COEFFICIENTS OF CATALAN STATES OF LATTICE CROSSING

by

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To my family

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DISSERTATION

Presented to the Faculty of

The University of Texas at Dallas

in Partial Fulfillment

of the Requirements

for the Degree of

DOCTOR OF PHILOSOPHY IN

MATHEMATICS

THE UNIVERSITY OF TEXAS AT DALLAS

May 2021

ACKNOWLEDGMENTS

I would like to express my gratitude to my advisor, Dr. Mieczyslaw K. Dabkowski, for his generous guidance during my doctoral study and advice on my dissertation work. I am thankful to Dr. Józef H. Przytycki for his valuable ideas on research and motivation on various topics in knot theory. I am grateful to my committee members, Dr. Vladimir Dragović, Dr. Mohammad A. Hooshyar, Dr. Viswanath Ramakrishna, and Dr. Anh T. Tran, for their help and advice during the years. Finally, I am thankful to my family and all my friends at UTD for their support during my PhD.

April 2021

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The University of Texas at Dallas, 2021

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Skein modules are algebraic invariants of oriented 3-manifolds motivated by knot theory. The Kauffman bracket skein module is the most extensively studied skein module due to its relation with the Jones polynomial. Results of this dissertation can naturally be regarded as contributions to further development of the theory of skein modules. The lattice crossing was first studied by Dabkowski, Li, and Przytycki in 2015 as a part of an effort to find closed-form formulas for the natural product in the Kauffman bracket skein algebra of a four-punctured sphere. In this dissertation, we focus on finding coefficients of Catalan states obtained from lattice crossing and derive some relations between the coefficients of two different Catalan states. In particular, we develop methods for computing coefficients of Catalan states and find closed-form formulas for the coefficients of Catalan states obtained from lattice crossing with 4 vertical strands. We also examine the unimodality property of the coefficients of Catalan states. The generalized crossing is an n-tangle obtained as a half-twist of n vertical strands related to the lattice crossing. In the last part of this dissertation, we present some results concerning coefficients of Catalan states obtained from generalized crossing.

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CHAPTER 1

INTRODUCTION

Skein modules are invariants of oriented 3-manifolds M that were introduced independently by Przytycki [24] and Turaev [29]. They are defined as quotients of a free module generated by the ambient isotopy classes of links modulo some selected skein relations. The Kauffman bracket skein module (KBSM) is the most studied skein module based on the Kauffman bracket skein relation [16] and it has been computed for several classes of 3-manifolds [24, 13, 14, 15, 18, 23, 21, 22, 19]. Unfortunately, there are no general methods for calculating KBSM of 3-manifolds. Hence, the set of examples of M for which KBSM is known remains still quite limited. This, in turn, limits our ability to understand important relations between the algebraic structure of KBSM of M and the geometry of M. However, when the underlying ring is a field, as first conjectured by E. Witten (see [3]) and then proved in 2019 by Gunningham, Jordan, and Safronov [12], the KBSM of any closed oriented 3-manifolds becomes a finite-dimensional vector space.

When a 3-manifold is a product of an oriented surface F and an interval $I = [0, 1]$, the KBSM has a natural structure of an algebra that is called Kauffman bracket skein algebra (KBSA) of F. The product of two links $L_1 * L_2$ in the KBSA of F is defined by placing L_1 above L_2. This algebra is commutative when F is a 2-dimensional sphere or a punctured 2-sphere with at most three boundary components. The first and the simplest example of a surface with a non-commutative KBSA is a 2-dimensional torus or a four-punctured sphere. In 2000, Frohman and Gelca [11] found an elegant formula – known as the product-to-sum formula – for the product of curves in KBSA of 2-torus. It is then quite natural to ask whether a similar formula can be found for KBSA of a four-punctured sphere. In [1], the authors considered the double branched cover of a four-punctured sphere to obtain an algorithm for the product of curves in KBSA of a four-punctured sphere. The algorithm is rather involved and quite difficult to apply which makes it not practical to use. Therefore, the idea used in [7] for finding closed-form formulas for the coefficients of the product of curves in KBSA of a four-punctured sphere seems to be a reasonable approach. Namely, we start by considering an $(m+n)$-tangle $L(m,n)$ called lattice crossing [7] and try to find formulas for the coefficients of the crossingless connections between $2(m+n)$ points on the circle. Such connections are called Catalan states and they form a basis of the Relative Kauffman Bracket Skein Module (RKBSM) of a 3-ball B^3 with $2(m+n)$ points fixed on its boundary. Therefore, in such a setting lattice crossing $L(m,n)$ can be expressed by a linear combination of Catalan states with coefficients in some commutative ring with identity. The first main result obtained by Dabkowski, Li, and Przytycki in [7] gives the necessary and sufficient conditions for a Catalan state to show up the linear combination for $L(m,n)$ in the RKBSM, and another one gives a counting formula for the number of Catalan states that appear in this linear combination. In the sequel paper, Dabkowski and Przytycki [9] introduced the plucking polynomial of a plane rooted tree with a delay function for computing coefficients of some particular classes of Catalan states (Catalan states with no returns on one side). The plucking polynomial of a plane rooted tree was introduced by Przytycki [26] and it was motivated by the problem of finding the coefficients of Catalan states that was considered in [7]. The strict unimodality of coefficients of the plucking polynomial of a plane rooted tree was studied later in [5]. Furthermore, the necessary and sufficient conditions for a polynomial to be obtained as a plucking polynomial of a plane rooted tree were found in [4], and finally, the problem of finding conditions for different plane rooted trees that have the same plucking polynomials was studied in [6].

The generalized crossing is an n-tangle obtained as a half-twist of n vertical parallel strands. This tangle was first studied in [20] in the context of finding a relationship with lattice crossing. As shown in [20], every Catalan state of generalized crossing is realizable. In this dissertation, we develop further results concerning coefficients of Catalan states obtained from generalized crossing.

The main contribution of this dissertation is the development of methods for computing coefficients of arbitrary Catalan states obtained from lattice crossing and analyzing their properties. The dissertation is organized as follows. In Chapter 2, we provide a summary of the necessary definitions and results that are needed in later chapters.

In Chapter 3, we focus on the development of methods for finding coefficients of Catalan states. In particular, we give general results that allow us to establish relations between the coefficient of a given Catalan state and coefficients of some other related Catalan states. This allows us to find the coefficient of a given state in terms of coefficients of some other Catalan states for which coefficients can be computed. The method we develop updates the connections of Catalan states near the top and bottom boundaries of the Catalan state but leaves the connections in the middle unchanged. This technique, called the *first-row expansion*, was used in its simpler version in [9] to find coefficients of Catalan states with no returns on one side. Another observation we make is that, after a finite number of the first-row expansions, we can get Catalan states with certain similar patterns of connections. These observations gave an idea for our method, that is, to find the coefficient of a given Catalan state, we can find the coefficient of some larger Catalan states with "good properties" (for instance those discussed in [9]) and then apply the first-row expansions till this given Catalan state appears.

In Section 3.1, after we introduce some relevant terminology, we give an algorithm that splits all Catalan states into groups according to the way they decompose into smaller pieces and the operations that can be applied to each piece. Then we show how to reduce problems on finding coefficients of general Catalan states into problems for which we already know the answer. Furthermore, once we know how to do such a reduction, we know how to compute coefficients for all other Catalan states in the same group. Thus, our algorithm is not just simply finding the coefficient of a particular Catalan state but rather finding coefficients for a family of Catalan states. In the same section, we prove a very useful lemma about plucking polynomial of a plane rooted tree with a delay function. Moreover, We establish a relation between coefficients of a Catalan state C and the Catalan state C' obtained from C by removing some of its arcs. These results play an important role in this dissertation as they will allow us to justify theorems in later sections. In Section 3.2, we study some classes of Catalan states for which our reduction has a particularly simple form. This allows us to find coefficients of such Catalan states using methods that are different than the one obtained from the algorithm in Section 3.1. Since the number of groups, into which we split Catalan states, depends only on n and it is finite, we list all such reduction formulas for small values of n.

In Chapter 4, we find closed-form formulas for coefficients of Catalan states obtained from lattice crossing $L(m, n)$ when n is small and we obtain results concerning unimodality of coefficients of Catalan states that admit a horizontal line that has four intersections with such a Catalan state.

In Section 4.1, we find closed-form formulas for $n = 3, 4$. Although such formulas for $n = 3$ were obtained in [9], we prove a stronger version of results given in there. The main idea for $n = 4$ in our proof is to decompose Catalan states into small pieces, use formulas developed in Chapter 3 to find the contribution of each piece to the coefficients of Catalan states. Furthermore, to reconstruct the Catalan state from such pieces one must follow rules described by walks in a directed graph. The unimodality of coefficients of Catalan states is discussed in Section 4.2. We prove that coefficients of Catalan states for $n \leq 3$ are Laurent polynomials with unimodal coefficients and when $n \geq 5$ we give examples of Catalan states with coefficients that are not unimodal. For the case $n = 4$, we show that if a Catalan state is a vertical product of some other Catalan states, then its coefficient is unimodal.

In the last chapter, based on the idea similar to the first-row expansion that was used for lattice crossing, we develop a method for computing coefficients of Catalan states obtained from generalized crossing and then we apply it to find coefficients for two infinite families of

Catalan states. Analyzing properties of those coefficients and developing efficient methods for finding them will be part of my future research projects.

CHAPTER 2

PRELIMINARIES

In this chapter, we recall all necessary definitions and results that we will use in the remaining chapters of this dissertation.

2.1 Kauffman Bracket Skein Modules

A *link* L in a 3-manifold M^3 is defined as the image of an embedding of a disjoint union of circles into M^3, and the image of each circle is called a *component* of the link. One component link is called a *knot*. A *framed link* is a link with integers assigned to its components that represent the twist numbers while regarding each component as a ribbon knot. We also may view a framed link L as the image of an embedding of a disjoint union of annuli into M^3 with $S^1 \times \{0\}$ of annulus $A^2 = S^1 \times [0, 1]$ regarded as L.

Links in $\mathbb{R}^3$ can be studied via their diagrams up to the natural moves that correspond to ambient isotopies of $\mathbb{R}^3$. Namely, given a link L in $\mathbb{R}^3$, its projection onto a plane which avoids diagrams shown in Figure 2.1 together with the information at each double point about which arc is above or below is called a *link diagram* of L.

Theorem 2.1.1 (Reidemeister [27]). *Given two links L_1, L_2 in $\mathbb{R}^3$ suppose that D_1 and D_2 are their link diagrams. Then L_1 is ambient isotopic to L_2 if and only if D_1 can be obtained from D_2 by a finite sequence of Reidemeister moves R_1, R_2, or R_3, as shown in the Figure 2.2.*

The above theorem allows us to work with the link diagrams in $\mathbb{R}^2$ rather than links in $\mathbb{R}^3$. In particular, we can define invariants of links in $\mathbb{R}^3$ using their diagrams. One of the most important invariant for us is the Kauffman bracket.

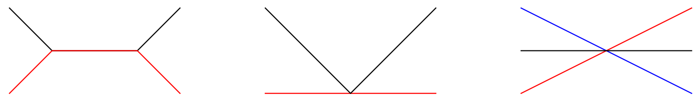


Figure 2.1. Diagrams are not allowed

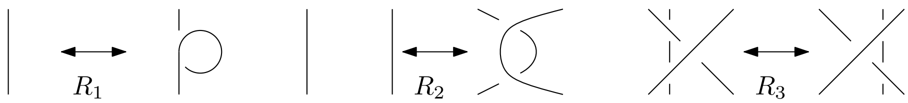


Figure 2.2. Reidemeister moves

Theorem 2.1.2 (Kauffman [16]). *Given a fixed invertible element A, define bracket polynomial of an unoriented link in $\mathbb{R}^3$ as follows:*

$$\langle \bigcirc \rangle = 1,$$

$$\langle \times \rangle = A \langle\,)\,(\, \rangle + A^{-1} \langle \asymp \rangle,$$

and

$$\langle \bigcirc \sqcup L \rangle = -(A^2 + A^{-2})\langle L \rangle,$$

where links involved in the second relation are identical outside of a neighborhood of the crossing, and the circle $\bigcirc$ in the first and third relations denotes the unknot. Then $\langle L \rangle$ is invariant under Reidemeister moves R_2 and R_3.

We will say that link diagrams D_1 and D_2 are related via *regular isotopy* of diagrams if D_1 can be obtained from D_2 by a finite sequence of R_2 and R_3 moves.

As we mentioned in Chapter 1, the theory of skein modules gives the natural context for the results of this dissertation. We recall the definition of the Kauffman bracket skein module (KBSM) of an oriented 3-manifold M^3. This invariant was defined by Przytycki as a generalization of the Kauffman bracket polynomial (defined for links in $\mathbb{R}^3$) to links in an arbitrary oriented 3-manifold. In particular, the KBSM of M^3 is a sound algebraic structure that allows us to study links in M^3.

Definition 2.1.3 (Przytycki [24]). *Given an oriented 3-manifold M^3, a commutative ring R, and an invertible element A in R. Denote by $\mathcal{L}_{fr}$ the set of all ambient isotopy classes of unoriented framed links in M^3 including the empty link $\emptyset$. Let $R\mathcal{L}_{fr}$ be the free R-module with basis $\mathcal{L}_{fr}$ and $\mathcal{S}_{2,\infty}$ be the submodule of $R\mathcal{L}_{fr}$ generated by*

$$\diagup\!\!\!\!\diagdown - A \asymp - A^{-1})\,(\quad \text{and} \quad \bigcirc \sqcup L + (A^2 + A^{-2})L, \tag{2.1}$$

where the skein triple in the first relation represents framed links which are identical outside of a neighborhood of the crossing, and the circle $\bigcirc$ in the second relation denotes the unknot. Then the Kauffman bracket skein module (KBSM) of M^3 is defined by

$$\mathcal{S}_{2,\infty}(M^3; R, A) = R\mathcal{L}_{fr}/\mathcal{S}_{2,\infty}.$$

Denote by $F_{g,b}$ an oriented surface of genus g with b boundary components and let $I = [0,1]$ be the unit interval. One defines a natural multiplication $*$ on $\mathcal{S}_{2,\infty}(F_{g,b} \times I; \mathbb{Z}[A^{\pm 1}], A)$ as follows: For links L_1 and L_2 in $F_{g,b} \times I$, let $L_1 * L_2$ be the link that obtained by placing L_1 above L_2, more precisely, we put L_1 in $F \times [\frac{1}{2}, 1]$ and L_2 in $F \times [0, \frac{1}{2}]$. The Kauffman bracket skein algebra (KBSA) of $F_{g,b}$ is then the KBSM $\mathcal{S}_{2,\infty}(F_{g,b} \times I; \mathbb{Z}[A^{\pm 1}], A)$ together with the multiplication $*$ and we denote it by $\mathcal{S}^*_{2,\infty}(F_{g,b} \times I)$. The following results concerning KBSA were obtained by Bullock and Przytycki. Since the main motivation for this work is to obtain closed-form formulas for the coefficients in the product of parallel copies of curves in a four-punctured sphere $F_{0,4}$, results of this dissertation are directly related to part (e) of the theorem below in which a presentation for KBSA of $F_{0,4}$ is given.

Theorem 2.1.4 (Bullock-Przytycki [2]). *Let $F_{g,b}$ be an oriented surface of genus g with b boundary components and $R = \mathbb{Z}[A^{\pm 1}]$.*

(a) $\mathcal{S}^*_{2,\infty}(F_{0,0} \times I) \simeq \mathcal{S}^*_{2,\infty}(F_{0,1} \times I) \simeq R.$

(b) $\mathcal{S}^*_{2,\infty}(F_{0,2} \times I) \simeq R[x]$, *where x is a curve parallel to a boundary component of $F_{0,2}$.*

(c) $\mathcal{S}^*_{2,\infty}(F_{0,3} \times I) \simeq R[x,y,z]$, *where* x, y, z *are curves parallel to boundary components of* $F_{0,3}$.

(d) $\mathcal{S}^*_{2,\infty}(F_{1,0} \times I) \simeq R\langle x,y,z\rangle/\mathcal{I}_{1,0}$, *where* x, y, z *are* $(1,0)$*-curve,* $(0,1)$*-curve, and* $(1,1)$*-curve, respectively, on* $F_{1,0}$ *and* $\mathcal{I}_{1,0}$ *is generated by*

$$Axy - A^{-1}yx - (A^2 - A^{-2})z,$$

$$Ayz - A^{-1}zy - (A^2 - A^{-2})x,$$

$$Azx - A^{-1}xz - (A^2 - A^{-2})y,$$

and

$$A^2x^2 + A^{-2}y^2 + A^2z^2 - Axyz - 2A^2 - 2A^{-2}.$$

(e) $\mathcal{S}^*_{2,\infty}(F_{0,4} \times I) \simeq R[a_1,a_2,a_3,a_4]\langle x,y,z\rangle/\mathcal{I}_{0,4}$, *where* x, y, z, a_1, a_2, a_3, a_4 *are* $(1,0)$*-curve,* $(0,1)$*-curve,* $(1,1)$*-curve, four curves parallel to different boundary components, respectively, on* $F_{0,4}$ *as shown in Figure 2.3, and* $\mathcal{I}_{0,4}$ *is generated by*

$$A^2xy - A^{-2}yx - (A^4 - A^{-4})z - (A^2 - A^{-2})(a_1a_3 + a_2a_4),$$

$$A^2yz - A^{-2}zy - (A^4 - A^{-4})x - (A^2 - A^{-2})(a_1a_4 + a_2a_3),$$

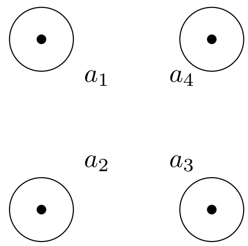
$$A^2zx - A^{-2}xz - (A^4 - A^{-4})y - (A^2 - A^{-2})(a_1a_2 + a_3a_4),$$

and

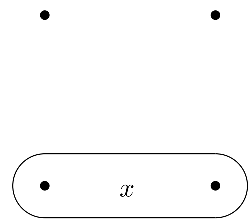
$$A^4x^2 + A^{-4}y^2 + A^4z^2 - A^2xyz - (A^2 + A^{-2})^2 + a_1a_2a_3a_4 + a_1^2 + a_2^2 + a_3^2 + a_4^2$$
$$+ A^2(a_1a_4 + a_2a_3)x + A^{-2}(a_1a_2 + a_3a_4)y + A^2(a_1a_3 + a_2a_4)z.$$

We note that $\mathcal{S}^*_{2,\infty}(F_{g,b} \times I)$ *is commutative if and only if* $(g,b) \in \{(0,0),(0,1),(0,2),(0,3)\}$.

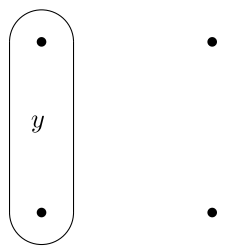
As we see from the above, when KBSA of $F_{g,b}$ is non-commutative, the product of curves is rather difficult to find. However, there is an elegant multiplicative formula for the product of curves on the 2-dimensional torus. Namely, there is a closed-form formula for the product in $\mathcal{S}^*_{2,\infty}(F_{1,0} \times I)$ that is known as the product-to-sum formula obtained by Frohman and



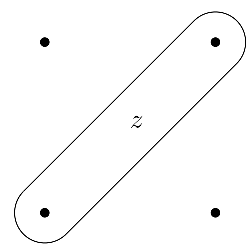
(a) Curves $a_1a_2a_3a_4$ on $F_{0,4}$



(b) Curve x on $F_{0,4}$



(c) Curve y on $F_{0,4}$



(d) Curve z on $F_{0,4}$

Figure 2.3. Curves on $F_{0,4}$

Gelca. This formula sparked an effort by Dabkowski, Li and Przytycki to find a similar formula for the product of curves on $F_{0,4}$ and motivated lots of new development, such as, for instance, plucking polynomial of a plane rooted tree with a delay function. We will recall the result by Frohman and Gelca mentioned before. Let $T_n(x)$ be the polynomials defined recursively by $T_0(x) = 2$, $T_1(x) = x$, $T_n(x) = xT_{n-1}(x) - T_{n-2}(x)$ for $n \geq 2$. For integers p and q, one defines the general (p,q)-curve by $(p,q)_T = T_{\gcd(p,q)}\left(\left(\frac{p}{\gcd(p,q)}, \frac{q}{\gcd(p,q)}\right)\right)$ with the convention $\gcd(0,0) = 0$.

Theorem 2.1.5 (Frohman-Gelca [11]). *For any integers p, q, r, s,*

$$(p,q)_T * (r,s)_T = A^{ps-qr}(p+r, q+s)_T + A^{-(ps-qr)}(p-r, q-s)_T$$

in $\mathcal{S}^*_{2,\infty}(F_{1,0} \times I)$.

In order to find the product $y^n * x^m$ of n parallel copies of the curve y and m parallel copies of the curve x in $\mathcal{S}^*_{2,\infty}(F_{0,4} \times I)$, we start by considering the diagram of $y^n * x^m$ on $F_{0,4}$

(see Figure 2.4(a)) locally. To make our considerations more precise, recall the definition of the relative Kauffman bracket skein module (RKBSM) of an oriented 3-manifold with $2n$ framed points (or equivalently intervals) on its boundary.

Definition 2.1.6 (Przytycki [25]). *Given an oriented 3-manifold M^3, a commutative ring R with identity, and an invertible element A in R. Let $\{x_i\}_{i=1}^{2n}$ be $2n$ framed points in ∂M^3, and denote by $\mathcal{L}_{fr}(n)$ the set of unoriented framed relative links in $(M^3, \partial M^3)$ such that $L \cap \partial M^3 = \partial L = \{x_i\}_{i=1}^{2n}$, up to ambient isotopy that fixes ∂M^3. Let $R\mathcal{L}_{fr}(n)$ be the R-module with basis $\mathcal{L}_{fr}(n)$ and $\mathcal{S}_{2,\infty}(n)$ be the submodule of $R\mathcal{L}_{fr}(n)$ generated by (2.1). Then the relative Kauffman bracket skein module (RKBSM) of M^3 is defined by*

$$\mathcal{S}_{2,\infty}(M^3, \{x_i\}_{i=1}^{2n}; R, A) = R\mathcal{L}_{fr}(n)/\mathcal{S}_{2,\infty}(n).$$

Let us consider the diagram $L(m, n)$ of $y^n * x^m$ which is inside the disk D^2 shown in Figure 2.4. As it can easily be seen this diagram is $(m + n)$-tangle obtained as a projection of the framed relative link in the cylinder $D^2 \times [0, 1]$ with $2(m + n)$ points on its boundary. The following results by Przytycki are of great importance to our further discussion of lattice crossing $L(m, n)$ as they give us the necessary mathematical setup for the problem we consider in this dissertation.

Proposition 2.1.7 (Przytycki [25]). *The RKBSM $\mathcal{S}_{2,\infty}(M^3, \{x_i\}_{i=1}^{2n}; R, A)$ depends only on the distribution points $\{x_i\}_{i=1}^{2n}$ among the boundary components of M, but not on the exact position of $\{x_i\}_{i=1}^{2n}$. In particular, if ∂M is connected, we can denote it by $\mathcal{S}_{2,\infty}(M^3, n; R, A)$.*

The following result describes a basis of the RKBSM in the case of $F_{g,b} \times I$ with $2n$ points on its boundary, which is relevant to our farther discussion.

Theorem 2.1.8 (Przytycki [25]). *$\mathcal{S}_{2,\infty}(F_{g,b} \times I, \{x_i\}_{i=1}^{2n}; R, A)$ is a free R-module for $b \geq 1$, and its basis consists of framed relative links on F without trivial components.*

Since the boundary of $D^2 \times I$ is certainly connected, the following result allows us to define the coefficient of a Catalan state C of lattice crossing $L(m,n)$ as we will discuss later. The main idea to keep in mind here is that a Catalan state is nothing but an element of the basis of the RKBSM of $D^2 \times I$ with $2(m+n)$ points described in Corollary 2.1.9 and the coefficient $C(A)$ of the Catalan state C is nothing but the coefficient of C when the relative link $L(m,n)$ is written in the basis of $\mathcal{S}_{2,\infty}(D^2 \times I, n; R, A)$. We will make it more precise in the section that follows.

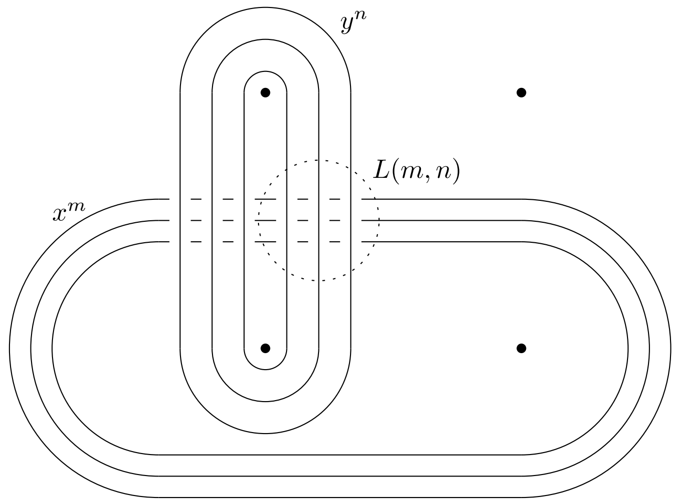
Corollary 2.1.9 (Przytycki [25]). *$\mathcal{S}_{2,\infty}(D^2 \times I, n; R, A)$ is a free R-module with $\frac{1}{n+1}\binom{2n}{n}$ basic elements, which consist of all framed crossingless connections between $2n$ boundary points.*

2.2 Lattice Crossing – Definition and Summary of Results

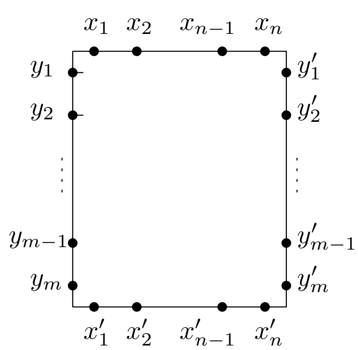
In [7], Dabkowski, Li, and Przytycki began their study of an $(m+n)$-tangle $L(m,n)$ called lattice crossing in order to analyze the product of n parallel copies of framed link y and m parallel copies of framed link x locally. More precisely, they considered the diagram of $y^n * x^m$ on $F_{0,4}$ inside the disk D^2 shown in Figure 2.4(a). For convenience, we consider the rectangle $R^2_{m,n}$ instead of the disk D^2 with $2(m+n)$ points $\mathfrak{X}_{m,n} = \{x_i, x'_i\}_{i=1}^{n} \cup \{y_j, y'_j\}_{j=1}^{m}$ fixed on its boundary and placed as shown in Figure 2.4(b). We define *lattice crossing* $L(m,n)$ as a framed relative link $\left(\bigsqcup_{i=1}^{n} \overline{x_i x'_i}\right) * \left(\bigsqcup_{j=1}^{m} \overline{y_j y'_j}\right)$ in $R^2_{m,n} \times I$ that consists of n parallel framed line segments $\overline{x_i x'_i}$ joining pairs of boundary points x_i and x'_i placed above the m parallel framed line segments $\overline{y_i y'_i}$ joining pairs of boundary points y_i and y'_i as shown in Figure 2.4(c). Let $\mathfrak{Cat}_{m,n}$ be the set of all crossingless connections between points of $\mathfrak{X}_{m,n}$. By Corollary 2.1.9, this set is a basis of $\mathcal{S}_{2,\infty}(R^2_{m,n} \times I, \mathfrak{X}_{m,n}; \mathbb{Z}[A^{\pm 1}], A)$. Hence

$$L(m,n) = \sum_{C \in \mathfrak{Cat}_{m,n}} C(A)\, C$$

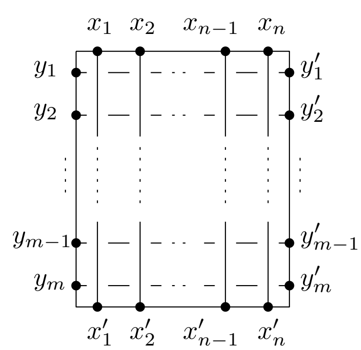
for some $C(A) \in \mathbb{Z}[A^{\pm 1}]$. Elements of $\mathfrak{Cat}_{m,n}$ are called *Catalan states* and the coefficient $C(A)$ of C in above linear combination is called the *coefficient* of the Catalan state C.



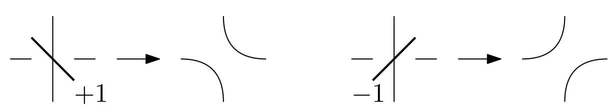
(a) Product $y^n * x^m$ in $\mathcal{S}^*_{2,\infty}(F_{0,4} \times I)$



(b) Rectangle $R^2_{m,n}$



(c) Lattice crossing $L(m,n)$



(d) $+1$ marker and -1 marker

Figure 2.4. $R^2_{m,n}$, $L(m,n)$, $+1$ marker and -1 markers

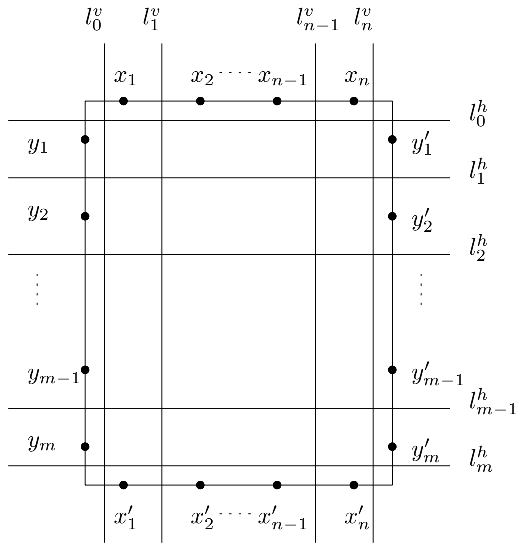


Figure 2.5. Straight lines l_i^v and l_j^h

Let $\mathrm{Mat}_{m,n}(\{\pm 1\})$ be the set of $m \times n$ matrices whose entries are either 1 or -1. A Kauffman state of $L(m,n)$ is an assignment of $+1$ or -1 markers on the mn crossings of $L(m,n)$ according to $s \in \mathrm{Mat}_{m,n}(\{\pm 1\})$. A Catalan state is *realized* by a Kauffman state s if it can be obtained by smoothing crossings of $L(m,n)$ that follow the rule in Figure 2.4(d) according to the markers determined by s and after removing all trivial components. A Catalan state is *realizable* if it can be realized by some Kauffman states. As one checks for $m = n = 2$ there are 14 Catalan states of $L(2,2)$ with 12 that are realizable.

Theorem below gives a complete characterization of realizable Catalan states.

Theorem 2.2.1 (Dabkowski-Li-Przytycki [7]). *A Catalan state C is realizable if and only if $|l_i^v \cap C| \leq m$ for all $1 \leq i \leq n-1$ and $|l_j^h \cap C| \leq n$ for all $1 \leq j \leq m-1$, where l_i^v and l_j^h are shown in Figure 2.5 and $|l \cap C|$ denotes the minimal number of intersections between curves l and C.*

Furthermore, one is able to give a formula that counts the realizable Catalan states.

Theorem 2.2.2 (Dabkowski-Li-Przytycki [7]). *The number of realizable Catalan states of $L(m,n)$ is*

$$\frac{1}{m+n+1}\binom{2m+2n}{m+n}$$
$$-\sum_{i=0}^{\infty}\left[\binom{2m+2n}{m-i(n+3)-2}-2\binom{2m+2n}{m-i(n+3)-3}+\binom{2m+2n}{m-i(n+3)-4}\right]$$
$$-\sum_{i=0}^{\infty}\left[\binom{2m+2n}{n-i(m+3)-2}-2\binom{2m+2n}{n-i(m+3)-3}+\binom{2m+2n}{n-i(m+3)-4}\right].$$

The edge of the rectangle $R^2_{m,n}$ that contains $\{x_i\}$ (respectively $\{x'_i\}$, $\{y_j\}$, and $\{y'_j\}$) is called the *top boundary* (respectively *bottom boundary*, *left boundary*, and *right boundary*) of $R^2_{m,n}$. A *return* of a Calatan state C is an arc of C with both endpoints lying on the same boundary of $R^2_{m,n}$. Let $\mathfrak{Cat}^F_{m,n}$ be the set of realizable Catalan states of $L(m,n)$ with no returns on the bottom boundary.

Definition 2.2.3 (Dabkowski-Przytycki [9]). *Given $C \in \mathfrak{Cat}^F_{m,n}$. Let $\mathfrak{b}(C)$ denote the set*

$$\left\{ \mathfrak{b} = (b_1, \ldots, b_m) \in \{0,1,\ldots,n\}^m \;\middle|\; \begin{bmatrix} s_b(b_1) \\ \vdots \\ s_b(b_m) \end{bmatrix} \text{ realizes } C \right\},$$

where the function s_b maps i to a $1 \times n$ row vector $[\underbrace{1,\ldots,1}_{i},\underbrace{-1,\ldots,-1}_{n-i}]$. The maximal sequence $\mathfrak{b}_M(C)$ is defined as the maximal element of the the set $\mathfrak{b}(C)$ ordered by the lexicographic order.

The $\mathfrak{b}_M(C)$ is well-defined as a consequence of the following result.

Theorem 2.2.4 (Dabkowski-Przytycki [9]). *The set $\mathfrak{b}(C)$ is nonempty for all $C \in \mathfrak{Cat}^F_{m,n}$.*

For a Catalan state C with no returns on the bottom boundary one defines the plane rooted tree with a delay function $\mathcal{T}(C)$ as follows. This tree will be used to compute the coefficient $C(A)$.

Definition 2.2.5 (Dabkowski-Przytycki [9]). *Given $C \in \mathfrak{Cat}^F_{m,n}$. Let C^F be a tangle obtained from C by removing all of its arcs with one of the endpoints on the bottom boundary of $R^2_{m,n}$. Denote by $T(C)$ the dual graph of C^F. There is a natural choice for the root v_0 of $T(C)$ that corresponds to the region of C^F containing points $\{x'_i\}$. Define a delay function f on the set of leaves of $T(C)$ different than v_0 that sends a leaf v to j if the points on the boundary of the region determined by C^F that corresponds to v are labeled by $\{y_{j-1}, y_j\}$ or $\{y'_{j-1}, y'_j\}$. Otherwise we define $f(v) = 1$. The plane rooted tree with a delay function determined by a Catalan state C with no returns on the bottom boundary is the triple $\mathcal{T}(C) = (T(C), v_0, f)$.*

For a Catalan state C with no returns on the bottom boundary we define a polynomial in terms of the graph $\mathcal{T}(C)$ associated to C. This polynomial is our main computational tool for $C(A)$.

Definition 2.2.6 (Dabkowski-Przytycki [9]). *Let (T, v_0, f) be a plane rooted tree T with root v_0 and a delay function f and let $L_1(T)$ be the set of all leaves v different than v_0 for which $f(v) = 1$. The plucking polynomial $Q(T, v_0, f)$ is a polynomial in q defined by $Q(T, v_0, f) = 1$ if T has no edges, and*

$$Q(T, v_0, f) = \sum_{v \in L_1(T)} q^{r(T,v_0,v)} Q(T - v, v_0, f_v)$$

otherwise, where $r(T, v_0, v)$ is the number of vertices of T to the right[1] *of the unique path from v to v_0, and f_v is a delay function defined on leaves of $T - v$ different than v_0 and such that $f_v(u) = \max\{1, f(u) - 1\}$ if u is a leaf of T and $f_v(u) = 1$ if u is a new leaf of $T - v$.*

The following result describes a very important relation between plucking polynomial of a plane rooted tree with a delay function and the coefficient of a Catalan state with no returns on the bottom boundary. In particular, the result below provides us with an effective method to compute $C(A)$.

[1]Consider the plane rooted tree (T, v_0) embedded in the xy-plane such that all vertices of T except v_0 have positive y coordinate, v_0 is the origin, and the unique path from v to v_0 are on the y-axis, then $r(T, v_0, v)$ is the number of vertices belonging the first quadrant.

Theorem 2.2.7 (Dabkowski-Przytycki [9])**.** *The coefficient $C(A)$ of $C \in \mathfrak{Cat}^F_{m,n}$ is given by*

$$C(A) = A^{2\|\mathfrak{b}_M(C)\| - mn + 4\cdot \operatorname{mindeg}_q Q(\mathcal{T}(C))} \cdot Q(\mathcal{T}(C))\big|_{q=A^{-4}},$$

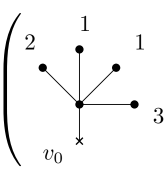
where $\|\mathfrak{b}_M(C)\| = \sum_{i=1}^m b_i$ if $\mathfrak{b}_M(C) = (b_1, \ldots, b_m)$ and $\operatorname{mindeg}_q Q(\mathcal{T}(C))$ denotes the minimum degree of q in $Q(\mathcal{T}(C))$.

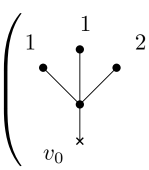
We note Theorem 2.2.7 is very important result that we will use in Chapter 3 and Chapter 4. The following example shows its application.

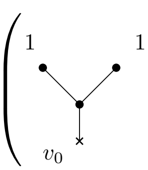
Example 2.2.8. *Given a Catalan state $C \in \mathfrak{Cat}^F_{5,3}$ shown in Figure 2.6(a). To find $C(A)$ we need both the plucking polynomial $Q(\mathcal{T}(C))$ of the plane rooted tree with a delay function $\mathcal{T}(C)$ and the maximal sequence $\mathfrak{b}_M(C)$ in order to apply the formula given in Theorem 2.2.7 that allows us to find $C(A)$. The plane rooted tree $(T(C), v_0)$ is shown in Figure 2.6(b), and the delay function f defined on the leaves of $T(C)$ different than v_0 is*

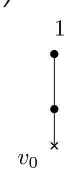
$$f(v) = \begin{cases} 2, & \text{if } v = v_1, \\ 1, & \text{if } v = v_2, v_3, \\ 3, & \text{if } v = v_4. \end{cases}$$

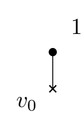
Therefore we can find $Q(\mathcal{T}(C))$ as follows:

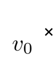
$Q(\mathcal{T}(C)) = Q$ 

$= (q+q^2)\, Q$ 

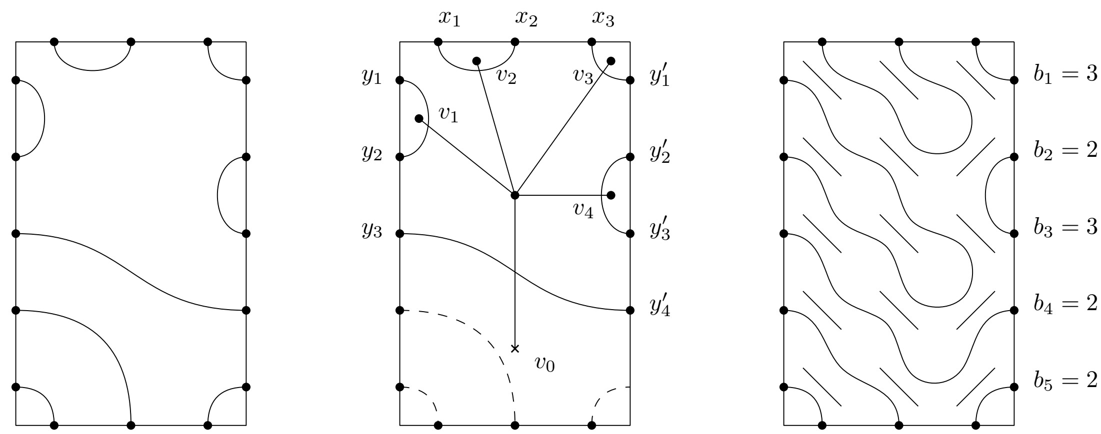
$= (q+q^2)^2\, Q$ 

$= q^2(1+q)^3\, Q$ 

$= q^2(1+q)^3\, Q$ 

$= q^2(1+q)^3\, Q$ 

$= q^2(1+q)^3.$



(a) Catalan state $C \in \mathfrak{Cat}^F_{5,3}$ (b) $T(C)$ and its root v_0 (c) The sequence $\mathfrak{b}_M(C)$

Figure 2.6. $C \in \mathfrak{Cat}^F_{m,n}$ in Example 2.2.8

The maximal sequence $\mathfrak{b}_M(C) = (3, 2, 3, 2, 2)$ *is shown in Figure 2.6(c). Hence,*

$$\begin{aligned} C(A) &= A^{2\|\mathfrak{b}_M(C)\| - mn + 4\cdot \mathrm{mindeg}_q Q(\mathcal{T}(C))} \cdot Q(\mathcal{T}(C))\big|_{q=A^{-4}} \\ &= A^{2(3+2+3+2+2)-5\cdot 3+4\cdot 2} \cdot (A^{-4})^2(1 + A^{-4})^3 \\ &= A^9(1 + A^{-4})^3. \end{aligned}$$

We note that the plucking polynomial of a plane rooted tree was first defined in [26]. It can be obtained by taking $f \equiv 1$ in Definition 2.2.6. Thus, we simply denoted it by $Q(T, v_0)$ and it can be used to compute coefficients of Catalan states with no returns on three sides.

Let $[n]_q = 1 + q + \ldots + q^{n-1}$, $[n]_q! = [1]_q\,[2]_q\,\ldots\,[n]_q$ and the q-multinomial coefficient

$$\binom{n_1 + n_2 + \ldots + n_k}{n_1, n_2, \ldots, n_k}_q = \frac{[n_1 + n_2 + \ldots + n_k]_q!}{[n_1]_q!\,[n_2]_q! \ldots [n_k]_q!}.$$

Theorem 2.2.9 (Przytycki [26])**.** *Suppose that a plane rooted tree* (T, v_0) *is a wedge product of plane rooted trees* (T_i, v_0)*,* $i = 1, \ldots, k$ *(see Figure 2.7). Then the plucking polynomial of* (T, v_0) *is given by*

$$Q(T, v_0) = \binom{|E(T)|}{|E(T_1)|, |E(T_2)|, \ldots, |E(T_k)|}_q \prod_{i=1}^{k} Q(T_i, v_0),$$

where $|E(T)|$ *denotes the number of edges of* T.

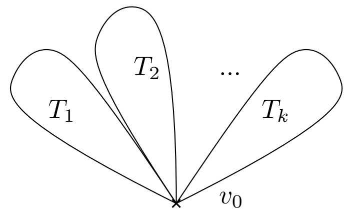


Figure 2.7. Wedge product of (T_i, v_0), $i = 1, \ldots, k$

There is no simple formula for $C(A)$, however, when there is a horizontal (respectively vertical) line that cuts C in n (respectively m) points, the computation of $C(A)$ becomes significantly easier. This is due to the following result that allows us to split C into the simpler Catalan states C_1 and C_2, find separately $C_1(A)$ and $C_2(A)$, and then $C(A) = C_1(A)C_2(A)$.

Theorem 2.2.10 (Dabkowski-Przytycki [9])**.** *Given $C \in \mathfrak{Cat}_{m,n}$. Suppose that $|l_s^h \cap C| = n$ for some $1 \leq s \leq m-1$, and let $\{z_1, \ldots, z_m\}$ be the intersection points. Splitting C into two Catalan states C_1 and C_2 according to the horizontal line l_s^h in the following way: C_1 is a Catalan state of $\mathfrak{Cat}_{s,n}$ such that its $2(s+n)$ endpoints are $\{x_i, z_i\}_{i=1}^n \cup \{y_j, y'_j\}_{j=1}^s$ and the connections are induced from C, and C_2 is a Catalan state of $\mathfrak{Cat}_{m-s,n}$ such that its $2(m-s+n)$ endpoints are $\{x'_i, z_i\}_{i=1}^n \cup \{y_j, y'_j\}_{j=s+1}^m$ and the connections are induced from C, see Figure 2.8. Then*

$$C(A) = C_1(A)\, C_2(A).$$

We will finish this section with the following result that gives a simple formula for coefficients of realizable Catalan states of $L(m,3)$. Furthermore, this description allows us to start our discussion concerning the properties of coefficients $C(A)$ of the Catalan state C.

Proposition 2.2.11 (Dabkowski-Przytycki [9])**.** *For a realizable $C \in \mathfrak{Cat}_{m,3}$ its coefficient $C(A)$ (up to a power of A) equals $[2]_{A^{-4}}^k\,[3]_{A^{-4}}$, $[2]_{A^{-4}}^k$, or $\frac{(A^2+A^{-2})^{2k}-1}{(A^2+A^{-2})^2-1}$ if $|l_i^h \cap C| < 3$ for all $1 \leq i \leq m-1$, and $[2]_{A^{-4}}^k\,[3]_{A^{-4}}^{k'}$ otherwise.*

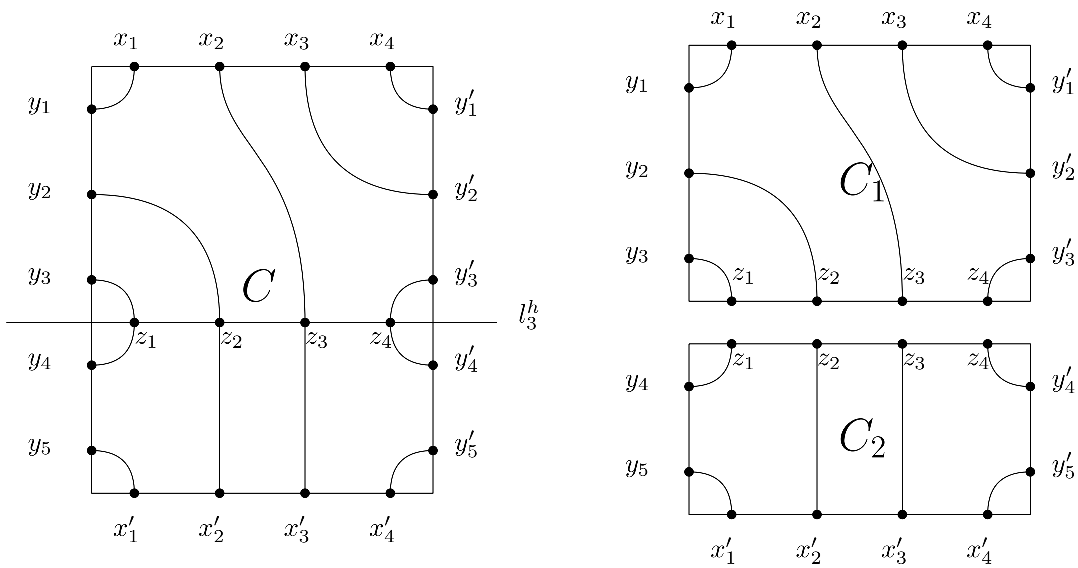


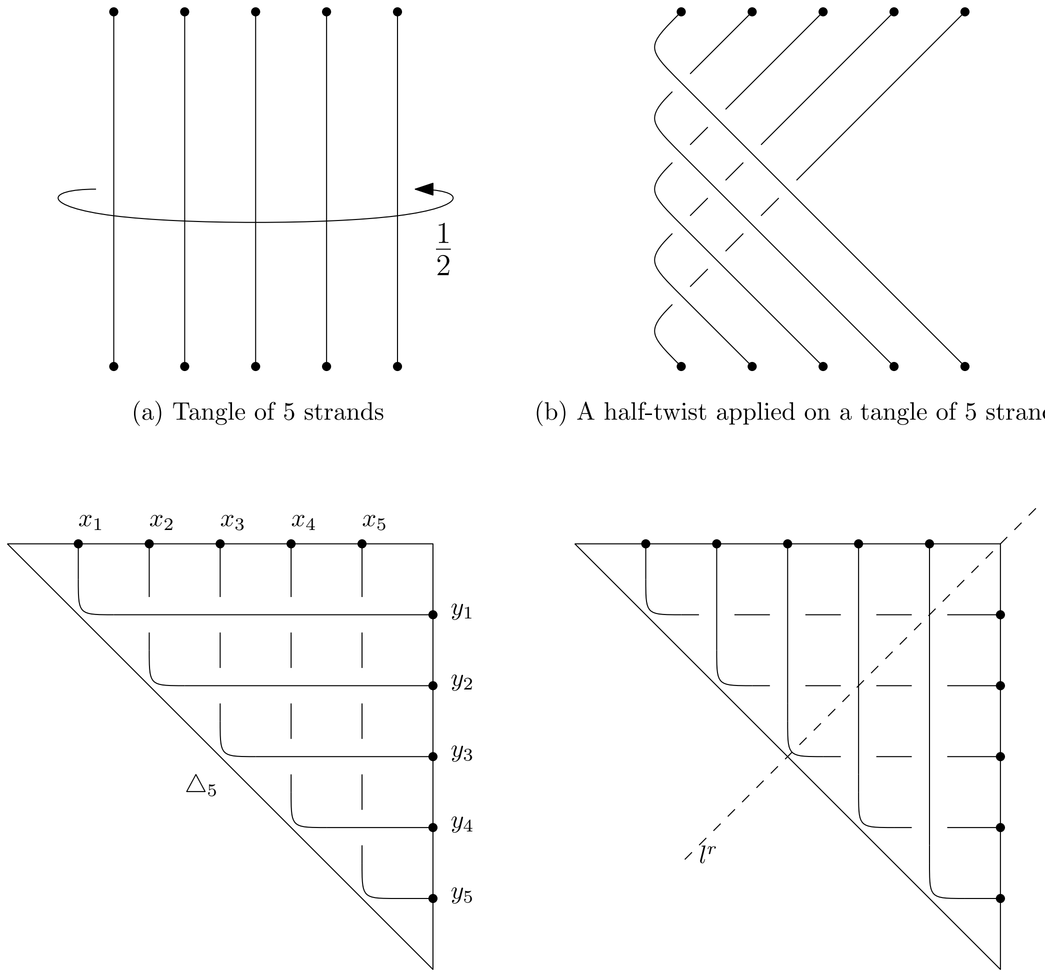
Figure 2.8. Splitting C into C_1 and C_2

2.3 Generalized Crossing – Definition and Summary of Results

The generalized crossing $G(k)$ is a k-tangle obtained a positive half-twist on k strands (see Figure 2.9(a) and (b) for the case $k = 5$), which was first considered by Li [20], though he used a negative half-twist. In this dissertation, we will use the diagram of $G(k)$ shown in Figure 2.9(c) instead of Figure 2.9(d) – the version used in Li [20]. We will denote by $\tilde{G}(k)$ the latter tangle. One can easily see that properties of $G(k)$ are same as of $\tilde{G}(k)$, since $G(k)$ is just a reflection of $\tilde{G}(k)$ about the line l^r, so there is a trivial correspondence between $G(k)$ and $\tilde{G}(k)$.

All questions that have been asked thus far for lattice crossing $L(m,n)$ have their equivalent version for $G(k)$. Let $\triangle_k$ be a triangle with $2k$ points $\mathfrak{X}_k = \{x_i, y_i\}_{i=1}^k$ on its boundary as shown in Figure 2.9(c). Denote by $\mathfrak{Cat}_k$ the set of all crossingless connections between points of $\mathfrak{X}_k$. By the Corollary 2.1.9, $\mathfrak{Cat}_k$ is a basis of $\mathcal{S}_{2,\infty}(\triangle_k \times I, \mathfrak{X}_k; \mathbb{Z}[A^{\pm 1}], A)$. Hence,

$$G(k) = \sum_{C \in \mathfrak{Cat}_k} C(A)\, C$$



(a) Tangle of 5 strands

(b) A half-twist applied on a tangle of 5 strands

(c) Generalized crossing $G(5)$

(d) Generalized crossing $\tilde{G}(5)$

Figure 2.9. Generalized crossing $G(5)$

for some $C(A) \in \mathbb{Z}[A^{\pm 1}]$. Elements of $\mathfrak{Cat}_k$ are called *Catalan states*, and the coefficient $C(A)$ of a Catalan state C in the linear combination above is called the *coefficient* of C.

We conclude this section with the definition of realizable Catalan states of generalized crossing. That is, let $\mathrm{Mat}_k^U(\{\pm 1\})$ be the set of $(k-1) \times (k-1)$ upper triangle matrices whose (i,j)-entry is either 1 or -1 for $i \leq j$. We say that a Kauffman state $s \in \mathrm{Mat}_k^U(\{\pm 1\})$ of $G(k)$ *realizes* a Catalan state $C \in \mathfrak{Cat}_k$ if smoothing all crossings of $G(k)$ according to s and removing all trivial components results in C. A Catalan state $C \in \mathfrak{Cat}_k$ is called *realizable* if there is a Kauffman state s that realizes C.

The following result describes realizable Catalan states of $G(k)$.

Theorem 2.3.1 (Li [20]). *Every Catalan state $C \in \mathfrak{Cat}_k$ of generalized crossing $G(k)$ is realizable.*

CHAPTER 3

METHODS FOR COMPUTING COEFFICIENTS

3.1 General Results and the Algorithm

Given a Catalan state $C \in \mathfrak{Cat}_{m,n}$, we can determine m and n from C, so there is no need to state m and n explicitly. We introduce some terminologies and notations related to Catalan states that will be used later. Let C^s denote the Catalan state obtained from C by reflecting C about a vertical line and $C^{r,\theta}$ the Catalan state obtained by a θ-rotation, where $\theta \in \{\pm\frac{\pi}{2}, \pi\}$, of C.

Lemma 3.1.1. *Let C be a Catalan state, then*

(a) $C^s(A) = C(A^{-1})$,

(b) $C^{r,\pi}(A) = C(A)$, *and*

(c) $C^{r,\pm\frac{\pi}{2}}(A) = C(A^{-1})$.

Proof. Suppose that $C \in \mathfrak{Cat}_{m,n}$. Consider the map sending $s = [s_{i,j}] \in \mathrm{Mat}_{m,n}(\{\pm 1\})$ to $s' = [-s_{i,n+1-j}] \in \mathrm{Mat}_{m,n}(\{\pm 1\})$. It is easy to see that this map is a bijection, if s realizes C then s' realizes C^s, and the number of trivial components after smoothing crossings according to s and s' are the same, so part (a) holds. Parts (b) and (c) can be proved in an analogous way. $\square$

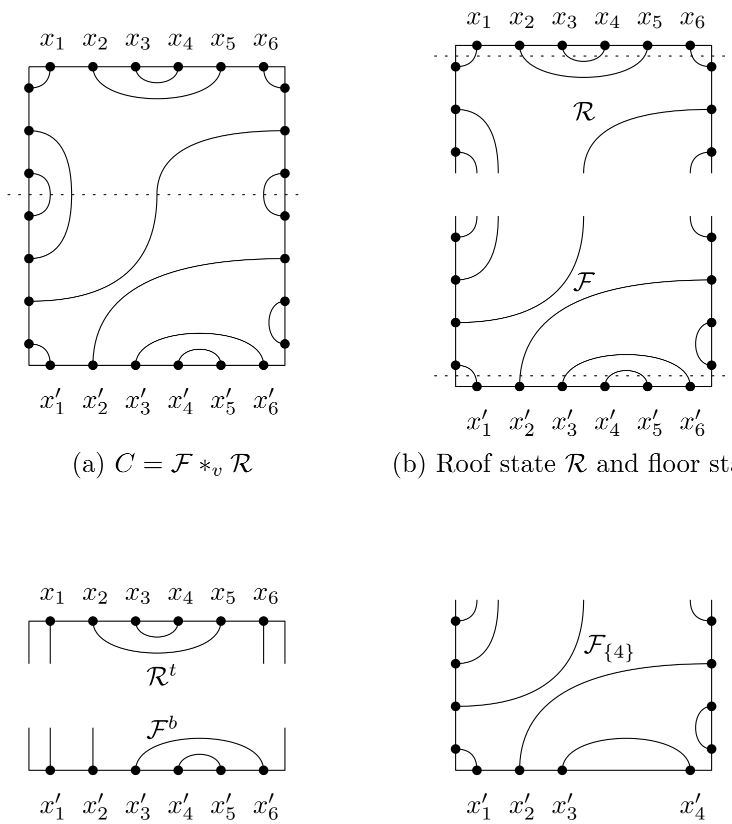
Each Catalan state C can be decomposed into two tangles by splitting it along a horizontal line l_j^h, $0 \le j \le m$ (see Figure 2.5) so that $l_j^h \cap C$ has the minimum number of intersections. The tangle that contains the top boundary (respectively the bottom boundary) of $R^2_{m,n}$ is called the *roof state* (respectively *floor state*). Roof and floor states have three boundaries with the points on them inherited from $R^2_{m,n}$. Arcs of C with both endpoints on the boundaries of a tangle are called *closed arcs* otherwise we will call them *open arcs*. The number of fixed points on the top boundary (respectively the bottom boundary) of the roof state (respectively the floor state) is called the *width* of the state while the number of fixed points on the left boundary is referred to as *height* of the state. We write $C = \mathcal{R} *_v \mathcal{F}$ when a Catalan state C is decomposed into a roof state $\mathcal{R}$ and a floor state $\mathcal{F}$. Conversely, given a roof state $\mathcal{R}$ and a floor state $\mathcal{F}$ with the same width and the same number of open arcs, we can concatenate them vertically to obtain a Catalan state $\mathcal{R} *_v \mathcal{F}$. In Figure 3.1 the tangle $\mathcal{R}$ is a roof state of width 6 and height 3 and the tangle $\mathcal{F}$ is a floor state of width 6 and height 4. The reflection and rotation operations defined as the above for Catalan states can also be applied to roof and floor states. For example, $\mathcal{R}^{r,\pi}$ yields a floor state for any roof state $\mathcal{R}$. Given roof $\mathcal{R}$ and floor $\mathcal{F}$ states, define $(\mathcal{R}, \mathcal{F}) = C(A)$ if $C = \mathcal{R} *_v \mathcal{F}$ is a Catalan state, and 0 otherwise. If $\mathcal{R} *_v \mathcal{F} = L(m, n)$ where $mn = 0$, then we set $(\mathcal{R}, \mathcal{F}) = 1$.

A floor state of height zero is called a *bottom state*. For a floor state $\mathcal{F}$, we denote by $\mathcal{F}^b$ the bottom state obtained from $\mathcal{F}$ as shown in Figure 3.1. More precisely, $\mathcal{F}^b$ is the floor state of $\mathcal{F}^{r,\pi} *_v \mathcal{F}$ whose height is zero.

Let $\mathbb{B}_n$ be the set of all bottom states of width n. Define a map Φ_n on $\mathbb{B}_n$ as follows. Given $\mathcal{F} \in \mathbb{B}_n$, suppose that the left endpoint of closed arcs of $\mathcal{F}$ are $x'_{i_1}, \ldots, x'_{i_k}$, then we set $\Phi_n(\mathcal{F}) = \{i_1, i_2, \ldots, i_k\}$. For example, for the floor state $\mathcal{F}$ shown in Figure 3.1, $\Phi_6(\mathcal{F}^b) = \{3, 4\}$. Let $\mathbb{V}_n := \{\Phi_n(\mathcal{F}) \mid \mathcal{F} \in \mathbb{B}_n\}$ denote the image of Φ_n, then $\Phi_n : \mathbb{B}_n \to \mathbb{V}_n$ is a bijection. Algorithm 3.1.2 gives a method for computing $\Phi_n^{-1}(I)$ for $I = \{i_1, \ldots, i_k\} \in \mathbb{V}_n$. In Figure 3.2 we showed an example of Algorithm 3.1.2 for finding $\Phi_n^{-1}(I)$.

Define an order $\preceq_F$ on $\mathbb{V}_n$ as follows. For $I, J \in \mathbb{V}_n$, $I \preceq_F J$ if all closed arcs of $\Phi_n^{-1}(I)$ are also closed arcs in $\Phi_n^{-1}(J)$. Posets $(\mathbb{V}_n, \preceq_F)$, for $n = 3, 4, 5, 6$, are shown in Figure 3.3. We see that, for $n \geq 4$, $\{2\} \preceq_F \{1, 2\}$ and $\{1\} \not\preceq_F \{1, 2\}$ in $\mathbb{V}_n$. Moreover, it follows from the definition of $(\mathbb{V}_n, \preceq_F)$ that, for all $n \in \mathbb{N}$, $(\mathbb{V}_n, \preceq_F)$ is a subposet of $(\mathbb{V}_{n+1}, \preceq_F)$.

Given a floor state $\mathcal{F}$ of width $n \geq 2$ and a set $I = \{i_1, \ldots, i_k\} \in \mathbb{V}_n$ define a floor state $\mathcal{F}_I$ as follows. If $I \preceq_F \Phi_n(\mathcal{F}^b)$, let $\mathcal{F}_I$ be the floor state obtained from $\mathcal{F}$ by removing



(a) $C = \mathcal{F} *_v \mathcal{R}$

(b) Roof state $\mathcal{R}$ and floor state $\mathcal{F}$

(c) Top state $\mathcal{R}^t$ and bottom state $\mathcal{F}^b$

(d) Floor state $\mathcal{F}_{\{4\}}$

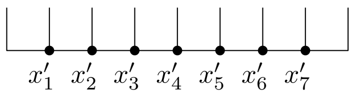
Figure 3.1. Notations on Catalan states in $L(m, n)$

Algorithm 3.1.2 Find the image of Φ_n^{-1}

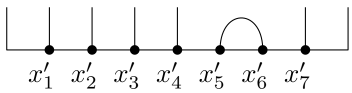
```
procedure FindPhiInverse(I, n)
    F ← a bottom state of width n and has no returns on its bottom boundary
    if I ≠ ∅ then
        Order the elements of I = {i_1 < ... < i_k}
        J ← {1, ..., n}
        for j ← k to 1 do
            i_* ← min{i ∈ J | i > i_j}
            Remove the open arcs whose endpoints are x'_{i_j} and x'_{i_*} in F
            Connect x'_{i_j} and x'_{i_*} by a closed arc in F
            J ← J \ {i_j, i_*}
    return F
```

Given $k = 3$, $I = \{i_1, i_2, i_3\} = \{1, 4, 5\}$, and $n = 7$.

Step 0 : $J = \{1, 2, 3, 4, 5, 6, 7\}$,

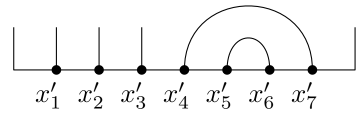
$\mathcal{F} =$ 

Step 1 : $j = 3$, $i_* = \min\{i \in J \mid i > i_3 = 5\} = 6$,

$\mathcal{F} =$ 

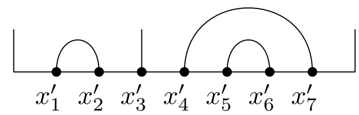
$J = \{1, 2, 3, 4, 5, 6, 7\} \setminus \{5, 6\} = \{1, 2, 3, 4, 7\}$.

Step 2 : $j = 2$, $i_* = \min\{i \in J \mid i > i_2 = 4\} = 7$,

$\mathcal{F} =$ 

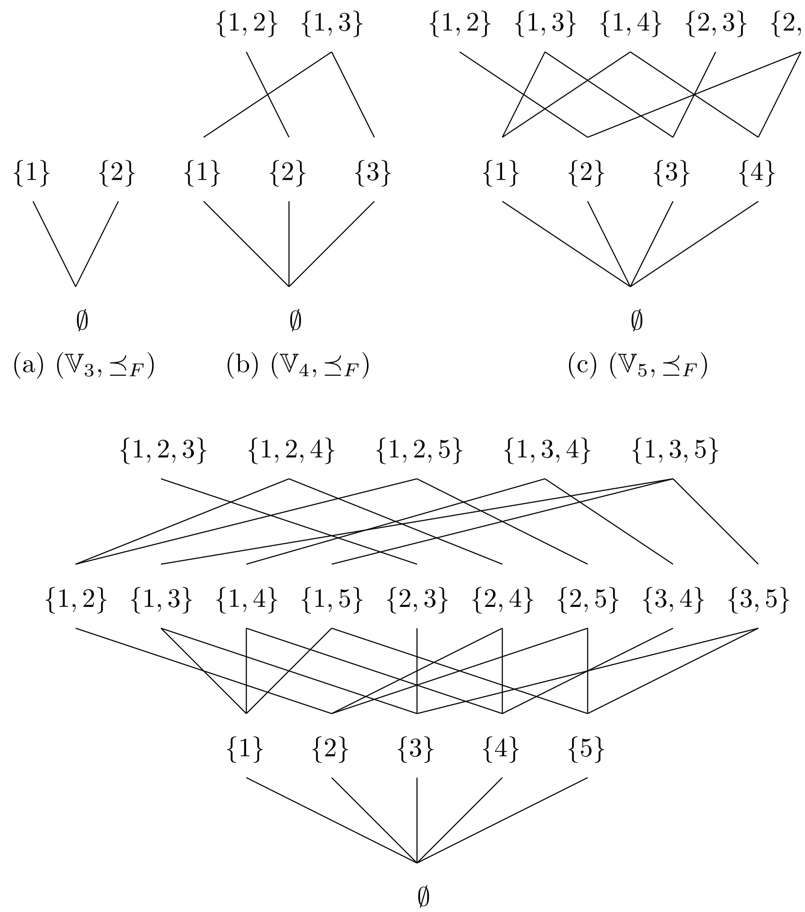
$J = \{1, 2, 3, 4, 7\} \setminus \{4, 7\} = \{1, 2, 3\}$.

Step 3 : $j = 1$, $i_* = \min\{i \in J \mid i > i_1 = 1\} = 2$,

$\mathcal{F} =$ 

$J = \{1, 2, 3\} \setminus \{1, 2\} = \{3\}$.

Figure 3.2. Example of Algorithm 3.1.2



(a) $(\mathbb{V}_3, \preceq_F)$ (b) $(\mathbb{V}_4, \preceq_F)$ (c) $(\mathbb{V}_5, \preceq_F)$

(d) $(\mathbb{V}_6, \preceq_F)$

Figure 3.3. Posets $(\mathbb{V}_n, \preceq_F)$

$|I|$ returns of $\mathcal{F}$ with left endpoint x'_i, $i \in I$, together with their corresponding endpoints. Otherwise, if $I \not\leq_F \Phi_n(\mathcal{F}^b)$, then we set $\mathcal{F}_I$ to be the empty graph K_0. It is easy to see that, when $\mathcal{F}_I \neq K_0$ then $\mathcal{F}_I$ is a floor state of same height as $\mathcal{F}$ and width is decreased by $2|I|$. Clearly, $\mathcal{F}_\emptyset = \mathcal{F}$, $(K_0)_I = K_0$, and $(\mathcal{R}, K_0) = 0$.

Lemma 3.1.3. *Let $\mathcal{F}$ be a floor state of width n. Suppose that $I = \{i_1 < \ldots < i_k\} \in \mathbb{V}_n$ and $J = \{j_1 < \ldots < j_l\} \in \mathbb{V}_{n-2|I|}$, then*

(a) $\left(\mathcal{F}_{\{i_1,\ldots,i_k\}}\right)_{\{j_1,\ldots,j_l\}} = \left(\left(\mathcal{F}_{\{i_1,\ldots,i_k\}}\right)_{\{j_l\}}\right)_{\{j_1,\ldots,j_{l-1}\}}$, *and*

(b) $\left(\mathcal{F}_{\{i_1,\ldots,i_k\}}\right)_{\{j\}} = \mathcal{F}_{\{i_1,\ldots,i_k,i_*\}}$, *where* $i_* = j + 2\sum_{t=1}^{k} \mathbb{1}_{\{i_t \leq j\}}$.

Proof. Part (a) follows directly from the definition of $\mathcal{F}_I$ (we can see it by simply drawing a floor state $\mathcal{F}$ and then removing its returns). For part (b), notice that before applying the operation of removing returns corresponding to $\{i_1, \ldots, i_k\}$, the original index of the point $p = x'_j$ in $\mathcal{F}_{\{i_1,\ldots,i_k\}}$ is x'_{i_*} in $\mathcal{F}$. Indeed, if $i_t \leq j$, then removing each return corresponding to each left endpoint x'_{i_t} decreases index of p by 2. Therefore, the original index of p is $i_* = j + 2\sum_{t=1}^{k} \mathbb{1}_{\{i_t \leq j\}}$ as claimed. $\square$

Let $I, J \subset \mathbb{N}$, define product $I \hat{*} J$ as follows:

$$I \hat{*} J = \begin{cases} I, & \text{if } J = \emptyset, \\ I \cup \{j + 2\sum_{t=1}^{k} \mathbb{1}_{\{i_t \leq j\}}\}, & \text{if } J = \{j\},\ I = \{i_1, \ldots, i_k\} \\ (I \hat{*} \{j_M\})\, \hat{*} (J \setminus \{j_M\}), & \text{otherwise,} \end{cases}$$

where j_M is the maximal element of J. One checks that $\emptyset \hat{*} J = J$ and by Lemma 3.1.3, if $I \in \mathbb{V}_n$ and $J \in \mathbb{V}_{n-2|I|}$, then $I \hat{*} J \in \mathbb{V}_n$, and $(\mathcal{F}_I)_J = \mathcal{F}_{I \hat{*} J}$ for all $I, J \subset \mathbb{N}$

Let a_0, a_n, a_j be arcs with endpoints $\{y_1, x_1\}$, $\{x_n, y'_1\}$, $\{x_j, x_{j+1}\}$ for $1 \leq i \leq n-1$, respectively. Given a roof state $\mathcal{R}$ of width $n \geq 1$, denote by $\mathcal{J}(\mathcal{R})$ the set $\{j \in \{0, 1, \ldots, n\} \mid a_j \in \mathcal{R}\}$. Let $J = \{j_1, \ldots, j_k\} \subseteq \mathcal{J}(\mathcal{R})$ and define $\mathcal{R}_J$ as the roof state obtained from $\mathcal{R}$ by moving points y_1 and y'_1 along boundaries to the top boundary of $\mathcal{R}$ and then removing arcs $a_{j_1}, \dots, a_{j_k}$ together with their endpoints from $\mathcal{R}$.

Our next result is based on the idea of analyzing the resolving tree of $L(m,n)$ obtained after smoothing of its first row of crossings according to all possible assignments of markers and applying regular isotopy. This idea was first used in [9] (see the proof of Proposition 2.1) and we call it the *first-row expansion* of $L(m,n)$. Consider $I = \{i_1, \dots, i_t\} \in \mathbb{V}_n$ and $J = \{j_1, \dots, j_{t+1}\} \subseteq \mathcal{J}(\mathcal{R})$, $t \geq 0$, if $j_1 < i_1 < j_2 < i_2 < \cdots < i_t < j_{t+1}$ then we write $I \propto J$.

Proposition 3.1.4. *Let $C = \mathcal{R} *_v \mathcal{F}$ be a Catalan state with the roof state $\mathcal{R}$ of width n and positive height. Then*

$$(\mathcal{R}, \mathcal{F}) = \sum_{\{(I,J):I \in \mathbb{V}_n,\, J \subseteq \mathcal{J}(\mathcal{R}),\, I \propto J\}} A^{-n+2(\|J\|-\|I\|)} (\mathcal{R}_J, \mathcal{F}_I), \tag{3.1}$$

where $\|I\|$ is the sum of elements of I.

Proof. In order to compute coefficient $C(A)$ of the Catalan state C, we proceed as follows. We start from the first row of crossings $L(m,n)$ (starting from its top) and consider all 2^n possible assignments of markers. Among such assignments, we choose these which, after a regular isotopy of the diagrams (one that gives the minimal number of crossings), has a potential to realize Catalan state C. For example, among all 2^5 assignments of markers to the first row of crossings of $L(6,5)$, we only choose three $(-1,-1,-1,-1,-1)$, $(+1,+1,+1,-1,-1)$, and $(-1,-1,+1,-1,-1)$ that give the first vertices of the computation tree for the coefficient of C shown in Figure 3.4. More precisely, we choose the following:

(1) Assignments with the first marker -1 or the last marker $+1$ that realize an arc of C with endpoints $\{x_1, y_1\}$ or $\{x_n, y'_1\}$.

(2) Assignments with consecutive markers $+1, -1$ that realize an arc of C with endpoints $\{x_i, x_{i+1}\}$ on its top boundary.

(3) Assignments with consecutive markers $-1, +1$ that realize, after regular isotopy of diagram, an arc of C with endpoints $\{x'_i, x'_{i+1}\}$ on its bottom boundary.

We observe that since the Kauffman bracket is an invariant of tangles, the coefficient $C(A)$ does not depend on the regular isotopy of diagrams and on the order in which crossings of $L(m,n)$ are being smoothed. Therefore, the procedure described above yields a recursive algorithm for finding $C(A)$. That is, after applying the first-row expansion to the first row of $L(m,n)$, the problem of finding $C(A)$ reduces to computing coefficients of Catalan states C' obtained from corresponding lattice crossings $L(m-1,n')$, $n' \leq n$. As shown in Figure 3.4, the problem of finding $C(A) = (\mathcal{R}, \mathcal{F})$ reduces to computing coefficients $(\mathcal{R}_{\{0\}}, \mathcal{F})$, $(\mathcal{R}_{\{3\}}, \mathcal{F})$, and $(\mathcal{R}_{\{0,3\}}, \mathcal{F}_{\{2\}})$. Thus, we note that equation (3.1) simply gives an algebraic description of the recursive procedure discussed above (called the first-row expansion). It then remains to show that the power of A is $-n + 2(\|J\| - \|I\|)$. Indeed, if $I = \{i_1, \ldots, i_t\}$ and $J = \{j_1, \ldots, j_{t+1}\}$, with $I \in \mathbb{V}_n$, $J \subseteq \mathcal{J}(\mathcal{R})$, and $I \propto J$, describe realization of arcs of C that are obtained using I and J, then the corresponding assignment of markers to the first row of $L(m,n)$ is shown in Figure 3.5. The power of A is then obtained as a result of first assigning $+1$ to all crossings in the first row of $L(m,n)$, and then changing markers of all crossings between j_k and $i_k + 1$ strands to -1, where $k = 1, \ldots, t+1$, and $i_{t+1} = n$. Therefore, the corresponding power of A is given by

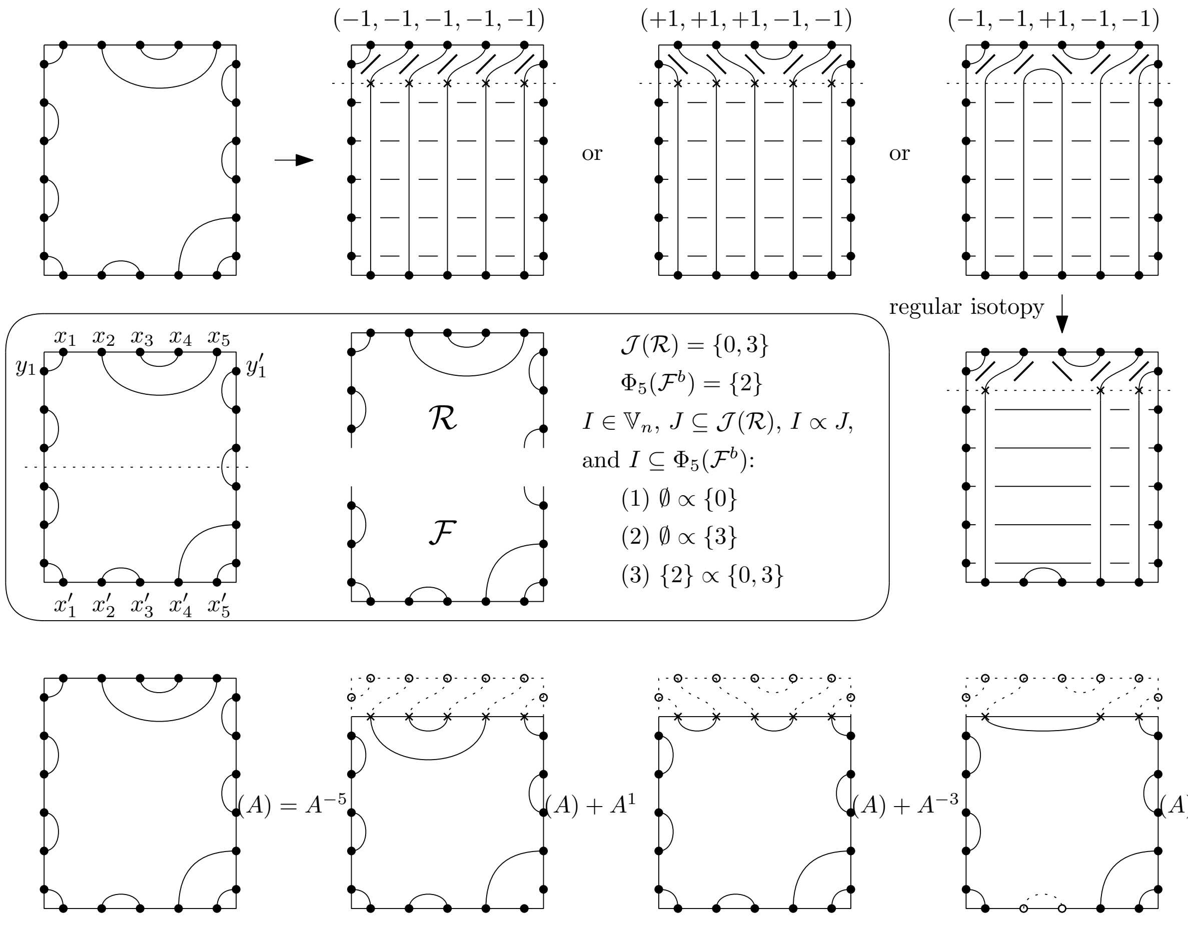
$$n - 2\sum_{k=1}^{t+1}(i_k - j_k) = n - 2\sum_{k=1}^{t} i_k - 2n + 2\sum_{k=1}^{t+1} j_k = -n + 2(\|J\| - \|I\|),$$

which completes our proof. $\square$

For a roof state $\mathcal{R}$ and $I \subset \mathbb{N}$, denote by $[\mathcal{R}, I]$ a function that assigns the coefficient $(\mathcal{R}, \mathcal{F}_I)$ of $\mathcal{R} *_v \mathcal{F}_I$ to each floor state $\mathcal{F}$, i.e.,

$$[\mathcal{R}, I](\mathcal{F}) = (\mathcal{R}, \mathcal{F}_I).$$

We are now ready to formulate our main result:



$$\Rightarrow (\mathcal{R}, \mathcal{F}) = A^{-5+2\cdot 0}\,(\mathcal{R}_{\{0\}}, \mathcal{F}) + A^{-5+2\cdot 3}\,(\mathcal{R}_{\{3\}}, \mathcal{F}) + A^{-5+2\cdot(0+3-2)}\,(\mathcal{R}_{\{0,3\}}, \mathcal{F}_{\{2\}})$$

Figure 3.4. Example of the first-row expansion

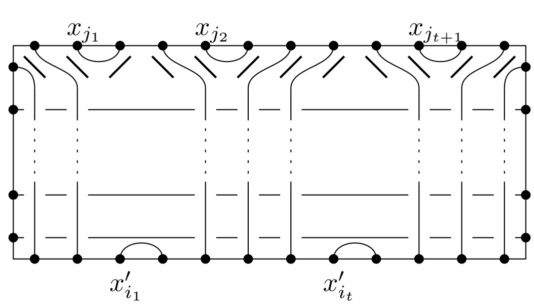


Figure 3.5. Proof of Proposition 3.1.4

Definition 3.1.5. *Let $\mathcal{R}$ be a roof state, $\mathcal{R}_i$ be roof states with the same number of open arcs as $\mathcal{R}$ and no returns on its top boundary, $I_i \subset \mathbb{N}$, $1 \leq i \leq k$, $k \in \mathbb{N}$. A relation in the form*

$$Z(A)\,[\mathcal{R}, \emptyset] = \sum_{i=1}^{k} Z_i(A)\,[\mathcal{R}_i, I_i], \tag{3.2}$$

where $Z(A), Z_i(A)$ are non-zero Laurent polynomials, is called an RF-formula for $\mathcal{R}$.

Remark 3.1.6. *We note that for a given roof state $\mathcal{R}$, there might be several RF-formulas. More importantly, we would like to point out here that if $\mathcal{R}$ has an RF-formula then the problem of finding the coefficient of $C = \mathcal{R} *_v \mathcal{F}$ reduces to a problem of finding coefficient for $\mathcal{R}_i *_v \mathcal{F}_{I_i}$. Since each such a state is either a Catalan state with no returns on the top boundary or it is not defined (hence its coefficient is* 0 *by the definition), we can always find $C(A)$. Notice that, since $Z(A) \neq 0$, we can divide* (3.2) *by $Z(A)$ to get*

$$[\mathcal{R}, \emptyset] = \sum_{i=1}^{k} Q_i(A)\,[\mathcal{R}_i, I_i], \tag{3.3}$$

where $Q_i(A) = Z_i(A)/Z(A)$ are rational functions of A. In what follows we use (3.3) *instead of* (3.2)*, and we will also refer to it as an RF-formula for $\mathcal{R}$. As we show later, for any $\mathcal{R}$ there is an RF-formula in the form given in* (3.2) *for which $Z(A)$ is a product of $[n] = [n]_{q=A^4} = 1 + A^4 + \ldots + A^{4(n-1)}$. Thus, if A is not a root of unity, the RF-formula* (3.3) *is equivalent to* (3.2)*. Furthermore, for any floor state $\mathcal{F}$, $[\mathcal{R}, \emptyset](\mathcal{F}) \in \mathbb{Z}[A^{\pm 1}]$. Since* (3.3) *holds for all but finite number of $A \in \mathbb{C}$, the right side of* (3.3) *is also a Laurent polynomial for any $\mathcal{F}$.*

Lemma 3.1.7. *Let $\mathcal{R}$ be a roof state and suppose that $\mathcal{R}$ has an RF-formula*

$$Z(A)\,[\mathcal{R}, \emptyset] = \sum_{i} Z_i(A)\,[\mathcal{R}_i, I_i],$$

then for all $I \subset \mathbb{N}$

$$Z(A)\,[\mathcal{R}, I] = \sum_{i} Z_i(A)\,[\mathcal{R}_i, I \hat{*} I_i].$$

Proof. Let $\mathcal{F}$ be a floor state. Then

$$\begin{aligned}Z(A)\,[\mathcal{R},I](\mathcal{F}) &= Z(A)\,(\mathcal{R},\mathcal{F}_I) = Z(A)\,[\mathcal{R},\emptyset](\mathcal{F}_I)\\ &= \sum_i Z_i(A)\,[\mathcal{R}_i,I_i](\mathcal{F}_I) = \sum_i Z_i(A)\,(\mathcal{R}_i,(\mathcal{F}_I)_{I_i})\\ &= \sum_i Z_i(A)\,(\mathcal{R}_i,\mathcal{F}_{I\hat{*}I_i}) = \sum_i Z_i(A)\,[\mathcal{R}_i,I\hat{*}I_i](\mathcal{F}).\end{aligned}$$

Hence the proof is completed. □

Given a Catalan state $C \in \mathfrak{Cat}_{m,n}$, its decomposition into roof and floor states is not unique. Define an order $\preceq_R$ on the set of all roof states of C as follows. Let $C = \mathcal{R}_i *_v \mathcal{F}_i$, where $\mathcal{R}_i$ and $\mathcal{F}_i$ are obtained by cutting C along the horizontal line l_i^h. Then $\mathcal{R}_i \prec_R \mathcal{R}_j$ if $i < j$. In particular, $\mathcal{R}_0 \prec_R \mathcal{R}_1 \prec_R \ldots \prec_R \mathcal{R}_n$.

Given a Catalan state $C \in \mathfrak{Cat}_{m,n}$, suppose that $C = \mathcal{R} *_v \mathcal{F}$, where $\mathcal{R}$ and $\mathcal{F}$ are obtained by cutting C along the horizontal line l_i^h and $C = \mathcal{R}' *_v \mathcal{F}'$, where $\mathcal{R}'$ and $\mathcal{F}'$ are obtained by cutting C along the horizontal line $l_{i'}^h$. Assume that $i' > i$ then $\mathcal{R} \prec_R \mathcal{R}'$ and the tangle $\mathcal{M}$ obtained by taking the part of C between lines l_i^h and $l_{i'}^h$ can be defined. That is, $\mathcal{M}$ consists of parts the left and right boundary of $R_{m,n}^2$, with same number of points on them, closed arcs, and open arcs resulted by cutting C along l_i^h and along $l_{i'}^h$ respectively. Therefore, $C = \mathcal{R} *_v \mathcal{M} *_v \mathcal{F}'$, $\mathcal{R}' = \mathcal{R} *_v \mathcal{M}$, $\mathcal{F} = \mathcal{M} *_v \mathcal{F}'$, and $\mathcal{M} = \mathcal{R}' \setminus \mathcal{R} = \mathcal{F} \setminus \mathcal{F}'$ (see Figure 3.6). Notice that if $\mathcal{F} = \mathcal{M} *_v \mathcal{F}'$ then $\mathcal{F}_I = (\mathcal{M} *_v \mathcal{F}')_I = \mathcal{M} *_v \mathcal{F}'_I$, so $\mathcal{M} = \mathcal{F} \setminus \mathcal{F}' = \mathcal{F}_I \setminus \mathcal{F}'_I$.

Lemma 3.1.8. *Suppose that $\mathcal{R}$ has an RF-formula. Then*

(a) $\mathcal{R}^s$ has an RF-formula, and

(b) if $\mathcal{R}'$ is a roof state, such that, $\mathcal{R} \preceq_R \mathcal{R}'$, then $\mathcal{R}'$ also has an RF-formula.

Proof. Let $\mathcal{F}$ be a floor state of width n. Then, by Lemma 3.1.1, $(\mathcal{R} *_v \mathcal{F}^s)(A) = (\mathcal{R}^s *_v \mathcal{F})^s(A) = (\mathcal{R}^s *_v \mathcal{F})(A^{-1})$. Let $I'_i = \Phi_n((\Phi_n^{-1}(I_i))^s)$, so $(\mathcal{R}_i^s *_v \mathcal{F}_{I'_i})(A) = (\mathcal{R}_i *_v \mathcal{F}_{I_i}^s)^s(A) =$

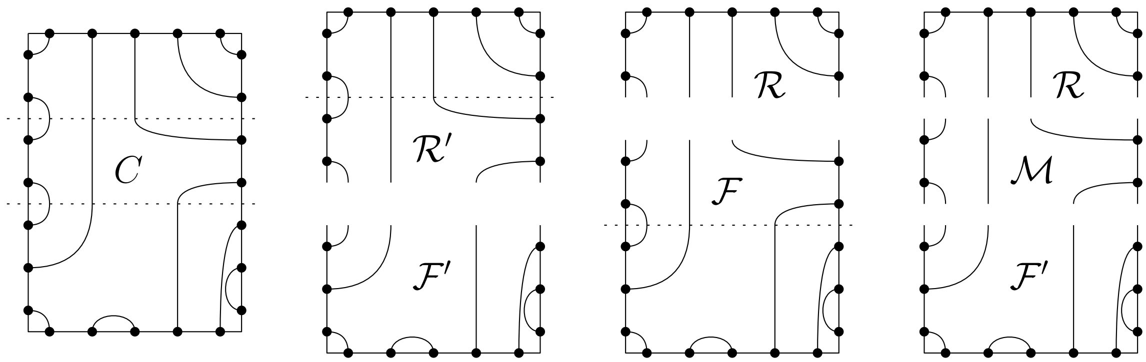


Figure 3.6. Decomposition of $C = \mathcal{R} *_v \mathcal{M} *_v \mathcal{F}'$

$(\mathcal{R}_i *_v \mathcal{F}^s_{I_i})(A^{-1})$. Then

$$\begin{aligned}
Z(A^{-1})\,[\mathcal{R}^s, \emptyset](\mathcal{F})(A) &= Z(A^{-1})\,(\mathcal{R}^s *_v \mathcal{F})(A) = Z(A^{-1})\,(\mathcal{R} *_v \mathcal{F}^s)(A^{-1}) \\
&= Z(A^{-1})\,[\mathcal{R}, \emptyset](\mathcal{F}^s)(A^{-1}) = \sum_i Z_i(A^{-1})\,[\mathcal{R}_i, I_i](\mathcal{F}^s)(A^{-1}) \\
&= \sum_i Z_i(A^{-1})\,(\mathcal{R}_i *_v \mathcal{F}^s_{I_i})(A^{-1}) = \sum_i Z_i(A^{-1})\,(\mathcal{R}^s_i *_v \mathcal{F}_{I'_i})(A) \\
&= \sum_i Z_i(A^{-1})\,[\mathcal{R}^s_i, I'_i](\mathcal{F})(A),
\end{aligned}$$

so $\mathcal{R}^s$ has an RF-formula

$$Z(A^{-1})\,[\mathcal{R}^s, \emptyset] = \sum_i Z_i(A^{-1})\,[\mathcal{R}^s_i, I'_i].$$

This proves the first statement.

Let $\mathcal{R} \preceq_R \mathcal{R}'$ and $\mathcal{M} = \mathcal{R}' \setminus \mathcal{R}$. Given a floor state $\mathcal{F}'$, define $\mathcal{F} = \mathcal{M} *_v \mathcal{F}'$ if the vertical product is defined or $\mathcal{F} = K_0$ otherwise, and we see that $(\mathcal{R}', \mathcal{F}') = (\mathcal{R}, \mathcal{F})$. Let $\mathcal{R}'_i = \mathcal{R}_i *_v \mathcal{M}$ and since $\mathcal{M} = \mathcal{F} \setminus \mathcal{F}' = \mathcal{F}_{I_i} \setminus \mathcal{F}'_{I_i}$ and $(\mathcal{R}'_i, \mathcal{F}'_{I_i}) = (\mathcal{R}_i, \mathcal{F}_{I_i})$,

$$\begin{aligned}
Z(A)\,[\mathcal{R}', \emptyset](\mathcal{F}') &= Z(A)\,(\mathcal{R}', \mathcal{F}') = Z(A)\,(\mathcal{R}, \mathcal{F}) = Z(A)\,[\mathcal{R}, \emptyset](\mathcal{F}) \\
&= \sum_i Z_i(A)\,[\mathcal{R}_i, I_i](\mathcal{F}) = \sum_i Z_i(A)\,(\mathcal{R}_i, \mathcal{F}_{I_i}) \\
&= \sum_i Z_i(A)\,(\mathcal{R}'_i, \mathcal{F}'_{I_i}) = \sum_i Z_i(A)\,[\mathcal{R}'_i, I_i](\mathcal{F}').
\end{aligned}$$

This shows that $\mathcal{R}'$ has also an RF-formula

$$Z(A)\,[\mathcal{R}',\emptyset] = \sum_i Z_i(A)\,[\mathcal{R}'_i, I_i].$$

Our proof is completed. $\square$

A *top state* is a roof state of height zero. Each roof state $\mathcal{R}$ yields a top state that we denote by $\mathcal{R}^t$ (see Figure 3.1). More precisely, $\mathcal{R}^t$ is defined as the roof state of $\mathcal{R} *_v \mathcal{R}^{r,\pi}$ whose height is zero. Notice that $\mathcal{R}^t \preceq_R \mathcal{R}$ for any roof state $\mathcal{R}$. Let $\mathbb{T}_n$ be the set of all top states of width n. Clearly, $\mathbb{T}_n$ is finite.

Lemma 3.1.9. *There are $\binom{n}{\lfloor n/2\rfloor}$ distinct top states of width n, that is, $|\mathbb{T}_n| = \binom{n}{\lfloor n/2\rfloor}$.*

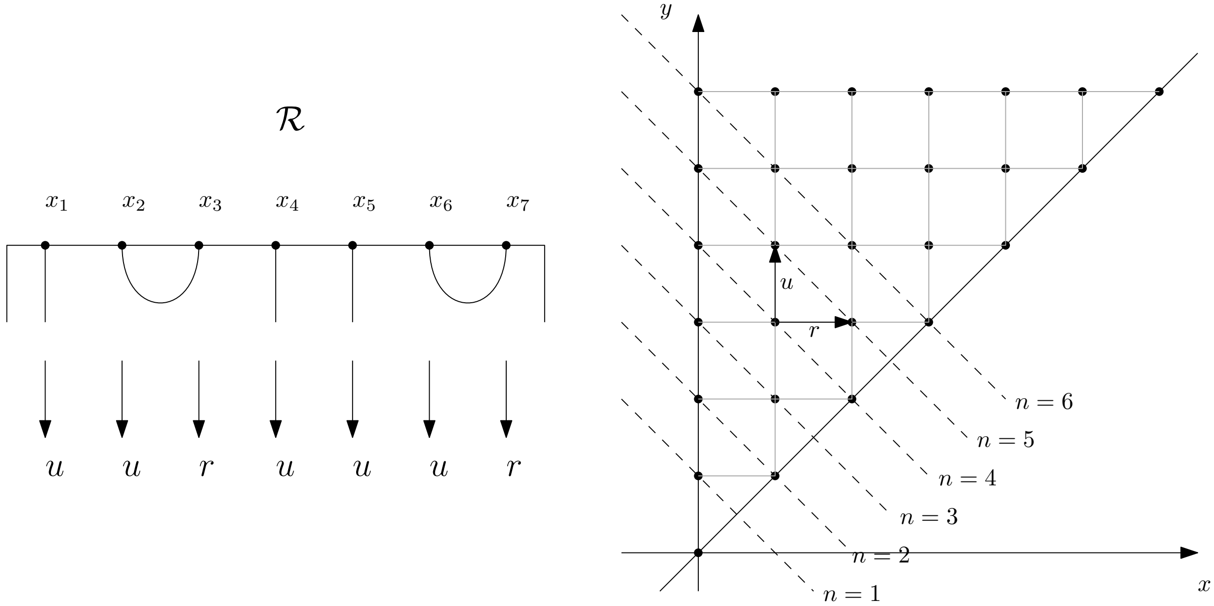
Proof. Let W_n be a set of all words w of length n on alphabet $\{u, r\}$ in which u occurs at least as many times as r in any prefix of w (initial segment of w), that is,

$$W_n = \{w_1 \dots w_n \mid w_i \in \{u, r\} \text{ and } N_u(w_1 \dots w_k) \geq N_r(w_1 \dots w_k),\ 1 \leq k \leq n\},$$

where $N_u(w)$ and $N_r(w)$ are numbers of u and r in w, respectively. There is a bijection f between $\mathbb{T}_n$ and W_n defined as follows. Let $\mathcal{R} \in \mathbb{T}_n$ and $x_1, x_2, \dots, x_n$ be points on the top boundary of $\mathcal{R}$ listed from left to right. We put $f(\mathcal{R}) = w_1 \dots w_n$, where $w_i = u$ if x_i is the left endpoint of a closed arc or an endpoint of an open arc, and $w_i = r$ otherwise (see Figure 3.7(a)). Now, it suffices to show that $|W_n| = \binom{n}{\lfloor n/2\rfloor}$. Indeed, let $u = (0, 1)$ and $r = (1, 0)$, then each word $w \in W_n$ can be represented as a lattice path that starts from $(0, 0)$ with steps u and r, end up at a point on $x + y = n$ and does not touch the line $x - y = 1$ (see Figure 3.7(b)). Let $a_n = |W_n|$, then a_n satisfies the following recursion

$$a_n = \begin{cases} 2a_{n-1}, & \text{if } 2 \mid n, \\ 2a_{n-1} - C_{(n-1)/2}, & \text{if } 2 \nmid n, \end{cases}$$

where $C_k = \frac{1}{k+1}\binom{2k}{k}$ is the k'th Catalan number. The result follows by induction.



(a) $f(\mathcal{R}) = uuruuur$

(b) u-move and r-move on the xy-plane

Figure 3.7. Proof of Lemma 3.1.9

If $n = 1$, then $a_1 = 1 = \binom{1}{\lfloor 1/2 \rfloor}$. Assume that $n = 2k$, $a_{2k} = \binom{2k}{k}$. Then

$$a_{2k+1} = 2a_{2k} - C_k = 2\binom{2k}{k} - \frac{1}{k+1}\binom{2k}{k} = \frac{2(2k)!}{k!k!} - \frac{(2k)!}{(k+1)!k!} = \binom{2k+1}{k}.$$

Assume that $n = 2k-1$, $a_{2k-1} = \binom{2k-1}{k-1}$. Then

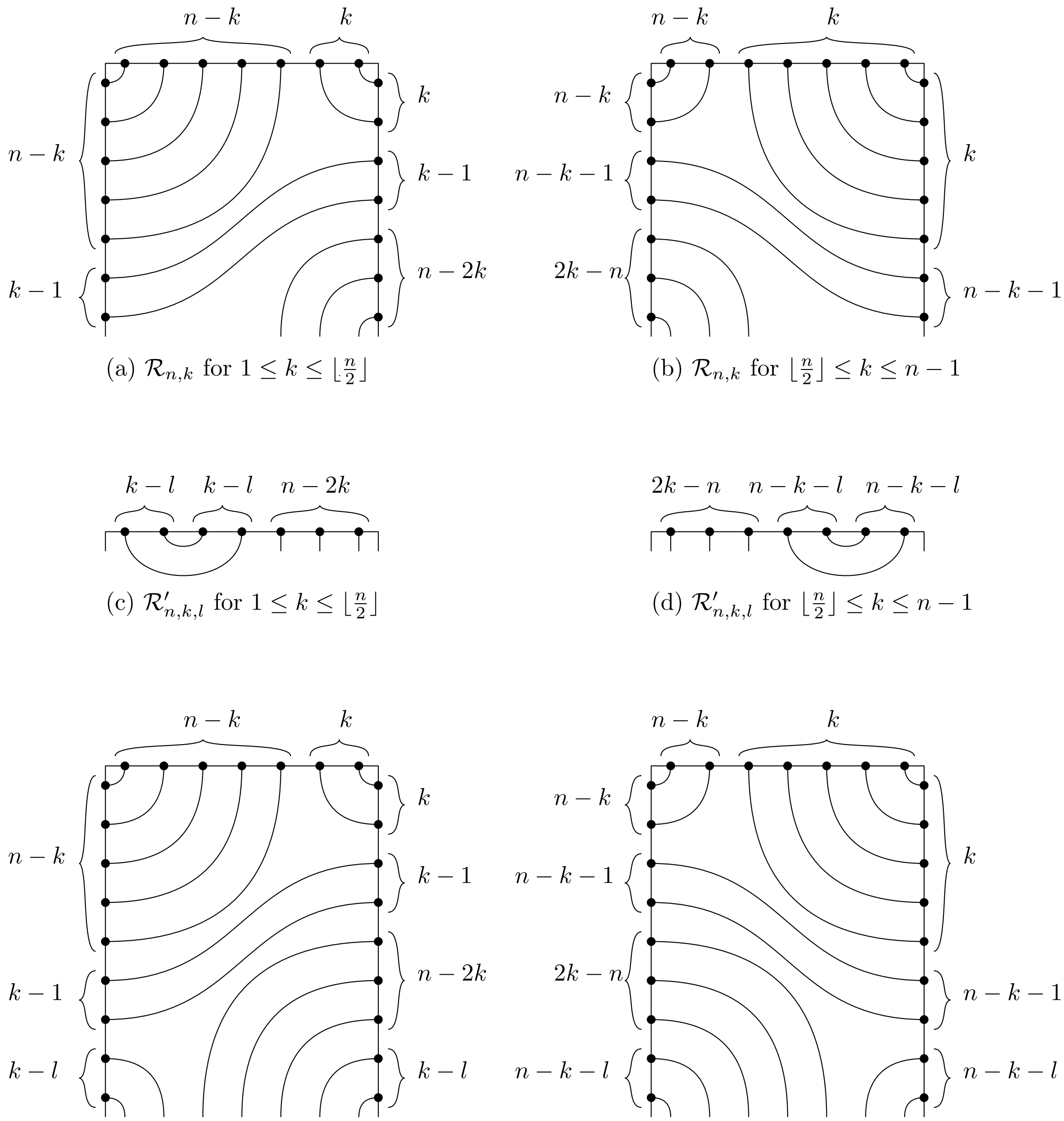
$$a_{2k} = 2a_{2k-1} = 2\binom{2k-1}{k-1} = \frac{2(2k-1)!}{k!(k-1)!} = \frac{(2k)!}{k!k!} = \binom{2k}{k}.$$

Therefore, $a_n = \binom{n}{\lfloor n/2 \rfloor}$ for all $n \in \mathbb{N}$. $\square$

Lemma 3.1.10. *Let $\mathcal{R}_{n,k}$ be the roof state as shown in Figure 3.8, $1 \le k \le n-1$. Then*

$$[\mathcal{R}_{n,k}, \emptyset] = \sum_{I \in \mathbb{V}_n,\, |I| \le \min\{k, n-k\}} Q_{n,k,I}(A)\ \left[\mathcal{R}'_{n,k,|I|}, I\right], \tag{3.4}$$

where $Q_{n,k,I}(A) = A^{(n-2k)(\min\{k,n-k\}-|I|)}\,(\mathcal{R}''_{n,k,|I|}, \Phi_n^{-1}(I))$ and the roof states $\mathcal{R}'_{n,k,l}$ and $\mathcal{R}''_{n,k,l}$ are shown in Figure 3.8.



(a) $\mathcal{R}_{n,k}$ for $1 \le k \le \lfloor \frac{n}{2} \rfloor$

(b) $\mathcal{R}_{n,k}$ for $\lfloor \frac{n}{2} \rfloor \le k \le n-1$

(c) $\mathcal{R}'_{n,k,l}$ for $1 \le k \le \lfloor \frac{n}{2} \rfloor$

(d) $\mathcal{R}'_{n,k,l}$ for $\lfloor \frac{n}{2} \rfloor \le k \le n-1$

(e) $\mathcal{R}''_{n,k,l}$ for $1 \le k \le \lfloor \frac{n}{2} \rfloor$

(f) $\mathcal{R}''_{n,k,l}$ for $\lfloor \frac{n}{2} \rfloor \le k \le n-1$

Figure 3.8. Roof states $\mathcal{R}_{n,k}$, $\mathcal{R}'_{n,k,l}$, $\mathcal{R}''_{n,k,l}$

Proof. We start by showing that after applying $(n-1)$ times the first-row expansions (3.1) on $\mathcal{R}_{n,k}$ results in the formula

$$[\mathcal{R}_{n,k}, \emptyset] = \sum_{I \in \mathbb{V}_n,\, |I| \leq \min\{k, n-k\}} Q'_{n,k,I}(A)\ \left[\mathcal{R}'_{n,k,|I|}, I\right], \tag{3.5}$$

for some $Q'_{n,k,I}(A)$. Assume that $1 \leq k \leq \lfloor \frac{n}{2} \rfloor$. One can easily see that, after applying m times first-row expansions (3.1) on $[\mathcal{R}_{n,k}, \emptyset]$, $1 \leq m \leq n-1$, results in the following formula

$$[\mathcal{R}_{n,k}, \emptyset] = \sum_i \tilde{Q}_i^{(m)}(A)\, [\mathcal{R}_i^{(m)}, I_i^{(m)}],$$

where $\tilde{Q}_i^{(m)}(A) \in \mathbb{Z}[A^{\pm 1}]$ are non-zero Laurent polynomials, and the following are satisfied

(1) $\mathcal{R}_i^{(m)}$ is of height $(n-1)-m$ and width $n-2|I_i^{(m)}|$,

(2) $1 \leq |\mathcal{J}(\mathcal{R}_i^{(m)})| \leq 2$, and

(3) if $\mathcal{J}(\mathcal{R}_i^{(m)}) = \{j\}$, then $j = k - |I_i^{(m)}|$, and if $\mathcal{J}(\mathcal{R}_i^{(m)}) = \{j_1, j_2\}$, then $|j_1 - j_2| = n - m - |I_i^{(m)}|$.

We prove the above statement by induction on m. The case $m=0$ is obvious, since $\mathcal{R}_i^{(0)} = \mathcal{R}_{n,k}$ and $I_i^{(0)} = \emptyset$. We show that applying first-row expansion $(m+1)$ times on $[\mathcal{R}_{n,k}, \emptyset]$ results in the same formula as the one obtained by applying a single first-row expansion on $\tilde{Q}_i^{(m)}(A)\, [\mathcal{R}_i^{(m)}, I_i^{(m)}]$. There are two cases of $\mathcal{R}_i^{(m)}$:

- If $|\mathcal{J}(\mathcal{R}_i^{(m)})| = 1$, then $\mathcal{J}(\mathcal{R}_i^{(m)}) = \{k - |I_i^{(m)}|\}$. One can check that $\mathcal{R}_i^{(m+1)} = (\mathcal{R}_i^{(m)})_{\{k-|I_i^{(m)}|\}}$ and $I_i^{(m+1)} = I_i^{(m)}$ satisfy (1)–(3).

- If $|\mathcal{J}(\mathcal{R}_i^{(m)})| = 2$, then $\mathcal{J}(\mathcal{R}_i^{(m)}) = \{j_1 < j_2\}$ for some $j_2 - j_1 = n - m - |I_i^{(m)}|$. We see that, after applying the first-row expansion,

$$\begin{aligned}
\left[\mathcal{R}_i^{(m)}, I_i^{(m)}\right] &= A^{-(n-2|I_i^{(m)}|)+2j_1} \left[(\mathcal{R}_i^{(m)})_{\{j_1\}}, I_i^{(m)}\right] \\
&+ A^{-(n-2|I_i^{(m)}|)+2j_2} \left[(\mathcal{R}_i^{(m)})_{\{j_2\}}, I_i^{(m)}\right] \\
&+ \sum_{l=j_1+1}^{j_2-1} A^{-(n-2|I_i^{(m)}|)+2(j_1+j_2-l)} \left[(\mathcal{R}_i^{(m)})_{\{j_1,j_2\}}, I_i^{(m)} \hat{*} \{l\}\right].
\end{aligned}$$

It is easy to see that for the first two terms, pairs $((\mathcal{R}_i^{(m)})_{\{j_1\}}, I_i^{(m)})$, $((\mathcal{R}_i^{(m)})_{\{j_2\}}, I_i^{(m)})$ satisfy (1)–(3). While for the last summation, we let $\mathcal{R}_i^{(m+1)} = (\mathcal{R}_i^{(m)})_{\{j_1,j_2\}}$ and $I_i^{(m+1)} = I_i^{(m)} \hat{*} \{l\}$. If $|\mathcal{J}(\mathcal{R}_i^{(m+1)})| = 2$ then $\mathcal{J}(\mathcal{R}_i^{(m+1)}) = \{j_1, j_2-2\}$, thus $(j_2-2)-j_1 = n-m-|I_i^{(m)}|-2 = n-(m+1)-|I_i^{(m+1)}|$, i.e., (3) holds. Statements (1) and (2) are obvious. Case $|\mathcal{J}(\mathcal{R}_i^{(m+1)})| = 1$ can be dealt with by similar argument as the above.

Since for $m = n-1$ in (3), $|j_1 - j_2| = n-(n-1)-|I_i| = 1-|I_i| \leq 1$ which is not possible, so it must be $|\mathcal{J}(\mathcal{R}_i^{(n-1)})| = 1$ and hence $\mathcal{J}(\mathcal{R}_i^{(n-1)}) = \{k - |I_i^{(n-1)}|\}$. Notice that the first-row expansions always realizes arcs on or near the top boundary, so if there is an arc l_1 below to l_2 that is parallel to l_2, then l_2 is realized earlier than l_1. Hence $\mathcal{R}_i^{(n-1)} = \mathcal{R}'_{n,k,|I_i^{(n-1)}|}$. The argument above proves the existence of $Q'_{n,k,I}(A)$ in (3.5). The case $\lfloor \frac{n}{2} \rfloor \leq k \leq n-1$ can be proved by applying Lemma 3.1.8.

Now we show that $Q'_{n,k,I}(A) = Q_{n,k,I}(A)$ for all $I \in \mathbb{V}_n$, $|I| \leq \min\{k, n-k\}$. Assume that $1 \leq k \leq \lfloor \frac{n}{2} \rfloor$ and define $\mathcal{M}_{n,k,l} = \mathcal{F}_{n,k,l} \setminus \Phi_{n-2l}^{-1}(\emptyset)$ for $l \leq \min\{k, n-k\}$, where $\mathcal{F}_{n,k,l}$ is shown in Figure 3.9(a). Given $I \in \mathbb{V}_n$, take $\mathcal{F}'_{n,k,I} = \mathcal{M}_{n,k,|I|} *_v \Phi_n^{-1}(I)$. Then for any $J \prec_R I$, $J \in \mathbb{V}_n$,

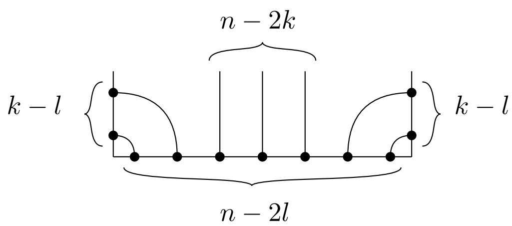
$$\left[\mathcal{R}'_{n,k,|J|}, J\right](\mathcal{F}'_{n,k,I}) = (\mathcal{R}'_{n,k,|J|}, \mathcal{M}_{n,k,|I|} *_v (\Phi_n^{-1}(I))_J) = 0,$$

because the line l^v in Figure 3.9(b) has the number of intersections $k-|J| > k-|I|$, the Catalan state $\mathcal{R}'_{n,k,|J|} *_v \mathcal{M}_{n,k,|I|} *_v (\Phi_n^{-1}(I))_J$ is not realizable by Theorem 2.2.1, so its coefficient is 0. Therefore, (3.5) reduces to

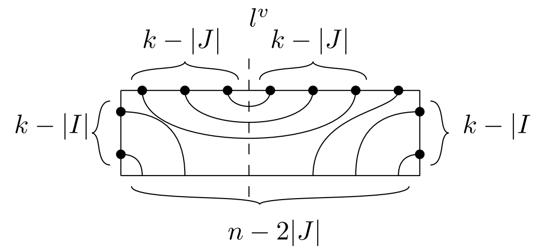
$$[\mathcal{R}_{n,k}, \emptyset](\mathcal{F}'_{n,k,I}) = Q'_{n,k,I}(A) \left[\mathcal{R}'_{n,k,|I|}, I\right](\mathcal{F}'_{n,k,I})$$

or

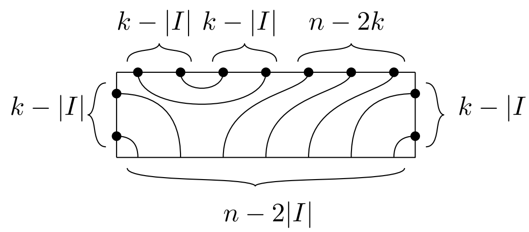
$$(\mathcal{R}_{n,k}, \mathcal{F}'_{n,k,I}) = Q'_{n,k,I}(A) \cdot A^{-(n-2k)(k-|I|)},$$



(a) Floor state $\mathcal{F}_{n,k,l}$



(b) Catalan state $\mathcal{R}'_{n,k,|J|} *_v \mathcal{M}_{n,k,|I|} *_v (\Phi_n^{-1}(I))_J$



(c) Catalan state $\mathcal{R}'_{n,k,|I|} *_v \mathcal{F}_{n,k,|I|}$

Figure 3.9. Proof of Lemma 3.1.10

since $(\mathcal{R}'_{n,k,|I|}, (\mathcal{F}'_{n,k,I})_I) = (\mathcal{R}'_{n,k,|I|}, \mathcal{F}_{n,k,I}) = A^{-(n-2k)(k-|I|)}$ as shown in Figure 3.9(c). Then

$$\begin{aligned} Q'_{n,k,I}(A) &= A^{(n-2k)(k-|I|)} \cdot (\mathcal{R}_{n,k}, \mathcal{M}_{n,k,|I|} *_v \Phi_n^{-1}(I)) \\ &= A^{(n-2k)(k-|I|)} \cdot \left(\mathcal{R}''_{n,k,|I|}, \Phi_n^{-1}(I)\right) \\ &= Q_{n,k,I}(A), \end{aligned}$$

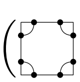
which completes our proof.

$\square$

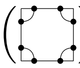
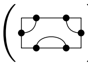
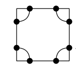
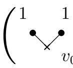
Example 3.1.11. *We find* (3.4) *for* $n = 2, 3$.

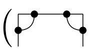
If $n = 2$, $k = 1$, *and* $\mathbb{V}_2 = \{\emptyset, \{1\}\}$. *Then*

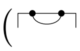
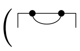
$$\begin{aligned} [\mathcal{R}_{2,1}, \emptyset] &= \sum_{I \in \mathbb{V}_2,\, |I| \leq \min\{1, 2-1\}} Q_{2,1,I}(A) \left[\mathcal{R}'_{2,1,|I|}, I\right] \\ &= Q_{2,1,\emptyset}(A) \left[\mathcal{R}'_{2,1,0}, \emptyset\right] + Q_{2,1,\{1\}}(A) \left[\mathcal{R}'_{2,1,1}, \{1\}\right] \end{aligned}$$

$= A^{(2-2)(1-0)} \Big($  $(A)\Big) \left[\mathcal{R}'_{2,1,0}, \emptyset\right] + A^{(2-2)(1-1)} \Big($  $(A)\Big) \left[\mathcal{R}'_{2,1,1}, \{1\}\right]$

$$= A^{-2}[2] \left[\mathcal{R}'_{2,1,0}, \emptyset\right] + \left[\mathcal{R}'_{2,1,1}, \{1\}\right],$$

since $\Big($$\Big)(A) = A^{-2}[2]$ *and clearly* $\Big($$\Big)(A) = 1$. *Indeed, for* $C =$ , *the corresponding plane rooted tree with a delay function* $\mathcal{T}(C) = \Big($$\Big)$. *Thus, by Theorem 2.2.9, its plucking polynomial is* $[2]_q$. *Furthermore, since the maximal sequence* $\mathfrak{b}_M(C) = (2,1)$, *it follows that* $C(A) = A^{2\cdot(2+1)-2\cdot 2}[2]_{q=A^{-4}} = A^{-2}[2]$ *by Theorem 2.2.7. We note that the above formula can also be obtained directly applying the first-row expansion* (3.1)*. Indeed, given a floor state* $\mathcal{F}$*, we see that*

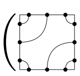
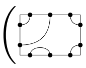
$[\mathcal{R}_{2,1},\emptyset](\mathcal{F}) = \Big($$,\mathcal{F}\Big)$

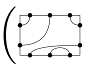
$= A^{-2+2\cdot 0}\Big($$,\mathcal{F}\Big) + A^{-2+2\cdot 2}\Big($$,\mathcal{F}\Big) + A^{-2+2(0+2-1)}\Big($$,\mathcal{F}_{\{1\}}\Big)$

$$= A^{-2}[2]\left(\mathcal{R}'_{2,1,0},\mathcal{F}_{\emptyset}\right) + \left(\mathcal{R}'_{2,1,1},\mathcal{F}_{\{1\}}\right).$$

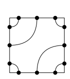
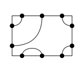
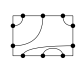
If $n = 3$, *then* $k = 1, 2$ *and* $\mathbb{V}_3 = \{\emptyset, \{1\}, \{2\}\}$. *Assume that* $k = 1$, *then*

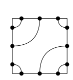
$$[\mathcal{R}_{3,1},\emptyset] = Q_{3,1,\emptyset}(A)\ \left[\mathcal{R}'_{3,1,0},\emptyset\right] + Q_{3,1,\{1\}}(A)\ \left[\mathcal{R}'_{3,1,1},\{1\}\right] + Q_{3,1,\{2\}}(A)\ \left[\mathcal{R}'_{3,1,1},\{2\}\right]$$

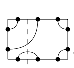
$= A^{(3-2)(1-0)}\Big($$(A)\Big)\ \left[\mathcal{R}'_{3,1,0},\emptyset\right] + A^{(3-2)(1-1)}\Big($$(A)\Big)\ \left[\mathcal{R}'_{3,1,1},\{1\}\right]$

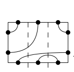
$+ A^{(3-2)(1-1)}\Big($$(A)\Big)\ \left[\mathcal{R}'_{3,1,1},\{2\}\right]$

$$= A^{-6}[3]\ \left[\mathcal{R}'_{3,1,0},\emptyset\right] + A^{-4}[2]\ \left[\mathcal{R}'_{3,1,1},\{1\}\right] + A^{-2}\ \left[\mathcal{R}'_{3,1,1},\{2\}\right],$$

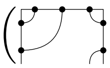
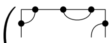
since coefficients $(A) = A^{-7}[3]$, $(A) = A^{-4}[2]$ *and* $(A) = A^{-2}$. *Indeed, we see that:*

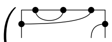
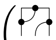
- *The plucking polynomial for* $C_1 =$  *is* $[3]_q$, *and the maximal sequence* $\mathfrak{b}_M(C_1) = (3,1,1)$, *so* $C_1(A) = A^{2(3+1+1)-3\cdot 3}\cdot[3]_{q=A^{-4}} = A^{-7}[3]$.

- *Since* , *the coefficient is* $A^{-2}\cdot A^{-2}[2]$.

- *Since* , *the coefficient is* $A^{-2}\cdot A^{-2}\cdot A^{2}$.

As before, we note that the formula for $[\mathcal{R}_{3,1},\emptyset]$ can be directly obtained using the first-row expansion (3.1)*, i.e.*

$[\mathcal{R}_{3,1},\emptyset]\,(\mathcal{F}) = \Big($  $,\mathcal{F}\Big) = A^{-3+2\cdot 0}\,\Big($  $,\mathcal{F}\Big)$

$+\,A^{-3+2\cdot 3}\,\Big($  $,\mathcal{F}\Big) + A^{-3+2\cdot(0+3-1)}\,\Big($  $,\mathcal{F}_{\{1\}}\Big) + A^{-3+2\cdot(0+3-2)}\,\Big($  $,\mathcal{F}_{\{2\}}\Big)$

$$
\begin{aligned}
&= A^{-3}\left(A^{-3+2\cdot 0}\left(\mathcal{R}'_{3,1,0},\mathcal{F}_{\emptyset}\right) + A^{-3+2\cdot 2}\left(\mathcal{R}'_{3,1,0},\mathcal{F}_{\emptyset}\right) + A^{-3+2\cdot(0+2-1)}\left(\mathcal{R}'_{3,1,1},\mathcal{F}_{\{1\}}\right)\right)\\
&+ A^{3}\cdot A^{-3+2\cdot 1}\left(\mathcal{R}'_{3,1,0},\mathcal{F}_{\emptyset}\right) + A\cdot A^{-1+2\cdot 0}\left(\mathcal{R}'_{3,1,1},\mathcal{F}_{\{1\}}\right) + A^{-1}\cdot A^{-1+2\cdot 0}\left(\mathcal{R}'_{3,1,1},\mathcal{F}_{\{2\}}\right)\\
&= A^{-6}[3]\left(\mathcal{R}'_{3,1,0},\mathcal{F}_{\emptyset}\right) + A^{-4}[2]\left(\mathcal{R}'_{3,1,1},\mathcal{F}_{\{1\}}\right) + A^{-2}\left(\mathcal{R}'_{3,1,1},\mathcal{F}_{\{2\}}\right).
\end{aligned}
$$

Now, from our proof of Lemma 3.1.8, we can easily see that $\Phi_3((\Phi_3^{-1}(\{1\}))^s) = \{2\}$ and $\Phi_3((\Phi_3^{-1}(\{2\}))^s) = \{1\}$, thus

$$
\begin{aligned}
&\left[\mathcal{R}^s_{3,1},\emptyset\right] = Q_{3,1,\emptyset}(A^{-1})\,\left[(\mathcal{R}'_{3,1,0})^s,\emptyset\right]\\
&+ Q_{3,1,\{1\}}(A^{-1})\,\left[(\mathcal{R}'_{3,1,1})^s,\{2\}\right] + Q_{3,1,\{2\}}(A^{-1})\,\left[(\mathcal{R}'_{3,1,1})^s,\{1\}\right].
\end{aligned}
$$

Furthermore, notice that $(\mathcal{R}'_{3,1,0})^s = \mathcal{R}'_{3,2,0}$ and $(\mathcal{R}'_{3,1,1})^s = \mathcal{R}'_{3,2,1}$. Hence, for $n = 3$ and $k = 2$,

$$
\begin{aligned}
[\mathcal{R}_{3,2},\emptyset] &= A^{6}[3]_{q=A^{-4}}\,\left[\mathcal{R}'_{3,2,0},\emptyset\right] + A^{4}[2]_{q=A^{-4}}\,\left[\mathcal{R}'_{3,2,1},\{2\}\right] + A^{2}\,\left[\mathcal{R}'_{3,2,1},\{1\}\right]\\
&= A^{-2}[3]\,\left[\mathcal{R}'_{3,2,0},\emptyset\right] + A^{2}\,\left[\mathcal{R}'_{3,2,1},\{1\}\right] + [2]\,\left[\mathcal{R}'_{3,2,1},\{2\}\right].
\end{aligned}
$$

Let $\mathcal{R}$ be a roof state of width n and $j_*(\mathcal{R}) = \min\{j \mid j \in \mathcal{J}(\mathcal{R})\}$. We construct new roof state $\mathcal{R}'$ using $\mathcal{R}$ via the process of *adding a cup at* i, $1 \le i \le n-1$, that first adds a closed arc together with its two endpoints p, q to the top boundary of $\mathcal{R}$ between x_i, x_{i+1}, and then moves points x_1 and x_n along the boundaries to left and right boundary, respectively. For instance, Figure 3.10 shows the new roof state $\mathcal{R}'$ obtained from $\mathcal{R}$ by adding a cup at 2. This process can be considered as an inverse operation of $\mathcal{R}_{\{i\}}$ on $\mathcal{R}$. Given a roof state $\mathcal{R}$ with width n and $n-2k$ open arcs, we say that $\mathcal{R}$ satisfies the *horizontal line condition* if $|(\mathcal{R} *_v \mathcal{F}_{n,k,0}) \cap l^h_i| \le n$ for all i, where the floor state $\mathcal{F}_{n,k,l}$ is shown in Figure 3.9.

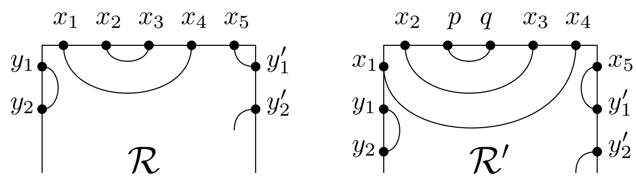


Figure 3.10. Adding a cup

Algorithm 3.1.12 Find an RF-formula for a roof state $\mathcal{R}$

---

procedure FindRFFormula($\mathcal{R}, I$)
  $n \leftarrow$ width of $\mathcal{R}$
  if $n < 2|I|$ **or** $\mathcal{R}$ does not satisfy the horizontal line conditions **then** ▷ Case 0
    $RF \leftarrow 0$
  else if $\mathcal{R}$ has no returns on its top boundary **then** ▷ Case 1
    $RF \leftarrow [\mathcal{R}, I]$
  else if the height of $\mathcal{R}$ is greater than zero **then** ▷ Case 2
    $\mathcal{M} \leftarrow \mathcal{R} \setminus \mathcal{R}^t$
    $\sum_i Q_i(A)\,[\mathcal{R}_i, I_i] \leftarrow$ FindRFFormula($\mathcal{R}^t, \emptyset$)
    $RF \leftarrow \sum_i Q_i(A)\,[\mathcal{R}_i *_v \mathcal{M}, I \hat{*} I_i]$
  else if $|\mathcal{J}(\mathcal{R})| = 1$ **then** ▷ Case 3
    $j_* \leftarrow j_*(\mathcal{R})$
    $k \leftarrow 0$
    $\mathcal{R}' \leftarrow \mathcal{R}$
    while $(\mathcal{R}')^t \neq \mathcal{R}'_{n,j_*,0}$ **do**
      $\mathcal{R}' \leftarrow$ the roof state obtained from $\mathcal{R}'$ by adding a cup at j_*
      $k \leftarrow k + 1$
    $\mathcal{M} \leftarrow \mathcal{R}' \setminus \mathcal{R}'_{n,j_*,0}$
    for $J \in \mathbb{V}_n$ with $|J| \leq \min\{j_*, n - j_*\}$ **do**
      $Q_J(A) \leftarrow Q_{n,j_*,J}(A)$ from Lemma 3.1.10
    $RF \leftarrow [\mathcal{R}_{n,j_*} *_v \mathcal{M}, I] - \sum_{J \neq \emptyset} Q_J(A) \cdot$ FindRFFormula($\mathcal{R}'_{n,j_*,|J|} *_v \mathcal{M}, I \hat{*} J$)
    $RF \leftarrow A^{(n-2j_*)k}\, Q_\emptyset(A)^{-1} \cdot RF$
  else ▷ Case 4
    $j'_* \leftarrow \min\{j \in \mathcal{J} \mid j > j_*\}$
    $\mathcal{R}' \leftarrow$ a roof state obtained from $\mathcal{R}$ by adding $n - j'_*$ cups at j_*
    Apply the first-row expansion (3.1) $n - j'_*$ times to get
    $[\mathcal{R}', \emptyset] = A^{(-n+2j_*)(n-j'_*)}[\mathcal{R}, \emptyset] + \sum_i Q_i(A)\,[\mathcal{R}_i, I_i]$
    $RF \leftarrow$ FindRFFormula($\mathcal{R}', I$) $- \sum_i Q_i(A) \cdot$ FindRFFormula($\mathcal{R}_i, I \hat{*} I_i$)
    $RF \leftarrow A^{(n-2j_*)(n-j'_*)} \cdot RF$
  return RF

Theorem 3.1.13. *Every roof state has an RF-formula. In particular, Algorithm 3.1.12 provides a method to find it.*

Proof. Our proof is by induction on the width n of a roof state $\mathcal{R}$ given as an input to Algorithm 3.1.12. For $n = 0$ or 1, the statement is obvious. Assume that the statement holds for all roof states of width less than n. Let $\mathcal{R}$ be a roof state of width n. If $\mathcal{R}$ is as in Case 1 of Algorithm 3.1.12, then F is clearly an RF-formula for $\mathcal{R}$. Assume that $\mathcal{R}$ is as in Case 2 of Algorithm 3.1.12. Using Lemma 3.1.8, we see that once we find an RF-formula for $\mathcal{R}^t$, then we can also find it for $\mathcal{R}$ (with an RF-formula is provided in the proof of the lemma).

Now, let $\mathcal{R}$ be a top state with a single return on its boundary, that is, $\mathcal{R}$ is as in Case 3. Assume that after adding k cups at j_*, the new roof state $\mathcal{R}'$ satisfies $(\mathcal{R}')^t = \mathcal{R}'_{n,j_*,0}$. We apply k times first-row expansions (3.1) and notice that, $|\mathcal{J}(\cdot)| = |\{j_*\}| = 1$ at each time the expansion is used, so there is only one term on the right hand side of (3.1). Therefore, $[\mathcal{R}', \emptyset] = A^{(-n+2j_*)k}\,[\mathcal{R}, \emptyset]$. Using Lemma 3.1.10 and our proof of Lemma 3.1.8 with $\mathcal{M} = \mathcal{R}' \setminus (\mathcal{R}')^t$, one obtains

$$[\mathcal{R}_{n,j_*} *_v \mathcal{M}, \emptyset] = Q_\emptyset(A)[\mathcal{R}', \emptyset] + \sum_{J\in\mathbb{V}_n,\, 0<|J|\le\min\{j_*, n-j_*\}} Q_J(A)\,[\mathcal{R}'_{n,j_*,|J|} *_v \mathcal{M}, J].$$

Hence,

$$[\mathcal{R}, \emptyset] = A^{(n-2j_*)k}\,Q_\emptyset(A)^{-1}\left([\mathcal{R}_{n,j_*} *_v \mathcal{M}, \emptyset] - \sum_{\substack{J\in\mathbb{V}_n,\\ 0<|J|\le\min\{j_* n-j_*\}}} Q_J(A)\,[\mathcal{R}'_{n,j_*,|J|} *_v \mathcal{M}, J]\right)$$

and, by Lemma 3.1.7

$$[\mathcal{R}, I] = A^{(n-2j_*)k}\,Q_\emptyset(A)^{-1}\left([\mathcal{R}_{n,j_*} *_v \mathcal{M}, I] - \sum_{\substack{J\in\mathbb{V}_n,\\ 0<|J|\le\min\{j_* n-j_*\}}} Q_J(A)\,[\mathcal{R}'_{n,j_*,|J|} *_v \mathcal{M}, I\hat{*}J]\right).$$

Each roof state $\mathcal{R}'_{n,j_*,|J|}$ has width $n-2|J| < n$, so it also has an RF-formula by the induction hypothesis. Hence for $\mathcal{R}$ as in Case 3 of Algorithm 3.1.12 there is an RF-formula.

If $\mathcal{R}$ is as in Case 4 of Algorithm 3.1.12, i.e., $\mathcal{R}$ is a top state with $|\mathcal{J}(\mathcal{R})| > 1$. One notices that each roof state $\mathcal{R}_i$ and I_i involved in the formulas that appear in the corresponding part of Algorithm 3.1.12 satisfies at least one of the following:

(1) $\mathcal{R}_i$ is either as in Case 1, Case 2, or Case 3,

(2) $|I_i| > 0$, or

(3) $j_*(\mathcal{R}_i^t) > j_*(\mathcal{R})$.

If in $[\mathcal{R}_i, I_i]$, $|I_i| > 0$, then since the width of $\mathcal{R}_i$ is $n - 2|I_i| < n$, we can apply the induction hypothesis. If $j_*(\mathcal{R}_i^t) > j_*(\mathcal{R})$, the index $j_*(\cdot)$ of the top state of $\mathcal{R}_i$ strictly increases but it is bounded by n. Hence after applying FindRFFormula at most $(n-1)$ times, we can eliminate roof states of such type. $\square$

The following result concerning plucking polynomials with a delay function generalizes Theorem 2.2.9, and it plays an important role in the remaining part of this chapter as well as in Chapter 4.

Lemma 3.1.14. *Let (T, v_0, f) be the plane rooted tree with a delay function. Suppose there is a subtree T' of T rooted at v_0' with $f(u) \leq f(w)$ for all $u \in L(T')$ and $w \in L(T) \setminus L(T')$, where $L(T)$ denotes the set of leaves of T different than its root. Let $f' = f|_{T'}$ be the restriction of f to $L(T')$ and T'' be the tree obtained from T by replacing T' by a simple path P_m of length $m = |E(T')|$ (see Figure 3.11). Denote by f'' the delay function defined on leaves of T'' whose value $f''(v)$ is $f(v)$ if $v \in L(T) \setminus L(T')$ and 1 otherwise. Then $Q(T, v_0, f) = Q(T', v_0', f') \cdot Q(T'', v_0, f'')$.*

Proof. When $m = 0$, there is nothing to prove. Thus, assume that $m \geq 1$. We will prove the statement by induction on the number of vertices $|V(T)|$. The case $|V(T)| = 2$ is trivial. Assume that the statement is true for $|V(T)| - 1$. Let T''' be the tree obtained from T by

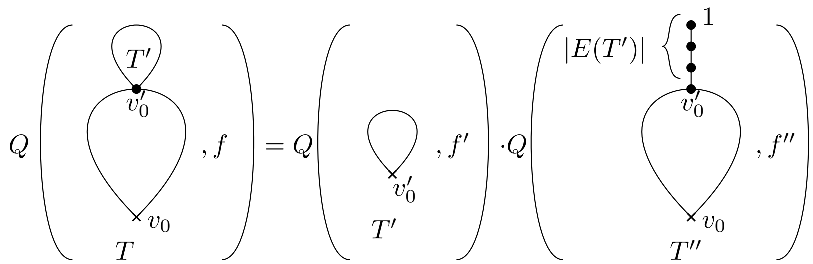


Figure 3.11. Product formula for plucking polynomials

replacing T' by a simple path P_{m-1} of length $m-1$ and f''' be the delay function defined on the leaves of T''' whose value for v is $\max\{f(v)-1,1\}$ if $v \in L(T) \setminus L(T')$ and 1 otherwise. We see that

$$\begin{aligned}\sum_{v\in L_1(T')} q^{r(T,v_0,v)}\, Q(T-v,v_0,f_v) &= \sum_{v\in L_1(T')} q^{r(T,v_0,v)}\, Q(T''',v_0,f''')\, Q(T'-v,v'_0,f'_v)\\ &= q^{r(T')}\, Q(T''',v_0,f''') \sum_{v\in L_1(T')} q^{r(T,v_0,v)-r(T')}\, Q(T'-v,v'_0,f'_v)\\ &= q^{r(T')}\, Q(T''',v_0,f''')\, Q(T',v'_0,f'),\end{aligned}$$

where $r(T') = \min\{r(T,v_0,v) \mid v \in L(T')\}$.

Let f'_* be the delay function defined on leaves of T' that takes value $\max\{f(v)-1,1\}$ for each $v \in L(T')$, then

$$\begin{aligned}\sum_{v\in L_1(T)\setminus L_1(T')} q^{r(T,v_0,v)}\, Q(T-v,v_0,f_v) &= \sum_{v\in L_1(T)\setminus L_1(T')} q^{r(T,v_0,v)}\, Q(T''-v,v_0,f''_v)\, Q(T',v'_0,f'_*)\\ &= Q(T',v'_0,f'_*) \sum_{v\in L_1(T)\setminus L_1(T')} q^{r(T,v_0,v)}\, Q(T''-v,v_0,f''_v).\end{aligned}$$

Hence,

$$\begin{aligned}Q(T,f) &= \sum_{v\in L_1(T')} q^{r(T,v_0,v)} Q(T-v,v_0,f_v) + \sum_{v\in L_1(T)\setminus L_1(T')} q^{r(T,v_0,v)}\, Q(T-v,v_0,f_v)\\ &= q^{r(T')}\, Q(T''',v_0,f''')\, Q(T',v'_0,f') + Q(T',v'_0,f'_*) \sum_{v\in L_1(T)\setminus L_1(T')} q^{r(T,v_0,v)}\, Q(T''-v,v_0,f''_v)\\ &= Q(T',v'_0,f')\, Q(T'',v_0,f''),\end{aligned}$$

where the last equality is either due to $L_1(T) \setminus L_1(T') = \emptyset$ or $f'_* = f|_{T'} \equiv 1$. $\square$

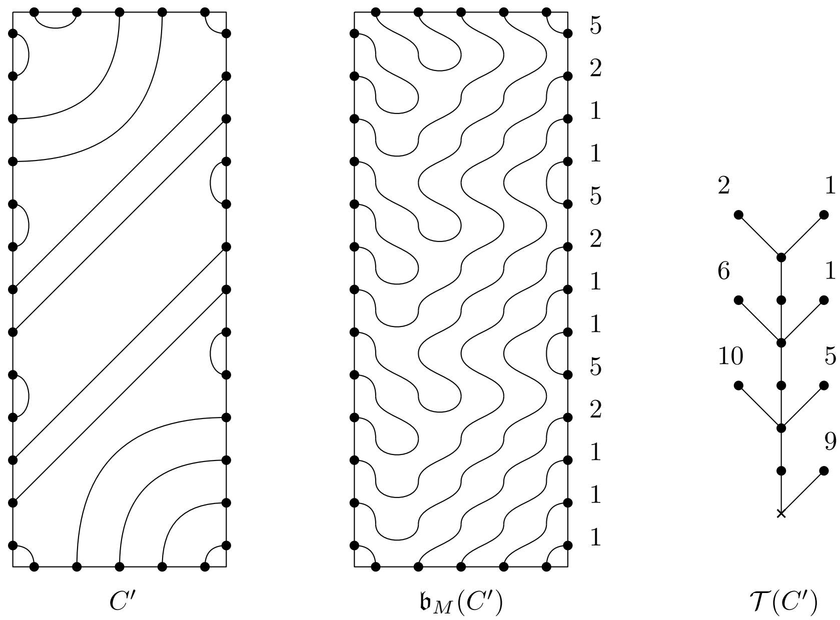
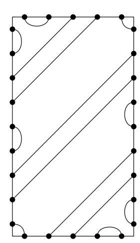


Figure 3.12. Computation of $C'(A)$

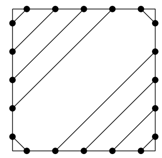
Example 3.1.15. *We find the coefficient $C(A)$ of the Catalan state*

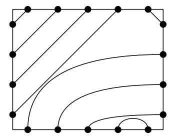
$$C = $$  $$\in \mathfrak{Cat}_{9,5}.$$

We see that its roof state $\mathcal{R} =$  *is as described in Case 2 of Algorithm 3.1.12. Therefore, applying Lemma 3.1.10 we find its RF-formula. Let $C = \mathcal{R} *_v \mathcal{F}$. Then $\mathcal{F}_I \neq K_0$ only if $I = \emptyset$ or $I = \{4\}$. Take $\mathcal{F}$ in* (3.4) *with $n = 5$ and $k = 1$, then*

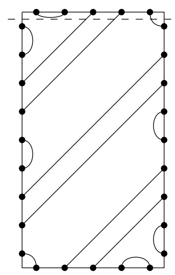
$$[\mathcal{R}_{5,1}, \emptyset]\,(\mathcal{F}) = Q_{5,1,\emptyset}(A)\ \left[\mathcal{R}'_{5,1,0}, \emptyset\right](\mathcal{F}) + Q_{5,1,\{4\}}(A)\ \left[\mathcal{R}'_{5,1,1}, \{4\}\right](\mathcal{F}),$$

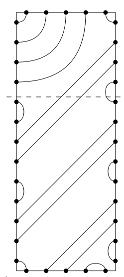
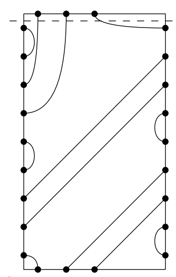
and we obtain the following coefficients:

$$Q_{5,1,\emptyset}(A) = A^{(5-2)(1-0)}$$  $$(A) = A^3 \cdot A^{2(5+1+1+1+1)-5\cdot 5} \cdot [5]_{q=A^{-4}} = A^{-20}[5],$$

$$Q_{5,1,\{4\}}(A) = A^{(5-2)(1-1)}$$  $$(A) = (A^{-3})^4 = A^{-12}.$$

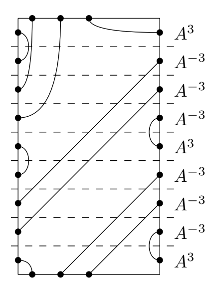
Hence,

$$C(A) =$$  $$(A) = (\mathcal{R}'_{5,1,0}, \mathcal{F}) = \frac{1}{Q_{5,1,\emptyset}(A)}(\mathcal{R}_{5,1}, \mathcal{F}) - \frac{Q_{5,1,\{4\}}(A)}{Q_{5,1,\emptyset}(A)}(\mathcal{R}'_{5,1,1}, \mathcal{F}_{\{4\}})$$

$$= \frac{A^{20}}{[5]}$$  $$(A) - \frac{A^8}{[5]}$$  $$(A).$$

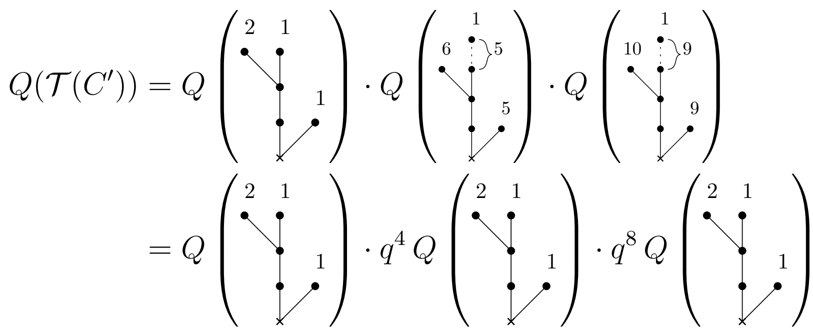
C' $\qquad$ C''

The coefficient of C'' is easy to find. Indeed,

$$C''(A) =$$ 

$$(A) = A^{-9}.$$

However, computing the coefficient of C' is much harder. Since $\mathfrak{b}_M(C')$ is the maximal sequence that realizes C' (see Figure 3.12(b)), $\|\mathfrak{b}_M(C')\| = 28$. Furthermore, one checks that the plane rooted tree with a delay function for C' is shown in Figure 3.12(c). Therefore, the *plucking polynomial of $\mathcal{T}(C')$ is:*



$$= q^{12}([2]_q + q[4]_q)^3 = q^{12}(1 + 2q + q^2 + q^3 + q^4)^3.$$

Hence,

$C'(A) = A^{2\cdot 28 - 13\cdot 5}\,(1 + 2q + q^2 + q^3 + q^4)^3\big|_{q=A^{-4}} = A^{-21}(A^4 + 2 + A^{-4} + A^{-8} + A^{-12})^3.$

Therefore, the coefficient of C is:

$$\begin{aligned}
C(A) &= \frac{A^{20}}{[5]} \cdot A^{-21}(A^4 + 2 + A^{-4} + A^{-8} + A^{-12})^3 - \frac{A^8}{[5]} \cdot A^{-9} \\
&= A^{-1}\frac{(A^4 + 2 + A^{-4} + A^{-8} + A^{-12})^3 - 1}{[5]} \\
&= A^{-13}\left((A^4 + 2 + A^{-4} + A^{-8} + A^{-12})^2 + (A^4 + 2 + A^{-4} + A^{-8} + A^{-12}) + 1\right) \\
&= A^{-37}(1 + 2A^4 + 3A^8 + 7A^{12} + 8A^{16} + 7A^{20} + 9A^{24} + 5A^{28} + A^{32}).
\end{aligned}$$

Theorem 3.1.16. *Let $C \in \mathfrak{Cat}_{m,n}$ and assume that there is an arc l of C connecting y_i and y'_j, where $|i - j| \leq 2$. Let C' be the Catalan state obtained from C by removing the arc l together with its two endpoints (see Figure 3.13). Then*

$$C(A) = A^{j-i}\, C'(A).$$

Proof. By Theorem 3.1.13, it suffices to show that the statement holds for Catalan states C with no returns on the bottom boundary. Assume that n is even and $j = i + 2$. Notice that after applying first-row expansions on both C and C' sufficient number of times (call it k) we may assume that the arc l connects x_1 and y'_2. Since there is a one-to-one correspondence between assignment of markers that realize C and C', terms of both expansions for C and C'

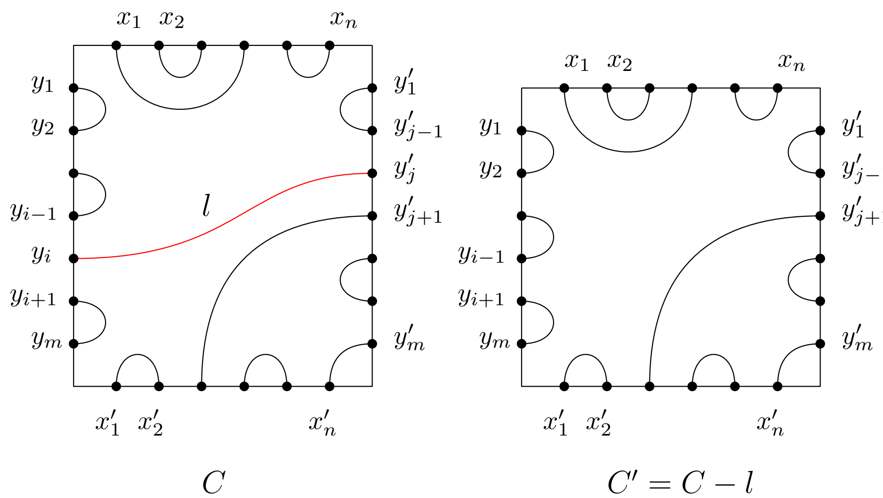


Figure 3.13. Removing an arc

coincide up to the stage k. Therefore, it suffices to prove that $C(A) = A^2\,C'(A)$ for Catalan states C with no returns on the bottom boundary and with an arc l that connects x_1 and y'_2, and the Catalan state $C' = C - l$ obtained by removing l from C together with its endpoints and then shifting y_1 to the top so it becomes a point on the top boundary.

Notice that, by Lemma 3.1.14, C and C' have the same plucking polynomial. Assume that $\mathfrak{b}_M(C) = (b_1, b_2, \ldots, b_m)$ and $\mathfrak{b}_M(C') = (b'_1, b'_2, \ldots, b'_{m-1})$ are maximal sequences that realize C and C', respectively. Let t be the index of b_i in $\mathfrak{b}_M(C)$ which corresponds to the realization of l in C and $1 < i_1 < \ldots < i_s < t$ be the indices of b_i in $\mathfrak{b}_M(C)$ that realize closed arcs of C on the right boundary before l is realized. Then

(1) $b_t = t$,

(2) $n = 2(t - s - 1)$,

(3) $\displaystyle\sum_{i=1}^{t-1} b'_i = \sum_{i=1}^{t-1} b_i + s$, and

(4) $b'_i = b_{i+1}$ for all $t \leq i \leq m-1$.

We say that the left (right) index of l is i if the left (right) endpoint of l is x_i, and we say that the right index of l is $n + j$ if the endpoint is y'_j for $j = 1, 2$.

We see that after the first arc of C is realized by the first term of the sequence $\mathfrak{b}_M(C)$, the left index of l (initially equals 1) increases by 1. Since the arc l is realized after $(t-1)$ steps using the first $(t-1)$ terms of $\mathfrak{b}_M(C)$, when l is realized, its left index becomes $1+(t-1)=t$, and it is same as b_t. This proves (1). Now we analyze the right index of l, which initially equals $n+2$. Notice that after the first arc of C is realized by the first term of the sequence $\mathfrak{b}_M(C)$, the right index decreases by 1 if the arc is inside of the regions bounded by l and the top boundary of the Catalan state, otherwise it increases by 1. Indices i_j's, $j=1,\ldots,s$, give terms of the sequence $\mathfrak{b}_M(C)$ that realize arcs which are not inside the region bounded by l and the top boundary. When l is realized, its right index is $(n+2)-(t-1)+2s$ and it is equal $b_t+1=t+1$. Hence, $n=2(t-s-1)$ and thus we completed our proof for (2). Notice that presence of l in C causes "delays" in realization of an arc from the top right corner of C by 1, hence the total number of such delays is exactly s, this proves (3). For example, for the Catalan states shown in Figure 3.14, $\mathfrak{b}_M(C)=(b_1,b_2,\ldots,b_9,\ldots)=(9,9,8,10,7,7,10,10,9,\ldots)$ and $\mathfrak{b}_M(C')=(b'_1,b'_2,\ldots,b'_8,\ldots)=(9,9,10,9,7,10,10,9,\ldots)$. We see that b_3 and b'_4 realize the same arc. Although $b'_3+b'_4$ differs from b_3+b_4 by 1 as $i_1=4$. Similarly, b_6 and b'_8 realize the same arc, while the sums $b'_6+b'_7+b'_8$ and $b_6+b_7+b_8$ differ by 2 as $i_2=7$ and $i_3=8$. After we realize the first t arcs of C following the first t terms of the maximal sequence $\mathfrak{b}_M(C)$ and realizing $(t-1)$ arcs of C' following the maximal sequence $\mathfrak{b}_M(C')$, we see that the remaining floor states are the same, hence our proof of (4) is completed.

Therefore, $(2\|\mathfrak{b}_M(C)\|-mn)-(2\|\mathfrak{b}_M(C')\|-(m-1)n)=2(-s+t)-n=2$, and the result follows by Theorem 2.2.7. Since the number of points between endpoints of l are even, it remains to show that the statement holds when (i) n is even and $i=j$ or $i=j+2$, or (ii) n is odd and $i=j+1$ or $j=i+1$. The cases (i) when $i=j$ and (ii) when $j=i+1$ follow by an argument similar to the above and (i) when $i=j+2$ and (ii) when $i=j+1$ follow by considering the Catalan state C^s instead of C and applying Lemma 3.1.1. $\square$

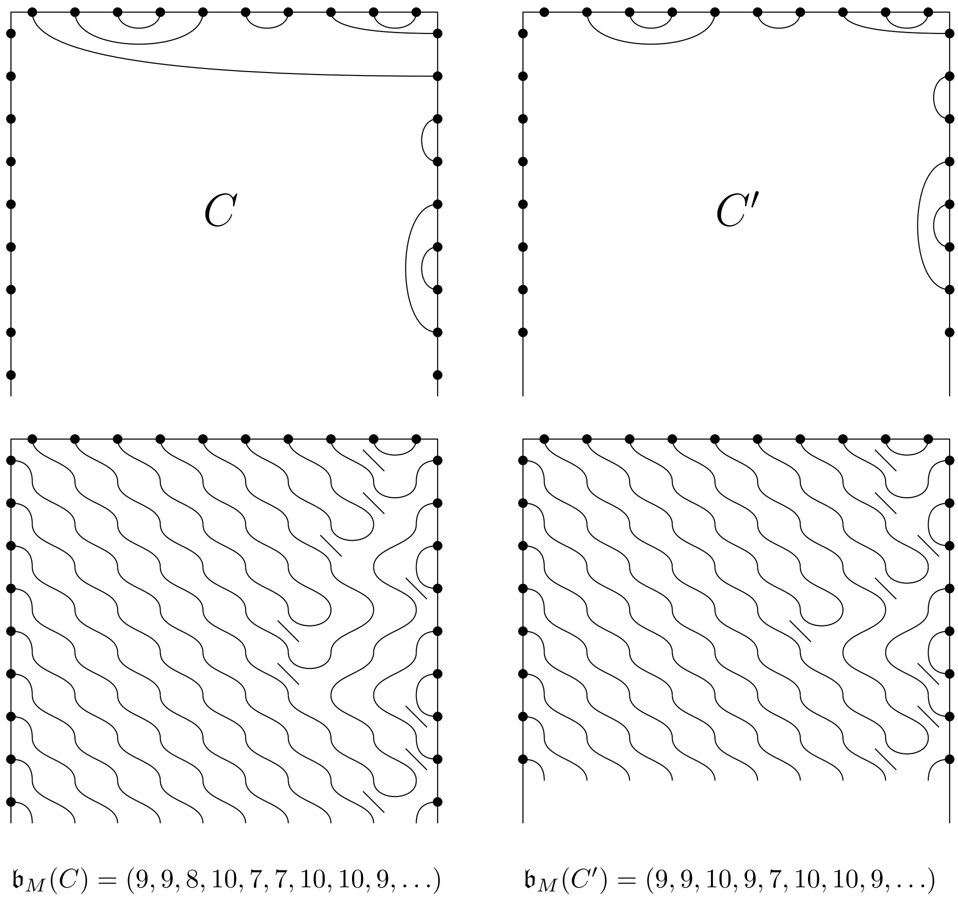


Figure 3.14. Proof of Theorem 3.1.16

3.2 Formulas for Particular Roof States

RF-formulas resulting from an application of Algorithm 3.1.12 are usually not simple. Therefore, for some roof states, alternative methods may be used to find them. Given n, let κ_1, κ_2 be non-negative integers such that $\nu = \frac{n-\kappa_1-\kappa_2}{2}$ is also a non-negative integer. Let $\mathbb{R}_n(\kappa_1, \kappa_2)$ be the set of roof states shown in Figure 3.15, where $\mathcal{C}$ is a crossingless connections inside the region bounded by the dotted curve.

Lemma 3.2.1. *Let κ_1, κ_2, k be non-negative integers such that $\nu = \frac{n-\kappa_1-\kappa_2}{2}$ is also a non-negative integer and $k \leq \mu = \frac{n+\kappa_1-\kappa_2}{2}$. Suppose that $\mathcal{C}_0, \ldots, \mathcal{C}_s$ are some crossingless connec-*

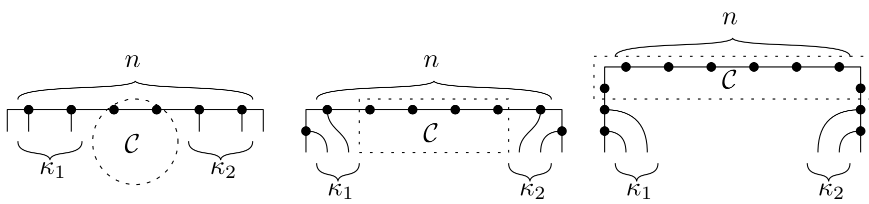
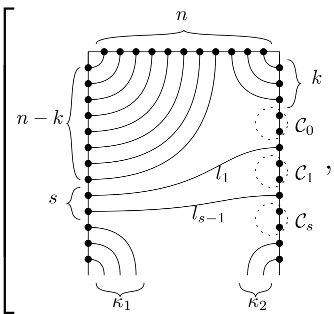
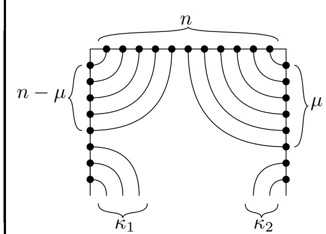
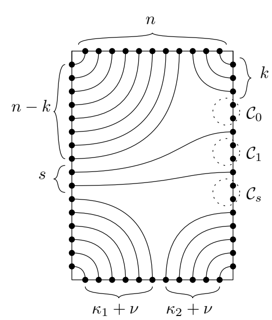
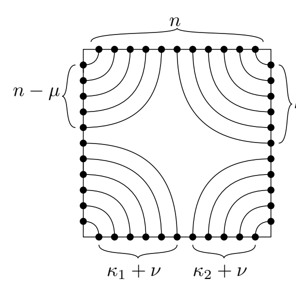


Figure 3.15. $\mathbb{R}_n(\kappa_1, \kappa_2)$

tions (possibly empty), inside of each dotted circle (see (3.6) *below),* $s \geq 0$. *Then*

[ , $\emptyset$] $= Q(A) \cdot$ [ , $\emptyset$] , (3.6)

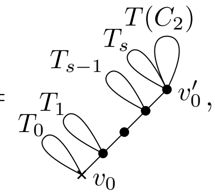
where

$Q(A) =$  (A) /  (A) . (3.7)

In particular, if for all i, $\mathcal{C}_i = \emptyset$ *then* $Q(A) = A^{s(2k-n)}$.

Proof. Denote by $\mathcal{R}_1$ the roof state shown on the left of (3.6) and by $\mathcal{R}_2$ the roof state on the right of (3.6). Consider a floor state $\mathcal{F}$ of width n and $\kappa_1 + \kappa_2$ open arcs. Let $C_1 = (\mathcal{R}_1 *_v \mathcal{F})^{r,\pi} \in \mathfrak{Cat}_{m_1,n}$ and $C_2 = (\mathcal{R}_2 *_v \mathcal{F})^{r,\pi} \in \mathfrak{Cat}_{m_2,n}$ and one notes that $m_1 = m_2 + (s + \mu - k)$.

By Lemma 3.1.1, $C_1(A) = (\mathcal{R}_1, \mathcal{F})$ and $C_2(A) = (\mathcal{R}_2, \mathcal{F})$. Let $\mathcal{T}(C_2) = (T(C_2), v_0', f_2)$ be the plane rooted tree with a delay function for C_2 and $\mathfrak{b}_M(C_2) = (b_1, b_2, \ldots, b_{m_2})$ be the maximal sequence that realizes C_2. Then the plane rooted tree of $\mathcal{T}(C_1) = (T(C_1), v_0, f_1)$ for C_1 is

$T(C_1) =$ 

where T_i is the dual graph of $\mathcal{C}_i$. Since $f_1 = f_2$ on the set of leaves of $T(C_2)$ and $f_1(v) \geq f_1(w)$, for all $v \in L_1(T(C_2)),\ w \in L_1(T(C_1)) \setminus L_1(T(C_2))$, by Lemma 3.1.14, plucking polynomials for $T(C_1)$ and $T(C_2)$ are related as follows $Q(\mathcal{T}(C_1)) = Q(\mathcal{T}(C_2)) \cdot \mathcal{Q}$ for some polynomial $\mathcal{Q}$ that depends only on $\bigcup_{i=0}^{s} T_i$. We note that the maximal sequence $\mathfrak{b}_M(C_1) = (b_1', b_2', \ldots, b_{m_1}')$ that realizes C_1 must realize arcs of $\mathcal{R}_1$ in the same order regardless of the choice made for $\mathcal{F}$. In fact, κ_1 open arcs of $\mathcal{R}_1$ are realized as first, and then κ_2 open arcs of $\mathcal{R}_1$, arcs of $\mathcal{C}_s$, the line l_{s-1}, arcs of $\mathcal{C}_{s-1}$, the line l_{s-2}, arcs of $\mathcal{C}_{s-2}$, etc. Therefore, $b_i' = b_i$ for $1 \leq i \leq m_2$ and b_i' depends only on $\mathcal{R}_1$, for $m_2 + 1 \leq i \leq m_1$. Let $B = \sum_{i=m_2+1}^{m_1} b_i'$, then by Theorem 2.2.7,

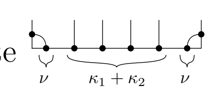
$$\begin{aligned}
C_1(A) &= A^{2\|\mathfrak{b}_M(C_1)\| - m_1 n + 4 \operatorname{mindeg}_q Q(\mathcal{T}(C_1))} \cdot Q(\mathcal{T}(C_1))\big|_{q=A^{-4}} \\
&= A^{2\|\mathfrak{b}_M(C_2)\| + 2B - (m_2+s+\mu-k)n + 4(\operatorname{mindeg}_q Q(\mathcal{T}(C_2)) + \operatorname{mindeg}_q \mathcal{Q})} \cdot Q(\mathcal{T}(C_2))\big|_{q=A^{-4}} \cdot \mathcal{Q}\big|_{q=A^{-4}} \\
&= A^{2B - (s+\mu-k)n + 4 \operatorname{mindeg}_q \mathcal{Q}} \cdot \mathcal{Q}\big|_{q=A^{-4}} \cdot C_2(A).
\end{aligned}$$

Hence, $[\mathcal{R}_1, \emptyset](\mathcal{F}) = C_1(A) = Q(A)\, C_2(A) = Q(A)\, [\mathcal{R}_2, \emptyset](\mathcal{F})$, for some

$$Q(A) = A^{2B - (s+\mu-k)n + 4 \operatorname{mindeg}_q \mathcal{Q}} \cdot \mathcal{Q}\big|_{q=A^{-4}}$$

which does not depend on $\mathcal{F}$. If $\mathcal{F}$ does not have width n or $\kappa_1 + \kappa_2$ open arcs, then

$$[\mathcal{R}_1, \emptyset](\mathcal{F}) = 0 = Q(A) \cdot 0 = [\mathcal{R}_2, \emptyset](\mathcal{F}).$$

Thus, for all $\mathcal{F}$, $[\mathcal{R}_1, \emptyset](\mathcal{F}) = Q(A)\, [\mathcal{R}_2, \emptyset](\mathcal{F})$. Choosing for the floor state 

in (3.6) yields (3.7). □

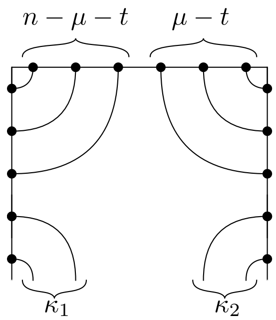


Figure 3.16. Roof state $\mathcal{R}_{n,\kappa_1,\kappa_2,t}$

Now we state the main result in this section.

Theorem 3.2.2. *Let n, κ_1, κ_2 be non-negative integers such that $\nu = \frac{n-\kappa_1-\kappa_2}{2}$ is also a non-negative integer. Then, for any roof state $\mathcal{R} \in \mathbb{R}_n(\kappa_1, \kappa_2)$, there is an RF-formula*

$$[\mathcal{R}, \emptyset] = \sum_{I \in \mathbb{V}_n,\, |I| \leq \nu} Q_{n,\mathcal{R},I}(A) \cdot \left[\mathcal{R}_{n,\kappa_1,\kappa_2,|I|}, I\right], \tag{3.8}$$

where $\mathcal{R}_{n,\kappa_1,\kappa_2,t}$ is shown in Figure 3.16 and $\mu = \frac{n+\kappa_1-\kappa_2}{2}$.

In particular, for every $I \in \mathbb{V}_n$, if $\mathcal{F}_{n,\kappa_1,\kappa_2,I}$ denotes a floor state of width n that satisfies $\Phi_n(\mathcal{F}^b_{n,\kappa_1,\kappa_2,I}) = I$ and $\left(\mathcal{R}_{n,\kappa_1,\kappa_2,|I|}, (\mathcal{F}_{n,\kappa_1,\kappa_2,I})_I\right) \neq 0$, then

$$Q_{n,\mathcal{R},I}(A) = \sum_{J \preceq_F I} \frac{(\mathcal{R}, \mathcal{F}_{n,\kappa_1,\kappa_2,J})}{\left(\mathcal{R}_{n,\kappa_1,\kappa_2,|I|}, (\mathcal{F}_{n,\kappa_1,\kappa_2,I})_I\right)} \cdot \mathcal{S}(n, \kappa_1, \kappa_2, I, J), \tag{3.9}$$

where

$$\mathcal{S}(n, \kappa_1, \kappa_2, I, J) = \sum_{J = I_0 \prec_F \cdots \prec_F I_s = I} (-1)^s \prod_{i=1}^{s} \frac{\left(\mathcal{R}_{n,\kappa_1,\kappa_2,|I_{i-1}|}, (\mathcal{F}_{n,\kappa_1,\kappa_2,I_i})_{I_{i-1}}\right)}{\left(\mathcal{R}_{n,\kappa_1,\kappa_2,|I_{i-1}|}, (\mathcal{F}_{n,\kappa_1,\kappa_2,I_{i-1}})_{I_{i-1}}\right)} \tag{3.10}$$

if $I \neq J$, and 1 otherwise.

Proof. First, we prove the existence of $Q_{n,\mathcal{R},I}(A)$'s in

$$[\mathcal{R}, \emptyset] = \sum_{I \in \mathbb{V}_n} Q_{n,\mathcal{R},I}(A) \cdot \left[\mathcal{R}_{n,\kappa_1,\kappa_2,|I|}, I\right]. \tag{3.11}$$

Without loss of generality, we may assume that $\kappa_1 \le \kappa_2$. Otherwise, since $\mathcal{R} \in \mathbb{R}_n(\kappa_1, \kappa_2)$ if and only if $\mathcal{R}^s \in \mathbb{R}_n(\kappa_2, \kappa_1)$ we can find an RF-formula for $\mathcal{R}^s$ and use our proof of Lemma 3.1.8 to find an RF-formula for $\mathcal{R}$. We first prove the existence of $Q_{n,\mathcal{R},I}(A)$ in (3.11), for all $\mathcal{R} \in \mathbb{R}_n(\kappa_1, \kappa_2) \cap \mathbb{T}_n$. Let $j_* = j_*(\mathcal{R})$, then $j_* - \kappa_1 \le \frac{n-\kappa_1-\kappa_2}{2} = \nu$ by the definition of j_*. We prove formula (3.11) by induction on n.

Consider the roof state $\mathcal{R}'$ obtained from $\mathcal{R}$ by adding $N \ge 0$ cups at j_*, such that $(\mathcal{R}')^t = \mathcal{R}'_{n,j_*,0}$, where $\mathcal{R}'_{n,k,l}$ is shown in Figure 3.8. Let $\mathcal{M} = \mathcal{R}' \setminus \mathcal{R}'_{n,j_*,0}$, then by Lemma 3.1.10,

$$
\begin{aligned}
&[\mathcal{R}', \emptyset] = \left[\mathcal{R}'_{n,j_*,0} *_v \mathcal{M}, \emptyset\right] \\
&= Q_{n,j_*,\emptyset}(A)^{-1} \left([\mathcal{R}_{n,j_*} *_v \mathcal{M}, \emptyset] - \sum_{I \in \mathbb{V}_n,\, 1 \le |I| \le j_*} Q_{n,j_*,I}(A) \left[\mathcal{R}'_{n,j_*,|I|} *_v \mathcal{M}, I\right] \right),
\end{aligned}
$$

where $Q_{n,k,I}(A)$ and $\mathcal{R}_{n,k}$ are defined in Lemma 3.1.10. Notice that $\mathcal{R}_{n,j_*} *_v \mathcal{M}$ is a roof state in the form that is on the left side of (3.6), hence

$$
[\mathcal{R}_{n,j_*} *_v \mathcal{M}, \emptyset] = Q(A) \cdot [\mathcal{R}_{n,\kappa_1,\kappa_2,0}, \emptyset]
$$

for some $Q(A)$ defined in Lemma 3.2.1. Since $\mathcal{R}'_{n,j_*,|I|} *_v \mathcal{M} \in \mathbb{R}_{n-2|I|}(\kappa_1, \kappa_2)$ and $|I| \ge 1$, the induction hypothesis applies for these roof states. It follows that $[\mathcal{R}', \emptyset]$ can be represented in the form given on the right hand side of (3.11).

Applying $N \ge 1$ times the first-row expansions (3.1) yields

$$
[\mathcal{R}', \emptyset] = A^{(-n+2j_*)N}[\mathcal{R}, \emptyset] + \sum_i [\mathcal{R}_i, I_i]
$$

for some $[\mathcal{R}_i, I_i]$, with either (1) $j_*(\mathcal{R}_i) > j_*$ and $\mathcal{R}_i \in \mathbb{R}_n(\kappa_1, \kappa_2) \cap \mathbb{T}_n$, or (2) $\mathcal{R}_i \in \mathbb{R}_{n-2|I_i|}(\kappa_1, \kappa_2)$ with $|I_i| > 0$. If $[\mathcal{R}_i, I_i]$ is as in (2), then by induction hypothesis, we can express it in the form as it is on the right hand side of (3.11). Thus, we only left those $[\mathcal{R}_i, I_i]$ for which $I_i = \emptyset$ and $j_*(\mathcal{R}_i) > j_*(\mathcal{R})$ increased by 1. In such a case, we apply the same process for $\mathcal{R}_i$ as we did for $\mathcal{R}'$. Since $j_*(\cdot)$ increases, this process finally reaches some roof states $\mathcal{R}''$ with $j_* = \mu$. Let $\mathcal{R}'''$ be the roof state obtained by adding N' cups at μ on

$\mathcal{R}''$ so that $(\mathcal{R}''')^t = \mathcal{R}'_{n,\mu,0}$. Define $\mathcal{M}' = \mathcal{R}''' \setminus \mathcal{R}'_{n,\mu,0}$, then using the first-row expansions and Lemma 3.1.10 gives

$$\begin{aligned}
[\mathcal{R}'', \emptyset] &= A^{(n-2\mu)N'}[\mathcal{R}''', \emptyset] = A^{(n-2\mu)N'}[\mathcal{R}'_{n,\mu,0} *_v \mathcal{M}', \emptyset] \\
&= A^{(n-2\mu)N'} Q_{n,\mu,\emptyset}(A)^{-1} \left([\mathcal{R}_{n,\mu} *_v \mathcal{M}, \emptyset] - \sum_{I \in \mathbb{V}_n,\, 1 \le |I| \le \mu} Q_{n,\mu,I}(A) \left[\mathcal{R}'_{n,\mu,|I|} *_v \mathcal{M}, I\right] \right).
\end{aligned}$$

Using the same argument as we did for $\mathcal{R}'$, we see that $[\mathcal{R}'', \emptyset]$ can be represented in the form of the right hand side of (3.11).

Therefore, our proof for the existence of (3.11) for all $\mathcal{R} \in \mathbb{R}_n(\kappa_1, \kappa_2) \cap \mathbb{T}_n$ is completed. Now, given $\mathcal{R} \in \mathbb{R}_n(\kappa_1, \kappa_2)$, applying the first-row expansion (3.1) on $\mathcal{R}$ sufficiently many times yields

$$[\mathcal{R}, \emptyset] = \sum_i Q_i(A) \cdot [\mathcal{R}_i, I_i],$$

where $\mathcal{R}_i \in \mathbb{R}_n(\kappa_1, \kappa_2) \cap \mathbb{T}_n$ if $I_i = \emptyset$ or $\mathcal{R}_i \in \mathbb{R}_{n-2|I_i|}(\kappa_1, \kappa_2)$ otherwise. Thus, the above observations and the induction hypothesis imply that (3.11) holds for all $\mathcal{R} \in \mathbb{R}_n(\kappa_1, \kappa_2)$. Notice that the width of $\mathcal{R}_{n,\kappa_1,\kappa_2,|I|}$ is $n-2|I|$, hence $[\mathcal{R}_{n,\kappa_1,\kappa_2,|I|}, I] \equiv 0$ unless $n-2|I| \ge \kappa_1+\kappa_2$ by Theorem 2.2.1, which means that $|I| \le \frac{n-\kappa_1-\kappa_2}{2} = \nu$, and therefore the proof for existence of $Q_{n,\mathcal{R},I}(A)$ in (3.8) is completed.

Now we prove (3.9) by induction on $|I|$. Indeed, choosing the floor state $\mathcal{F}_{n,\kappa_1,\kappa_2,\emptyset}$ in (3.8) gives:

$$Q_{n,\mathcal{R},\emptyset}(A) = \frac{(\mathcal{R}, \mathcal{F}_{n,\kappa_1,\kappa_2,\emptyset})}{\left(\mathcal{R}_{n,\kappa_1,\kappa_2,0}, (\mathcal{F}_{n,\kappa_1,\kappa_2,\emptyset})_\emptyset\right)},$$

i.e., (3.9) holds when $|I| = 0$. Assume that (3.9) holds for all $J \in \mathbb{V}_n$ with $|J| < |I|$ and notice that $(\mathcal{F}_{n,\kappa_1,\kappa_2,I})_{I'} = K_0$ unless $I' \preceq_F I$. So, taking the floor state $\mathcal{F}_{n,\kappa_1,\kappa_2,I}$ in (3.8), yields

$$\begin{aligned}
Q_{n,\mathcal{R},I}(A) &= \frac{(\mathcal{R}, \mathcal{F}_{n,\kappa_1,\kappa_2,I})}{\left(\mathcal{R}_{n,\kappa_1,\kappa_2,|I|}, (\mathcal{F}_{n,\kappa_1,\kappa_2,I})_I\right)} \\
&\quad - \sum_{J \in \mathbb{V}_n, J \prec_F I} Q_{n,\mathcal{R},J}(A) \cdot \frac{\left(\mathcal{R}_{n,\kappa_1,\kappa_2,|J|}, (\mathcal{F}_{n,\kappa_1,\kappa_2,I})_J\right)}{\left(\mathcal{R}_{n,\kappa_1,\kappa_2,|I|}, (\mathcal{F}_{n,\kappa_1,\kappa_2,I})_I\right)},
\end{aligned}$$

and then (3.9) follows by the induction hypothesis on $Q_{n,\mathcal{R},J}(A)$'s. $\square$

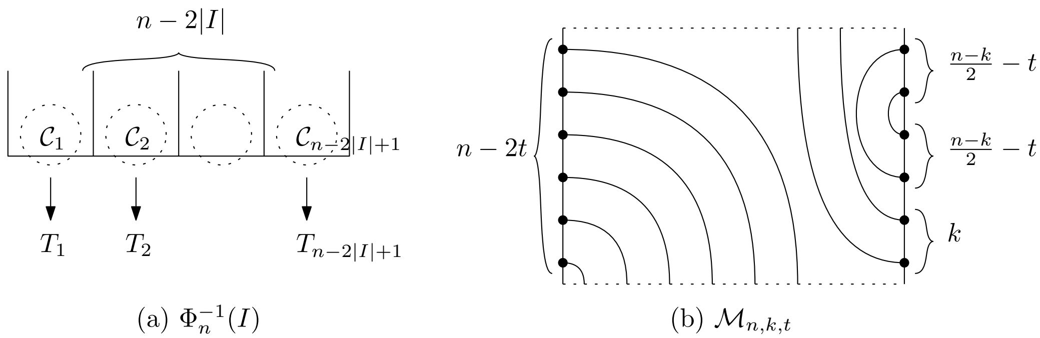


Figure 3.17. $\Phi_n^{-1}(I)$ and $\mathcal{M}_{n,k,t}$

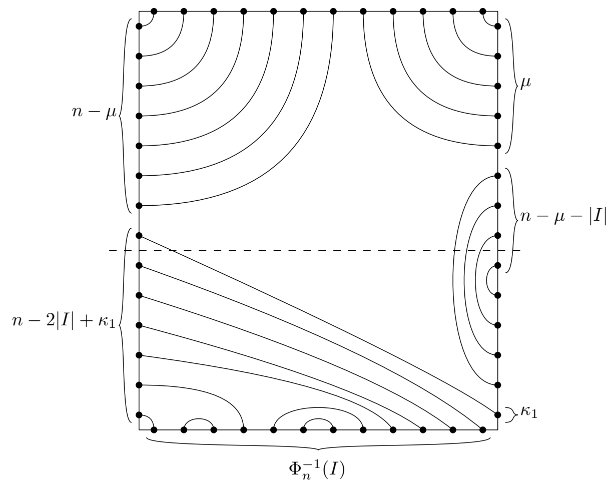
According to the theorem, we can find $Q_{n,\mathcal{R},I}(A)$ by choosing an appropriate $\mathcal{F}_{n,\kappa_1,\kappa_2,I}$'s and then computing (3.9) and (3.10). One such a choice for floor states is given in the next lemma.

Lemma 3.2.3. *Let n, κ_1, κ_2 be non-negative integers such that $\nu = \frac{n-\kappa_1-\kappa_2}{2}$ is also a non-negative integer. Given $I \in \mathbb{V}_n$, suppose that $\Phi_n^{-1}(I)$ has the form that is shown in Figure 3.17(a), where $\mathcal{C}_i$ is some crossingless connections inside dotted circle, and T_i is the plane rooted tree obtained as the dual graph of $\mathcal{C}_i^{r,\pi}$. Then the coefficient of $\mathcal{R}_{n,\kappa_1,\kappa_2,0} *_v \mathcal{M}_{n,\kappa_1+\kappa_2,|I|} *_v \Phi_n^{-1}(I)$ is*

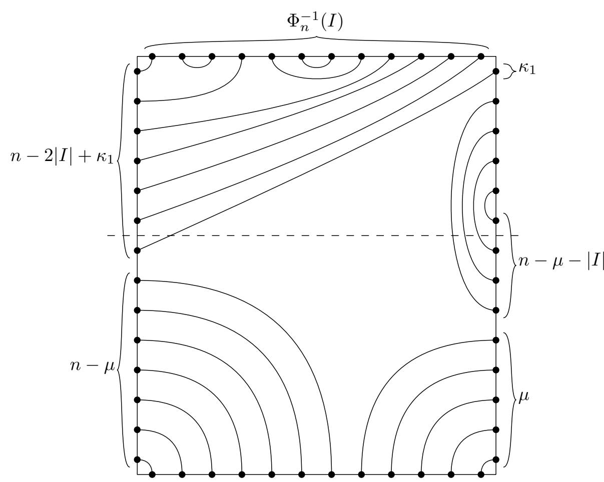
$$A^{-2\|I\|+|I|\cdot(3|I|-n-\kappa_1-\kappa_2+1)+\frac{n}{2}(3\kappa_1-\kappa_2)+\frac{1}{2}(\kappa_1-\kappa_2)^2}\begin{bmatrix} n-|I| \\ \mu \end{bmatrix}\prod_{i=1}^{n-2|I|+1} Q(T_i)\big|_{q=A^4}\begin{bmatrix} \sum_{j=1}^{i}|V(T_j)|-1 \\ |V(T_i)|-1 \end{bmatrix},$$

where $\mathcal{R}_{n,\kappa_1,\kappa_2,t}$ and $\mathcal{M}_{n,k,t}$ are shown in Figure 3.16 and Figure 3.17(b), respectively, $\mu = \frac{n+\kappa_1-\kappa_2}{2}$, $\|I\| = \sum_{i\in I} i$, and $Q(T_i)$ is the plucking polynomial of T_i.

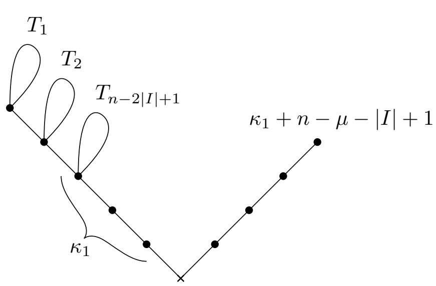
Proof. Instead of calculating the coefficient of

$$C = \mathcal{R}_{n,\kappa_1,\kappa_2,0} *_v \mathcal{M}_{n,\kappa_1+\kappa_2,|I|} *_v \Phi_n^{-1}(I) =$$ ,

we calculate the coefficient of

$$C' = (C^{r,\pi})^s =$$ .

One can easily see that

$$\mathcal{T}(C') =$$ ,

thus by Lemma 3.1.14,

$$Q^*(\mathcal{T}(C')) = q^{-\operatorname{mindeg}_q \mathcal{T}(C')} Q(\mathcal{T}(C')) = \begin{bmatrix} n-|I| \\ \mu \end{bmatrix}_q \prod_{i=1}^{n-2|I|+1} Q(T_i) \cdot \begin{bmatrix} \sum_{j=1}^{i} |V(T_j)| - 1 \\ |V(T_i)| - 1 \end{bmatrix}_q .$$

Notice that $\|\mathfrak{b}_M(C')\| = \|I\| + \frac{|I|(|I|-1)}{2} + |I|(n-2|I|+\kappa_1-\mu) + n(n-\mu-|I|) + (n-\mu)\mu$. Indeed, this is true since

(1) We first realize all the arcs corresponding to elements of I, and if $I = \{i_1 < i_2 < \ldots < i_t\}$, then $(b_1, \ldots, b_t)$ must be $(i_t, i_{t-1}+1, \ldots, i_1+t-1)$. Hence, the total contribution of $\mathfrak{b}_M(C')$ is $\|I\| + \frac{|I|(|I|-1)}{2}$.

(2) Then, as the second, we realize all arcs with an endpoint y_j, $1 \leq j \leq n-2|I|+\kappa_1-\mu$, after realizing $|I|$ returns on the top boundary. For these arcs $b_i = |I|$ in the maximal sequence $\mathfrak{b}_M(C')$, thus the total contribution of these arcs to $\|\mathfrak{b}_M(C')\|$ is $|I|(n-2|I|+\kappa_1-\mu)$.

(3) For arcs with an endpoint y'_{κ_1+j}, $1 \leq j \leq n-\mu-|I|$, $b_i = n$ in the maximal sequence $\mathfrak{b}_M(C')$. Therefore, the total contribution of these arcs to $\|\mathfrak{b}_M(C')\|$ is $n(n-\mu-|I|)$.

(4) The remaining μ arcs contribute $(n-\mu)\mu$ to $\|\mathfrak{b}_M(C')\|$.

Therefore,

$$\begin{aligned} C'(A) &= A^{2\|\mathfrak{b}_M(C')\| - n\cdot(2n-2|I|-\mu+\kappa_1)} \cdot Q^*(\mathcal{T}(C'))\big|_{q=A^{-4}} \\ &= A^{2\|I\| - |I|\cdot(3|I|-n-\kappa_1-\kappa_2+1) - \frac{n}{2}(3\kappa_1-\kappa_2) - \frac{1}{2}(\kappa_1-\kappa_2)^2} \cdot Q^*(\mathcal{T}(C'))\big|_{q=A^{-4}} \end{aligned}$$

by Theorem 2.2.7, and the result follows since $C(A) = C'(A^{-1})$. $\square$

When $|I|$ is small, we can compute (3.10) directly.

Lemma 3.2.4. *Suppose all assumptions of Theorem 3.2.2 hold and let*

$$\mathcal{F}_{n,\kappa_1,\kappa_2,I} = \mathcal{M}_{n,\kappa_1+\kappa_2,|I|} *_v \Phi_n^{-1}(I),$$

where $\mathcal{M}_{n,k,t}$ is shown in Figure 3.17(b). Then, for all $J \preceq_F I$ with $0 \leq |I|-|J| \leq 3$,

$$\mathcal{S}(n,\kappa_1,\kappa_2,I,J) = (-1)^{|I|-|J|} \cdot A^{-2(|I|-|J|)(|I|-|J|-1)} \frac{\begin{bmatrix} 2(n-|I|-|J|)+1 \\ n-|I|-|J| \end{bmatrix} \left(\mathcal{R}_{n,\kappa_1,\kappa_2,|J|}, (\mathcal{F}_{n,\kappa_1,\kappa_2,I})_J\right)}{\begin{bmatrix} 2(n-|I|-|J|)+1 \\ n-2|J| \end{bmatrix} \left(\mathcal{R}_{n,\kappa_1,\kappa_2,|J|}, (\mathcal{F}_{n,\kappa_1,\kappa_2,J})_J\right)}. \tag{3.12}$$

Proof. We simply consider all possible J and I, and then use Lemma 3.2.3 to compute (3.10). In order to simplify notations, let $C_{I,J} = (\mathcal{R}_{n,\kappa_1,\kappa_2,|J|}, (\mathcal{F}_{n,\kappa_1,\kappa_2,I})_J)$. When $I = J$, our statement is obviously true. Thus, consider $I \neq J$ and denote by

$$\mathcal{S}(I,J) = \sum_{J=I_0 \prec_F \cdots \prec_F I_s = I} (-1)^s \prod_{i=1}^{s} \frac{C_{I_i,I_{i-1}}}{C_{I_{i-1},I_{i-1}}}$$

and

$$\mathcal{S}'(I,J) = (-1)^{|I|-|J|} \cdot A^{-2(|I|-|J|)(|I|-|J|-1)} \frac{\begin{bmatrix} 2(n-|I|-|J|)+1 \\ n-|I|-|J| \end{bmatrix} C_{I,J}}{\begin{bmatrix} 2(n-|I|-|J|)+1 \\ n-2|J| \end{bmatrix} C_{J,J}}.$$

We show that $\mathcal{S}(I,J) = \mathcal{S}'(I,J)$, for all $J \prec_F I$ with $I, J \in \mathbb{V}_n$. If $|I| - |J| = 1$, then

$$\mathcal{S}(I,J) = -\frac{C_{I,J}}{C_{J,J}} = (-1)^1 \cdot A^{-2\cdot 1\cdot(1-1)} \frac{\begin{bmatrix} 2n-4|J|-1 \\ n-2|J|-1 \end{bmatrix} C_{I,J}}{\begin{bmatrix} 2n-4|J|-1 \\ n-2|J| \end{bmatrix} C_{J,J}} = \mathcal{S}'(I,J).$$

Now, when $|I| - |J| = 2$, there are two cases to consider: $I \setminus J = \{i_1 < i_2\}$ with $i_2 = i_1 + 1$ or $i_2 > i_1 + 1$. Let $n' = n - 2|J|$, $\mu' = \mu - |J|$, $i_1' = i_1 - 2|\{j \in J \mid j < i_1\}|$, $i_2' = i_2 - 2|\{j \in J \mid j < i_2\}|$, $I' = \{i_1', i_2'\}$.

For $i_2 = i_1 + 1$, we obtain

- $J \prec_F I$ or
- $J \prec_F J \cup \{i_1 + 1\} \prec_F I$.

So,

$$\mathcal{S}(I,J) = -\frac{C_{I,J}}{C_{J,J}} + \frac{C_{J\cup\{i_1+1\},J}}{C_{J,J}} \cdot \frac{C_{I,J\cup\{i_1+1\}}}{C_{J\cup\{i_1+1\},J\cup\{i_1+1\}}}.$$

By Lemma 3.2.3,

$$C_{I,J} = A^{-2(2i_1'+1)+2(6-n'-\kappa_1-\kappa_2+1)+\frac{n'}{2}(3\kappa_1-\kappa_2)+\frac{1}{2}(\kappa_1-\kappa_2)^2} \begin{bmatrix} n'-2 \\ \mu' \end{bmatrix} \begin{bmatrix} i_1'+1 \\ 2 \end{bmatrix},$$

$$C_{J\cup\{i_1+1\},J} = A^{-2(i_1'+1)+(3-n'-\kappa_1-\kappa_2+1)+\frac{n'}{2}(3\kappa_1-\kappa_2)+\frac{1}{2}(\kappa_1-\kappa_2)^2} \begin{bmatrix} n'-1 \\ \mu' \end{bmatrix} [i_1'+1],$$

$$C_{I,J\cup\{i_1+1\}} = A^{-2(i_1')+(3-(n'-2)-\kappa_1-\kappa_2+1)+\frac{n'-2}{2}(3\kappa_1-\kappa_2)+\frac{1}{2}(\kappa_1-\kappa_2)^2} \begin{bmatrix} n'-3 \\ \mu'-1 \end{bmatrix} [i_1'],$$

and

$$C_{J\cup\{i_1+1\},J\cup\{i_1+1\}} = A^{\frac{n'-2}{2}(3\kappa_1-\kappa_2)+\frac{1}{2}(\kappa_1-\kappa_2)^2}\begin{bmatrix} n'-2 \\ \mu'-1 \end{bmatrix}.$$

Then,

$$\begin{aligned}
&C_{J\cup\{i_1+1\},J} \cdot \frac{C_{I,J\cup\{i_1+1\}}}{C_{J\cup\{i_1+1\},J\cup\{i_1+1\}}} - C_{I,J} \\
&= A^{-2\|I'\|+2(6-n'-\kappa_1-\kappa_2+1)+\frac{n'}{2}(3\kappa_1-\kappa_2)+\frac{1}{2}(\kappa_1-\kappa_2)^2} \\
&\quad \cdot \frac{[i_1'][i_1'+1][n'-3]!}{[2][\mu']![n'-\mu'-2]!}\left(A^{-4}[n'-1][2]-[n'-2]\right) \\
&= C_{I,J}\cdot\frac{A^{-4}[n']}{[n'-2]} = C_{I,J}\cdot A^{-2\cdot 2\cdot(2-1)}\cdot\frac{\begin{bmatrix} 2n'-3 \\ n'-2 \end{bmatrix}}{\begin{bmatrix} 2n'-3 \\ n' \end{bmatrix}}.
\end{aligned}$$

Therefore,

$$\mathcal{S}(I,J) = (-1)^2A^{-2\cdot 2\cdot(2-1)}\cdot\frac{\begin{bmatrix} 2n-4|J|-3 \\ n-2|J|-2 \end{bmatrix}C_{I,J}}{\begin{bmatrix} 2n-4|J|-3 \\ n-2|J| \end{bmatrix}C_{J,J}} = \mathcal{S}'(I,J).$$

For $i_2 > i_1+1$, one obtains

- $J \prec_F I$,
- $J \prec_F J\cup\{i_1\} \prec_F I$, or
- $J \prec_F J\cup\{i_2\} \prec_F I$.

So

$$\mathcal{S}(I,J) = -\frac{C_{I,J}}{C_{J,J}} + \frac{C_{J\cup\{i_1\},J}}{C_{J,J}}\cdot\frac{C_{I,J\cup\{i_1\}}}{C_{J\cup\{i_1\},J\cup\{i_1\}}} + \frac{C_{J\cup\{i_2\},J}}{C_{J,J}}\cdot\frac{C_{I,J\cup\{i_2\}}}{C_{J\cup\{i_2\},J\cup\{i_2\}}}.$$

By Lemma 3.2.3,

$$C_{I,J} = A^{-2(i_1'+i_2')+2(6-n'-\kappa_1-\kappa_2+1)+\frac{n'}{2}(3\kappa_1-\kappa_2)+\frac{1}{2}(\kappa_1-\kappa_2)^2}\begin{bmatrix} n'-2 \\ \mu' \end{bmatrix}[i_1'][i_2'-1],$$

$$C_{J\cup\{i_1\},J} = A^{-2(i_1')+(3-n'-\kappa_1-\kappa_2+1)+\frac{n'}{2}(3\kappa_1-\kappa_2)+\frac{1}{2}(\kappa_1-\kappa_2)^2}\begin{bmatrix} n'-1 \\ \mu' \end{bmatrix}[i_1'],$$

$$C_{I,J\cup\{i_1\}} = A^{-2(i_2'-2)+(3-(n'-2)-\kappa_1-\kappa_2+1)+\frac{n'-2}{2}(3\kappa_1-\kappa_2)+\frac{1}{2}(\kappa_1-\kappa_2)^2}\begin{bmatrix} n'-3 \\ \mu'-1 \end{bmatrix}[i_2'-2],$$

$$C_{J\cup\{i_2\},J} = A^{-2(i_2')+(3-n'-\kappa_1-\kappa_2+1)+\frac{n'}{2}(3\kappa_1-\kappa_2)+\frac{1}{2}(\kappa_1-\kappa_2)^2} \begin{bmatrix} n'-1 \\ \mu' \end{bmatrix} [i_2'],$$

$$C_{I,J\cup\{i_2\}} = A^{-2(i_1')+(3-(n'-2)-\kappa_1-\kappa_2+1)+\frac{n'-2}{2}(3\kappa_1-\kappa_2)+\frac{1}{2}(\kappa_1-\kappa_2)^2} \begin{bmatrix} n'-3 \\ \mu'-1 \end{bmatrix} [i_1'],$$

and

$$C_{J\cup\{i_1\},J\cup\{i_1\}} = C_{J\cup\{i_2\},J\cup\{i_2\}} = A^{\frac{n'-2}{2}(3\kappa_1-\kappa_2)+\frac{1}{2}(\kappa_1-\kappa_2)^2} \begin{bmatrix} n'-2 \\ \mu'-1 \end{bmatrix}.$$

Then,

$$\begin{aligned}
&C_{J\cup\{i_1\},J} \cdot \frac{C_{I,J\cup\{i_1\}}}{C_{J\cup\{i_1\},J\cup\{i_1\}}} + C_{J\cup\{i_2\},J} \cdot \frac{C_{I,J\cup\{i_2\}}}{C_{J\cup\{i_2\},J\cup\{i_2\}}} - C_{I,J} \\
&= A^{-2\|I'\|+2(6-n'-\kappa_1-\kappa_2+1)+\frac{n'}{2}(3\kappa_1-\kappa_2)+\frac{1}{2}(\kappa_1-\kappa_2)^2} [i_1'] \\
&\quad \cdot \left(\frac{\begin{bmatrix} n'-1 \\ \mu' \end{bmatrix} \begin{bmatrix} n'-3 \\ \mu'-1 \end{bmatrix} [i_2'-2] + A^{-4} \begin{bmatrix} n'-1 \\ \mu' \end{bmatrix} \begin{bmatrix} n'-3 \\ \mu'-1 \end{bmatrix} [i_2']}{\begin{bmatrix} n'-2 \\ \mu'-1 \end{bmatrix}} - \begin{bmatrix} n'-2 \\ \mu' \end{bmatrix} [i_2'-1] \right) \\
&= A^{-2\|I'\|+2(6-n'-\kappa_1-\kappa_2+1)+\frac{n'}{2}(3\kappa_1-\kappa_2)+\frac{1}{2}(\kappa_1-\kappa_2)^2} \\
&\quad \cdot \frac{[i_1'][n'-3]!}{[\mu']![n'-2-\mu']!} \left([n'-1][i_2'-2] + A^{-4}[n'-1][i_2'] - [n'-2][i_2'-1] \right) \\
&= C_{I,J} \cdot \frac{A^{-4}[n']}{[n'-2]} = C_{I,J} \cdot A^{-2\cdot 2\cdot(2-1)} \cdot \frac{\begin{bmatrix} 2n'-3 \\ n'-2 \end{bmatrix}}{\begin{bmatrix} 2n'-3 \\ n' \end{bmatrix}},
\end{aligned}$$

since

$$\begin{aligned}
&[n-1]_q\,[k-2]_q + q^{-1}[n-1]_q\,[k]_q - [n-2]_q\,[k-1]_q \\
=&[n-1]_q\,(1+q+\ldots+q^{k-3}+q^{-1}+1+q+\ldots+q^{k-2}) - [n-2]_q\,[k-1]_q \\
=&[n-1]_q\,q^{-1}\,[2]_q\,[k-1]_q - [n-2]_q\,[k-1]_q \\
=&[k-1]_q\,\left((q^{-1}+2+2q+\ldots+2q^{n-3}+q^{n-2}) - (1+q+\ldots+q^{n-3})\right) \\
=&[k-1]_q\,q^{-1}\,[n]_q.
\end{aligned}$$

Therefore,

$$\mathcal{S}(I,J) = (-1)^2 A^{-2\cdot 2\cdot(2-1)} \cdot \frac{\begin{bmatrix} 2n-4|J|-3 \\ n-2|J|-2 \end{bmatrix} C_{I,J}}{\begin{bmatrix} 2n-4|J|-3 \\ n-2|J| \end{bmatrix} C_{J,J}} = \mathcal{S}'(I,J).$$

Above discussions show that $\mathcal{S}(I,J) = \mathcal{S}'(I,J)$ for all $|I|-|J|=2$.

The cases $|I|-|J|=3$ is rather long, so its proof is given in Appendix A. $\square$

There might not be a general method for simplifying (3.10). However, according to the above calculations, we believe that it is true for all $J \preceq_F I$ and leave it here as the following open problem.

Conjecture 3.2.5. *Suppose all assumptions of Theorem 3.2.2 hold and let*

$$\mathcal{F}_{n,\kappa_1,\kappa_2,I} = \mathcal{M}_{n,\kappa_1+\kappa_2,|I|} *_v \Phi_n^{-1}(I),$$

where $\mathcal{M}_{n,k,t}$ is shown in Figure 3.17(b). Then for all $J \preceq_F I$,

$$\mathcal{S}(n,\kappa_1,\kappa_2,I,J) = (-1)^{|I|-|J|} \cdot A^{-2(|I|-|J|)(|I|-|J|-1)} \frac{\begin{bmatrix} 2(n-|I|-|J|)+1 \\ n-|I|-|J| \end{bmatrix} \left(\mathcal{R}_{n,\kappa_1,\kappa_2,|J|}, \left(\mathcal{F}_{n,\kappa_1,\kappa_2,I}\right)_J\right)}{\begin{bmatrix} 2(n-|I|-|J|)+1 \\ n-2|J| \end{bmatrix} \left(\mathcal{R}_{n,\kappa_1,\kappa_2,|J|}, \left(\mathcal{F}_{n,\kappa_1,\kappa_2,J}\right)_J\right)}.$$

Now, we list all RF-formulas for the top states of width $n = 2, 3, 4$.

Corollary 3.2.6. *The RF-formula for the top state (with a return) in $\mathbb{T}_2$ is*

$$\left[\quad, \emptyset\right] = \frac{A^2}{[2]}\left[\quad, \emptyset\right] - \frac{A^2}{[2]}\left[\quad, \{1\}\right].$$

Proof. The RF formula in this case was already found in Example 3.1.11. $\square$

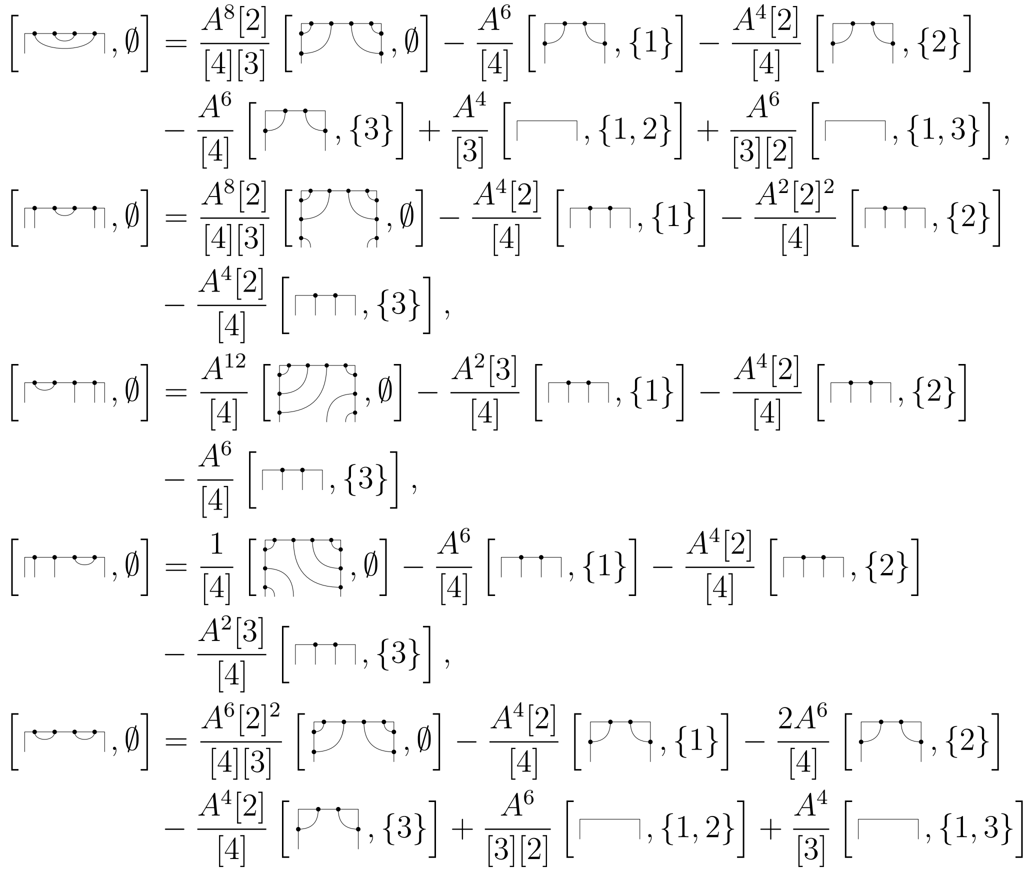
Corollary 3.2.7. *The RF-formulas for top states (with returns) in $\mathbb{T}_3$ are*

$$\left[\quad, \emptyset\right] = \frac{A^6}{[3]}\left[\quad, \emptyset\right] - \frac{A^2[2]}{[3]}\left[\quad, \{1\}\right] - \frac{A^4}{[3]}\left[\quad, \{2\}\right],$$

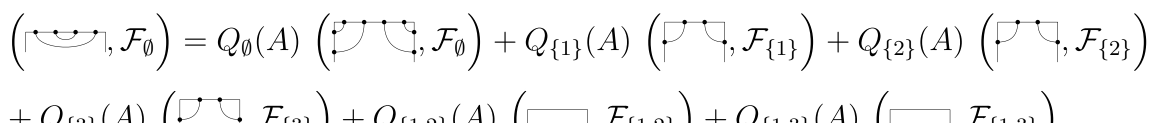
$$\left[\quad, \emptyset\right] = \frac{A^2}{[3]}\left[\quad, \emptyset\right] - \frac{A^4}{[3]}\left[\quad, \{1\}\right] - \frac{A^2[2]}{[3]}\left[\quad, \{2\}\right].$$

Proof. The RF formulas in this case were already found in Example 3.1.11. $\square$

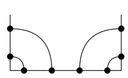
Corollary 3.2.8. *The formulas for top states (with returns) in $\mathbb{T}_4$ are*

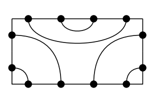
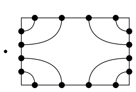


Proof. By Theorem 3.2.2,



for some $Q_I(A)$, $I \preceq_F \mathbb{V}_4$.

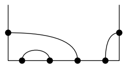
Take $\mathcal{F} =$ . Then

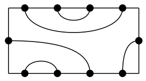
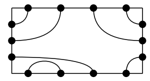
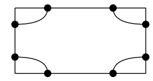
 $(A) = Q_\emptyset(A) \cdot$  (A)

or

$$1 = Q_\emptyset(A) \cdot A^{-8} \begin{bmatrix} 4 \\ 2 \end{bmatrix}.$$

Hence, $Q_\emptyset(A) = \frac{A^8[2]}{[4][3]}$.

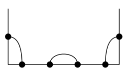
Take $\mathcal{F} =$ . Then

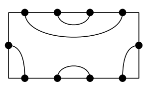
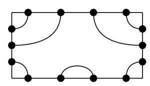
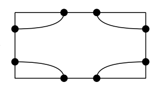
 $(A) = Q_{\emptyset}(A) \cdot$  $(A) + Q_{\{1\}}(A) \cdot$  (A)

or

$$0 = \frac{A^8[2]}{[4][3]} \cdot A^{-4}[3] + Q_{\{1\}}(A) \cdot A^{-2}[2].$$

Hence, $Q_{\{1\}}(A) = -\frac{A^6}{[4]}$.

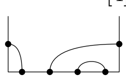
Take $\mathcal{F} =$ . Then

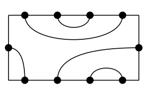
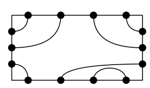
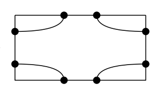
 $(A) = Q_{\emptyset}(A) \cdot$  $(A) + Q_{\{2\}}(A) \cdot$  (A)

or

$$0 = \frac{A^8[2]}{[4][3]} \cdot A^{-6}[3][2] + Q_{\{2\}}(A) \cdot A^{-2}[2].$$

Hence, $Q_{\{2\}}(A) = -\frac{A^4[2]}{[4]}$.

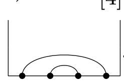
Take $\mathcal{F} =$ . Then

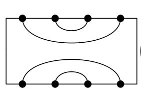
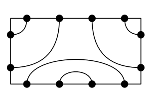
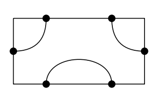
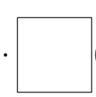
 $(A) = Q_{\emptyset}(A) \cdot$  $(A) + Q_{\{3\}}(A) \cdot$  (A)

or

$$0 = \frac{A^8[2]}{[4][3]} \cdot A^{-4}[3] + Q_{\{3\}}(A) \cdot A^{-2}[2].$$

Hence, $Q_{\{3\}}(A) = -\frac{A^6}{[4]}$.

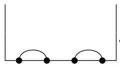
Take $\mathcal{F} =$ . Then

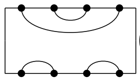
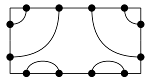
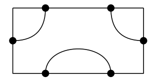
 $(A) = Q_{\emptyset}(A) \cdot$  $(A) + Q_{\{2\}}(A) \cdot$  $(A) + Q_{\{1,2\}}(A) \cdot$  (A)

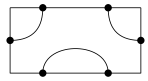
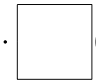
or

$$0 = \frac{A^8[2]}{[4][3]} \cdot 1 - \frac{A^4[2]}{[4]} \cdot 1 + Q_{\{1,2\}}(A) \cdot 1.$$

Hence, $Q_{\{1,2\}}(A) = \frac{A^4}{[3]}$.

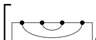
Take $\mathcal{F} =$ . Then

 $(A) = Q_{\emptyset}(A) \cdot$  $(A) + Q_{\{1\}}(A) \cdot$  (A)

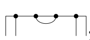
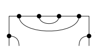
$+ Q_{\{3\}}(A) \cdot$  $(A) + Q_{\{1,3\}}(A) \cdot$  (A)

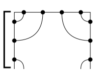
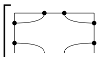
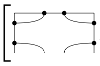
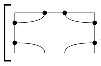
or

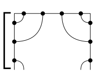
$$0 = \frac{A^8[2]}{[4][3]} \cdot A^{-2}[2] - \frac{A^6}{[4]} \cdot 1 - \frac{A^6}{[4]} \cdot 1 + Q_{\{1,3\}}(A) \cdot 1.$$

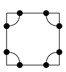
Hence, $Q_{\{1,3\}}(A) = \frac{A^6}{[3][2]}$. Therefore, the RF-formula for $\Big[$  $, \emptyset\Big]$ follows.

According to the Algorithm 3.1.12,

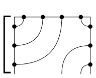
$\Big[$  $, \emptyset\Big] = \Big[$  $, \emptyset\Big]$

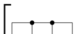
$= \frac{A^8[2]}{[4][3]} \Big[$  $, \emptyset\Big] - \frac{A^6}{[4]} \Big[$  $, \{1\}\Big] - \frac{A^4[2]}{[4]} \Big[$  $, \{2\}\Big] - \frac{A^6}{[4]} \Big[$  $, \{3\}\Big]$

$= \frac{A^8[2]}{[4][3]} \Big[$  $, \emptyset\Big] - \frac{A^4[2]}{[4]} \Big[$  $, \{1\}\Big] - \frac{A^2[2]^2}{[4]} \Big[$  $, \{2\}\Big] - \frac{A^4[2]}{[4]} \Big[$  $, \{3\}\Big]$,

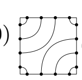
since  $(A) = A^{-2}[2]$.

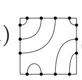
Taking $n = 4$, $k = 1$ in Lemma 3.1.10 yields, $0 \leq |I| \leq \min\{1, 4-1\} = 1$, i.e., $I = \emptyset, \{1\}, \{2\}, \{3\}$, and

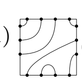
$\Big[$  $, \emptyset\Big] = Q_{4,1,\emptyset}(A) \Big[$  $, \emptyset\Big] + Q_{4,1,\{1\}}(A) \Big[$  $, \{1\}\Big] + Q_{4,1,\{2\}}(A) \Big[$  $, \{2\}\Big]$

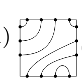
$+ Q_{4,1,\{3\}}(A) \Big[$  $, \{3\}\Big]$,

where

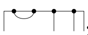
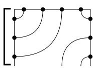
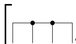
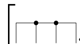
$Q_{4,1,\emptyset}(A) = A^{(4-2)(1-0)}$  $(A) = A^{-12}[4]$,

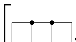
$Q_{4,1,\{1\}}(A) = A^{(4-2)(1-1)}$  $(A) = A^{-10}[3]$,

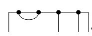
$Q_{4,1,\{2\}}(A) = A^{(4-2)(1-1)}$  $(A) = A^{-8}[2]$,

$Q_{4,1,\{3\}}(A) = A^{(4-2)(1-1)}$  $(A) = A^{-6}$.

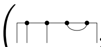
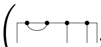
Thus,

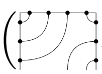
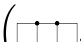
$\Big[$  $, \emptyset \Big] = \frac{1}{Q_{4,1,\emptyset}(A)} \Big[$  $, \emptyset \Big] - \frac{Q_{4,1,\{1\}}(A)}{Q_{4,1,\emptyset}(A)} \Big[$  $, \{1\} \Big] - \frac{Q_{4,1,\{2\}}(A)}{Q_{4,1,\emptyset}(A)} \Big[$  $, \{2\} \Big]$

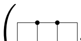
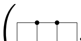
$- \frac{Q_{4,1,\{3\}}(A)}{Q_{4,1,\emptyset}(A)} \Big[$  $, \{3\} \Big],$

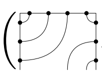
which is the RF-formula for .

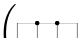
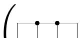
By Lemma 3.1.8, for a floor state $\mathcal{F}$:

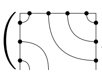
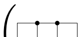
$\Big($  $, \mathcal{F} \Big)(A) = \Big($  $, \mathcal{F}^s \Big)(A^{-1})$

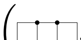
$= \frac{A^{-12}}{[4]_{q=A^{-4}}} \Big($  $, \mathcal{F}^s \Big)(A^{-1}) - \frac{A^{-2}[3]_{q=A^{-4}}}{[4]_{q=A^{-4}}} \Big($  $, (\mathcal{F}^s)_{\{1\}} \Big)(A^{-1})$

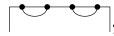
$- \frac{A^{-4}[2]_{q=A^{-4}}}{[4]_{q=A^{-4}}} \Big($  $, (\mathcal{F}^s)_{\{2\}} \Big)(A^{-1}) - \frac{A^{-6}}{[4]_{q=A^{-4}}} \Big($  $, (\mathcal{F}^s)_{\{3\}} \Big)(A^{-1})$

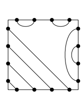
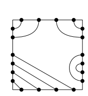
$= \frac{1}{[4]} \Big($  $, \mathcal{F}^s \Big)(A^{-1}) - \frac{A^2[3]}{[4]} \Big($  $, (\mathcal{F}_{\{3\}})^s \Big)(A^{-1})$

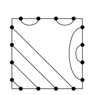
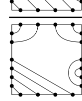
$- \frac{A^4[2]}{[4]} \Big($  $, (\mathcal{F}_{\{2\}})^s \Big)(A^{-1}) - \frac{A^6}{[4]} \Big($  $, (\mathcal{F}_{\{1\}})^s \Big)(A^{-1})$

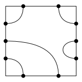
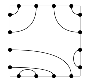
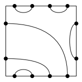
$= \frac{1}{[4]} \Big($  $, \mathcal{F} \Big)(A) - \frac{A^6}{[4]} \Big($  $, \mathcal{F}_{\{1\}} \Big)(A) - \frac{A^4[2]}{[4]} \Big($  $, \mathcal{F}_{\{2\}} \Big)(A)$

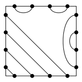
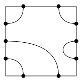
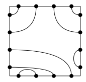
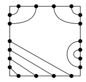
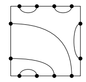
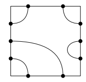
$- \frac{A^2[3]}{[4]} \Big($  $, \mathcal{F}_{\{3\}} \Big)(A).$

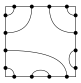
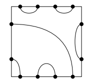
To find an RF-formula for $\mathcal{R} =$ , we use Theorem 3.2.2, Lemma 3.2.3, and Lemma 3.2.4. Since $n = 4$, $\kappa_1 = \kappa_2 = 0$, $\mathbb{V}_4 = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}\}$. Then $Q_{4,\mathcal{R},I}(A)$ can be calculated as follows.

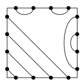
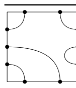
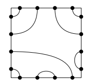
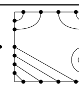
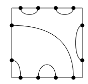
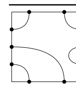
Since $(A) = [2]_{q=A^{-4}} \cdot A^{2(3+2+4+4)-4\cdot 4} = A^6[2]$ and $(A) = A^{-2\cdot 0} \begin{bmatrix} 4 \\ 2 \end{bmatrix} = \frac{[4][3]}{[2]}$,

$Q_{4,\mathcal{R},\emptyset}(A) =$  (A) /  (A) $= \frac{A^6[2]^2}{[4][3]}.$

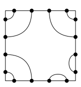
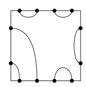
Since  $(A) = A^2 \cdot A^{-2}[2] = [2]$,  $(A) = A^{-2\cdot 1+(3-4-0-0+1)}\begin{bmatrix}4-1\\2\end{bmatrix}[1] = A^{-2}[3]$ and  $(A) = 0$, it follows that

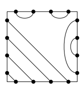
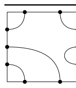
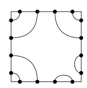
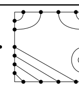
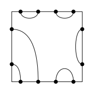
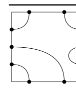
$Q_{4,\mathcal{R},\{1\}}(A) =$  (A) /  (A) $\cdot (-1)^1 A^{-2\cdot 1\cdot(1-1)}$ $\begin{bmatrix}7\\3\end{bmatrix} \cdot$  (A) / $\begin{bmatrix}7\\4\end{bmatrix} \cdot$  (A) $+$  (A) /  (A) $\cdot 1 = -\dfrac{A^4[2]}{[4]}.$

Since  $(A) = A^{-2\cdot 2+(3-4-0-0+1)}\begin{bmatrix}4-1\\2\end{bmatrix}[2] = A^{-4}[3][2]$ and  $(A) = A^2$, it follows that

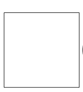
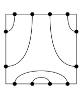
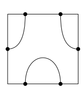
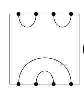
$Q_{4,\mathcal{R},\{2\}}(A) =$  (A) /  (A) $\cdot (-1)^1 A^{-2\cdot 1\cdot(1-1)}$ $\begin{bmatrix}7\\3\end{bmatrix} \cdot$  (A) / $\begin{bmatrix}7\\4\end{bmatrix} \cdot$  (A) $+$  (A) /  (A) $\cdot 1$

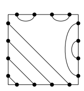
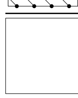
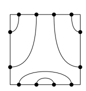
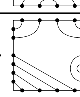
$$= -\frac{A^2[2]^2}{[4]} + \frac{A^2}{[2]} = -\frac{2A^6}{[4]}.$$

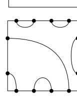
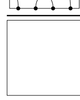
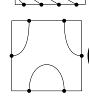
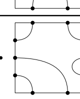
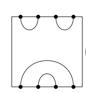
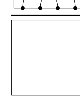
Since  $(A) = A^{-2\cdot 3+(3-4-0-0+1)}\begin{bmatrix}4-1\\2\end{bmatrix}[3] = A^{-6}[3]^2$ and  $(A) = A^2 \cdot A^{-2}[2] = [2]$, it follows that

$Q_{4,\mathcal{R},\{3\}}(A) =$  (A) /  (A) $\cdot (-1)^1 A^{-2\cdot 1\cdot(1-1)}$ $\begin{bmatrix}7\\3\end{bmatrix} \cdot$  (A) / $\begin{bmatrix}7\\4\end{bmatrix} \cdot$  (A) $+$  (A) /  (A) $\cdot 1$

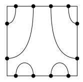
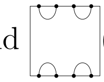
$$= -\frac{[3][2]}{[4]} + 1 = -\frac{A^4[2]}{[4]}.$$

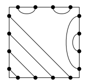
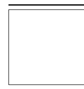
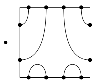
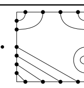
Since  $(A) =$  $(A) =$  $(A) = 1$ and  $(A) = 0$,

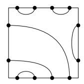
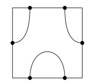
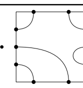
$Q_{4,\mathcal{R},\{1,2\}}(A) =$  (A) /  (A) $\cdot (-1)^2 A^{-2\cdot 2\cdot(2-1)}$ $\begin{bmatrix}5\\2\end{bmatrix} \cdot$  (A) / $\begin{bmatrix}5\\4\end{bmatrix} \cdot$  (A)

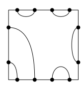
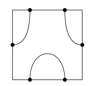
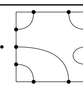
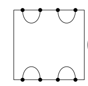
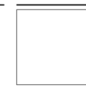
$+$  (A) /  (A) $\cdot (-1)^1 A^{-2\cdot 1\cdot(1-1)}$ $\begin{bmatrix}3\\1\end{bmatrix} \cdot$  (A) / $\begin{bmatrix}3\\2\end{bmatrix} \cdot$  (A) $+$  (A) /  (A) $\cdot 1$

$$= \frac{A^2[2]}{[3]} - \frac{A^2}{[2]} = \frac{A^6}{[3][2]}.$$

Since  $(A) = A^{-2\cdot(1+3)+2(6-4-0-0+1)} \left[\begin{smallmatrix}4-2\\2\end{smallmatrix}\right][1][3-1] = A^{-2}[2]$ and  $(A) = 0$,

$Q_{4,\mathcal{R},\{1,3\}}(A) =$  (A) /  (A) $\cdot (-1)^2 A^{-2\cdot 2\cdot(2-1)}$ $\left[\begin{smallmatrix}5\\2\end{smallmatrix}\right] \cdot$  (A) / $\left[\begin{smallmatrix}5\\4\end{smallmatrix}\right] \cdot$  (A)

$+$  (A) /  (A) $\cdot (-1)^1 A^{-2\cdot 1\cdot(1-1)}$ $\left[\begin{smallmatrix}3\\1\end{smallmatrix}\right] \cdot$  (A) / $\left[\begin{smallmatrix}3\\2\end{smallmatrix}\right] \cdot$  (A)

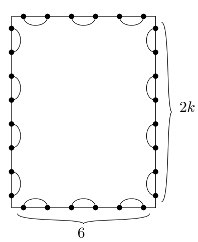
$+$  (A) /  (A) $\cdot (-1)^1 A^{-2\cdot 1\cdot(1-1)}$ $\left[\begin{smallmatrix}3\\1\end{smallmatrix}\right] \cdot$  (A) / $\left[\begin{smallmatrix}3\\2\end{smallmatrix}\right] \cdot$  (A) $+$  (A) /  (A) $\cdot 1$

$$= \frac{[2]^2}{[3]} - 1 = \frac{A^4}{[3]}.$$

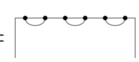
□

We finish our discussions by giving an examples of applications of results obtained in this section.

Example 3.2.9. *We find the coefficient of*

$C =$ 

$\in \mathfrak{Cat}_{2k,6}$

for $k \geq 1$. We start by finding an RF-formula for $\mathcal{R} =$ *. By Theorem 3.2.2, there are $Q_I = Q_{6,\mathcal{R},I}$'s, for $I \in \mathbb{V}_6$, such that*

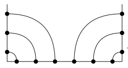
$$[\mathcal{R}, \emptyset] = \sum_{I\in\mathbb{V}_6, |I|\leq 3} Q_I(A)\,[\mathcal{R}_{6,0,0,|I|}, I],$$

where $\mathcal{R}_{n,\kappa_1,\kappa_2,t}$ is shown in Figure 3.16. Let $\hat{\mathcal{F}}$ be the floor state such that $C = \mathcal{R} *_v \hat{\mathcal{F}}$. Since $\Phi_6(\hat{\mathcal{F}}^b) = \{1,3,5\}$,

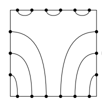
$$C(A) = [\mathcal{R}, \emptyset](\hat{\mathcal{F}}) = \sum_{I\in\mathbb{V}_6, |I|\leq 3} Q_I(A)\,[\mathcal{R}_{6,0,0,|I|}, I](\hat{\mathcal{F}}) = \sum_{I\in\mathbb{V}_6, I\preceq_F\{1,3,5\}} Q_I(A)\left(\mathcal{R}_{6,0,0,|I|}, \hat{\mathcal{F}}_I\right),$$

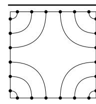
and one can easily see that $\left(\mathcal{R}_{6,0,0,3}, \hat{\mathcal{F}}_{\{1,3,5\}}\right) = 0$. *Thus, we just need to find* Q_I*'s for* $I \prec_F \{1,3,5\}$. *For brevity, let* $\mathcal{Z}(\mathcal{F}) = \sum_{I \in \mathbb{V}_6,\ I \not\prec_F \{1,3,5\}} Q_I(A)\ (\mathcal{R}_{6,0,0,3}, \mathcal{F}_I)$, *then*

$$(\mathcal{R}, \mathcal{F}) = \sum_{I \in \mathbb{V}_6, I \prec_F \{1,3,5\}} Q_I(A)\ \left(\mathcal{R}_{6,0,0,|I|}, \mathcal{F}_I\right) + \mathcal{Z}(\mathcal{F}). \tag{3.13}$$

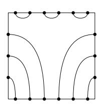
Take $\mathcal{F} =$ , *then*

$$Q_{\emptyset}(A) = \frac{\text{(diagram)}(A)}{\text{(diagram)}(A)} = \frac{A^{12}[3]^2[2]^2}{[6][5][4]},$$



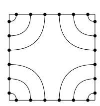


since

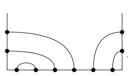


$$(A) = A^{-2(1+3+5)+3(9-6+1)} \begin{bmatrix} 6-3 \\ 3 \end{bmatrix} [1][2][3] = A^{-6}[3][2]$$

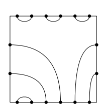
and

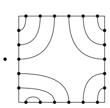


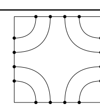
$$(A) = A^{2(6+6+6+3+3+3)-6\cdot 6} \begin{bmatrix} 6 \\ 3 \end{bmatrix}_{q=A^{-4}} = A^{-18}\frac{[6][5][4]}{[3][2]}.$$

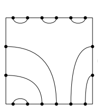
Take $\mathcal{F} =$ , *then*

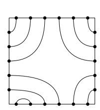
$$Q_{\{1\}}(A) = \frac{\text{(diagram)}(A) - Q_{\emptyset}(A) \cdot \text{(diagram)}(A)}{\text{(diagram)}(A)} = -\frac{A^8[3][2]^2}{[6][4]},$$





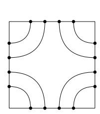


since  $(A) = 0$,

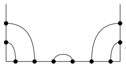


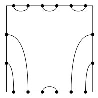
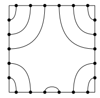
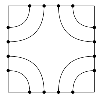
$$(A) = A^{2(5+5+5+3+3)-6\cdot 5} \begin{bmatrix} 5 \\ 2 \end{bmatrix}_{q=A^{-4}} = A^{-12}\frac{[5][4]}{[2]},$$

and

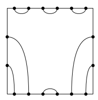


$$(A) = A^{2(4+4+2+2)-4\cdot 4} \begin{bmatrix} 4 \\ 2 \end{bmatrix}_{q=A^{-4}} = A^{-8}\frac{[4][3]}{[2]}.$$

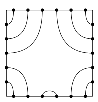
Take $\mathcal{F} =$ , *then*

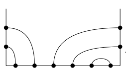
$Q_{\{3\}}(A) =$ [ $(A) - Q_{\emptyset}(A)\cdot$  (A)] / [ (A)] $= -\frac{A^{8}[2]^{2}(1+3A^{4}+A^{8})}{[6][4]},$

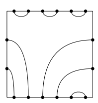
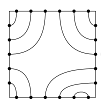
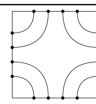
since

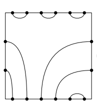
 $(A) = (A^{-2}[2])^{2} = A^{-4}[2]^{2}$

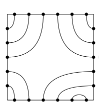
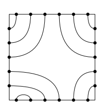
and

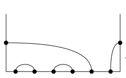
 $(A) = A^{2(6+6+5+3+3)-6\cdot 5}[3]_{q=A^{-4}}\begin{bmatrix}5\\2\end{bmatrix}_{q=A^{-4}} = A^{-16}\frac{[5][4][3]}{[2]}.$

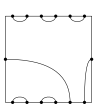
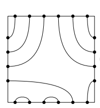
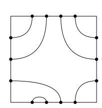
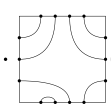
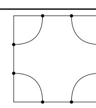
Take $\mathcal{F} =$ , *then*

$Q_{\{5\}}(A) =$ [ $(A) - Q_{\emptyset}(A)\cdot$  (A)] / [ (A)] $= -\frac{A^{8}[3][2]^{2}}{[6][4]},$

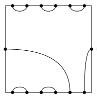
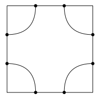
since  $(A) = 0$ *and*

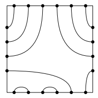
 $(A) =$  $(A^{-1}) = A^{-12}\frac{[5][4]}{[2]}.$

Take $\mathcal{F} =$ , *then*

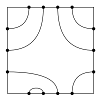
$Q_{\{1,3\}}(A) =$ [ $(A) - Q_{\emptyset}(A)\cdot$  $(A) - Q_{\{1\}}(A)\cdot$  $(A) - Q_{\{3\}}(A)\cdot$  (A)] / [ (A)]

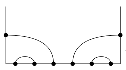
$$= \frac{0 - \frac{A^{12}[3]^{2}[2]^{2}}{[6][5][4]}\cdot A^{-8}[4][2] + \left(\frac{A^{8}[3][2]^{2}}{[6][4]} + \frac{A^{8}[2]^{2}(1+3A^{4}+A^{8})}{[6][4]}\right)\cdot A^{-4}[3]}{A^{-2}[2]} = \frac{A^{6}[2]^{4}}{[5][4]},$$

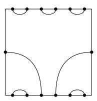
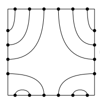
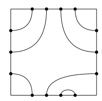
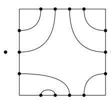
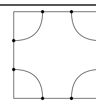
since  $(A) = 0,$  $(A) = A^{-2}[2],$

 $(A) = A^{2(5+4+4+3)-6\cdot 4}[2]_{q=A^{-4}}[4]_{q=A^{-4}} = A^{-8}[4][2],$

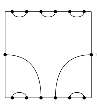
and

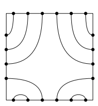
 $(A) = A^{2(3+3+2)-4\cdot 3}[3]_{q=A^{-4}} = A^{-4}[3].$

Take $\mathcal{F} =$ , *then*

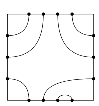
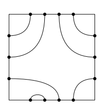
$Q_{\{1,5\}}(A) =$ ( $(A) - Q_{\emptyset}(A)\cdot$  $(A) - Q_{\{1\}}(A)\cdot$  $(A) - Q_{\{5\}}(A)\cdot$  (A)) / ( (A))

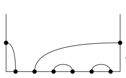
$$= \frac{0 - \frac{A^{12}[3]^2[2]^2}{[6][5][4]}\cdot A^{-8}\frac{[4][3]}{[2]} + \left(\frac{A^8[3][2]^2}{[6][4]} + \frac{A^8[3][2]^2}{[6][4]}\right)\cdot A^{-4}[3]}{A^{-2}[2]} = \frac{A^6[3][2]^2}{[5][4]},$$

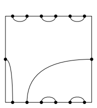
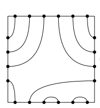
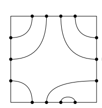
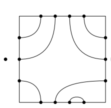
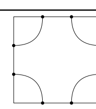
since  $(A) = 0,$

 $(A) = A^{2(5+5+3+3)-6\cdot 4}\begin{bmatrix}4\\2\end{bmatrix}_{q=A^{-4}} = A^{-8}\frac{[4][3]}{[2]},$

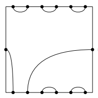
and

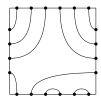
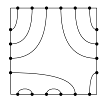
 $(A) =$  $(A^{-1}) = A^{-4}[3].$

Take $\mathcal{F} =$ , *then*

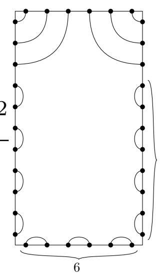
$Q_{\{3,5\}}(A) =$ ( $(A) - Q_{\emptyset}(A)\cdot$  $(A) - Q_{\{3\}}(A)\cdot$  $(A) - Q_{\{5\}}(A)\cdot$  (A)) / ( (A))

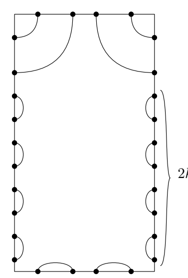
$$= \frac{0 - \frac{A^{12}[3]^2[2]^2}{[6][5][4]}\cdot A^{-8}[4][2] + \left(\frac{A^8[2]^2(1+3A^4+A^8)}{[6][4]} + \frac{A^8[3][2]^2}{[6][4]}\right)\cdot A^{-4}[3]}{A^{-2}[2]} = \frac{A^6[2]^4}{[5][4]},$$

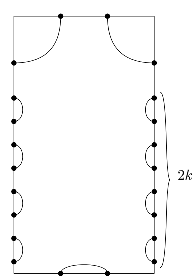
since  $(A) = 0$ *and*

 $(A) =$  $(A^{-1}) = A^{-8}[4][2].$

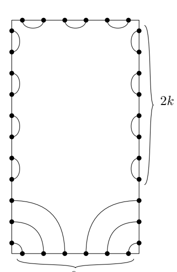
Now take $\mathcal{F} = \hat{\mathcal{F}}$ *in* (3.13), *thus*

$C(A) = \dfrac{A^{12}[3]^2[2]^2}{[6][5][4]}$  (A)

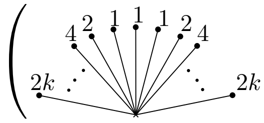
$- \left(\dfrac{A^8[3][2]^2}{[6][4]} + \dfrac{A^8[2]^2(1 + 3A^4 + A^8)}{[6][4]} + \dfrac{A^8[3][2]^2}{[6][4]} \right)$  (A)

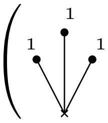
$+ \left(\dfrac{A^6[2]^4}{[5][4]} + \dfrac{A^6[3][2]^2}{[5][4]} + \dfrac{A^6[2]^4}{[5][4]} \right)$  $(A).$

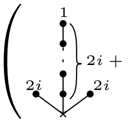
We find the coefficient of

$C' =$  $\in \mathfrak{Cat}_{2k+3,6}.$

Indeed, its plucking polynomial is

$Q(\mathcal{T}(C')) = Q\Bigg($ 

$\Bigg) = Q\Bigg($ 

$\Bigg) \cdot \prod_{i=1}^{k} Q\Bigg($ 

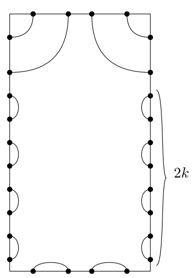
$\Bigg)$

$$= [3]_q[2]_q \cdot \prod_{i=1}^{k} \left(q^{2i-1}[4]_q[3]_q\right) = q^{k^2}[4]_q^k[3]_q^{k+1}[2]_q,$$

and its maximal sequence $\mathfrak{b}_M(C') = (5,6,5,6,\ldots,5,6,5,4,3) \Rightarrow \|\mathfrak{b}_M(C')\| = 11k+12$. *Therefore, by Theorem 2.2.7, the coefficient is*

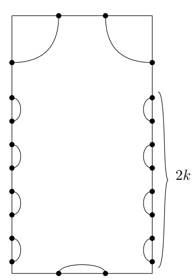
$$C'(A) = A^{2(11k+12)-6(2k+3)}[4]^k_{q=A^{-4}}[3]^{k+1}_{q=A^{-4}}[2]_{q=A^{-4}} = A^{-10k-6}[4]^k[3]^{k+1}[2].$$

Similarly, one finds



$$(A) = A^{2(3+4+3+4+\ldots+4+3+2)-4(2k+2)}[3]^k_{q=A^{-4}}[2]^{k+1}_{q=A^{-4}} = A^{-6k-2}[3]^k[2]^{k+1}$$

and



$$(A) = (A^{-2}[2])^k = A^{-2k}[2]^k.$$

Therefore, $C(A)$ *is*

$$\frac{A^{-2k+6}[2]^3}{[6][5][4]}\left(A^{-8k}[4]^k[3]^{k+3} - A^{-4k}[5][3]^k[2]^k(3+5A^4+3A^8) + [6][2]^{k-1}(3+5A^4+3A^8)\right).$$

Remark 3.2.10. *The formula for* $C(A)$ *in Example 3.2.9 is different than the one given in [8]. In particular,* $C(A)$ *in [8] was obtained directly as a rather involved recursive formula comparing to the one given in here. However,* $C(A)$ *obtained in [8] can be used to give a simple argument that shows unimodality of* $C(A)$ *(see Section 4.2 for the definition), while ours is not.*

CHAPTER 4

APPLICATIONS

In this chapter, we discuss some applications of the results obtained in the previous chapter. These results, in particular, allow us to compute coefficients of realizable Catalan states of $L(m,4)$ and study their properties. We regard it as a first step toward developing a new method for finding closed-form formulas for coefficients of realizable Catalan states of $L(m,n)$.

4.1 Coefficients of Realizable Catalan States of L(m, 4)

In this section, we find the closed-form formulas for coefficients of $L(m,n)$ when $n \leq 4$. As the cases $n=1$ or $n=2$ are both quite simple, we discuss them here only very briefly. We also note that in [9], authors analyzed coefficients of realizable Catalan states when $n=3$ and they showed that these coefficients are product of some Laurent polynomials. In this dissertation, we give closed-form formulas for them. The most interesting case is, of course, when $n=4$, which is the main result of this section.

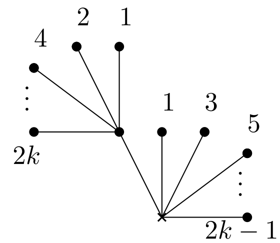
We start by finding plucking polynomials of some plane rooted trees with a delay function that are needed later.

Lemma 4.1.1. *Let $\mathcal{T}_k^{(i)}$, $i=1,2,3,4$, be plane rooted trees with a delay function shown in Figure 4.1. Then their plucking polynomials are given by*

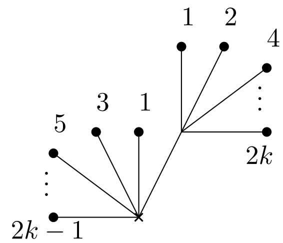
$$Q(\mathcal{T}_k^{(1)}) = q^{k^2-k}\tilde{\mathcal{B}}_k(q),\ Q(\mathcal{T}_k^{(2)}) = q^{k^2+k-1}\tilde{\mathcal{A}}_k(q),\ Q(\mathcal{T}_k^{(3)}) = q^{k^2}\tilde{\mathcal{C}}_k(q),\ Q(\mathcal{T}_k^{(4)}) = q^{k^2+2k}\tilde{\mathcal{C}}_k(q),$$

where

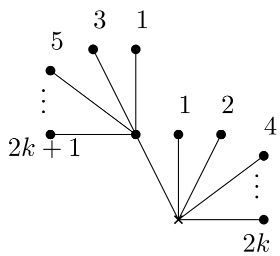
$$\tilde{\mathcal{A}}_k(q) = \frac{[2]_q^{2k}[4]_q - q^k}{[3]_q},\ \tilde{\mathcal{B}}_k(q) = \frac{[2]_q^{2k}[4]_q - q^{k+3}}{[3]_q},\ \tilde{\mathcal{C}}_k(q) = \frac{[2]_q^{2k+1}[4]_q + q^{k+2}}{[3]_q}.$$



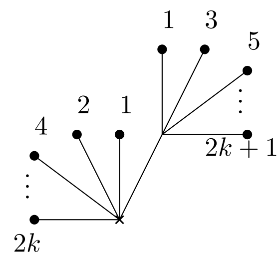
(a) $\mathcal{T}_k^{(1)}$



(b) $\mathcal{T}_k^{(2)}$



(c) $\mathcal{T}_k^{(3)}$



(d) $\mathcal{T}_k^{(4)}$

Figure 4.1. $\mathcal{T}_k^{(1)}, \mathcal{T}_k^{(2)}, \mathcal{T}_k^{(3)}, \mathcal{T}_k^{(4)}$

Proof. For brevity, let $Q_k^{(i)} = Q(\mathcal{T}_k^{(i)})$ for $i = 1, 2, 3, 4$. Then $Q_1^{(1)} = [2]_q + q\,[3]_q = 1 + 2q + q^2 + q^3$ and $Q_1^{(3)} = q([2]_q + q\,[3]_q) + q^2\,[2]_q\,[3]_q = q + 3q^2 + 3q^3 + 3q^4 + q^5$. By the definition, for $k \geq 2$:

$$
\begin{aligned}
Q_k^{(1)} &= q^{k-1}\,[2]_q\,q^{k-1}\,Q_{k-1}^{(1)} + q^k\,Q_{k-1}^{(3)} = q^{2k-2}\,[2]_q\,Q_{k-1}^{(1)} + q^k\,Q_{k-1}^{(3)},\\
Q_k^{(3)} &= q^k\,Q_k^{(1)} + q^{k+1}\,[2]_q\,q^{k-1}\,Q_{k-1}^{(3)} = q^{3k-2}\,[2]_q\,Q_{k-1}^{(1)} + q^{2k}\,(2+q)\,Q_{k-1}^{(3)}.
\end{aligned}
$$

Let $\tilde{Q}_k^{(1)} = q^{-k^2+k}\,Q_k^{(1)}$ and $\tilde{Q}_k^{(3)} = \frac{q^{-k^2}}{[2]_q}\,Q_k^{(3)}$, then the above equations become

$$
\begin{aligned}
\tilde{Q}_k^{(1)} &= (1+q)\,\tilde{Q}_{k-1}^{(1)} + q(1+q)\,\tilde{Q}_{k-1}^{(3)},\\
\tilde{Q}_k^{(3)} &= \tilde{Q}_{k-1}^{(1)} + q(2+q)\,\tilde{Q}_{k-1}^{(3)}.
\end{aligned}
$$

Notice that

$$
\begin{aligned}
\tilde{Q}_k^{(1)} - \tilde{Q}_k^{(3)} &= q(\tilde{Q}_{k-1}^{(1)} - \tilde{Q}_{k-1}^{(3)}) = \ldots = q^{k-1}(\tilde{Q}_1^{(1)} - \tilde{Q}_1^{(3)})\\
&= q^{k-1}\left((1 + 2q + q^2 + q^3) - \frac{q^{-1}(q + 3q^2 + 3q^3 + 3q^4 + q^4)}{1+q}\right)\\
&= -\frac{q^{k+2}}{1+q}
\end{aligned}
$$

and

$$\begin{aligned}\tilde{Q}_k^{(1)} + q(1+q)\,\tilde{Q}_k^{(3)} &= [2]_q^2(\tilde{Q}_{k-1}^{(1)} + q(1+q)\,\tilde{Q}_{k-1}^{(3)}) = \ldots = [2]_q^{2k-2}(\tilde{Q}_1^{(1)} + q(1+q)\,\tilde{Q}_1^{(3)}) \\ &= [2]_q^{2k-2}\left((1+2q+q^2+q^3) + q(1+q)\frac{q^{-1}(q+3q^2+3q^3+3q^4+q^4)}{1+q}\right) \\ &= [2]_q^{2k}\,[4]_q.\end{aligned}$$

Finally, we find $\tilde{Q}_k^{(1)} = \dfrac{[2]_q^{2k}\,[4]_q - q^{k+3}}{[3]_q}$ and $\tilde{Q}_k^{(3)} = \dfrac{[2]_q^{2k+1}\,[4]_q + q^{k+2}}{[3]_q[2]_q}$, and hence

$$Q_k^{(1)} = q^{k^2-k}\,\tilde{Q}_k^{(1)} = q^{k^2-k}\tilde{\mathcal{B}}_k(q),\;\; Q_k^{(3)} = q^{k^2}\,[2]_q\,\tilde{Q}_k^{(3)} = q^{k^2}\tilde{\mathcal{C}}_k(q).$$

Similarly, we can show that

$$Q(\mathcal{T}_k^{(2)}) = q^{k^2+k-1}\tilde{\mathcal{A}}_k(q),\; Q(\mathcal{T}_k^{(4)}) = q^{k^2+2k}\tilde{\mathcal{C}}_k(q).$$

$\square$

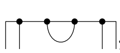
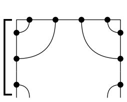
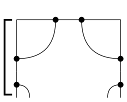
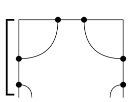
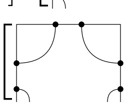
Define the Laurent polynomials

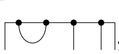
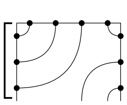
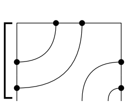
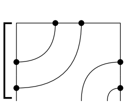
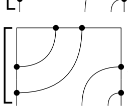
$$\mathcal{A}_k(q) = q^{-k-1}\frac{[2]_q^{2k}[4]_q - q^k}{[3]_q},\; \mathcal{B}_k(q) = q^{-k}\frac{[2]_q^{2k}[4]_q - q^{k+3}}{[3]_q},\; \mathcal{C}_k(q) = q^{-k-1}\frac{[2]_q^{2k+1}[4]_q + q^{k+2}}{[3]_q},$$

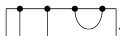
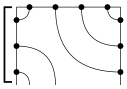
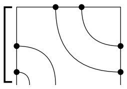
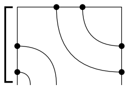
for $k \in \mathbb{Z}_{\geq 0}$.

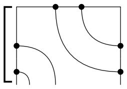
Now, we rewrite RF-formulas for cases $n=4$ as follows.

Lemma 4.1.2. *For the four top states in $\mathbb{T}_4$, RF-formulas are given by*

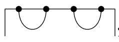
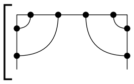
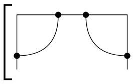
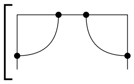
[, $\emptyset$] $= \dfrac{A^8[2]}{[4][3]}$ [, $\emptyset$] $- \dfrac{A^6}{[4]}$ [, $\{1\}$] $- \dfrac{A^4[2]}{[4]}$ [, $\{2\}$]
$- \dfrac{A^6}{[4]}$ [, $\{3\}$],

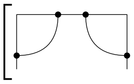
[, $\emptyset$] $= \dfrac{A^{12}}{[4]}$ [, $\emptyset$] $- \dfrac{A^6[3]}{[4]}$ [, $\{1\}$] $- \dfrac{A^8[2]}{[4]}$ [, $\{2\}$]
$- \dfrac{A^{10}}{[4]}$ [, $\{3\}$],

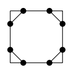
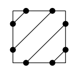
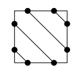
$\Big[$  $, \emptyset\Big] = \frac{1}{[4]}\Big[$  $, \emptyset\Big] - \frac{A^2}{[4]}\Big[$  $, \{1\}\Big] - \frac{[2]}{[4]}\Big[$  $, \{2\}\Big]$

$- \frac{A^{-2}[3]}{[4]}\Big[$  $, \{3\}\Big],$

and

$\Big[$  $, \emptyset\Big] = \frac{A^6[2]^2}{[4][3]}\Big[$  $, \emptyset\Big] - \frac{A^4[2]}{[4]}\Big[$  $, \{1\}\Big] - \frac{2A^6}{[4]}\Big[$  $, \{2\}\Big]$

$- \frac{A^4[2]}{[4]}\Big[$  $, \{3\}\Big] + \frac{A^6}{[3][2]}\Big[$  $, \{1,2\}\Big] + \frac{A^4}{[3]}\Big[$  $, \{1,3\}\Big].$

Proof. The above RF-formulas are a consequence of Corollary 3.2.8 after applying Theorem 2.2.10 and using the formulas  $(A) = A^{-2}[2]$,  $(A) = A^{-4}$ and  $(A) = A^4$ on the right hand sides of the first three formulas. The last formula is unchanged. □

Let $C \in \mathfrak{Cat}_{m,n}$, we call C *vertically irreducible* if $|C \cap l_i^h| < n$, for all $1 \leq i \leq m-1$. Now we state our main result:

Theorem 4.1.3. *Let C be a realizable Catalan state of $L(m,4)$. Then $C(A)$ is given by one of the following closed-form formulas:*

$$A^{2a}[2]^b[3]^c[4]^d \prod_{i=1}^{r} \mathcal{A}_{\alpha_i}(A^4) \prod_{j=1}^{s} \mathcal{B}_{\beta_j}(A^4) \prod_{k=1}^{t} \mathcal{C}_{\gamma_k}(A^4), \tag{4.1}$$

$$(a,b,c,d,r,s,t) \in \Omega_0,$$

$$\frac{A^{2a}[2]^{u+b}}{[4]} \left(\mathcal{C}_0(A^4)^b \prod_{i=1}^{r} \mathcal{A}_{\alpha_i}(A^4) \prod_{j=1}^{s} \mathcal{B}_{\beta_j}(A^4) \prod_{k=1}^{t} \mathcal{C}_{\gamma_k}(A^4) - 1 \right),$$

$$u \in \{0,1,2\},\ (a,b,r,s,t) \in \Omega_1^{(u)},$$

$$\frac{A^{2a}[2]^{b}}{[4]} \left(\mathcal{C}_0(A^4)^b \prod_{i=1}^{r} \mathcal{A}_{\alpha_i}(A^4) \prod_{j=1}^{s} \mathcal{B}_{\beta_j}(A^4) \prod_{k=1}^{t} \mathcal{C}_{\gamma_k}(A^4) - A^{-4}[3] \right),$$

$$(a,b,r,s,t) \in \Omega_2,$$

and

$$\frac{A^{2a}[2]^{u+b}}{[4]} \left(A^{-4}[2]^2\mathcal{C}_0(A^4)^b \prod_{i=1}^{r} \mathcal{A}_{\alpha_i}(A^4) \prod_{j=1}^{s} \mathcal{B}_{\beta_j}(A^4) \prod_{k=1}^{t} \mathcal{C}_{\gamma_k}(A^4) - 2 \right),$$

$$u \in \{1,2\},\ (a,b,r,s,t) \in \Omega_3^{(u)},$$

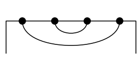
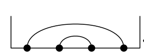
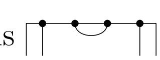
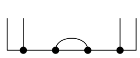
where $\alpha_i, \beta_j \in \mathbb{N}, \gamma_k \in \mathbb{Z}_{\geq 0}$, $\Omega_0 = \{(a, b, c, d, r, s, t) \in \mathbb{Z}^2 \times (\mathbb{Z}_{\geq 0})^5 \mid b \geq -\min\{c, d\}\}$, *and sets* $\Omega_1^{(u)}, \Omega_2, \Omega_3^{(u)}$ *are defined in the proof.*

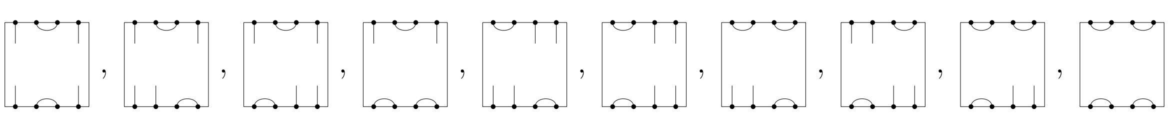
Furthermore, the converse is also true, i.e., if $P(A) \in \mathbb{Z}[A^{\pm}]$ *equals to one of the above rational functions in* A, *then there is* $m \in \mathbb{N}$ *and a realizable Catalan state* C *of* $L(m, 4)$, *such that* $C(A) = P(A)$.

Proof. Our proof is divided into six steps. In the first four steps, we prove that $C(A)$ must be given by one of the above formulas when C is vertically irreducible and has returns on both top and bottom boundaries. In the fifth step, we prove the converse of our statement in this case. Finally, in the last step, we consider all the remaining cases of realizable Catalan states C of $L(m, 4)$.

Step 1: Classification. If C is not vertically irreducible, then C can be decomposed into vertically irreducible Catalan states $C_1, C_2, \ldots, C_k$, and by Theorem 2.2.10, $C(A) = \prod_{i=1}^{k} C_i(A)$. We will consider the cases of Catalan states C with no returns on top or bottom boundary later, so now we make the following assumption:

$$\boxed{\textbf{Assumption 1. } C \text{ satisfies } |C \cap l_i^h| < 4 \text{ for all } 0 \leq i \leq m.}$$

Under our assumption, there are returns on both top and bottom boundaries, there are 5 possibilities for the top and bottom states. Moreover, we do not need to consider neither the top state  nor the bottom state . This is because a Catalan state with top (respectively bottom) state has same coefficient as the Catalan state obtained after a finite number of first-row expansions with the top state is  (respectively bottom state is ). Since $C(A) = C^{r,\pi}(A)$, so it suffices to consider the following 10 cases:



Step 2: Decomposing Catalan States into Tangles According to Theorem 3.1.16, we can simplify the Catalan state C by removing all of its arcs that connect y_i and y'_j with $|i-j| \leq 2$, since they only change $C(A)$ by a factor of $A^{\pm 2}$. Similarly, we can remove arcs of two endpoints $\{x_2, y'_1\}$, $\{x_3, y_1\}$, $\{x'_2, y'_m\}$, or $\{x'_3, y_m\}$ according to Lemma 4.1.2 and Theorem 3.1.16. Therefore, we can make the following assumption:

Assumption 2. No arcs of C that connect y_i and y'_j with $|i-j| \leq 2$, or $\{x_2, y'_1\}$, $\{x_3, y_1\}$, $\{x'_2, y'_m\}$, $\{x'_3, y_m\}$.

By Lemma 4.1.2, to find $C(A)$ for a Catalan state C that satisfies both Assumptions 1 and 2, it is sufficient to find $C'(A)$ for some Catalan states C' (those which appear on the right hand side of the RF-formulas given in the lemma) whose roof and floor states are shown in the first column of Table 4.1. Hence, we consider the Catalan state $C^\star$ on the right hand side of the RF-formula that has the same width as C and it satisfies Assumption 2 and additionally

Assumption 3. For all $1 \leq i \leq m-1$, $|C^\star \cap l_i^h| < 4$ and the roof and floor states of $C^\star$ are shown in the first column of Table 4.1.

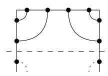
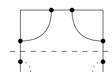
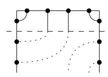
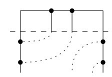
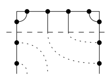
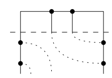
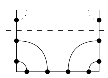
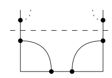
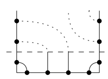
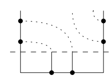
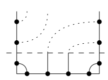
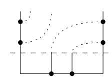
An arc a of a Catalan state C is called a *long arc* if its endpoints $\{y_i, y_j\}$, $\{y_i, y'_j\}$, or $\{y'_i, y'_j\}$ satisfy $|i-j| > 2$. Let $i_h(a) = \min\{i, j\}$ and $i_l(a) = \max\{i, j\}$.

Algorithm 4.1.4 gives a decomposition of $C^\star$ after removing its roof and floor states (shown in the first column of Table 4.1) into tangles shown in Table 4.2 and then it outputs a word on the alphabet

$$\mathbb{A} = \{N_0, N_1, \ldots, N_6, M_1^{(k+1)}, M_2^{(k+1)}, M_3^{(k)}, M_4^{(k)}, E\}_{k \in \mathbb{Z}_{\geq 0}}.$$

Figure 4.2 gives examples of decomposition of Catalan states $C^\star$ after removing their corresponding roof and floor states shown in the first column in Table 4.1. Table 4.2 shows

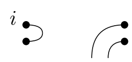
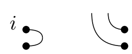
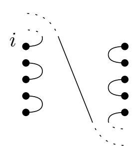
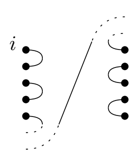
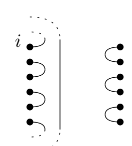
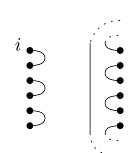
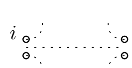
Table 4.1. Contributions of roof and floor states to $C'(A)$

Tangle	$(b_1, b_2, \ldots)$	$Q(\cdot)$	$C'(\cdot)$	Tangle	$(b_1, b_2, \ldots)$	$Q(\cdot)$	$C'(\cdot)$
	$(4,4,2)$	$\frac{[4]_q[3]_q}{[2]_q}$	$A^{-8}\frac{[4][3]}{[2]}$		$(2,1)$	$[2]_q$	$A^{-2}[2]$
	$(4,1)$	$[4]_q$	$A^{-10}[4]$		(0)	1	A^{-2}
	$(4,4)$	$[4]_q$	$A^{-4}[4]$		(2)	1	A^{2}
	N/A	N/A	1		N/A	N/A	1
	N/A	N/A	1		N/A	N/A	A^{2}
	N/A	N/A	A^{-6}		N/A	N/A	A^{-2}

Algorithm 4.1.4 Decompose $C^\star \in \mathfrak{Cat}_{m,4}$

procedure DecomposeCatalanState($C^\star$)
    Remove roof and floor states shown in the first column of Table 4.1 from $C^\star$
    for $k \leftarrow 2$ to $m-2$ **do**
        if $|C^\star \cap l_k| = 0$ **then** ▷ A virtual tangle
            Split $C^\star$ along l_k^h and assign E between the two tangles determined by l_k^h
        else if l_k^h intersects a long arc a of $C^\star$ **then**
            if $k - i_h(a) == 1$ **or** $i_l(a) - k == 2$ **then**
                Split $C^\star$ along l_k^h
            if $k - i_h(a) == 1$ **and** $i_l(a) - k == 2$ **then** ▷ Another virtual tangle
                Assign E' between the two tangles determined by l_k^h
    Replace each tangle by a letter according to Table 4.2
    Replace E' by $M_3^{(0)}$ if N_1 or N_3 appears before E'
    Replace E' by $M_4^{(0)}$ if N_2 or N_4 appears before E'
    return The word formed by concatenating letters starting from the top to the bottom

Table 4.2. Contributions of tangles to $C'(A)$

Label	Tangle	$n=4$			$n=2$		
		$(b_{i+1}, b_{i+2}, \ldots)$	$Q^*(\cdot)$	$\mathcal{P}_4(\cdot)$	$(b_{i+1}, b_{i+2}, \ldots)$	$Q^*(\cdot)$	$\mathcal{P}_2(\cdot)$
N_0		$(4,3)$	$[3]_q[2]_q$	$A^{-6}[3][2]$	$(2,1)$	$[2]_q$	$A^{-2}[2]$
N_1		$(4,3)$	$[2]_q$	$A^2[2]$	$(2,0)$	1	1
N_2		$(2,4)$	$[2]_q$	$[2]$	$(0,2)$	1	1
N_3		$(4,3)$	1	A^6	$(2,0)$	1	1
N_4		$(1,4)$	1	A^2	$(0,2)$	1	1
N_5		$(4,3)$	$[3]_q$	$A^{-2}[3]$	$(2,1)$	$[2]_q$	$A^{-2}[2]$
N_6		$(4,2)$	$[3]_q$	$A^{-4}[3]$	$(2,1)$	$[2]_q$	$A^{-2}[2]$
$M_1^{(k)}$		$(\underbrace{4,2,\ldots,4}_{2k-1})$	$\tilde{\mathcal{A}}_k(q)$	$A^{-4}\mathcal{B}_k(A^4)$	$(\underbrace{2,0,\ldots,2}_{2k-1})$	1	A^2
$M_2^{(k)}$		$(\underbrace{2,4,\ldots,2}_{2k-1})$	$\tilde{\mathcal{B}}_k(q)$	$A^{-4}\mathcal{A}_k(A^4)$	$(\underbrace{0,2,\ldots,0}_{2k-1})$	1	A^{-2}
$M_3^{(k)}$		$(\underbrace{4,2,\ldots,2}_{2k})$	$\tilde{\mathcal{C}}_k(q)$	$A^{-4}\mathcal{C}_k(A^4)$	$(\underbrace{2,0,\ldots,0}_{2k})$	1	1
$M_4^{(k)}$		$(\underbrace{2,4,\ldots,4}_{2k})$	$\tilde{\mathcal{C}}_k(q)$	$A^{-4}\mathcal{C}_k(A^4)$	$(\underbrace{0,2,,\ldots,2}_{2k})$	1	1
E		$(\ )$	1	1	$(\ )$	1	1

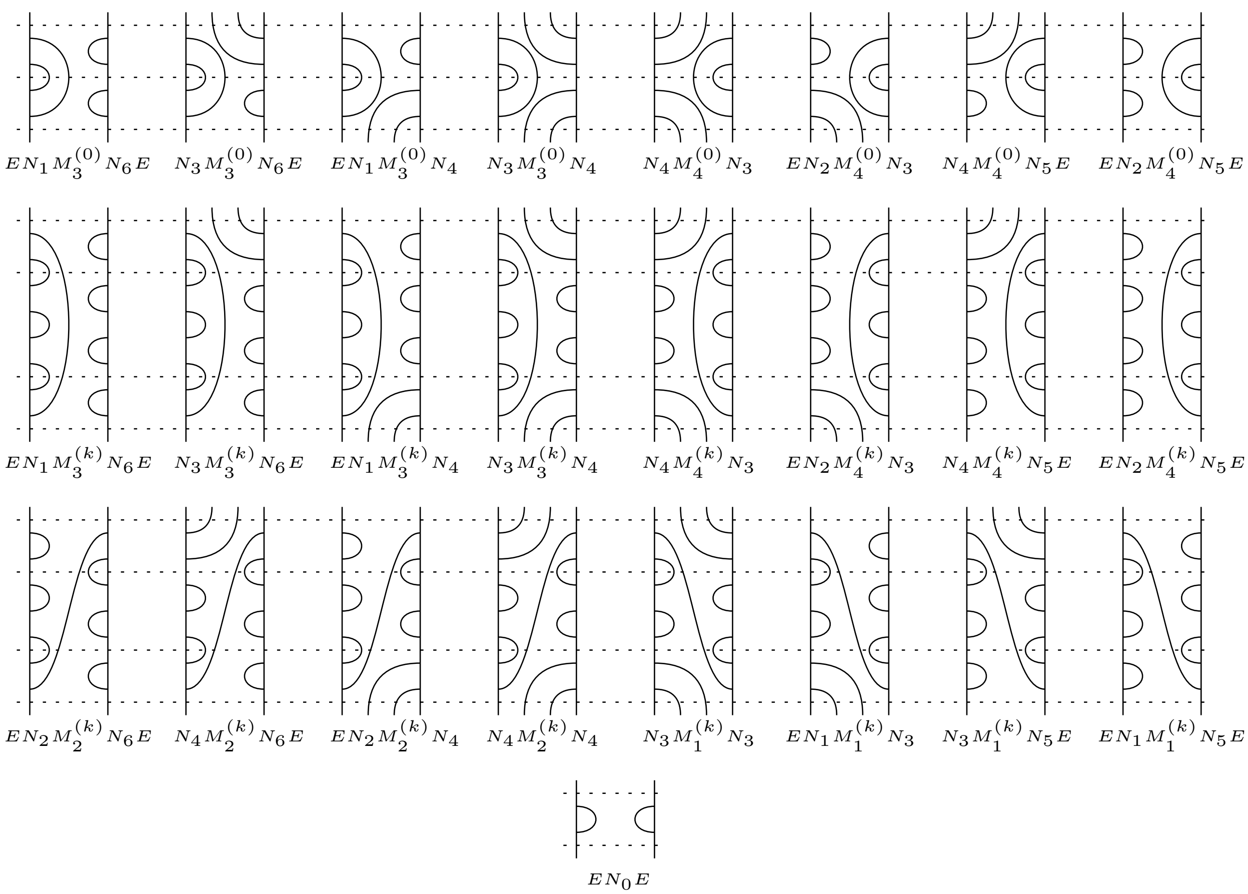


Figure 4.2. Decomposing Catalan states by Algorithm 4.1.4

contribution of each tangle to the plucking polynomial of $\mathcal{T}(C')$ and to the maximal sequence $\mathfrak{b}_M(C')$. Let $Q^*(\cdot) = q^{-\operatorname{mindeg}_q(Q)}Q(\cdot)$, where Q is the plucking polynomial (see Definition 2.2.6). Using Theorem 2.2.7, for $T \in \mathbb{A}$ of height h that is a subtangle of C' of width n, we define the Laurent polynomial

$$\mathcal{P}_n(T) = A^{2(\sum_{j=1}^{h} b_{i+j})-nh} Q^*(\mathcal{T}_n)\big|_{q=A^{-4}},$$

where $\mathcal{T}_n$ is the plane rooted tree with a delay function corresponding to T. Polynomials $\mathcal{P}_n(T)$ are the building blocks of coefficient $C'(A)$ as we will see it later.

Consider a Catalan state $C^\star$ with $M_1^{(k)}$ as its subtangle. Figure 4.3 shows $\mathcal{T}(C^\star)$ and the subsequence of $\mathfrak{b}_M(C^\star)$ corresponding to $M_1^{(k)}$. Using Lemma 3.1.14 and Lemma 4.1.1,

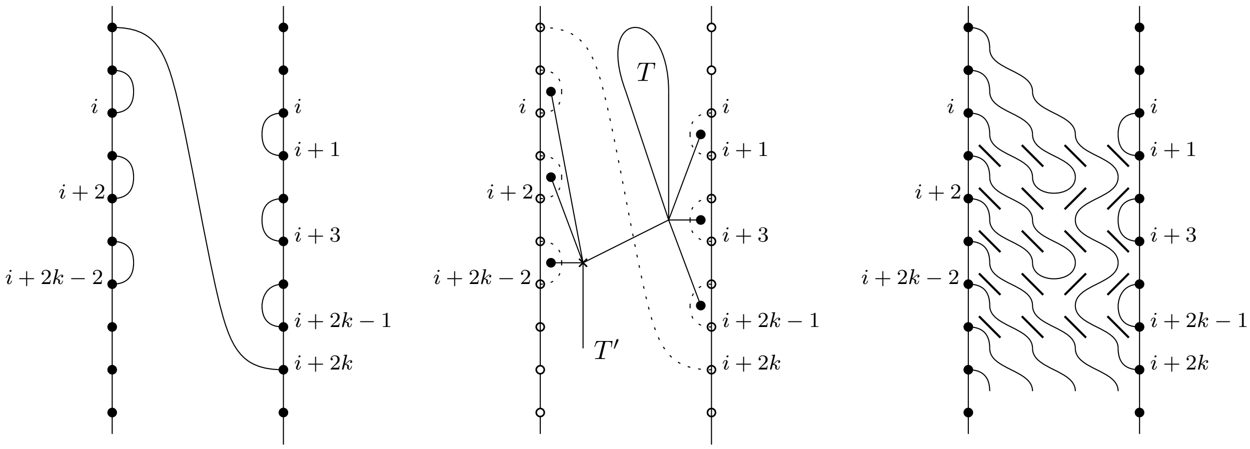
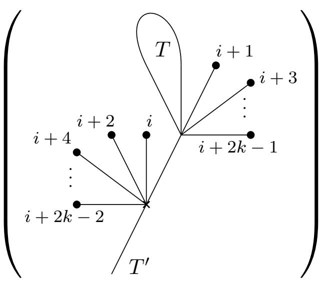


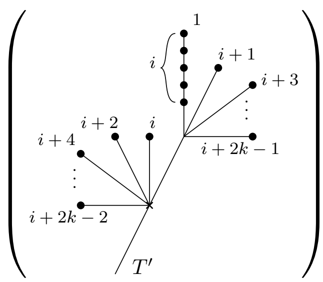
Figure 4.3. $\mathcal{T}(C^\star)$ and subsequence of $\mathfrak{b}_M(C^\star)$ corresponding to $M_1^{(k)}$

since $|V(T)| = i+1$, the plucking polynomial for a Catalan state $C^\star$ is

$$Q$$



$$= Q(T) \cdot Q$$



$$= Q(T) \cdot q^{k(i-1)}\, Q\left(T_k^{(2)}\right) \cdot Q(T'') = Q(T) \cdot q^{k(i-1)+k^2+k-1} \tilde{\mathcal{A}}_k(q) \cdot Q(T''),$$

where $T_k^{(2)}$ is the tree shown in Figure 4.1 and T'' is obtained by appending T' by a simple path described in Lemma 3.1.14.

Words on the alphabet $\mathbb{A} = \{N_0, N_1, \ldots, N_6, M_1^{(k+1)}, M_2^{(k+1)}, M_3^{(k)}, M_4^{(k)}, E\}_{k \in \mathbb{Z}_{\geq 0}}$ that are obtained from Algorithm 4.1.4 are not arbitrary, but rather they satisfy certain rules. Namely, such words are walks in the directed graph G shown in Figure 4.4 starting and ending at vertices N_3, N_4 or E. We also note that $M_3^{(0)}$ and $M_4^{(0)}$ were introduced in Algorithm 4.1.4 for the convenience of our description of outputs of Algorithm 4.1.4 by the graph in Figure 4.4. Furthermore, we also would like to add that although $M_3^{(0)}$ and $M_4^{(0)}$ have no contribution to $\mathfrak{b}_M(C^\star)$, they still both contribute to the plucking polynomial of $\mathcal{T}(C^\star)$ by a factor $\tilde{\mathcal{C}}_0(q) = [3]_q$.

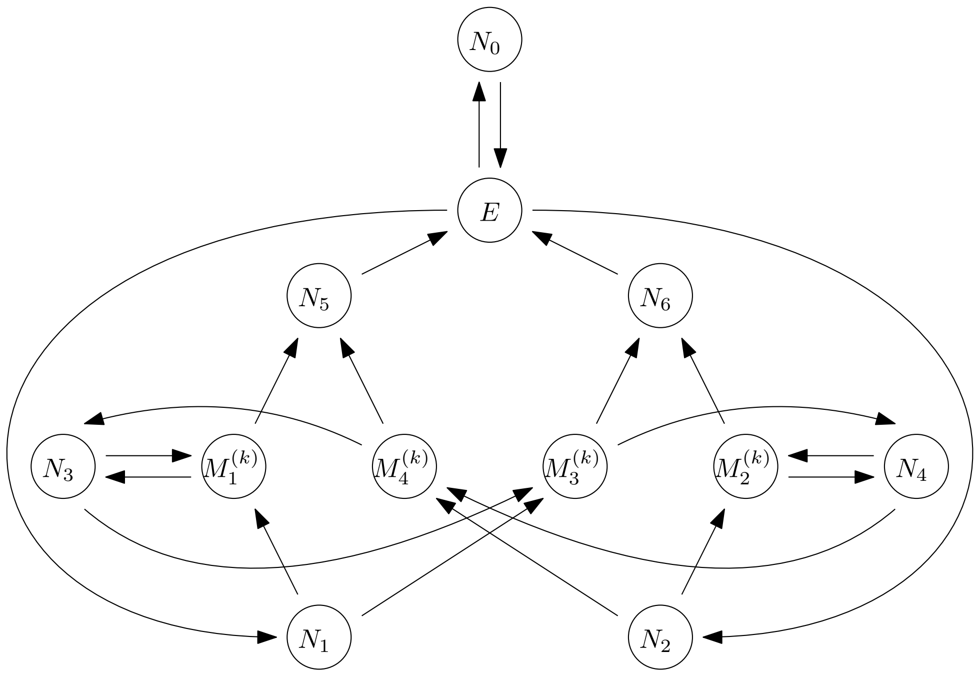
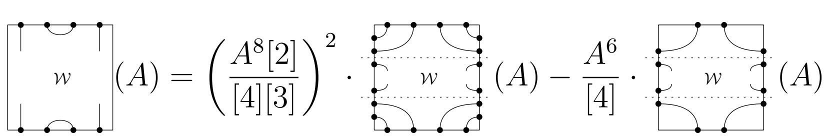


Figure 4.4. Directed graph G for words on $\mathbb{A}$

Step 3: Expressing $C(A)$ in terms of $\mathcal{P}_n(\mathcal{W})$. Recall, the function $\mathcal{P}_n : \mathbb{A} \to \mathbb{Z}[A^{\pm 1}]$ assigns to each letter of $\mathbb{A}$ a unique Laurent polynomial according to Table 4.2 for $n = 2, 4$. We can extend $\mathcal{P}_n$ to the set of non-empty words $\mathbb{W} = \{\mathcal{W} = a_1 a_2 \ldots a_l \mid a_i \in \mathbb{A}\}$ on alphabet $\mathbb{A}$ recursively as follows: $\mathcal{P}_n(\mathcal{W}_1\mathcal{W}_2) = \mathcal{P}_n(\mathcal{W}_1) \cdot \mathcal{P}_n(\mathcal{W}_2)$ for $n = 2, 4$ and $\mathcal{W}_1, \mathcal{W}_2 \in \mathbb{W}$. Let $\mathcal{W}_{v_0 \to v_k}$ be a walk in G starting at vertex v_0 and ending at vertex v_k.

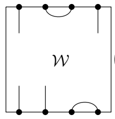
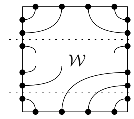
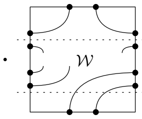
Using Lemma 4.1.2, Steps 2, and Table 4.1 for the roof and floor states, we obtain the following formulas.

(1)



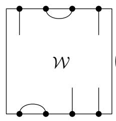
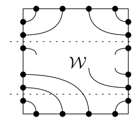
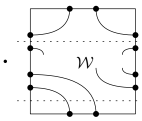
$$= \frac{A^8[2]}{[4]}\left(\frac{\mathcal{P}_4(\mathcal{W}_{E\to E})}{[3]} - A^{-4}\,\mathcal{P}_2(\mathcal{W}_{E\to E})\right).$$

(2)

 $(A) = \frac{A^{12}}{[4]} \cdot \frac{A^8[2]}{[4][3]} \cdot$  $(A) - \frac{A^{10}}{[4]} \cdot$  (A)

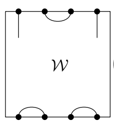
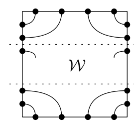
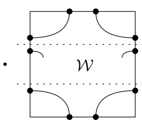
$$= \frac{A^6}{[4]} \left(\mathcal{P}_4(\mathcal{W}_{E\to N_4}) - [2]\,\mathcal{P}_2(\mathcal{W}_{E\to N_4})\right).$$

(3)

 $(A) = \frac{1}{[4]} \cdot \frac{A^8[2]}{[4][3]} \cdot$  $(A) - \frac{A^2}{[4]} \cdot$  (A)

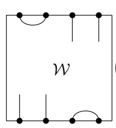
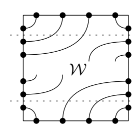
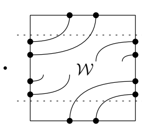
$$= \frac{1}{[4]} \big(\mathcal{P}_4(\mathcal{W}_{E\to N_3}) - A^2[2]\,\mathcal{P}_2(\mathcal{W}_{E\to N_3})\big).$$

(4)

 $(A) = \frac{A^6[2]^2}{[4][3]} \cdot \frac{A^8[2]}{[4][3]} \cdot$  $(A) - \frac{2A^8}{[4][2]} \cdot$  (A)

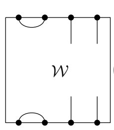
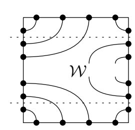
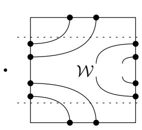
$$= \frac{A^6}{[4]} \left(\frac{[2]^2\,\mathcal{P}_4(\mathcal{W}_{E\to E})}{[3]} - 2\,\mathcal{P}_2(\mathcal{W}_{E\to E})\right).$$

(5)

 $(A) = \left(\frac{A^{12}}{[4]}\right)^2 \cdot$  $(A) - \frac{A^{14}}{[4]} \cdot$  (A)

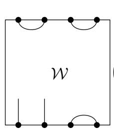
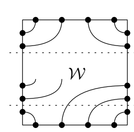
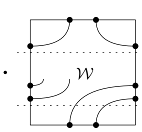
$$= \frac{A^8}{[4]} \big(\mathcal{P}_4(\mathcal{W}_{N_4\to N_4}) - A^2\,\mathcal{P}_2(\mathcal{W}_{N_4\to N_4})\big).$$

(6)

 $(A) = \frac{A^{12}}{[4]} \cdot \frac{1}{[4]} \cdot$  $(A) - \frac{A^2[3]}{[4]} \cdot$  (A)

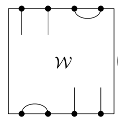
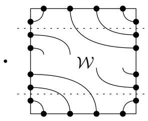
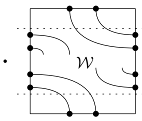
$$= \frac{A^2}{[4]} \left(\mathcal{P}_4(\mathcal{W}_{N_4\to N_3}) - [3]\,\mathcal{P}_2(\mathcal{W}_{N_4\to N_3})\right).$$

(7)

 $(A) = \frac{A^{12}}{[4]} \cdot \frac{A^6[2]^2}{[4][3]} \cdot$  $(A) - \frac{A^8[2]}{[4]} \cdot$  (A)

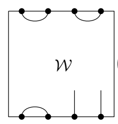
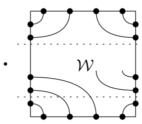
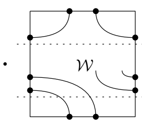
$$= \frac{A^4[2]}{[4]} \left(\mathcal{P}_4(\mathcal{W}_{E\to N_4}) - [2]\,\mathcal{P}_2(\mathcal{W}_{E\to N_4})\right).$$

(8)

 $(A) = \left(\frac{1}{[4]}\right)^2 \cdot$  $(A) - \frac{A^{-2}}{[4]} \cdot$  (A)

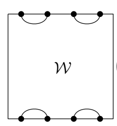
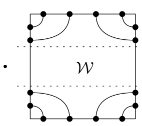
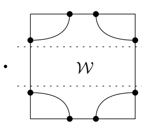
$$= \frac{A^{-4}}{[4]} \big(\mathcal{P}_4(\mathcal{W}_{N_3 \to N_3}) - A^6\, \mathcal{P}_2(\mathcal{W}_{N_3 \to N_3})\big)\,.$$

(9)

 $(A) = \frac{1}{[4]} \cdot \frac{A^6[2]^2}{[4][3]} \cdot$  $(A) - \frac{[2]}{[4]} \cdot$  (A)

$$= \frac{A^{-2}[2]}{[4]} \big(\mathcal{P}_4(\mathcal{W}_{E \to N_3}) - A^2[2]\, \mathcal{P}_2(\mathcal{W}_{E \to N_3})\big)\,.$$

(10)

 $(A) = \left(\frac{A^6[2]^2}{[4][3]}\right)^2 \cdot$  $(A) - \frac{2A^6}{[4]} \cdot$  (A)

$$= \frac{A^4[2]}{[4]} \left(\frac{[2]^2\, \mathcal{P}_4(\mathcal{W}_{E \to E})}{[3]} - 2\, \mathcal{P}_2(\mathcal{W}_{E \to E})\right).$$

Step 4: Weights Adjustment. Formulas in Step 3 are rather inconvenient to use, so we will adjust weights $\mathcal{P}_n(v)$, $v \in V(G)$ to make the computation of $C(A)$ simpler. Such an adjustment cannot be arbitrary, but it needs to follow some rules determined by G. For instance, we notice that, if $v_i = M_1^{(k)}$ or $M_3^{(k)}$ is a vertex of $\mathcal{W} = v_0 v_1 \dots v_l$ then v_{i-1} must be N_1 or N_3. Therefore, when weights of $M_1^{(k)}$ and $M_3^{(k)}$ are multiplied by A^4, then weights of N_1 and N_3 need to be multiplied by A^{-4}. Therefore, we make an adjustment of weights $\mathcal{P}_n(v)$, $v \in V(G)$ so that, after this modification, the total weight of a directed walk $\mathcal{W}$ in G is preserved if the weights of its first and last vertices are not changed. However, if weights of these two vertices are modified, we can multiply the new weight of $\mathcal{W}$ by some factor (depending only on these two vertices) to preserve the total weight of $\mathcal{W}$. Figure 4.5 and Figure 4.6 show the adjustments of weights for each $v \in V(G)$, and Table 4.3 gives the new

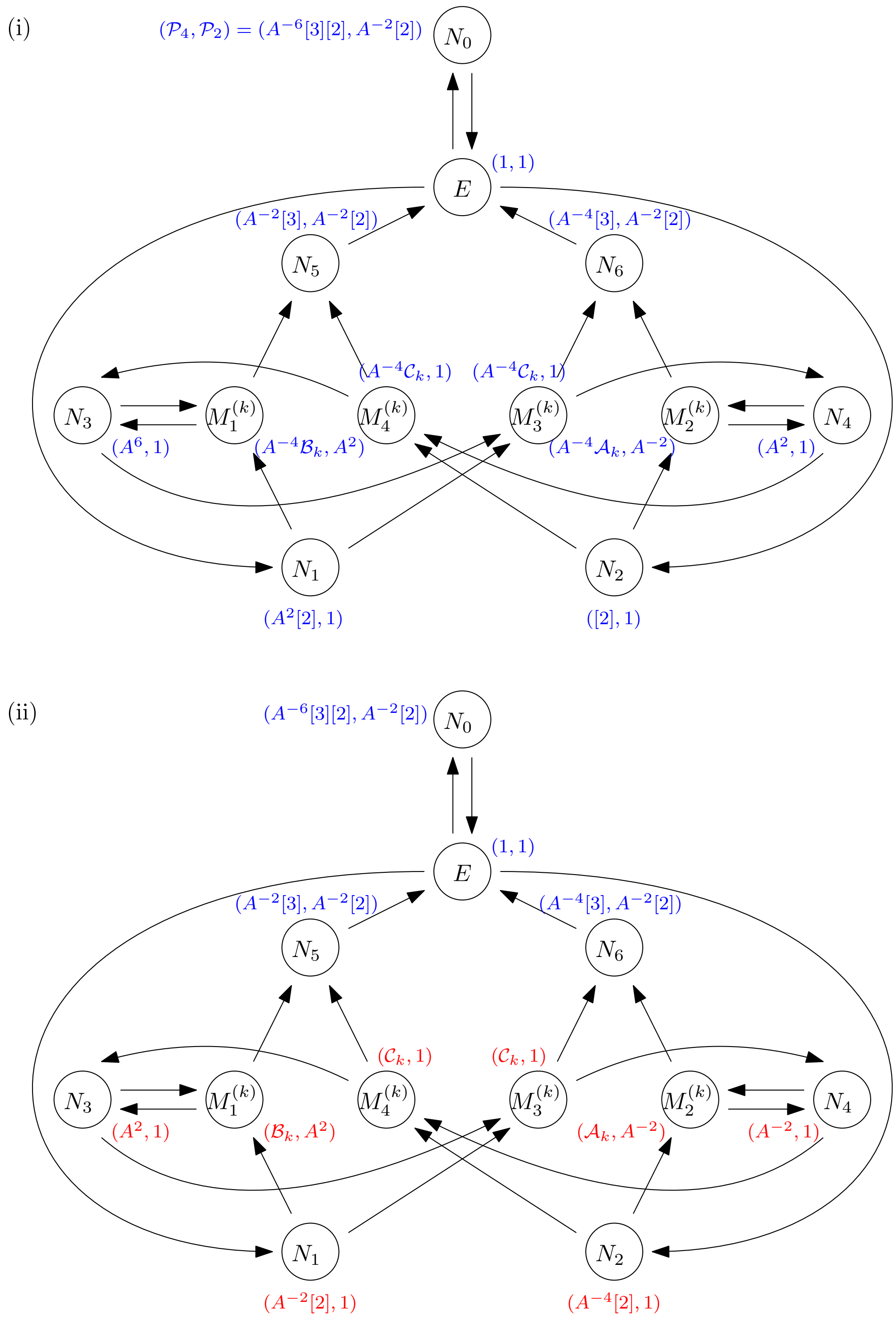
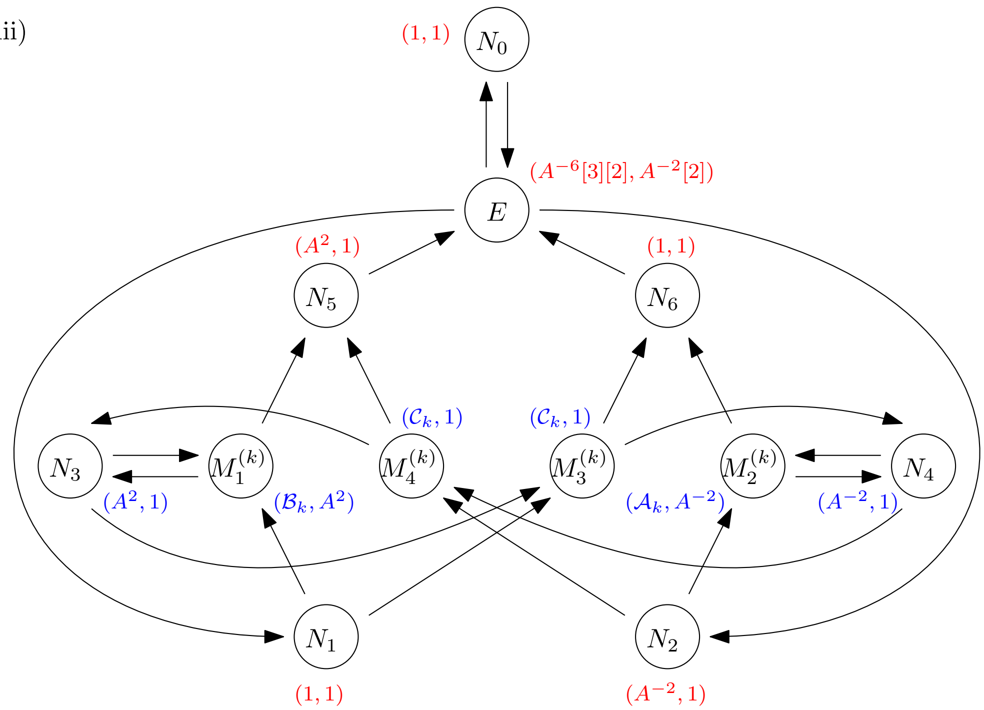


Figure 4.5. Weights adjustments on G

(iii)



(iv)

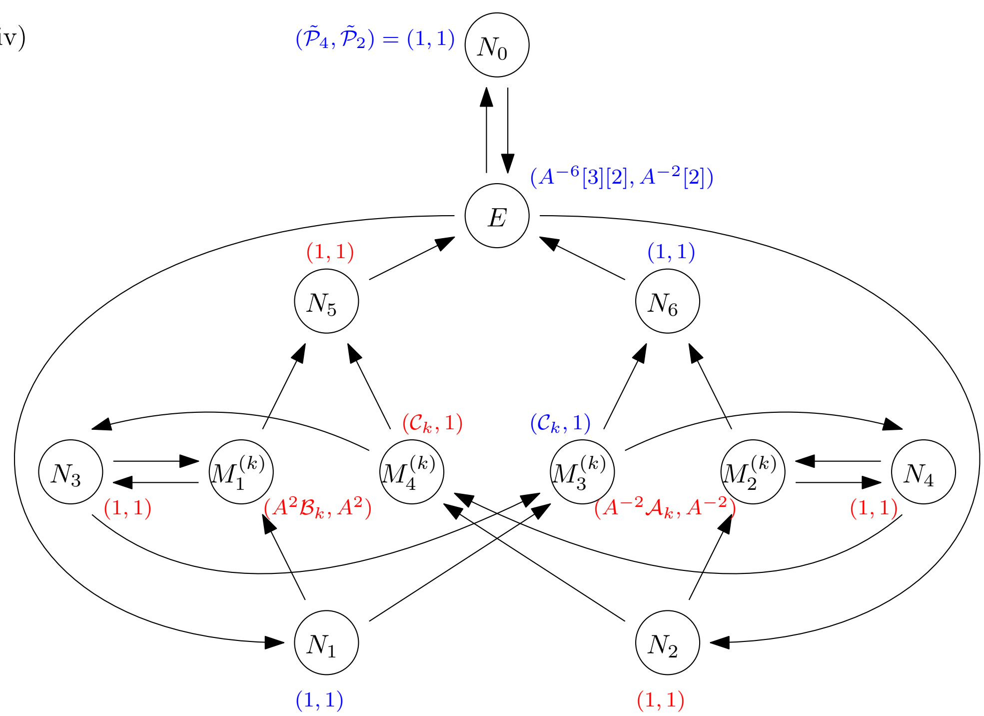


Figure 4.6. Weights adjustments on G (continued)

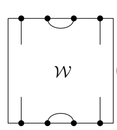
Table 4.3. Correction factors on walks

Name	$\tilde{\mathcal{P}}_4(\cdot)$	$\tilde{\mathcal{P}}_2(\cdot)$
N_i for $i = 0, 1, \ldots, 6$	1	1
$M_1^{(k)}$	$A^2 \mathcal{B}_k(A^4)$	A^2
$M_2^{(k)}$	$A^{-2} \mathcal{A}_k(A^4)$	A^{-2}
$M_3^{(k)}$	$\mathcal{C}_k(A^4)$	1
$M_4^{(k)}$	$\mathcal{C}_k(A^4)$	1
E	$A^{-6}[3][2]$	$A^{-2}[2]$
Correction factor on $N_3 \to$	A^2	1
Correction factor on $N_4 \to$	1	1
Correction factor on $E \to$	$A^4/[3]$	$A^2/[2]$
Correction factor on $\to N_3$	A^4	1
Correction factor on $\to N_4$	A^2	1
Correction factor on $\to E$	$A^2/[2]$	1

values of weights together with the some correction factors for N_3, N_4, E when these vertices are the first or last in a walk.

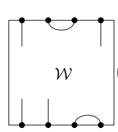
Let $\mathcal{W} = v_0 v_1 \ldots v_l$ and assume that $l \geq 1$. Denote by $\mathcal{W}^*$ the walk obtained from $\mathcal{W}$ by removing its first and last vertex, i.e., $\mathcal{W}^* = v_1 v_2 \ldots v_{l-1}$. Then the formulas in Step 3 can be written as:

(1)



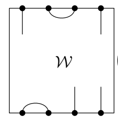
$$(A) = \frac{A^8[2]}{[4]} \left(\frac{A^6\, \tilde{\mathcal{P}}_4(\mathcal{W}_{E \to E})}{[3]^2\,[2]} - \frac{A^{-2}\, \tilde{\mathcal{P}}_2(\mathcal{W}_{E \to E})}{[2]} \right)$$
$$= \frac{A^2[2]^2}{[4]} \left(\tilde{\mathcal{P}}_4(\mathcal{W}^*_{E \to E}) - \tilde{\mathcal{P}}_2(\mathcal{W}^*_{E \to E}) \right).$$

(2)



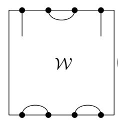
$$(A) = \frac{A^6}{[4]} \left(\frac{A^6\, \tilde{\mathcal{P}}_4(\mathcal{W}_{E \to N_4})}{[3]} - A^2\, \tilde{\mathcal{P}}_2(\mathcal{W}_{E \to N_4}) \right)$$
$$= \frac{A^6[2]}{[4]} \left(\tilde{\mathcal{P}}_4(\mathcal{W}^*_{E \to N_4}) - \tilde{\mathcal{P}}_2(\mathcal{W}^*_{E \to N_4}) \right).$$

(3)



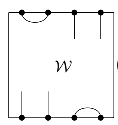
$$
\begin{aligned}
(A) &= \frac{1}{[4]}\left(\frac{A^8\,\tilde{\mathcal{P}}_4(\mathcal{W}_{E\to N_3})}{[3]} - A^4\,\tilde{\mathcal{P}}_2(\mathcal{W}_{E\to N_3})\right)\\
&= \frac{A^2[2]}{[4]}\left(\tilde{\mathcal{P}}_4(\mathcal{W}^*_{E\to N_3}) - \tilde{\mathcal{P}}_2(\mathcal{W}^*_{E\to N_3})\right).
\end{aligned}
$$

(4)



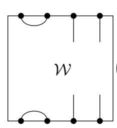
$$
\begin{aligned}
(A) &= \frac{A^6}{[4]}\left(\frac{A^6\,[2]\,\tilde{\mathcal{P}}_4(\mathcal{W}_{E\to E})}{[3]^2} - \frac{2A^2\,\tilde{\mathcal{P}}_2(\mathcal{W}_{E\to E})}{[2]}\right)\\
&= \frac{A^4[2]}{[4]}\left(A^{-4}\,[2]^2\,\tilde{\mathcal{P}}_4(\mathcal{W}^*_{E\to E}) - 2\,\tilde{\mathcal{P}}_2(\mathcal{W}^*_{E\to E})\right).
\end{aligned}
$$

(5)



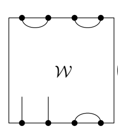
$$
\begin{aligned}
(A) &= \frac{A^8}{[4]}\left(A^2\,\tilde{\mathcal{P}}_4(\mathcal{W}_{N_4\to N_4}) - A^2\,\tilde{\mathcal{P}}_2(\mathcal{W}_{N_4\to N_4})\right)\\
&= \frac{A^{10}}{[4]}\left(\tilde{\mathcal{P}}_4(\mathcal{W}^*_{N_4\to N_4}) - \tilde{\mathcal{P}}_2(\mathcal{W}^*_{N_4\to N_4})\right).
\end{aligned}
$$

(6)



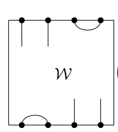
$$
\begin{aligned}
(A) &= \frac{A^2}{[4]}\left(A^4\,\tilde{\mathcal{P}}_4(\mathcal{W}_{N_4\to N_3}) - [3]\,\tilde{\mathcal{P}}_2(\mathcal{W}_{N_4\to N_3})\right)\\
&= \frac{A^6}{[4]}\left(\tilde{\mathcal{P}}_4(\mathcal{W}^*_{N_4\to N_3}) - A^{-4}[3]\,\tilde{\mathcal{P}}_2(\mathcal{W}^*_{N_4\to N_3})\right).
\end{aligned}
$$

(7)



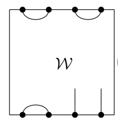
$$
\begin{aligned}
(A) &= \frac{A^4[2]}{[4]}\left(\frac{A^6\,\tilde{\mathcal{P}}_4(\mathcal{W}_{E\to N_4})}{[3]} - A^2\,\tilde{\mathcal{P}}_2(\mathcal{W}_{E\to N_4})\right)\\
&= \frac{A^4[2]^2}{[4]}\left(\tilde{\mathcal{P}}_4(\mathcal{W}^*_{E\to N_4}) - \tilde{\mathcal{P}}_2(\mathcal{W}^*_{E\to N_4})\right).
\end{aligned}
$$

(8)



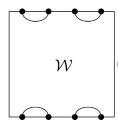
$$
\begin{aligned}
(A) &= \frac{A^{-4}}{[4]}\left(A^6\,\tilde{\mathcal{P}}_4(\mathcal{W}_{N_3\to N_3}) - A^6\,\tilde{\mathcal{P}}_2(\mathcal{W}_{N_3\to N_3})\right)\\
&= \frac{A^2}{[4]}\left(\tilde{\mathcal{P}}_4(\mathcal{W}^*_{N_3\to N_3}) - \tilde{\mathcal{P}}_2(\mathcal{W}^*_{N_3\to N_3})\right).
\end{aligned}
$$

(9)



$$(A) = \frac{A^{-2}[2]}{[4]}\left(\frac{A^8\,\tilde{\mathcal{P}}_4(\mathcal{W}_{E\to N_3})}{[3]} - A^4\,\tilde{\mathcal{P}}_2(\mathcal{W}_{E\to N_3})\right)$$
$$= \frac{[2]^2}{[4]}\left(\tilde{\mathcal{P}}_4(\mathcal{W}^*_{E\to N_3}) - \tilde{\mathcal{P}}_2(\mathcal{W}^*_{E\to N_3})\right).$$

(10)



$$(A) = \frac{A^{4}[2]}{[4]}\left(\frac{A^6[2]\,\tilde{\mathcal{P}}_4(\mathcal{W}_{E\to E})}{[3]^2} - \frac{2A^2\,\tilde{\mathcal{P}}_2(\mathcal{W}_{E\to E})}{[2]}\right)$$
$$= \frac{A^2[2]^2}{[4]}\left(A^{-4}[2]^2\,\tilde{\mathcal{P}}_4(\mathcal{W}^*_{E\to E}) - 2\,\tilde{\mathcal{P}}_2(\mathcal{W}^*_{E\to E})\right).$$

Since possible values for $\tilde{\mathcal{P}}_{4/2}(\cdot) = \tilde{\mathcal{P}}_4(\cdot)/\tilde{\mathcal{P}}_2(\cdot)$ are: $1, \mathcal{A}_k(A^4), \mathcal{B}_k(A^4), \mathcal{C}_k(A^4), \mathcal{C}_0(A^4)$ (= $A^{-4}[3]$) and $\tilde{\mathcal{P}}_2 = A^c[2]^d$ for some c, d. We conclude that the coefficient of C satisfying Assumptions 1 and 2 is given by one of the following closed-form formulas:

$$\frac{A^c[2]^d}{[4]}\left(\mathcal{C}_0(A^4)^b\prod_{i=1}^{r}\mathcal{A}_{\alpha_i}(A^4)\prod_{j=1}^{s}\mathcal{B}_{\beta_j}(A^4)\prod_{k=1}^{t}\mathcal{C}_{\gamma_k}(A^4) - 1\right), \tag{4.2}$$

$$\frac{A^c[2]^d}{[4]}\left(\mathcal{C}_0(A^4)^b\prod_{i=1}^{r}\mathcal{A}_{\alpha_i}(A^4)\prod_{j=1}^{s}\mathcal{B}_{\beta_j}(A^4)\prod_{k=1}^{t}\mathcal{C}_{\gamma_k}(A^4) - A^{-4}[3]\right) \tag{4.3}$$

and

$$\frac{A^c[2]^d}{[4]}\left(A^{-4}[2]^2\mathcal{C}_0(A^4)^b\prod_{i=1}^{r}\mathcal{A}_{\alpha_i}(A^4)\prod_{j=1}^{s}\mathcal{B}_{\beta_j}(A^4)\prod_{k=1}^{t}\mathcal{C}_{\gamma_k}(A^4) - 2\right), \tag{4.4}$$

where b (respectively r, s, t) is the number of times E (respectively $M_2^{(k)}, M_1^{(k)}$ and $M_3^{(k)}$ or $M_4^{(k)}$) appears in $\mathcal{W}^*$, $c \in \mathbb{Z}$, $d, \gamma_k \in \mathbb{Z}_{\geq 0}$ and $\alpha_i, \beta_j \in \mathbb{N}$.

Step 5: Sets $\boldsymbol{\Omega_1^{(u)}, \Omega_2, \Omega_3^{(u)}}$. Given a Catalan state C satisfying Assumptions 1 and 2 introduced in Step 2, we showed that its coefficient $C(A)$ must be given by one of the closed-form formulas (4.2)–(4.4) for some $(c, d, b, r, s, t) \in \mathbb{Z} \times (\mathbb{Z}_{\geq 0})^5$. Now, we consider the questions which Laurent polynomials above are coefficients of some realizable Catalan states.

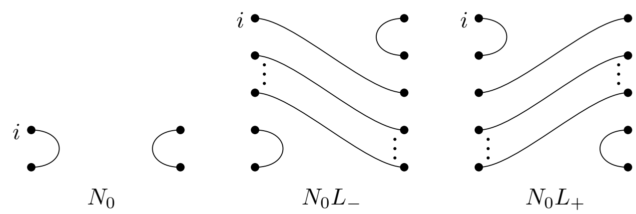


Figure 4.7. L_- and L_+

Therefore, we would like to find the converse of statement given in Step 4. That is, we want to find conditions for c, d, b, r, s, t which assure that there is a Catalan state $C \in \mathfrak{Cat}_{m,4}$ for which $C(A)$ is one of (4.2)–(4.4).

Since each Catalan state, after removing its roof and floor states shown in Table 4.1, corresponds to a unique walk in Figure 4.4, so we focus on walks in the form $E \to E$, $E \to N_3$, $E \to N_4$, $N_4 \to N_4$, $N_4 \to N_3$, and $N_3 \to N_3$. We introduce two letters L_+ and L_-, where L_+ (respectively L_-) represents parallel copies of arcs that connect y_j, y'_{j-2} (respectively y_{j-2}, y'_j), see Figure 4.7.

The letter L_+ can be inserted into each walk $\mathcal{W}$ after N_0, N_2, N_4 and N_6. Analogously, the letter and L_- can be inserted into $\mathcal{W}$ after N_0, N_1, N_3 and N_5. Notice that, each arc of L_+ (respectively L_-) changes the coefficient $C(A)$ by a factor of A^{-2} (respectively A^2). Since L_+ and L_- can be inserted into a walk $\mathcal{W}$ without changing the sequence of letters in $\mathcal{W}$ that are in $\mathbb{A}$, so we will still use the same notation for the words on the extended alphabet $\mathbb{A} \cup \{L_+, L_-\}$. Let

$$b(\mathcal{W}^*) = |\{w \in \mathcal{W}^* : w = E\}|$$

$$r(\mathcal{W}^*) = |\{w \in \mathcal{W}^* : w = M_2^{(k)} \text{ for some } k\}|$$

$$s(\mathcal{W}^*) = |\{w \in \mathcal{W}^* : w = M_1^{(k)} \text{ for some } k\}|$$

$$t(\mathcal{W}^*) = |\{w \in \mathcal{W}^* : w = M_3^{(k)} \text{ or } M_4^{(k)} \text{ for some } k\}|$$

$$e(\mathcal{W}^*) = (\#\text{ of arcs in } L_-) - (\#\text{ of arcs in } L_+)$$

and

$$\omega(\mathcal{W}^*) = (e(\mathcal{W}^*), b(\mathcal{W}^*), r(\mathcal{W}^*), s(\mathcal{W}^*), t(\mathcal{W}^*)).$$

We want to find relations among tuples (e, b, r, s, t) that assure existence of $\mathcal{W}^*$ with $\omega(\mathcal{W}^*) = (e, b, r, s, t)$. We analyze each of the six types of walks and check which tuples (e, b, r, s, t) can be realized by them. Define the set $\Omega = \mathbb{Z} \times (\mathbb{Z}_{\geq 0})^4$.

(i) $E \to E$:

- $t = 2k + 1 \geq 1$ and $r, s, b \geq 0$:

  $\mathcal{W} = (EN_0)^b EN_2 L_+ (M_2 N_4)^r M_4 (N_3 M_3 N_4 M_4)^{\lfloor t/2 \rfloor} (N_3 M_1)^s N_5 L_- E;$
- $t = 2k \geq 2$ and $r, s, b \geq 0$:

  $\mathcal{W} = (EN_0)^b EN_1 L_- M_3 (N_4 M_2)^r N_4 L_+ M_4 (N_3 M_3 N_4 M_4)^{\lfloor t/2 \rfloor - 1} (N_3 M_1)^s N_5 E;$
- $s = t = b = 0$ and $r \geq 1$:

  $\mathcal{W} = EN_2 L_+ (M_2 N_4)^{r-1} M_2 N_6 E;$
- $r = t = b = 0$ and $s \geq 1$:

  $\mathcal{W} = EN_1 L_- (M_1 M_3)^{s-1} M_1 N_5 E;$
- $t = 0$ and $r, s, b \geq 1$:

  $\mathcal{W} = EN_2 L_+ (M_2 N_4)^{r-1} M_2 N_6 (EN_0)^{b-1} EN_1 L_- (M_1 N_3)^{s-1} M_1 N_5 E;$
- $s = t = 0$ and $r, b \geq 1$:

  $\mathcal{W} = (EN_0)^b L_\pm EN_2 (M_2 N_4)^{r-1} M_2 N_6 E;$
- $r = t = 0$ and $s, b \geq 1$:

  $\mathcal{W} = (EN_0)^b L_\pm EN_1 (M_1 N_3)^{s-1} M_1 N_5 E;$
- $r = s = t = 0$ and $b \geq 0$:

  $\mathcal{W} = (EN_0)^{b+1} L_\pm E;$
- $(e, b, r, s, t) \in V_{E \to E}$, where

$$\begin{aligned} V_{E\to E} = \{&(e,0,r,s,0) \mid (e,0,r,s,0) \in \Omega,\ r,s \geq 1\} \\ \cup \{&(e,0,r,0,0) \mid (e,0,r,0,0) \in \Omega,\ r \geq 1,\ e \geq 1\} \\ \cup \{&(e,0,0,s,0) \mid (e,0,0,s,0) \in \Omega,\ s \geq 1,\ e \leq -1\}. \end{aligned}$$

In this case, $\mathcal{W}$ does not exist.

(ii) $E \to N_3$:

- $t = 2k+1 \geq 1$ and $b, r, s \geq 0$:

  $\mathcal{W} = (EN_0)^b EN_2L_+(M_2N_4)^r M_4(N_3M_3N_4M_4)^{\lfloor t/2 \rfloor}(N_3M_1)^s N_3L_-$;

- $t = 2k \geq 2$ and $b, r, s \geq 0$:

  $\mathcal{W} = (EN_0)^b EN_1L_-M_3N_4L_+(M_2N_4)^r M_4(N_3M_3N_4M_4)^{\lfloor t/2 \rfloor - 1}(N_3M_1)^s N_3$;

- $b = r = t = 0$ and $s \geq 1$:

  $\mathcal{W} = EN_1L_-(M_1N_3)^s$;

- $t = 0$ and $b, r, s \geq 1$:

  $\mathcal{W} = EN_2L_+(M_2N_4)^{r-1}M_2N_6(EN_0)^{b-1}EN_1L_-(M_1N_3)^s$;

- $r = t = 0$ and $b, s \geq 1$:

  $\mathcal{W} = (EN_0)^b L_\pm EN_1(M_1N_3)^s$;

- $(e, b, r, s, t) \in V_{E \to N_3}$, where

$$\begin{aligned} V_{E \to N_3} = {} & \{(e, 0, r, s, 0) \mid (e, 0, r, s, 0) \in \Omega,\ r, s \geq 1\} \\ & \cup \{(e, 0, 0, s, 0) \mid (e, 0, 0, s, 0) \in \Omega,\ s \geq 1,\ e \leq -1\} \\ & \cup \{(e, b, r, 0, 0) \mid (e, b, r, 0, 0) \in \Omega\}. \end{aligned}$$

  In this case, $\mathcal{W}$ does not exist.

(iii) $E \to N_4$:

- $t = 2k+1 \geq 1$ and $b, r, s \geq 0$:

  $\mathcal{W} = (EN_0)^b EN_1L_-(M_1N_3)^s M_3(N_4M_4N_3M_3)^{\lfloor t/2 \rfloor}(N_4M_2)^r N_4L_+$;

- $t = 2k \geq 2$ and $b, r, s \geq 0$:

  $\mathcal{W} = (EN_0)^b EN_2L_+M_4N_3L_-(M_1N_3)^s M_3(N_4M_4N_3M_3)^{\lfloor t/2 \rfloor - 1}(N_4M_2)^r N_4$;

- $b = s = t = 0$ and $r \geq 1$:

  $\mathcal{W} = EN_2L_+(M_2N_4)^r$;

- $t = 0$ and $b, r, s \geq 1$:

  $\mathcal{W} = EN_1L_-(M_1N_3)^{s-1}M_1N_5(EN_0)^{b-1}EN_2L_+(M_2N_4)^r$;

- $s = t = 0$ and $b, r \geq 1$:

  $\mathcal{W} = (EN_0)^bL_\pm EN_2(M_2N_4)^r$;

- $(e, b, r, s, t) \in V_{E\to N_4}$, where

$$\begin{aligned} V_{E\to N_4} = &\{(e,0,r,s,0) \mid (e,0,r,s,0) \in \Omega,\ r,s \geq 1\} \\ &\cup \{(e,0,r,0,0) \mid (e,0,r,0,0) \in \Omega, r \geq 1,\ e \geq 1\} \\ &\cup \{(e,b,0,s,0) \mid (e,b,0,s,0) \in \Omega\}. \end{aligned}$$

  In this case, $\mathcal{W}$ does not exist.

(iv) $N_4 \to N_4$:

- $b = 0$, $t = 2k \geq 2$ and $r, s \geq 0$:

  $\mathcal{W} = (N_4M_2)^rN_4L_+M_4(N_3M_1)^s(N_3M_3N_4M_4)^{\lfloor t/2\rfloor-1}N_3L_-M_3N_4$;

- $b = s = t = 0$ and $r \geq 1$:

  $\mathcal{W} = (N_4M_2)^rN_4L_+$;

- $t = 2k \geq 2$, $b \geq 1$ and $r, s \geq 0$:

  $\mathcal{W} = (N_4M_2)^rN_4L_+M_4N_5(EN_0)^{b-1}EN_1L_-(M_1N_3)^sM_3(N_4M_4N_3M_3)^{\lfloor t/2\rfloor-1}N_4$;

- $t = 2k+1 \geq 3$, $b \geq 1$, $r = 0$ and $s \geq 0$:

  $\mathcal{W} = N_4M_4(N_3M_1)^sN_3L_-M_3N_6L_+(EN_0)^{b-1}EN_1M_3(N_4M_4N_3M_3)^{\lfloor t/2\rfloor-1}N_4$;

- $t = 2k+1 \geq 1$, $s \geq 0$ and $b, r \geq 1$:

  $\mathcal{W} = (N_4M_2)^rN_6L_+(EN_0)^{b-1}EN_1L_-(M_1N_3)^sM_3(N_4M_4N_3M_3)^{\lfloor t/2\rfloor}N_4$;

- $t = 0$, $s \geq 1$ and $b, r \geq 2$:

  $\mathcal{W} = N_4 M_2 N_6 L_+ E N_1 L_- (M_1 N_3)^{s-1} M_1 N_5 (E N_0)^{b-2} E N_2 (M_2 N_4)^{r-1}$;

- $s = t = 0$, $b = 1$ and $r \geq 2$:

  $\mathcal{W} = N_4 M_2 N_6 L_+ E N_2 (M_2 N_4)^{r-1}$;

- $s = t = 0$ and $b, r \geq 2$:

  $\mathcal{W} = N_4 M_2 N_6 (E N_0)^{b-1} L_\pm E N_2 (M_2 N_4)^{r-1}$;

- $(e, b, r, s, t) \in V_{N_4 \to N_4}$, where

$$\begin{aligned} V_{N_4 \to N_4} = &\{(e, 0, r, s, t) \mid (e, 0, r, s, t) \in \Omega,\ t \text{ is odd}\} \\ &\cup \{(e, b, 0, s, 1) \mid (e, b, 0, s, 1) \in \Omega,\ b \geq 1\} \\ &\cup \{(e, b, 0, s, 0) \mid (e, b, 0, s, 0) \in \Omega\} \\ &\cup \{(e, 0, r, 0, 0) \mid (e, 0, r, 0, 0) \in \Omega,\ r \geq 1,\ e \geq 1\} \\ &\cup \{(e, 1, r, 0, 0) \mid (e, 1, r, 0, 0) \in \Omega,\ r \geq 2,\ e \geq 1\} \\ &\cup \{(e, b, r, s, 0) \mid (e, b, r, s, 0) \in \Omega,\ b \leq 1,\ r, s \geq 1\} \\ &\cup \{(e, b, 1, s, 0) \mid (e, b, 1, s, 0) \in \Omega,\ b \geq 1\}. \end{aligned}$$

  In this case, $\mathcal{W}$ does not exist.

(v) $N_4 \to N_3$:

- $b = 0$, $t = 2k + 1 \geq 1$ and $r, s \geq 0$:

  $\mathcal{W} = (N_4 M_2)^r N_4 L_+ M_4 (N_3 M_1)^s (N_3 M_3 N_4 M_4)^{\lfloor t/2 \rfloor} N_3 L_-$;

- $t = 1$, $r \geq 0$ and $b, s \geq 1$:

  $\mathcal{W} = (N_4 M_2)^r N_4 L_+ M_4 N_5 L_- (E N_0)^{b-1} E N_1 (M_1 N_3)^s$;

- $t = 2k + 1 \geq 3$, $b \geq 1$ and $r, s \geq 0$:

  $\mathcal{W} = (N_4 M_2)^r N_4 L_+ M_4 N_5 L_- (E N_0)^{b-1} E N_1 (M_3 N_4 M_4 N_3)^{\lfloor t/2 \rfloor} (M_1 N_3)^s$;

- $t = 2k \geq 2$, $b \geq 1$ and $r, s \geq 0$:

  $\mathcal{W} = (N_4M_2)^r N_4 L_+ M_4 N_5 L_- (EN_0)^{b-1} EN_2 M_4 (N_3M_3N_4M_4)^{\lfloor t/2 \rfloor -1} (N_3M_1)^s N_3;$

- $t = 1$, $s \geq 0$ and $b, r \geq 1$:

  $\mathcal{W} = (N_4M_2)^r N_6 L_+ (EN_0)^{b-1} EN_2 M_4 (N_3M_1)^s N_3 L_-;$

- $t = 0$ and $b, r, s \geq 1$:

  $\mathcal{W} = (N_4M_2)^r N_6 L_+ (EN_0)^{b-1} EN_1 L_- (M_1N_3)^s;$

- $(e, b, r, s, t) \in V_{N_4 \to N_3}$, where

$$\begin{aligned} V_{N_4 \to N_3} = &\{(e, 0, r, s, t) \mid (e, 0, r, s, t) \in \Omega,\ t \text{ is even}\} \\ &\cup \{(e, b, 0, 0, 1) \mid (e, b, 0, 0, 1) \in \Omega,\ b \geq 1\} \\ &\cup \{(e, b, r, s, 0) \mid (e, b, r, s, 0) \in \Omega,\ rs = 0\}. \end{aligned}$$

  In this case, $\mathcal{W}$ does not exist.

(vi) $N_3 \to N_3$:

- $b = 0$, $t = 2k \geq 2$ and $r, s \geq 0$:

  $\mathcal{W} = (N_3M_1)^s N_3 L_- M_3 (N_4M_2)^r (N_4M_4N_3M_3)^{\lfloor t/2 \rfloor -1} N_4 L_+ M_4 N_3;$

- $b = r = t = 0$ and $s \geq 1$:

  $\mathcal{W} = (N_3M_1)^s N_3 L_-;$

- $t = 2k \geq 2$, $b \geq 1$ and $r, s \geq 0$:

  $\mathcal{W} = (N_3M_1)^s N_3 L_- M_3 N_6 (EN_0)^{b-1} EN_2 L_+ (M_2N_4)^r M_4 (N_3M_3N_4M_4)^{\lfloor t/2 \rfloor -1} N_3;$

- $t = 2k+1 \geq 3$, $b \geq 1$, $s = 0$ and $r \geq 0$:

  $\mathcal{W} = N_3M_3 (N_4M_2)^r N_4 L_+ M_4 N_5 L_- (EN_0)^{b-1} EN_2 M_4 (N_3M_3N_4M_4)^{\lfloor t/2 \rfloor -1} N_3;$

- $t = 2k+1 \geq 1$, $r \geq 0$ and $b, s \geq 1$:

  $\mathcal{W} = (N_3M_1)^s N_5 L_- (EN_0)^{b-1} EN_2 L_+ (M_2N_4)^r M_4 (N_3M_3N_4M_4)^{\lfloor t/2 \rfloor} N_3;$

- $t = 0$, $r \geq 1$ and $b, s \geq 2$:

  $\mathcal{W} = N_3 M_1 N_5 L_- E N_2 L_+ (M_2 N_4)^{r-1} M_2 N_6 (E N_0)^{b-2} E N_1 (M_1 N_3)^{s-1}$;

- $r = t = 0$, $b = 1$ and $s \geq 2$:

  $\mathcal{W} = N_3 M_1 N_5 L_- E N_1 (M_1 N_3)^{s-1}$;

- $r = t = 0$ and $b, s \geq 2$:

  $\mathcal{W} = N_3 M_1 N_5 (E N_0)^{b-1} L_\pm E N_1 (M_1 N_3)^{s-1}$;

- $(e, b, r, s, t) \in V_{N_3 \to N_3}$, where

$$\begin{aligned} V_{N_3 \to N_3} = &\{(e, 0, r, s, t) \mid (e, 0, r, s, t) \in \Omega,\ t \text{ is odd}\} \\ &\cup \{(e, b, r, 0, 1) \mid (e, b, r, 0, 1) \in \Omega,\ b \geq 1\} \\ &\cup \{(e, b, r, 0, 0) \mid (e, b, r, 0, 0) \in \Omega\} \\ &\cup \{(e, 0, 0, s, 0) \mid (e, 0, 0, s, 0) \in \Omega,\ s \geq 1,\ e \leq -1\} \\ &\cup \{(e, 1, 0, s, 0) \mid (e, 1, 0, s, 0) \in \Omega,\ s \geq 2,\ e \leq -1\} \\ &\cup \{(e, b, r, s, 0) \mid (e, b, r, s, 0) \in \Omega,\ b \leq 1,\ r, s \geq 1\} \\ &\cup \{(e, b, r, 1, 0) \mid (e, b, r, 1, 0) \in \Omega,\ b \geq 1\}. \end{aligned}$$

  In this case, $\mathcal{W}$ does not exist.

Now we can consider formulas (4.2)–(4.4).

1. For formula (4.2): We see that this formula results from one of the cases (1), (2), (3), (5), (7), (8) and (9) in Step 4.

   - Case (1): $A^{2+2s-2r+2e-2b}[2]^{2+b} \left(\dfrac{\tilde{\mathcal{P}}_{4/2}(\mathcal{W}^*_{E \to E}) - 1}{[4]} \right)$. Define the set

   $$I_1 = \{(1 + s - r + e - b, b, r, s, t) \mid (e, b, r, s, t) \in \Omega \setminus V_{E \to E}\}.$$

   - Case (2): $A^{6+2s-2r+2e-2b}[2]^{1+b} \left(\dfrac{\tilde{\mathcal{P}}_{4/2}(\mathcal{W}^*_{E \to N_4}) - 1}{[4]} \right)$. Define the set

   $$I_2 = \{(3 + s - r + e - b, b, r, s, t) \mid (e, b, r, s, t) \in \Omega \setminus V_{E \to N_4}\}.$$

- Case (3): $A^{2+2s-2r+2e-2b}[2]^{1+b}\left(\dfrac{\tilde{\mathcal{P}}_{4/2}(\mathcal{W}^*_{E\to N_3})-1}{[4]}\right)$. Define the set

$$I_3=\{(1+s-r+e-b,b,r,s,t)\mid(e,b,r,s,t)\in\Omega\setminus V_{E\to N_3}\}.$$

- Case (5): $A^{10+2s-2r+2e-2b}[2]^{b}\left(\dfrac{\tilde{\mathcal{P}}_{4/2}(\mathcal{W}^*_{N_4\to N_4})-1}{[4]}\right)$. Define the set

$$I_5=\{(5+s-r+e-b,b,r,s,t)\mid(e,b,r,s,t)\in\Omega\setminus V_{N_4\to N_4}\}.$$

- Case (7): $A^{4+2s-2r+2e-2b}[2]^{2+b}\left(\dfrac{\tilde{\mathcal{P}}_{4/2}(\mathcal{W}^*_{E\to N_4})-1}{[4]}\right)$. Define the set

$$I_7=\{(2+s-r+e-b,b,r,s,t)\mid(e,b,r,s,t)\in\Omega\setminus V_{E\to N_4}\}.$$

- Case (8): $A^{2+2s-2r+2e-2b}[2]^{b}\left(\dfrac{\tilde{\mathcal{P}}_{4/2}(\mathcal{W}^*_{N_3\to N_3})-1}{[4]}\right)$. Define the set

$$I_8=\{(1+s-r+e-b,b,r,s,t)\mid(e,b,r,s,t)\in\Omega\setminus V_{N_3\to N_3}\}.$$

- Case (9): $A^{2s-2r+2e-2b}[2]^{2+b}\left(\dfrac{\tilde{\mathcal{P}}_{4/2}(\mathcal{W}^*_{E\to N_3})-1}{[4]}\right)$. Define the set

$$I_9=\{(s-r+e-b,b,r,s,t)\mid(e,b,r,s,t)\in\Omega\setminus V_{E\to N_3}\}.$$

Denote by $\Omega_1^{(0)}=I_5\cup I_8$, $\Omega_1^{(1)}=I_2\cup I_3$, and $\Omega_1^{(2)}=I_1\cup I_7\cup I_9$.

2. For formula (4.3): This formula is obtained from the case (6) in Step 4:

$$A^{6+2s-2r+2e-2b}[2]^{b}\left(\frac{\tilde{\mathcal{P}}_{4/2}(\mathcal{W}^*_{N_4\to N_3})-A^{-4}[3]}{[4]}\right).$$

Denote by $\Omega_2=\{(3+s-r+e-b,b,r,s,t)\mid(e,b,r,s,t)\in\Omega\setminus V_{N_4\to N_3}\}$.

3. For formula (4.4): This formula results from cases (4) or (10) in Step 4:

- Case (4): $A^{4+2s-2r+2e-2b}[2]^{1+b}\left(\dfrac{A^{-4}[2]^2\tilde{\mathcal{P}}_{4/2}(\mathcal{W}^*_{E\to E})-2}{[4]}\right)$. Define the set

$$\Omega_3^{(1)}=\{(2+s-r+e-b,b,r,s,t)\mid(e,b,r,s,t)\in\Omega\setminus V_{E\to E}\}.$$

- Case (10): $A^{2+2s-2r+2e-2b}[2]^{2+b}\left(\dfrac{A^{-4}[2]^2\tilde{\mathcal{P}}_{4/2}(\mathcal{W}^*_{E\to E})-2}{[4]}\right)$. Define the set

$$\Omega_3^{(2)}=\{(1+s-r+e-b,b,r,s,t)\mid (e,b,r,s,t)\in\Omega\setminus V_{E\to E}\}.$$

Then the sets $\Omega_1^{(u)},\Omega_2,\Omega_3^{(u')}$ for $u\in\{0,1,2\},u'\in\{1,2\}$ defined above are indeed all feasible tuples (a,b,r,s,t) for the formulas. This finishes our discussion of all cases that satisfy Assumption 1.

Step 6: Set $\boldsymbol{\Omega_0}$. Consider the Catalan states C with $|C\cap l_i^h|=4$, for some $0\leq i\leq m$. Using lines l_i^h we split C into a finite number of vertically irreducible Catalan states $C_1,C_2,\ldots,C_l$.

Since the number of crossings of $L(m,4)$ is $4m$, the exponent of A in the formula (4.1) must be even. We notice that each C_i has no returns on its bottom boundary except possibly for C_l. However, after a π-rotation of C_k we may assume that all C_i have no returns on the bottom boundary. Figure 4.8 shows the list of all infinite families of roof sates except those with an arc that have endpoints x_i and x'_j. These were obtained by considering, in particular, all possible roof states that have one arc with one of its endpoints x_i and the other either y_k, y'_k, $k\geq 4$, and the roof states with all of its arcs with an endpoint x_i and the other either y_k or y'_k, $k\leq 3$, or x_j. We notice that if a Catalan state C has an arc with endpoints x_i and x'_j then the plucking polynomial of $\mathcal{T}(C)$ is a product of $[2]_q$'s and $[3]_q$'s. Moreover, one shows that the plucking polynomial corresponding to families of the roof states in Figure 4.8 are, up to a power of q, products of $\frac{[4]_q[3]_q}{[2]_q}$, $[2]_q$, $[3]_q$, $[4]_q$ and $\tilde{\mathcal{A}}_\alpha(q)$, $\tilde{\mathcal{B}}_\beta(q)$, $\tilde{\mathcal{C}}_\gamma(q)$. To compute $C(A)$, notice that by Theorem 3.1.16, we can remove from C all arcs with endpoints y_i and y'_j with $|i-j|\leq 2$. Cutting off the roof state above the line l_k^h and one of the floor states shown in the first column of Table 4.1 results in tangle $\mathcal{M}$ for which we can use an algorithm this similar to Algorithm 4.1.4 and decompose $\mathcal{M}$ into tangles in Table 4.2. Then using Step 2, we translate this decomposition of $\mathcal{M}$ into the corresponding directed walk $\mathcal{W}$ in

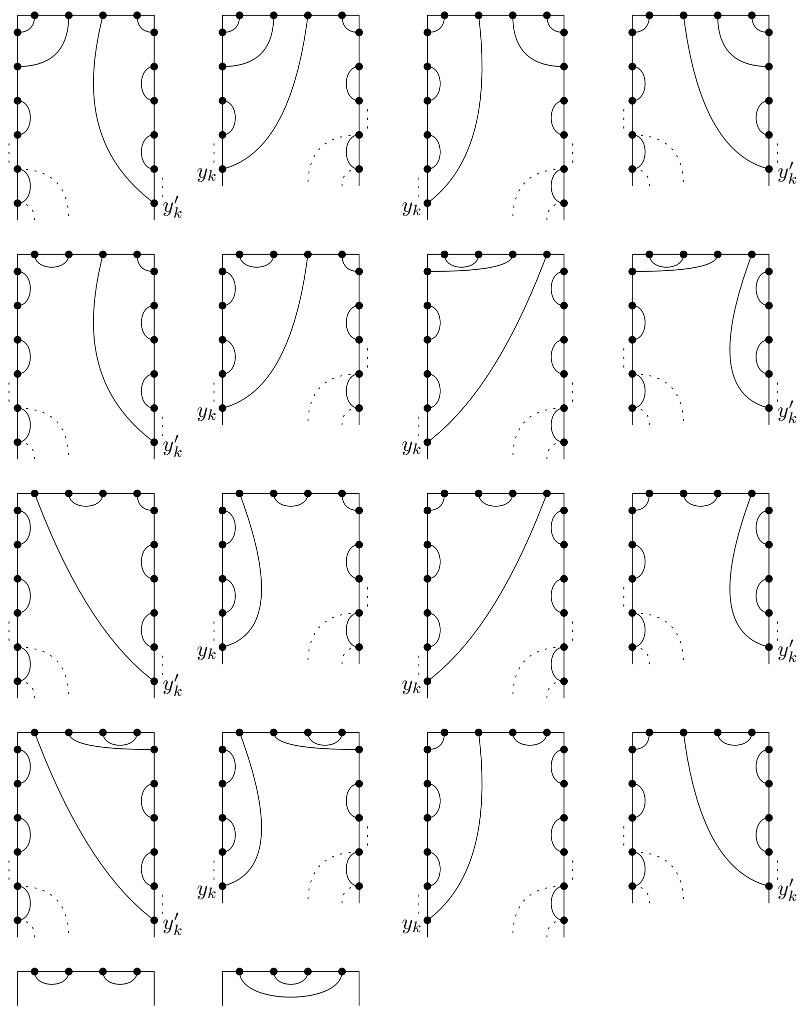


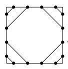
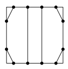
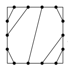
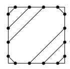
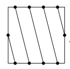
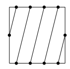
Figure 4.8. Roof states for vertically irreducible Catalan states of $L(m, 4)$

the directed graph G (see Figure 4.4). Using Step 3, we compute the corresponding Laurent polynomial $\mathcal{P}_4(\mathcal{W})$. Therefore, contribution of $\mathcal{M}$ to $C_i(A)$ is a product of [2], [3], $\mathcal{A}_\alpha(A^4)$, $\mathcal{B}_\beta(A^4)$ and $\mathcal{C}_\gamma(A^4)$. Hence $C(A)$ must be of the form:

$$A^{2a}\left(\frac{[4][3]}{[2]}\right)^{\bar{a}}[2]^{\bar{b}}[3]^{\bar{c}}[4]^{\bar{d}}\prod_{i=1}^{r}\mathcal{A}_{\alpha_i}(A^4)\prod_{j=1}^{s}\mathcal{B}_{\beta_j}(A^4)\prod_{k=1}^{t}\mathcal{C}_{\gamma_k}(A^4)$$

$$=A^{2a}[2]^{b}[3]^{c}[4]^{d}\prod_{i=1}^{r}\mathcal{A}_{\alpha_i}(A^4)\prod_{j=1}^{s}\mathcal{B}_{\beta_j}(A^4)\prod_{k=1}^{t}\mathcal{C}_{\gamma_k}(A^4)$$

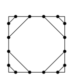
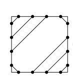
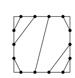
for $\bar{a}, \bar{b}, \bar{c}, \bar{d} \geq 0$. Notice that in the above, $b = \bar{b} - \bar{a}, c = \bar{c} + \bar{a}, d = \bar{d} + \bar{a}$. This implies that $b + c \geq 0, b + d \geq 0$, so $b \geq -\min\{c, d\}$.

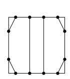
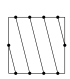
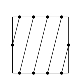
To prove the converse, we construct a Catalan state C for which $C(A)$ is the Laurent polynomial as the above. Such C can be constructed by vertical product of Catalan state $C^{\dagger}$, , , , , , , and a π-rotation of $C^{\ddagger}$, where $C^{\dagger}$ and $C^{\ddagger}$ are defined below. Let $\hat{T}$, $\hat{T}^s$, $\hat{B}$, $L_{-}\hat{B}$, $\hat{B}^s$ be roof and floor states shown in Figure 4.9. Notice that, if the roof state $\hat{T}$ (respectively $\hat{T}^s$) is used to construct $C^{\dagger}$ (respectively $C^{\ddagger}$) then it needs to be followed by a walk $N_4 \rightarrow$ (respectively $N_3 \rightarrow$) that originates at N_4 (respectively N_3). Moreover, the roof state $\hat{T}$ (respectively $\hat{T}^s$) contributes a factor A^{-2} (respectively A^2) to $C^{\dagger}(A)$ (respectively $C^{\ddagger}(A)$). If the floor state $\hat{B}$ (respectively $L_{-}\hat{B}$, $\hat{B}^s$) is used in the construction of $C^{\dagger}$ or $C^{\ddagger}$ then the walk $\rightarrow N_3$ (respectively $\rightarrow N_3$, $\rightarrow N_4$) we use needs to end with N_3 (respectively N_3, N_4). The floor state $\hat{B}$ (respectively $L_{-}\hat{B}$, $\hat{B}^s$) contribute a factor 1 (respectively A^2, A^{-6}) to $C^{\dagger}(A)$ or $C^{\ddagger}(A)$. Given $(a,b,c,d,r,s,t) \in \Omega_0$, we construct Catalan states $C^{\dagger}, C^{\ddagger}$ as follows:

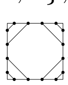
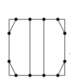
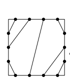
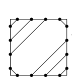
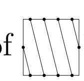
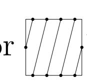
- If t is even, then $C^{\dagger}$ is the Catalan state obtained as concatenation of $\hat{T}$ with the tangle corresponding to the walk $\mathcal{W} = N_4(M_2N_4)^r$, and the floor state $\hat{B}^s$ is used; If t is odd, we use $\hat{T}$, $\mathcal{W} = N_4(M_2N_4)^r M_4 N_3$ and $\hat{B}$ to construct $C^{\dagger}$.

- $C^{\ddagger}$ is constructed by concatenation of the roof state $\hat{T}^s$ with the tangle corresponding to walk $\mathcal{W} = N_3(M_1N_3)^s(M_3N_4M_4N_3)^{\lfloor t/2 \rfloor}$ and one of the floor states $L_{-}^{u}\hat{B}$ $(u \in \{0,1\})$.

One shows that:

$(A) = A^{-8}\dfrac{[4][3]}{[2]}$, $(A) = A^{-14}[4]$, $(A) = A^{-8}[3]$,

$(A) = A^{-2}[2]$, $(A) = A^{4}$, $(A) = A^{-4}$.

Let $\bar{a} = \max\{-b, 0\}, \bar{b} = b + \bar{a}, \bar{c} = c - \bar{a}, \bar{d} = d - \bar{a}$. Then we construct C by concatenating $C^{\dagger}$ with $\bar{a}, \bar{b}, \bar{c}, \bar{d}$ of , , , 's, suitable number of 's or 's and a π-rotation of $C^{\ddagger}$ with $u \in \{0,1\}$ suitably chosen. One checks that $C(A)$ is the Laurent polynomial of the form (4.1) determined by $(a,b,c,d,r,s,t) \in \Omega_0$.

$\square$

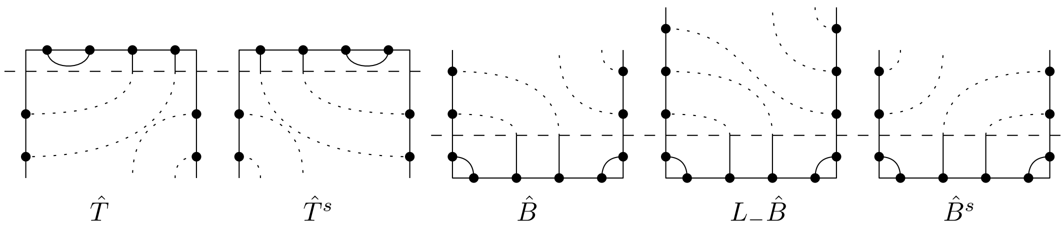


Figure 4.9. Roof and floor states for $C^\dagger$ and $C^\ddagger$

Remark 4.1.5. *Theorem 4.1.3 gives a closed-form formulas for coefficients of realizable Catalan states of a lattice crossing with four vertical strands. However, when m is fixed, finding a similar formula for realizable Catalan states of $L(m,4)$ is rather difficult task. This is because, sets $\Omega_0, \Omega_1^{(u)}, \Omega_2, \Omega_3^{(u)}$ need to be modified appropriately. In particular, to make such modifications we define a height function $h(\cdot)$ that assigns to each tangle from Table 4.2 (and some other) the number of points on the right (or left) boundary as its value. For example, $h(N_0) = 2, h(M_1^{(k)}) = 2k-1, h(E) = 0$. Then one adds an additional condition in the Step 5-6 of our proof: if C can be decomposed into tangles $t_1, \ldots, t_k$, then $\sum_{i=1}^k h(t_i) = m$.*

Similar results for $n = 1, 2, 3$ can be obtained. Indeed, for $n = 1$, $C(A)$ equals A^a for some $a \in \mathbb{Z}$. When $n = 2$, $C(A) = A^{2a}[2]^b$, for some $a \in \mathbb{Z}, b \in \mathbb{Z}_{\geq 0}$. For $n = 3$, $C(A)$ is given as a result of following theorem.

Theorem 4.1.6. *Let C be a realizable Catalan state of $L(m,3)$. Then $C(A)$ is given by one of the following closed-form formulas:*

$$A^a[2]^b[3]^c, \ (a,b,c) \in \mathbb{Z} \times \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}$$

and

$$\frac{A^a[2]^u}{[3]}\left(A^{-4b}[2]^{2b} - 1\right), \ u \in \{0,1\}, \ (a,b) \in \mathbb{Z} \times \mathbb{N}.$$

Furthermore, the converse is true, i.e., suppose that $P(A) \in \mathbb{Z}[A^{\pm}]$ is in the form above, then there is $m \in \mathbb{N}$ and a realizable Catalan state C of $L(m,3)$, such that $C(A) = P(A)$.

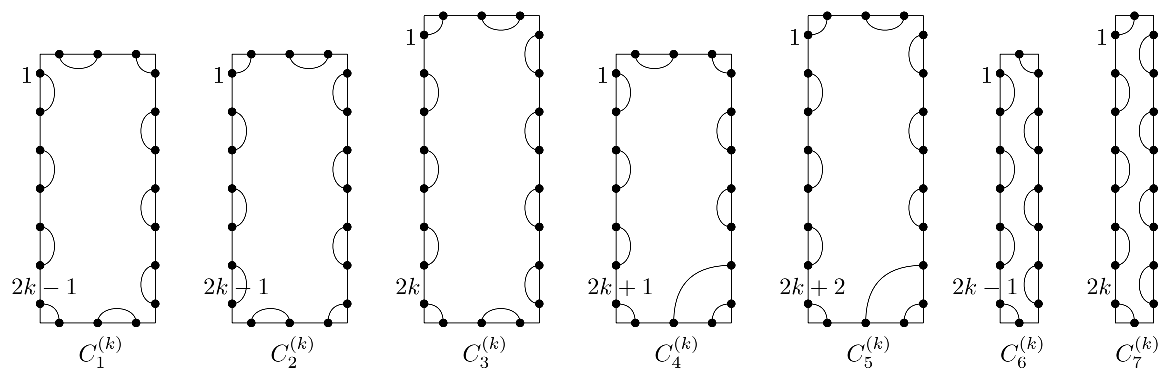


Figure 4.10. Proof of Theorem 4.1.6

Proof. Our proof is similar to the case $n = 4$. First we consider the cases $C \in \mathfrak{Cat}_{m,3}$ satisfying $|C \cap l_i^h| < 3$, for $0 \leq i \leq m$. When $n = 3$, there are only two possible top or bottom states. After removing all arcs that connect y_j and $y'_{j\pm1}$, up to a π-rotation, we are left with the following Catalan states $C_1^{(k)}$, $C_2^{(k)}$, $C_3^{(k)}$ shown in Figure 4.10. As one checks that $C_4^{(k)}(A) = A^{-4k-1}[2]^{2k}$, $C_5^{(k)}(A) = A^{-4k-4}[2]^{2k+1}$, $C_6^{(k)}(A) = A$, $C_7^{(k)}(A) = 1$, thus by Corollary 3.2.7,

$$C_1^{(k)}(A) = \frac{A^6}{[3]}C_4^{(k)}(A) - \frac{A^4}{[3]}C_6^{(k)}(A) = \frac{A^5}{[3]}\left(A^{-4k}[2]^{2k} - 1\right),$$

$$C_2^{(k)}(A) = \frac{A^2}{[3]}(C_4^{(k)})^s(A) - \frac{A^4}{[3]}(C_6^{(k)})^s(A) = \frac{A^3}{[3]}\left(A^{-4k}[2]^{2k} - 1\right),$$

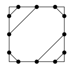
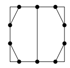
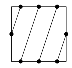
and

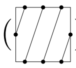
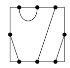
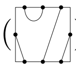
$$C_3^{(k)}(A) = \frac{A^6}{[3]}C_5^{(k)}(A) - \frac{A^2[2]}{[3]}C_7^{(k)}(A) = \frac{A^2[2]}{[3]}\left(A^{-4k}[2]^{2k} - 1\right).$$

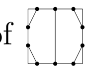
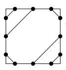
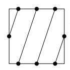
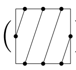
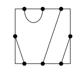
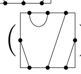
Putting back arcs connecting y_j and $y'_{j\pm1}$ to $C_1^{(k)}$, $C_2^{(k)}$, $C_3^{(k)}$, when $k \geq 2$, results in the second formula for $(a, b) \in \mathbb{Z} \times (\mathbb{N} \setminus \{1\})$. The case $b = 1$ can be dealt as the next case discussed below.

Now we consider cases of Catalan states for which there is $0 \leq i \leq m$, such that $|C \cap l_i^h| = 3$. In such a case, like in the case $n = 4$, we split C into vertically irreducible Catalan states, and after classifying all possible tops and bottoms, for each irreducible state, one shows that $C(A)$ must a Laurent polynomial in the form $A^a[2]^b[3]^c$, $(a, b, c) \in \mathbb{Z} \times \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}$.

For the converse, one sees that

 $(A) = A^{-7}[3]$,  $(A) = A^{-2}[2]$,  $(A) = A^{-3}$,

( $)^s(A) = A^3$,  $(A) = A^{-1}$, ( $)^s(A) = A$.

So given $(a, b, c) \in \mathbb{Z} \times \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}$, one constructs a Catalan state C using b of 's, c of 's, some number of 's or ( $)^s$, and possibly one of the states  or ( $)^s$. Clearly, for such C, $C(A) = A^a[2]^b[3]^c$. The second formula is already discussed above. □

4.2 The Unimodality of Coefficients of Catalan States

Recall, a sequence $(a_i)_{i=0}^n = (a_0, a_1, \ldots, a_n)$ of real numbers is *unimodal* if $a_0 \leq a_1 \leq \ldots \leq a_i \geq a_{i+1} \geq \ldots \geq a_n$, for some $0 \leq i \leq n$. A polynomial $P(q) = a_0 + a_1 q + \ldots + a_n q^n$ is called unimodal if the sequence of its coefficients $(a_0, a_1, \ldots, a_n)$ is unimodal. Unimodality of the coefficient $C(A)$ of a Catalan state C is defined as follows:

Definition 4.2.1. *Let C be a realizable Catalan state of $L(m, n)$ and*

$$C(A) = A^k \sum_{i=0}^{l} a_i A^{4i}$$

be its coefficient, $k \in \mathbb{Z}$ and $l \in \mathbb{Z}_{\geq 0}$. We say that $C(A)$ is unimodal if the sequence $(a_i)_{i=0}^l$ is unimodal.

The main results of this section give partial answers to the following questions:

Question 4.2.2.

- *Are coefficients of realizable Catalan states of $L(m, 4)$ unimodal?*
- *For which pairs (m, n), coefficients of all realizable Catalan states of $L(m, n)$ are unimodal?*

Recall, the following definitions: Let $(a_i)_{i=0}^n = (a_0, a_1, \ldots, a_n)$ be a unimodal sequence of real numbers, the index $0 \le i \le n$ such that $a_0 \le a_1 \le \ldots \le a_i \ge a_{i+1} \ge \ldots \ge a_n$ is called a *mode* of $(a_i)_{i=0}^n$. A sequence $(a_i)_{i=0}^n$ is *log-concave* if $a_i^2 \ge a_{i-1}a_{i+1}$, for all $1 \le i \le n-1$. A sequence $(a_i)_{i=0}^n$ has *no internal zeros* if there is no integers $0 \le i < j < k \le n$ satisfying $a_i \neq 0$, $a_j = 0$, and $a_k \neq 0$. A sequence $(a_i)_{i=0}^n$ is *symmetric* if $a_i = a_{n-i}$, for all $0 \le i \le n$. A polynomial $P(q) = a_0 + a_1 q + \ldots + a_n q^n$ is log-concave (respectively symmetric) if the sequence of its coefficients $(a_0, a_1, \ldots, a_n)$ is log-concave (respectively symmetric). It can easily be seen that positive log-concave sequences are unimodal.

Theorem 4.2.3 (Stanley [28], Keilson-Gerber [17])**.** *Given polynomials $A(q)$ and $B(q)$ with nonnegative coefficients.*

(a) If coefficients of $A(q)$ and $B(q)$ are log-concave and have no internal zeros, then coefficients of $A(q)\,B(q)$ have the same properties.

(b) If $A(q)$ is unimodal and $B(q)$ is log-concave, then $A(q)\,B(q)$ is unimodal.

Recall, polynomials $\tilde{\mathcal{A}}_k(q)$, $\tilde{\mathcal{B}}_k(q)$, and $\tilde{\mathcal{C}}_k(q)$ are defined by:

$$\tilde{\mathcal{A}}_k(q) = \frac{[2]_q^{2k}[4]_q - q^k}{[3]_q} = a_0 + a_1 q + \ldots + a_{2k+1} q^{2k+1},$$

$$\tilde{\mathcal{B}}_k(q) = \frac{[2]_q^{2k}[4]_q - q^{k+3}}{[3]_q} = b_0 + b_1 q + \ldots + b_{2k+1} q^{2k+1},$$

and

$$\tilde{\mathcal{C}}_k(q) = \frac{[2]_q^{2k+1}[4]_q + q^{k+2}}{[3]_q} = c_0 + c_1 q + \ldots + c_{2k+2} q^{2k+2},$$

for $k \ge 0$.

Lemma 4.2.4. *Polynomials $\tilde{\mathcal{A}}_k(q)$, $\tilde{\mathcal{B}}_k(q)$, and $\tilde{\mathcal{C}}_k(q)$ satisfy the following properties*

(a) $a_{2k+1-i} = a_i$ for all $i \neq k, k+1$ and $a_{k+1} = a_k + 1$,

(b) $b_i = a_{2k+1-i}$ for all i, and

(c) $c_i = a_i + a_{i-1}$ *for all* $i \neq k$ *and* $c_k = a_k + a_{k-1} + 1$, *where* $a_{-1} = a_{2k+2} = 0$.

Proof. It is sufficient to notice that properties (a)–(c) are the consequences of the following observations:

$$\tilde{\mathcal{A}}_k(q) - \tilde{\mathcal{B}}_k(q) = \frac{q^{k+3} - q^k}{[3]_q} = q^{k+1} - q^k,$$

$$\tilde{\mathcal{A}}_k(q^{-1}) = \frac{q^{-2k-3}[2]_q^{2k}[4]_q - q^{-k}}{q^{-2}[3]_q} = q^{-2k-1}\tilde{\mathcal{B}}_k(q),$$

and

$$\tilde{\mathcal{C}}_k(q) - \tilde{\mathcal{A}}_k(q)[2]_q = q^k.$$

We see that substituting for $\tilde{\mathcal{B}}_k(q)$ from the first equation into the second and comparing the coefficients yields (a). Properties (b) and (c) follow from (a) and the above identities. □

Lemma 4.2.5. *Let* $\tilde{\mathcal{A}}_k(q)$, $\tilde{\mathcal{B}}_k(q)$, *and* $\tilde{\mathcal{C}}_k(q)$ *be polynomials defined above. Then*

(a) Coefficients of $\tilde{\mathcal{A}}_k(q)$ *are non-negative and unimodal with the mode* $k+1$,

(b) Coefficients of $\tilde{\mathcal{B}}_k(q)$ *are non-negative and unimodal with the mode* k, *and*

(c) Coefficients of $\tilde{\mathcal{C}}_k(q)$ *are non-negative, symmetric and unimodal with a mode* $k+1$.

Proof. We prove (a) by induction on k. When $k = 0$ or $k = 1$, $\tilde{\mathcal{A}}_0(q) = q$ and $\tilde{\mathcal{A}}_1(q) = 1 + q + 2q^2 + q^3$ both polynomials have non-negative unimodal coefficients with the mode 1 or 2 respectively.

Assume that $\tilde{\mathcal{A}}_k(q) = a_0 + a_1 q + \ldots + a_{2k+1}q^{2k+1}$ and $a_0 \leq a_1 \leq \ldots \leq a_k \leq a_{k+1} \geq a_{k+2} \geq \ldots \geq a_{2k+1}$. Since

$$\tilde{\mathcal{A}}_{k+1}(q) = \frac{[2]_q^{2k+2}[4]_q - q^{k+1}}{[3]_q} = \frac{[2]_q^2([3]_q\tilde{\mathcal{A}}_k(q) + q^k) - q^{k+1}}{[3]_q} = [2]_q^2\tilde{\mathcal{A}}_k(q) + q^k,$$

if $\tilde{\mathcal{A}}_{k+1}(q) = a'_0 + a'_1 q + \ldots + a'_{2k+3}q^{2k+3}$, then $a'_i = a_i + 2a_{i-1} + a_{i-2}$, for all $0 \leq i \leq 2k+3$ except when $i = k$, $a'_k = a_k + 2a_{k-1} + a_{k-2} + 1$, where $a_{-1} = a_{-2} = a_{2k+2} = a_{2k+3} = 0$. By

Lemma 4.2.4, it suffices to show that $a'_0 \le a'_1 \le \ldots \le a'_k \le a'_{k+1}$. Since $a'_i = a_i + 2a_{i-1} + a_{i-2} \le a_{i+1} + 2a_i + a_{i-1} \le a'_{i+1}$, for all $0 \le i \le k-1$, and $a'_k = a_k + 2a_{k-1} + a_{k-2} + 1 = a_{k+1} + 2a_{k-1} + a_{k-2} \le a_{k+1} + 2a_k + a_{k-1} \le a'_{k+1}$, statement given in (a) follows.

Statement (b) is a consequence of (a) and Lemma 4.2.4.

Since

$$\tilde{\mathcal{C}}_k(q^{-1}) = \frac{q^{-2k-4}[2]_q^{2k+1}[4]_q + q^{-k-2}}{q^{-2}[3]_q} = q^{-2k-2}\tilde{\mathcal{C}}_k(q),$$

polynomial $\tilde{\mathcal{C}}_k(q)$ is symmetric. By Lemma 4.2.4, $c_{i-1} = a_{i-1} + a_{i-2} \le a_i + a_{i-1} \le c_i$, for all $1 \le i \le k$ and $c_k = a_k + a_{k-1} + 1 = a_{k+1} + a_{k-1} \le a_{k+1} + a_k \le c_{k+1}$. Since $\tilde{\mathcal{C}}_k(q)$ is symmetric, statement (c) follows. □

Lemma 4.2.6. *$\tilde{\mathcal{A}}_k(q)$ and $\tilde{\mathcal{B}}_k(q)$ are log-concave for $k \ge 2$ and $\tilde{\mathcal{C}}_k(q)$ is log-concave for $k \ge 1$.*

Proof. We prove that $\tilde{\mathcal{A}}_k(q)$ is log-concave for $k \ge 2$ by induction on k. One checks that $\tilde{\mathcal{A}}_2(q) = 1 + 4q + 5q^2 + 6q^3 + 4q^4 + q^5$ is log-concave. Let $\tilde{\mathcal{A}}_k(q) = a_0 + a_1 q + \ldots + a_{2k+1}q^{2k+1}$ and $\tilde{\mathcal{A}}_{k+1}(q) = a'_0 + a'_1 q + \ldots + a'_{2k+3}q^{2k+3}$ and assume that $\tilde{\mathcal{A}}_k(q)$ is log-concave. In the proof of Lemma 4.2.5 we showed $a'_i = a_i + 2a_{i-1} + a_{i-2}$ for all $0 \le i \le 2k+3$ except $i = k$, which is $a'_k = a_k + 2a_{k-1} + a_{k-2} + 1$, where $a_{-1} = a_{-2} = a_{2k+2} = a_{2k+3} = 0$. Moreover, by the induction hypothesis $a_i^2 - a_{i+1}a_{i-1} \ge 0$ for all $0 \le i \le 2k+1$.

- For $0 \le i \le k-2$ and $k+2 \le i \le 2k+3$:

$$\begin{aligned}
(a'_i)^2 - a'_{i+1}a'_{i-1} &= (a_i + 2a_{i-1} + a_{i-2})^2 - (a_{i+1} + 2a_i + a_{i-1})(a_{i-1} + 2a_{i-2} + a_{i-3}) \\
&= (a_i^2 + 4a_{i-1}^2 + a_{i-2}^2 + 4a_ia_{i-1} + 2a_ia_{i-2} + 4a_{i-1}a_{i-2}) \\
&- (a_{i+1}a_{i-1} + 2a_{i+1}a_{i-2} + a_{i+1}a_{i-3} + 2a_ia_{i-1} + 4a_ia_{i-2} \\
&+ 2a_ia_{i-3} + a_{i-1}a_{i-1} + 2a_{i-1}a_{i-2} + a_{i-1}a_{i-3}) \\
&= a_i^2 + 3a_{i-1}^2 + a_{i-2}^2 + 2a_ia_{i-1} - 2a_ia_{i-2} + 2a_{i-1}a_{i-2} \\
&- a_{i+1}a_{i-1} - 2a_{i+1}a_{i-2} - a_{i+1}a_{i-3} - 2a_ia_{i-3} - a_{i-1}a_{i-3} \ge 0,
\end{aligned}$$

  because $a_ia_{i-1} \ge a_{i+1}a_{i-2}$ and $a_{i-1}^2 \ge a_{i+1}a_{i-3}$.

- For $i = k-1$,

$$
\begin{aligned}
&(a'_{k-1})^2 - a'_k a'_{k-2} \\
=&(a_{k-1} + 2a_{k-2} + a_{k-3})^2 - (a_k + 2a_{k-1} + a_{k-2} + 1)(a_{k-2} + 2a_{k-3} + a_{k-4}) \\
=&(a_{k+2} + 2a_{k+3} + a_{k+4})^2 - (a_{k+1} + 2a_{k+2} + a_{k+3})(a_{k+3} + 2a_{k+4} + a_{k+5}) \\
=&(a'_{k+4})^2 - a'_{k+3} a'_{k+5} \geq 0,
\end{aligned}
$$

  by Lemma 4.2.4 and previous case when $i = k+4$.

- For $i = k$,

$$
\begin{aligned}
&(a'_k)^2 - a'_{k+1} a'_{k-1} \\
=&(a_k + 2a_{k-1} + a_{k-2} + 1)^2 - (a_{k+1} + 2a_k + a_{k-1})(a_{k-1} + 2a_{k-2} + a_{k-3}) \\
\geq&(a_k + 2a_{k-1} + a_{k-2})^2 - (a_{k+1} + 2a_k + a_{k-1})(a_{k-1} + 2a_{k-2} + a_{k-3}) \geq 0,
\end{aligned}
$$

  using arguments similar to the first case.

- For $i = k+1$,

$$
\begin{aligned}
&(a'_{k+1})^2 - a'_{k+2} a'_k = (a_{k+1} + 2a_k + a_{k-1})^2 - (a_{k+2} + 2a_{k+1} + a_k)(a_k + 2a_{k-1} + a_{k-2} + 1) \\
&= (3a_k + a_{k-1} + 1)^2 - (3a_k + a_{k-1} + 2)(a_k + 2a_{k-1} + a_{k-2} + 1) \\
&= (6a_k^2 + a_k) - (3a_k a_{k-2} + a_k a_{k-1} + a_{k-1}^2 + a_{k-1} a_{k-2} + 3a_{k-1} + 2a_{k-2} + 1) \\
&\geq (5a_k a_{k-1} + 3a_{k-1} + a_k) - (3a_k a_{k-2} + a_{k-1}^2 + a_{k-1} a_{k-2} + 2a_{k-2} + 1) \geq 0,
\end{aligned}
$$

  since by Lemma 4.2.4, $a_k^2 \geq a_{k+1} a_{k-1} = (a_k + 1) a_{k-1}$ and $a_k \geq 1$ (which is clearly true for $\tilde{\mathcal{A}}_k(q)$ when $k \geq 1$).

By Lemma 4.2.4, it follows that $\tilde{\mathcal{B}}_k(q)$ for $k \geq 2$ is log-concave.

Clearly, $\tilde{\mathcal{C}}_1(q) = 1 + 3q + 3q^2 + 3q^3 + q^4$ is log-concave. For $k \geq 2$, to show that $\tilde{\mathcal{C}}_k(q)$ is log-concave, we use the relation $c_i = a_i + a_{i-1}$, for $i \neq k$, and $c_k = a_k + a_{k-1} + 1$.

- For $0 \leq i \leq k-2$ and $k+2 \leq i \leq 2k+2$

$$\begin{aligned} c_i^2 - c_{i+1}c_{i-1} &= (a_i + a_{i-1})^2 - (a_{i+1} + a_i)(a_{i-1} + a_{i-2}) \\ &= (a_i^2 + a_{i-1}^2 + 2a_ia_{i-1}) - (a_{i+1}a_{i-1} + a_{i+1}a_{i-2} + a_ia_{i-1} + a_ia_{i-2}) \geq 0, \end{aligned}$$

  because $a_ia_{i-1} \geq a_{i+1}a_{i-2}$.

- For $i = k-1$,

$$\begin{aligned} &c_{k-1}^2 - c_kc_{k-2} \\ =&(a_{k-1} + a_{k-2})^2 - (a_k + a_{k-1} + 1)(a_{k-2} + a_{k-3}) \\ =&(a_{k+2} + a_{k+3})^2 - (a_{k+1} + a_{k+2})(a_{k+3} + a_{k+4}) \\ =&c_{k+3}^2 - c_{k+2}c_{k+4} \geq 0, \end{aligned}$$

  using Lemma 4.2.4 and previous case with $i = k+3$.

- For $i = k$,

$$\begin{aligned} &c_k^2 - c_{k+1}c_{k-1} \\ =&(a_k + a_{k-1} + 1)^2 - (a_{k+1} + a_k)(a_{k-1} + a_{k-2}) \\ \geq&(a_k + a_{k-1})^2 - (a_{k+1} + a_k)(a_{k-1} + a_{k-2}) \geq 0, \end{aligned}$$

  by arguments similar as in the first case.

- For $i = k+1$,

$$\begin{aligned} c_{k+1}^2 - c_{k+2}c_k &= (a_{k+1} + a_k)^2 - (a_{k+2} + a_{k+1})(a_k + a_{k-1} + 1) \\ &= (a_{k+1} + a_k)^2 - (a_{k+1} + a_{k-1})^2 \geq 0, \end{aligned}$$

  since $a_k \geq a_{k-1}$.

This completes our proof. $\square$

Theorem 4.2.7. *Let C be a realizable Catalan state of $L(m,4)$. If $|C \cap l_i^h| = 4$ for some $0 \leq i \leq m$, then $C(A)$ is unimodal.*

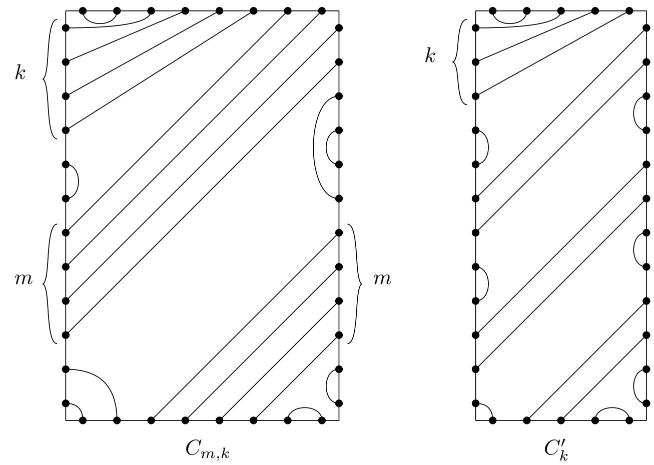


Figure 4.11. Examples of non-unimodal cases

Proof. According to Theorem 4.1.3, it suffices to show that

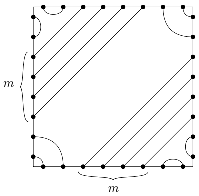
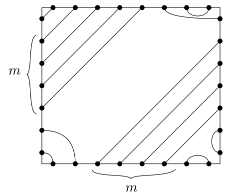
$$\begin{bmatrix}4\\2\end{bmatrix}_q^a [2]_q^b\,[3]_q^c\,[4]_q^d \prod_{i=1}^{r} \tilde{\mathcal{A}}_{\alpha_i}(q) \prod_{j=1}^{s} \tilde{\mathcal{B}}_{\beta_j}(q) \prod_{k=1}^{t} \tilde{\mathcal{C}}_{\gamma_k}(q),$$

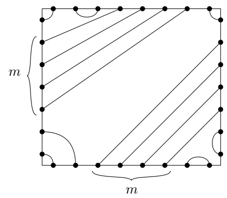
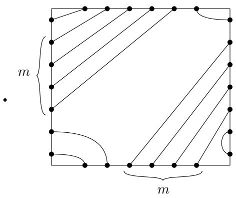
is unimodal, for all $a, b, c, d \in \mathbb{Z}_{\geq 0}$ and $\alpha_i, \beta_j, \gamma_k \in \mathbb{N}$. Polynomials $[2]_q$, $[3]_q$, $[4]_q$ are log-concave and by Lemma 4.2.6 $\tilde{\mathcal{A}}_k(q)$, $\tilde{\mathcal{B}}_k(q)$, $\tilde{\mathcal{C}}_{k'}(q)$ are also log-concave (and their coefficients are non-negative and have no internal zeros), for $k \geq 2$ and $k' \geq 1$. One checks that $\tilde{\mathcal{A}}_1(q)^n$ is log-concave for $n = 3, 4, 5$. Since for $n \geq 6$, $n = 3n_1 + 4n_2$, for some $n_1, n_2 \geq 0$, it follows by Theorem 4.2.3 that $\tilde{\mathcal{A}}_1(q)^n = (\tilde{\mathcal{A}}_1(q)^3)^{n_1}(\tilde{\mathcal{A}}_1(q)^4)^{n_2}$ is log-concave. Therefore, for all $n \geq 3$, $\tilde{\mathcal{A}}_1(q)^n$ is log-concave. The above argument also applies for $\tilde{\mathcal{B}}_1(q)^n$ and $\left[\begin{smallmatrix}4\\2\end{smallmatrix}\right]_q^n$. Finally, one checks that $\left[\begin{smallmatrix}4\\2\end{smallmatrix}\right]_q^a\tilde{\mathcal{A}}_1(q)^n\tilde{\mathcal{B}}_1(q)^{n'}$ are unimodal, for all $0 \leq a, n, n' \leq 3$. Therefore, we conclude our statement by applying Theorem 4.2.3. $\square$

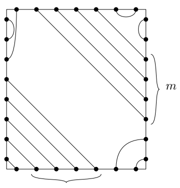
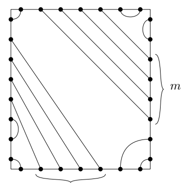
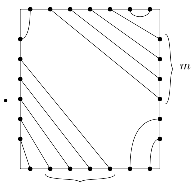
The following lemma partially answers the second part of the Question 4.2.2.

Lemma 4.2.8. *Coefficients $C_{m,k}(A)$ and $C'_k(A)$ of Catalan states $C_{m,k} \in \mathfrak{Cat}_{4+m+k,4+m}$ and $C'_k \in \mathfrak{Cat}_{9+k,5}$ are not unimodal, for all $m \geq 2, k \geq 0$, where $C_{m,k}$ and C'_k are shown in Figure 4.11.*

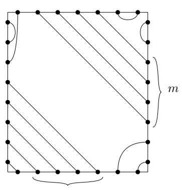
Proof. We find $C_{m,0}(A)$ by applying the first-row expansion (3.1) and Lemma 3.1.1,

 $(A) = A^{-(m+4)+2\cdot 1} \cdot$  (A)

$+ A^{-(m+4)+2(m+4)} \cdot$  $(A) + A^{-(m+4)+2(1+(m+4)-(m+3))} \cdot$  (A)

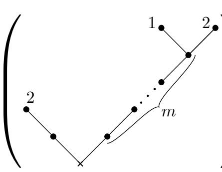
$= A^{-m-2} \cdot$  $(A^{-1}) + A^{m+4} \cdot$  $(A^{-1}) + A^{-m} \cdot$  (A^{-1}).

The last three Catalan states have no returns on the bottom boundary, thus we can compute their coefficients using Theorem 2.2.7:

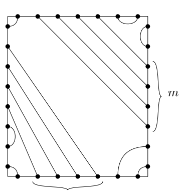
- For $C =$ , one checks that

$$\mathfrak{b}_M(C) = (m+2, \underbrace{m+3, \ldots, m+3}_{m+1}, m+1, m+1) \Rightarrow \|\mathfrak{b}_M(C)\| = m^2 + 7m + 7$$

  and the plucking polynomial with a delay function associated to C is given by

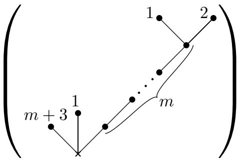
  $Q\Bigg($  $\Bigg) = q \begin{bmatrix} m+3 \\ 2 \end{bmatrix}_q.$

  Hence, $C(A) = A^{2(m^2+7m+7)-(m+3)(m+4)} \begin{bmatrix} m+3 \\ 2 \end{bmatrix}_{q=A^{-4}}$.

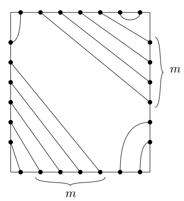
- For $C =$ , one verifies that

$$\mathfrak{b}_M(C) = (m+2, \underbrace{m+3, \ldots, m+3}_{m+1}, m+2, 1) \Rightarrow \|\mathfrak{b}_M(C)\| = m^2 + 6m + 8$$

and the plucking polynomial with a delay function associated with C is given by

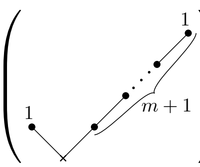
$Q\Bigg($  $\Bigg) = q\left([m+2]_q + q^{m+1}[2]_q\right)[2]_q.$

Hence, $C(A) = A^{2(m^2+6m+8)-(m+3)(m+4)}\left(\left([m+2]_q + q^{m+1}[2]_q\right)[2]_q\right)_{q=A^{-4}}$.

- For $C =$  , one checks that

$$\mathfrak{b}_M(C) = (\underbrace{m+2,\ldots,m+2}_{m+1}, m+1) \Rightarrow \|\mathfrak{b}_M(C)\| = m^2+4m+3$$

and the plucking polynomial with a delay function associated with C is given by

$Q\Bigg($  $\Bigg) = [m+2]_q.$

Hence, $C(A) = A^{2(m^2+4m+3)-(m+3)(m+2)}[m+2]_{q=A^{-4}}$.

It follows that the coefficient of $C_{m,0}$ is

$$\begin{aligned} C_{m,0}(A) &= A^{-m^2-8m-4}\left(\frac{[m+3][m+2]}{[2]} + A^{4m+4}[2]([m+2] + A^{4m+4}[2]) + A^{4m+4}[m+2]\right) \\ &= A^{-m^2-8m-4}\left(A^{8m+16} + 3A^{8m+12} + 5A^{8m+8} + 4A^{8m+4} + 5A^{8m} + Q(A)\right), \end{aligned}$$

where $Q(A)$ is a polynomial of A with $\deg_A Q(A) \leq 8m-4$. This shows that $C_{m,0}(A)$ is not unimodal. Using the first-row expansion (3.1), we see that $C_{m,k}(A) = A^{-(m+2)k}C_{m,0}(A)$. Therefore, $C_{m,k}(A)$ is also not unimodal.

The coefficient $C'_0(A)$ was found in Example 3.1.15 and it is not unimodal. Since using the first-row expansion (3.1), $C'_k(A) = A^{-3k}C'_0(A)$, it follows that $C'_k(A)$ is also not unimodal. □

It is known that coefficients of realizable Catalan states of $L(m,n)$, $n = 1,2,3$, are Laurent polynomials with unimodal coefficients. Using a computer, one verifies that all coefficients of realizable Catalan states of $L(m,5)$, where $1 \leq m \leq 8$, are also unimodal. Hence, to answer the second part of Question 4.2.2 we only need to answer its first part. Using Theorem 4.1.3, it is then suffices to prove the following conjecture:

Conjecture 4.2.9. *If the rational functions given by* (4.2), (4.3) *and* (4.4) *are Laurent polynomials, then they are unimodal.*

CHAPTER 5

GENERALIZED CROSSING AND FUTURE WORK

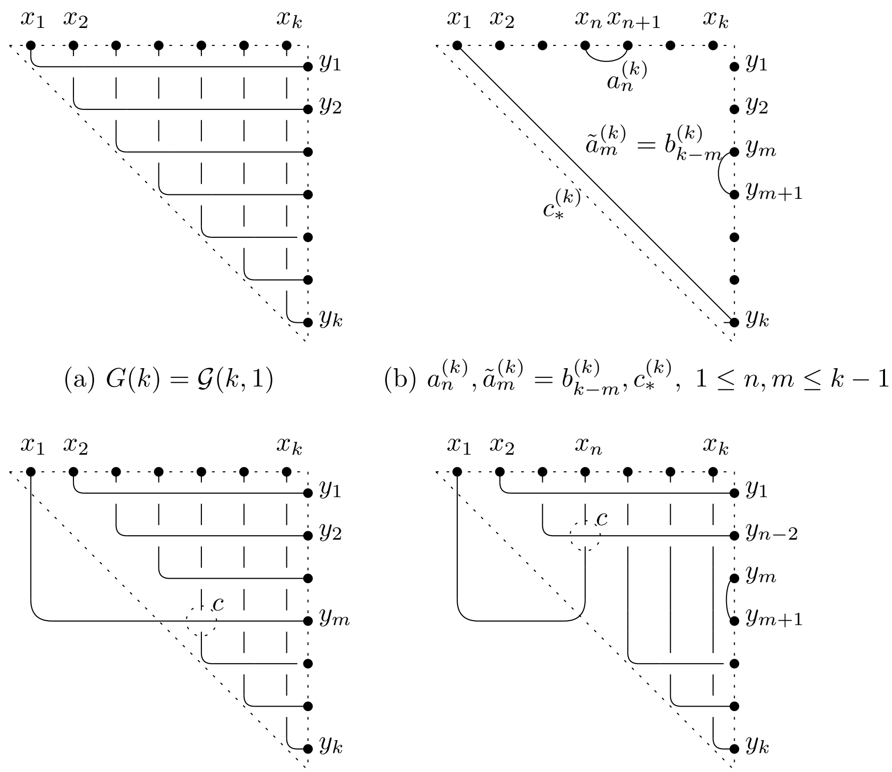
5.1 Coefficients of Catalan States of Generalized Crossing

Let $\mathcal{G}(k,m)$ and $\tilde{\mathcal{G}}(k,n,m)$ be tangles shown in Figure 5.1(c) and Figure 5.1(d), and let $a_n^{(k)}$, $\tilde{a}_m^{(k)} = b_{k-m}^{(k)}$, $c_*^{(k)}$ be arcs with endpoints $\{x_n, x_{n+1}\}$, $\{y_m, y_{m+1}\}$, $\{x_1, y_k\}$, respectively, as in Figure 5.1(b). We see that $\mathcal{G}(k,1) = G(k)$ and $\mathcal{G}(k,k) = G(k-1)\, c_*^{(k)}$, where $G(k-1)\, c_*^{(k)}$ is the generalized crossing $G(k-1)$ with the arc $c_*^{(k)}$ and its two endpoints added. For convenience, let $\mathcal{G}(0,1) = G(0)$. After smoothing the crossing c of $\mathcal{G}(k,m)$ and $\tilde{\mathcal{G}}(k,n,m)$, in the RKBSM, one obtains

$$\begin{aligned}
\mathcal{G}(k,m) &= A\,\tilde{\mathcal{G}}(k,m+1,m) + A^{-1}\,\mathcal{G}(k,m+1) \\
&= A\,\tilde{\mathcal{G}}(k,m+1,m) + A^{-1}\,\left[A\,\tilde{\mathcal{G}}(k,m+2,m+1) + A^{-1}\,\mathcal{G}(k,m+2)\right] \\
&= \left[\sum_{j=1}^{k-m} A^{2-j}\,\tilde{\mathcal{G}}(k,m+j,m+j-1)\right] + A^{m-k}\,\mathcal{G}(k,k)
\end{aligned}$$

and

$$\begin{aligned}
\tilde{\mathcal{G}}(k,n,m) &= A\,\mathcal{G}(k-2,n-2)\,a_{n-1}^{(k)}\,\tilde{a}_m^{(k)} + A^{-1}\,\tilde{\mathcal{G}}(k,n-1,m) \\
&= A\,\mathcal{G}(k-2,n-2)\,a_{n-1}^{(k)}\,\tilde{a}_m^{(k)} \\
&+ A^{-1}\,\left[A\,\mathcal{G}(k-2,n-3)\,a_{n-2}^{(k)}\,\tilde{a}_m^{(k)} + A^{-1}\,\tilde{\mathcal{G}}(k,n-2,m)\right] \\
&= \left[\sum_{i=1}^{n-2} A^{3-n+i}\,\mathcal{G}(k-2,i)\,a_{i+1}^{(k)}\,\tilde{a}_m^{(k)}\right] + A^{2-n}\,\mathcal{G}(k-2,1)\,a_1^{(k)}\,\tilde{a}_m^{(k)} \\
&= \sum_{i=0}^{n-2} A^{3-n+i-\mathbb{1}_{\{i=0\}}}\,\mathcal{G}(k-2,\max\{i,1\})\,a_{i+1}^{(k)}\,\tilde{a}_m^{(k)},
\end{aligned}$$



(a) $G(k) = \mathcal{G}(k,1)$ (b) $a_n^{(k)}, \tilde{a}_m^{(k)} = b_{k-m}^{(k)}, c_*^{(k)},\ 1 \leq n, m \leq k-1$

(c) $\mathcal{G}(k,m),\ 1 \leq m \leq k$ (d) $\tilde{\mathcal{G}}(k,n,m),\ 2 \leq n \leq m+1 \leq k$

Figure 5.1. Tangles $G(k), \mathcal{G}(k,m), \tilde{\mathcal{G}}(k,n,m)$ and arcs $a_n^{(k)}, \tilde{a}_m^{(k)}, c_*^{(k)}$

where $G(k-2,i)\, a_n^{(k)}\, \tilde{a}_m^{(k)}$ is the tangle $G(k-2,i)$ with the arcs $a_n^{(k)}$ and $\tilde{a}_m^{(k)}$ together with their two endpoints added (see Figure 5.1(b)). Thus,

$$
\begin{aligned}
&\mathcal{G}(k,m) \\
&= \left[\sum_{j=1}^{k-m} A^{2-j} \left[\sum_{i=0}^{(m+j)-2} A^{3-(m+j)+i-\mathbb{1}_{\{i=0\}}}\, \mathcal{G}(k-2, \max\{i,1\})\, a_{i+1}^{(k)}\, \tilde{a}_{m+j-1}^{(k)}\right]\right] + A^{m-k}\, \mathcal{G}(k,k),
\end{aligned}
$$

and, after the change of indices: $i \to k-i-1$ and $j \to j-m+1$, one obtains:

$$
\begin{aligned}
&\mathcal{G}^*(k,m) = A^{-m}\mathcal{G}(k,m) \\
&= \left[\sum_{(i,j)\in\mathbb{I}(k,m)} A^{1+2(k-i-j)}\, \mathcal{G}^*(k-2, \max\{k-i-1,1\})\, a_{k-i}^{(k)}\, b_{k-j}^{(k)}\right] + A^{1-k}\, \mathcal{G}^*(k-1,1)\, c_*^{(k)},
\end{aligned}
\tag{5.1}
$$

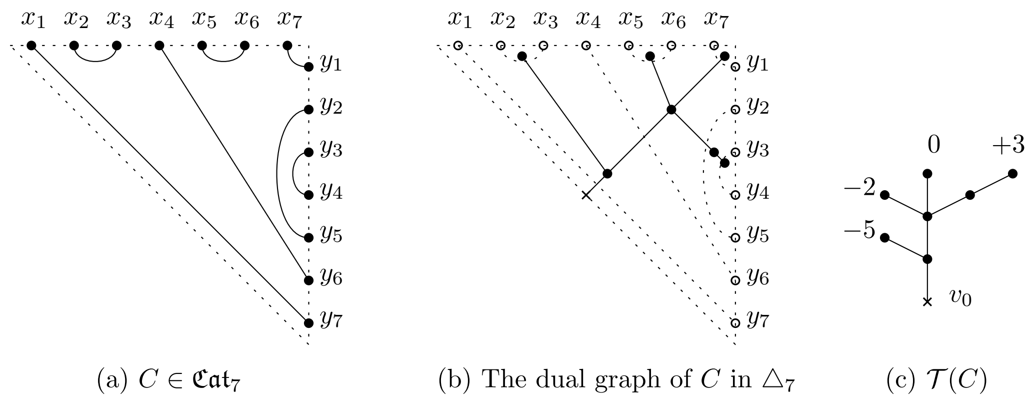


Figure 5.2. Dual tree $\mathcal{T}(C)$ for $C \in \mathfrak{Cat}_k$

where $\mathbb{I}(k,m) = \{(i,j) \in \mathbb{N} \times \mathbb{N} \mid i \leq k-1,\ m \leq j \leq k-1,\ i+j \geq k\}$ is the feasible set of indices for $a_{k-i}^{(k)}$ and $b_{k-j}^{(k)}$.

Using the above equation, we see that there is a recursion that computes coefficients of Catalan states of $G(k)$ that is an analog of the first-row expansion for lattice crossing $L(m,n)$.

Definition 5.1.1. *Define a plane rooted tree with a label function associated to a Catalan state $C \in \mathfrak{Cat}_k$ as follows. Denote by $T(C)$ the dual graph of C in $\triangle_k$. The root v_0 of $T(C)$ is the vertex that corresponds to the region containing the hypotenuse of $\triangle_k$. Define the label function f from the set of leaves $L(T(C))$ of $T(C)$ different than v_0 as follows. For a leaf v let $f(v) = -i$ if v corresponds to the region bounded by the arc with boundary $\{x_{k-i}, x_{k-i+1}\}$, and $f(v) = j$ if v corresponds to the region determined by the arc with boundary $\{y_j, y_{j+1}\}$, respectively, for $1 \leq i,j \leq k-1$, and we put $f(v) = 0$ if the region corresponding to v is determined by the arc with endpoints $\{x_k, y_1\}$ (see Figure 5.2). The plane rooted tree with a delay function determined by a Catalan state C is the triple $\mathcal{T}(C) = (T(C), v_0, f)$.*

Definition 5.1.2. *Let $\mathcal{T} = (T, v_0, f)$ be a plane rooted tree (T, v_0) with the root v_0 and a label function f defined on the set $L(T)$ of all leaves v different that v_0 and let $\mathbb{F}(k,m) =$*

$\{(v,u) \in L(T) \mid (-f(v), f(u)) \in \mathbb{I}(k,m)\}$ *be the feasible set of leaves. Define a polynomial* $\tilde{Q}_{k,m}(\mathcal{T}(C))$ *of* A, $k \geq 0$ *and* $1 \leq m \leq \max\{k,1\}$, *recursively as follows.*

Let $\tilde{Q}_{k,m}(T, v_0, f) = 0$ *if* T *is not a tree,* $\tilde{Q}_{0,1}(T, v_0, f) = 1$ *if* T *has no edges, and*

$$\begin{aligned}\tilde{Q}_{k,m}(T, v_0, f) &= A^{1-k}\,\tilde{Q}_{k-1,1}(T - v_0, v_0', f_{v_0}) \\ &\quad + \sum_{(v,u)\in\mathbb{F}(k,m)} A^{1+2(k+f(v)-f(u))}\,\tilde{Q}_{k-2,\max\{k+f(v)-1,1\}}(T - \{v,u\}, v_0, f_{v,u}),\end{aligned} \tag{5.2}$$

where v_0' *is the vertex that is incident to the unique edge that is incident to* v_0 *in* T, *provided that* $T - v_0$ *is a tree,* $f_{v_0}(w) = f(w)$ *for all* $w \in L(T - v_0)$, *and*[1]

$$f_{v,u}(w) = \begin{cases} f(w), & \text{if } f(v) < f(w) < f(u), \\ \operatorname{sgn}(f(w')) \cdot (|f(w')| - 1), & \text{if } w \text{ is a new leaf}, \{w, w'\} \in E(T) \text{ for } w' \in \{v,u\}, \\ \operatorname{sgn}(f(w)) \cdot (|f(w)| - 2), & \text{otherwise,} \end{cases}$$

for all $w \in L(T - \{v,u\})$.

Proposition 5.1.3. *Given a Catalan state* C *of* $G(k)$. *Let* $\mathcal{T}(C)$ *be the plane rooted tree with a label function associated with* C. *Then*

$$C(A) = \tilde{Q}_{k,1}(\mathcal{T}(C)),$$

where $\tilde{Q}_{k,1}(\cdot)$ *is defined as in Definition 5.1.2.*

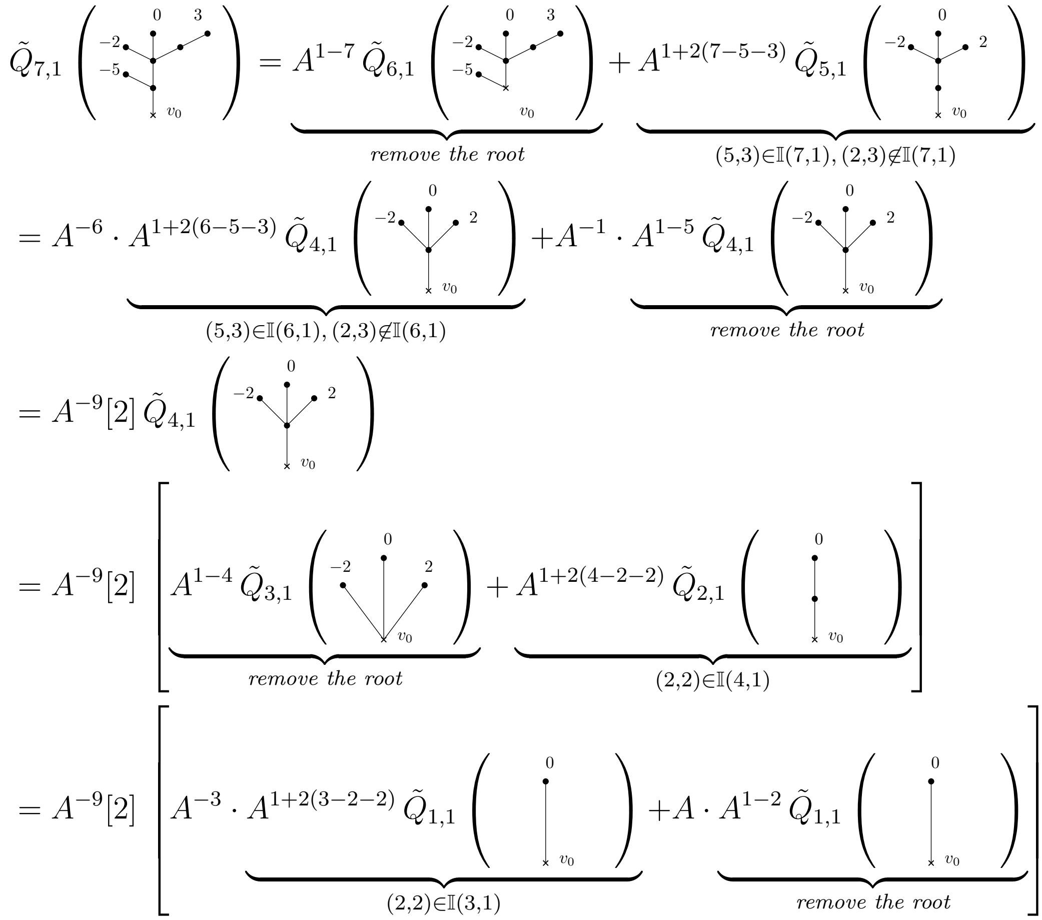
Proof. Let C be a Catalan state of $G(k)$, and let $(C)_{k,m}$ be the coefficient of Catalan state C of $\mathcal{G}^*(k,m) = A^{-m} \cdot \mathcal{G}(k,m)$. According to (5.1),

$$(C)_{k,m} = \left[\sum_{(i,j)\in\mathbb{I}(k,m)} A^{1+2(k-i-j)} \left(C - \{a^{(k)}_{k-i}, b^{(k)}_{k-j}\}\right)_{k-2,\max\{k-i-1,1\}} \right] + A^{1-k} \left(C - c^{(k)}_{*}\right)_{k-1,1}.$$

Let $\tilde{Q}_{k,m}(\mathcal{T}(C)) = A \cdot (C)_{k,m}$. It is easy to see that (5.2) is same as the equation above, and the initial condition $\tilde{Q}_{0,1}(\mathcal{T}(C)) = 1 = A \cdot A^{-1} = A \cdot (C)_{0,1}$ since $(C)_{0,1}$ is the coefficient of $\mathcal{G}^*(0,1) = A^{-1}\mathcal{G}(0,1)$. Therefore, the $C(A) = A \cdot (C)_{k,1} = \tilde{Q}_{k,1}(\mathcal{T}(C))$. $\square$

[1] The sgn function defined as: $\operatorname{sgn}(i) = 1$ if $i > 0$, $\operatorname{sgn}(i) = -1$ if $i < 0$, and $\operatorname{sgn}(0) = 0$.

Example 5.1.4. *We can calculate the Catalan state $C \in \mathfrak{Cat}_7$ shown in Figure 5.2, as follows.*



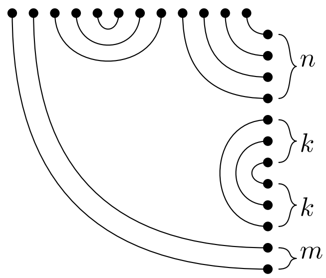
$$= A^{-9}[2] \cdot A^{-4}[2] \cdot (A^{1-1} \cdot 1) = A^{-13}[2]^2.$$

An immediate consequence of Proposition 5.1.3 is the following theorem.

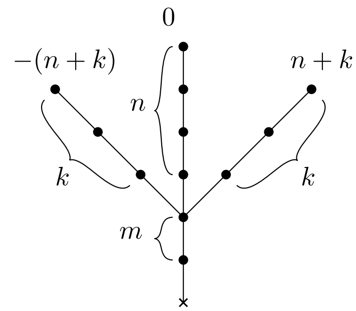
Theorem 5.1.5. *The coefficients of Catalan states of $G(k)$ are Laurent polynomials with non-negative coefficients.*

In the remaining part of this section, we find coefficients for two families of Catalan states of $G(k)$ for some k.

Example 5.1.6. *Given a Catalan state $C_{n,k,m} \in \mathfrak{Cat}_{n+2k+m}$ shown in Figure 5.3(a). Let $P_l^{(n,k,m)} = \tilde{Q}_{n+2k+m,l}(\mathcal{T}(C_{n,k,m}))$, where the plane rooted tree with a label function $\mathcal{T}(C_{n,k,m})$*



(a) $C_{n,k,m}$



(b) $\mathcal{T}(C_{n,k,m})$

Figure 5.3. $C_{n,k,m}$ and $\mathcal{T}(C_{n,k,m})$ in Example 5.1.6

is shown in Figure 5.3(b). We consider the case $n = 0$ first. For $m \geq 1$,

$$P_l^{(0,k,m)} = A^{1-(2k+m)}\, P_1^{(0,k,m-1)} = \ldots = A^{\sum_{i=1}^{m}(1-(2k+i))} P_1^{(0,k,0)} = A^{-2km-\frac{m(m-1)}{2}} P_1^{(0,k,0)},$$

and, for $k \geq 1$

$$P_1^{(0,k,0)} = A^{1+2(2k-k-k)} P^{(0,k-1,0)}_{\max\{k-1,1\}} = A^1 P^{(0,k-1,0)}_{\max\{k-1,1\}} = \ldots = A^k P_1^{(0,0,0)} = A^k.$$

Hence, $P_l^{(0,k,m)} = A^{k(1-2m)-\frac{m(m-1)}{2}}$. Then $P_l^{(n,0,m)} = P_l^{(0,0,n+m)} = A^{-\frac{(n+m)(n+m-1)}{2}}$.

For $m = 0$ and $n, k \geq 1$,

$$\begin{aligned} P_1^{(n,k,0)} &= A^{1+2(n+2k-(n+k)-(n+k))} P^{(n,k-1,0)}_{\max\{k-1,1\}} = A^{1-2n} P^{(n,k-1,0)}_{\max\{k-1,1\}} = \ldots \\ &= A^{k(1-2n)} P_1^{(n,0,0)} = A^{k(1-2n)-\frac{n(n-1)}{2}}. \end{aligned}$$

Now, for $n, k, m \in \mathbb{N}$,

$$\begin{aligned} P_1^{(n,k,m)} &= A^{1-(n+2k+m)} P_1^{(n,k,m-1)} \cdot \mathbb{1}_{\{m\geq 1\}} \\ &\quad + A^{1+2((n+2k+m)-(n+k)-(n+k))} P^{(n,k-1,m)}_{\max\{k+m-1,1\}} \cdot \mathbb{1}_{\{n\geq m,\, k+m\geq 1\}} \\ &= A^{1-(n+2k+m)} P_1^{(n,k,m-1)} \cdot \mathbb{1}_{\{m\geq 1\}} + A^{1+2m-2n} P^{(n,k-1,m)}_{\max\{k+m-1,1\}} \cdot \mathbb{1}_{\{n\geq m,\, k+m\geq 1\}} \end{aligned}$$

and

$$\begin{aligned} P^{(n,k-1,m)}_{\max\{k+m-1,1\}} &= A^{1-(n+2(k-1)+m)} P_1^{(n,k-1,m-1)} \cdot \mathbb{1}_{\{m\geq 1\}} \\ &\quad + A^{1+2((n+2(k-1)+m)-(n+k-1)-(n+k-1))} P^{(n,k-2,m)}_{\max\{k+m-2,1\}} \cdot \mathbb{1}_{\{n\geq m,\, k+m\geq 2\}} \\ &= A^{1-(n+2k+m)+2} P_1^{(n,k-1,m-1)} \cdot \mathbb{1}_{\{m\geq 1\}} + A^{1+2m-2n} P^{(n,k-2,m)}_{\max\{k+m-2,1\}} \cdot \mathbb{1}_{\{n\geq m,\, k+m\geq 2\}}. \end{aligned}$$

So if $m > n \geq 1$, then

$$\begin{aligned} P_1^{(n,k,m)} &= A^{1-(n+2k+m)} P_1^{(n,k,m-1)} = A^{1-(n+2k+m)} A^{1-(n+2k+m-1)} P_1^{(n,k,m-2)} = \dots \\ &= A^{(1-n-2k-m)(m-n)+\sum_{i=0}^{m-n-1} i} P_1^{(n,k,n)} = A^{(1-n-2k-m)(m-n)+\frac{(m-n-1)(m-n)}{2}} P_1^{(n,k,n)}, \end{aligned}$$

and if $n \geq m \geq 1$, then

$$\begin{aligned} P_1^{(n,k,m)} &= A^{1-(n+2k+m)} \sum_{i=0}^{k-1} A^{(3+2m-2n)i} P_1^{(n,k-i,m-1)} + A^{(1+2m-2n)k} P_{\max\{m,1\}}^{(n,0,m)} \\ &= A^{1-(n+2k+m)} \sum_{i=1}^{k} A^{(3+2m-2n)(k-i)} P_1^{(n,i,m-1)} + A^{(1+2m-2n)k} P_{\max\{m,1\}}^{(n,0,m)}. \end{aligned} \tag{5.3}$$

However,

$$P_{\max\{m,1\}}^{(n,0,m)} = P_{\max\{m,1\}}^{(0,0,n+m)} = A^{1-(n+m)} P_1^{(0,0,n+m-1)} = P_1^{(0,0,n+m)} = P_1^{(n,0,m)},$$

if we define $\tilde{P}_1^{(n,k,m)} = A^{-(1+2m-2n)k} P_1^{(n,k,m)}$, then (5.3) becomes

$$\tilde{P}_1^{(n,k,m)} = \tilde{P}_1^{(n,0,m)} + A^{1-n-m} \sum_{i=1}^{k} A^{-4i} \tilde{P}_1^{(n,i,m-1)}.$$

So,

$$\begin{aligned} \tilde{P}_1^{(n,k,m)} &= \tilde{P}_1^{(n,0,m)} + A^{1-n-m} \sum_{i_1=1}^{k} A^{-4i_1} \left(\tilde{P}_1^{(n,0,m-1)} + A^{1-n-(m-1)} \sum_{i_2=1}^{i_1} A^{-4i_2} \tilde{P}_1^{(n,i_2,m-2)} \right) \\ &= \tilde{P}_1^{(n,0,m)} + A^{1-n-m} \tilde{P}_1^{(n,0,m-1)} \left(\sum_{i_1=1}^{k} A^{-4i_1} \right) + A^{2(1-n-m)+1} \left(\sum_{i_1=1}^{k} \sum_{i_2=1}^{i_1} A^{-4i_1-4i_2} \tilde{P}_1^{(n,i_2,m-2)} \right) \\ &= \dots \\ &= \tilde{P}_1^{(n,0,m)} + A^{1-n-m} \tilde{P}_1^{(n,0,m-1)} \left(\sum_{i_1=1}^{k} A^{-4i_1} \right) + A^{2(1-n-m)+1} \tilde{P}_1^{(n,0,m-2)} \left(\sum_{i_1=1}^{k} \sum_{i_2=1}^{i_1} A^{-4i_1-4i_2} \right) \\ &+ \dots + A^{s(1-n-m)+\sum_{i=0}^{s-1} i} \tilde{P}_1^{(n,0,m-s)} \left(\sum_{i_1=1}^{k} \sum_{i_2=1}^{i_1} \cdots \sum_{i_s=1}^{i_{s-1}} A^{-4i_1-4i_2-\ldots-4i_s} \right) + \dots \\ &+ A^{m(1-n-m)+\sum_{i=0}^{m-1} i} \left(\sum_{i_1=1}^{k} \sum_{i_2=1}^{i_1} \cdots \sum_{i_m=1}^{i_{m-1}} A^{-4i_1-4i_2-\ldots-4i_m} \tilde{P}_1^{(n,i_m,0)} \right). \end{aligned}$$

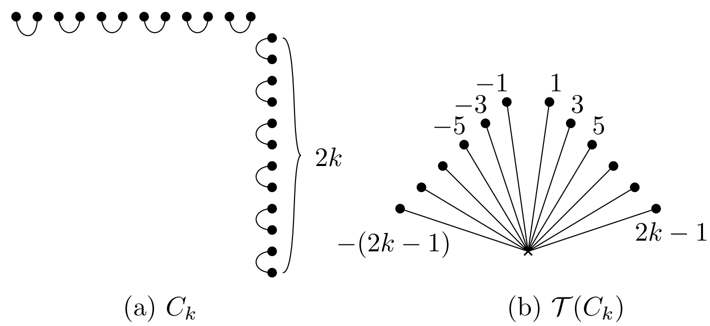


Figure 5.4. C_k and $\mathcal{T}(C_k)$ in Example 5.1.7

Since

$$\sum_{i_1=1}^{k}\sum_{i_2=1}^{i_1}\cdots\sum_{i_s=1}^{i_{s-1}} q^{i_1+i_2+\ldots+i_s} = \sum_{0\le i_s-1\le i_{s-1}-1\le\ldots\le i_1-1\le k-1} q^{(i_1-1)+(i_2-1)+\ldots+(i_s-1)}\cdot q^s$$
$$= q^s \begin{bmatrix} s+k-1 \\ s \end{bmatrix}_q,$$

and

$$\tilde{P}_1^{(n,i_m,0)} = A^{-(1+0-2n)i_m} P_1^{(n,i_m,0)} = A^{-(1-2n)i_m}\cdot A^{i_m(1-2n)-\frac{n(n-1)}{2}} = A^{-\frac{n(n-1)}{2}},$$

then

$$\tilde{P}_1^{(n,k,m)} = \sum_{s=0}^{m} A^{s(1-n-m)+\frac{s(s-1)}{2}} A^{-4s} \begin{bmatrix} s+k-1 \\ s \end{bmatrix}_{q=A^{-4}} \cdot A^{-\frac{(n+m-s)(n+m-s-1)}{2}}$$
$$= A^{-\frac{(m+n)(m+n-1)}{2}} \sum_{s=0}^{m} A^{-4s} \begin{bmatrix} s+k-1 \\ s \end{bmatrix}_{q=A^{-4}}$$

and hence

$$P_1^{(n,k,m)} = A^{(1+2m-2n)k}\tilde{P}_1^{(n,k,m)} = A^{(1+2m-2n)k-\frac{(m+n)(m+n-1)}{2}} \sum_{s=0}^{m} A^{-4s} \begin{bmatrix} s+k-1 \\ s \end{bmatrix}_{q=A^{-4}}.$$

Then, we conclude that, for all $m,n,k\in\mathbb{N}$,

$$C_{n,k,m}(A) = P_1^{(n,k,m)} = A^{(1-2|m-n|)k-\frac{(m+n)(m+n-1)}{2}} \sum_{s=0}^{\min\{m,n\}} A^{-4s} \begin{bmatrix} s+k-1 \\ s \end{bmatrix}_{q=A^{-4}}.$$

Example 5.1.7. *Given a Catalan state $C_k \in \mathfrak{Cat}_{2k}$ shown in Figure 5.4(a). Let $P_l^{(k)} = \tilde{Q}_{2k,l}(\mathcal{T}(C_k))$, where the plane rooted tree with a label function $\mathcal{T}(C_k)$ is shown in Figure 5.4(b). Let $\mathcal{T}(C_k) = (T(C_k), v_0, f)$, then $\{f(v) \mid v \in L(T(C))\} = \{2i-1 \mid -k+1 \leq i \leq k\}$, so*

$$\begin{aligned} P_1^{(k)} &= \sum_{i_1=0}^{k-1} \sum_{j_1=k-i_1}^{k} A^{1+2(2k-(2i_1+1)-(2j_1-1))} P^{(k-1)}_{\max\{2k-(2i_1+1)-1,1\}} \\ &= \sum_{i_1=0}^{k-1} A^{1-4i_1}[i_1+1] P^{(k-1)}_{\max\{2k-2i_1-2,1\}} \end{aligned}$$

and

$$\begin{aligned} &P^{(k-s)}_{\max\{2k-2i_s-2s,1\}} \\ &= \sum_{i_{s+1}=0}^{k-s-1} \sum_{j_{s+1}=\max\{k-s-i_{s+1},k-s-i_s+1\}}^{k-s} A^{1+2(2(k-s)-(2i_{s+1}+1)-(2j_{s+1}-1))} P^{(k-s-1)}_{\max\{2(k-s)-(2i_{s+1}+1)-1,1\}} \\ &= \sum_{i_{s+1}=0}^{k-s-1} A^{1-4i_{s+1}}[\min\{i_{s+1}+1, i_s\}] P^{(k-s-1)}_{\max\{2k-2i_{s+1}-2s-2,1\}} \end{aligned}$$

for $1 \leq s \leq k$, where $[n] = [n]_{q=A^4} = 1 + A^4 + \ldots + A^{4(n-1)}$. Hence

$$P_1^{(k)} = \sum_{i_1=0}^{k-1} \sum_{i_2=0}^{k-2} \cdots \sum_{i_k=0}^{0} A^{k-4(i_1+i_2+\ldots+i_k)}[i_1+1] \prod_{l=1}^{k-1} [\min\{i_{l+1}+1, i_l\}].$$

One shows after some small computations that

$$\begin{aligned} P_1^{(1)} &= A \\ P_1^{(2)} &= A^{-2}[2] \\ P_1^{(3)} &= A^{-9}[2]^3 \\ P_1^{(4)} &= A^{-20}[2]^3(1+2A^4+3A^8+A^{12}) \\ P_1^{(5)} &= A^{-35}[2]^5(1+2A^4+5A^8+5A^{12}+5A^{16}+A^{20}). \end{aligned}$$

Proposition 5.1.8. *Given $C \in \mathfrak{Cat}_k$. Let C' be the Catalan state obtained from C by a reflection about line l^r (see Figure 2.9). Then*

$$C'(A) = C(A).$$

Proof. Notice that, if $s = (s_{i,j}) \in \mathrm{Mat}_k^U(\{\pm 1\})$ realizes C, then $s' = (s_{k-j,k-i})$ realizes C' obtained by the reflection about the l^r. Since the map $s = (s_{i,j}) \mapsto s' = (s_{k-j,k-i})$ is invertible, s, s' have same number of positive and negative markers, and they create the same number of trivial components when realizing C and C', respectively, it follows that $C'(A) = C(A)$. □

5.2 Future Work

There are several questions that remain still unanswered and I plan to address them in my future work.

1. Questions related to lattice crossing:

    - *Conjecture 3.2.5.* In order to solve the general cases we may need to understand some properties of posets involved and to understand how to expand appropriately $\big(\mathcal{R}_{n,\kappa_1,\kappa_2,|J|}, (\mathcal{F}_{n,\kappa_1,\kappa_2,I})_J\big)$ for $J \preceq_F I$. This conjecture is rather a technical result that worth of addressing since it will make calculations of $Q_{n,\mathcal{R},I}$ in Theorem 3.2.2 much simpler.

    - *What can we say on coefficients of realizable Catalan states of $L(m,n)$, $n \geq 5$?* Unfortunately, our methods used in case $n = 4$ do not apply when $n \geq 5$ since plucking polynomials corresponding to Catalan states of $L(m,n)$ do not factor into plucking polynomials of some simple tangles that could be easily analyzed. The hope is to find other relations between plucking polynomials of plane rooted trees with a delay function or calculate coefficients of Catalan states using different methods.

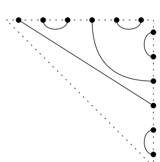
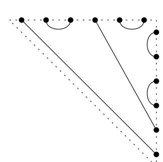
    - *Conjecture 4.2.9.* The problem of determining the unimodality of a sum of Laurent polynomials is not a simple problem. This is even more difficult in our case, since the closed-form formulas are not even such kinds of expressions.

- *Other properties of coefficients of realizable Catalan states of $L(m,n)$.* Given $C \in \mathfrak{Cat}_{m,n}$, one may ask whether C is realizable if and only if $C(A) \neq 0$, and if $C(A) = A^k \sum_{i=0}^{l} a_i A^{4i}$ with $a_0 a_l \neq 0$ then $a_0 = a_l = 1$ and all a_i's are positive. One may also ask whether the coefficient of a symmetric Catalan state C, that is $C = C^s$, or a Catalan state with returns on at most three sides is unimodal?

2. Questions related to generalized crossing:

- *Formulas or efficient methods for finding coefficients of Catalan states of $G(k)$.* We see that all formulas and results obtained for the lattice crossing are developed based on the method of the first-row expansion Proposition 3.1.4. We showed an analog of this method for generalized crossing (see Proposition 5.1.3). Therefore, one might wonder whether we can use similar ideas to those presented in Chapter 3 in order to find formulas or efficient methods for computing coefficients of the Catalan states of generalized crossing. Unfortunately, this is not that straight forward due to the difficulties related to the lack of good methods to deal with index set $\mathbb{I}(k,m)$.

- *Properties on the coefficients of Catalan states of $G(k)$.* Given $C \in \mathfrak{Cat}_k$, we know that the leading coefficient of $C(A)$ is not always 1 and the coefficients of $C(A)$ are not always positive (hence the coefficients of $C(A)$ are not unimodal). For instance

 $(A) = 2A^{-5} + A^{-9}$,  $(A) = A^{-1} + A^{-9}$.

However, it is still unknown if $C(A) \neq 0$ for all $C \in \mathfrak{Cat}_k$.

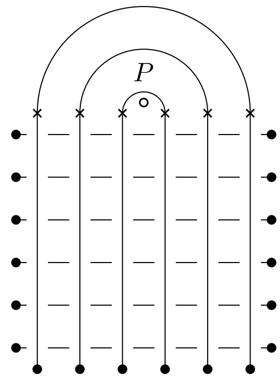
- *Applications.* Generalized crossing is obtained by a half twist of n parallel strands and it is a tangle often mentioned in context of some other problems in knot theory. However, finding more direct applications of results that we obtained for generalized crossing in knot theory problems requires further research investigations. In particular, finding a relation between coefficients of Catalan states of lattice crossing and the coefficients of Catalan states of generalized crossing is another open problem worth further considerations.

3. Closure of tangles:

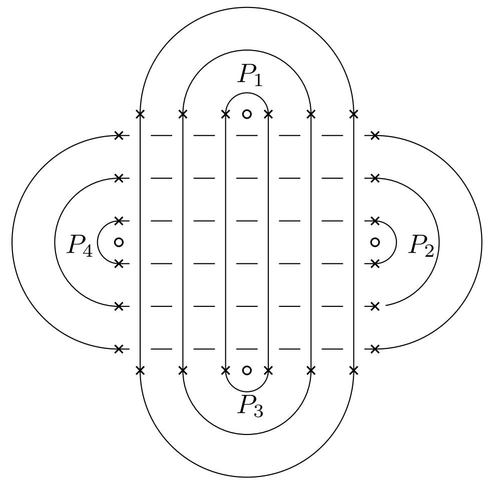
   - *B-type lattice crossing $L_B(m,n)$.* Let $L_B(m,n)$ be the tangle shown in Figure 5.5(a). Some results concerning counting realizable Catalan states of $L_B(m,n)$ and computing their coefficients were obtained by M. Dabkowski and M. Rakotomalala. However, no general methods for finding closed-form formulas for coefficients of Catalan states are known thus far. I had found an analog of the first-row expansion for Catalan states of $L_B(m,n)$. This might be regarded as the first step to toward solving this problem. The immediate consequence of this expansion is that $C(A) \in \mathbb{Z}_{\geq 0}[A^{\pm 1}]$.

   - *Closure $L_C(m,n)$ of $L(m,n)$ with four fixed punctures P_1, P_2, P_3, P_4 (see Figure 5.5(b)).* Given the standard basis of $\mathcal{S}^*_{2,\infty}(F_{0,4} \times I)$, we consider the problem of finding coefficients of basis elements that are obtained from $L_C(m,n)$. This problem is similar to our original one, that is, to the problem of finding the coefficients of the standard basis of $y^n * x^m$ in $\mathcal{S}^*_{2,\infty}(F_{0,4} \times I)$. It appears that this problem is related to the famous open problem known as *the meander problem.*

   - *Other perspective.* The A-type Gram determinant is invertible when A is not a root of unity, see [10]. In such a case, let $\{C_i\}$ be the basis of $\mathcal{S}_{2,\infty}(D^2 \times I, n+m; R, A)$ described in Corollary 2.1.9 and $N = \frac{1}{m+n+1}\binom{2(m+n)}{m+n}$. If $L(m,n) = \sum_i C_i(A) \cdot C_i$, $\langle a, b\rangle$ is the product of tangles a, b with $2(m+n)$ fixed points on the boundary defined in a natural way, $V = [\langle C_1, L(m,n)\rangle \ \langle C_2, L(m,n)\rangle \ \cdots \ \langle C_N, L(m,n)\rangle]$,



(a) $L_B(m,n)$



(b) $L_C(m,n)$

Figure 5.5. $L_B(m,n)$ and $L_C(m,n)$

and $U = [C_1(A)\ C_2(A)\ \cdots\ C_N(A)]$, then

$$V = U \cdot D_N,$$

where D_N is the A-type Gram determinant. Thus, $C_i(A) = [V \cdot D_N^{-1}]_i$. This idea can also be applied to find coefficients of Catalan states of generalized crossing. It is worth checking if this approach leads to an efficient method for finding coefficients of Catalan states of $L(m,n)$ and $G(k)$.

APPENDIX A

PROOF OF LEMMA 3.2.4

For $|I| - |J| = 3$, there are five cases $I \setminus J = \{i_1 < i_2 < i_3\}$ with

(1) $i_3 = i_2 + 1 = i_1 + 2$,

(2) $i_3 = i_2 + 2 = i_1 + 3$,

(3) $i_3 = i_2 + 1 > i_1 + 2$,

(4) $i_3 > i_2 + 2 = i_1 + 3$, or

(5) $i_3 > i_2 + 1 > i_1 + 2$.

Let $n' = n - 2|J|$, $\mu' = \mu - |J|$, $i'_1 = i_1 - 2|\{j \in J \mid j < i_1\}|$, $i'_2 = i_2 - 2|\{j \in J \mid j < i_2\}|$, $i'_3 = i_3 - 2|\{j \in J \mid j < i_3\}|$, $I' = \{i'_1, i'_2, i'_3\}$.

For the case (1), $I = \{i_1, i_1 + 1, i_1 + 2\}$, and then

- $J \prec_F I$,
- $J \prec_F J \cup \{i_1 + 2\} \prec_F I$,
- $J \prec_F J \cup \{i_1 + 1, i_1 + 2\} \prec_F I$, or
- $J \prec_F J \cup \{i_1 + 2\} \prec_F J \cup \{i_1 + 1, i_1 + 2\} \prec_F I$.

So,

$$\begin{aligned}\mathcal{S}(I,J) = &-\frac{C_{I,J}}{C_{J,J}} + \frac{C_{J\cup\{i_1+2\},J}}{C_{J,J}} \cdot \frac{C_{I,J\cup\{i_1+2\}}}{C_{J\cup\{i_1+2\},J\cup\{i_1+2\}}} \\ &+ \frac{C_{J\cup\{i_1+1,i_1+2\},J}}{C_{J,J}} \cdot \frac{C_{I,J\cup\{i_1+1,i_1+2\}}}{C_{J\cup\{i_1+1,i_1+2\},J\cup\{i_1+1,i_1+2\}}} \\ &- \frac{C_{J\cup\{i_1+2\},J}}{C_{J,J}} \cdot \frac{C_{J\cup\{i_1+1,i_1+2\},J\cup\{i_1+2\}}}{C_{J\cup\{i_1+2\},J\cup\{i_1+2\}}} \cdot \frac{C_{I,J\cup\{i_1+1,i_1+2\}}}{C_{J\cup\{i_1+1,i_1+2\},J\cup\{i_1+1,i_1+2\}}}.\end{aligned}$$

By Lemma 3.2.3,

$$C_{I,J} = A^{-2(3i_1'+3)+3(9-n'-\kappa_1-\kappa_2+1)+\frac{n'}{2}(3\kappa_1-\kappa_2)+\frac{1}{2}(\kappa_1-\kappa_2)^2} \cdot \begin{bmatrix} n'-3 \\ \mu' \end{bmatrix} \begin{bmatrix} i_1'+2 \\ 3 \end{bmatrix},$$

$$C_{J\cup\{i_1+2\},J} = A^{-2(i_1'+2)+(3-n'-\kappa_1-\kappa_2+1)+\frac{n'}{2}(3\kappa_1-\kappa_2)+\frac{1}{2}(\kappa_1-\kappa_2)^2} \cdot \begin{bmatrix} n'-1 \\ \mu' \end{bmatrix} [i_1'+2],$$

$$C_{I,J\cup\{i_1+2\}} = A^{-2(2i_1'+1)+2(6-(n'-2)-\kappa_1-\kappa_2+1)+\frac{n'-2}{2}(3\kappa_1-\kappa_2)+\frac{1}{2}(\kappa_1-\kappa_2)^2} \cdot \begin{bmatrix} n'-4 \\ \mu'-1 \end{bmatrix} \begin{bmatrix} i_1'+1 \\ 2 \end{bmatrix},$$

$$C_{J\cup\{i_1+2\},J\cup\{i_1+2\}} = A^{\frac{n'-2}{2}(3\kappa_1-\kappa_2)+\frac{1}{2}(\kappa_1-\kappa_2)^2} \cdot \begin{bmatrix} n'-2 \\ \mu'-1 \end{bmatrix},$$

$$C_{J\cup\{i_1+1,i_1+2\},J} = A^{-2(2i_1'+3)+2(6-n'-\kappa_1-\kappa_2+1)+\frac{n'}{2}(3\kappa_1-\kappa_2)+\frac{1}{2}(\kappa_1-\kappa_2)^2} \cdot \begin{bmatrix} n'-2 \\ \mu' \end{bmatrix} \begin{bmatrix} i_1'+2 \\ 2 \end{bmatrix},$$

$$C_{I,J\cup\{i_1+1,i_1+2\}} = A^{-2(i_1')+(3-(n'-4)-\kappa_1-\kappa_2+1)+\frac{n'-4}{2}(3\kappa_1-\kappa_2)+\frac{1}{2}(\kappa_1-\kappa_2)^2} \cdot \begin{bmatrix} n'-5 \\ \mu'-2 \end{bmatrix} [i_1'],$$

$$C_{J\cup\{i_1+1,i_1+2\},J\cup\{i_1+1,i_1+2\}} = A^{\frac{n'-4}{2}(3\kappa_1-\kappa_2)+\frac{1}{2}(\kappa_1-\kappa_2)^2} \cdot \begin{bmatrix} n'-4 \\ \mu'-2 \end{bmatrix},$$

and

$$C_{J\cup\{i_1+1,i_1+2\},J\cup\{i_1+2\}} = A^{-2(i_1'+1)+(3-(n'-2)-\kappa_1-\kappa_2+1)+\frac{n'-2}{2}(3\kappa_1-\kappa_2)+\frac{1}{2}(\kappa_1-\kappa_2)^2} \cdot \begin{bmatrix} n'-3 \\ \mu'-1 \end{bmatrix} [i_1'+1].$$

Then,

$$\begin{aligned}
C_{J,J} \cdot \mathcal{S}(I,J) &= C_{I,J} \cdot \left(-1 + \frac{A^{-8} \begin{bmatrix} n'-1 \\ \mu' \end{bmatrix} [i_1'+2] \begin{bmatrix} n'-4 \\ \mu'-1 \end{bmatrix} \begin{bmatrix} i_1'+1 \\ 2 \end{bmatrix}}{\begin{bmatrix} n'-2 \\ \mu'-1 \end{bmatrix} \begin{bmatrix} n'-3 \\ \mu' \end{bmatrix} \begin{bmatrix} i_1'+2 \\ 3 \end{bmatrix}} \right. \\
&\quad \left. + \frac{A^{-8} \begin{bmatrix} n'-2 \\ \mu' \end{bmatrix} \begin{bmatrix} i_1'+2 \\ 2 \end{bmatrix} \begin{bmatrix} n'-5 \\ \mu'-2 \end{bmatrix} [i_1']}{\begin{bmatrix} n'-4 \\ \mu'-2 \end{bmatrix} \begin{bmatrix} n'-3 \\ \mu' \end{bmatrix} \begin{bmatrix} i_1'+2 \\ 3 \end{bmatrix}} - \frac{A^{-12} \begin{bmatrix} n'-1 \\ \mu' \end{bmatrix} [i_1'+2] \begin{bmatrix} n'-3 \\ \mu'-1 \end{bmatrix} [i_1'+1] \begin{bmatrix} n'-5 \\ \mu'-2 \end{bmatrix} [i_1']}{\begin{bmatrix} n'-2 \\ \mu'-1 \end{bmatrix} \begin{bmatrix} n'-4 \\ \mu'-2 \end{bmatrix} \begin{bmatrix} n'-3 \\ \mu' \end{bmatrix} \begin{bmatrix} i_1'+2 \\ 3 \end{bmatrix}} \right) \\
&= C_{I,J} \cdot \left(-1 + A^{-8} \frac{[3][n'-1]}{[n'-3]} + A^{-8} \frac{[3][n'-2]}{[n'-4]} - A^{-12} \frac{[2][3][n'-1]}{[n'-4]} \right) \\
&= C_{I,J} \cdot \left(-1 + A^{-8} \frac{[3][n'-1]}{[n'-3]} - A^{-12} \frac{[3][n']}{[n'-4]} \right) \\
&= C_{I,J} \cdot \left(A^{-8} \frac{[2][n']}{[n'-3]} - A^{-12} \frac{[3][n']}{[n'-4]} \right) \\
&= -C_{I,J} \cdot \frac{A^{-12}[n'][n'-1]}{[n'-3][n'-4]} = (-1)^3 A^{-2\cdot 3\cdot(3-1)} \cdot \frac{\begin{bmatrix} 2n'-5 \\ n'-3 \end{bmatrix}}{\begin{bmatrix} 2n'-5 \\ n' \end{bmatrix}} C_{I,J},
\end{aligned}$$

since

$$
\begin{aligned}
[n-2]_q - q^{-1}[2]_q[n-1]_q &= (1+q+\ldots+q^{n-3}) - q^{-1}(1+q)(1+q+\ldots+q^{n-2}) \\
&= (1+q+\ldots+q^{n-3}) - (q^{-1}+2+2q+\ldots+2q^{n-3}+q^{n-2}) \\
&= -(q^{-1}+1+\ldots+q^{n-3}+q^{n-2}) \\
&= -q^{-1}[n]_q,
\end{aligned}
$$

$$
\begin{aligned}
q^{-2}[3]_q[n-1]_q - [n-3]_q =& (q^{-2}+2q^{-1}+3+3q+\ldots+3q^{n-4}+2q^{n-3}+q^{n-2}) \\
& - (1+q+\ldots+q^{n-4}) \\
=& q^{-2}+2q^{-1}+2+\ldots+2q^{n-3}+q^{n-2} \\
=& q^{-2}[2]_q[n]_q,
\end{aligned}
$$

and

$$
\begin{aligned}
[2]_q[n-4]_q - q^{-1}[3]_q[n-3]_q =& (1+2q+2q^2+\ldots+2q^{n-5}+q^{n-4}) \\
& - (q^{-1}+2+3q+3q^2+\ldots+3q^{n-5}+2q^{n-4}+q^{n-3}) \\
=& -(q^{-1}+1+\ldots+q^{n-4}+q^{n-3}) \\
=& -q^{-1}[n-1]_q.
\end{aligned}
$$

Therefore,

$$
\mathcal{S}(I,J) = (-1)^3 A^{-2\cdot 3\cdot(3-1)} \cdot \frac{\begin{bmatrix} 2n-4|J|-5 \\ n-2|J|-3 \end{bmatrix} C_{I,J}}{\begin{bmatrix} 2n-4|J|-5 \\ n-2|J| \end{bmatrix} C_{J,J}} = \mathcal{S}'(I,J).
$$

For the case (2), $I = \{i_1, i_1+1, i_1+3\}$, and then

- $J \prec_F I$,
- $J \prec_F J \cup \{i_1+1\} \prec_F I$,
- $J \prec_F J \cup \{i_1+3\} \prec_F I$,
- $J \prec_F J \cup \{i_1+1, i_1+3\} \prec_F I$,

- $J \prec_F J \cup \{i_1+1\} \prec_F J \cup \{i_1+1, i_1+3\} \prec_F I$, or

- $J \prec_F J \cup \{i_1+3\} \prec_F J \cup \{i_1+1, i_1+3\} \prec_F I$.

So,

$$\begin{aligned}\mathcal{S}(I,J) = &-\frac{C_{I,J}}{C_{J,J}} + \frac{C_{J\cup\{i_1+1\},J}}{C_{J,J}} \cdot \frac{C_{I,J\cup\{i_1+1\}}}{C_{J\cup\{i_1+1\},J\cup\{i_1+1\}}} + \frac{C_{J\cup\{i_1+3\},J}}{C_{J,J}} \cdot \frac{C_{I,J\cup\{i_1+3\}}}{C_{J\cup\{i_1+3\},J\cup\{i_1+3\}}} \\ &+ \frac{C_{J\cup\{i_1+1,i_1+3\},J}}{C_{J,J}} \cdot \frac{C_{I,J\cup\{i_1+1,i_1+3\}}}{C_{J\cup\{i_1+1,i_1+3\},J\cup\{i_1+1,i_1+3\}}} \\ &- \frac{C_{J\cup\{i_1+1\},J}}{C_{J,J}} \cdot \frac{C_{J\cup\{i_1+1,i_1+3\},J\cup\{i_1+1\}}}{C_{J\cup\{i_1+1\},J\cup\{i_1+1\}}} \cdot \frac{C_{I,J\cup\{i_1+1,i_1+3\}}}{C_{J\cup\{i_1+1,i_1+3\},J\cup\{i_1+1,i_1+3\}}} \\ &- \frac{C_{J\cup\{i_1+3\},J}}{C_{J,J}} \cdot \frac{C_{J\cup\{i_1+1,i_1+3\},J\cup\{i_1+3\}}}{C_{J\cup\{i_1+3\},J\cup\{i_1+3\}}} \cdot \frac{C_{I,J\cup\{i_1+1,i_1+3\}}}{C_{J\cup\{i_1+1,i_1+3\},J\cup\{i_1+1,i_1+3\}}}.\end{aligned}$$

By Lemma 3.2.3,

$$C_{I,J} = A^{-2(3i_1'+4)+3(9-n'-\kappa_1-\kappa_2+1)+\frac{n'}{2}(3\kappa_1-\kappa_2)+\frac{1}{2}(\kappa_1-\kappa_2)^2} \cdot \begin{bmatrix} n'-3 \\ \mu' \end{bmatrix} \begin{bmatrix} i_1'+2 \\ 3 \end{bmatrix} [2],$$

$$C_{J\cup\{i_1+1\},J} = A^{-2(i_1'+1)+(3-n'-\kappa_1-\kappa_2+1)+\frac{n'}{2}(3\kappa_1-\kappa_2)+\frac{1}{2}(\kappa_1-\kappa_2)^2} \cdot \begin{bmatrix} n'-1 \\ \mu' \end{bmatrix} [i_1'+1],$$

$$\begin{aligned}&C_{I,J\cup\{i_1+1\}} = C_{I,J\cup\{i_1+3\}} \\ =&A^{-2(2i_1'+1)+2(6-(n'-2)-\kappa_1-\kappa_2+1)+\frac{n'-2}{2}(3\kappa_1-\kappa_2)+\frac{1}{2}(\kappa_1-\kappa_2)^2} \cdot \begin{bmatrix} n'-4 \\ \mu'-1 \end{bmatrix} \begin{bmatrix} i_1'+1 \\ 2 \end{bmatrix},\end{aligned}$$

$$C_{J\cup\{i_1+1\},J\cup\{i_1+1\}} = C_{J\cup\{i_1+3\},J\cup\{i_1+3\}} = A^{\frac{n'-2}{2}(3\kappa_1-\kappa_2)+\frac{1}{2}(\kappa_1-\kappa_2)^2} \cdot \begin{bmatrix} n'-2 \\ \mu'-1 \end{bmatrix},$$

$$C_{J\cup\{i_1+3\},J} = A^{-2(i_1'+3)+(3-n'-\kappa_1-\kappa_2+1)+\frac{n'}{2}(3\kappa_1-\kappa_2)+\frac{1}{2}(\kappa_1-\kappa_2)^2} \cdot \begin{bmatrix} n'-1 \\ \mu' \end{bmatrix} [i_1'+3],$$

$$C_{J\cup\{i_1+1,i_1+3\},J} = A^{-2(2i_1'+4)+2(6-n'-\kappa_1-\kappa_2+1)+\frac{n'}{2}(3\kappa_1-\kappa_2)+\frac{1}{2}(\kappa_1-\kappa_2)^2} \cdot \begin{bmatrix} n'-2 \\ \mu' \end{bmatrix} [i_1'+1][i_1'+2],$$

$$C_{I,J\cup\{i_1+1,i_1+3\}} = A^{-2(i_1')+(3-(n'-4)-\kappa_1-\kappa_2+1)+\frac{n'-4}{2}(3\kappa_1-\kappa_2)+\frac{1}{2}(\kappa_1-\kappa_2)^2} \cdot \begin{bmatrix} n'-5 \\ \mu'-2 \end{bmatrix} [i_1'],$$

$$C_{J\cup\{i_1+1,i_1+3\},J\cup\{i_1+1,i_1+3\}} = A^{\frac{n'-4}{2}(3\kappa_1-\kappa_2)+\frac{1}{2}(\kappa_1-\kappa_2)^2} \cdot \begin{bmatrix} n'-4 \\ \mu'-2 \end{bmatrix},$$

and

$$
\begin{aligned}
&C_{J\cup\{i_1+1,i_1+3\},J\cup\{i_1+1\}} = C_{J\cup\{i_1+1,i_1+3\},J\cup\{i_1+3\}}\\
=&A^{-2(i_1'+1)+(3-(n'-2)-\kappa_1-\kappa_2+1)+\frac{n'-2}{2}(3\kappa_1-\kappa_2)+\frac{1}{2}(\kappa_1-\kappa_2)^2}\cdot\begin{bmatrix}n'-3\\ \mu'-1\end{bmatrix}[i_1'+1].
\end{aligned}
$$

Thus, using previously established results, it follows that

$$
\begin{aligned}
&C_{J,J}\cdot\mathcal{S}(I,J)=C_{I,J}\cdot\left(-1+\frac{A^{-4}\begin{bmatrix}n'-1\\ \mu'\end{bmatrix}[i_1'+1]\begin{bmatrix}n'-4\\ \mu'-1\end{bmatrix}\begin{bmatrix}i_1'+1\\ 2\end{bmatrix}}{\begin{bmatrix}n'-2\\ \mu'-1\end{bmatrix}\begin{bmatrix}n'-3\\ \mu'\end{bmatrix}\begin{bmatrix}i_1'+2\\ 3\end{bmatrix}[2]}\right.\\
&+\frac{A^{-8}\begin{bmatrix}n'-1\\ \mu'\end{bmatrix}[i_1'+3]\begin{bmatrix}n'-4\\ \mu'-1\end{bmatrix}\begin{bmatrix}i_1'+1\\ 2\end{bmatrix}}{\begin{bmatrix}n'-2\\ \mu'-1\end{bmatrix}\begin{bmatrix}n'-3\\ \mu'\end{bmatrix}\begin{bmatrix}i_1'+2\\ 3\end{bmatrix}[2]}+\frac{A^{-8}\begin{bmatrix}n'-2\\ \mu'\end{bmatrix}[i_1'+1][i_1'+2]\begin{bmatrix}n'-5\\ \mu'-2\end{bmatrix}[i_1']}{\begin{bmatrix}n'-4\\ \mu'-2\end{bmatrix}\begin{bmatrix}n'-3\\ \mu'\end{bmatrix}\begin{bmatrix}i_1'+2\\ 3\end{bmatrix}[2]}\\
&\left.-\frac{A^{-8}\begin{bmatrix}n'-1\\ \mu'\end{bmatrix}[i_1'+1]\begin{bmatrix}n'-3\\ \mu'-1\end{bmatrix}[i_1'+1]\begin{bmatrix}n'-5\\ \mu'-2\end{bmatrix}[i_1']}{\begin{bmatrix}n'-2\\ \mu'-1\end{bmatrix}\begin{bmatrix}n'-4\\ \mu'-2\end{bmatrix}\begin{bmatrix}n'-3\\ \mu'\end{bmatrix}\begin{bmatrix}i_1'+2\\ 3\end{bmatrix}[2]}-\frac{A^{-12}\begin{bmatrix}n'-1\\ \mu'\end{bmatrix}[i_1'+3]\begin{bmatrix}n'-3\\ \mu'-1\end{bmatrix}[i_1'+1]\begin{bmatrix}n'-5\\ \mu'-2\end{bmatrix}[i_1']}{\begin{bmatrix}n'-2\\ \mu'-1\end{bmatrix}\begin{bmatrix}n'-4\\ \mu'-2\end{bmatrix}\begin{bmatrix}n'-3\\ \mu'\end{bmatrix}\begin{bmatrix}i_1'+2\\ 3\end{bmatrix}[2]}\right)\\
&=C_{I,J}\cdot\left(-1+A^{-8}\frac{[3][n'-1]}{[n'-3]}+A^{-8}\frac{[3][n'-2]}{[n'-4]}-A^{-12}\frac{[2][3][n'-1]}{[n'-4]}\right)\\
&=(-1)^3A^{-2\cdot3\cdot(3-1)}\cdot\frac{\begin{bmatrix}2n'-5\\ n'-3\end{bmatrix}}{\begin{bmatrix}2n'-5\\ n'\end{bmatrix}}C_{I,J}.
\end{aligned}
$$

Therefore,

$$
\mathcal{S}(I,J)=(-1)^3A^{-2\cdot3\cdot(3-1)}\cdot\frac{\begin{bmatrix}2n-4|J|-5\\ n-2|J|-3\end{bmatrix}C_{I,J}}{\begin{bmatrix}2n-4|J|-5\\ n-2|J|\end{bmatrix}C_{J,J}}=\mathcal{S}'(I,J).
$$

For the case (3), $I=\{i_1,i_2,i_2+1\}$ with $i_1+1<i_2$, and then

- $J\prec_F I$,
- $J\prec_F J\cup\{i_1\}\prec_F I$,
- $J\prec_F J\cup\{i_2+1\}\prec_F I$,
- $J\prec_F J\cup\{i_1,i_2+1\}\prec_F I$,
- $J\prec_F J\cup\{i_2,i_2+1\}\prec_F I$,
- $J\prec_F J\cup\{i_1\}\prec_F J\cup\{i_1,i_2+1\}\prec_F I$,

- $J \prec_F J \cup \{i_2+1\} \prec_F J \cup \{i_1, i_2+1\} \prec_F I$, or
- $J \prec_F J \cup \{i_2+1\} \prec_F J \cup \{i_2, i_2+1\} \prec_F I$.

So,

$$\begin{aligned}\mathcal{S}(I,J) = &-\frac{C_{I,J}}{C_{J,J}} + \frac{C_{J\cup\{i_1\},J}}{C_{J,J}} \cdot \frac{C_{I,J\cup\{i_1\}}}{C_{J\cup\{i_1\},J\cup\{i_1\}}} + \frac{C_{J\cup\{i_2+1\},J}}{C_{J,J}} \cdot \frac{C_{I,J\cup\{i_2+1\}}}{C_{J\cup\{i_2+1\},J\cup\{i_2+1\}}} \\ &+ \frac{C_{J\cup\{i_1,i_2+1\},J}}{C_{J,J}} \cdot \frac{C_{I,J\cup\{i_1,i_2+1\}}}{C_{J\cup\{i_1,i_2+1\},J\cup\{i_1,i_2+1\}}} + \frac{C_{J\cup\{i_2,i_2+1\},J}}{C_{J,J}} \cdot \frac{C_{I,J\cup\{i_2,i_2+1\}}}{C_{J\cup\{i_2,i_2+1\},J\cup\{i_2,i_2+1\}}} \\ &- \frac{C_{J\cup\{i_1\},J}}{C_{J,J}} \cdot \frac{C_{J\cup\{i_1,i_2+1\},J\cup\{i_1\}}}{C_{J\cup\{i_1\},J\cup\{i_1\}}} \cdot \frac{C_{I,J\cup\{i_1,i_2+1\}}}{C_{J\cup\{i_1,i_2+1\},J\cup\{i_1,i_2+1\}}} \\ &- \frac{C_{J\cup\{i_2+1\},J}}{C_{J,J}} \cdot \frac{C_{J\cup\{i_1,i_2+1\},J\cup\{i_2+1\}}}{C_{J\cup\{i_2+1\},J\cup\{i_2+1\}}} \cdot \frac{C_{I,J\cup\{i_1,i_2+1\}}}{C_{J\cup\{i_1,i_2+1\},J\cup\{i_1,i_2+1\}}} \\ &- \frac{C_{J\cup\{i_2+1\},J}}{C_{J,J}} \cdot \frac{C_{J\cup\{i_2,i_2+1\},J\cup\{i_2+1\}}}{C_{J\cup\{i_2+1\},J\cup\{i_2+1\}}} \cdot \frac{C_{I,J\cup\{i_2,i_2+1\}}}{C_{J\cup\{i_2,i_2+1\},J\cup\{i_2,i_2+1\}}}.\end{aligned}$$

By Lemma 3.2.3,

$$C_{I,J} = A^{-2(i_1'+2i_2'+1)+3(9-n'-\kappa_1-\kappa_2+1)+\frac{n'}{2}(3\kappa_1-\kappa_2)+\frac{1}{2}(\kappa_1-\kappa_2)^2} \cdot \begin{bmatrix} n'-3 \\ \mu' \end{bmatrix} [i_1'] \begin{bmatrix} i_2' \\ 2 \end{bmatrix},$$

$$C_{J\cup\{i_1\},J} = A^{-2(i_1')+(3-n'-\kappa_1-\kappa_2+1)+\frac{n'}{2}(3\kappa_1-\kappa_2)+\frac{1}{2}(\kappa_1-\kappa_2)^2} \cdot \begin{bmatrix} n'-1 \\ \mu' \end{bmatrix} [i_1'],$$

$$C_{I,J\cup\{i_1\}} = A^{-2(2i_2'-3)+2(6-(n'-2)-\kappa_1-\kappa_2+1)+\frac{n'-2}{2}(3\kappa_1-\kappa_2)+\frac{1}{2}(\kappa_1-\kappa_2)^2} \cdot \begin{bmatrix} n'-4 \\ \mu'-1 \end{bmatrix} \begin{bmatrix} i_2'-1 \\ 2 \end{bmatrix},$$

$$C_{J\cup\{i_1\},J\cup\{i_1\}} = C_{J\cup\{i_2+1\},J\cup\{i_2+1\}} = A^{\frac{n'-2}{2}(3\kappa_1-\kappa_2)+\frac{1}{2}(\kappa_1-\kappa_2)^2} \cdot \begin{bmatrix} n'-2 \\ \mu'-1 \end{bmatrix},$$

$$C_{J\cup\{i_2+1\},J} = A^{-2(i_2'+1)+(3-n'-\kappa_1-\kappa_2+1)+\frac{n'}{2}(3\kappa_1-\kappa_2)+\frac{1}{2}(\kappa_1-\kappa_2)^2} \cdot \begin{bmatrix} n'-1 \\ \mu' \end{bmatrix} [i_2'+1],$$

$$C_{I,J\cup\{i_2+1\}} = A^{-2(i_1'+i_2')+2(6-(n'-2)-\kappa_1-\kappa_2+1)+\frac{n'-2}{2}(3\kappa_1-\kappa_2)+\frac{1}{2}(\kappa_1-\kappa_2)^2} \cdot \begin{bmatrix} n'-4 \\ \mu'-1 \end{bmatrix} [i_1'][i_2'-1],$$

$$C_{J\cup\{i_1,i_2+1\},J} = A^{-2(i_1'+i_2'+1)+2(6-n'-\kappa_1-\kappa_2+1)+\frac{n'}{2}(3\kappa_1-\kappa_2)+\frac{1}{2}(\kappa_1-\kappa_2)^2} \cdot \begin{bmatrix} n'-2 \\ \mu' \end{bmatrix} [i_1'][i_2'],$$

$$C_{I,J\cup\{i_1,i_2+1\}} = A^{-2(i_2'-2)+(3-(n'-4)-\kappa_1-\kappa_2+1)+\frac{n'-4}{2}(3\kappa_1-\kappa_2)+\frac{1}{2}(\kappa_1-\kappa_2)^2} \cdot \begin{bmatrix} n'-5 \\ \mu'-2 \end{bmatrix} [i_2'-2],$$

$$C_{J\cup\{i_1,i_2+1\},J\cup\{i_1,i_2+1\}} = C_{J\cup\{i_2,i_2+1\},J\cup\{i_2,i_2+1\}} = A^{\frac{n'-4}{2}(3\kappa_1-\kappa_2)+\frac{1}{2}(\kappa_1-\kappa_2)^2} \cdot \begin{bmatrix} n'-4 \\ \mu'-2 \end{bmatrix},$$

$$C_{J\cup\{i_2,i_2+1\},J} = A^{-2(2i_2'+1)+2(6-n'-\kappa_1-\kappa_2+1)+\frac{n'}{2}(3\kappa_1-\kappa_2)+\frac{1}{2}(\kappa_1-\kappa_2)^2} \cdot \begin{bmatrix} n'-2 \\ \mu' \end{bmatrix} \begin{bmatrix} i_2'+1 \\ 2 \end{bmatrix},$$

$$C_{I,J\cup\{i_2,i_2+1\}} = A^{-2(i_1')+(3-(n'-4)-\kappa_1-\kappa_2+1)+\frac{n'-4}{2}(3\kappa_1-\kappa_2)+\frac{1}{2}(\kappa_1-\kappa_2)^2} \cdot \begin{bmatrix} n'-5 \\ \mu'-2 \end{bmatrix} [i_1'],$$

$$C_{J\cup\{i_1,i_2+1\},J\cup\{i_1\}} = A^{-2(i_2'-1)+(3-(n'-2)-\kappa_1-\kappa_2+1)+\frac{n'-2}{2}(3\kappa_1-\kappa_2)+\frac{1}{2}(\kappa_1-\kappa_2)^2} \cdot \begin{bmatrix} n'-3 \\ \mu'-1 \end{bmatrix} [i_2'-1],$$

$$C_{J\cup\{i_1,i_2+1\},J\cup\{i_2+1\}} = A^{-2(i_1')+(3-(n'-2)-\kappa_1-\kappa_2+1)+\frac{n'-2}{2}(3\kappa_1-\kappa_2)+\frac{1}{2}(\kappa_1-\kappa_2)^2} \cdot \begin{bmatrix} n'-3 \\ \mu'-1 \end{bmatrix} [i_1'],$$

and

$$C_{J\cup\{i_2,i_2+1\},J\cup\{i_2+1\}} = A^{-2(i_2')+(3-(n'-2)-\kappa_1-\kappa_2+1)+\frac{n'-2}{2}(3\kappa_1-\kappa_2)+\frac{1}{2}(\kappa_1-\kappa_2)^2} \cdot \begin{bmatrix} n'-3 \\ \mu'-1 \end{bmatrix} [i_2'].$$

Thus, from the previous results, it follows that

$$\begin{aligned}
C_{J,J} \cdot \mathcal{S}(I,J) &= C_{I,J} \cdot \Bigg(-1 + \frac{\begin{bmatrix} n'-1 \\ \mu' \end{bmatrix} [i_1'] \begin{bmatrix} n'-4 \\ \mu'-1 \end{bmatrix} \begin{bmatrix} i_2'-1 \\ 2 \end{bmatrix}}{\begin{bmatrix} n'-2 \\ \mu'-1 \end{bmatrix} \begin{bmatrix} n'-3 \\ \mu' \end{bmatrix} [i_1'] \begin{bmatrix} i_2' \\ 2 \end{bmatrix}} + \frac{A^{-8} \begin{bmatrix} n'-1 \\ \mu' \end{bmatrix} [i_2'+1] \begin{bmatrix} n'-4 \\ \mu'-1 \end{bmatrix} [i_1'][i_2'-1]}{\begin{bmatrix} n'-2 \\ \mu'-1 \end{bmatrix} \begin{bmatrix} n'-3 \\ \mu' \end{bmatrix} [i_1'] \begin{bmatrix} i_2' \\ 2 \end{bmatrix}} \\
&+ \frac{A^{-4} \begin{bmatrix} n'-2 \\ \mu' \end{bmatrix} [i_1'][i_2'] \begin{bmatrix} n'-5 \\ \mu'-2 \end{bmatrix} [i_2'-2]}{\begin{bmatrix} n'-4 \\ \mu'-2 \end{bmatrix} \begin{bmatrix} n'-3 \\ \mu' \end{bmatrix} [i_1'] \begin{bmatrix} i_2' \\ 2 \end{bmatrix}} + \frac{A^{-8} \begin{bmatrix} n'-2 \\ \mu' \end{bmatrix} \begin{bmatrix} i_2'+1 \\ 2 \end{bmatrix} \begin{bmatrix} n'-5 \\ \mu'-2 \end{bmatrix} [i_1']}{\begin{bmatrix} n'-4 \\ \mu'-2 \end{bmatrix} \begin{bmatrix} n'-3 \\ \mu' \end{bmatrix} [i_1'] \begin{bmatrix} i_2' \\ 2 \end{bmatrix}} \\
&- \frac{A^{-4} \begin{bmatrix} n'-1 \\ \mu' \end{bmatrix} [i_1'] \begin{bmatrix} n'-3 \\ \mu'-1 \end{bmatrix} [i_2'-1] \begin{bmatrix} n'-5 \\ \mu'-2 \end{bmatrix} [i_2'-2]}{\begin{bmatrix} n'-2 \\ \mu'-1 \end{bmatrix} \begin{bmatrix} n'-4 \\ \mu'-2 \end{bmatrix} \begin{bmatrix} n'-3 \\ \mu' \end{bmatrix} [i_1'] \begin{bmatrix} i_2' \\ 2 \end{bmatrix}} - \frac{A^{-8} \begin{bmatrix} n'-1 \\ \mu' \end{bmatrix} [i_2'+1] \begin{bmatrix} n'-3 \\ \mu'-1 \end{bmatrix} [i_1'] \begin{bmatrix} n'-5 \\ \mu'-2 \end{bmatrix} [i_2'-2]}{\begin{bmatrix} n'-2 \\ \mu'-1 \end{bmatrix} \begin{bmatrix} n'-4 \\ \mu'-2 \end{bmatrix} \begin{bmatrix} n'-3 \\ \mu' \end{bmatrix} [i_1'] \begin{bmatrix} i_2' \\ 2 \end{bmatrix}} \\
&- \frac{A^{-12} \begin{bmatrix} n'-1 \\ \mu' \end{bmatrix} [i_2'+1] \begin{bmatrix} n'-3 \\ \mu'-1 \end{bmatrix} [i_2'] \begin{bmatrix} n'-5 \\ \mu'-2 \end{bmatrix} [i_1']}{\begin{bmatrix} n'-2 \\ \mu'-1 \end{bmatrix} \begin{bmatrix} n'-4 \\ \mu'-2 \end{bmatrix} \begin{bmatrix} n'-3 \\ \mu' \end{bmatrix} [i_1'] \begin{bmatrix} i_2' \\ 2 \end{bmatrix}} \Bigg) \\
&= C_{I,J} \cdot \Bigg(-1 + \frac{[n'-1]}{[n'-3][i_2']} \big([i_2'-2] + A^{-8}[i_2'+1][2] \big) + \frac{A^{-8}[n'-2]}{[n'-4][i_2'-1]} \big(A^4[i_2'-2][2] \\
&+ [i_2'+1] \big) - \frac{A^{-12}[n'-1][2]}{[n'-4]} \bigg(\frac{A^8[i_2'-2]}{[i_2']} + \frac{A^4[i_2'-2][i_2'+1]}{[i_2'][i_2'-1]} + \frac{[i_2'+1]}{[i_2'-1]} \bigg) \Bigg) \\
&= C_{I,J} \cdot \left(-1 + \frac{A^{-8}[n'-1][3]}{[n'-3]} + \frac{A^{-8}[n'-2][3]}{[n'-4]} - \frac{A^{-12}[n'-1][3][2]}{[n'-4]} \right) \\
&= (-1)^3 A^{-2\cdot 3\cdot(3-1)} \cdot \frac{\begin{bmatrix} 2n'-5 \\ n'-3 \end{bmatrix}}{\begin{bmatrix} 2n'-5 \\ n' \end{bmatrix}} C_{I,J},
\end{aligned}$$

since

$$\begin{aligned}
[n-2]_q + q^{-2}[n+1]_q[2]_q &= (1+q+\ldots+q^{n-3}) + q^{-2}(1+2q+2q^2+\ldots+2q^n+q^{n+1}) \\
&= q^{-2}(1+2q+3q^2+3q^3+\ldots+3q^{n-1}+2q^n+q^{n+1}) \\
&= q^{-2}[n]_q[3]_q,
\end{aligned}$$

$$\begin{aligned} q\,[n-2]_q\,[2]_q + [n+1]_q =& q(1+2q+2q^2+\ldots+2q^{n-3}+q^{n-2}) + (1+q+\ldots+q^n) \\ =& 1+2q+3q^2+3q^3+\ldots+3q^{n-2}+2q^{n-1}+q^n \\ =& [n-1]_q\,[3]_q, \end{aligned}$$

and

$$\begin{aligned} & \frac{q^2\,[n-2]_q}{[n]_q} + \frac{q\,[n-2]_q\,[n+1]_q}{[n]_q\,[n-1]_q} + \frac{[n+1]_q}{[n-1]_q} \\ =& \frac{q^2\,[n-2]_q}{[n]_q} + \frac{[n+1]_q}{[n]_q\,[n-1]_q}(q(1+q+q^2+\ldots+q^{n-3}) + (1+q+\ldots+q^{n-1})) \\ =& \frac{q^2\,[n-2]_q}{[n]_q} + \frac{[n+1]_q}{[n]_q\,[n-1]_q}[2]_q\,[n-1]_q \\ =& \frac{1}{[n]_q}\left(q^2(1+q+\ldots+q^{n-3}) + (1+q)(1+q+\ldots+q^n)\right) \\ =& \frac{1}{[n]_q}[3]_q\,[n]_q = [3]_q. \end{aligned}$$

Therefore,

$$\mathcal{S}(I,J) = (-1)^3 A^{-2\cdot 3\cdot(3-1)} \cdot \frac{\begin{bmatrix} 2n-4|J|-5 \\ n-2|J|-3 \end{bmatrix} C_{I,J}}{\begin{bmatrix} 2n-4|J|-5 \\ n-2|J| \end{bmatrix} C_{J,J}} = \mathcal{S}'(I,J).$$

For the case (4), $I = \{i_1, i_1+1, i_3\}$ with $i_1+3 < i_3$, and then

- $J \prec_F I$,
- $J \prec_F J \cup \{i_1+1\} \prec_F I$,
- $J \prec_F J \cup \{i_3\} \prec_F I$,
- $J \prec_F J \cup \{i_1, i_1+1\} \prec_F I$,
- $J \prec_F J \cup \{i_1+1, i_3\} \prec_F I$,
- $J \prec_F J \cup \{i_1+1\} \prec_F J \cup \{i_1, i_1+1\} \prec_F I$,
- $J \prec_F J \cup \{i_1+1\} \prec_F J \cup \{i_1+1, i_3\} \prec_F I$, or

- $J \prec_F J \cup \{i_3\} \prec_F J \cup \{i_1+1, i_3\} \prec_F I$.

So,

$$\begin{aligned}
\mathcal{S}(I,J) = &-\frac{C_{I,J}}{C_{J,J}} + \frac{C_{J\cup\{i_1+1\},J}}{C_{J,J}} \cdot \frac{C_{I,J\cup\{i_1+1\}}}{C_{J\cup\{i_1+1\},J\cup\{i_1+1\}}} + \frac{C_{J\cup\{i_3\},J}}{C_{J,J}} \cdot \frac{C_{I,J\cup\{i_3\}}}{C_{J\cup\{i_3\},J\cup\{i_3\}}} \\
&+ \frac{C_{J\cup\{i_1,i_1+1\},J}}{C_{J,J}} \cdot \frac{C_{I,J\cup\{i_1,i_1+1\}}}{C_{J\cup\{i_1,i_1+1\},J\cup\{i_1,i_1+1\}}} + \frac{C_{J\cup\{i_1+1,i_3\},J}}{C_{J,J}} \cdot \frac{C_{I,J\cup\{i_1+1,i_3\}}}{C_{J\cup\{i_1+1,i_3\},J\cup\{i_1+1,i_3\}}} \\
&- \frac{C_{J\cup\{i_1+1\},J}}{C_{J,J}} \cdot \frac{C_{J\cup\{i_1,i_1+1\},J\cup\{i_1\}}}{C_{J\cup\{i_1\},J\cup\{i_1\}}} \cdot \frac{C_{I,J\cup\{i_1,i_1+1\}}}{C_{J\cup\{i_1,i_1+1\},J\cup\{i_1,i_1+1\}}} \\
&- \frac{C_{J\cup\{i_1+1\},J}}{C_{J,J}} \cdot \frac{C_{J\cup\{i_1+1,i_3\},J\cup\{i_1+1\}}}{C_{J\cup\{i_1+1\},J\cup\{i_1+1\}}} \cdot \frac{C_{I,J\cup\{i_1+1,i_3\}}}{C_{J\cup\{i_1+1,i_3\},J\cup\{i_1+1,i_3\}}} \\
&- \frac{C_{J\cup\{i_3\},J}}{C_{J,J}} \cdot \frac{C_{J\cup\{i_1+1,i_3\},J\cup\{i_3\}}}{C_{J\cup\{i_3\},J\cup\{i_3\}}} \cdot \frac{C_{I,J\cup\{i_1+1,i_3\}}}{C_{J\cup\{i_1+1,i_3\},J\cup\{i_1+1,i_3\}}}.
\end{aligned}$$

By Lemma 3.2.3,

$$C_{I,J} = A^{-2(2i_1'+i_3'+1)+3(9-n'-\kappa_1-\kappa_2+1)+\frac{n'}{2}(3\kappa_1-\kappa_2)+\frac{1}{2}(\kappa_1-\kappa_2)^2} \cdot \begin{bmatrix} n'-3 \\ \mu' \end{bmatrix} \begin{bmatrix} i_1'+1 \\ 2 \end{bmatrix} [i_3'-2],$$

$$C_{J\cup\{i_1+1\},J} = A^{-2(i_1'+1)+(3-n'-\kappa_1-\kappa_2+1)+\frac{n'}{2}(3\kappa_1-\kappa_2)+\frac{1}{2}(\kappa_1-\kappa_2)^2} \cdot \begin{bmatrix} n'-1 \\ \mu' \end{bmatrix} [i_1'+1],$$

$$C_{I,J\cup\{i_1+1\}} = A^{-2(i_1'+i_3'-2)+2(6-(n'-2)-\kappa_1-\kappa_2+1)+\frac{n'-2}{2}(3\kappa_1-\kappa_2)+\frac{1}{2}(\kappa_1-\kappa_2)^2} \cdot \begin{bmatrix} n'-4 \\ \mu'-1 \end{bmatrix} [i_1'][i_3'-3],$$

$$C_{J\cup\{i_1+1\},J\cup\{i_1+1\}} = C_{J\cup\{i_3\},J\cup\{i_3\}} = A^{\frac{n'-2}{2}(3\kappa_1-\kappa_2)+\frac{1}{2}(\kappa_1-\kappa_2)^2} \cdot \begin{bmatrix} n'-2 \\ \mu'-1 \end{bmatrix},$$

$$C_{J\cup\{i_3\},J} = A^{-2(i_3')+(3-n'-\kappa_1-\kappa_2+1)+\frac{n'}{2}(3\kappa_1-\kappa_2)+\frac{1}{2}(\kappa_1-\kappa_2)^2} \cdot \begin{bmatrix} n'-1 \\ \mu' \end{bmatrix} [i_3'],$$

$$C_{I,J\cup\{i_3\}} = A^{-2(2i_1'+1)+2(6-(n'-2)-\kappa_1-\kappa_2+1)+\frac{n'-2}{2}(3\kappa_1-\kappa_2)+\frac{1}{2}(\kappa_1-\kappa_2)^2} \cdot \begin{bmatrix} n'-4 \\ \mu'-1 \end{bmatrix} \begin{bmatrix} i_1'+1 \\ 2 \end{bmatrix},$$

$$C_{J\cup\{i_1,i_1+1\},J} = A^{-2(2i_1'+1)+2(6-n'-\kappa_1-\kappa_2+1)+\frac{n'}{2}(3\kappa_1-\kappa_2)+\frac{1}{2}(\kappa_1-\kappa_2)^2} \cdot \begin{bmatrix} n'-2 \\ \mu' \end{bmatrix} \begin{bmatrix} i_1'+1 \\ 2 \end{bmatrix},$$

$$C_{I,J\cup\{i_1,i_1+1\}} = A^{-2(i_3'-4)+(3-(n'-4)-\kappa_1-\kappa_2+1)+\frac{n'-4}{2}(3\kappa_1-\kappa_2)+\frac{1}{2}(\kappa_1-\kappa_2)^2} \cdot \begin{bmatrix} n'-5 \\ \mu'-2 \end{bmatrix} [i_3'-4],$$

$$C_{J\cup\{i_1,i_1+1\},J\cup\{i_1,i_1+1\}} = C_{J\cup\{i_1+1,i_3\},J\cup\{i_1+1,i_3\}} = A^{\frac{n'-4}{2}(3\kappa_1-\kappa_2)+\frac{1}{2}(\kappa_1-\kappa_2)^2} \cdot \begin{bmatrix} n'-4 \\ \mu'-2 \end{bmatrix},$$

$$C_{J\cup\{i_1+1,i_3\},J} = A^{-2(i_1'+i_3'+1)+2(6-n'-\kappa_1-\kappa_2+1)+\frac{n'}{2}(3\kappa_1-\kappa_2)+\frac{1}{2}(\kappa_1-\kappa_2)^2} \cdot \begin{bmatrix} n'-2 \\ \mu' \end{bmatrix} [i_1'+1][i_3'-1],$$

$$C_{I,J\cup\{i_1+1,i_3\}} = A^{-2(i'_1)+(3-(n'-4)-\kappa_1-\kappa_2+1)+\frac{n'-4}{2}(3\kappa_1-\kappa_2)+\frac{1}{2}(\kappa_1-\kappa_2)^2}\cdot\begin{bmatrix}n'-5\\ \mu'-2\end{bmatrix}[i'_1],$$

$$C_{J\cup\{i_1,i_1+1\},J\cup\{i_1+1\}} = A^{-2(i'_1)+(3-(n'-2)-\kappa_1-\kappa_2+1)+\frac{n'-2}{2}(3\kappa_1-\kappa_2)+\frac{1}{2}(\kappa_1-\kappa_2)^2}\cdot\begin{bmatrix}n'-3\\ \mu'-1\end{bmatrix}[i'_1],$$

$$C_{J\cup\{i_1+1,i_3\},J\cup\{i_1+1\}} = A^{-2(i'_3-2)+(3-(n'-2)-\kappa_1-\kappa_2+1)+\frac{n'-2}{2}(3\kappa_1-\kappa_2)+\frac{1}{2}(\kappa_1-\kappa_2)^2}\cdot\begin{bmatrix}n'-3\\ \mu'-1\end{bmatrix}[i'_3-2]$$

and

$$C_{J\cup\{i_1+1,i_3\},J\cup\{i_3\}} = A^{-2(i'_1+1)+(3-(n'-2)-\kappa_1-\kappa_2+1)+\frac{n'-2}{2}(3\kappa_1-\kappa_2)+\frac{1}{2}(\kappa_1-\kappa_2)^2}\cdot\begin{bmatrix}n'-3\\ \mu'-1\end{bmatrix}[i'_1+1].$$

Then, using previous results, it follows that

$$\begin{aligned}
C_{J,J}\cdot\mathcal{S}(I,J) &= C_{I,J}\cdot\Bigg(-1+\frac{A^{-4}\left[\begin{smallmatrix}n'-1\\ \mu'\end{smallmatrix}\right][i'_1+1]\left[\begin{smallmatrix}n'-4\\ \mu'-1\end{smallmatrix}\right][i'_1][i'_3-3]}{\left[\begin{smallmatrix}n'-2\\ \mu'-1\end{smallmatrix}\right]\left[\begin{smallmatrix}n'-3\\ \mu'\end{smallmatrix}\right]\left[\begin{smallmatrix}i'_1+1\\ 2\end{smallmatrix}\right][i'_3-2]}+\frac{A^{-8}\left[\begin{smallmatrix}n'-1\\ \mu'\end{smallmatrix}\right][i'_3]\left[\begin{smallmatrix}n'-4\\ \mu'-1\end{smallmatrix}\right]\left[\begin{smallmatrix}i'_1+1\\ 2\end{smallmatrix}\right]}{\left[\begin{smallmatrix}n'-2\\ \mu'-1\end{smallmatrix}\right]\left[\begin{smallmatrix}n'-3\\ \mu'\end{smallmatrix}\right]\left[\begin{smallmatrix}i'_1+1\\ 2\end{smallmatrix}\right][i'_3-2]}\\
&+\frac{\left[\begin{smallmatrix}n'-2\\ \mu'\end{smallmatrix}\right]\left[\begin{smallmatrix}i'_1+1\\ 2\end{smallmatrix}\right]\left[\begin{smallmatrix}n'-5\\ \mu'-2\end{smallmatrix}\right][i'_3-4]}{\left[\begin{smallmatrix}n'-4\\ \mu'-2\end{smallmatrix}\right]\left[\begin{smallmatrix}n'-3\\ \mu'\end{smallmatrix}\right]\left[\begin{smallmatrix}i'_1+1\\ 2\end{smallmatrix}\right][i'_3-2]}+\frac{A^{-8}\left[\begin{smallmatrix}n'-2\\ \mu'\end{smallmatrix}\right][i'_1+1][i'_3-1]\left[\begin{smallmatrix}n'-5\\ \mu'-2\end{smallmatrix}\right][i'_1]}{\left[\begin{smallmatrix}n'-4\\ \mu'-2\end{smallmatrix}\right]\left[\begin{smallmatrix}n'-3\\ \mu'\end{smallmatrix}\right]\left[\begin{smallmatrix}i'_1+1\\ 2\end{smallmatrix}\right][i'_3-2]}\\
&-\frac{A^{-4}\left[\begin{smallmatrix}n'-1\\ \mu'\end{smallmatrix}\right][i'_1+1]\left[\begin{smallmatrix}n'-3\\ \mu'-1\end{smallmatrix}\right][i'_1]\left[\begin{smallmatrix}n'-5\\ \mu'-2\end{smallmatrix}\right][i'_3-4]}{\left[\begin{smallmatrix}n'-2\\ \mu'-1\end{smallmatrix}\right]\left[\begin{smallmatrix}n'-4\\ \mu'-2\end{smallmatrix}\right]\left[\begin{smallmatrix}n'-3\\ \mu'\end{smallmatrix}\right]\left[\begin{smallmatrix}i'_1+1\\ 2\end{smallmatrix}\right][i'_3-2]}-\frac{A^{-8}\left[\begin{smallmatrix}n'-1\\ \mu'\end{smallmatrix}\right][i'_1+1]\left[\begin{smallmatrix}n'-3\\ \mu'-1\end{smallmatrix}\right][i'_3-2]\left[\begin{smallmatrix}n'-5\\ \mu'-2\end{smallmatrix}\right][i'_1]}{\left[\begin{smallmatrix}n'-2\\ \mu'-1\end{smallmatrix}\right]\left[\begin{smallmatrix}n'-4\\ \mu'-2\end{smallmatrix}\right]\left[\begin{smallmatrix}n'-3\\ \mu'\end{smallmatrix}\right]\left[\begin{smallmatrix}i'_1+1\\ 2\end{smallmatrix}\right][i'_3-2]}\\
&-\frac{A^{-12}\left[\begin{smallmatrix}n'-1\\ \mu'\end{smallmatrix}\right][i'_3]\left[\begin{smallmatrix}n'-3\\ \mu'-1\end{smallmatrix}\right][i'_1+1]\left[\begin{smallmatrix}n'-5\\ \mu'-2\end{smallmatrix}\right][i'_1]}{\left[\begin{smallmatrix}n'-2\\ \mu'-1\end{smallmatrix}\right]\left[\begin{smallmatrix}n'-4\\ \mu'-2\end{smallmatrix}\right]\left[\begin{smallmatrix}n'-3\\ \mu'\end{smallmatrix}\right]\left[\begin{smallmatrix}i'_1+1\\ 2\end{smallmatrix}\right][i'_3-2]}\Bigg)\\
&= C_{I,J}\cdot\Bigg(-1+\frac{A^{-8}[n'-1]}{[n'-3][i'_3-2]}\left(A^4[i'_3-3][2]+[i'_3]\right)+\frac{[n'-2]}{[n'-4][i'_3-2]}\big([i'_3-4]\\
&+A^{-8}[i'_3-1][2]\big)-\frac{A^{-12}[n'-1][2]}{[n'-4]}\left(\frac{A^8[i'_3-4]}{[i'_3-2]}+A^4+\frac{[i'_3]}{[i'_3-2]}\right)\Bigg)\\
&= C_{I,J}\cdot\left(-1+\frac{A^{-8}[n'-1][3]}{[n'-3]}+\frac{A^{-8}[n'-2][3]}{[n'-4]}-\frac{A^{-12}[n'-1][3][2]}{[n'-4]}\right)\\
&= (-1)^3A^{-2\cdot 3\cdot(3-1)}\cdot\frac{\left[\begin{smallmatrix}2n'-5\\ n'-3\end{smallmatrix}\right]}{\left[\begin{smallmatrix}2n'-5\\ n'\end{smallmatrix}\right]}C_{I,J},
\end{aligned}$$

since

$$\begin{aligned}
q\,[n-3]_q\,[2]_q+[n]_q =& q(1+2q+2q^2+\ldots+2q^{n-4}+q^{n-3})+(1+q+\ldots+q^{n-1})\\
=& 1+2q+3q^2+3q^3+\ldots+3q^{n-3}+2q^{n-2}+q^{n-1})\\
=& [n-2]_q\,[3]_q,
\end{aligned}$$

$$\begin{aligned}
[n-4]_q + q^{-2}\,[n-1]_q\,[2]_q =&(1+q+\ldots+q^{n-5}) + q^{-2}(1+2q+2q^2+\ldots+2q^{n-2}+q^{n-3}) \\
=&q^{-2}(1+2q+3q^2+3q^3+\ldots+3q^{n-3}+2q^{n-2}+q^{n-1}) \\
=&q^{-2}\,[n-2]_q\,[3]_q,
\end{aligned}$$

and

$$\begin{aligned}
q^2\,[n-4]_q + q\,[n-2]_q\,[n]_q =&q^2(1+q+\ldots+q^{n-5}) + q(1+q+\ldots+q^{n-3}) \\
&+(1+q+\ldots+q^{n-1}) \\
=&1+2q+3q^2+3q^3+\ldots+3q^{n-3}+2q^{n-2}+q^{n-1} \\
=&[n-2]_q\,[3]_q.
\end{aligned}$$

Therefore,

$$\mathcal{S}(I,J) = (-1)^3 A^{-2\cdot 3\cdot(3-1)} \cdot \frac{\left[{2n-4|J|-5 \atop n-2|J|-3}\right] C_{I,J}}{\left[{2n-4|J|-5 \atop n-2|J|}\right] C_{J,J}} = \mathcal{S}'(I,J).$$

For the case (5), $I=\{i_1,i_2,i_3\}$ with $i_1+2<i_2+1<i_3$, and then

- $J \prec_F I$,
- $J \prec_F J\cup\{i_1\} \prec_F I$,
- $J \prec_F J\cup\{i_2\} \prec_F I$,
- $J \prec_F J\cup\{i_3\} \prec_F I$,
- $J \prec_F J\cup\{i_1,i_2\} \prec_F I$,
- $J \prec_F J\cup\{i_1,i_3\} \prec_F I$,
- $J \prec_F J\cup\{i_2,i_3\} \prec_F I$,
- $J \prec_F J\cup\{i_1\} \prec_F J\cup\{i_1,i_2\} \prec_F I$,
- $J \prec_F J\cup\{i_2\} \prec_F J\cup\{i_1,i_2\} \prec_F I$,

- $J \prec_F J \cup \{i_1\} \prec_F J \cup \{i_1, i_3\} \prec_F I$,
- $J \prec_F J \cup \{i_3\} \prec_F J \cup \{i_1, i_3\} \prec_F I$,
- $J \prec_F J \cup \{i_2\} \prec_F J \cup \{i_2, i_3\} \prec_F I$, or
- $J \prec_F J \cup \{i_3\} \prec_F J \cup \{i_2, i_3\} \prec_F I$.

So,

$$\begin{aligned}\mathcal{S}(I,J) = &-\frac{C_{I,J}}{C_{J,J}} + \frac{C_{J\cup\{i_1\},J}}{C_{J,J}} \cdot \frac{C_{I,J\cup\{i_1\}}}{C_{J\cup\{i_1\},J\cup\{i_1\}}} + \frac{C_{J\cup\{i_2\},J}}{C_{J,J}} \cdot \frac{C_{I,J\cup\{i_2\}}}{C_{J\cup\{i_2\},J\cup\{i_2\}}} \\
&+ \frac{C_{J\cup\{i_3\},J}}{C_{J,J}} \cdot \frac{C_{I,J\cup\{i_3\}}}{C_{J\cup\{i_3\},J\cup\{i_3\}}} + \frac{C_{J\cup\{i_1,i_2\},J}}{C_{J,J}} \cdot \frac{C_{I,J\cup\{i_1,i_2\}}}{C_{J\cup\{i_1,i_2\},J\cup\{i_1,i_2\}}} \\
&+ \frac{C_{J\cup\{i_1,i_3\},J}}{C_{J,J}} \cdot \frac{C_{I,J\cup\{i_1,i_3\}}}{C_{J\cup\{i_1,i_3\},J\cup\{i_1,i_3\}}} + \frac{C_{J\cup\{i_2,i_3\},J}}{C_{J,J}} \cdot \frac{C_{I,J\cup\{i_2,i_3\}}}{C_{J\cup\{i_2,i_3\},J\cup\{i_2,i_3\}}} \\
&- \frac{C_{J\cup\{i_1\},J}}{C_{J,J}} \cdot \frac{C_{J\cup\{i_1,i_2\},J\cup\{i_1\}}}{C_{J\cup\{i_1\},J\cup\{i_1\}}} \cdot \frac{C_{I,J\cup\{i_1,i_2\}}}{C_{J\cup\{i_1,i_2\},J\cup\{i_1,i_2\}}} \\
&- \frac{C_{J\cup\{i_2\},J}}{C_{J,J}} \cdot \frac{C_{J\cup\{i_1,i_2\},J\cup\{i_2\}}}{C_{J\cup\{i_2\},J\cup\{i_2\}}} \cdot \frac{C_{I,J\cup\{i_1,i_2\}}}{C_{J\cup\{i_1,i_2\},J\cup\{i_1,i_2\}}} \\
&- \frac{C_{J\cup\{i_1\},J}}{C_{J,J}} \cdot \frac{C_{J\cup\{i_1,i_3\},J\cup\{i_1\}}}{C_{J\cup\{i_1\},J\cup\{i_1\}}} \cdot \frac{C_{I,J\cup\{i_1,i_3\}}}{C_{J\cup\{i_1,i_3\},J\cup\{i_1,i_3\}}} \\
&- \frac{C_{J\cup\{i_3\},J}}{C_{J,J}} \cdot \frac{C_{J\cup\{i_1,i_3\},J\cup\{i_3\}}}{C_{J\cup\{i_3\},J\cup\{i_3\}}} \cdot \frac{C_{I,J\cup\{i_1,i_3\}}}{C_{J\cup\{i_1,i_3\},J\cup\{i_1,i_3\}}} \\
&- \frac{C_{J\cup\{i_2\},J}}{C_{J,J}} \cdot \frac{C_{J\cup\{i_2,i_3\},J\cup\{i_2\}}}{C_{J\cup\{i_2\},J\cup\{i_2\}}} \cdot \frac{C_{I,J\cup\{i_2,i_3\}}}{C_{J\cup\{i_2,i_3\},J\cup\{i_2,i_3\}}} \\
&- \frac{C_{J\cup\{i_3\},J}}{C_{J,J}} \cdot \frac{C_{J\cup\{i_2,i_3\},J\cup\{i_3\}}}{C_{J\cup\{i_3\},J\cup\{i_3\}}} \cdot \frac{C_{I,J\cup\{i_2,i_3\}}}{C_{J\cup\{i_2,i_3\},J\cup\{i_2,i_3\}}}.\end{aligned}$$

By Lemma 3.2.3,

$$C_{I,J} = A^{-2(i_1'+i_2'+i_3')+3(9-n'-\kappa_1-\kappa_2+1)+\frac{n'}{2}(3\kappa_1-\kappa_2)+\frac{1}{2}(\kappa_1-\kappa_2)^2} \cdot \begin{bmatrix} n'-3 \\ \mu' \end{bmatrix} [i_1'][i_2'-1][i_3'-2],$$

$$C_{J\cup\{i_1\},J} = A^{-2(i_1')+(3-n'-\kappa_1-\kappa_2+1)+\frac{n'}{2}(3\kappa_1-\kappa_2)+\frac{1}{2}(\kappa_1-\kappa_2)^2} \cdot \begin{bmatrix} n'-1 \\ \mu' \end{bmatrix} [i_1'],$$

$$C_{I,J\cup\{i_1\}} = A^{-2(i_2'+i_3'-4)+2(6-(n'-2)-\kappa_1-\kappa_2+1)+\frac{n'-2}{2}(3\kappa_1-\kappa_2)+\frac{1}{2}(\kappa_1-\kappa_2)^2} \cdot \begin{bmatrix} n'-4 \\ \mu'-1 \end{bmatrix} [i_2'-2][i_3'-3],$$

$$C_{J\cup\{i_1\},J\cup\{i_1\}} = C_{J\cup\{i_2\},J\cup\{i_2\}} = C_{J\cup\{i_3\},J\cup\{i_3\}} = A^{\frac{n'-2}{2}(3\kappa_1-\kappa_2)+\frac{1}{2}(\kappa_1-\kappa_2)^2} \cdot \begin{bmatrix} n'-2 \\ \mu'-1 \end{bmatrix},$$

$$C_{J\cup\{i_2\},J} = A^{-2(i_2')+(3-n'-\kappa_1-\kappa_2+1)+\frac{n'}{2}(3\kappa_1-\kappa_2)+\frac{1}{2}(\kappa_1-\kappa_2)^2} \cdot \begin{bmatrix} n'-1 \\ \mu' \end{bmatrix} [i_2'],$$

$$C_{I,J\cup\{i_2\}} = A^{-2(i_1'+i_3'-2)+2(6-(n'-2)-\kappa_1-\kappa_2+1)+\frac{n'-2}{2}(3\kappa_1-\kappa_2)+\frac{1}{2}(\kappa_1-\kappa_2)^2} \cdot \begin{bmatrix} n'-4 \\ \mu'-1 \end{bmatrix} [i_1'][i_3'-3],$$

$$C_{J\cup\{i_3\},J} = A^{-2(i_3')+(3-n'-\kappa_1-\kappa_2+1)+\frac{n'}{2}(3\kappa_1-\kappa_2)+\frac{1}{2}(\kappa_1-\kappa_2)^2} \cdot \begin{bmatrix} n'-1 \\ \mu' \end{bmatrix} [i_3'],$$

$$C_{I,J\cup\{i_3\}} = A^{-2(i_1'+i_2')+2(6-(n'-2)-\kappa_1-\kappa_2+1)+\frac{n'-2}{2}(3\kappa_1-\kappa_2)+\frac{1}{2}(\kappa_1-\kappa_2)^2} \cdot \begin{bmatrix} n'-4 \\ \mu'-1 \end{bmatrix} [i_1'][i_2'-1],$$

$$C_{J\cup\{i_1,i_2\},J} = A^{-2(i_1'+i_2')+2(6-n'-\kappa_1-\kappa_2+1)+\frac{n'}{2}(3\kappa_1-\kappa_2)+\frac{1}{2}(\kappa_1-\kappa_2)^2} \cdot \begin{bmatrix} n'-2 \\ \mu' \end{bmatrix} [i_1'][i_2'-1],$$

$$C_{I,J\cup\{i_1,i_2\}} = A^{-2(i_3'-4)+(3-(n'-4)-\kappa_1-\kappa_2+1)+\frac{n'-4}{2}(3\kappa_1-\kappa_2)+\frac{1}{2}(\kappa_1-\kappa_2)^2} \cdot \begin{bmatrix} n'-5 \\ \mu'-2 \end{bmatrix} [i_3'-4],$$

$$C_{J\cup\{i_1,i_2\},J\cup\{i_1,i_2\}} = C_{J\cup\{i_1,i_3\},J\cup\{i_1,i_3\}} = C_{J\cup\{i_2,i_3\},J\cup\{i_2,i_3\}} = A^{\frac{n'-4}{2}(3\kappa_1-\kappa_2)+\frac{1}{2}(\kappa_1-\kappa_2)^2} \cdot \begin{bmatrix} n'-4 \\ \mu'-2 \end{bmatrix},$$

$$C_{J\cup\{i_1,i_3\},J} = A^{-2(i_1'+i_3')+2(6-n'-\kappa_1-\kappa_2+1)+\frac{n'}{2}(3\kappa_1-\kappa_2)+\frac{1}{2}(\kappa_1-\kappa_2)^2} \cdot \begin{bmatrix} n'-2 \\ \mu' \end{bmatrix} [i_1'][i_3'-1],$$

$$C_{I,J\cup\{i_1,i_3\}} = A^{-2(i_2'-2)+(3-(n'-4)-\kappa_1-\kappa_2+1)+\frac{n'-4}{2}(3\kappa_1-\kappa_2)+\frac{1}{2}(\kappa_1-\kappa_2)^2} \cdot \begin{bmatrix} n'-5 \\ \mu'-2 \end{bmatrix} [i_2'-2],$$

$$C_{J\cup\{i_2,i_3\},J} = A^{-2(i_2'+i_3')+2(6-n'-\kappa_1-\kappa_2+1)+\frac{n'}{2}(3\kappa_1-\kappa_2)+\frac{1}{2}(\kappa_1-\kappa_2)^2} \cdot \begin{bmatrix} n'-2 \\ \mu' \end{bmatrix} [i_2'][i_3'-1],$$

$$C_{I,J\cup\{i_2,i_3\}} = A^{-2(i_1')+(3-(n'-4)-\kappa_1-\kappa_2+1)+\frac{n'-4}{2}(3\kappa_1-\kappa_2)+\frac{1}{2}(\kappa_1-\kappa_2)^2} \cdot \begin{bmatrix} n'-5 \\ \mu'-2 \end{bmatrix} [i_1'],$$

$$C_{J\cup\{i_1,i_2\},J\cup\{i_1\}} = A^{-2(i_2'-2)+(3-(n'-2)-\kappa_1-\kappa_2+1)+\frac{n'-2}{2}(3\kappa_1-\kappa_2)+\frac{1}{2}(\kappa_1-\kappa_2)^2} \cdot \begin{bmatrix} n'-3 \\ \mu'-1 \end{bmatrix} [i_2'-2],$$

$$C_{J\cup\{i_1,i_2\},J\cup\{i_2\}} = A^{-2(i_1')+(3-(n'-2)-\kappa_1-\kappa_2+1)+\frac{n'-2}{2}(3\kappa_1-\kappa_2)+\frac{1}{2}(\kappa_1-\kappa_2)^2} \cdot \begin{bmatrix} n'-3 \\ \mu'-1 \end{bmatrix} [i_1'],$$

$$C_{J\cup\{i_1,i_3\},J\cup\{i_1\}} = A^{-2(i_3'-2)+(3-(n'-2)-\kappa_1-\kappa_2+1)+\frac{n'-2}{2}(3\kappa_1-\kappa_2)+\frac{1}{2}(\kappa_1-\kappa_2)^2} \cdot \begin{bmatrix} n'-3 \\ \mu'-1 \end{bmatrix} [i_3'-2],$$

$$C_{J\cup\{i_1,i_3\},J\cup\{i_3\}} = A^{-2(i_1')+(3-(n'-2)-\kappa_1-\kappa_2+1)+\frac{n'-2}{2}(3\kappa_1-\kappa_2)+\frac{1}{2}(\kappa_1-\kappa_2)^2} \cdot \begin{bmatrix} n'-3 \\ \mu'-1 \end{bmatrix} [i_1'],$$

$$C_{J\cup\{i_2,i_3\},J\cup\{i_2\}} = A^{-2(i_3'-2)+(3-(n'-2)-\kappa_1-\kappa_2+1)+\frac{n'-2}{2}(3\kappa_1-\kappa_2)+\frac{1}{2}(\kappa_1-\kappa_2)^2} \cdot \begin{bmatrix} n'-3 \\ \mu'-1 \end{bmatrix} [i_3'-2],$$

and

$$C_{J\cup\{i_2,i_3\},J\cup\{i_3\}} = A^{-2(i_2')+(3-(n'-2)-\kappa_1-\kappa_2+1)+\frac{n'-2}{2}(3\kappa_1-\kappa_2)+\frac{1}{2}(\kappa_1-\kappa_2)^2} \cdot \begin{bmatrix} n'-3 \\ \mu'-1 \end{bmatrix} [i_2'].$$

Thus, from the previous results, it follows that

$$
\begin{aligned}
C_{J,J}\cdot\mathcal{S}(I,J) &= C_{I,J}\cdot\Bigg(-1+\frac{\left[{n'-1\atop \mu'}\right][i_1']\left[{n'-4\atop \mu'-1}\right][i_2'-2][i_3'-3]}{\left[{n'-2\atop \mu'-1}\right]\left[{n'-3\atop \mu'}\right][i_1'][i_2'-1][i_3'-2]}+\frac{A^{-4}\left[{n'-1\atop \mu'}\right][i_2']\left[{n'-4\atop \mu'-1}\right][i_1'][i_3'-3]}{\left[{n'-2\atop \mu'-1}\right]\left[{n'-3\atop \mu'}\right][i_1'][i_2'-1][i_3'-2]}\\
&+\frac{A^{-8}\left[{n'-1\atop \mu'}\right][i_3']\left[{n'-4\atop \mu'-1}\right][i_1'][i_2'-1]}{\left[{n'-2\atop \mu'-1}\right]\left[{n'-3\atop \mu'}\right][i_1'][i_2'-1][i_3'-2]}+\frac{\left[{n'-2\atop \mu'}\right][i_1'][i_2'-1]\left[{n'-5\atop \mu'-2}\right][i_3'-4]}{\left[{n'-4\atop \mu'-2}\right]\left[{n'-3\atop \mu'}\right][i_1'][i_2'-1][i_3'-2]}\\
&+\frac{A^{-4}\left[{n'-2\atop \mu'}\right][i_1'][i_3'-1]\left[{n'-5\atop \mu'-2}\right][i_2'-2]}{\left[{n'-4\atop \mu'-2}\right]\left[{n'-3\atop \mu'}\right][i_1'][i_2'-1][i_3'-2]}+\frac{A^{-8}\left[{n'-2\atop \mu'}\right][i_2'][i_3'-1]\left[{n'-5\atop \mu'-2}\right][i_1']}{\left[{n'-4\atop \mu'-2}\right]\left[{n'-3\atop \mu'}\right][i_1'][i_2'-1][i_3'-2]}\\
&-\frac{\left[{n'-1\atop \mu'}\right][i_1']\left[{n'-3\atop \mu'-1}\right][i_2'-2]\left[{n'-5\atop \mu'-2}\right][i_3'-4]}{\left[{n'-2\atop \mu'-1}\right]\left[{n'-4\atop \mu'-2}\right]\left[{n'-3\atop \mu'}\right][i_1'][i_2'-1][i_3'-2]}-\frac{A^{-4}\left[{n'-1\atop \mu'}\right][i_2']\left[{n'-3\atop \mu'-1}\right][i_1']\left[{n'-5\atop \mu'-2}\right][i_3'-4]}{\left[{n'-2\atop \mu'-1}\right]\left[{n'-4\atop \mu'-2}\right]\left[{n'-3\atop \mu'}\right][i_1'][i_2'-1][i_3'-2]}\\
&-\frac{A^{-4}\left[{n'-1\atop \mu'}\right][i_1']\left[{n'-3\atop \mu'-1}\right][i_3'-2]\left[{n'-5\atop \mu'-2}\right][i_2'-2]}{\left[{n'-2\atop \mu'-1}\right]\left[{n'-4\atop \mu'-2}\right]\left[{n'-3\atop \mu'}\right][i_1'][i_2'-1][i_3'-2]}-\frac{A^{-8}\left[{n'-1\atop \mu'}\right][i_3']\left[{n'-3\atop \mu'-1}\right][i_1']\left[{n'-5\atop \mu'-2}\right][i_2'-2]}{\left[{n'-2\atop \mu'-1}\right]\left[{n'-4\atop \mu'-2}\right]\left[{n'-3\atop \mu'}\right][i_1'][i_2'-1][i_3'-2]}\\
&-\frac{A^{-8}\left[{n'-1\atop \mu'}\right][i_2']\left[{n'-3\atop \mu'-1}\right][i_3'-2]\left[{n'-5\atop \mu'-2}\right][i_1']}{\left[{n'-2\atop \mu'-1}\right]\left[{n'-4\atop \mu'-2}\right]\left[{n'-3\atop \mu'}\right][i_1'][i_2'-1][i_3'-2]}-\frac{A^{-12}\left[{n'-1\atop \mu'}\right][i_3']\left[{n'-3\atop \mu'-1}\right][i_2']\left[{n'-5\atop \mu'-2}\right][i_1']}{\left[{n'-2\atop \mu'-1}\right]\left[{n'-4\atop \mu'-2}\right]\left[{n'-3\atop \mu'}\right][i_1'][i_2'-1][i_3'-2]}\Bigg)\\
&= C_{I,J}\cdot\Bigg(-1+\frac{A^{-8}[n'-1]}{[n'-3][i_2'-1][i_3'-2]}\left(A^8[i_2'-2][i_3'-3]+A^4[i_2'][i_3'-3]+[i_2'-1][i_3']\right)\\
&+\frac{A^{-8}[n'-2]}{[n'-4][i_2'-1][i_3'-2]}\left(A^8[i_2'-1][i_3'-4]+A^4[i_2'-2][i_3'-1]+[i_2'][i_3'-1]\right)\\
&-\frac{A^{-8}[n'-1][i_2'-2]}{[n'-4][i_2'-1][i_3'-2]}\left(A^8[i_3'-4]+A^4[i_3'-2]+[i_3']\right)\\
&-\frac{A^{-12}[n'-1][i_2']}{[n'-4][i_2'-1][i_3'-2]}\left(A^8[i_3'-4]+A^4[i_3'-2]+[i_3']\right)\Bigg)\\
&= C_{I,J}\cdot\left(-1+\frac{A^{-8}[n'-1][3]}{[n'-3]}+\frac{A^{-8}[n'-2][3]}{[n'-4]}-\frac{A^{-12}[n'-1][3][2]}{[n'-4]}\right)\\
&= (-1)^3A^{-2\cdot3\cdot(3-1)}\cdot\frac{\left[{2n'-5\atop n'-3}\right]}{\left[{2n'-5\atop n'}\right]}C_{I,J},
\end{aligned}
$$

Therefore,

$$
\mathcal{S}(I,J)=(-1)^3A^{-2\cdot3\cdot(3-1)}\cdot\frac{\left[{2n-4|J|-5\atop n-2|J|-3}\right]C_{I,J}}{\left[{2n-4|J|-5\atop n-2|J|}\right]C_{J,J}}=\mathcal{S}'(I,J).
$$

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