\
0 & \widehat{\mathrm{c}} & 0 & 0 & 0 & -\widehat{\mathrm{s}} & 0 & 0\\
0 & 0 & \widehat{\mathrm{c}} & 0 & 0 & 0 & -\widehat{\mathrm{s}} & 0\\
0 & 0 & 0 & \widehat{\mathrm{c}} & 0 & 0 & 0 & \widehat{\mathrm{s}}\\
-\widehat{\mathrm{s}} & 0 & 0 & 0 & \widehat{\mathrm{c}} & 0 & 0 & 0\\
0 & \widehat{\mathrm{s}} & 0 & 0 & 0 & \widehat{\mathrm{c}} & 0 & 0\\
0 & 0 & \widehat{\mathrm{s}} & 0 & 0 & 0 & \widehat{\mathrm{c}} & 0\\
0 & 0 & 0 & -\widehat{\mathrm{s}} & 0 & 0 & 0 & \widehat{\mathrm{c}}
\end{pmatrix}
$$

So the inverse image is $\phi_{3,2}^{-1}(R_{1,3})=\pm\begin{pmatrix}
\widehat{\mathrm{c}} & 0 & \widehat{\mathrm{s}} & 0\\
0 & \widehat{\mathrm{c}} & 0 & -\widehat{\mathrm{s}}\\
-\widehat{s} & 0 & \widehat{c} & 0\\
0 & \widehat{s} & 0 & \widehat{c}
\end{pmatrix}$

- $R_{2,3}$ case

  We solve the system of quadratic equations which arise when the target is standard Givens $R_{2,3}$.

$$
M=R_{2,3}
$$

If $\theta\neq 0,\pi,2\pi,$ and $\theta\in(0,\pi)$, then

$$\mathrm{S}\left(R_{2,3}\right)=\mp\begin{pmatrix} \widehat{\mathrm{c}} & 0 & \widehat{\mathrm{s}} & 0 & 0 & 0 & 0 & 0 \\ 0 & \widehat{\mathrm{c}} & 0 & -\widehat{\mathrm{s}} & 0 & 0 & 0 & 0 \\ -\widehat{\mathrm{s}} & 0 & \widehat{\mathrm{c}} & 0 & 0 & 0 & 0 & 0 \\ 0 & \widehat{\mathrm{s}} & 0 & \widehat{\mathrm{c}} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \widehat{\mathrm{c}} & 0 & \widehat{\mathrm{s}} & 0 \\ 0 & 0 & 0 & 0 & 0 & \widehat{\mathrm{c}} & 0 & -\widehat{\mathrm{s}} \\ 0 & 0 & 0 & 0 & -\widehat{\mathrm{s}} & 0 & \widehat{\mathrm{c}} & 0 \\ 0 & 0 & 0 & 0 & 0 & \widehat{\mathrm{s}} & 0 & \widehat{\mathrm{c}} \end{pmatrix}$$

So the inverse image is $\phi_{3,2}^{-1}(R_{2,3})=\mp\begin{pmatrix} \widehat{\mathrm{c}} & \widehat{\mathrm{s}} & 0 & 0 \\ -\widehat{\mathrm{s}} & \widehat{\mathrm{c}} & 0 & 0 \\ 0 & 0 & \widehat{c} & \widehat{s} \\ 0 & 0 & -\widehat{s} & \widehat{c} \end{pmatrix}$

If $\theta\neq 0,\pi,2\pi$, and $\theta\in(\pi,2\pi)$, then

$$\mathrm{S}\left(R_{2,3}\right)=\pm\begin{pmatrix} \widehat{\mathrm{c}} & 0 & \widehat{\mathrm{s}} & 0 & 0 & 0 & 0 & 0 \\ 0 & \widehat{\mathrm{c}} & 0 & -\widehat{\mathrm{s}} & 0 & 0 & 0 & 0 \\ -\widehat{\mathrm{s}} & 0 & \widehat{\mathrm{c}} & 0 & 0 & 0 & 0 & 0 \\ 0 & \widehat{\mathrm{s}} & 0 & \widehat{\mathrm{c}} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \widehat{\mathrm{c}} & 0 & \widehat{\mathrm{s}} & 0 \\ 0 & 0 & 0 & 0 & 0 & \widehat{\mathrm{c}} & 0 & -\widehat{\mathrm{s}} \\ 0 & 0 & 0 & 0 & -\widehat{\mathrm{s}} & 0 & \widehat{\mathrm{c}} & 0 \\ 0 & 0 & 0 & 0 & 0 & \widehat{\mathrm{s}} & 0 & \widehat{\mathrm{c}} \end{pmatrix}$$

Therefore the inverse image is $\phi_{3,2}^{-1}(R_{2,3})=\pm\begin{pmatrix} \widehat{\mathrm{c}} & \widehat{\mathrm{s}} & 0 & 0 \\ -\widehat{\mathrm{s}} & \widehat{\mathrm{c}} & 0 & 0 \\ 0 & 0 & \widehat{c} & \widehat{s} \\ 0 & 0 & -\widehat{s} & \widehat{c} \end{pmatrix}$

- $R_{4,5}$ case

  We solve the system of quadratic equations which arise when the target is standard Givens $R_{4,5}$.

$$M = R_{4,5}$$

If $\theta \neq 0, \pi, 2\pi$, and $\theta \in (0, \pi)$, then

$$\mathrm{S}\left(R_{4,5}\right) = \mp \begin{pmatrix} \widehat{\mathrm{c}} & 0 & 0 & 0 & 0 & 0 & -\widehat{\mathrm{s}} & 0 \\ 0 & \widehat{\mathrm{c}} & 0 & 0 & 0 & 0 & 0 & -\widehat{\mathrm{s}} \\ 0 & 0 & \widehat{\mathrm{c}} & 0 & \widehat{\mathrm{s}} & 0 & 0 & 0 \\ 0 & 0 & 0 & \widehat{\mathrm{c}} & 0 & \widehat{\mathrm{s}} & 0 & 0 \\ 0 & 0 & -\widehat{\mathrm{s}} & 0 & \widehat{\mathrm{c}} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\widehat{\mathrm{s}} & 0 & \widehat{\mathrm{c}} & 0 & 0 \\ \widehat{\mathrm{s}} & 0 & 0 & 0 & 0 & 0 & \widehat{\mathrm{c}} & 0 \\ 0 & \widehat{\mathrm{s}} & 0 & 0 & 0 & 0 & 0 & \widehat{\mathrm{c}} \end{pmatrix}$$

So the inverse image is $\phi_{3,2}^{-1}(R_{4,5}) = \mp \begin{pmatrix} \widehat{\mathrm{c}} & 0 & 0 & -\widehat{\mathrm{s}} \\ 0 & \widehat{\mathrm{c}} & \widehat{\mathrm{s}} & 0 \\ 0 & -\widehat{s} & \widehat{c} & 0 \\ \widehat{s} & 0 & 0 & \widehat{c} \end{pmatrix}$

If $\theta \neq 0, \pi, 2\pi$, and $\theta \in (\pi, 2\pi)$, then

$$
\mathrm{S}\left(R_{4,5}\right) = \pm \begin{pmatrix}
\widehat{\mathrm{c}} & 0 & 0 & 0 & 0 & 0 & -\widehat{\mathrm{s}} & 0 \\
0 & \widehat{\mathrm{c}} & 0 & 0 & 0 & 0 & 0 & -\widehat{\mathrm{s}} \\
0 & 0 & \widehat{\mathrm{c}} & 0 & \widehat{\mathrm{s}} & 0 & 0 & 0 \\
0 & 0 & 0 & \widehat{\mathrm{c}} & 0 & \widehat{\mathrm{s}} & 0 & 0 \\
0 & 0 & -\widehat{\mathrm{s}} & 0 & \widehat{\mathrm{c}} & 0 & 0 & 0 \\
0 & 0 & 0 & -\widehat{\mathrm{s}} & 0 & \widehat{\mathrm{c}} & 0 & 0 \\
\widehat{\mathrm{s}} & 0 & 0 & 0 & 0 & 0 & \widehat{\mathrm{c}} & 0 \\
0 & \widehat{\mathrm{s}} & 0 & 0 & 0 & 0 & 0 & \widehat{\mathrm{c}}
\end{pmatrix}
$$

So the inverse image is $\phi_{3,2}^{-1}(R_{4,5}) = \pm \begin{pmatrix} \widehat{\mathrm{c}} & 0 & 0 & -\widehat{\mathrm{s}} \\ 0 & \widehat{\mathrm{c}} & \widehat{\mathrm{s}} & 0 \\ 0 & -\widehat{s} & \widehat{c} & 0 \\ \widehat{s} & 0 & 0 & \widehat{c} \end{pmatrix}$

- $H_{1,4}$ case

  We solve the system of quadratic equations which arise when the target is hyperbolic Givens $H_{1,4}$.

$$M = H_{1,4}$$

$$\mathrm{S}\left(H_{1,4}\right)=\mp\begin{pmatrix} e^{\beta/2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & e^{\beta/2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & e^{\beta/2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & e^{\beta/2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & e^{-\beta/2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & e^{-\beta/2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & e^{-\beta/2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & e^{-\beta/2} \end{pmatrix},$$

So the inverse image is $\phi_{3,2}^{-1}(H_{1,4})=\mp\begin{pmatrix} e^{\frac{\beta}{2}} & 0 & 0 & 0 \\ 0 & e^{\frac{\beta}{2}} & 0 & 0 \\ 0 & 0 & e^{-\frac{\beta}{2}} & 0 \\ 0 & 0 & 0 & e^{-\frac{\beta}{2}} \end{pmatrix}$

- $H_{2,5}$ case

  We solve the system of quadratic equations which arise when the target is hyperbolic Givens $H_{2,5}$.

$$M=H_{2,5}$$

$$\mathrm{S}\left(H_{2,5}\right)=\mp\begin{pmatrix} e^{\beta/2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & e^{\beta/2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & e^{-\beta/2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & e^{-\beta/2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & e^{\beta/2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & e^{\beta/2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & e^{-\beta/2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & e^{-\beta/2} \end{pmatrix}$$

So the inverse image is $\phi_{3,2}^{-1}(H_{1,4})=\mp\begin{pmatrix} e^{\beta/2} & 0 & 0 & 0 \\ 0 & e^{-\beta/2} & 0 & 0 \\ 0 & 0 & e^{\beta/2} & 0 \\ 0 & 0 & 0 & e^{-\beta/2} \end{pmatrix}$

We need to address the special cases $\theta \neq 0,\ \pi,\ 2\pi$. In such cases we have the following matrices

- $R_{1,2}\left(0\right)=R_{1,3}\left(0\right)=R_{2,3}\left(0\right)=R_{4,5}\left(0\right)=\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$;

We obtain: $\mathrm{S}\left(R_{1,2}\left(0\right)\right) = \mp \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$

So the inverse image is $\phi_{3,2}^{-1}(R_{1,2}(0)) = \mp \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

- $R_{1,2}\left(\pi\right) = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$, $\mathrm{S}\left(R_{1,2}\left(\pi\right)\right) = \pm \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$

So the inverse image is $\phi_{3,2}^{-1}(R_{1,2}(\pi)) = \pm \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$

- $R_{1,3}(\pi) = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$

$$\mathrm{S}\left(R_{1,3}(\pi)\right) = \pm \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

So the inverse image is $\phi_{3,2}^{-1}(R_{1,3}(\pi)) = \pm \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$

- $R_{2,3}(\pi) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$,

$$S\left(R_{2,3}\left(\pi\right)\right)=\pm\begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

So the inverse image is $\phi_{3,2}^{-1}(R_{2,3}(\pi))=\pm\begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$

- $R_{4,5}\left(\pi\right)=\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix}$

$$S\left(R_{4,5}\left(\pi\right)\right)=\pm\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

So the inverse image is $\phi_{3,2}^{-1}(R_{4,5}(\pi)) = \pm \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$

4.5 From $SO^+(3,3)$ to $Spin^+(3,3)$

Following Example (2.4) every matrix in $SO^+(3,3)$ is a product of

$R_{1,2}$, $R_{2,3}$, $R_{4,5}$, $R_{5,6}$, $H_{1,4}$, $H_{2,4}$, $H_{3,4}$, where

$$R_{1,2} = \begin{pmatrix} c & -s & 0 & 0 & 0 & 0 \\ s & c & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, R_{2,3} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & c & -s & 0 & 0 & 0 \\ 0 & s & c & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, R_{4,5} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & c & -s & 0 \\ 0 & 0 & 0 & s & c & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$R_{5,6} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & c & -s \\ 0 & 0 & 0 & 0 & s & c \end{pmatrix}, H_{1,4} = \begin{pmatrix} a & 0 & 0 & b & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ b & 0 & 0 & a & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, H_{2,4} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & a & 0 & 0 & b & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & b & 0 & 0 & a & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$H_{3,4} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & a & b & 0 & 0 \\ 0 & 0 & b & a & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix};$$

- $R_{1,2}$ case

  We solve the system of quadratic equations which arise when the target is standard Givens $R_{1,2}$.

$$M = R_{1,2}$$

If $\theta \in (0, \pi)$, then we have

$$\mathrm{S}\left(R_{1,2}\right) = \mp \begin{pmatrix} \widehat{\mathrm{c}} & -\widehat{\mathrm{s}} & 0 & 0 & 0 & 0 & 0 & 0 \\ \widehat{\mathrm{s}} & \widehat{\mathrm{c}} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \widehat{\mathrm{c}} & -\widehat{\mathrm{s}} & 0 & 0 & 0 & 0 \\ 0 & 0 & \widehat{\mathrm{s}} & \widehat{\mathrm{c}} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \widehat{\mathrm{c}} & -\widehat{\mathrm{s}} & 0 & 0 \\ 0 & 0 & 0 & 0 & \widehat{\mathrm{s}} & \widehat{\mathrm{c}} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \widehat{\mathrm{c}} & -\widehat{\mathrm{s}} \\ 0 & 0 & 0 & 0 & 0 & 0 & \widehat{\mathrm{s}} & \widehat{\mathrm{c}} \end{pmatrix}$$

So the inverse image is $\phi_{3,3}^{-1}(R_{1,2}) = \mp \begin{pmatrix} \widehat{\mathrm{c}} & -\widehat{\mathrm{s}} & 0 & 0 \\ \widehat{\mathrm{s}} & \widehat{\mathrm{c}} & 0 & 0 \\ 0 & 0 & \widehat{c} & -\widehat{s} \\ 0 & 0 & \widehat{s} & \widehat{c} \end{pmatrix}$

If $\theta \in (\pi, 2\pi)$, then

$$\mathrm{S}\left(R_{1,2}\right) = \pm \begin{pmatrix} \widehat{\mathrm{c}} & -\widehat{\mathrm{s}} & 0 & 0 & 0 & 0 & 0 & 0 \\ \widehat{\mathrm{s}} & \widehat{\mathrm{c}} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \widehat{\mathrm{c}} & -\widehat{\mathrm{s}} & 0 & 0 & 0 & 0 \\ 0 & 0 & \widehat{\mathrm{s}} & \widehat{\mathrm{c}} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \widehat{\mathrm{c}} & -\widehat{\mathrm{s}} & 0 & 0 \\ 0 & 0 & 0 & 0 & \widehat{\mathrm{s}} & \widehat{\mathrm{c}} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \widehat{\mathrm{c}} & -\widehat{\mathrm{s}} \\ 0 & 0 & 0 & 0 & 0 & 0 & \widehat{\mathrm{s}} & \widehat{\mathrm{c}} \end{pmatrix},$$

So the inverse image is $\phi_{3,3}^{-1}(R_{1,2}) = \pm \begin{pmatrix} \widehat{\mathrm{c}} & -\widehat{\mathrm{s}} & 0 & 0 \\ \widehat{\mathrm{s}} & \widehat{\mathrm{c}} & 0 & 0 \\ 0 & 0 & \widehat{c} & -\widehat{s} \\ 0 & 0 & \widehat{s} & \widehat{c} \end{pmatrix}$

- $R_{2,3}$ case

  We solve the system of quadratic equations which arise when the target is standard Givens $R_{2,3}$.

$$M = R_{2,3}$$

If $\theta \in (0, 2\pi)$, then

$$\mathrm{S}\left(R_{2,3}\right) = \mp \begin{pmatrix} \widehat{\mathrm{c}} & 0 & 0 & -\widehat{\mathrm{s}} & 0 & 0 & 0 & 0 \\ 0 & \widehat{\mathrm{c}} & -\widehat{\mathrm{s}} & 0 & 0 & 0 & 0 & 0 \\ 0 & \widehat{\mathrm{s}} & \widehat{\mathrm{c}} & 0 & 0 & 0 & 0 & 0 \\ \widehat{\mathrm{s}} & 0 & 0 & \widehat{\mathrm{c}} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \widehat{\mathrm{c}} & 0 & 0 & -\widehat{\mathrm{s}} \\ 0 & 0 & 0 & 0 & 0 & \widehat{\mathrm{c}} & -\widehat{\mathrm{s}} & 0 \\ 0 & 0 & 0 & 0 & 0 & \widehat{\mathrm{s}} & \widehat{\mathrm{c}} & 0 \\ 0 & 0 & 0 & 0 & \widehat{\mathrm{s}} & 0 & 0 & \widehat{\mathrm{c}} \end{pmatrix}$$

So the inverse image is $\phi_{3,3}^{-1}(R_{2,3}) = \pm \begin{pmatrix} \widehat{\mathrm{c}} & 0 & 0 & -\widehat{\mathrm{s}} \\ 0 & \widehat{\mathrm{c}} & -\widehat{\mathrm{s}} & 0 \\ 0 & \widehat{s} & \widehat{c} & 0 \\ \widehat{s} & 0 & 0 & \widehat{c} \end{pmatrix}$

- $R_{4,5}$ case

  We solve the system of quadratic equations which arise when the target is standard Givens $R_{4,5}$.

$$M = R_{4,5}$$

If $\theta \in (0, \pi)$, then we have

$$S\left(R_{4,5}\right)=\mp\begin{pmatrix} \widehat{\mathrm{c}} & \widehat{\mathrm{s}} & 0 & 0 & 0 & 0 & 0 & 0 \\ -\widehat{\mathrm{s}} & \widehat{\mathrm{c}} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \widehat{\mathrm{c}} & -\widehat{\mathrm{s}} & 0 & 0 & 0 & 0 \\ 0 & 0 & \widehat{\mathrm{s}} & \widehat{\mathrm{c}} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \widehat{\mathrm{c}} & -\widehat{\mathrm{s}} & 0 & 0 \\ 0 & 0 & 0 & 0 & \widehat{\mathrm{s}} & \widehat{\mathrm{c}} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \widehat{\mathrm{c}} & \widehat{\mathrm{s}} \\ 0 & 0 & 0 & 0 & 0 & 0 & -\widehat{\mathrm{s}} & \widehat{\mathrm{c}} \end{pmatrix}.$$

So the inverse image is $\phi_{3,3}^{-1}(R_{4,5})=\mp\begin{pmatrix} \widehat{\mathrm{c}} & \widehat{\mathrm{s}} & 0 & 0 \\ -\widehat{\mathrm{s}} & \widehat{\mathrm{c}} & 0 & 0 \\ 0 & 0 & \widehat{c} & -\widehat{s} \\ 0 & 0 & \widehat{s} & \widehat{c} \end{pmatrix}$

If $\theta\in(\pi,2\pi)$, then

$$S\left(R_{4,5}\right)=\pm\begin{pmatrix} \widehat{\mathrm{c}} & \widehat{\mathrm{s}} & 0 & 0 & 0 & 0 & 0 & 0 \\ -\widehat{\mathrm{s}} & \widehat{\mathrm{c}} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \widehat{\mathrm{c}} & -\widehat{\mathrm{s}} & 0 & 0 & 0 & 0 \\ 0 & 0 & \widehat{\mathrm{s}} & \widehat{\mathrm{c}} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \widehat{\mathrm{c}} & -\widehat{\mathrm{s}} & 0 & 0 \\ 0 & 0 & 0 & 0 & \widehat{\mathrm{s}} & \widehat{\mathrm{c}} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \widehat{\mathrm{c}} & \widehat{\mathrm{s}} \\ 0 & 0 & 0 & 0 & 0 & 0 & -\widehat{\mathrm{s}} & \widehat{\mathrm{c}} \end{pmatrix}.$$

Therefore the inverse image is $\phi_{3,3}^{-1}(R_{4,5})=\pm\begin{pmatrix} \widehat{\mathrm{c}} & \widehat{\mathrm{s}} & 0 & 0 \\ -\widehat{\mathrm{s}} & \widehat{\mathrm{c}} & 0 & 0 \\ 0 & 0 & \widehat{c} & -\widehat{s} \\ 0 & 0 & \widehat{s} & \widehat{c} \end{pmatrix}$

- $R_{5,6}$ case

  We solve the system of quadratic equations which arise when the target is standard Givens $R_{5,6}$.

$$M = R_{5,6}$$

If $\theta \in (0, 2\pi)$, then

$$\mathrm{S}\left(R_{5,6}\right) = \mp \begin{pmatrix} \widehat{\mathrm{c}} & 0 & 0 & \widehat{\mathrm{s}} & 0 & 0 & 0 & 0 \\ 0 & \widehat{\mathrm{c}} & -\widehat{\mathrm{s}} & 0 & 0 & 0 & 0 & 0 \\ 0 & \widehat{\mathrm{s}} & \widehat{\mathrm{c}} & 0 & 0 & 0 & 0 & 0 \\ -\widehat{\mathrm{s}} & 0 & 0 & \widehat{\mathrm{c}} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \widehat{\mathrm{c}} & 0 & 0 & \widehat{\mathrm{s}} \\ 0 & 0 & 0 & 0 & 0 & \widehat{\mathrm{c}} & -\widehat{\mathrm{s}} & 0 \\ 0 & 0 & 0 & 0 & 0 & \widehat{\mathrm{s}} & \widehat{\mathrm{c}} & 0 \\ 0 & 0 & 0 & 0 & -\widehat{\mathrm{s}} & 0 & 0 & \widehat{\mathrm{c}} \end{pmatrix},$$

So the inverse image is $\phi_{3,3}^{-1}(R_{5,6}) = \mp \begin{pmatrix} \widehat{\mathrm{c}} & 0 & 0 & \widehat{\mathrm{s}} \\ 0 & \widehat{\mathrm{c}} & -\widehat{\mathrm{s}} & 0 \\ 0 & \widehat{s} & \widehat{c} & 0 \\ -\widehat{s} & 0 & 0 & \widehat{c} \end{pmatrix}$

- $H_{1,4}$ case

  We solve the system of quadratic equations which arise when the target is hyperbolic Givens $H_{1,4}$.

$$M = H_{1,4}$$

$$
\mathrm{S}\left(H_{1,4}\right)=\mp\begin{pmatrix}
\mathrm{e}^{\beta/2} & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & \mathrm{e}^{-\beta/2} & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & \mathrm{e}^{-\beta/2} & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & \mathrm{e}^{\beta/2} & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & \mathrm{e}^{-\beta/2} & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & \mathrm{e}^{\beta/2} & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & \mathrm{e}^{\beta/2} & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathrm{e}^{-\beta/2}
\end{pmatrix},
$$

So the inverse image is $\phi_{3,3}^{-1}(H_{1,4})=\mp\begin{pmatrix}
e^{\beta/2} & 0 & 0 & 0\\
0 & e^{-\beta/2} & 0 & 0\\
0 & 0 & e^{-\beta/2} & 0\\
0 & 0 & 0 & e^{\beta/2}
\end{pmatrix}$

- $H_{2,4}$ case

  We solve the system of quadratic equations which arise when the target is hyperbolic Givens $H_{2,4}$.

$$
M=H_{2,4}
$$

$$
\mathrm{S}\left(H_{2,4}\right)=\mp\begin{pmatrix}
\mathrm{e}^{\beta/2} & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & \mathrm{e}^{-\beta/2} & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & \mathrm{e}^{\beta/2} & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & \mathrm{e}^{-\beta/2} & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & \mathrm{e}^{\beta/2} & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & \mathrm{e}^{-\beta/2} & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & \mathrm{e}^{\beta/2} & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathrm{e}^{-\beta/2}
\end{pmatrix},
$$

So the inverse image is $\phi_{3,3}^{-1}(H_{2,4}) = \mp \begin{pmatrix} e^{\beta/2} & 0 & 0 & 0 \\ 0 & e^{-\beta/2} & 0 & 0 \\ 0 & 0 & e^{\beta/2} & 0 \\ 0 & 0 & 0 & e^{-\beta/2} \end{pmatrix}$

- $H_{3,4}$ case

  We solve the system of quadratic equations which arise when the target is hyperbolic Givens $H_{3,4}$.

$$M = H_{3,4}$$

$$\mathrm{S}\left(H_{3,4}\right) = \pm \begin{pmatrix} -\widehat{\mathrm{ch}} & 0 & -\widehat{\mathrm{sh}} & 0 & 0 & 0 & 0 & 0 \\ 0 & -\widehat{\mathrm{ch}} & 0 & -\widehat{\mathrm{sh}} & 0 & 0 & 0 & 0 \\ -\widehat{\mathrm{sh}} & 0 & -\widehat{\mathrm{ch}} & 0 & 0 & 0 & 0 & 0 \\ 0 & -\widehat{\mathrm{sh}} & 0 & -\widehat{\mathrm{ch}} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \widehat{\mathrm{ch}} & 0 & \widehat{\mathrm{sh}} & 0 \\ 0 & 0 & 0 & 0 & 0 & \widehat{\mathrm{ch}} & 0 & \widehat{\mathrm{sh}} \\ 0 & 0 & 0 & 0 & \widehat{\mathrm{sh}} & 0 & \widehat{\mathrm{ch}} & 0 \\ 0 & 0 & 0 & 0 & 0 & \widehat{\mathrm{sh}} & 0 & \widehat{\mathrm{ch}} \end{pmatrix}.$$

So the inverse image is $\phi_{3,3}^{-1}(H_{3,4}) = \pm \begin{pmatrix} -\widehat{\mathrm{ch}} & 0 & -\widehat{\mathrm{sh}} & 0 \\ 0 & -\widehat{\mathrm{ch}} & 0 & -\widehat{\mathrm{sh}} \\ -\widehat{sh} & 0 & -\widehat{ch} & 0 \\ 0 & -\widehat{sh} & 0 & -\widehat{ch} \end{pmatrix}$

CHAPTER 5

DIRECT APPROACH TO SOME INDEFINITE SPIN GROUPS

It is well known that $Spin^+(p,q)$ is isomorphic to $Spin^+(q,p)$. However, $Cl(p,q)$ is not isomorphic to $Cl(q,p)$. Therefore, we look at some of the cases not considered in Emily Herzig and Viswanath Ramakrishna [1]. One motivation is that will provide us new concrete realizations of the classical groups. We will construct the indefinite spin group $Spin^+(p,q)$ as a matrix subalgebra of the matrix algebra that $Cl(p,q)$ is isomorphic to, for the pairs $(p,q) \in \{(1,2),(1,3),(2,3)\}$. In view of this, we need to provide explicit constructions for Clifford conjugation and reversion map as this will be useful in defining the grade map with respect to the basis set of 1-vectors use. We then characterize the elements of
$Spin^+(p,q) = \{X \in Cl(p,q) : X^{gr} = X\ ,\ XX^{cc} = I\ ,\ XVX^{cc}$ is a 1-vector for any 1-vector $V\}$. Having defined the Clifford conjugation and reversion on any Clifford algebra $Cl(p,q)$ realized as an algebra of matrices, we can explicitly define Clifford conjugation, reversion and grade map on $Cl(p+1,q+1)$. Therefore, we start with low dimensional p,q and use the iteration **IC** for higher dimensions.

5.1 $Spin^+(1,2)$

We construct $Cl(1,2)$ as follows:
Let $Cl(0,1) = \mathbb{C}$ via $\{i\}$, then, the Clifford conjugation and reversion are defined as follows cc is $z \to \bar{z}$ and rev is $z \to z$ respectively. So $Cl(1,2) = M(2,\mathbb{C})$ and $X = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ an element of it.

We obtain

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{rev} = \begin{pmatrix} \bar{d} & \bar{b} \\ \bar{c} & \bar{a} \end{pmatrix};\ \text{and} \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{cc} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = adj(A).$$

So the grade of X is $\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{gr} = \begin{pmatrix} \bar{a} & -\bar{b} \\ -\bar{c} & \bar{d} \end{pmatrix}$

Thus $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is even iff $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \bar{a} & -\bar{b} \\ -\bar{c} & \bar{d} \end{pmatrix}$, i.e., if and only if $a, d \in \mathbb{R};\ \ b, c \in i\mathbb{R}$.

Thus, $Spin^+(1,2) = \left\{ \begin{pmatrix} \alpha & i\beta \\ i\gamma & \delta \end{pmatrix} \text{ s.t } \begin{pmatrix} \alpha & i\beta \\ i\gamma & \delta \end{pmatrix} \begin{pmatrix} \delta & -i\beta \\ -i\gamma & \alpha \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\}$

Now if $A \in Spin^+(1,2)$, then $Aadj(A) = I_2$. So $det(A) = 1$.

So $Spin^+(1,2) \subseteq SL(2,\mathbb{C})$.

Next, we will show that this group of matrices is, in fact, isomorphic to $SL(2,\mathbb{R})$ via $\phi\left[\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}\right] = \begin{pmatrix} \alpha & i\beta \\ -i\gamma & \delta \end{pmatrix}$.

Indeed, $\alpha\delta - \beta\gamma = 1$ and $adj\phi\left[\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}\right] = \begin{pmatrix} \delta & -i\beta \\ i\gamma & \alpha \end{pmatrix}$.

We calculate:

$$\begin{pmatrix} \alpha & i\beta \\ -i\gamma & \delta \end{pmatrix} \begin{pmatrix} \delta & -i\beta \\ i\gamma & \alpha \end{pmatrix} = \begin{pmatrix} \alpha\delta - \beta\gamma & -i\alpha\beta + i\beta\alpha \\ -i\gamma\delta + i\delta\gamma & -\beta\gamma + \alpha\delta \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Therefore $\alpha\delta - \beta\gamma = 1$

Thus, the image of ϕ is indeed in S$pin^+(1,2)$. it is trivially onto and injective. Further, it is a group homomorphism and thus a group isomorphism:

Indeed,

$$\begin{aligned}
\phi\left[\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}\begin{pmatrix}a & b\\ c & d\end{pmatrix}\right] &= \phi\left[\begin{pmatrix}\alpha a+\beta c & \alpha b+\beta d\\ \gamma a+\delta c & \gamma b+\delta d\end{pmatrix}\right]\\
&= \begin{pmatrix}\alpha a+\beta c & i(\alpha b+\beta d)\\ -i(\gamma a+\delta c) & \gamma b+\delta d\end{pmatrix}\\
&= \begin{pmatrix}\alpha & i\beta\\ -i\gamma & \delta\end{pmatrix}\begin{pmatrix}a & ib\\ -ic & d\end{pmatrix}\\
&= \begin{pmatrix}\alpha a+\beta c & i(\alpha b+\beta d)\\ -i(\gamma a+\delta c) & \gamma b+\delta d\end{pmatrix}\\
&= \phi\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}\phi\begin{pmatrix}a & b\\ c & d\end{pmatrix}
\end{aligned}$$

5.2 $Spin^{+}(0,2)$

As preparation for studiying S$pin^{+}(1,3)$ we first need to look at $Cl(0,2)$ and then use **IC**. As a bonus, we will obtain an explicit form of reversion on $Cl(0,2)$, which is perhaps folklore, but has not been explicitly recorded elsewhere.

Let $Cl(0,2) = \mathbb{H}$ via the basis of 1-vectors $\{i,j\}$. Clifford conjugation cc is same as quaternionic conjugation. i.e. $q^{cc} = \bar{q}$.

Let us next calculate reversion. To that end recall, the association $q \to M_q = \begin{pmatrix} z & w\\ -\bar{w} & \bar{z}\end{pmatrix}$, where q is written as $z + wj$.

Now observe that

$$
\begin{aligned}
(i) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\
= \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\
= \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \to M_i,
\end{aligned}
$$

$$
\begin{aligned}
(ii) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}^* \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\
= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\
= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\
= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \to M_j,
\end{aligned}
$$

Now the matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ is itself not an M_q for any quaternion q. However, it equals $\frac{1}{i}M_k$ where i as viewed as a complex number and k as a quaternion).

One can thus conclude that

$$q^{rev} = -ik\bar{q}(-ik) = -k\bar{q}k$$

Therefore,

$$\begin{aligned} q^{gr} &= (q^{rev})^{cc} \\ &= (\overline{-k\bar{q}k}) \\ &= \bar{k}q(-\bar{k}) \\ &= -kqk \end{aligned}$$

Lemma 1.

$Cl(0,2)$ for the choice of 1-vectors $\{i, j\}$, the Clifford

(i) conjugation: $q^{cc} = \bar{q}$.

(ii) reversion: $q^{rev} = -k\bar{q}k$.

(iii) grade: $q^{gr} = -kqk$.

Though not needed for understanding S$pin^+(1,3)$, it is interesting to look at what the previous lemma yields regarding S$pin^+(1,2)$ (which is abstractly isomorphic to $SO(2,\mathbb{R})$). First, q is even $\Longleftrightarrow$

$$\begin{aligned} -kqk &= q \\ -kq &= q(-k). \end{aligned}$$

$$\begin{aligned} Note: \ -k\bar{k}k& \\ = -k(-k)k& \\ = -k.& \end{aligned}$$

Thus, $Cl^+(0,2) = \{q : qk = kq\}$.

So if $q = a + bi + cj + dk$, then $qk = ak - bj + ci - d$ and $kq = ak + bj - ci - d$.

So $qk = kq \Longleftrightarrow b = c = 0$. So $Cl^+(0,2) = \{q : a + dk\}$.

Thus $Spin^+(0,2) = \{q : a + dk; (a + dk)(a - dk) = 1\}$

5.3 $Spin^+(1,3)$

Now that we have all the details of the $Cl(0,2)$ from section 5.2, we can use the iteration **IC** to proceed as follows:

Let $Cl(1,3) = M(2,\mathbb{H})$, then

$$\begin{pmatrix} q_1 & q_2 \\ q_3 & q_4 \end{pmatrix}^{rev} = \begin{pmatrix} \bar{q}_4 & \bar{q}_2 \\ \bar{q}_3 & \bar{q}_1 \end{pmatrix}$$

$$\begin{pmatrix} q_1 & q_2 \\ q_3 & q_4 \end{pmatrix}^{cc} = \begin{pmatrix} -k\bar{q}_4k & k\bar{q}_2k \\ k\bar{q}_3k & -k\bar{q}_1k \end{pmatrix}$$

$$\begin{aligned}\begin{pmatrix} q_1 & q_2 \\ q_3 & q_4 \end{pmatrix}^{gr} &= \begin{pmatrix} -k\bar{q}_4k & k\bar{q}_2k \\ k\bar{q}_3k & -k\bar{q}_1k \end{pmatrix}^{rev} \\ &= \begin{pmatrix} -\overline{k\bar{q}_1k} & \overline{k\bar{q}_2k} \\ \overline{k\bar{q}_3k} & -\overline{k\bar{q}_4k} \end{pmatrix} \\ &= \begin{pmatrix} -kq_1k & -kq_2(-k) \\ kq_3k & -kq_4k \end{pmatrix}\end{aligned}$$

So $q_1 = \alpha_1 + \delta_1 k$, $q_4 = \alpha_4 + \delta_4 k$ and $kq_2k = q_2 \Longleftrightarrow kq_2 = q_2(-k)$.

Now, if $q_2 = \alpha_2 + \beta_2 i + \gamma_2 j + \delta_2 k$, then

$kq_2 = \alpha_2 k + \beta_2 j - \gamma_2 i - \delta_2$ and

$q_2(-k) = -(\alpha_2 k - \beta_2 j + \gamma_2 i - \delta_2) = -\alpha_2 k + \beta_2 j - \gamma_2 i + \delta_2.$

Hence, $\begin{pmatrix} q_1 & q_2 \\ q_3 & q_4 \end{pmatrix}$ is even $\Longleftrightarrow$ it is of the form $\begin{pmatrix} \alpha_1 + \delta_1 k & \beta_2 \mathrm{i} + \gamma_2 j \\ \beta_3 i + \gamma_3 j & \alpha_4 + \delta_4 k \end{pmatrix}$.

Therefore $Spin^+(1,3) = \{P \in M(2,\mathbb{H}),\ P \ \text{ even} : PP^{cc} = I_2\}$.

Now for $P = \begin{pmatrix} \alpha_1 + \delta_1 k & \beta_2 i + \gamma_2 j \\ \beta_3 i + \gamma_3 j & \alpha_4 + \delta_4 k \end{pmatrix}$, we get

$$\begin{aligned} P^{cc} &= \begin{pmatrix} -k(\alpha_4 - \delta_4 k)k & k(-\beta_2 i - \gamma_2 j)k \\ k(-\beta_3 i - \gamma_3 j)k & -k(\alpha_1 - \delta_1 k)k \end{pmatrix} \\ &= \begin{pmatrix} -k(\alpha_4 k + \delta_4) & k(\beta_2 i - \gamma_2 j) \\ k(\beta_3 i - \gamma_3 j) & -k(\alpha_1 k + \delta_1) \end{pmatrix} \\ &= \begin{pmatrix} \alpha_4 - \delta_4 k & -\beta_2 i - \gamma_2 j \\ -\beta_3 i - \gamma_3 j & \alpha_1 - \delta_1 k \end{pmatrix} \end{aligned}$$

Hence $PP^{cc} = I_2$ is equivalent to: $\begin{pmatrix} \alpha_1 + \delta_1 k & \beta_2 \mathrm{i} + \gamma_2 j \\ \beta_3 \mathrm{i} + \gamma_3 \mathrm{j} & \alpha_4 + \delta_4 \mathrm{k} \end{pmatrix} \begin{pmatrix} \alpha_4 - \delta_4 k & -\beta_2 i - \gamma_2 j \\ -\beta_3 i - \gamma_3 j & \alpha_1 - \delta_1 k \end{pmatrix}$

At the level of the entries of PP^{cc} this is equivalent to the following equations:

(1,1) is

$$\alpha_1\alpha_4 + \delta_1\delta_4 + k(\delta_1\alpha_4 - \delta_4\alpha_1 + \beta_2\beta_3 + \gamma_2\gamma_3 + k(-\beta_2\gamma_3 + \gamma_2\beta_3)$$

So

$$\alpha_1\alpha_4 + \delta_1\delta_4 + \beta_2\beta_3 + \gamma_2\gamma_3 = 1$$

and

$$\delta_1\alpha_4 - \delta_4\alpha_1 + \gamma_2\beta_3 - \beta_2\gamma_3 = 0$$

Therefore

$$\delta_1\alpha_4 - \delta_4\alpha_1 = \beta_2\gamma_3 - \gamma_2\beta_3$$

(2,2) is

$$\beta_3\beta_2 + \gamma_3\gamma_2 + k(\beta_2\gamma_3 - \gamma_1\beta_3) + \alpha_4\alpha_1 + \delta_1\delta_4 + k(-\delta_1\alpha_4 + \delta_4\alpha_1)$$

So

$$\delta_4\alpha_1 - \delta_1\alpha_4 = \beta_2\gamma_3 - \gamma_2\beta_3$$

$(1, 2)$ is

$$-\alpha_1\beta_2 i - \alpha_1\gamma_2 j - \delta_1\beta_2 j + \delta_1\gamma_2 i + \beta_2\alpha_1 i + \beta_2\delta_1 j + \gamma_2\alpha_1 j - \gamma_2\delta_1 i = 0$$

.

(2,1) is

$$\beta_3\alpha_4 i + \beta_3\delta_4 j + \gamma_3\alpha_4 j - \gamma_3\delta_4 i - \alpha_4\beta_3 i - \alpha_4\gamma_3 j - \alpha_4\beta_3 j + \gamma_3\delta_4 i = 0$$

So $PP^{cc} = I \iff$

$$\alpha_1\alpha_4 + \delta_1\delta_4 + \beta_2\beta_3 + \gamma_2\gamma_3 = 1$$

and

$$\delta_1\alpha_4 - \delta_4\alpha_1 + \beta_3\gamma_2 - \beta_2\gamma_3 = 0$$

Our next goal is to display en explicit isomorphism between $SL(2, \mathbf{C})$ and $\mathrm{S}pin^{+}(1,3)$ as obtained above.

To that end let $Y = \begin{pmatrix} \alpha_1 + \delta_1 i & \beta_2 + \gamma_2 i \\ -\beta_3 + \gamma_3 i & \alpha_4 - \delta_4 i \end{pmatrix}$,

then

$$det(Y) = (\alpha_1 + \delta_1 i)(\alpha_4 - \delta_4 i) - (\beta_2 + \gamma_2 i)(-\beta_3 + \gamma_3 i)$$

$$= \alpha_1\alpha_4 + \delta_1\delta_4 + i(\delta_1\alpha_4 - \delta_4\alpha_1) - [(-\beta_2\beta_3 - \gamma_2\gamma_3) + i(\beta_2\gamma_3 - \gamma_2\beta_3)]$$

$$= \alpha_1\alpha_4 + \delta_1\delta_4 + \beta_2\beta_3 + \gamma_2\gamma_3 + i(\delta_1\alpha_4 - \delta_4\alpha_1 - \beta_2\gamma_3 + \beta_3\gamma_2)$$

This motivates us to define $\Phi : SL(2, \mathbf{C}) \to Cl(1,3)$ via:

$$\Phi\left[\begin{pmatrix} \alpha_1 + \delta_1 i & \beta_2 + \gamma_2 i \\ -\beta_3 + \gamma_3 i & \alpha_4 - \delta_4 i \end{pmatrix}\right] = \begin{pmatrix} \alpha_1 + \delta_1 k & \beta_2 i + \gamma_2 j \\ \beta_3 i + \gamma_3 j & \alpha_4 + \delta_4 k \end{pmatrix}$$

Is this a group isomorphism onto $\mathrm{S}pin^{+}(1,3)$?.

To answer the above question, let calculate:

$$\phi(Y_1 Y_2) = \phi\left[\begin{pmatrix} \alpha_1 + \delta_1 i & \beta_2 + \gamma_2 i \\ -\beta_3 + \gamma_3 i & \alpha_4 - \delta_4 i \end{pmatrix}\begin{pmatrix} a_1 + d_1 i & b_2 + c_2 i \\ -b_3 + c_3 i & a_4 - d_4 i \end{pmatrix}\right]$$

$$= \left(\begin{array}{c|c} \begin{array}{c} \alpha_1 a_1 - d_1\delta_1 + i(\delta_1 a_1 + d_1\alpha_1 \\ +(-\beta_2 b_3 - \gamma_2 c_3) + i(-\gamma_2 b_3 + \beta_2 c_3) \end{array} & \alpha_1 b_2 - c_2\delta_1 + i(\delta_1 b_2 + c_2\alpha_1) + \beta_2 a_4 + \gamma_2 d_4 + i(\gamma_2 a_4 - d_4\beta_2) \\ \hline \begin{array}{c} -\beta_3 a_1 - \gamma_3 d_1 + i(-d_1\beta_3 + \gamma_3 a_1) \\ -\alpha_4 b_3 + \delta_4 c_3 + i(\alpha_4 c_3 + \delta_4 b_3) \end{array} & -\beta_3 b_2 - \gamma_3 c_2 + i(\gamma_3 b_2 - \beta_3 c_2) + \alpha_4 a_4 - \delta_4 d_4 + i(-\delta_4 a_4 - \alpha_4 d_4) \end{array}\right)$$

$$= \left(\begin{array}{c|c} \begin{array}{c} \alpha_1 a_1 - d_1\delta_1 - \beta_2 b_3 - \gamma_2 c_3 \\ +(\delta_1 a_1 + d_1\alpha_1 - \gamma_2 b_3 + \beta_2 c_3)k \end{array} & (\alpha_1 b_2 - c_2\delta_1 + \beta_2 a_4 + d_4\gamma_2)i + (\delta_1 b_2 + c_2\alpha_1 + \gamma_2 a_4 - d_4\beta_2)j \\ \hline \begin{array}{c} (\beta_3 a_1 + \gamma_3 d_1 + \alpha_4 b_3 - \delta_4 b_3)i \\ +(-d_1)b_3 + \gamma_3 a_1 + \alpha_4 c_3 + \delta_4 b_3)j \end{array} & (\alpha_4 a_4 - \delta_4 d_4 - \beta_3 b_2 - \gamma_3 c_2) - (\gamma_3 b_2 - \beta_3 c_2 - \delta_4 a_4 - d_4\alpha_4)k \end{array}\right)$$

$$= \begin{pmatrix} \alpha_1 + \delta_1 k & \beta_2 i + \gamma_2 j \\ \beta_3 i + \gamma_3 j & \alpha_4 + \delta_4 k \end{pmatrix} \begin{pmatrix} a_1 + d_1 k & b_2 i + c_2 j \\ b_3 i + c_3 j & a_4 + d_4 k \end{pmatrix}$$

$$= \left(\begin{array}{c|c} \begin{array}{c} \alpha_1 a_1 - d_1\delta_1 - \beta_2 b_3 - \gamma_2 c_3 \\ +(\delta_1 a_1 + d_1\alpha_1 - \gamma_2 b_3 + \beta_2 c_3)k \end{array} & (\alpha_1 b_2 - c_2\delta_1 + \beta_2 a_4 + d_4\gamma_2)i + (\delta_1 b_2 + c_2\alpha_1 + \gamma_2 a_4 - d_4\beta_2)j \\ \hline \begin{array}{c} (\beta_3 a_1 + \gamma_3 d_1 + \alpha_4 b_3 - \delta_4 c_3)i \\ +(-d_1 b_3 + \gamma_3 a_1 + \alpha_4 c_3 + \delta_4 b_3)j \end{array} & (\alpha_4 a_4 - \delta_4 d_4 - \beta_3 b_2 - \gamma_3 c_2) - (\gamma_3 b_2 - \beta_3 c_2 - \delta_4 a_4 - d_4\alpha_4) \end{array}\right)$$

5.4 $Spin^+(2,3)$

Since $Cl(0,1) = \mathbb{C}$, it follows from **IC** that $Cl(2,3) = M(4,\mathbb{C})$. Accordingly let $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ be an element of $Cl(2,3)$. Then

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}^{rev} = \begin{pmatrix} D^{cc} & B^{cc} \\ C^{cc} & A^{cc} \end{pmatrix}, \text{ and } \begin{pmatrix} A & B \\ C & D \end{pmatrix}^{cc} = \begin{pmatrix} D^{rev} & -B^{rev} \\ -C^{rev} & A^{rev} \end{pmatrix}$$

So using the results of Section 5.1, we find

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}^{rev} = \begin{pmatrix} adjD & adjB \\ adjC & adjA \end{pmatrix}$$

Note that if $X = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is 2×2, then

$$\begin{aligned} adj \begin{pmatrix} a & b \\ c & d \end{pmatrix} &= \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \\ &= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a & c \\ b & d \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} -b & -d \\ a & c \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \\ &= J_2^{-1} X^T J_2 \end{aligned}$$

So

$$\begin{aligned} \begin{pmatrix} A & B \\ C & D \end{pmatrix}^{rev} &= \begin{pmatrix} 0 & I_2 \\ I_2 & 0 \end{pmatrix}^T \begin{pmatrix} J_2^{-1} A^T J_2 & J_2^{-1} C^T J_2 \\ J_2^{-1} B^T J_2 & J_2^{-1} D^T J_2 \end{pmatrix} \begin{pmatrix} 0 & I_2 \\ I_2 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & I_2 \\ I_2 & 0 \end{pmatrix}^T \begin{pmatrix} J_2^T & 0 \\ 0 & J_2^T \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix}^T \begin{pmatrix} J_2 & 0 \\ 0 & J_2 \end{pmatrix} \begin{pmatrix} 0 & I_2 \\ I_2 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & J_2 \\ J_2 & 0 \end{pmatrix}^T \begin{pmatrix} A & B \\ C & D \end{pmatrix}^T \begin{pmatrix} 0 & J_2 \\ J_2 & 0 \end{pmatrix} \\ &= M_{1\otimes k}^T X^T M_{1\otimes k} \end{aligned}$$

Next,

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}^{cc} = \begin{pmatrix} D^{rev} & -B^{rev} \\ -C^{rev} & A^{rev} \end{pmatrix}$$
$$= J_4^T \begin{pmatrix} A^{rev} & C^{rev} \\ B^{rev} & D^{rev} \end{pmatrix} J_4$$

Now

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{rev} = \begin{pmatrix} \bar{d} & \bar{b} \\ \bar{c} & \bar{a} \end{pmatrix}$$
$$= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \bar{a} & \bar{c} \\ \bar{b} & \bar{d} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$
$$= \begin{pmatrix} \bar{d} & \bar{b} \\ \bar{c} & \bar{a} \end{pmatrix}$$
$$= \sigma_x^T \begin{pmatrix} a & b \\ c & d \end{pmatrix}^* \sigma_x$$
$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}^{cc} = \begin{pmatrix} 0 & I_2 \\ -I_2 & 0 \end{pmatrix}^T \begin{pmatrix} \sigma_x^T A^* \sigma_x & \sigma_x^T C^* \sigma_x \\ \sigma_x^T B^* \sigma_x & \sigma_x^T D^* \sigma_x \end{pmatrix} \begin{pmatrix} 0 & I_2 \\ -I_2 & 0 \end{pmatrix}$$
$$= J_4^T \begin{pmatrix} \sigma_x & 0 \\ 0 & \sigma x \end{pmatrix}^T \begin{pmatrix} A & B \\ C & D \end{pmatrix}^* \begin{pmatrix} \sigma_x & 0 \\ 0 & \sigma_x \end{pmatrix} J_4$$

Thus $\begin{pmatrix} A & B \\ C & D \end{pmatrix}^{cc} = \begin{pmatrix} 0 & \sigma_x \\ -\sigma_x & 0 \end{pmatrix}^T X^* \begin{pmatrix} 0 & \sigma_x \\ -\sigma_x & 0 \end{pmatrix}$

Now $\begin{pmatrix} 0 & \sigma_x \\ -\sigma_x & 0 \end{pmatrix} = M_{k\otimes 1}$ where $M_{k\otimes 1} = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$

$\begin{pmatrix} \sigma_x & 0 \\ 0 & \sigma_x \end{pmatrix} \begin{pmatrix} 0 & I_2 \\ -I_2 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \sigma_x \\ -\sigma_x & 0 \end{pmatrix}$. So as $kj = -i$, it follows that

$$k \begin{pmatrix} 0 & \sigma_x \\ -\sigma_x & 0 \end{pmatrix} = M_{-k\otimes 1}$$

Therefore,

$$\begin{aligned} X^{cc} &= (M_{-k\otimes 1})^T X^* M_{-K\otimes 1} \\ &= M_{k\otimes 1} X^* M_{-k\otimes 1} \end{aligned}$$

Hence, the grade automorphism is:

$$\begin{aligned} X^{gr} &= (X^{rev})^{cc} \\ &= M^T_{-k\otimes 1} \left[M^T_{1\otimes k} X^T M_{1\otimes k} \right]^* M_{-k\otimes 1} \\ &= M_{k\otimes 1} M_{1\otimes -k} \bar{X} M_{1\otimes k} M_{-k\otimes 1} \\ &= M_{k\otimes -k} \bar{X} M_{-k\otimes k} \\ &= M_{k\otimes k} \bar{X} M_{k\otimes k} \end{aligned}$$

Thus $X^{gr} = M_{k\otimes -k} \bar{X} M_{-k\otimes k}$

Therefore, $X = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ is even $\iff \; M_{k\otimes k} \bar{X} = X M_{k\otimes k}$.

Now

$M_{k\otimes k} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$. Therefore the condition that X is even becomes:

$$\begin{pmatrix} \sigma_z & 0 \\ 0 & -\sigma_z \end{pmatrix} \begin{pmatrix} \bar{A} & \bar{B} \\ \bar{C} & \bar{D} \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} \sigma_z & 0 \\ 0 & -\sigma_z \end{pmatrix}$$

$$\begin{pmatrix} \sigma_z\bar{A} & \sigma_z\bar{B} \\ -\sigma_z\bar{C} & -\sigma_z\bar{D} \end{pmatrix} = \begin{pmatrix} A\sigma_z & -B\sigma_z \\ C\sigma_z & -D\sigma_z \end{pmatrix}$$

This implies

$\sigma_z\bar{A} = A\sigma_z$

$\sigma_z\bar{D} = D\sigma_z$

$\sigma_z\bar{B} = -B\sigma_z$

$\sigma_z\bar{C} = -C\sigma_z$

Let $Y = \begin{pmatrix} z & w \\ \zeta & \omega \end{pmatrix}$,

then $\sigma_z\bar{Y} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \bar{z} & \bar{w} \\ \bar{\zeta} & \bar{\omega} \end{pmatrix} = \begin{pmatrix} \bar{z} & \bar{w} \\ -\bar{\zeta} & -\bar{\omega} \end{pmatrix} = A\sigma_z = \begin{pmatrix} z & -w \\ \zeta & -\omega \end{pmatrix}$

So $z,\ w \in \mathbb{R},\ \omega, \zeta \in i\mathbb{R}$

Let $Z = \begin{pmatrix} z & w \\ \zeta & \omega \end{pmatrix}$; $\sigma_z\bar{Z} = -Z\sigma_z$.

So $\begin{pmatrix} \bar{z} & \bar{w} \\ -\bar{\zeta} & -\bar{\omega} \end{pmatrix} = \begin{pmatrix} -z & w \\ -\zeta & \omega \end{pmatrix} \zeta,\ w \in \mathbb{R}\ \omega, z \in i\mathbb{R}$

Thus $X \in Cl(2,3)$ is even if it has the form

$$\begin{pmatrix} x_1 & iy_1 & ix_2 & y_2 \\ iz_1 & w_1 & z_2 & iw_2 \\ ix_3 & y_3 & x_4 & iy_4 \\ z_3 & iw_3 & iz_4 & w_4 \end{pmatrix}$$

Let X $= \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ and $M_{k\otimes 1} = \begin{pmatrix} 0 & -\sigma_x \\ \sigma_x & 0 \end{pmatrix}$ we claim that:

Now an even X is in S$pin^+(2,3)$ iff it additionally satisfies

$$M_{k\otimes 1}X^*M_{-k\otimes 1}X^* = I$$

$$\Rightarrow X^*M_{-k\otimes 1}X = M_{-k\otimes 1}$$

$$\Rightarrow X^*M_{k\otimes 1}X = M_{k\otimes 1}$$

To see what this last condition entails we first note that: $M_{k\otimes 1} \begin{pmatrix} 0 & -\sigma_x \\ \sigma_x & 0 \end{pmatrix}$

So

$$\begin{pmatrix} A^* & C^* \\ B^* & D* \end{pmatrix} \begin{pmatrix} 0 & -\sigma_x \\ \sigma_x & 0 \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 0 & -\sigma_x \\ \sigma_x & 0 \end{pmatrix}$$

$$\begin{pmatrix} C^*\sigma_x & -A^*\sigma_x \\ D^*\sigma_x & -B^*\sigma_x \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 0 & -\sigma_x \\ \sigma_x & 0 \end{pmatrix}$$

$$\begin{pmatrix} C^*\sigma_x A - A^*\sigma_x C & C^*\sigma_x B - A^*\sigma_x D \\ D^*\sigma_x A - B^*\sigma_x C & D^*\sigma_x B - B^*\sigma_x D \end{pmatrix} = \begin{pmatrix} 0 & -\sigma_x \\ \sigma_x & 0 \end{pmatrix}$$

Thus,

1. $C^*\sigma_x A - A^*\sigma_x C = 0$ and
2. $D^*\sigma_x B - B^*\sigma_x D = 0$.
3. $C^*\sigma_x B - A^*\sigma_x C = -\sigma_x$
4. $D^*\sigma_x A - B^*\sigma_x C = \sigma_x$

The first condition says that $A^*\sigma_x C$ is Hermitian. The second condition says $D^*\sigma_x B$ is Hermitian. The third and fourth conditions are equivalent to onether, since σ_x is Hermitian.

Theorem 5.1.

$Spin^+(2,3)$ is the set of matrices in $M(4,\mathbb{C})$, of the form $\begin{pmatrix} x_1 & iy_1 & ix_2 & y_2 \\ iz_1 & w_1 & z_2 & iw_2 \\ ix_3 & y_3 & x_4 & iy_4 \\ z_3 & iw_3 & iz_4 & w_4 \end{pmatrix}$, *which when written in block form* $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ *satisfies additionally*

i) $A^*\sigma_x C$ *is Hermitian*

ii) $D^*\sigma_x B$ *is Hermitian*

iii) $D^*\sigma_x A - B^*\sigma_x C = \sigma_x$

CHAPTER 6

CONCLUSIONS AND FUTURE WORK

In this dissertation, we completely addressed the essential questions of developing explicit quadratic formulae for the covering map $\phi_{p,q}$ and then inverting this map $\phi_{p,q} : Spin^{+}(p,q) \to SO^{+}(p,q,\mathbb{R})$, for $(p,q) \in \{(2,1),(2,2),(3,1),(3,2),(3,3)\}$. Finally, we produced details of a first principles approach, in the spirit of [1], of the indefinite spin groups when $(p,q) \in \{(1,2),(1,3),(2,3)\}$.

Future work will concentrate on calculating the inversion of the remaining covers of $\phi_{p,q}$, where $p+q \leq 6$. It is particularly interesting to do this for $\phi_{5,1}$ since $Spin^{+}(5,1)$ is $M(2,\mathbb{H})$. Finally it would be useful to develop a first principles approach to $Spin^{+}(9,1)$ due to it relations to the Octonions.

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BIOGRAPHICAL SKETCH

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