PERIODIC SOLUTIONS TO REVERSIBLE SECOND ORDER AUTONOMOUS DDES IN PRESCRIBED SYMMETRIC NONCONVEX DOMAINS

by

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To the memory of my father and to my mother.

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The existence of periodic solutions to second order differential systems is a classical problem that has been studied by many authors using different methods and techniques. In this Thesis, the existence and spatio-temporal patterns of 2π-periodic solutions to second order reversible equivariant autonomous systems with commensurate delays are studied using the Brouwer $O(2) \times \Gamma \times \mathbb{Z}_2$-equivariant degree theory. The solutions are supposed to take their values in a prescribed symmetric domain D, while $O(2)$ is related to the reversal symmetry combined with the autonomous form of the system. The group Γ reflects symmetries of D and/or possible coupling in the corresponding network of identical oscillaltors, and $\mathbb{Z}_2$ is related to the oddness of the right-hand side. Abstract results, based on the use of Gauss curvature of ∂D, Hartman-Nagumo type *a priori bounds* and Brouwer equivariant degree techniques, are supported by a concrete example with $\Gamma = D_8$ – the dihedral group of order 16.

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CHAPTER 1

INTRODUCTION

1.1 Subject and goal

Existence of periodic solutions to equivariant dynamical systems together with describing their spatio-temporal symmetries constitute an important problem of equivariant dynamics (see, for example, [24, 23] for the equivariant singularity theory based methods and [9, 8, 31] for the equivariant degree treatment). As is well-known, second order systems of ODEs with no friction term exhibit an extra symmetry – the so-called reversal symmetry, i.e. if $x(t)$ is a solution to the system, then so is $x(-t)$. We refer to [36] for a comprehensive exposition of (equivariant) reversible ODEs as well as their applications in natural sciences (see also [2]). It should be stressed that in the context relevant to spatio-temporal symmetries of periodic solutions, the reversal symmetry gives rise to extra subgroups of the non-abelian group $O(2)$.

Simple examples show that, in contrast to their ODEs counterparts, second order delay differential equations (in short, DDEs) with no friction term are not reversible, in general. In [10] (see also [33]), we considered *space reversible* equivariant mixed DDEs of the form

$$\ddot{v}(y) = g(\alpha, v(y)) + a(v(y-\alpha) + v(y+\alpha)), \quad a, \alpha \in \mathbb{R}, \tag{1.1}$$

with equivariant $g : \mathbb{R}^n \to \mathbb{R}^n$ (one can think of equations governing steady-state solutions to PDEs, cf. [36] and references therein). Note that by replacing y by t in (1.1), one obtains *time-reversible* DDEs. However, such systems involve using the information from the future by "traveling back in time", which is difficult to justify from a commonsensical viewpoint.

Time delay systems with *commensurate delays* play an important role in robust control theory (see, for example, [29] and references therein). A class of such systems exhibiting a reversal symmetry is the main *subject* of the present thesis. To be more specific, we are interested in the periodic problem

$$\begin{cases} \ddot{x}(t) = f\left(x(t), x\left(t-\tau_1\right), \ldots, x\left(t-\tau_{m-1}\right), \dot{x}(t)\right), & t \in \mathbb{R},\ x(t) \in \mathbf{V} = \mathbb{R}^n, \\ x(t) = x(t+2\pi), \quad \dot{x}(t) = \dot{x}(t+2\pi) \end{cases} \tag{1.2}$$

(where $\tau_k := \frac{2\pi k}{m}$, $k = 1, 2, \ldots, m-1$) under the following assumption on $f : \mathbf{V}\times\mathbf{V}^{m-1}\times\mathbf{V} \to \mathbf{V}$ providing (together with some additional assumptions), the time reversibility of system (1.2) on periodic solutions:

(R) $f(x, y^1, y^2, \cdots, y^{m-2}, y^{m-1}, z) = f(x, y^{m-1}, y^{m-2}, \cdots, y^2, y^1, z)$, for all $(x, y^1, \cdots, y^{m-1}, z) \in \mathbf{V}^{m+1}$

Assume, in addition, that $\mathbf{V}$ is an orthogonal representation of a finite group Γ.

Put $\mathbf{u} := (x, y^1, \cdots, y^{m-1}, z) \in \mathbf{V}^{m+1}$ and define on $\mathbf{V}^{m+1}$ the diagonal Γ-action by

$$\gamma\mathbf{u} := (\gamma x, \gamma y^1, ..., \gamma y^{m-1}, \gamma z).$$

We make the following symmetry and regularity assumptions:

(A_1) f is Γ-equivariant, i.e., f is continuous and $f(\gamma\mathbf{u}) = \gamma f(\mathbf{u})$ for all $\gamma \in \Gamma$ and $\mathbf{u} \in \mathbf{V}^{m+1}$;

(A_2) for all $x, z \in \mathbf{V}$ and $\mathbf{y} \in \mathbf{V}^{m-1}$, one has:

(i) $f(x, \mathbf{y}, -z) = f(x, \mathbf{y}, z)$,

(ii) $f(-x, -\mathbf{y}, z) = -f(x, \mathbf{y}, z)$;

(A_3) The derivative $A := Df(0) = [A_0, A_1, \ldots, A_{m-1}, 0]$ exists and $A_jA_s = A_sA_j$ for $j, s = 0, 1, \ldots, m-1$.

Furthermore, we will be looking for periodic solutions “living" in a prescribed compact Γ-invariant domain. More formally, let $\eta : \mathbf{V} \to \mathbb{R}$ be a function such that:

(η_1) η is C^2-smooth;

(η_2) $\eta(\gamma x) = \eta(x)$ for all $x \in \mathbf{V}$ and $\gamma \in \Gamma$;

(η_3) $\eta(-x) = \eta(x)$ for all $x \in \mathbf{V}$;

(η_4) $\eta(0) < 0$;

(η_5) 0 is a regular value of η;

(η_6) there exists $R > 0$ such that $D := \eta^{-1}(-\infty, 0) \subset B_R(0)$, where $B_R(0)$ stands for the open ball of radius R centered at the origin.

Clearly, $\overline{D}$ is a smooth compact (oriented) Γ-invariant manifold with boundary

$$C := \partial D = \eta^{-1}(0) \tag{1.3}$$

being a smooth Γ-submanifold of $\mathbf{V}$. Moreover, $-\overline{D} = \overline{D}$ and $0 \in D$.

A starting point for our discussion is the work [1], where the authors considered (non-equivariant) non-autonomous systems without delays. As a matter of fact, the results obtained in [1], being applied in the autonomous setting, do not guarantee that the detected periodic solutions are *non-constant*. At the same time, by combining the reversibility of the system in question with other symmetries, we are able to refine the results of [1] in such a way that the existence of *non-constant* periodic solutions together with their symmetric classification can be provided.

Following [1], we will use the concept of second fundamental form in order to formulate curvature/growth conditions on f generalizing the classical Hartman-Nagumo conditions originally formulated for $D = B_R(0)$ (cf. [27, 39]). Recall the definition of the second fundamental form associated with C. For every $x \in C$, denote by n_x the outer normal vector to C at x i.e.

$$n_x = \frac{\nabla\eta(x)}{|\nabla\eta(x)|}, \tag{1.4}$$

and let $\nu : C \to S^{n-1}$ be the *Gauss map* given by $\nu(x) := n_x$. Obviously, for any $x \in C$, the tangent spaces $T_x(C)$ and $T_{n_x}(S^{n-1})$ are parallel, and as such can be identified. This way, for any $x \in C$, the tangent map $d\nu_x$ (as well as its negative known as a *Weingarten map* or *shape operator* (see, for example, [42])) can be considered as a linear map from $T_x(C)$ into itself. The function $\kappa(x) := \det(-d\nu(x))$ is called the *Gauss curvature* of C. It is well-known (and easy to see) that $-d\nu_x$ is a self-adjoint operator with respect to the standard inner product $\langle \cdot, \cdot \rangle$ in $\mathbb{R}^n = \mathbf{V}$. The quadratic form associated with $-d\nu_x$ and denoted $\mathbb{I}_x(v) := -\langle d\nu_x(v), v\rangle$ is called the *second fundamental form* of C. We will use the notation $\mathbb{I}_x(v, w)$ for the bilinear form associated with $\mathbb{I}_x(v)$. In particular, for two smooth curves c, $d : (-\varepsilon, \varepsilon) \to C$, $c(0) = d(0) = x$ and $\dot{c}(0) = v$, $\dot{d}(0) = w$, one has

$$\mathbb{I}_x(v, w) = -\left\langle \tfrac{d}{dt}\nu(c(t)), \dot{d}(t)\right\rangle \Big|_{t=0}. \tag{1.5}$$

We are now in a position to formulate curvature/growth conditions on f (cf. [1]; see also [20, 11, 21, 38]):

(A_4) for any $x \in C$, $\mathbf{y} \in \mathbf{V}^{m-1}$ and $z \in \mathbf{V}$ such that $|\mathbf{y}| \le R$ and $z \perp n_x$, one has

$$\langle f(x, \mathbf{y}, z), n_x\rangle > \mathbb{I}_x(z) \tag{1.6}$$

(cf. (η_1)–(η_6), (1.3) and (1.4));

(A_5) there exist constants A, $B > 0$ such that the function $\phi(s) := A + Bs^2$, $s \in \mathbb{R}$, satisfies

$$|f(x, \mathbf{y}, z)| \le \phi(|z|)$$

for any $(x, \mathbf{y}, z) \in \mathbf{V} \times \mathbf{V}^{m-1} \times \mathbf{V}$ with $|x|, |\mathbf{y}| \le R$;

(A_6) there exists a constant $K > 0$ such that for any $(x, \mathbf{y}, z) \in \mathbf{V} \times \mathbf{V}^{m-1} \times \mathbf{V}$ with $|x|, |\mathbf{y}| \le R$, one has

$$|f(x, \mathbf{y}, z)| \le \nabla^2\eta(x)(z, z) + \langle f(x, \mathbf{y}, z), \nabla\eta(x)\rangle + K.$$

(A_6') There are constants $\alpha > 0$, $K > 0$ such that

$$\forall_{|x|\le R}\ \forall_{|\mathbf{y}|\le R}\ \forall_{z\in V} \quad |f(x, \mathbf{y}, z)| \le \alpha(\langle x, f(x, \mathbf{y}, z)\rangle + |z|^2) + K$$

Given η satisfying (η_1)–(η_6) and f satisfying (R) along with (A_1)–(A_6) (or (R) along with (A_1)–(A_5) and (A_6')), the *goal* of the present thesis is to study the existence and spatio-temporal patterns of solutions to problem (1.2) living in D. Some remarks are in order:

(i) Under the assumptions that f is continuos and satisfies (A_4)–(A_6), problem (1.2) was considered for *non-autonomous ODEs* in [1], with *no symmetry* conditions on f and D being imposed. The method we are using in the present thesis allows us to treat equivariant non-autonomous DDEs the same way as the autonomous ones with cosmetic modifications only. On the other hand, equivariant autonomous systems satisfying condition (R) allow us to study the impact of the orthogonal group $O(2)$ on spatio-temporal patterns of periodic solutions (versus $D_1 = \{1, \kappa\} < O(2)$ in the non-autonomous case). Also, one can easily adopt the method to treat BVPs rather than periodic problems.

(ii) Since $\mathbb{I}_x(z) \geq -\lambda_{min}(x)$ for every $x \in C$ and $z \perp n_x$ (here $\lambda_{min}(x)$ stands for the minimal eigenvalue of the self-adjoint operator $d\nu_x$), one can replace condition (A_4) by the more verifiable one: (A_4') for every $x \in C$, $\mathbf{y} \in \mathbf{V}^{m-1}$ and $z \in \mathbf{V}$ such that $|\mathbf{y}| \leq R$ and $z \perp n_x$, one has

$$\langle f(x, \mathbf{y}, z), n_x \rangle \geq -\lambda_{min}(x). \tag{1.7}$$

1.2 Method

Observe that given an orthogonal $\mathfrak{G}$-representation V (here $\mathfrak{G}$ stands for a compact Lie group) and an admissible $\mathfrak{G}$-pair (f, Ω) in V (i.e. $\Omega \subset V$ is an open bounded $\mathfrak{G}$-invariant set and $f : V \to V$ is a $\mathfrak{G}$-equivariant map without zeros on $\partial\Omega$), the Brouwer degree $d_{\mathscr{H}} := \deg(f^{\mathscr{H}}, \Omega^{\mathscr{H}})$ is well-defined for any $\mathscr{H} \leq \mathfrak{G}$ (here $\Omega^{\mathscr{H}} := \{x \in \Omega : hx = x \; \forall h \in \mathscr{H}\}$ and $f^{\mathscr{H}} := f|_{\Omega^{\mathscr{H}}}$). If for some $\mathscr{H}$, one has $d_{\mathscr{H}} \neq 0$, then the existence of solutions with symmetry at least $\mathscr{H}$ to equation $f(x) = 0$ in Ω, can be predicted. Although this approach provides a way to determine the existence of solutions in Ω, and even to distinguish their different orbit types, nevertheless, it comes at a price of elaborate $\mathscr{H}$-fixed-point space computations which can be a rather challenging task.

Our method is based on the usage of the Brouwer equivariant degree theory; for the detailed exposition of this theory, we refer to the monographs [9, 33, 31, 35] and survey [8] (see also [6, 7, 4]). In short, the equivariant degree is a topological tool allowing "counting" orbits of solutions to symmetric equations in the same way as the usual Brouwer degree does, but according to their symmetry properties.

To be more explicit, the equivariant degree $\mathfrak{G}$-Deg(f,Ω) is an element of the free $\mathbb{Z}$-module $A(\mathfrak{G})$ generated by the conjugacy classes $(\mathscr{H})$ of subgroups $\mathscr{H}$ of $\mathfrak{G}$ with a finite Weyl group $W(\mathscr{H})$:

$$\mathfrak{G}\text{-Deg}(f,\Omega)=\sum_{(\mathscr{H})} n_{\mathscr{H}}\,(\mathscr{H}),\quad n_{\mathscr{H}}\in\mathbb{Z}, \tag{1.8}$$

where the coefficients $n_{\mathscr{H}}$ are given by the following Recurrence Formula

$$n_{\mathscr{H}}=\frac{d_{\mathscr{H}}-\sum_{(\mathscr{L})>(\mathscr{H})} n_{\mathscr{L}}\, n(\mathscr{H},\mathscr{L})\,|W(\mathscr{L})|}{|W(\mathscr{H})|}, \tag{1.9}$$

and $n(\mathscr{H},\mathscr{L})$ denotes the number of subgroups $\mathscr{L}'$ in $(\mathscr{L})$ such that $\mathscr{H}\leq\mathscr{L}'$ (see [9]). One can immediately recognize a connection between the two collections: $\{d_{\mathscr{H}}\}$ and $\{n_{\mathscr{H}}\}$, where $\mathscr{H}\leq\mathfrak{G}$ and $W(\mathscr{H})$ is finite. As a matter of fact, $\mathfrak{G}$-Deg(f,Ω) satisfies the standard properties expected from any topological degree. However, there is one additional functorial property, which plays a crucial role in computations, namely the *product property*. In fact, $A(\mathfrak{G})$ has a natural structure of a ring (which is called the *Burnside ring* of $\mathfrak{G}$), where the multiplication $\cdot : A(\mathfrak{G})\times A(\mathfrak{G})\to A(\mathfrak{G})$ is defined on generators by

$$(\mathscr{H})\cdot(\mathscr{K})=\sum_{(\mathscr{L})} m_{\mathscr{L}}\,(\mathscr{L})\qquad (W(\mathscr{L})\text{ is finite}), \tag{1.10}$$

where the integer $m_{\mathscr{L}}$ represents the number of $(\mathscr{L})$-orbits contained in the space $\mathfrak{G}/\mathscr{H}\times\mathfrak{G}/\mathscr{K}$ equipped with the natural diagonal $\mathfrak{G}$-action. The product property for two admissible

$\mathfrak{G}$-pairs (f_1, Ω_1) and (f_2, Ω_2) means the following equality:

$$\mathfrak{G}\text{-Deg}(f_1 \times f_2, \Omega_1 \times \Omega_2) = \mathfrak{G}\text{-Deg}(f_1, \Omega_1) \cdot \mathfrak{G}\text{-Deg}(f_2, \Omega_2). \tag{1.11}$$

Given a $\mathfrak{G}$-equivariant linear isomorphism $A : V \to V$, formula (1.11) combined with the equivariant spectral decomposition of A, reduces the computations of $\mathfrak{G}$-Deg$(A, B(V))$ to the computation of the so-called basic degrees $\deg_{\mathcal{V}_k}$, which can be 'prefabricated' in advance for any group $\mathfrak{G}$ (here $\deg_{\mathcal{V}_k} := \mathfrak{G}\text{-Deg}(-\mathrm{Id}\,, B(\mathcal{V}_k))$ with $\mathcal{V}_k$ being an irreducible $\mathfrak{G}$-representation and $B(X)$ stands for the unit ball in X). In many cases, the equivariant degree based method can be easily assisted by computer (its usage seems to be unavoidable for large symmetry groups).

In the present thesis, to establish the abstract results on the existence and symmetric properties of periodic solutions, we use the $\mathfrak{G}$-equivariant Brouwer degree with $\mathfrak{G} := O(2) \times \Gamma \times \mathbb{Z}_2$, where $O(2)$ is related to the reversal symmetry combined with the autonomous form of the system, Γ reflects symmetries of D and/or possible coupling in the corresponding network of identical oscillaltors, and $\mathbb{Z}_2$ is related to the oddness of f. We also present a concrete illustrating example with $\Gamma := D_8$, where D_8 stands for the dihedral group of order 16. Our computations are essentially based on new group-theoretical computational algorithms, which were implemented in the specially created Hao-Pin Wu (see [43]) package *EquiDeg* for the GAP system.

Overview. After the Introduction, the dissertation is organized as follows. In the second chapter we present the concepts, constructions and basic facts from compact Lie group theory, representation theory, degree theory and differential geometry widely used through this dissertation. Special focus is given to the Brouwer equivariant degree theory and its infinite dimensional generalization (Leray-Schauder equivariant degree).

In the third chapter we formulate and prove our main results. In Subsection 3.0.1, we establish a priori bounds for solutions to problem (1.2) in the space $C^2(S^1; \mathbf{V})$ (actually, we assume that values of solutions “live" in a given open bounded symmetric domain $D \subset \mathbf{V}$; cf. (3.1) for the precise formulation). In particular, we generalize the corresponding results of Hartman [27] and Nagumo [39], who considred the nonsymmetric case for ODEs. In Section 3.1, we reformulate problem (3.1) as an $O(2) \times \Gamma \times \mathbb{Z}_2$-equivariant fixed point problem in $C^2(S^1; \mathbf{V})$ and present an *abstract* equivariant degree based result. This result can be effectively applied to concrete symmetric systems only if a “workable" formula for the degrees associated can be elaborated. The latter is a subject of Sections 3.2 and 3.3. In Section 3.2, we combine the product property of the equivariant degree with equivariant spectral data of the linearization of the operator equation at the origin in order to reduce the degree computations to products of appropriate basic degrees. In Section 3.3, we compute the degree of the operator involved on the boundary of the domain provided by the a priori bound. Actually, this is the place where the curvature of ∂D and $\mathfrak{G}$-equivariant degree come together: here we essentially use admissible homotopies considered in [1]. In Section 3.4, based on the results of Sections 3.0.1–3.3, we present our main results (see Theorems 3.4.1 and 3.4.2) expressed in terms of the function η (cf. (η_1)–(η_6)) and right-hand side of (3.1) only. As an example, we consider $\mathbf{V} = \mathbb{R}^2$ equipped with the natural $\Gamma := D_8$-representation and explicitly describe a D_8-invariant function $\eta : \mathbf{V} \to \mathbb{R}$ giving rise to the D_8-invariant domain D with ∂D admitting points with both positive and negative curvature. Using $\nabla \eta$, we explicitly describe f in (3.1) satisfying (R), (A_1)–(A_5), (A_6').

CHAPTER 2

PRELIMINARIES

2.1 Compact Lie groups and their actions

2.1.1 Basic definitions

In this subsection, we will provide basic definitions related to Lie groups together with several examples. Let $\mathfrak{G}$ be simultaneously a group and (smooth) manifold. Then, $\mathfrak{G}$ is called a Lie group if the group multiplication $p : \mathfrak{G} \times \mathfrak{G} \to \mathfrak{G}$ given by $p(u, v) = uv$ and inversion $i : \mathfrak{G} \to \mathfrak{G}$ given by $i(u) = u^{-1}$ are both smooth maps. In what follows, $\mathfrak{G}$ stands for a compact Lie group. Next, we provide examples of Lie groups.

Example 2.1.1.

1. Any finite group is a zero-dimensional compact Lie group. The following group will be important for our consideration. The dihedral group D_n of $2n$ elements is described by the following relations

$$D_n := \langle \gamma, \kappa | \gamma^n = \kappa^2 = e, \gamma\kappa\gamma = \kappa^{-1} \rangle,$$

between the two generators, the rotation

$$\gamma = \begin{bmatrix} cos(2\pi/n) & -sin(2\pi/n) \\ sin(2\pi/n) & cos(2\pi/n) \end{bmatrix} \tag{2.1}$$

and reflection

$$\kappa = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \tag{2.2}$$

and its elements are

$$\{e, \gamma, \gamma^2, \dots, \gamma^{n-1}, \kappa, \kappa\gamma, \kappa\gamma^2, \dots, \kappa\gamma^{n-1}\},$$

where e denotes the identity element.

2. The general linear groups $GL(n,\mathbb{R})$ and $GL(n,\mathbb{C})$ of degree n are formed by $n \times n$ invertible matrices with real (respectively complex) entries, together with the matrix multiplication as an operation. They are noncompact Lie groups.

3. Any closed subgroup of $GL(n,\mathbb{R})$ and $GL(n,\mathbb{C})$ is a Lie group.

4. As examples of compact Lie groups we mention $O(n), SO(n), U(n)$ and $SU(n)$. To be more specific, the $n \times n$ orthogonal matrices over the field $\mathbb{R}$ form a subgroup, denoted $O(n)$, of the general linear group $GL(n,\mathbb{R})$, defined as

$$O(n) := \{Q \in GL(n,\mathbb{R}) | Q^T Q = \mathrm{Id}\},$$

where Q^T denotes the transpose matrix.

For $n = 2$, this group is generated by the rotations

$$\gamma = \begin{bmatrix} cos(\theta) & -sin(\theta) \\ sin(\theta) & cos(\theta) \end{bmatrix}, \tag{2.3}$$

where $\theta \in [0, 2\pi)$, together with the reflection (2.2). Therefore,

$$O(2) := \{\gamma\} \cup \{\kappa\gamma\},$$

where γ and κ have been defined in (2.3) and (2.2), respectively. The subgroup of $O(n)$ which consists of the $n \times n$ orthogonal matrices with determinant 1 is called the special orthogonal group $SO(n)$, defined as

$$SO(n) := \{Q \in GL(n,\mathbb{R}) | Q^T Q = \mathrm{Id} \text{ and } \det(Q) = 1\}.$$

For $n = 2, SO(2) \cong S^1$, where

$$S^1 := \{\gamma = e^{i\theta}, \theta \in [0, 2\pi)\}.$$

The unitary group of degree n, denoted $U(n)$, is the subgroup of $GL(n, \mathbb{C})$ formed by $n \times n$ unitary matrices:

$$U(n) := \{Q \in GL(n, \mathbb{C}) | Q^H = Q^{-1}\},$$

where the letter H means conjugate transpose.

For $n = 2$, the general expression of a 2×2 unitary matrix is

$$Q = \begin{bmatrix} a & b \\ -e^{i\varphi} b^* & e^{i\varphi} a^* \end{bmatrix}, \quad |a|^2 + |b|^2 = 1,$$

where a^* and b^* are the conjugates of the complex numbers a and b, respectively.

The determinant of such a matrix is

$$\det(Q) = e^{i\varphi}, \ |\det(Q)| = 1.$$

The closed subgroup of $U(n)$ formed by the unitary matrices with $\det(Q) = 1$ is called the special unitary group, defined as

$$SU(n, \mathbb{C}) := \{Q \in GL(n, \mathbb{C}) | Q^H = Q^{-1} \text{ and } \det(Q) = 1\}.$$

In what follows, $\mathfrak{G}$ stands for a compact Lie group and all subgroups of $\mathfrak{G}$ are supposed to be closed.

Two subgroups $\mathscr{H}$ and $\mathscr{K}$ of $\mathfrak{G}$ are said to be conjugate in $\mathfrak{G}$ if $\mathscr{H} = g\mathscr{K}g^{-1}$ for some $g \in \mathfrak{G}$. Conjugacy is an equivalence and it partitions $\mathfrak{G}$ into equivalence classes.

The partial order on the conjugacy classes on $\mathfrak{G}$ is given by

$$(\mathscr{H}) \leq (\mathscr{K}) \Longleftrightarrow \exists_{g \in \mathfrak{G}} \ g\mathscr{H}g^{-1} \leq \mathscr{K}.$$

Let $\mathscr{H}$ be a subgroup of $\mathfrak{G}$. Then,

$$N(\mathscr{H}) := \{g \in \mathfrak{G} : g\mathscr{H}g^{-1} = \mathscr{H}\}$$

denotes the normalizer of $\mathscr{H}$ in $\mathfrak{G}$ and

$$W(\mathscr{H}) := N(\mathscr{H})/\mathscr{H}$$

denotes the Weyl group of $\mathscr{H}$ in $\mathfrak{G}$. The normalizer of a subgroup of $\mathfrak{G}$ and the Weyl group of a subgroup of $\mathfrak{G}$ are also compact Lie groups.

The set $\Phi(\mathfrak{G})$ of all conjugacy classes in $\mathfrak{G}$ can be stratified by the subsets

$$\Phi_k(\mathfrak{G}) := \{(\mathscr{H}) \in \Phi(\mathfrak{G}) : \dim W(\mathscr{H}) = k\}.$$

Example 2.1.2. We will calculate Φ_0 and Φ_1 for $\mathfrak{G} \cong O(2)$. The conjugacy classes of the subgroups of $O(2)$ are $(O(2)), (SO(2), (D_k)$ and $(\mathbb{Z}_k)$. Their normalizers, Weyl groups and their corresponding dimensions are given in Table 2.1. Therefore,

Table 2.1. Weyl group of subgroups of $O(2)$.

Conjugacy class	normalizer	Weyl group	dimension of Weyl group
$(O(2))$	$O(2)$	$\mathbb{Z}_1$	0
$(SO(2))$	$O(2)$	$\mathbb{Z}_2$	0
(D_k)	D_{2k}	$\mathbb{Z}_2$	0
$(\mathbb{Z}_k)$	$O(2)$	$O(2)$	1

$$\Phi_0(\mathfrak{G}) = \{(O(2)),\ (S^1),\ (D_k); k = 1, 2, 3, \ldots\},$$

and

$$\Phi_1(\mathfrak{G}) = \{(\mathbb{Z}_k); k = 1, 2, 3, \ldots\}.$$

2.1.2 Amalgamated notation

The direct product of two Lie groups is a Lie group (R.L. Bishop, and R.J. Crittenden, 1964). Given a direct product group $\mathcal{G} \cong \mathcal{G}_1 \times \mathcal{G}_2$, the natural question is to describe its subgroups. Intuitively, any subgroup of $\mathcal{G}$ would be of the form $\widetilde{\mathcal{G}_1} \times \widetilde{\mathcal{G}_2}$, where $\widetilde{\mathcal{G}_1}$ and $\widetilde{\mathcal{G}_2}$ are subgroups of $\mathcal{G}_1$ and $\mathcal{G}_2$, respectively. It turns out that this is not true in general.

The description of subgroups of $\mathcal{G} \cong \mathcal{G}_1 \times \mathcal{G}_2$ is given by É. Goursat's Lemma [26] as follows.

Given two groups $\mathcal{G}_1$ and $\mathcal{G}_2$, there exist subgroups $\mathscr{H} \leq \mathcal{G}_1$ and $\mathscr{K} \leq \mathcal{G}_2$, a group $\mathscr{L}$, and two epimorphisms $\varphi : \mathscr{H} \to \mathscr{L}$ and $\psi : \mathscr{K} \to \mathscr{L}$ such that

$$\mathscr{M} = \{(h, k) \in \mathscr{H} \times \mathscr{K} : \varphi(h) = \psi(k)\}.$$

Then $\mathscr{M}$ is called an *amalgamated* subgroup of $\mathcal{G}_1 \times \mathcal{G}_2$, and the used notation for $\mathscr{M}$ is

$$\mathscr{M} := \mathscr{H}^{\varphi} \times_{\mathscr{L}}^{\psi} \mathscr{K}. \tag{2.4}$$

The simplest example of an amalgamated subgroup is the direct product of two subgroups; in this case, $\mathscr{L}$ is $\mathbb{Z}_1$, while φ and ψ are the trivial epimorphisms.

In order to make notation (2.4) simpler, it is convenient to indicate $\mathscr{L}$, $Z \cong \ker(\varphi)$ and $R \cong \ker(\psi)$. Hence, instead of (2.4), we use the following notation:

$$\mathscr{M} := \mathscr{H}^{Z} \times_{\mathscr{L}}^{R} \mathscr{K}. \tag{2.5}$$

Remark 2.1.3. In general, the knowledge of a kernel of an epimorphism is not enough to describe the epimorphism itself. More explicitely, given two groups $\mathcal{G}_1$ and $\mathcal{G}_2$, such a situation may arise when two different epimorphisms

$$\alpha, \beta : \mathcal{G}_1 \to \mathcal{G}_2,$$

have the same kernel. Take for example,

$$\mathcal{G}_1 \cong \mathcal{G}_2 \cong \mathbb{Z}_3 := \left\{1, \gamma, \gamma^2, \gamma = e^{\frac{2\pi i}{3}}\right\},$$

and the epimorphisms (in fact automorphisms), are defined as follows:

$$\begin{aligned} \alpha(1) &= 1, & \beta(1) &= 1, \\ \alpha(\gamma) &= \gamma, & \beta(\gamma) &= \gamma^2, \\ \alpha(\gamma) &= \gamma^2, & \beta(\gamma) &= \gamma. \end{aligned}$$

Clearly, both epimorphisms have the same trivial kernel, which makes notation (2.5) useful but not complete in this particular example. However, this notation is enough for the purpose of this Thesis (symmetric classification of periodic solutions to dynamical systems).

The group used with special interest in this Thesis is $O(2) \times D_8 \times \mathbb{Z}_2$, and the examples of amalgamates subgroups discussed in the following are closely related to it.

Example 2.1.4.

1. For the simplest case of an amalgamated subgroup, one can mention the direct-product subgroup $\mathbb{Z}_2 \times \mathbb{Z}_2$ of the direct-product group $O(2) \times D_8$. Here $\mathfrak{G}_1 \cong O(2)$, $\mathfrak{G}_2 \cong D_8$, $\mathscr{H} \cong \mathscr{K} \cong \mathbb{Z}_2$, $\mathscr{H} = \{\mathbf{-1}, \mathbf{-1}\}$ and $\mathscr{K} = \{-1, 1\}$, and $\mathscr{L} \cong \mathbb{Z}_1$.

   We notice that $\mathscr{H}$ and $\mathscr{K}$ are subgroups of both $O(2)$ and D_8. Moreover, the two considered epimorphisms

   $$\varphi : \mathscr{H} \to \mathscr{L}, \quad \psi : \mathscr{K} \to \mathscr{L},$$

   are given by

   $$\varphi(\mathbf{1}) = e, \qquad \psi(1) = e,$$

   $$\varphi(\mathbf{-1}) = e, \qquad \psi(-1) = e,$$

   where e is the identity in $\mathscr{L}$. Hence, $Z \cong \ker(\varphi) = \{\mathbf{-1}, \mathbf{1}\} \cong \mathbb{Z}_2, R \cong \ker(\psi) = \{-1, 1\} \cong \mathbb{Z}_2$, and the elements of $\mathbb{Z}_2^{\varphi} \times_{\mathbb{Z}_1}^{\psi} \mathbb{Z}_2$ are $\{(\mathbf{1}, 1), (\mathbf{-1}, -1), (\mathbf{-1}, 1), (\mathbf{1}, -1)\}$. Since both epimorphisms φ and ψ are trivial, the amalgamated notation $\mathbb{Z}_2^{\varphi} \times_{\mathbb{Z}_1}^{\psi} \mathbb{Z}_2$ is simplified to $\mathbb{Z}_2 \times \mathbb{Z}_2$.

2. Consider the group $\mathfrak{G}_1 \times S^1$, or in our usual notation $\mathfrak{G}_1 \times \mathfrak{G}_2$, where $\mathfrak{G}_2 \cong S^1$. A particular case of the amalgamated subgroup is a twisted subgroup, which is defined as follows [9], [24].

Let $\mathscr{H} \leq \mathfrak{G}_1$. Then, given an integer $l \geq 0$ and a homomorphism $\Theta : \mathscr{H} \to S^1$, a twisted subgroup $\mathscr{H}^{\Theta,l}$ of $\mathfrak{G}_1 \times S^1$ is

$$\mathscr{H}^{\Theta,l} := \Big\{(\gamma, z) \in \mathfrak{G} \times S^1 : \Theta(\gamma) = z^l\Big\}. \tag{2.6}$$

Now we give an interpretation of (2.6) for $\mathscr{H}^{\Theta,l} \leq \mathfrak{G}_1 \times S^1$ in terms of the Goursat's Lemma. In this Thesis we will be dealing with the situation where $\mathfrak{G}_1$ is finite.

Claim 2.1.5. *In "amalgamated symbols", one has*

$$\mathscr{H}^{\Theta,l} \cong \mathscr{H}^{Z} \times^{R}_{\mathbb{Z}_m} \mathbb{Z}_r,$$

where $Z \cong \ker\varphi, R \cong \mathbb{Z}_l$ *and* $r = ml$.

Proof: Let $\mathscr{H} \leq \mathfrak{G}_1, \mathscr{K} \leq S^1, \mathscr{L} \leq S^1$. Since $\mathscr{H}$ is finite and S^1 is not, it follows that there cannot be an epimorphism φ from $\mathscr{H}$ onto S^1. Moreover, since $\mathscr{H}$ is finite, it follows that $\varphi(\mathscr{H})$ is finite. Therefore, we must assume that $\mathscr{L}$ is finite. Since all finite subgroups of S^1 are cyclic, we may assume that $\mathscr{L}$ is isomorphic to a cyclic subgroup of S^1, say $\mathbb{Z}_m,\ m \in \mathbb{N}$. Next we have to find a group $\mathscr{K}$ satisfying two conditions:

- $\mathscr{K}$ has to be a subgroup of S^1;
- there is an epimorphism $\psi : \mathscr{K} \to \mathscr{L}$.

There cannot be an epimorphism from S^1 onto $\mathbb{Z}_m$, because the image of the connected manifold S^1 through the continuous map ψ cannot be the disconnected set $\mathbb{Z}_m$. Therefore, $\mathscr{K}$ has to be also isomorphic to a cyclic subgroup of S^1, say $\mathscr{K} \cong \mathbb{Z}_r,\ r \in \mathbb{N}$.

Then we identify the homomorphism

$$\Theta : \mathscr{H} \to S^1, \Theta(\gamma) = z^l,$$

with the epimorphism

$$\varphi : \mathscr{H} \to \mathscr{L} \cong \mathbb{Z}_m, \quad \varphi(\mathscr{H}) \cong \mathbb{Z}_m,$$

with $r = ml$.

The epimorphism

$$\psi : \mathscr{K} \to \mathscr{L}, \quad \psi(\mathbb{Z}_r) = \mathbb{Z}_m,$$

has kernel $\ker\psi \cong \mathbb{Z}_l$. Next we apply twice (to the epimorphisms ϕ and ψ, respectively), the First Isomorphism Theorem to obtain:

$$\mathscr{H}/\ker\varphi \cong \mathbb{Z}_m \cong \mathscr{K}/\ker\psi.$$

Therefore,

$$|\mathscr{K}/\ker\psi| = |\mathbb{Z}_m| \Longrightarrow r = ml.$$

The explicit formula for ψ is

$$\psi(z_r) = (z_m)^l, \quad z_r \in \mathscr{K}, \quad z_m \in \mathscr{L}.$$

□ With this description in mind, we will provide an example of a twisted subgroup of $D_8 \times S^1$. Therefore, in the next example, $\mathfrak{G}_1 \cong D_8$, $\mathfrak{G}_2 \cong S^1$, while $\mathscr{H} \cong V_4 \le \mathfrak{G}_1$ and $\mathscr{K} \cong \mathscr{L} \cong \mathbb{Z}_2 := \{-1, 1\} \le \mathfrak{G}_2$, are specified as follows:

$$D_4^{\Theta,l} := \Big\{(\gamma, z) \in D_4 \times S^1 : \Theta(\gamma) = z^l\Big\},$$

where $\mathscr{H} \cong D_4$, with

$$D_4 := \{1, i, -1, -i, \kappa, \kappa i, -\kappa, -\kappa i\}, \quad \mathbb{Z}_4 := \{1, i, -1, -i\}, \tag{2.7}$$

and $\mathscr{K} \cong \mathscr{L} \cong \mathbb{Z}_2 := \{-1, 1\}$. Here $\mathfrak{G}_1 \cong D_8$, $\mathfrak{G}_2 \cong S^1$, while

$$\varphi : D_4 \to \mathbb{Z}_2, \quad \psi : \mathbb{Z}_4 \to \mathbb{Z}_2,$$

and $l = 2$.

The epimorphisms φ and ψ are given by

$$\begin{aligned}
&\varphi(1) = 1, && \psi(1) = 1,\\
&\varphi(i) = 1, && \psi(-1) = 1,\\
&\varphi(-1) = 1, && \psi(i) = -1,\\
&\varphi(-i) = 1, && \psi(-i) = -1,\\
&\varphi(\kappa) = -1, &&\\
&\varphi(\kappa i) = -1, &&\\
&\varphi(-\kappa) = -1, &&\\
&\varphi(-\kappa i) = -1. &&
\end{aligned}$$

Therefore, $Z \cong \mathbb{Z}_4, R \cong \mathbb{Z}_2$ and the corresponding amalgamated subgroup is

$$D_4^{\mathbb{Z}_4} \times_{\mathbb{Z}_2}^{\mathbb{Z}_2} \mathbb{Z}_4 = \left\{ \begin{array}{l} (1,1),(1,-1),(i,1),(i,-1),(-1,1),(-1,-1),\\ (-i,1),(-i,-1),(\kappa,-1),(\kappa,-i),(\kappa i,-1),\\ (\kappa i,-i),(-\kappa,-1),(-\kappa,-i),(-\kappa i,-1),(-\kappa i,-i) \end{array} \right\}.$$

3. Finally, we provide an example of an amalgamated subgroup of a direct-product group, which is neither direct product nor a twisted subgroup. In particular we seek for such a subgroup in $D_8 \times O(2)$. This is

$$D_4^{\mathbb{Z}_2} \times_{D_2^d}^{\mathbb{Z}_1} D_2^d,$$

where $\mathfrak{G}_1 \cong D_8, \mathfrak{G}_1 \cong O(2), D_2^d := \{(1,1),(-1,-1),(\kappa,1),(-\kappa,-1)\}$ where κ is defined as in (2.2), and $\mathscr{H} \cong D_4$ is defined in (2.7). The elements which belong to $\mathscr{K} \cong D_2^d$ are indicated in bold, to distinguish them from the ones which belong to $\mathscr{L} \cong D_2^d$.

The epimorphisms

$$\varphi : D_4 \to D_2^d, \quad \text{and} \quad \psi : D_2^d \to D_2^d,$$

are given by

$$\begin{aligned}
&\varphi(1) = (1,1), && \psi((\mathbf{1,1})) = (1,1),\\
&\varphi(i) = (1,1), && \psi((\mathbf{-1,-1})) = (-1-,1),\\
&\varphi(-1) = (-1,-1), && \psi((\boldsymbol{\kappa}\mathbf{,1})) = (\kappa,1),\\
&\varphi(-i) = (-1,-1), && \psi((\mathbf{-}\boldsymbol{\kappa}\mathbf{,-1})) = (-\kappa,-1),\\
&\varphi(\kappa) = (\kappa,1), &&\\
&\varphi(\kappa i) = (\kappa,1), &&\\
&\varphi(-\kappa) = (-\kappa,-1), &&\\
&\varphi(-\kappa i) = (-\kappa,-1). &&
\end{aligned}$$

Therefore, $Z = \{1, i\} \cong \mathbb{Z}_2$, $R \cong \mathbb{Z}_1$ and the corresponding amalgamated subgroup is

$$D_4^{\mathbb{Z}_2} \times_{D_2^d}^{\mathbb{Z}_1} D_2^d = \left\{ \begin{array}{c} (1,(\mathbf{1,1})),(i,(\mathbf{1,1})),(-1,(\mathbf{-1,-1})),(-i,(\mathbf{-1,-1})), \\ (\kappa,(\boldsymbol{\kappa}\mathbf{,1})),(\kappa i,(\boldsymbol{\kappa}\mathbf{,1})),(-\kappa,(\mathbf{-}\boldsymbol{\kappa}\mathbf{,-1})),(-\kappa i,(\mathbf{-}\boldsymbol{\kappa}\mathbf{,-1})) \end{array} \right\},$$

where the component of each pair coming from $\mathscr{K}$ is again specified in bold letters.

2.1.3 Haar measure and Haar Integral

In this subsection, we give a definition of the Haar measure and state the theorem about its existence and uniqueness. In close relationship with the Haar measure is the Haar integral, whose properties are then exposed, with emphasis on the translation invariance. Finally, we provide examples of Haar integrals on the groups related to this Thesis. The Haar integral will be used in what follows for averaging purposes.

Since $\mathfrak{G}$ is a compact Lie group, which is also a manifold, take the Borel algebra, i.e. the σ-algebra generated by all open subsets of $\mathfrak{G}$. Let $\mu : \mathcal{B} \to [0, \infty]$ be the corresponding Borel measure.

In order to formulate the existence and uniqueness result, we will need the following two additional properties of the Borel measure. We define the left and right translation maps as follows:

$$\begin{aligned} L_\gamma &: \mathfrak{G} \to \mathfrak{G}, \quad L_\gamma(g) = \gamma g,\ \forall g, \gamma \in \mathfrak{G}, \\ R_\gamma &: \mathfrak{G} \to \mathfrak{G}, \quad R_\gamma(g) = g\gamma,\ \forall g, \gamma \in \mathfrak{G}. \end{aligned} \tag{2.8}$$

From (2.8) it follows that the image of a Borel set is a Borel set as well. A Borel measure is called left-translation invariant if for all Borel subsets $S \subseteq \mathfrak{G}$ and all $\gamma \in \mathfrak{G}$, one has

$$\mu(L_\gamma(S)) = \mu(S).$$

A Borel measure is called right-translation invariant if for all Borel subsets $S \subseteq \mathfrak{G}$ and all $\gamma \in \mathfrak{G}$, one has

$$\mu(R_\gamma(S)) = \mu(S).$$

When $\mu(\mathfrak{G}) = 1$, we say that the measure is normalized.

Definition 2.1.6. A Borel measure $\mathcal{B}$ which is both left-translation invariant and right-translation invariant, is called Haar measure.

Next, we state the existence and uniqueness of the Haar measure together with its properties.

Theorem 2.1.7 (G. Hochschild, 1965)**.** *There exists a unique, normalized Haar measure μ on the Borel subsets of $\mathfrak{G}$, satisfying the following properties:*

- *μ is left-translation invariant and right-translation invariant for every $\gamma \in \mathfrak{G}$ and all Borel sets $S \subseteq \mathfrak{G}$.*
- *μ is finite on every compact set: $\mu(K) < \infty$ for any compact $K \subseteq \mathfrak{G}$.*
- *μ is outer regular on Borel sets $S \subseteq \mathfrak{G}$:*

$$\mu(S) = \inf\{\mu(U) : S \subseteq U,\ U \text{ open}\}.$$

- *μ is inner regular on open sets $U \subseteq \mathfrak{G}$:*

$$\mu(U) = \sup\{\mu(K) : K \subseteq U,\ K \text{ compact}\}.$$

It can be shown as a consequence of the above properties, that $\mu(U) > 0$ for every non-empty open subset $U \subseteq \mathfrak{G}$.

The existence of the Haar measure implies the existence of the Haar integral.

Theorem 2.1.8. *Let μ be the Haar measure on $\mathfrak{G}$ provided by Theorem 2.1.7, and let*

$$\int_{\mathfrak{G}} f(\gamma) d\mu(\gamma),$$

be the corresponding integral (Haar integral). This integral satisfies the following properties:

(i) $\displaystyle\int_{\mathfrak{G}} (f_1(\gamma) + f_2(\gamma))\, d\mu(\gamma) = \int_{\mathfrak{G}} f_1(\gamma) d\mu(\gamma) + \int_{\mathfrak{G}} f_2(\gamma) d\mu(\gamma);$

(ii) $\displaystyle\int_{\mathfrak{G}} cf(\gamma) d\mu(\gamma) = c\int_{\mathfrak{G}} f(\gamma) d\mu(\gamma),$ *where* $c \in \mathbb{R}$;

(iii) $\displaystyle\int_{\mathfrak{G}} d\mu(\gamma) = 1;$

(iv) $\displaystyle\int_{\mathfrak{G}} f\left(h^{-1}\gamma\right) d\mu(\gamma) = \int_{\mathfrak{G}} f(\gamma h) d\mu(\gamma) = \int_{\mathfrak{G}} f\left(\gamma^{-1}\right) d\mu(\gamma) = \int_{\mathfrak{G}} f(\gamma)\, d\mu(\gamma),$ *for all* $h \in \mathfrak{G}$;

(v) $\displaystyle\int_{\mathfrak{G}} f(\gamma)\, d\mu(\gamma) \geq 0,$ *for* $f(\gamma) \geq 0$, $\forall \gamma \in \mathfrak{G}$.

From the property (*iv*) above, it is clear that the Haar integral is invariant under the translation by elements of $\mathfrak{G}$.

In the following we give three examples of Haar integral on different groups (related to this Thesis).

Example 2.1.9.

1. In the case of a finite group, the Haar integral has the form

$$\int_{\mathfrak{G}} f d\mu(\gamma) \equiv \frac{1}{|\mathfrak{G}|} \sum_{\gamma \in \mathfrak{G}} f(\gamma).$$

2. For S^1, the Haar integral takes the following form

$$\int_{\mathfrak{G}} f d\mu(\gamma) \equiv \frac{1}{2\pi} \int_0^{2\pi} \widetilde{f}(\theta) d(\theta),$$

where given $f : S^1 \to \mathbb{R}$, $\widetilde{f}(\theta)$ is a canonically defined continuous 2π-periodic function $\widetilde{f} : \mathbb{R} \to \mathbb{R}$, by the formula $\widetilde{f}(\theta) = f\left(e^{2\pi i\theta}\right)$.

3. The $O(2)$-invariant Haar integral has the form

$$\int_{\mathfrak{G}} f d\mu(\gamma) \equiv \frac{1}{4\pi} \int_0^{2\pi} \left(f\left(e^{2\pi i\theta}\right) + f\left(\overline{e^{2\pi i\theta}}\right)\right) d(\theta).$$

2.1.4 Group actions and equivariant jargon

A topological transformation group is a triple $(\mathfrak{G}, X, \phi)$, where X is a Hausdorff topological space and $\phi : \mathfrak{G} \times X \to X$ is a continuous map such that:

(i) $\phi(g, \phi(h, x)) = \phi(gh, x)$ for all $g, h \in \mathfrak{G}$ and $x \in X$;

(ii) $\phi(e, x) = x$ for all $x \in X$, where e is the identity element of $\mathfrak{G}$.

The map ϕ is called a $\mathfrak{G}$-action on X and the space X, together with a given action ϕ of $\mathfrak{G}$, is called a $\mathfrak{G}$-space.

Let X be a $\mathfrak{G}$-space. For any $x \in X$, the subgroup $\mathfrak{G}_x = \{g \in \mathfrak{G} : gx = x\}$ of $\mathfrak{G}$ is called the isotropy group of x and if $\mathfrak{G}_x = \{e\}$, $\forall x \in X$, then the action of $\mathfrak{G}$ is called free. The subspace $\mathfrak{G}(x) := \{gx : g \in \mathfrak{G}\}$ of X is called the orbit of x. The orbit space (denoted $X/\mathfrak{G}$), is the set of all orbits for the action of $\mathfrak{G}$ on X equipped with the quotient topology.

For $x \in X$, one has $\mathfrak{G}_{gx} = g\mathfrak{G}_x g^{-1}$. This gives rise to the notion of the orbit type of x defined as the conjugacy class $(\mathfrak{G}_x)$. The proof of the following result can be found in [37].

Theorem 2.1.10 (L.N. Mann)**.** *If X is a contractible or compact manifold, then any compact Lie group acts on X with finitely many orbit types.*

Let $\Phi(\mathfrak{G})$ be the set of all conjugacy classes in $\mathfrak{G}$, and define by

$$\Phi(\mathfrak{G};X) := \{(\mathscr{H}) \in \Phi(\mathfrak{G}) : \mathscr{H} = \mathfrak{G}_x \text{ for some } x \in X\},$$

the set of all orbit types occurring in X. Put $\Phi_k(\mathfrak{G}.;X) := \Phi(\mathfrak{G};X) \cap \Phi_k(\mathfrak{G})$. We illustrate the concept of orbit types with an example.

Example 2.1.11. Let us describe the orbit types of the dihedral group $\mathfrak{G} = D_8$ acting on $\mathbb{R}^2 \cong \mathbb{C}$ with the action

$$\begin{aligned} &\gamma^k z = e^{\frac{2\pi i k}{8}} \cdot z, \quad k = 0, \dots, 7, \\ &\kappa z = \overline{z}, \end{aligned} \tag{2.9}$$

where the dot means complex multiplication. Clearly the origin is fixed by the whole group so the orbit type of $\{0\}$ is $(\mathfrak{G})$.

If a rotation $\gamma^k \in \mathfrak{G}_z$, with $z \neq 0$, then $\gamma^k z = z \Leftrightarrow \gamma^k = 1$, i.e. $k = 0$. Moreover, $\kappa\gamma^k z = z$ holds if and only if $\gamma^k z = \overline{z}$. Then, if $z = |z|e^{i\theta}$, we get $\frac{2\pi k}{8} + \theta = -\theta + 2\pi l$ for some $l \in \mathbb{Z}$, which means $\theta = r\frac{\pi}{8}$, $r \in \{0, 1, \dots, 7\}$. For $z_k = |z|e^{i\pi k/8}$, $k = 0, 1, \dots, 7$, we get $\mathfrak{G}_{z_0} = \mathfrak{G}_{z_4} = D_1 = \{1, \kappa\}$, $\mathfrak{G}_{z_1} = \mathfrak{G}_{z_5} = \widetilde{D_1} = \{1, \kappa\gamma^2\}$, $\mathfrak{G}_{z_2} = \mathfrak{G}_{z_6} = \{1, \kappa\gamma^4\}$, $\mathfrak{G}_{z_3} = \mathfrak{G}_{z_7} = \{1, \kappa\gamma^6\}$, with $\mathfrak{G}_{z_0} = \kappa\gamma^4\mathfrak{G}_{z_0}\left(\kappa\gamma^4\right)^{-1}$ and $\mathfrak{G}_{z_1} = \kappa\gamma^6\mathfrak{G}_{z_1}\left(\kappa\gamma^6\right)^{-1}$. All other points are fixed only by the identity. Therefore, there are four orbit types, $(D_8), (D_1), (\widetilde{D_1})$ and $(\mathbb{Z}_1)$.

A subset K of X is called $\mathfrak{G}$-invariant if it contains all orbits of points of K.

Adopt the notations

$$X^{\mathscr{H}} := \{x \in X : \mathscr{H} \subset \mathfrak{G}_x\},$$

$$X_{\mathscr{H}} := \{x \in X : \mathscr{H} = \mathfrak{G}_x\},$$

and

$$X_{(\mathscr{H})} := \{x \in X : (\mathfrak{G}_x) = (\mathscr{H})\}.$$

The group $W(\mathscr{H})$ acts freely on $X_{\mathscr{H}}$, as it is shown in [14].

Let X and Y be two $\mathfrak{G}$-spaces. A continuous map $f : X \to Y$ is said to be equivariant if $f(gx) = gf(x)$ for all $x \in X$ and $g \in \mathfrak{G}$.

For any subgroup $\mathscr{H} \subset \mathfrak{G}$ and equivariant map $f : X \to Y$, the map $f^{\mathscr{H}} : X^{\mathscr{H}} \to Y^{\mathscr{H}}$, with $f^{\mathscr{H}} := f|_{X^{\mathscr{H}}}$, is well defined.

An example for a D_8-equivariant map for the action of D_8 on $\mathbb{R}^2 \cong \mathbb{C}$, is

$$f(z) = \alpha z + \beta \overline{z}^7,$$

where $z \in \mathbb{C}$ and $\alpha, \beta \in \mathbb{R}$ and the action of D_8 is detailed in (2.9). One only needs to check that f commutes with κ and γ, the generators of D_8. Indeed, $f(\kappa z) = f(\overline{z}) = \alpha\overline{z} + \beta z^7 = \kappa f(z)$ and

$$f(\gamma^k z) = \alpha\gamma^k z + \beta\gamma^k \overline{z}^7 = \gamma^k f(z).$$

Thus, f is D_8-equivariant.

2.1.5 Numbers $n(\mathscr{L}, \mathscr{H})$

The numbers $n(\mathscr{L}, \mathscr{H})$ we are going to discuss below will play an important role in the practical computations of the equivariant degree.

Consider two subgroups $\mathscr{L} \subset \mathscr{H}$ of $\mathfrak{G}$. Put

$$N(\mathscr{L}, \mathscr{H}) := \{g \in \mathfrak{G} : g\mathscr{L}g^{-1} \subset \mathscr{H}\}. \tag{2.10}$$

One can easily see that the set $N(\mathscr{L}, \mathscr{H})$ is a compact subset of $\mathfrak{G}$, but it is not a subgroup of $\mathfrak{G}$ in general.

Let $h \in N(\mathscr{H}),\ x \in N(\mathscr{L}, \mathscr{H})$, and consider the map

$$(h, x) \to h \cdot x. \tag{2.11}$$

One can check that

$$hx\mathscr{L}(hx)^{-1} = hx\mathscr{L}x^{-1}h^{-1} = h\mathscr{M}h^{-1} \subset N(\mathscr{L}, \mathscr{H}), \tag{2.12}$$

where $\mathscr{M} = x\mathscr{L}x^{-1} \subset \mathscr{H}$ (the dot has been omitted for simplicity). Therefore, the set $N(\mathscr{L}, \mathscr{H})$ is $N(\mathscr{H})$-invariant and the map (2.11) represents the action of $N(\mathscr{H})$ on $N(\mathscr{L}, \mathscr{H})$.

The symbol $n(\mathscr{L}, \mathscr{H})$ is defined by the formula

$$n(\mathscr{L}, \mathscr{H}) := \left| \frac{N(\mathscr{L}, \mathscr{H})}{N(\mathscr{H})} \right|, \tag{2.13}$$

where $|X|$ stands for the cardinality of the set X.

Proposition 2.1.12 (Z. Balanov, W. Krawcewicz and H. Steinlein, 2006)**.** *Let $\mathscr{L} \subset \mathscr{H}$ be two subgroups of a compact Lie group $\mathfrak{G}$ such that* $\dim W(\mathscr{L}) = \dim W(\mathscr{H})$*. Then, $n(\mathscr{L}, \mathscr{H})$ is finite.*

The main ingredient of the proof of Proposition 2.1.12 is the following classical fact.

Proposition 2.1.13 (G.E. Bredon, 1972)**.** *Let $\mathscr{L} \subset \mathscr{H}$ be two subgroups of the compact Lie group $\mathfrak{G}$. Then, the orbit space*

$$\frac{(\mathfrak{G}/\mathscr{H})^{\mathscr{L}}}{W(\mathscr{L})} \quad ((\mathfrak{G}/\mathscr{H})^{\mathscr{L}} \text{ is considered as the left } W(\mathscr{L}) - \text{space})$$

is finite.

Proposition 2.1.12 will be systematically used in what follows.

The numbers $n(\mathscr{L}, \mathscr{H})$ admit simple algebraic and geometric interpretation. The algebraic meaning is that numbers $n(\mathscr{L}, \mathscr{H})$ represent the number of different copies of $\widetilde{\mathscr{H}}$ in the conjugacy class of $(\mathscr{H})$ containing $\mathscr{L}$.

To describe the geometric meaning, let X be a metric space on which $\mathfrak{G}$ acts with finitely many orbit types, and let $(\mathscr{H}), (\mathscr{L}) \in \Phi(\mathfrak{G}; X)$. Then, $X^{\mathscr{L}} \cap X_{(\mathscr{H})}$ is a disjoint union of exactly $n(\mathscr{L}, \mathscr{H})$ sets $X_{\mathscr{H}_j}$, $j = 1, 2, \ldots, n(\mathscr{L}, \mathscr{H})$.

2.2 Finite-dimensional representations

2.2.1 Basic definitions

Consider a finite-dimensional real (resp. complex) vector space W. We say that W is a real (resp. complex) *representation* of $\mathfrak{G}$ (in short, $\mathfrak{G}$-*representation*), if W is a $\mathfrak{G}$-space such that the *translation map* $T_g : W \to W$, defined by $T_g(v) := gv$ for $v \in W$, is a $\mathbb{R}$-linear (resp. $\mathbb{C}$-linear) operator for every $g \in \mathfrak{G}$. In what follows, in case the underlying field is not essential, we will omit the adjective "real" or "complex".

It is obvious that for a $\mathfrak{G}$-representation W, the map $T : \mathfrak{G} \to GL(W)$ given by $T(g) := T_g$, is a continuous homomorphism. Once a basis in W is chosen, one can associate a matrix with each T_g. In this case, a continuous homomorphism $T : \mathfrak{G} \to GL(n;\mathbb{R})$ (resp. $T : \mathfrak{G} \to GL(n;\mathbb{C})$) is called a *real* (resp. *complex*) *matrix* $\mathfrak{G}$-*representation.*

Considering two real (resp. complex) $\mathfrak{G}$-representations W_1 and W_2, we say that W_1 and W_2 are equivalent and write $W_1 \cong W_2$, if there is a $\mathfrak{G}$-equivariant real (resp. complex) isomorphism $W_1 \to W_2$. From the point of view of the representation theory, equivalent representations are indistinguishable.

Definition 2.2.1. Let W be a real (resp. complex) finite-dimensional $\mathfrak{G}$-representation. An inner product (resp. Hermitian inner product) $\langle \cdot, \cdot \rangle_{\mathfrak{G}} : W \oplus W \to \mathbb{R}$ (resp. $\langle \cdot, \cdot \rangle_{\mathfrak{G}} : W \oplus W \to \mathbb{C}$) is called $\mathfrak{G}$-*invariant* if

$$\langle gu, gv \rangle_{\mathfrak{G}} = \langle u, v \rangle_{\mathfrak{G}}, \tag{2.14}$$

for all $g \in \mathfrak{G},\ u, v \in W$.

Proposition 2.2.2. *Given a $\mathfrak{G}$-representation W, there exists a $\mathfrak{G}$-invariant inner product on W, such that T_g is orthogonal for all $g \in \mathfrak{G}$.*

Proof: To prove that such a $\mathfrak{G}$-invariant inner product on W exists, we will make use of the Haar integral. Take any inner product $\langle \cdot, \cdot \rangle$ on W and define

$$\langle u, v \rangle_{\mathfrak{G}} = \int_{\mathfrak{G}} \langle gu, gv \rangle d\mu(g), \tag{2.15}$$

which is also an inner product by linearity property of the Haar integral, i.e. items $(i) - (ii)$ in Theorem 2.1.8. Moreover, the invariance of the Haar integral under the translation by elements of $\mathfrak{G}$, i.e. property (iv) in Theorem 2.1.8, shows that the inner product (2.15) satisfies (2.14). $\square$

A $\mathfrak{G}$-representation together with a $\mathfrak{G}$-invariant inner product is called *orthogonal* (resp. *unitary*) $\mathfrak{G}$-representation.

Remark 2.2.3. Proposition 2.2.2 implies that any real (resp. complex) finite-dimensional $\mathfrak{G}$-representation is equivalent to an orthogonal (resp. unitary) one.

2.2.2 Invariant subspaces and irreducible representations

The structure of a representation essentially depends on the structure of its invariant subspaces. Given a representation W, a subspace $\widetilde{W} \subset W$ is called a *subrepresentation* of W if it is invariant under all operators of this representation. Clearly, a sum and intersection of invariant subspaces is again an invariant subspace. We say that W is *irreducible* if it has no subrepresentation different from $\{0\}$ and W. Otherwise, W is called *reducible.*

Example 2.2.4.

(i) Any one-dimensional representation is irreducible.

(ii) A real two-dimensional $\mathbb{Z}_2$-representation given by $(x, y) \to (-x, -y)$ is reducible: any one-dimensional subspace of $\mathbb{R}^2$ is an irreducible subrepresentation.

(iii) The real two-dimensional S^1 -representation $R_n : S^1 \to GL(2, \mathbb{R})$ given by

$$R_n\left(e^{i\varphi}\right) = \begin{bmatrix} \cos(n\varphi) & -\sin(n\varphi) \\ \sin(n\varphi) & \cos(n\varphi) \end{bmatrix}, \quad n \in \mathbb{N}, \tag{2.16}$$

is an irreducible orthogonal representation.

(iv) The complex two-dimensional S^1-representation $C_n : S^1 \to GL(2, \mathbb{C})$ given by

$$C_n\left(e^{i\varphi}\right) = \begin{bmatrix} e^{in\varphi} & 0 \\ 0 & e^{-in\varphi} \end{bmatrix}, \quad n \in \mathbb{N}, \tag{2.17}$$

is a reducible representation.

(v) The real irreducible representations of D_n are:

(a_0) The trivial one-dimensional representation $\mathcal{M}_0$;

(a_1) For each integer $1 \leq j < n/2$, there is an orthogonal representation $\mathcal{M}_j$ of D_n on $\mathbb{R}^2 \cong \mathbb{C}$, given by the generators of D_n (cf. (2.1), (2.2))

$$\begin{cases} \gamma z := \gamma^j \cdot z, \quad z \in \mathbb{C}; \\ \kappa z := \overline{z}, \end{cases} \tag{2.18}$$

where γ^j corresponds to the complex number $e^{\frac{2\pi j}{n}}$ and "$\cdot$" stands for the complex multiplication.

(a_2) The representation $\mathcal{M}_{j_n}$ determined by the homomorphism $c : D_n \to \mathbb{Z}_2 \cong O(1)$, such that $\ker c = \mathbb{Z}_n$;

(a_3) For n even, there is an irreducible representation $\mathcal{M}_{j_n+1}$ given by $d : D_n \to \mathbb{Z}_2 \cong O(1)$ such that $\ker d = D_{n/2}$;

(a_4) The irreducible representation $\mathcal{M}_{j_n+2}$ given by $\hat{d} : D_n \to \mathbb{Z}_2 \cong O(1)$ such that $\ker \hat{d} = \widetilde{D}_{n/2}$.

(vi) Real irreducible representations of $O(2)$ are described as follows. Denote by $\mathcal{V}_0 \cong \mathbb{R}$ the trivial representation of $O(2)$, by $\mathcal{V}_{\frac{1}{2}} \cong \mathbb{R}$ the one-dimensional irreducible real representation, where $O(2)$ acts on $\mathbb{R}$ through the homomorphism $O(2) \to O(2)/SO(2) \cong \mathbb{Z}_2$, and by $\mathcal{V}_m = \mathbb{R}^2 \cong \mathbb{C}$ the two-dimensional irreducible real representation of $O(2)$, where the action of $O(2)$ is given by (cf. (2.2), (2.3))

$$\begin{cases} \gamma z := \gamma^m \cdot z, \text{ for } z \in \mathcal{V}_m; \\ \kappa z := \overline{z}, \end{cases}, \tag{2.19}$$

where γ^m corresponds to the complex number $e^{im\theta}$ and "$\cdot$" stands for the complex multiplication.

Proposition 2.2.5. *Let W be a finite-dimensional $\mathfrak{G}$-representation. Let $U \subset W$ be a G-invariant subspace. Then there exists a G-invariant complementary subspace $V \subset W$ such that*

$$W = U \oplus V.$$

Proof: Proposition 2.2.2 shows that there exists a $\mathfrak{G}$-invariant inner product $\langle \cdot, \cdot \rangle$ on W. Let $V = U^{\perp}$ where

$$U^{\perp} = \{w \in W : \langle u, w \rangle = 0, \forall u \in U\}.$$

Since the inner product is $\mathfrak{G}$-invariant, it follows that $U^{\perp}$ is a $\mathfrak{G}$-invariant complement to U. □

It follows from this proposition that every representation of $\mathfrak{G}$ may be written as a direct sum of irreducible subspaces.

Proposition 2.2.6 (Complete Reducibility Theorem)**.** *Every (finite-dimensional) $\mathfrak{G}$-representation W is a (not necessarily unique) direct sum of irreducible subrepresentations of W, i.e. there exist irreducible subrepresentations $\mathcal{W}^1, \dots, \mathcal{W}^m$ of W such that*

$$W = \mathcal{W}^1 \oplus \mathcal{W}^2 \oplus \dots \oplus \mathcal{W}^m. \tag{2.20}$$

Proof: Start by assuming that $W \neq 0$. If W is irreducible, then put $W = \mathcal{W}^1$ and we are done.

Assume W is not irreducible. Then find its orthogonal complement using Proposition 2.2.5. Since W is finite-dimensional, the process must come to an end yielding the decomposition (2.20). $\square$

2.2.3 Tensor product of representations

In order to describe the irreducible representations of a direct-product group, it is useful to define the tensor product of two representations. We start with a brief description of the tensor product of two finite-dimensional vector spaces.

Let V, W be $\mathbb{K}$-vector spaces. Let $v_1, \ldots, v_n$ be a basis in V and $w_1, \ldots, w_n$ be a basis in W, then a basis in the space $V \oplus W$ is given by

$$v_1 \otimes w_1, \ldots, v_n \otimes w_1, v_1 \otimes w_2, \ldots, v_n \otimes w_2, \ldots, v_1 \otimes w_n, \ldots, v_n \otimes w_n. \tag{2.21}$$

The vector space $V \otimes W$ is spanned by (2.21). Then the following identities hold for any scalar α.

$$\begin{aligned}
&(\mathbf{v}_1 + \mathbf{v}_2) \otimes \mathbf{w} = \mathbf{v}_1 \otimes \mathbf{w} + \mathbf{v}_2 \otimes \mathbf{w}, \quad \mathbf{v}_1, \mathbf{v}_2 \in \mathbf{V},\ \mathbf{w} \in \mathbf{W} \\
&\mathbf{v} \otimes (\mathbf{w}_1 + \mathbf{w}_2) = \mathbf{v} \otimes \mathbf{w}_1 + \mathbf{v} \otimes \mathbf{w}_2, \quad \mathbf{w}_1, \mathbf{w}_2 \in \mathbf{W},\ \mathbf{v} \in \mathbf{V} \\
&\alpha(\mathbf{v} \otimes \mathbf{w}) = (\alpha \mathbf{v}) \otimes \mathbf{w} = \mathbf{v} \otimes (\alpha \mathbf{w}).
\end{aligned} \tag{2.22}$$

Assume that V and W are representations of the compact Lie groups $\mathfrak{G}$ and $\mathfrak{H}$, respectively. Then, the tensor product $V \otimes W$ is a representation of $\mathfrak{G} \times \mathfrak{H}$. The action of an element $(g, h) \in \mathfrak{G} \times \mathfrak{H}$ on a basis element $v \otimes w \in V \otimes W$ is given by

$$(g, h)(v \otimes w) = gv \otimes hw.$$

The action of an element $(g,h) \in \mathfrak{G} \times \mathfrak{H}$ on an element $x = \sum_{i,j=1}^{n} \alpha_{ij}\mathbf{v}_i \otimes \mathbf{w}_j \in \mathbf{V} \otimes \mathbf{W}$ is obtained by extending the action on the basis elements to all vectors in $\mathbf{V} \otimes \mathbf{W}$ by bilinearity:

$$\begin{aligned}(g,h)(x) = (g,h)\left(\sum_{i,j=1}^{n} \alpha_{ij}\mathbf{v}_i \otimes \mathbf{w}_j\right) = \\ \sum_{i,j=1}^{n}(g,h)\left(\alpha_{ij}v_i \otimes w_j\right) = \sum_{i,j=1}^{n} \alpha_{ij} g v_i \otimes h w_j.\end{aligned} \tag{2.23}$$

Each matrix in the tensor product representation is a Kronecker product of the corresponding matrices.

In general it is not true that the tensor product of two real irreducible representations is an irreducible representation.

Take for example, the standard two-dimensional real representations of $\mathbb{Z}_3$ and $\mathbb{Z}_5$ given by

$$\rho_1 = \begin{bmatrix} \cos(2k_1\pi/3) & -\sin(2k_1\pi/3) \\ \sin(2k_1\pi/3) & \cos(2k_1\pi/3) \end{bmatrix}, \quad \rho_2 = \begin{bmatrix} \cos(2k_2\pi/5) & -\sin(2k_2\pi/5) \\ \sin(2k_2\pi/5) & \cos(2k_2\pi/5) \end{bmatrix},$$

where $k_1 = 0, 1, 2$ and $k_2 = 0, \dots, 4$.

Then ρ_1 and ρ_2 are irreducible representations, but $\rho_1 \otimes \rho_2$ corresponding to $\mathbb{Z}_{15} \cong \mathbb{Z}_3 \times \mathbb{Z}_5$ is not (all real irreducible representations of $\mathbb{Z}_k$, $k > 2$, have dimension less than or equal to 2).

2.2.4 Character of a Representation

For a finite-dimensional real (resp. complex) $\mathfrak{G}$-representation W, with the corresponding homomorphism $T : \mathfrak{G} \to GL(W)$, the function $\chi_W : \mathfrak{G} \to \mathbb{R}$ (resp. $\chi_W : \mathfrak{G} \to \mathbb{C}$), defined by

$$\chi_W(g) = \mathrm{Tr}(T(g)), \quad g \in \mathfrak{G},$$

where Tr stands for the trace, is called the character of W.

Proposition 2.2.7.

(a) Let V, W be two $\mathfrak{G}$-representations over $\mathbb{K}$. Then

$$\chi_{V \oplus W} = \chi_V + \chi_W.$$

(b) Let V be a $\mathfrak{G}$-representation and W a $\mathfrak{H}$-representation of compact Lie groups $\mathfrak{G}$ and $\mathfrak{H}$, respectively, over $\mathbb{K}$. Then

$$\chi_{V \otimes W}(g, h) = \chi_V(g) \cdot \chi_W(h),$$

for all $g \in \mathfrak{G}$ and all $h \in \mathfrak{H}$.

Proof: It follows from the properties of the trace of matrices and (2.23). $\square$

A character of an irreducible $\mathfrak{G}$-representation is called irreducible. The proof of the following theorem can be found in [25].

Theorem 2.2.8. *The following properties hold for the characters of complex representations.*

1. *A character χ is irreducible if and only if $\langle \chi, \chi \rangle = 1$.*
2. *For a finite group, the number of irreducible characters is equal to the number of conjugacy classes of elements in the group.*
3. *Two representations are equivalent if and only if their characters are equal.*
4. *Two irreducible representations are nonequivalent if they have different and orthogonal characters.*
5. *Two (not-necessarily irreducible) representations are equivalent if they have the same character.*

Table 2.2. Character table for the real irreducible D_8-representations

Irreps	Id	κ	$\gamma\kappa$	γ	γ^2	γ^3	γ^4
$\chi_{\mathcal{M}_0}$	1	1	1	1	1	1	1
$\chi_{\mathcal{M}_1}$	2	0	0	$\sqrt{2}$	0	$-\sqrt{2}$	-2
$\chi_{\mathcal{M}_2}$	2	0	0	0	-2	0	2
$\chi_{\mathcal{M}_3}$	2	0	0	$-\sqrt{2}$	0	$\sqrt{2}$	-2
$\chi_{\mathcal{M}_{j_8}}$	1	-1	-1	1	1	1	1
$\chi_{\mathcal{M}_{j_8+1}}$	1	-1	1	-1	1	-1	1
$\chi_{\mathcal{M}_{\hat{j}_8+2}}$	1	1	-1	-1	1	-1	1

2.2.5 Schur's Lemma and types of irreducible representations

Assume that W_1 and W_2 are two $\mathfrak{G}$-representations over a field $\mathbb{K}$. Next, let us denote by $L_{\mathbb{K}}^{\mathfrak{G}}(W_1, W_2)$ the space of all $\mathbb{K}$-linear $\mathfrak{G}$-equivariant maps $W_1 \to W_2$, and $GL_{\mathbb{K}}^{\mathfrak{G}}(W_1, W_2)$ its subset of all $\mathfrak{G}$-equivariant $\mathbb{K}$-isomorphisms. Then we use the notation $L_{\mathbb{K}}^{\mathfrak{G}}(W) := L_{\mathbb{K}}^{\mathfrak{G}}(W, W)$ and $GL_{\mathbb{K}}^{\mathfrak{G}}(W) := GL_{\mathbb{K}}^{\mathfrak{G}}(W, W)$.

Proposition 2.2.9 (Schur's Lemma)**.** *Assume that $\mathcal{W}_1$ and $\mathcal{W}_2$ are two irreducible $\mathfrak{G}$-representations over a field $\mathbb{K}$ and $A \in L_{\mathbb{K}}^{\mathfrak{G}}(\mathcal{W}_1, \mathcal{W}_2)$. Then A is either an isomorphism or the zero map.*

Recall that an associative $\mathbb{K}$-algebra $\mathfrak{A}$ is a vector space over $\mathbb{K}$, with an additional operation – multiplication – satisfying the following conditions:

$$x \cdot (y \cdot z) = (x \cdot y) \cdot z,$$

$$x \cdot (y + z) = x \cdot y + x \cdot z,$$

$$x \cdot (\alpha y) = \alpha(x \cdot y),$$

for $x, y, z \in \mathfrak{A}$ and $\alpha \in \mathbb{K}$.

The set of all linear maps commuting with a $\mathfrak{G}$-representation W (reducible, in general) over a field $\mathbb{K}$, is an associative algebra.

A non-zero associative algebra over $\mathbb{K}$ is called a division $\mathbb{K}$-algebra if and only if it has a multiplicative identity element and every non-zero element has a multiplicative inverse.

Schur's Lemma assures that the algebra formed by all linear maps commuting with a given irreducible representation is a division algebra.

Theorem 2.2.10 (Frobenius Theorem)**.** *Any division* $\mathbb{R}$*-algebra is isomorphic to either* $\mathbb{R}, \mathbb{C}$ *or* $\mathbb{H}$.

Frobenius' Theorem – whose proof can be found in [14] – together with Schur's Lemma imply the following result.

Proposition 2.2.11. *Let* $\mathcal{V}$ *be a real irreducible* $\mathfrak{G}$*-representation. Then, the algebra* $L^{\mathfrak{G}}_{\mathbb{R}}(\mathcal{V})$ *is isomorphic to either* $\mathbb{R}$, *or* $\mathbb{C}$ *or* $\mathbb{H}$.

Definition 2.2.12. A real irreducible $\mathfrak{G}$-representation $\mathcal{V}$ is called of *real* (resp. *complex* or *quaternionic*) *type*, if $L^{\mathfrak{G}}(\mathcal{V}) \cong \mathbb{R}$ (resp. $L^{\mathfrak{G}}(\mathcal{V}) \cong \mathbb{C}$ or $L^{\mathfrak{G}}(\mathcal{V}) \cong \mathbb{H}$).

Example 2.2.13.

(a) *Types of real irreducible representations of* S^1.

All real nontrivial irreducible S^1-representations are of the form (2.16). The only matrices commuting with (2.16) are of the form

$$\begin{bmatrix} a & b \\ -b & a \end{bmatrix}, \quad a, b \in \mathbb{R}. \tag{2.24}$$

The set of matrices of the form (2.24) is a division algebra isomorphic to $\mathbb{C}$ with basis formed by the elements

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \tag{2.25}$$

where the first matrix in (2.25) corresponds to 1, and the second one to i.

Therefore, any real nontrivial irreducible S^1-representation is of complex type.

(b) *Types of real irreducible representations of* D_n.

The only matrices commuting with the real nontrivial two-dimensional irreducible D_n-representations (2.18) are of the form

$$\begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}, \quad a \in \mathbb{R}. \tag{2.26}$$

The set of matrices of the form (2.26) is the one-dimensional division algebra isomorphic to $\mathbb{R}$. Therefore, the nontrivial irreducible D_n-representations (2.18) are of real type.

(c) *Types of real irreducible representations of* $O(2)$.

The only matrices commuting with the real nontrivial two-dimensional irreducible $O(2)$-representations (2.19) are of the form (2.26).

Therefore, the nontrivial real irreducible $O(2)$-representations (2.19) are of real type.

2.2.6 Complexification, conjugation and type of irreducible representation

We start with reminding the concept of complexification of a vector space.

Let V be a real vector space. Then the *complexification* of V- denoted by V^c- is given by

$$V^c := \mathbb{C} \otimes_{\mathbb{R}} V.$$

The subscript $\mathbb{R}$ in the tensor product means that the tensor product is taken over the real numbers. According to the definition, V^c is a real vector space. Let us introduce a complex structure on it as follows.

Let $\{1, i\}$ be a basis in $\mathbb{C}$ and $\{v_1, \ldots, v_n\}$ be a basis in V. Then,

$$\{1 \otimes v_1, \ldots 1 \otimes v_n, i \otimes v_1, \ldots, i \otimes v_n\}$$

is a basis for $\mathbb{C} \otimes_{\mathbb{R}} V$, where $\{1 \otimes v_1, \ldots, 1 \otimes v_n\}$ generates V and $\{i \otimes v_1, \ldots, i \otimes v_n\}$ generates another copy of V which is denoted iV.

Therefore, V^c can be represented as the direct sum

$$V^c \cong V \oplus iV. \tag{2.27}$$

So far, iV stands as a symbol of the second copy of V. The complex multiplication of a vector $v = v_1 + iv_2 \in V^c$ by a complex number $a + ib \in \mathbb{C}$ is given by the usual rule

$$(a + ib)(v_1 + iv_2) = (av_1 - bv_2) + i(bv_1 + av_2). \tag{2.28}$$

We have i as a complex number and we can also associate with i an operator $\mathcal{I}$ which assigns to every vector v the vector iv, according to (2.28), where $a = 0$ and $b = 1$. This is a real operator. However, since it satisfies the property

$$\mathcal{I}^2 = -\mathrm{Id}, \tag{2.29}$$

it determines the complex structure of V^c.

Next we describe the complexification of linear operators. Assume we have a real linear operator $A : V \to V$. The question is under what conditions can A be used to define a complex linear operator $A^c : V^c \to V^c$ on V^c, equipped with the complex structure given by (2.29).

Define $A^c : V^c \to V^c$ by

$$A^c(u + iv) := Au + \mathcal{I}Av, \tag{2.30}$$

for any vector $w = u + iv \in V^c$.

The operator A^c defined in (2.30), with $\mathcal{I}$ satisfying (2.29), such that A^c *commutes* with $\mathcal{I}$, is a *complex linear operator* on V^c.

Given a $\mathfrak{G}$-representation $T : \mathfrak{G} \to GL(V, \mathbb{R})$, in order to define its complexification, take V^c, for any $T(g)$ take its complexification, and get the complexification T^c of T.

Example 2.2.14. An example of complexification of the two-dimensional irreducible real S^1-representation is provided by (2.17). We look for the the complexification of

$$A := R_n\left(e^{i\varphi}\right),$$

where $R_n\left(e^{i\varphi}\right)$ has been defined in (2.16).

This is achieved by using the eigenvector basis of A in $\mathbb{C}^2$. The complex eigenvalues of the matrix of A are $e^{in\varphi}$ and $e^{-in\varphi}$. The complex eigenvectors associated to these eigenvalues are

$$v_1 = (1, i) \text{ and } v_2 = (1, -i). \tag{2.31}$$

If we choose (2.31) as a basis for the complexification of (2.16), then we obtain

$$\begin{bmatrix} e^{in\varphi} & 0 \\ 0 & e^{-in\varphi} \end{bmatrix} = \begin{bmatrix} 1 & i \\ 1 & -i \end{bmatrix} \begin{bmatrix} \cos(n\varphi) & -\sin(n\varphi) \\ \sin(n\varphi) & \cos(n\varphi) \end{bmatrix} \begin{bmatrix} 1 & i \\ 1 & -i \end{bmatrix}^{-1}, \tag{2.32}$$

and the complexification of the real operator A is

$$A^c := \begin{bmatrix} e^{in\varphi} & 0 \\ 0 & e^{-in\varphi} \end{bmatrix}.$$

Therefore, the complexification of the real two-dimensional representation (2.17) of S^1 is a complex two-dimensional representation. Since from the Schur's Lemma all complex irreducible representations of the abelian groups are one-dimensional, it follows that the complexification of (2.17) is a complex reducible representation. It can be expressed as

$$C_n(e^{in\varphi}) = V_1 \oplus V_2, \tag{2.33}$$

where $V_1 = R_n(e^{in\varphi})$ and $V_2 = R_n(e^{-in\varphi})$ are equivalent as real representations but non-equivalent as complex representations. Therefore, it is of complex type, as shown in Example 2.2.13.

Example 2.2.15. The complexification of the real irreducible two-dimensional representation of D_n (2.18), is given by

$$\begin{cases} \gamma(z_1, z_2)^T := \left(e^{\frac{2\pi i}{n}} \cdot z_1, e^{-\frac{2\pi i}{n}} \cdot z_1\right)^T, \text{ for } e^{\frac{2\pi i}{n}} \in \mathbb{Z}_n \text{ and } (z_1, z_2)^T \in \mathbb{C}^2; \\ \kappa(z_1, z_2)^T := (z_2, z_1)^T, \text{ where } (z_1, z_2)^T \in \mathbb{C}^2 \text{ and } \kappa \in D_n \end{cases}. \tag{2.34}$$

The matrix representation for κ acting on $\mathbb{C}^2$, is obtained by using the same eigenbasis for $\mathbb{C}^2$ formed by the eigenvectors (2.31) obtained at the diagonalization of A, in (2.32):

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & i \\ 1 & -i \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & i \\ 1 & -i \end{bmatrix}^{-1}. \tag{2.35}$$

There does not exist a proper invariant subspace common to γ and κ. This explains the irreducibility of the complexification and its real type.

Example 2.2.16. The complexification of the irreducible real $O(2)$-representation (2.19) is given by

$$\begin{cases} \gamma(z_1, z_2)^T := \left(e^{i\varphi} \cdot z_1, e^{-i\varphi} \cdot z_1\right)^T, \text{ for } e^{i\varphi} \in SO(2) \text{ and } (z_1, z_2)^T \in \mathbb{C}^2; \\ \kappa(z_1, z_2)^T := (z_2, z_1)^T, \end{cases}, \tag{2.36}$$

where $\varphi \in [0, 2\pi)$, γ and κ have been defined in (2.3) and (2.2) respectively, and the dot in (2.36) denotes complex multiplication. Using the same arguments as above, we conclude that the complexification of the real irreducible $O(2)$-representation is irreducible.

Examples (2.2.14), (2.2.15) and (2.2.16) suggest to consider the concept of complex conjugate representations. Let us recall a concept of complex conjugate space.

Let U be a complex vector space. Let $\overline{U}$ denote the space with same vectors as U, where the complex multiplication is given by $z \cdot u := \bar{z}u, z \in \mathbb{C}, u \in U$. This space is called conjugate to U and is denoted by $\overline{U}$.

In the case of a complex matrix $\mathfrak{G}$-representation $T : \mathfrak{G} \to GL(n, \mathbb{C})$, the $\mathfrak{G}$- representation conjugate to T is given by $\overline{T} : \mathfrak{G} \to GL(n, \mathbb{C})$, where $\overline{T}(g)$ denotes the matrix obtained from $T(g)$ by replacing its entries with their conjugates. Clearly, $GL(U) = GL(\overline{U})$.

As an example of a conjugate representation, consider the irreducible one-dimensional complex representation of S^1, as a homomorphism $f : S^1 \to GL(1, \mathbb{C}) \cong \mathbb{C}\backslash\{0\}, \varphi \in [0, 2\pi)$, $n \in \mathbb{N}$, defined by $f\left(e^{i\varphi}\right) = e^{in\varphi}$. Its conjugate representation is $e^{-in\varphi}$.

The following result links the type of an irreducible representation with its complexification; its proof can be found in [14].

Proposition 2.2.17. *Let V be a real irreducible $\mathfrak{G}$-representation.*

(a) Then, the complex $\mathfrak{G}$-representation V^c is irreducible if and only if V is of real type.

Assume V^c is reducible. Then

$$V^c = U \oplus \overline{U},$$

where U and $\overline{U}$ are irreducible complex representations and U is conjugate to $\overline{U}$. In addition,

(b) V is of complex type if U and $\overline{U}$ are nonequivalent representations;

(c) V is of quaternionic type if U and $\overline{U}$ are equivalent representations.

It is easy to see that the types of irreducible real representations of S^1, D_n and $O(2)$, as well as the structure of their complexifications are in a complete agreement with Proposition 2.2.17.

Theorem 2.2.8 lists some useful properties of the character of irreducible complex representations. Since in this Thesis we will be mainly dealing with real irreducible representations of real type, it is important to know which of the properties listed in Theorem 2.2.8 apply to the real irreducible representations.

In this Thesis we restrict to groups $\mathfrak{G}$ whose all real irreducible representations are of real type. Therefore, all properties of the real irreducible representations of these groups can be understood from the corresponding properties of the complex irreducible representations.

Remark 2.2.18.

(a) It is not true in general that a character χ of a real representation is irreducible if and only if $\langle \chi, \chi \rangle = 1$, as it can be seen for example from $\mathbb{Z}_3$ acting on the plane, where $\langle \chi, \chi \rangle = 2$ for any nontrivial character χ (all nontrivial real irreducible representations of the cyclic groups are of complex type). However, it is true for any irreducible real representation of real type, as a consequence of 2.2.17 (a).

(b) It is not true in general that for a finite group $\mathfrak{G}$, the number of real irreducible characters is equal to the number of conjugacy classes of the elements in $\mathfrak{G}$. It follows directly from Proposition 2.2.17, that in general, the number of complex irreducible representations is greater or equal than the the number of real irreducible representations.

2.2.7 Isotypic Decomposition

Consider a real $\mathfrak{G}$-representation V and its decomposition into a direct sum

$$V = \mathcal{V}^1 \oplus \mathcal{V}^2 \oplus \cdots \oplus \mathcal{V}^m \tag{2.37}$$

Among the irreducible subrepresentations $\mathcal{V}^j$ of V, some of them may be equivalent. Denote by V_j the minimal invariant subspace containing all subrepresentations that are equivalent to $\mathcal{V}^j$.

Then the sum

$$V = V_0 \oplus \cdots \oplus V_r, \tag{2.38}$$

is called the isotypic decomposition (2.37) of V. In contrast to the decomposition (2.37), the isotypic decomposition (2.38) is unique.

The isotypic components V_j, $j = 0, 1, 2, \dots, r$, can be also described in another way, which is more useful for infinite-dimensional generalizations. Denote by $\chi_j : \mathfrak{G} \to \mathbb{R}$ the real character of the irreducible representation $\mathcal{V}_j$, $j = 0, 1, 2, \dots, r$. We define the linear map $P_j : V \to V$ by

$$P_j x = n(\mathcal{V}_j) \int_{\mathfrak{G}} \chi_j(g)\, gx\, d\mu(g), \quad x \in V, \tag{2.39}$$

where

$$n(\mathcal{V}_j) = \begin{cases} \dim_{\mathbb{R}} \mathcal{V}_j, & \text{if } \mathcal{V}_j \text{ is of real type,} \\ \frac{\dim_{\mathbb{R}} \mathcal{V}_j}{2}, & \text{if } \mathcal{V}_j \text{ is of complex type,} \\ \frac{\dim_{\mathbb{R}} \mathcal{V}_j}{4}, & \text{if } \mathcal{V}_j \text{ is of quaternionic type.} \end{cases} \tag{2.40}$$

The number $n(\mathcal{V}_j)$ in (2.40) is called the *intristic dimension* of $\mathcal{V}_j$. Then, we have :

(a) $x \in V_j \iff P_j(x) = x$;

(b) $x \in V_l$, $l \neq j \implies P_j(x) = 0$;

(c) $P_j \circ P_j(x) = P_j(x)$ for all $x \in V$,

(d) $P_j : V \to V$ is $\mathfrak{G}$-equivariant.

Therefore, every $x \in V$ can be written as

$$x = \sum_{j=0}^{r} P_j(x),$$

where $P_j(x) \in V_j$, so $\mathrm{Id} = \sum_{j=0}^{r} P_j$.

As an example of isotypic decomposition, consider the D_n-representation $V \cong \mathbb{R}^n$ ($n > 2$, even) given by

$$\gamma(x_1, x_2, \dots, x_{n-1}, x_n) = (x_n, x_1, \dots, x_{n-2}, x_{n-1}),$$

$$\kappa(x_1, x_2, \dots, x_{n-1}, x_n) = (x_n, x_{n-1}, \dots, x_2, x_1).$$

The isotypic decomposition is given by

$$V = \mathcal{M}_0 \oplus \mathcal{M}_1 \oplus \ldots \oplus \mathcal{M}_{\frac{n-2}{2}} \oplus \mathcal{M}_{j_n+1},$$

with the notations of Example 2.2.4.

2.3 Representations in infinite-dimensional spaces

2.3.1 Banach representation and its isotypic decomposition

Let W be a real Banach space. We say that W is a real Banach representation of $\mathfrak{G}$ (in short, *Banach $\mathfrak{G}$-representation*) if the space W is a $\mathfrak{G}$-space such that the translation map $T_g : W \to W$, defined by $T_g(w) = gw$ for $w \in W$, is a bounded $\mathbb{R}$-linear operator for every $g \in \mathfrak{G}$.

Clearly, every finite-dimensional $\mathfrak{G}$-representation is a Banach $\mathfrak{G}$-representation. We say that a Banach $\mathfrak{G}$-representation W is isometric if for each $g \in \mathfrak{G}$, the translation operator T_g is an isometry, i.e. $\|T_g w\| = \|w\|$ for all $w \in W$, and we call the norm $\|\cdot\|$ a $\mathfrak{G}$-invariant norm.

It can be shown that given a Banach $\mathfrak{G}$-representation W, it is possible to construct a new $\mathfrak{G}$-invariant norm on W, equivalent to the initial one, denoted $\|\cdot\|_{\mathfrak{G}}$, using the Haar integral.

By a Banach subrepresentation it is meant a closed $\mathfrak{G}$-invariant linear subspace V of W.

All irreducible Banach $\mathfrak{G}$-representations (i.e. representations that do not contain any proper nontrivial Banach G-subrepresentations) are finite-dimensional (see [30]. Notice that for a closed subgroup $\mathscr{H} \subset \mathfrak{G}$, the set $W^{\mathscr{H}}$ is a closed linear subspace of W.

The inner product $\langle \cdot, \cdot \rangle$ on W is called $\mathfrak{G}$-*invariant* if $\langle gv, gw \rangle = \langle v, w \rangle$ for all $g \in \mathfrak{G}$, $v, w \in W$. In this case W is called an isometric Hilbert $\mathfrak{G}$-*representation*.

Given a Banach $\mathfrak{G}$-representation consider the complete list $\{\mathcal{V}_k : k = 0, 1, 2, \ldots\}$ of all irreducible $\mathfrak{G}$-representations and let $\chi_k : \mathfrak{G} \to \mathbb{R}$ be the corresponding character of $\mathcal{V}_k$ for

$k = 0, 1, 2, \dots$. Define the linear maps

$$P_k x = n(\mathcal{V}_k) \int_{\mathfrak{G}} \chi_k(g)\, gx\, d\mu(g), \quad g \in \mathfrak{G}, \quad x \in V, \quad k = 1, 2, \dots, \tag{2.41}$$

where $n(\mathcal{V}_k)$ denotes the intrinsic dimension of $\mathcal{V}_k$. Then, similarly to the finite-dimensional case, we have that $P_k : V \to V$ is a $\mathfrak{G}$-equivariant (i.e. $P_k(gv) = gP_k(v)$ for $g \in \mathfrak{G}$ and $v \in V$) bounded linear projection onto the subspace $V_k := P_k(V)$.

We have the following result whose proof can be found in [33].

Theorem 2.3.1. *Let V be a real isometric Banach representation of a compact Lie group $\mathfrak{G}$, $\mathcal{V}$ a real irreducible representation of $\mathfrak{G}$ and χ the character of $\mathcal{V}$. Then, the linear operator $P_{\mathcal{V}} : V \to V$ defined by*

$$P_{\mathcal{V}} x = n(\mathcal{V}) \int_{\mathfrak{G}} \chi(g)\, gx\, d\mu(g), \quad x \in V \tag{2.42}$$

is a bounded $\mathfrak{G}$-equivariant projection on the subspace $P_{\mathcal{V}}(V)$, which satisfies the following properties:

(i) If $x \in V$ belongs to an irreducible subrepresentation of V that is equivalent to $\mathcal{V}$, then $P_{\mathcal{V}} x = x$;

(ii) If $x \in V$ belongs to an irreducible subrepresentation of V that is not equivalent to $\mathcal{V}$, then $P_{\mathcal{V}} x = 0$.

It is an immediate consequence of Theorem 2.3.1 that every irreducible subrepresentation of V, which is equivalent to $\mathcal{V}_k$, is contained in V_k. The $\mathfrak{G}$-invariant subspace V_k is called the *isotypic component* of V corresponding to $\mathcal{V}_k$. We define the subspace

$$V_\infty := \bigoplus_k V_k \tag{2.43}$$

which is clearly dense in V, i.e. we have $\overline{V}_\infty = V$. Consequently, the decomposition

$$V_\infty := \overline{\bigoplus_k V_k} \tag{2.44}$$

is called the isotypic decomposition of V. Moreover, for every $\mathfrak{G}$-equivariant linear operator $A : V \to V$ we have that $A(V_k) \subseteq V_k$ for all $k = 1, 2, \ldots$.

2.3.2 Representations in a Banach space of periodic functions

As an example of a Banach $\mathfrak{G}$-representation, consider the Banach space $W = W(S^1; \mathbf{V})$ of "reasonable" 2π-periodic functions from S^1 to $\mathbf{V}$. The group $O(2) \times \Gamma \times \mathbb{Z}_2$ acts on W by

$$\left(e^{i\theta}, \gamma, \pm 1\right) x(t) := \pm\gamma x(t + \theta), \tag{2.45}$$

$$\left(e^{i\theta}\kappa, \gamma, \pm 1\right) x(t) := \pm\gamma x(-t + \theta), \tag{2.46}$$

where $x \in W, e^{i\theta}, \kappa \in O(2)$, $\gamma \in \Gamma$ and $\pm 1 \in \mathbb{Z}_2$. Clearly, W is an *isometric* Banach $\mathfrak{G}$-representation.

Consider first W as an $O(2)$-representation corresponding to its Fourier modes:

$$W = \overline{\bigoplus_{k=0}^{\infty} \mathbb{V}_k}, \quad \mathbb{V}_k := \{\cos(kt)u + \sin(kt)v : \; u, v \in \mathbf{V}\}, \tag{2.47}$$

where each $\mathbb{V}_k$, for $k \in \mathbb{N}$, is equivalent to the complexification $\mathbf{V}^c := \mathbf{V} \oplus i\mathbf{V}$ (as a real $O(2)$-representation) of $\mathbf{V}$, and the rotations $e^{i\theta} \in SO(2)$ act on vectors $\mathbf{z} \in \mathbf{V}^c$ by $e^{i\theta}(\mathbf{z}) := e^{-ik\theta} \cdot \mathbf{z}$ (here " $\cdot$ " stands for complex multiplication) and $\kappa\mathbf{z} := \overline{\mathbf{z}}$. Indeed, the linear isomorphism $\varphi_k : \mathbf{V}^c \to \mathbb{V}_k$ given by

$$\varphi_k(x + iy) := \cos(kt)u + \sin(kt)v, \;\; u, v \in \mathbf{V}, \tag{2.48}$$

is $O(2)$-equivariant.

Clearly, $\mathbb{V}_0$ can be identified with $\mathbf{V}$ with the trivial $O(2)$-action, while $\mathbb{V}_k$, $k = 1, 2, \ldots$, is modeled on the irreducible $O(2)$-representation $\mathcal{W}_k \cong \mathbb{R}^2$, where $SO(2)$ acts by k-folded rotations and κ acts by complex conjugation. One notices that each $\mathbb{V}_k, k = 0, 1, 2, \ldots$, is also $\Gamma \times \mathbb{Z}_2$-invariant.

Assume that we have the complete list of all irreducible orthogonal $\Gamma \times \mathbb{Z}_2-$representations on which the $\Gamma \times \mathbb{Z}_2$-isotypic components of $\mathbf{V} \cong \mathbb{V}_0$ are modeled, namely $\mathcal{V}_0^-, \mathcal{V}_1^-, \mathcal{V}_2^-, \ldots, \mathcal{V}_\tau^-$, where the symbol " $-$ " indicates the antipodal $\mathbb{Z}_2$-action and $\mathcal{V}_0^-$ corresponds to the trivial Γ-action.

On the other hand, we have the complete list of the irreducible orthogonal $O(2)$- representations $\mathcal{W}_k$. The action of the direct product $O(2) \times \Gamma \times \mathbb{Z}_2$, is defined in the tensor product space

$$\mathcal{V}_{k,l}^- := \mathcal{W}_k \otimes \mathcal{V}_l^- . \tag{2.49}$$

Remark 2.3.2.

(i) In what follows, we will assume that all real irreducible representations of Γ are of real type. Moreover, Example 2.2.13 shows that all two-dimensional real irreducible representations of $O(2)$ are also of real type. It is not true, in general, that the tensor product of two real irreducible representations is an irreducible representation. This is true, however, when the two real irreducible representations are of *real type.* Therefore, $\mathcal{V}_{k,l}^-$ is irreducible, in the cases of pure interest.

(ii) The assumption that all real irreducible representations of Γ are of real type does not lead to the loss of generality. Indeed, the set of complex linear equivariant operators on an irreducible representation of non-real type has only one connected component. Therefore, any such operator is equivariantly homotopic to the identity operator and as such does not contribute to the corresponding equivariant degree.

Example 2.3.3. The tensor product space $\mathcal{V}_{k,l}^-$ defined in (2.49) is an irreducible real representation of $O(2)\times\Gamma\times\mathbb{Z}_2$, where $\Gamma \cong D_n,\ n > 2$. This follows from the fact that all real irreducible representations of $O(2)$ and D_n are of real type, as shown in Remark 2.3.2 and Example 2.2.13, respectively. The additional $\mathbb{Z}_2$ action by ±1 doesn't affect the irreducibility of the representation.

Since $\mathcal{V}_{k,l}^-$ is an irreducible orthogonal $\mathfrak{G}$-representation, it follows that $\mathbb{V}_0$ and $\mathbb{V}_k$ (cf. (2.48)) admit the following $\mathfrak{G}$-isotypic decompositions:

$$\mathbb{V}_0 = V_0^- \oplus V_1^- \oplus \ldots \oplus V_\tau^-$$

(with the trivial $O(2)$-action) and

$$\mathbb{V}_k = V_{k,0}^- \oplus V_{k,1}^- \oplus \ldots \oplus V_{k,\tau}^-$$

where V_l^- (resp. $V_{k,l}^-$) is modeled on $\mathcal{V}_{0,l}^-$ (resp. $\mathcal{V}_{k,l}^-$ with $k > 0$) .

2.4 Local Brouwer Degree

Let us suppose that V is a finite-dimensional normed space (assumed without loss of generality to be the Euclidean space $V := \mathbb{R}^n$). Let $\Omega \subset V$ be an open, bounded, nonempty set, and let $f : V \to V$ be a continuous function, such that $f(x) \neq 0$, $\forall x \in \partial\Omega$, in which case a pair (Ω, f) is called an *admissible pair* in V. In such a case we will also say that f is an Ω-*admissible map.* One can extend this definition to a homotopy in a natural way. Suppose that $h : [0,1] \times V \to V$ is a continuous map, and is Ω-admissible for every $t \in [0,1]$ (i.e. the map $h_t(x) := h(t,x)$, $x \in V$, is Ω-admissible). Then, we say that h_0 and h_1 are Ω-*admissibly homotopic.* We denote by $\mathfrak{M}(V)$ the set of all admissible pairs in V, and we put

$$\mathfrak{M} := \bigcup_V \mathfrak{M}(V) \ .$$

Definition 2.4.1. We call a function $\deg : \mathfrak{M} \to \mathbb{Z}$ a (*local Brouwer*) *degree* if it satisfies the following three conditions:

(P1) (*Additivity*) For every $(f, \Omega) \in \mathfrak{M}$ and every pair of open, bounded, nonempty, disjoint subsets Ω_1 and Ω_2 of V, such that $f^{-1}(0) \cap \Omega \subset \Omega_1 \cup \Omega_2$, one has $\deg(f, \Omega) = \deg(f, \Omega_1) + \deg(f, \Omega_2)$.

(P2) (*Homotopy Invariance*) Let $\Omega \subset V$ be an open, bounded and nonempty set and let $h(t,x) : [0,1] \times V \to V$ be a continuous map such that $h(t,x) \neq 0$ for $x \in \partial\Omega$ and $t \in [0,1]$. Then, $\deg(h_t, \Omega) =$ constant for all $t \in [0,1]$.

(P3) (*Normalization*) For any open, bounded, nonempty set $\Omega \subset V$ and $x_o \in V$ such that $x_o \notin \partial\Omega$ we have

$$\deg(\mathrm{Id} - x_o, \Omega) = \begin{cases} 1 & \text{if } x_o \in \Omega \\ 0 & \text{if } x_o \notin \Omega \end{cases}.$$

The proof of the following result can be found in [18].

Proposition 2.4.2. *The function* $\deg : \mathfrak{M} \to \mathbb{Z}$ *satisfying the properties* $(P1)$, $(P2)$ *and* $(P3)$ *exists and is unique.*

Proposition 2.4.3. *Using Properties* $(P1) - (P3)$, *one can prove the following results.*

(P4) (*Existence*) *For every* $(f, \Omega) \in \mathfrak{M}$, *if* $\deg(f, \Omega) \neq 0$, *then, there exists* $x_o \in \Omega$ *such that* $f(x_o) = 0$.

(P5) (*Excision*) *Suppose that* $(f, \Omega) \in \mathfrak{M}$ *and let* Ω_o *be an open subset of* Ω *such that* $f^{-1}(0) \cap \Omega \subset \Omega_o$. *Then,* $\deg(f, \Omega) = \deg(f, \Omega_o)$.

(P6) (*Rouché property*) *If* $(f, \Omega) \in \mathfrak{M}$ *and* $g : V \to V$ *is a continuous map such that*

$$\sup_{x \in \partial\Omega} |f(x) - g(x)| < \inf_{x \in \partial\Omega} |f(x)|,$$

then $(g, \Omega) \in \mathfrak{M}$, *and* $\deg(f, \Omega) = \deg(g, \Omega)$.

(P7) (*Boundary Values Dependence*) *Let* $\Omega \subset V$ *be an open, bounded and nonempty set. Then, for every* $f, g \in C(\overline{\Omega}; \partial\Omega)$ *such that* $f(x) = g(x)$ *for all* $x \in \partial\Omega$, *we have* $\deg(f, \Omega) = \deg(g, \Omega)$.

(P8) (*Product*) *Let* $(f, \Omega) \in \mathfrak{M}$ *and* $(g, U) \in \mathfrak{M}$. *Then,* $\deg(f \times g, \Omega \times U) = \deg(f, \Omega) \cdot \deg(g, U)$.

(P9) (*Linear map*) *Let Ω be an open and bounded neighborhood of zero in V and let $f : V \to V$ be an invertible linear map. Then*

$$\deg(f, \Omega) = (-1)^{\lambda}, \tag{2.50}$$

where λ is the sum of algebraic multiplicities of the real negative eigenvalues of f.

Let $f : V \to V$ be at least a C^1-map, where $V \cong \mathbb{R}^n$. Then, we say that $x \in V$ is a *critical point* if $Df(x)$ is singular. Moreover, we say that $x \in V$ is a *regular point* if $Df(x)$ is non-singular. Finally, we say that $z \in V$ is a *regular value* if z is such that $f^{-1}(z)$ is either empty or it consists only of regular points.

Theorem 2.4.4. *Assume that (f, Ω) is an admissible pair in $V := \mathbb{R}^n$ such that f is a C^1-map and such that 0 is a regular value of $f|_{\Omega}$. Then, the set $f^{-1}(0) \cap \Omega = \{x_1, x_2, \dots, x_m\}$ is finite and we have*

$$\deg(f, \Omega) = \sum_{j=1}^{m} \operatorname{sign}\left(\det Df(x_j)\right). \tag{2.51}$$

2.5 Leray-Schauder Degree

The Leray-Schauder Degree extends the Brouwer Degree to compact vector fields in infinite dimensional Banach spaces. Recall that the Schauder's approximation is a prerequisite for defining the Leray-Schauder degree for compact fields. We will follow the exposition in [33].

Definition 2.5.1. Assume that V is a Banach space and $T : V \to V$ is a continuous map. Then,

(a) If $T(V)$ is contained in a compact subset of V, then T is called compact.

(b) If $T(V)$ is contained in a finite-dimensional linear subspace of V, then the map $T : V \to V$ is called finite-dimensional.

Construction 2.5.2 (Schauder approximation)**.** *Let $\mathcal{S} = \{y_1, \ldots, y_n\}$ be a finite set in the Banach space V and $\epsilon > 0$ be a fixed real number. Define $B(\epsilon, \mathcal{S}) := \bigcup_{i=1}^{n} B_\epsilon(y_i)$. Let $\mu_i : B(\epsilon, \mathcal{S}) \to \mathbb{R},\ i = 1, \ldots, n,$ be given by $\mu_i(x) = \max\{0, \epsilon - \|x - y_i\|\}$.*

Then, the Schauder projection $\pi_\epsilon : B(\epsilon, \mathcal{S}) \to \operatorname{conv}(\mathcal{S})$ is given by

$$\pi_\epsilon = \frac{1}{\sum\limits_{i=1}^{n} \mu_i(x)} \sum_{i=1}^{n} \mu_i(x) y_i,$$

where $\operatorname{conv}(\mathcal{S})$ denotes the convex hull of the set $\mathcal{S}$.

Theorem 2.5.3 (Schauder's approximation theorem)**.** *Assume that $C \subseteq V$ is a closed convex subset, $X \subseteq V$, and $T : X \to C$ is a compact map. Then for each $\epsilon > 0$, there is a finite set $\mathcal{S} = \{y_1, \ldots, y_n\} \subseteq T(X) \subseteq C$ and a finite-dimensional map $T_\epsilon : V \to C$ satisfying*

(a) $\|T_\epsilon(x) - T(x)\| < \epsilon\ \forall x \in X$;

(b) $T_\epsilon(X) \subseteq \operatorname{conv}(\mathcal{S}) \subseteq C$.

Assume $\Omega \subseteq V$ and $T : \Omega \to V$ is a compact map. Then a map $f : \Omega \to V,\ f(x) = x - T(x)$ is called a compact field on Ω. We say that $h : [0,1] \times \Omega \to V,\ h(t,x) = x - H(t,x)$, is a homotopy of compact fields, if $H : [0,1] \times \Omega \to V$ is a compact map.

Assume that $\Omega \subseteq V$ is an open, bounded, nonempty subset. If a map $f : \overline{\Omega} \to V$ is a compact field on $\overline{\Omega}$ and $f(x) \neq 0\ \forall x \in \partial\Omega$, then we say that f is an Ω-admissible compact field. A homotopy $h : [0,1] \times \overline{\Omega} \to X$ of compact fields is called Ω-admissible homotopy if $h(t,x) \neq 0\ \forall\ (t,x) \in [0,1] \times \partial\Omega$.

Proposition 2.5.4. *Let $\Omega \subseteq V$ be an open, bounded, nonempty subset and let $h : [0,1] \times \overline{\Omega} \to V$ be an Ω-admissible homotopy of compact fields. Then,*

$$\inf\{\|h(t,x)\|;\ x \in \partial\Omega,\ t \in [0,1]\} > 0.$$

Assume that $\Omega \subset V$ is an open, bounded and nonempty subset. Let $T : \overline{\Omega} \to V$ be a compact map (cf. Definition 2.5.1). Put $f(x) = x - T(x)$ and let $\epsilon = \inf\{\|f(x)\|,\ x \in \partial\Omega\}$. From Proposition 2.5.4, we know that $\epsilon > 0$. From Theorem 2.5.3 it follows that there exists a finite-dimensional map $T_\epsilon : \overline{\Omega} \to V$, called an ϵ-approximation of T, such that

$$\|T_\epsilon(x) - T(x)\| < \epsilon, \quad x \in \overline{\Omega}.$$

Then, $f_\epsilon : \overline{\Omega} \to V$ defined as $f_\epsilon(x) = x - T_\epsilon(x)$, is also an Ω-admissible compact field and one obtains

$$\|f_\epsilon(x) - f(x)\| = \|T_\epsilon(x) - T(x)\| < \epsilon, \quad x \in \overline{\Omega}.$$

Hence, $\|f_\epsilon(x) - f(x)\| > 0,\ \forall x \in \partial\Omega$.

Let $V_o \subseteq V$ be a finite-dimensional subspace of V such that $\Omega \cap V_o \neq \emptyset$ and $T_\epsilon(\overline{\Omega}) \subseteq V_o$. Then, the Leray-Schauder degree of f on Ω is defined as

$$\deg(f, \Omega) := \deg(f_o, \Omega_o), \tag{2.52}$$

where

$$f_o := f_\epsilon\Big|_{\overline{\Omega \cap V_o}} : \overline{\Omega \cap V_o} \to V_o, \quad \Omega_o = \Omega \cap V_o.$$

The proof of the following result can be found in [33].

Proposition 2.5.5. *The Leray-Schauder degree defined in* (2.52) *is independent either of the choice of* Ω_o *or the finite-dimensional approximation* T_ϵ.

Theorem 2.5.6. *Let* $\Omega \subset V$ *be a nonempty open and bounded subset, and* $f = \mathrm{Id} - T : \overline{\Omega} \to V$ *be an* Ω*-admissible compact field. Then, the Leray-Schauder degree defined in* (2.52), $\deg(f, \Omega)$, *satisfies the following properties:*

(P1) (*Normalization*) *Let* $x_o \in V$, *be such that* $x_o \notin \partial\Omega$. *Then,*

$$\deg(\mathrm{Id} - x_o, \Omega) = \begin{cases} 1 & \text{if } x_o \in \Omega \\ 0 & \text{if } x_o \notin \Omega \end{cases}.$$

(P2) (*Additivity*) *Let* $\Omega_1, \Omega_2 \subseteq \Omega$ *be two nonempty, open and disjoint subsets. If* $f^{-1}(0) \subseteq \Omega_1 \cup \Omega_2$, *then*

$$\deg(f, \Omega) = \deg(f, \Omega_1) + \deg(f, \Omega_2).$$

(P3) (*Homotopy*) *Let* $h : [0,1] \times \overline{\Omega} \to V$ *be an* Ω*-admissible homotopy of compact fields. Then,* $\deg(h_t, \Omega) =$ *constant for all* $t \in [0,1]$.

(P4) (*Existence*) *Let* $T : \overline{\Omega} \to V$ *be a compact map and* $f = \mathrm{Id} - T : \overline{\Omega} \to V$ *be an* Ω*-admissible compact field. If the Leray-Schauder degree defined in* (2.52) *satisfies* $\deg(f, \Omega) \neq 0$, *then there exists* $x_o \in \Omega$ *such that* $f(x_o) = 0$.

(P5) (*Excision*) *Let* $T : \overline{\Omega} \to V$ *be a compact map and* $f = \mathrm{Id} - T : \overline{\Omega} \to V$ *be an* Ω*-admissible compact field. If* Ω_1 *is an open subset of* Ω *such that* $f^{-1}(0) \subseteq \Omega_1$, *then* $\deg(f, \Omega) = \deg(f, \Omega_1)$.

(P6) (*Continuity*) *Let* $g : \overline{\Omega} \to V$ *be a compact field such that*

$$\|f - g\|_\infty := \sup\{\|f(x) - g(x)\|;\ x \in \overline{\Omega}\} < \inf\{\|f(x)\|;\ x \in \partial\Omega\}.$$

Then, $\deg(f, \Omega) = \deg(g, \Omega)$.

(P7) (*Boundary Values Dependence*) *If* $g : \overline{\Omega} \to V$ *is a compact field such that* $f(x) = g(x)$ *for all* $x \in \partial\Omega$, *then* $\deg(f, \Omega) = \deg(g, \Omega)$.

(P8) (*Linear map*) *Let* Ω *be the unit ball, and let* $T : V \to V$ *be a compact linear operator. Assume that* $\lambda \in \mathbb{R}\backslash\{0\}$ *is such that* λ^{-1} *is not an eigenvalue of* T. *Then, if* $0 \in \Omega$, *we have*

$$\deg(\mathrm{Id} - \lambda f, \Omega) = (-1)^{m(\lambda)}, \tag{2.53}$$

where $m(\lambda)$ *is the sum of the algebraic multiplicities of the real eigenvalues* μ *satisfying* $\mu\lambda > 1$.

2.6 Equivariant Brouwer and Leray-Schauder Degrees

In this section we describe the Brouwer Equivariant Degree, which is a generalization of the local Brouwer degree for $\mathfrak{G}$-equivariant maps. The range $A(\mathfrak{G})$ of this degree admits a natural ring structure called the *Burnside ring* of $\mathfrak{G}$.

2.6.1 Burnside Ring

Let us denote by $A(G)$ the free $\mathbb{Z}$-module generated by the conjugacy classes $(\mathscr{H}) \in \Phi_0(\mathfrak{G})$, i.e.

$$A(\mathfrak{G}) := \mathbb{Z}[\Phi_0(\mathfrak{G})].$$

Elements $\alpha \in A(\mathfrak{G})$ are finite sums

$$\alpha = n_{\mathscr{H}_1}(\mathscr{H}_1) + \ldots + n_{\mathscr{H}_m}(\mathscr{H}_m),$$

with $n_{\mathscr{H}_i} \in \mathbb{Z}$ and $(\mathscr{H}_i) \in \Phi_0(\Gamma)$. Then, define the multiplication $\cdot : A(\mathfrak{G}) \times A(\mathfrak{G}) \to A(\mathfrak{G})$, on generators $(\mathscr{H})$, $(\mathscr{K}) \in \Phi_0(\mathfrak{G})$ by

$$(\mathscr{H}) \cdot (\mathscr{K}) := \sum_{(\mathscr{L}) \in \Phi_0(\mathfrak{G})} m_{\mathscr{L}}(\mathscr{L}), \tag{2.54}$$

where

$$m_{\mathscr{L}} := \text{ number of } (\mathscr{L})\text{-orbits in } \frac{\mathfrak{G}}{\mathscr{H}} \times \frac{\mathfrak{G}}{\mathscr{K}}$$

with respect to the diagonal $\mathfrak{G}$-action given by $g(h\mathscr{H}, k\mathscr{K}) = (gh\mathscr{H}, gk\mathscr{K})$, $\forall g \in \mathfrak{G}, h \in \mathscr{H}, k \in \mathscr{K}$. One can show that formula (2.54) is well-defined, see for example [5]. Extending by distributivity, one gets the ring structure called the *Burnside ring* of $\mathfrak{G}$ and denoted by $A(\mathfrak{G})$.

Formula (2.54) is not suitable for computations. One can use the *recurrence formula* for the Burnside ring multiplication

$$m_{\mathscr{L}} = \frac{n(\mathscr{L},\mathscr{H})|W(\mathscr{H})|n(\mathscr{L},\mathscr{K})|W(\mathscr{K})| - \sum_{(\widetilde{\mathscr{L}})>(\mathscr{L})} m_{\widetilde{\mathscr{L}}}\, n(\mathscr{L},\widetilde{\mathscr{L}})\,|W(\widetilde{\mathscr{L}})|}{|W(\mathscr{L})|}. \tag{2.55}$$

Table 2.3. Numbers $n(\mathscr{L},\mathscr{H})$ for the generators of $A(D_8)$.

$\mathscr{L}$	$\mathscr{H}$	$n(\mathscr{L},\mathscr{H})$	$\mathscr{L}$	$\mathscr{H}$	$n(\mathscr{L},\mathscr{H})$	$\mathscr{L}$	$\mathscr{H}$	$n(\mathscr{L},\mathscr{H})$
D_8	D_8	1	$\mathbb{Z}_2$	D_4	1	$\mathbb{Z}_1$	D_2	2
D_4	D_8	1	$\mathbb{Z}_1$	D_4	1	$\widetilde{D}_2$	$\widetilde{D}_2$	1
$\mathbb{Z}_8$	D_8	1	$\mathbb{Z}_8$	$\mathbb{Z}_8$	1	$\widetilde{D}_1$	$\widetilde{D}_2$	1
$\widetilde{D}_4$	D_8	1	$\mathbb{Z}_4$	$\mathbb{Z}_8$	1	$\mathbb{Z}_2$	$\widetilde{D}_2$	2
D_2	D_8	1	$\mathbb{Z}_2$	$\mathbb{Z}_8$	1	$\mathbb{Z}_1$	$\widetilde{D}_2$	2
$\widetilde{D}_2$	D_8	1	$\mathbb{Z}_1$	$\mathbb{Z}_8$	1	$\mathbb{Z}_4$	$\mathbb{Z}_4$	1
$\mathbb{Z}_4$	D_8	1	$\widetilde{D}_4$	$\widetilde{D}_4$	1	$\mathbb{Z}_2$	$\mathbb{Z}_4$	1
D_1	D_8	1	$\widetilde{D}_2$	$\widetilde{D}_4$	1	$\mathbb{Z}_1$	$\mathbb{Z}_4$	1
$\widetilde{D}_1$	D_8	1	$\mathbb{Z}_4$	$\widetilde{D}_4$	1	D_1	D_1	1
$\mathbb{Z}_2$	D_8	1	$\widetilde{D}_1$	$\widetilde{D}_4$	1	$\mathbb{Z}_1$	D_1	4
$\mathbb{Z}_1$	D_8	1	$\mathbb{Z}_2$	$\widetilde{D}_4$	1	$\widetilde{D}_1$	$\widetilde{D}_1$	1
D_4	D_4	1	$\mathbb{Z}_1$	$\widetilde{D}_4$	1	$\mathbb{Z}_1$	$\widetilde{D}_1$	4
D_2	D_4	1	D_2	D_2	1	$\mathbb{Z}_2$	$\mathbb{Z}_2$	1
$\mathbb{Z}_4$	D_4	1	D_1	D_2	1	$\mathbb{Z}_1$	$\mathbb{Z}_2$	1
D_1	D_4	1	$\mathbb{Z}_2$	D_2	2	$\mathbb{Z}_1$	$\mathbb{Z}_1$	1

It can be verified that $(\mathfrak{G})\cdot(\mathscr{L}) = (\mathscr{L})$ for every $(\mathscr{L}) \in \Phi_0(\mathfrak{G})$, therefore $(\mathfrak{G})$ is the unity in $A(\mathfrak{G})$.

Example 2.6.1. Consider now the dihedral group D_n, where $n \geq 2$ is an even natural number. We have the following conjugacy classes of subgroups in D_n

$$\Phi(D_n) = \{(D_k),\, (\mathbb{Z}_k) : k \mid n\} \cup \{(\widetilde{D}_k) : 2k \mid n\},$$

where

$$\widetilde{D}_k = \{1, \gamma^s, \dots, \gamma^{s(k-1)}, \gamma\kappa, \gamma^{s+1}\kappa, \dots, \gamma^{s(k-1)+1}\kappa\}.$$

For example consider $n = 8$. The numbers $n(\mathscr{L},\mathscr{H})$ for D_8 are given in Table 2.3. As an example, let us compute $(D_2)\cdot(D_2) = x_1(D_2) + x_2(\mathbb{Z}_2)$. We have

$$x_1 = \frac{n(D_2,D_2)|W(D_2)|n(D_2,D_2)|W(D_2)|}{|W(D_2)|} = \frac{1\cdot 2\cdot 1\cdot 2}{2} = 2$$

and

$$\begin{aligned} x_2 &= \frac{n(\mathbb{Z}_2,D_2)|W(D_2)|n(\mathbb{Z}_2,D_2)|W(D_2)| - x_1\cdot n(\mathbb{Z}_2,D_2)|W(D_2)|}{|W(\mathbb{Z}_2)|} \\ &= \frac{2\cdot 2\cdot 2\cdot 2 - 2\cdot 2\cdot 2}{8} = 1. \end{aligned}$$

Similarly, $(\widetilde{D}_1) \cdot (\widetilde{D}_1) = x_3(\widetilde{D}_1) + x_4(\mathbb{Z}_1)$, where

$$x_3 = \frac{n(\widetilde{D}_1, \widetilde{D}_1)|W(\widetilde{D}_1)|n(\widetilde{D}_1, \widetilde{D}_1)|W(\widetilde{D}_1)|}{|W(\widetilde{D}_1)|} = \frac{1 \cdot 2 \cdot 1 \cdot 2}{2} = 2$$

and

$$\begin{aligned} x_4 &= \frac{n(\mathbb{Z}_1, \widetilde{D}_1)|W(\widetilde{D}_1)|n(\mathbb{Z}_1, \widetilde{D}_1)|W(\widetilde{D}_1)| - x_3 \cdot n(\mathbb{Z}_1, \widetilde{D}_1)|W(\widetilde{D}_1)|}{|W(\mathbb{Z}_1)|} \\ &= \frac{4 \cdot 2 \cdot 4 \cdot 2 - 2 \cdot 4 \cdot 2}{16} = 3. \end{aligned}$$

Finally, $(D_1) \cdot (D_1) = x_5(\mathbb{Z}_1) + x_6(D_1)$, where

$$x_5 = \frac{n(D_1, D_1)|W(D_1)|n(D_1, D_1)|W(\widetilde{D}_1)|}{|W(D_1)|} = \frac{1 \cdot 2 \cdot 1 \cdot 2}{2} = 2$$

and

$$\begin{aligned} x_6 &= \frac{n(\mathbb{Z}_1, D_1)|W(D_1)|n(\mathbb{Z}_1, D_1)|W(D_1)| - x_5 \cdot n(\mathbb{Z}_1, D_1)|W(D_1)|}{|W(\mathbb{Z}_1)|} \\ &= \frac{4 \cdot 2 \cdot 4 \cdot 2 - 2 \cdot 4 \cdot 2}{16} = 3, \end{aligned}$$

all these cases being in agreement with Tables 2.3. Using the recurrence formula (2.55), one can compute the multiplication Table 2.4. The computations can be assisted by the GAP software by using the "EquiDeg" package [43].

Table 2.4. Multiplication table for $A(D_8)$.

$\mathbf{A(D_8)}$	(D_8)	(D_4)	$(\mathbb{Z}_8)$	$(\widetilde{D}_4)$	(D_2)	$(\widetilde{D}_2)$	$(\mathbb{Z}_4)$	(D_1)	$(\widetilde{D}_1)$	$(\mathbb{Z}_2)$	$(\mathbb{Z}_1)$
(D_8)	(D_8)	(D_4)	$(\mathbb{Z}_8)$	$(\widetilde{D}_4)$	(D_2)	$(\widetilde{D}_2)$	$(\mathbb{Z}_4)$	(D_1)	$(\widetilde{D}_1)$	$(\mathbb{Z}_2)$	$(\mathbb{Z}_1)$
(D_4)	(D_4)	$2(D_4)$	$(\mathbb{Z}_4)$	$(\mathbb{Z}_4)$	$2(D_2)$	$(\mathbb{Z}_2)$	$2(\mathbb{Z}_4)$	$2(D_1)$	$(\mathbb{Z}_1)$	$2(\mathbb{Z}_2)$	$2(\mathbb{Z}_1)$
$(\mathbb{Z}_8)$	$(\mathbb{Z}_8)$	$(\mathbb{Z}_4)$	$2(\mathbb{Z}_8)$	$(\mathbb{Z}_4)$	$(\mathbb{Z}_2)$	$(\mathbb{Z}_2)$	$2(\mathbb{Z}_4)$	$(\mathbb{Z}_1)$	$(\mathbb{Z}_1)$	$2(\mathbb{Z}_2)$	$2(\mathbb{Z}_1)$
$(\widetilde{D}_4)$	$(\widetilde{D}_4)$	$(\mathbb{Z}_4)$	$(\mathbb{Z}_4)$	$2(\widetilde{D}_4)$	$(\mathbb{Z}_2)$	$2(\widetilde{D}_2)$	$2(\mathbb{Z}_4)$	$(\mathbb{Z}_1)$	$2(\widetilde{D}_1)$	$2(\mathbb{Z}_2)$	$2(\mathbb{Z}_1)$
(D_2)	(D_2)	$2(D_2)$	$(\mathbb{Z}_2)$	$(\mathbb{Z}_2)$	$(\mathbb{Z}_2)+2(D_2)$	$2(\mathbb{Z}_2)$	$2(\mathbb{Z}_2)$	$(\mathbb{Z}_1)+2(D_1)$	$2(\mathbb{Z}_1)$	$4(\mathbb{Z}_2)$	$4(\mathbb{Z}_1)$
$(\widetilde{D}_2)$	$(\widetilde{D}_2)$	$(\mathbb{Z}_2)$	$(\mathbb{Z}_2)$	$2(\widetilde{D}_2)$	$2(\mathbb{Z}_2)$	$(\mathbb{Z}_2)+2(\widetilde{D}_2)$	$2(\mathbb{Z}_2)$	$2(\mathbb{Z}_1)$	$(\mathbb{Z}_1)+2(\widetilde{D}_1)$	$4(\mathbb{Z}_2)$	$4(\mathbb{Z}_1)$
$(\mathbb{Z}_4)$	$(\mathbb{Z}_4)$	$2(\mathbb{Z}_4)$	$2(\mathbb{Z}_4)$	$2(\mathbb{Z}_4)$	$2(\mathbb{Z}_2)$	$2(\mathbb{Z}_2)$	$4(\mathbb{Z}_4)$	$2(\mathbb{Z}_1)$	$2(\mathbb{Z}_1)$	$4(\mathbb{Z}_2)$	$4(\mathbb{Z}_1)$
(D_1)	(D_1)	$2(D_1)$	$(\mathbb{Z}_1)$	$(\mathbb{Z}_1)$	$(\mathbb{Z}_1)+2(D_1)$	$2(\mathbb{Z}_1)$	$2(\mathbb{Z}_1)$	$3(\mathbb{Z}_1)+2(D_1)$	$4(\mathbb{Z}_1)$	$4(\mathbb{Z}_1)$	$8(\mathbb{Z}_1)$
$(\widetilde{D}_1)$	$(\widetilde{D}_1)$	$(\mathbb{Z}_1)$	$(\mathbb{Z}_1)$	$2(\widetilde{D}_1)$	$2(\mathbb{Z}_1)$	$(\mathbb{Z}_1)+2(\widetilde{D}_1)$	$2(\mathbb{Z}_1)$	$4(\mathbb{Z}_1)$	$3(\mathbb{Z}_1)+2(\widetilde{D}_1)$	$4(\mathbb{Z}_1)$	$8(\mathbb{Z}_1)$
$(\mathbb{Z}_2)$	$(\mathbb{Z}_2)$	$2(\mathbb{Z}_2)$	$2\mathbb{Z}_2)$	$2(\mathbb{Z}_2)$	$4(\mathbb{Z}_2)$	$4(\mathbb{Z}_2)$	$4(\mathbb{Z}_2)$	$4(\mathbb{Z}_1)$	$4(\mathbb{Z}_1)$	$8(\mathbb{Z}_2)$	$8(\mathbb{Z}_1)$
$(\mathbb{Z}_1)$	$(\mathbb{Z}_1)$	$2(\mathbb{Z}_1)$	$2(\mathbb{Z}_1)$	$2(\mathbb{Z}_1)$	$4(\mathbb{Z}_1)$	$4(\mathbb{Z}_1)$	$4(\mathbb{Z}_1)$	$8(\mathbb{Z}_1)$	$8(\mathbb{Z}_1)$	$8(\mathbb{Z}_1)$	$16(\mathbb{Z}_1)$

GAP Code: After selecting the group `G := pDihedralGroup( 8 ),` one needs to label the corresponding generators of $A(G)$. A possibility is `H[1]` $= (\mathbb{Z}_1)$, `H[2]` $= (\mathbb{Z}_2)$, `H[3]` $= (\widetilde{D}_1)$, `H[4]` $= (D_1)$, `H[5]` $= (\mathbb{Z}_4)$, `H[6]` $= (\widetilde{D}_2)$, `H[7]` $= (D_2)$, `H[8]` $= (\widetilde{D}_4)$, `H[9]` $= (\mathbb{Z}_8)$, `H[10]` $= (D_4)$, `H[11]` $= (D_8)$. These generators of the Burnside ring are the conjugacy classes of D_8. The complete GAP code for D_8 would then look as follows.

```
LoadPackage( "EquiDeg" );
G := pDihedralGroup( 8 );
ccs:=ConjugacyClassesSubgroups( G );
ccs_names:=["Z1", "Z2", "tD1", "D1", "Z4", "tD2", "D2", "tD4",
   "Z8", "D4", "D8"];
ListA(ccs, ccs_names, SetName);
AG := BurnsideRing( G );
```

A direct way to identify a conjugacy class representative is using the GAP functions "*Representative*" and "*StructureDescription*", respectively. For example, the group corresponding to the element "*ccs*[9]" of the conjugacy classes list, is identified by the GAP command "$StructureDescription(Representative(ccs[9]));$".

2.6.2 Properties of the Brouwer Equivariant Degree

To define axiomatically the equivariant degree, we need some preliminaries.

Definition 2.6.2. Suppose that V is a finite-dimensional orthogonal $\mathfrak{G}$-representation. Let $\Omega \subset V$ be a $\mathfrak{G}$-invariant open, bounded and nonempty subset, and let $f : V \to V$ be a continuous $\mathfrak{G}$-equivariant function, such that $f(x) \neq 0$, $\forall x \in \partial\Omega$. In this case, (f, Ω) is called a $\mathfrak{G}$-admissible pair.

(a) We denote by $\mathfrak{M}^{\mathfrak{G}}(V)$ the set of all $\mathfrak{G}$-admissible pairs in V, and we put

$$\mathfrak{M}^{\mathfrak{G}} := \bigcup_V \mathfrak{M}^{\mathfrak{G}}(V);$$

(b) one can define a $\mathfrak{G}$-equivariant homotopy as follows. Suppose that $h : [0,1] \times V \to V$ is a $\mathfrak{G}$-equivariant continuous map, and is Ω-admissible for every $t \in [0,1]$ (i.e. the map $h_t(x) := h(t,x)$, $x \in V$, is Ω-admissible). Then, we say that h_0 and h_1 are $\mathfrak{G}$-*equivariantly* Ω-*admissibly homotopic.*

Definition 2.6.3.

(a) f is said to be *normal* in Ω if for each $\alpha = (\mathscr{H}) \in \Phi(\mathfrak{G};\Omega)$ and each $x \in f^{-1}(0) \cap \Omega_{\mathscr{H}}$, there exists a $\delta_x > 0$ such that for all $w \in \nu_x(\Omega_\alpha)$ with $\|w\| < \delta_x$,

$$f(x+w) = f(x) + w = w,$$

where "$\nu_x(\Omega_\alpha)$" stands for the normal space to the manifold Ω_α at x;

(b) f is said to be *regular normal* in Ω if

1. f is C^1-smooth;
2. f is normal in Ω;
3. for every $(\mathscr{H}) \in \Phi(\mathfrak{G},\Omega)$, zero is a regular value of

$$f_{\mathscr{H}} := f|_{\Omega_{\mathscr{H}}} : \Omega_{\mathscr{H}} \to W^{\mathscr{H}}.$$

The role of regular normal maps for equivariant degree theory is parallel to the role of regular maps in nonequivariant Brouwer degree theory. Namely, any equivariant map of our interest can be approximated by a regular normal map. Therefore, the regular normal maps serve as "nice" representatives of equivariant homotopy classes. We describe the axioms of the Brouwer equivariant degree in a parallel way to the axiomatic description of the Brouwer Degree presented in Propositions 2.4.2 and 2.4.3, respectively.

Theorem 2.6.4. *Let* $V := \mathbb{R}^n$, $\Omega \subset V$ *and let* $(f,\Omega) \in \mathfrak{M}^{\mathfrak{G}}$. *There exists a unique function* $\mathfrak{G}\text{-Deg} : \mathfrak{M}^{\mathfrak{G}} \to A(\mathfrak{G})$ *called the Brouwer* $\mathfrak{G}$*-equivariant degree, which assigns to every* (f,Ω)

an element $\mathfrak{G}$-Deg(f,Ω) *in the Burnside ring* $A(\mathfrak{G})$*, i.e.*

$$\mathfrak{G}\text{-Deg}(f,\Omega) = n_1(\mathscr{H}_1) + n_2(\mathscr{H}_2) + \cdots + n_m(\mathscr{H}_m) \tag{2.56}$$

and satisfies the following properties:

(P1) **Existence:** *If* $\mathfrak{G}$-Deg$(f,\Omega) \neq 0$*, i.e if there is* $n_j \neq 0$ *for some* $j \in \{1,2,\dots,m\}$*, then there exists* $x \in \Omega$ *with* $f(x) = 0$ *and* $\mathfrak{G}_x \geq \mathscr{H}_j$*.*

(P2) **Additivity:** *Assume that* Ω_1 *and* Ω_2 *are two* $\mathfrak{G}$*-invariant open disjoint subsets of* Ω *such that* $f^{-1}(0) \cap \Omega \subset \Omega_1 \cup \Omega_2$*. Then,*

$$\mathfrak{G}\text{-Deg}(f,\Omega) = \mathfrak{G}\text{-Deg}(f,\Omega_1) + \mathfrak{G}\text{-Deg}(f,\Omega_2).$$

(P3) **Homotopy:** *Suppose that* $h : [0,1] \times V \to V$ *is an* Ω*-admissible* $\mathfrak{G}$*-equivariant homotopy. Then,*

$$\mathfrak{G}\text{-Deg}(h_t,\Omega) = \text{const}$$

for all $t \in [0,1]$*.*

(P4) **Suspension:** *Suppose that* W *is another orthogonal* $\mathfrak{G}$*-representation and let* U *be an open bounded* $\mathfrak{G}$*-invariant neighborhood of* 0 *in* W*. Then,*

$$\mathfrak{G}\text{-Deg}(f \times \text{Id}, \Omega \times U) = \mathfrak{G}\text{-Deg}(f,\Omega).$$

(P5) **Normalization:** *Suppose that* f *is a regular normal map in* Ω *such that* $f^{-1}(0) \cap \Omega = \mathfrak{G}(x_o)$ *and* $\mathscr{H} := \mathfrak{G}_{x_o}$*. Then,*

$$\mathfrak{G}\text{-Deg}(f,\Omega) = n_o(\mathscr{H}),$$

where

$$n_o := \text{sign det}\left(Df^{\mathscr{H}}(x_o)\right).$$

(P6) **Elimination:** *Suppose that f is a regular normal map in Ω such that $f^{-1}(0) \cap \Omega$ does not contain any orbit type from $\Phi_0(\mathfrak{G}; \Omega)$. Then,*

$$\mathfrak{G}\text{-Deg}(f, \Omega) = 0.$$

(P7) **Excision:** *If $f^{-1}(0) \cap \Omega \subset \Omega_0$, where $\Omega_0 \subset \Omega$ is an open invariant subset, then*

$$\mathfrak{G}\text{-Deg}(f, \Omega) = \mathfrak{G}\text{-Deg}(f, \Omega_0).$$

(P8) **(Recurrence Formula):**

$$\mathfrak{G}\text{-Deg}(f, \Omega) = \sum_{(\mathscr{H}) \in \Phi_0(\mathfrak{G}; V)} n_H \,(\mathscr{H}),$$

where

$$n_H = \frac{\deg(f^{\mathscr{H}}, \Omega^{\mathscr{H}}) - \sum_{(\mathscr{K}) > (\mathscr{H})} n_K \, n(\mathscr{H}, \mathscr{K}) \, |W(\mathscr{K})|}{|W(\mathscr{H})|}. \tag{2.57}$$

(P9) **(Multiplicativity Property):** *Assume that (f_1, Ω_1), $(f_2, \Omega_2) \in \mathfrak{M}^{\mathfrak{G}}$. Then, we have*

$$\mathfrak{G}\text{-Deg}(f_1 \times f_2, \Omega_1 \times \Omega_2) = \mathfrak{G}\text{-Deg}(f_1, \Omega_1) \cdot \mathfrak{G}\text{-Deg}(f_2, \Omega_2), \tag{2.58}$$

where the product "$\cdot$" is taken in the Burnside ring $A(\mathfrak{G})$.

2.6.3 Basic Degree and Degree of Linear Equivariant Operators

Let $\{\mathcal{V}_j\}$ be a list of all irreducible $\mathfrak{G}$-representations, and let $B(\mathcal{V}_j)$ be the unit ball in $\{\mathcal{V}_j\}$. Clearly, every linear $\mathfrak{G}$-equivariant isomorphism $A : \mathcal{V}_j \to \mathcal{V}_j$ is a $B(\mathcal{V}_j)$-admissible $\mathfrak{G}$-equivariant map. Hence, these maps can be considered as the most elementary for which $\mathfrak{G}\text{-Deg}(A, B(\mathcal{V}_j))$ is defined.

Definition 2.6.5. Take a list of irreducible $\mathfrak{G}$-representations $\{\mathcal{V}_j\}$. For any $\mathcal{V}_j$, the map $-\text{Id} : \mathcal{V}_j \to \mathcal{V}_j$ is called a *basic map.* Then, the Brouwer $\mathfrak{G}$-equivariant degree

$$\deg_{\mathcal{V}_j} := \mathfrak{G}\text{-Deg}(-\text{Id}\,, B(\mathcal{V}_j))$$

is called the $\mathcal{V}_j$*-basic Brouwer* $\mathfrak{G}$*-degree*, or simply *basic Brouwer* $\mathfrak{G}$*-degree*.

Let V be a finite-dimensional $\mathfrak{G}$-representation. Take $A \in GL^{\mathfrak{G}}(V)$ and consider the $\mathfrak{G}$-isotypic decomposition of V

$$V = V_0 \oplus V_1 \oplus \ldots \oplus V_r.$$

By the multiplicativity property (2.58), we have

$$\mathfrak{G}\text{-Deg}(A, B(V)) = \prod_{j=1}^{r} \mathfrak{G}\text{-Deg}(A_j, B(V_j)),$$

where $B(V_j)$ is the unit ball in V_j and $A_j := A|_{V_j} : V_j \to V_j$. Next, keeping in mind formulas (2.50) and (2.53), denote by $\sigma_-(A)$ the set of all negative real eigenvalues of the operator A, choose $\mu \in \sigma_-(A)$ and let

$$E(\mu) := \bigcup_{k=1}^{\infty} \ker(A - \mu \mathrm{Id})^k$$

denote the generalized eigenspace of A corresponding to μ. Put

$$m_j(\mu) := \dim(E(\mu) \cap V_j)/\dim \mathcal{V}_j,$$

and call it the $\mathcal{V}_j$-isotypic multiplicity of the eigenvalue μ of A. Then, by using (2.58), we have the following result.

Theorem 2.6.6. *Let V be an orthogonal $\mathfrak{G}$-representation with the isotypic decomposition* (2.6.3) *and let $A \in GL^{\mathfrak{G}}(V)$. Then,*

$$\mathfrak{G}\text{-Deg}(A, B(V)) = \prod_{\mu \in \sigma_-(A)} \prod_{j=0}^{r} (\deg_{\mathcal{V}_j})^{m_j(\mu)},$$

where the product is taken in $A(\mathfrak{G})$.

Take an irreducible $\mathfrak{G}$-representation $\mathcal{V}_j$. Since $\mathrm{Id} \times \mathrm{Id} : \mathcal{V}_j \oplus \mathcal{V}_j \to \mathcal{V}_j \oplus \mathcal{V}_j$ is $B(\mathcal{V}_j \oplus \mathcal{V}_j)$-admissible $\mathfrak{G}$-homotopic to $\mathrm{Id}|_{\mathcal{V}_j \oplus \mathcal{V}_j}$, by using formula (2.58), one obtains for the basic degree $\deg_{\mathcal{V}_j}$:

$$\left(\deg_{\mathcal{V}_j} \right)^2 = (\mathfrak{G}). \tag{2.59}$$

The identity (2.59) is valid for any $\mathcal{V}_j$-basic degree $\deg_{\mathcal{V}_j}$. It leads to the following consequence.

Corollary 2.6.7. *Let V be an orthogonal $\mathfrak{G}$-representation and suppose that $A : V \to V$ is a $\mathfrak{G}$-equivariant linear isomorphism and $\Omega := B_1(0)$ is the unit ball in V. Then, we have*

$$\mathfrak{G}\text{-Deg}(A, \Omega) = \prod_{\mu \in \sigma_-(A)} \prod_{j=0}^{r} \left(\deg_{\mathcal{V}_j} \right)^{\varepsilon_j(\mu)}, \tag{2.60}$$

where

$$\varepsilon_j(\mu) := \begin{cases} 1 & \text{if } m_j(\mu) \text{ is odd} \\ 0 & \text{if } m_j(\mu) \text{ is even.} \end{cases}$$

Formula (2.60) reduces the computation of an equivariant linear map to the computation of the basic degree. In turn, the computation of the basic degree can be done using formula (2.57). Below we give a few examples (cf. [9]).

Example 2.6.8. In the following we compute the basic degree related to the real irreducible representations of D_8 described in Example 2.2.4. We will use the notations introduced in that example.

(a): Take $\mathcal{M}_0$ from Example 2.2.4. Then, the only orbit type is (D_8). Hence, we have $\mathcal{M}_0^{\mathscr{H}} = \mathcal{M}_0 \cong \mathbb{R}$, and

$$\deg(f^{\mathscr{H}}, B(\mathcal{M}_0)) = \deg(-\mathrm{Id}\,, B(\mathbb{R})) = -1.$$

Thus, we obtain

$$\deg_{\mathcal{M}_0} = -(D_8).$$

(b): To compute the basic degree of the orthogonal representations $\mathcal{M}_1$ and $\mathcal{M}_3$ (see Example 2.2.4) of D_8, one needs the set of orbit types for the action of D_8 on $\mathbb{C} = \mathbb{R}^2$. This set is given by

$$\Phi_0(D_8; \mathcal{M}_1) = \{(D_8),\ (D_1),\ (\widetilde{D}_1),\ (\mathbb{Z}_1)\} = \Phi_0(D_8; \mathcal{M}_3),$$

see [9]. On the other hand,

$$
\begin{gathered}
\dim \mathcal{M}_1^{D_1} = \dim \mathcal{M}_3^{D_1} = 1, \quad \dim \mathcal{M}_1^{\widetilde{D}_1} = \dim \mathcal{M}_j^{D_3} = 1, \quad \dim \mathcal{M}_j^{\mathbb{Z}_1} = 2, \\
\dim \mathcal{M}_2^{D_2} = \dim \mathcal{M}_2^{\widetilde{D}_2} = 1, \quad \dim \mathcal{M}_j^{\mathbb{Z}_2} = 1.
\end{gathered} \tag{2.61}
$$

We have $\mathfrak{G}\text{-Deg}_{\mathcal{M}_1} = \mathfrak{G}\text{-Deg}_{\mathcal{M}_3} = x(D_8) - y(D_1) - z(\widetilde{D}_1) + w(\mathbb{Z}_1)$, where clearly $x = 1$,

$$
\begin{aligned}
y &= \frac{\deg(-\mathrm{Id}\,, B(\mathcal{M}_1^{D_1})) - x}{|W(D_1)|} = \frac{-1-1}{2} = -1, \\
z &= \frac{\deg(-\mathrm{Id}\,, B(\mathcal{M}_1^{\widetilde{D}_1})) - x}{|W(\widetilde{D}_1)|} = \frac{-1-1}{2} = -1,
\end{aligned}
$$

and

$$
\begin{aligned}
w &= \frac{\deg(-\mathrm{Id}\,, B(\mathcal{M}_1)) - x - yn(\mathbb{Z}_1, D_1)|W(D_1)| - zn(\mathbb{Z}_1, \widetilde{D}_1)|W(\widetilde{D}_1)|}{|W(\mathbb{Z}_1)|} \\
&= \frac{1 - 1 + 1 \cdot 4 \cdot 2 + 1 \cdot 4 \cdot 2}{16} = 1,
\end{aligned}
$$

where $n(\mathbb{Z}_1, D_1)$ and $n(\mathbb{Z}_1, \widetilde{D}_1)$ have been calculated in Table 2.3. Therefore, we obtain

$$
\deg_{\mathcal{M}_1} = \deg_{\mathcal{M}_3} = (D_8) - (D_1) - (\widetilde{D}_1) + (\mathbb{Z}_1),
$$

see [9]. To compute the basic degree of the orthogonal representation $\mathcal{M}_2$, the set of orbit types is given by

$$
\Phi_0(D_8, \mathcal{M}_2) = \{(D_8), (D_2), (\widetilde{D}_2), (\mathbb{Z}_2)\}, \tag{2.62}
$$

see [9]. Then, $\deg_{\mathcal{M}_2}$ can be calculated using (2.61) and (2.62), and one obtains $\deg_{\mathcal{M}_2} = x(D_8) - y(D_2) - z(\widetilde{D}_2) + w(\mathbb{Z}_2)$, where $x = 1$,

$$
\begin{aligned}
y &= \frac{\deg(-\mathrm{Id}\,, B(\mathcal{M}_2^{D_2})) - x}{|W(D_2)|} = \frac{-1-1}{2} = -1, \\
z &= \frac{\deg(-\mathrm{Id}\,, B(\mathcal{M}_2^{\widetilde{D}_2})) - x}{|W(\widetilde{D}_2)|} = \frac{-1-1}{2} = -1,
\end{aligned}
$$

and

$$
\begin{aligned}
w &= \frac{\deg(-\mathrm{Id}\,, B(\mathcal{M}_2)) - x - yn(\mathbb{Z}_2, D_2)|W(D_2)| - zn(\mathbb{Z}_2, \widetilde{D}_2)|W(\widetilde{D}_2)|}{|W(\mathbb{Z}_2)|} \\
&= \frac{1 - 1 + 1 \cdot 2 \cdot 2 + 1 \cdot 2 \cdot 2}{8} = 1,
\end{aligned}
$$

where $n(\mathbb{Z}_2, D_2)$ and $n(\mathbb{Z}_2, \widetilde{D}_2)$ have been calculated in Table 2.3. Therefore, we obtain

$$\deg_{\mathcal{M}_2} = (D_8) - (D_2) - (\widetilde{D}_2) + (\mathbb{Z}_2),$$

see [9].

(c): The degree of the one-dimensional representation $\mathcal{M}_{j_8}$, given by the homomorphism $c : D_8 \to \mathbb{Z}_2$, with $\ker c = \mathbb{Z}_8$ can be calculated as follows. In this case, since $\Phi_0(D_8; \mathcal{M}_{j_8}) = \{(D_8), (\mathbb{Z}_8)\}$ the basic degree is

$$\deg_{\mathcal{M}_{j_8}} = (D_8) - (\mathbb{Z}_8),$$

see [9].

(d): For the irreducible one-dimensional representation $\mathcal{M}_{j_8+1}$ given by $d : D_8 \to \mathbb{Z}_2$ with $\ker d = D_4$, we have the basic degree

$$\deg_{\mathcal{M}_{j_8+1}} = (D_8) - (D_4),$$

see [9].

(*e*): Finally, for the irreducible one-dimensional representation $\mathcal{M}_{j_8+2}$ given by $\hat{d} : D_8 \to \mathbb{Z}_2$, where $\ker \hat{d} = \widetilde{D}_4$, and

$$\deg_{\mathcal{M}_{j_8+2}} = (D_8) - (\widetilde{D}_4),$$

see [9].

Example 2.6.9. Here we present the basic degrees for irreducible representations of the group $\mathfrak{G} := D_8 \times \mathbb{Z}_2$. It is convenient to denote the irreducible $\mathfrak{G}$-representations as follows:

• the irreducible D_8-representations, where $\mathbb{Z}_2$ acts trivially, i.e. $\mathcal{M}_0$, $\mathcal{M}_j, \mathcal{M}_{j_8}$, $\mathcal{M}_{j_8+1}$ and $\mathcal{M}_{j_8+2}$, as described in Example 2.6.8.

• the irreducible D_8-representations with non-trivial $\mathbb{Z}_2$-action on which $\mathbb{Z}_2$ acts antipodally, namely $\mathcal{M}_j^-, \mathcal{M}_{j_8}^-$, $\mathcal{M}_{j_8+1}^-$ and $\mathcal{M}_{j_8+2}^-$.

Case 1: For $\mathbb{Z}_2$ acting trivially, we immediately have the basic degrees from Example 2.6.8.

- $\mathcal{M}_0 \cong \mathbb{R}$ (trivial representation) with $\deg_{\mathcal{M}_0} = -(D_8 \times \mathbb{Z}_2)$;
- the one-dimensional representations $\mathcal{M}_{j_8}, \mathcal{M}_{j_8+1}, \mathcal{M}_{j_8+2}$, with

$$\deg_{\mathcal{M}_{j_8}} = (D_8 \times \mathbb{Z}_2) - (\mathbb{Z}_8 \times \mathbb{Z}_2),$$

$$\deg_{\mathcal{M}_{j_8+1}} = (D_8 \times \mathbb{Z}_2) - (D_4 \times \mathbb{Z}_2),$$

$$\deg_{\mathcal{M}_{j_8+2}} = (D_8 \times \mathbb{Z}_2) - (\widetilde{D}_4 \times \mathbb{Z}_2);$$

- the two-dimensional representations $\mathcal{M}_1, \mathcal{M}_2, \mathcal{M}_3$, with

$$\deg_{\mathcal{M}_1} = \deg_{\mathcal{M}_3} = (D_8 \times \mathbb{Z}_2) - (D_2) - (\widetilde{D}_2) - (\mathbb{Z}_2),$$

$$\deg_{\mathcal{M}_2} = (D_8 \times \mathbb{Z}_2) - (D_2 \times \mathbb{Z}_2) - (\widetilde{D}_2 \times \mathbb{Z}_2) - (D_2);$$

Case 2: For $\mathbb{Z}_2$ acting antipodally, we have:

- the one-dimensional representations $\mathcal{M}_{j_8}, \mathcal{M}_{j_8+1}, \mathcal{M}_{j_8+2}$,

$$\begin{aligned} \deg_{\mathcal{M}_0^-} &= (D_8 \times \mathbb{Z}_2) - (D_8), \\ \deg_{\mathcal{M}_{j_8+1}^-} &= (D_8 \times \mathbb{Z}_2) - (D_8^z), \\ \deg_{\mathcal{M}_{j_8+2}^-} &= (D_8 \times \mathbb{Z}_2) - (D_8^d), \\ \deg_{\mathcal{M}_{j_8+2}^-} &= (D_8 \times \mathbb{Z}_2) - (\widetilde{D}_8^d), \end{aligned} \tag{2.63}$$

  and we refer to [9], Section 5.3 for the structure of the groups $D_8^d, D_8^z, \widetilde{D}_8^d$.

- the two-dimensional representations $\mathcal{M}_1^-, \mathcal{M}_2^-, \mathcal{M}_3^-$,

$$\deg_{\mathcal{M}_1^-} = \deg_{\mathcal{M}_3} = (D_8 \times \mathbb{Z}_2) - (D_2^z) - (\widetilde{D}_2^d) - (\mathbb{Z}_2),$$

$$\deg_{\mathcal{M}_2^-} = (D_8 \times \mathbb{Z}_2) - (D_4) - (\widetilde{D}_4) - (\mathbb{Z}_4),$$

  and we refer to [9], Section 5.3 for the structure of the groups $D_8^d, D_8^z, \widetilde{D}_8^d$. See [43] for the EquiDeg package.

2.6.4 Leray-Schauder Equivariant Degree

The construction of the Leray-Schauder equivariant degree is similar to the nonequivariant case. The only difference is the usage of equivariant projection. Let V be a Banach space and let $T : V \to V$ be an equivariant compact map. Define a finite set $\mathcal{S} \subseteq C$ (where C is a closed, convex and $\mathfrak{G}$-invariant subset of V), $B(\epsilon, \mathcal{S})$, μ_i and π_ϵ as in Construction 2.5.2. Take a real number $\epsilon > 0$ such that $\|f(x)\| > \epsilon$ for $x \in \partial\Omega$, and use the Haar measure to construct the equivariant projection map $p_\epsilon : \overline{\Omega} \to C$, defined by

$$p_\epsilon(x) := \int_{\mathfrak{G}} g\pi_\epsilon(g^{-1}x)d\mu(g).$$

Then, there exists a finite-dimensional map $T_\epsilon : \overline{\Omega} \to C$, satisfying

(a) $\|T_\epsilon(x) - T(x)\| < \epsilon \ \forall x \in \overline{\Omega}$

(b) $T_\epsilon(\overline{\Omega}) \subseteq \operatorname{conv}(\mathcal{S}) \subseteq C$.

Then, the equivariant Schauder approximation theorem can be formulated as follows.

Theorem 2.6.10 (Equivariant Schauder's Approximation Theorem)**.** *Let $\Omega \subset V$ be a $\mathfrak{G}$-invariant bounded subset and $T : \overline{\Omega} \to V$ a $\mathfrak{G}$-equivariant compact map. Then, for every $\epsilon > 0$, there exists a $\mathfrak{G}$-equivariant finite-dimensional map $T_\epsilon : \overline{\Omega} \to V$ (i.e. the image $T_\epsilon(\Omega)$ is contained in a finite-dimensional subrepresentation of V) such that*

$$\|T_\epsilon(x) - T(x)\| < \epsilon \text{ for all } x \in \Omega.$$

Then, the Leray-Schauder equivariant degree is defined using formula (2.52) where $f_\epsilon := x - T_\epsilon$, $\mathcal{S}$, $\operatorname{conv}(\mathcal{S})$ and C are defined in Construction 2.5.2.

The Leray-Schauder equivariant degree satisfies properties analogues to those of the Brouwer equivariant degree, listed in Theorem 2.6.4.

2.7 Gauss Curvature

Let $\mathbf{V} := \mathbb{R}^n$ and let $\eta : \mathbf{V} \to \mathbb{R}$ be a C^2-smooth function such that 0 is a regular value of η. Take the smooth submanifold

$$C := \partial D = \eta^{-1}(0). \tag{2.64}$$

Recall the definition of the Gauss curvature of C. For every $x \in C$, denote by n_x the outer normal vector to C at x i.e.

$$n_x = \frac{\nabla \eta(x)}{|\nabla \eta(x)|}, \tag{2.65}$$

and let $\nu : C \to S^{n-1}$ be the *Gauss map* given by $\nu(x) := n_x$. Obviously, for any $x \in C$, the tangent spaces $T_x(C)$ and $T_{n_x}(S^{n-1})$ are parallel, and as such can be identified. This way, for any $x \in C$, the tangent map $d\nu_x$ (as well as its negative known as a *Weingarten map* or *shape operator* (see, for example, [42])) can be considered as a linear map from $T_x(C)$ into itself. The function $\kappa(x) := \det(-d\nu(x))$ is called the *Gauss curvature* of C. It is well-known that $-d\nu_x$ is a self-adjoint operator with respect to the standard inner product $\langle \cdot, \cdot \rangle$ in $\mathbf{V}$. The quadratic form associated with $-d\nu_x$ and denoted $\mathbb{I}_x(v) := -\langle d\nu_x(v), v \rangle$ is called the *second fundamental form* of C.

The sign of the Gauss curvature allows one to characterize the surface.

If the Gauss curvature is positive (resp. negative, zero) at certain point x, then x is said to be an *elliptic point* (resp. *hyperbolic point*, *parabolic point*).

It is useful to recall computational formulas for the Gauss curvature for different settings. An explanation about the curvature sign computed by these formulas can be found in [22].

Remark 2.7.1. The curve $\eta(x, y) = 0$ is identical to the curve $c\eta(x, y) = 0$, $\forall c \neq 0$. Therefore one expects that the curvature of $\eta(x, y) = 0$ should be the same as the curvature of $c\eta(x, y) = 0$. This is true whenever $c > 0$, and in this case the sign of the curvature remains the same. If however, one replaces η by $-\eta$, the sign of the numerator in (2.66) changes while the sign of the denominator remains unchanged; therefore, the sign of the curvature changes.

Replacing η by $-\eta$ also changes the direction of the unit normal vector. Similar reasoning applies to the Gauss curvature formula for implicit surfaces (2.68).

1. First, consider the case $n = 2$. Let C be given by (2.64). Since the gradient $\nabla\eta = (\eta_x, \eta_y)$ of $\eta(x, y)$ is perpendicular to the level curve $\eta(x, y) = 0$, $\nabla\eta$ is parallel to the normal of $\eta(x, y) = 0$. The unit normal at (x, y) is

$$\nu(x, y) = \frac{\nabla\eta}{\|\nabla\eta\|} = \frac{(\eta_x, \eta_y)}{\sqrt{\eta_x^2 + \eta_y^2}},$$

and one obtains for the Gauss curvature

$$\kappa = \frac{-\eta_y^2\eta_{xx} + 2\eta_x\eta_y\eta_{xy} - \eta_x^2\eta_{yy}}{(\eta_x^2 + \eta_y^2)^{\frac{3}{2}}} \tag{2.66}$$

see [22], [46].

Example 2.7.2.

(a) Take the circle $\eta(x, y) = x^2 + y^2 - R^2 = 0$. Applying formula (2.66) yields $\kappa = 1/R$ and the normal vector $\nabla\eta = (2x, 2y)$ is pointing outward (see Remark 2.7.1). All points are elliptic.

(b) Let $\eta = xy - R = 0$. Formula (2.66) yields

$$\kappa = \frac{2Rx^3}{(R^2 + x^4)^{\frac{3}{2}}}.$$

All points with $x < 0$ are hyperbolic points.

(c) Let $\eta = y + ax + b = 0,\ a, b \in \mathbb{R}$. Then, formula (2.66) yields $\kappa = 0$. Clearly, all points are parabolic.

A special case is when the curve is given explicitly as a graph of a function, i.e. $y = f(x)$. Take $\eta = x^2 - y = 0$. Then, if $f'(x)$ and $f''(x)$ exist, formula (2.66) gives

$$\kappa = \frac{f''(x)}{(1 + (f'(x))^2)^{\frac{3}{2}}}.$$

Take for example, $\eta = y - x^2 = 0$. Then, formula (2.66) yields

$$\kappa = -\frac{2}{(1+(f'(x))^2)^{\frac{3}{2}}}.$$

If a two-dimensional curve $r = r(\theta)$ is given in polar coordinates, then the Gauss curvature is

$$\kappa = \frac{r^2 + 2r_\theta^2 - rr_{\theta\theta}}{(r^2 + r_\theta^2)^{\frac{3}{2}}}. \tag{2.67}$$

Example 2.7.3. Let $r(\theta) \equiv R$, where the equation of the circle $\eta = x^2 + y^2 - R^2 = 0$ has been rewritten in polar coordinates using $x = r\cos(\theta),\ y = r\sin(\theta)$. Then, formula (2.67) yields

$$\kappa = 1/R.$$

2. The expression for the Gauss curvature of implicit 2-dimensional surfaces

$$f(x, y, z) = 0,$$

is given by

$$\kappa = -\frac{\begin{vmatrix} H(\eta) & \nabla\eta^T \\ \nabla\eta & 0 \end{vmatrix}}{|\nabla\eta|^4} = -\frac{\begin{vmatrix} \eta_{xx} & \eta_{xy} & \eta_{xz} & \eta_x \\ \eta_{xy} & \eta_{yy} & \eta_{yz} & \eta_y \\ \eta_{xz} & \eta_{yz} & \eta_{zz} & \eta_z \\ \eta_x & \eta_y & \eta_z & 0 \end{vmatrix}}{|\nabla\eta|^4}, \tag{2.68}$$

where H is the Hessian matrix [22].

Example 2.7.4. (a) Take a plane $\eta = ax + by + cz = 0$. Then, formula (2.68) yields $\kappa = 0$. All points are parabolic.

(b) Take a cylinder $\eta = x^2 + y^2 - R = 0$. Then, formula (2.68) yields $\kappa = 0$. All points are parabolic.

(c) Take a sphere $\eta = x^2 + y^2 + z^2 - R^2 = 0$. Then, formula (2.68) yields $\kappa = 1/R^2$. All points are elliptic (see Remark 2.7.1).

(d) Let $\eta = \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} - \dfrac{z^2}{c^2} - 1, 0 < a < b$ and $c > 0$. Then, formula (2.68) yields

$$\kappa = -\frac{1}{a^2b^2c^2}\frac{1}{\left(\frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4}\right)^2}.$$

Clearly, all points are hyperbolic.

A special case is the explicit surfaces

$$z = f(x, y).$$

Let $\eta = f(x, y) - z$. Then, the Gauss curvature is

$$\kappa = \frac{f_{xx}f_{yy} - 2f_{xy}^2}{(1 + f_x^2 + f_y^2)^2}, \tag{2.69}$$

(see [22], [41]).

CHAPTER 3

RESULTS

The material of this chapter is taken from the paper "Periodic Solutions to Reversible Second Order Autonomous DDEs in Prescribed Symmetric Nonconvex Domains", by Z. Balanov, N. Hirano, W. Krawcewicz, F. Liao and A.C. Murza, published in Nonlinear Differ. Equ., **24**, p. 40-76 (2021).

3.0.1 A priori bound for the first derivative

In this subsection, we establish a priori bounds for the first and second derivatives of solutions to problem (1.2) living in D. The lemma following below can be traced back to [27], where the case of ODEs was studied for $D = B_R(0)$. In our proof, we combine the ideas from [1] (where Hartman's result was extended to arbitrary D) with [4] (where the case of equivariant DDEs and $D = B_R(0)$ was considered). To simplify our notations, given a function $x : \mathbb{R} \to \mathbf{V}$, put $\mathbf{x}_t := (x(t-\tau_1), \dots, x(t-\tau_{m-1}))$, so that we are interested in the problem

$$\begin{cases} \ddot{x}(t) = f(x(t), \mathbf{x}_t, \dot{x}(t)), \quad t \in \mathbb{R},\ x(t) \in \overline{D} \subset \mathbf{V} = \mathbb{R}^n, \\ x(t) = x(t+p), \quad \dot{x}(t) = \dot{x}(t+p), \end{cases} \tag{3.1}$$

where $p := 2\pi$.

Lemma 3.0.1. *Let $\eta : \mathbf{V} \to \mathbb{R}$ satisfy (η_1), (η_4)–(η_6), and let $f : \mathbf{V} \times \mathbf{V}^{m-1} \times \mathbf{V} \to \mathbf{V}$ be a continuous map satisfying (A_5) and (A_6) (resp. (A_5) and (A_6')). If $x = x(t)$ is a solution to* (3.1) *such that $|x(t)| \le R$ for $t \in \mathbb{R}$, then there exists a constant $M := M(\phi, \eta, K, p, R)$ (resp. $M := M(\phi, \alpha, K, p, R)$) such that*

$$\forall_{t \in \mathbb{R}} \quad |\dot{x}(t)| \le M. \tag{3.2}$$

Proof: We only prove Lemma 3.0.1 assuming that f satisfies (A_5) and (A_6). The case when f satisfies (A_5) and (A_6') was treated in [6] (see also Remark 3.0.2).

Let $x = x(t)$ be a C^2-smooth solution to (3.1). Since $|x(t)| \le R$, one has $|x(t-\tau_k)| \le R$ for all $k = 1, ..., m-1$, so that $|\mathbf{x}_t| \le R$. Put $\boldsymbol{\eta}(t) := \eta(x(t))$, $t \in \mathbb{R}$. Then by (A_6), one has

$$\begin{aligned}|\ddot{x}(t)| = |f(x(t), \mathbf{x}_t, \dot{x}(t))| &\le \nabla^2\eta(x(t))(\dot{x}(t), \dot{x}(t)) + \langle f(x(t), \mathbf{x}_t, \dot{x}(t)), \nabla\eta(x(t))\rangle + K \\ &= \boldsymbol{\eta}''(t) + K.\end{aligned}$$

Thus,

$$\forall_{t\in\mathbb{R}} \quad |\ddot{x}(t)| \le \boldsymbol{\eta}''(t) + K. \tag{3.3}$$

Next, by using integration by parts and the fact that $x(t)$ is p-periodic, one calculates:

$$\begin{aligned}\textstyle\int_t^{t+p}(t+p-s)\ddot{x}(s)ds = (t+p-s)\ddot{x}(s)\Big|_t^{t+p} + \\ \textstyle\int_t^{t+p}\dot{x}(s)ds = x(t+p) - x(t) - p\dot{x}(t) = -p\dot{x}(t)\end{aligned}$$

i.e.

$$\forall_{t\in\mathbb{R}} \quad p\dot{x}(t) = -\int_t^{t+p} (t+p-s)\ddot{x}(s)ds. \tag{3.4}$$

Similarly,

$$p\dot{x}(t) = x(t) - x(t-p) - \int_{t-p}^{t}(t-p-s)\ddot{x}(s)ds = -\int_{t-p}^{t}(t-p-s)\ddot{x}(s)ds,$$

i.e.

$$p\dot{x}(t) = -\int_{t-p}^{t}(t-p-s)\ddot{x}(s)ds. \tag{3.5}$$

Then by (3.4), one obtains

$$p\dot{x}(0) \ = \ -\int_0^p (p-s)\ddot{x}(s)ds,$$

and by (3.3) and p-periodicity of x, one has:

$$\begin{aligned}p|\dot{x}(0)| \ &\le \ \int_0^p (p-s)|\ddot{x}(s)|ds \le \int_0^p (p-s)\Big(\boldsymbol{\eta}''(s) + K\Big)ds \\ &= \int_0^p (p-s)\boldsymbol{\eta}''(s)ds + K\int_0^p (p-s)ds = -p\boldsymbol{\eta}'(0) + \frac{1}{2}Kp^2,\end{aligned}$$

i.e.

$$p|\dot{x}(0)| \ \leq -p\boldsymbol{\eta}'(0) + \frac{1}{2}Kp^2. \tag{3.6}$$

Similarly, by (3.5), one obtains

$$p|\dot{x}(0)| \leq p\boldsymbol{\eta}'(0) + \frac{1}{2}Kp^2. \tag{3.7}$$

By adding inequalities (3.6) and (3.7), one obtains

$$2p|\dot{x}(0)| \leq Kp^2 \quad \Leftrightarrow \quad |\dot{x}(0)| \leq \frac{1}{2}Kp. \tag{3.8}$$

Moreover (see (3.4) and (3.3)), one has:

$$p|\dot{x}(t)| \leq \int_t^{t+p} (t+p-s)|\ddot{x}(s)|ds \leq \int_t^{t+p} (t+p-s)\Big(\boldsymbol{\eta}''(s) + K\Big)ds = -p\boldsymbol{\eta}'(t) + \frac{1}{2}Kp^2.$$

The last inequality, together with condition (A_5) imply

$$\frac{\langle \dot{x}(t), \ddot{x}(t)\rangle}{\phi(|\dot{x}(t)|)} \leq \frac{|\langle \dot{x}(t), \ddot{x}(t)\rangle|}{\phi(|\dot{x}(t)|)} \leq \frac{|\dot{x}(t)||\ddot{x}(t)|}{\phi(|\dot{x}(t)|)} \leq |\dot{x}(t)| \leq \frac{1}{2}Kp - \boldsymbol{\eta}'(t). \tag{3.9}$$

Next, by integrating inequality (3.9), one obtains for $t \in [0, p]$:

$$\left|\int_0^t \frac{\langle \dot{x}(s), \ddot{x}(s)\rangle}{\phi(|\dot{x}(t)|}ds\right| \leq \int_0^t \left[\frac{1}{2}Kp - \boldsymbol{\eta}'(s)\right] ds = \frac{1}{2}Kpt - \boldsymbol{\eta}(t) + \boldsymbol{\eta}(0) \leq \frac{K}{2}p^2 + 2\widetilde{R}, \tag{3.10}$$

where $\widetilde{R} := \max\{|\eta(x)| : x \in \overline{D}\}$. On the other hand, by making substitution $u = |\dot{x}(s)|$, one obtains:

$$\int_0^t \frac{\langle \dot{x}(s), \ddot{x}(s)\rangle}{\phi(|\dot{x}(s)|)}ds = \int_{|\dot{x}(0)|}^{|\dot{x}(t)|} \frac{udu}{\phi(u)}. \tag{3.11}$$

Put $\Phi(w) := \displaystyle\int_0^w \frac{udu}{\phi(u)}$, then

$$\left|\int_{|\dot{x}(0)|}^{|\dot{x}(t)|} \frac{udu}{\phi(u)}\right| = |\Phi(|\dot{x}(t)|) - |\Phi(|\dot{x}(0)|)|. \tag{3.12}$$

Therefore (cf. (3.10)–(3.12)),

$$|\Phi(|\dot{x}(t)|) - \Phi(|\dot{x}(0)|)| \leq \frac{K}{2}p^2 + 2\widetilde{R},$$

in particular,

$$\Phi(|\dot{x}(t)|) \leq \frac{1}{2}Kp^2 + 2\widetilde{R} + \Phi(|\dot{x}(0)|)|. \tag{3.13}$$

By (A_5), $\lim\limits_{w\to\infty} \Phi(w) = \infty$, hence, $\Phi : [0,\infty) \to [0,\infty)$ is a continuous monotonic bijective function. Therefore (see (3.8) and (3.13)), the inequality

$$\Phi(|\dot{x}(t)|) \leq \frac{1}{2}Kp^2 + 2\widetilde{R} + \Phi\left(\frac{1}{2}Kp\right),$$

implies

$$|\dot{x}(t)| \leq \Phi^{-1}\left[\frac{1}{2}Kp^2 + 2\widetilde{R} + \Phi\left(\frac{1}{2}Kp\right)\right] =: M, \tag{3.14}$$

and the required estimate follows. □

Remark 3.0.2. Observe that if (A_6) is replaced with (A_6'), then the following estimate for $\dot{x}(t)$ was established in [6]:

$$|\dot{x}(t)| \leq \Phi^{-1}\left[\frac{1}{2}Kp^2 + \alpha R^2 + \Phi\left(\frac{1}{2}Kp\right)\right] =: M \tag{3.15}$$

One has the following immediate consequence of Lemma 3.0.1.

Lemma 3.0.3. *Under the assumptions of Lemma 3.0.1, there exists $N > 0$ such that for any C^2-smooth solution $x = x(t)$ to* (3.1)*, one has*

$$\forall_{t\in\mathbb{R}} \quad |\ddot{x}(t)| \leq N. \tag{3.16}$$

Proof: *Let M be a constant provided by Lemma 3.0.1. Put*

$$N := \max_{x\in\overline{D},\, \mathbf{y}\in\overline{D}^{m-1},\, |z|\leq M} |f(x,\mathbf{y},z)|. \tag{3.17}$$

Let $x = x(t)$ be a C^2-smooth solution to (3.1)*. Then,*

$$\forall_{t\in\mathbb{R}} \quad |\ddot{x}(t)| = |f(x(t),\mathbf{x}_t,\dot{x}(t)| \leq N.$$

□

3.0.2 C-Touching

In this subsection, essentially following [1] and [15], we show that solutions to problem (3.1) cannot touch $C := \partial D$ provided that f satisfies (A_4). More precisely,

Lemma 3.0.4. *Let $\eta : \mathbf{V} \to \mathbb{R}$ satisfy (η_1), (η_4)–(η_6), and let $f : \mathbf{V} \times \mathbf{V}^{m-1} \times \mathbf{V} \to \mathbf{V}$ be a continuous map satisfying (A_4). Let $x : \mathbb{R} \to \mathbf{V}$ be a C^2-smooth 2π-periodic function such that:*

(i) $x(t) \in \overline{D}$ for all $t \in \mathbb{R}$;

(ii) $x(t_o) \in C$ for some $t_o \in \mathbb{R}$.

Then, x is not a solution to problem (3.1)*.*

Proof: Assume for contradiction that $x : \mathbb{R} \to \mathbf{V}$ is a C^2-smooth 2π-periodic solution to problem (3.1) satisfying (i) and (ii). Take a tubular neighborhood of C around the point $x(t_o) \in C$. Then, for a sufficiently small $\varepsilon > 0$, one can represent x as follows:

$$x(t) = \alpha(t) + \beta(t)\nu(\alpha(t)) \qquad (\alpha(t) \in C, \ \ \beta(t) \le 0, \ \ t \in (t_o - \varepsilon, t_o + \varepsilon)). \tag{3.18}$$

Combining (3.18) with the fact that $x = x(t)$ is a solution to (3.1) and using $n_{\alpha(t)} \perp \dot{\alpha}(t)$ and $n_{\alpha(t)} \perp \frac{d}{dt}(\nu(\alpha(t)))$, one obtains:

$$\begin{aligned}
\langle f(x(t), \mathbf{x}_t, \dot{x}(t)), n_{\alpha(t)} \rangle &= \langle \ddot{x}(t), n_{\alpha(t)} \rangle = \tfrac{d}{dt}\langle \dot{x}(t), n_{\alpha(t)} \rangle - \langle \tfrac{d}{dt}\nu(\alpha(t)), \dot{x}(t) \rangle \\
&= \frac{d}{dt}\Big\langle \frac{d}{dt}\Big[\alpha(t) + \beta(t)\nu(\alpha(t))\Big], n_{\alpha(t)} \Big\rangle - \langle \tfrac{d}{dt}\nu(\alpha(t)), \dot{x}(t) \rangle \\
&= \frac{d}{dt}\Big\{ \langle \dot{\alpha}(t), n_{\alpha(t)} \rangle + \dot{\beta}(t)\langle \nu(\alpha(t), n_{\alpha(t)} \rangle + \\
&\qquad\qquad \beta(t)\langle \tfrac{d}{dt}\nu(\alpha(t)), n_{\alpha(t)} \rangle \Big\} - \langle \tfrac{d}{dt}\nu(\alpha(t)), \dot{x}(t) \rangle \\
&= \frac{d}{dt}\Big\{ \dot{\beta}(t)\|n_{\alpha(t)}\|^2 \Big\} - \langle \tfrac{d}{dt}\nu(\alpha(t)), \dot{x}(t) \rangle \\
&= \ddot{\beta}(t) - \langle \tfrac{d}{dt}\nu(\alpha(t)), \dot{x}(t) \rangle.
\end{aligned}$$

Since $\beta(t)$ achieves its local maximum at t_o, one has $\ddot{\beta}(t_o) \leq 0$. Combining this with (3.19) yields:

$$\langle f(x(t_o), \mathbf{x}_{t_o}, \dot{x}(t_o)), n_{x(t_o)} \rangle \leq -\langle \tfrac{d}{dt}\nu(\alpha(t)), \dot{x}(t) \rangle \Big|_{t=t_o} = \mathbb{I}_{x(t_o)}(\dot{x}(t_o)),$$

which contradicts condition (A_4). $\square$

3.1 Operator Reformulation in Function Spaces

3.1.1 Spaces

Denote by $C_{2\pi}(\mathbb{R};\mathbf{V})$ the space of continuous 2π-periodic functions equipped with the norm

$$\|x\|_\infty = \sup_{t\in\mathbb{R}} |x(t)|, \quad x \in C_{2\pi}(\mathbb{R};\mathbf{V}). \tag{3.19}$$

Denote by $\mathscr{E} := C^2_{2\pi}(\mathbb{R},\mathbf{V})$ the space of C^2-smooth 2π-periodic functions from $\mathbb{R}$ to $\mathbf{V}$ equipped with the norm

$$\|x\|_{\infty,2} = \max\{\|x\|_\infty, \|\dot{x}\|_\infty, \|\ddot{x}\|_\infty\}. \tag{3.20}$$

Let $O(2)$ denote the group of orthogonal 2×2 matrices. Notice that $O(2) = SO(2) \cup SO(2)\kappa$, where $\kappa = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, and $SO(2)$ denotes the group of rotations $\begin{bmatrix} \cos\tau & -\sin\tau \\ \sin\tau & \cos\tau \end{bmatrix}$ which can be identified with $e^{i\tau} \in S^1 \subset \mathbb{C}$. Notice that $\kappa e^{i\tau} = e^{-i\tau}\kappa$.

Put $G := O(2) \times \Gamma \times \mathbb{Z}_2$ and define the G-action on $\mathscr{E}$ by

$$(e^{i\theta}, \gamma, \pm 1)x(t) := \pm\gamma x(t+\theta), \tag{3.21}$$

$$(e^{i\theta}\kappa, \gamma, \pm 1)x(t) := \pm\gamma x(-t+\theta), \tag{3.22}$$

where $x \in \mathscr{E}$, $e^{i\theta}, \kappa \in O(2)$, $\gamma \in \Gamma$ and $\pm 1 \in \mathbb{Z}_2$. Clearly, $\mathscr{E}$ is an isometric Banach G-representation. In a standard way, one can identify a 2π-periodic function $x : \mathbb{R} \to \mathbf{V}$ with a function $\tilde{x} : S^1 \to \mathbf{V}$, so one can write $C^2(S^1,\mathbf{V})$ instead of $C^2_{2\pi}(\mathbb{R},\mathbf{V})$. Similar to

(3.21)-(3.22) formulas define isometric G-representations on the spaces of periodic functions $C_{2\pi}(\mathbb{R}, \mathbf{V})$ and $L^2_{2\pi}(\mathbb{R}; V)$ to which appropriate identifications are applied.

Let us describe the G-isotypic decomposition of $\mathscr{E}$. Consider, first, $\mathscr{E}$ as an $O(2)$-representation corresponding to its Fourier modes:

$$\mathscr{E} = \overline{\bigoplus_{k=0}^{\infty} \mathbb{V}_k}, \quad \mathbb{V}_k := \{\cos(kt)u + \sin(kt)v : u,\, v \in \mathbf{V}\}, \tag{3.23}$$

where each $\mathbb{V}_k$, for $k \in \mathbb{N}$, is equivalent to the complexification $\mathbf{V}^c := \mathbf{V} \oplus i\mathbf{V}$ (as a *real* $O(2)$-representation) of $\mathbf{V}$, where the rotations $e^{i\theta} \in SO(2)$ act on vectors $\mathbf{z} \in \mathbf{V}^c$ by $e^{i\theta}(\mathbf{z}) := e^{-ik\theta} \cdot \mathbf{z}$ (here '$\cdot$' stands for complex multiplication) and $\kappa\mathbf{z} := \overline{\mathbf{z}}$. Indeed, the linear isomorphism $\varphi_k : \mathbf{V}^c \to \mathbb{V}_k$ given by

$$\varphi_k(x + iy) := \cos(kt)u + \sin(kt)v, \quad u,\, v \in \mathbf{V}, \tag{3.24}$$

is $O(2)$-equivariant. Clearly, $\mathbb{V}_0$ can be identified with $\mathbf{V}$ with the trivial $O(2)$-action, while $\mathbb{V}_k$, $k = 1, 2, \ldots$, is modeled on the irreducible $O(2)$-representation $\mathcal{W}_k \simeq \mathbb{R}^2$, where $SO(2)$ acts by k-folded rotations and κ acts by complex conjugation.

Next, each $\mathbb{V}_k$, $k = 0, 1, 2, \ldots$, is also $\Gamma \times \mathbb{Z}_2$-invariant. Let $\mathcal{V}_0^-, \mathcal{V}_1^-, \mathcal{V}_2^-, \ldots, \mathcal{V}_{\mathfrak{r}}^-$ be a complete list of all irreducible orthogonal $\Gamma \times \mathbb{Z}_2$-representations on which $\Gamma \times \mathbb{Z}_2$-isotypic components of $\mathbf{V} \simeq \mathbb{V}_0$ are modeled (here "$^-$" stands to indicate the antipodal $\mathbb{Z}_2$-action and $\mathcal{V}_0^-$ corresponds to the trivial Γ-action). Since $\mathcal{V}_{k,l}^- := \mathcal{W}_k \otimes \mathcal{V}_l^-$ is an irreducible orthogonal G-representation, it follows that $\mathbb{V}_0$ and $\mathbb{V}_k$ (cf. (3.23)) admit the following G-isotypic decompositions:

$$\mathbb{V}_0 = V_0^- \oplus V_1^- \oplus \cdots \oplus V_{\mathfrak{r}}^- \tag{3.25}$$

(with the trivial $O(2)$-action) and

$$\mathbb{V}_k = V_{k,0}^- \oplus V_{k,1}^- \oplus \cdots \oplus V_{k,\mathfrak{r}}^-, \tag{3.26}$$

where V_l^- (resp. $V_{k,l}^-$) is modeled on $\mathcal{V}_{0,l}^-$ (resp. $\mathcal{V}_{k,l}^-$ with $k > 0$).

3.1.2 Operators

Define the following operators:

$$
\begin{aligned}
\mathfrak{i} : \mathscr{E} &\to C(S^1, \mathbf{V}), & (\mathfrak{i}x)(t) &:= x(t) \\
L : \mathscr{E} &\to C(S^1, \mathbf{V}), & (Lx)(t) &:= \ddot{x}(t) - (\mathfrak{i}x)(t) \\
j : \mathscr{E} &\to C(S^1, \mathbf{V}^{m+1}), & (jx)(t) &:= (x(t), x(t-\tau), \dots, x(t-(m-1)\tau), \dot{x}(t))
\end{aligned}
$$

and the Nemytskii operator $N_f :\ C(S^1, \mathbf{V}^{m+1}) \to C(S^1, \mathbf{V})$ given by

$$(N_f(x, \mathbf{y}, z))(t) := f(x(t), y^1(t), \dots, y^{m-1}(t), z(t)).$$

The above operators are illustrated on the (non-commutative) diagram following below:

$$
\begin{tikzcd}
\mathscr{E} \arrow[rr, "{L,\ \mathfrak{i}}"] \arrow[dr, "j"'] & & C(S^1, \mathbf{V}) \\
& C(S^1, \mathbf{V}^{m+1}) \arrow[ur, "N_f"'] &
\end{tikzcd}
$$

Figure 3.1. Operators involved

System (1.2) is equivalent to

$$Lx = N_f(jx) - \mathfrak{i}(x), \quad x \in \mathscr{E}. \tag{3.27}$$

Since L is an isomorphism, equation (3.27) can be reformulated as follows:

$$\mathscr{F}(x) := x - L^{-1}(N_f(jx) - \mathfrak{i}(x)) = 0, \;\; x \in \mathscr{E}. \tag{3.28}$$

Proposition 3.1.1. *Suppose that f satisfies conditions (R), (A_1)–(A_3), and the nonlinear operator $\mathscr{F} : \mathscr{E} \to \mathscr{E}$ is given by* (3.28)*. Then, the map $\mathscr{F}$ is a $\mathfrak{G}$-equivariant completely continuous field.*

Proof: Combining (3.23) and (3.24) with the definition of L yields:

$$L|_{\mathbb{V}_k} = -(k^2+1)\mathrm{Id} : V^c \to V^c \quad \text{and} \quad \mathrm{L}|_{\mathbb{V}_0} = -\mathrm{Id} \quad (\mathrm{k} > 0). \tag{3.29}$$

In particular, L (and, therefore, L^{-1}) is $\mathfrak{G}$-equivariant. Since j is the embedding, it is $\mathfrak{G}$-equivariant as well. Since L and N_f are continuous and j is a compact operator, it follows that $\mathscr{F}$ is a completely continuous field. Also, by assumption (A_1) (resp. (A_2)), $\mathscr{F}$ is Γ-equivariant (resp. $\mathbb{Z}_2$-equivariant). Since system (1.2) is autonomous, it follows that $\mathscr{F}$ is $SO(2)$-equivariant. To complete the proof of part (i), one only needs to show that $\mathscr{F}$ commutes with the κ-action. In fact, for all $t \in \mathbb{R}$ and $x \in \mathscr{E}$, one has (we skip $\mathfrak{i}$ to simplify notations):

$$\begin{aligned}
\mathscr{F}(\kappa x)(t) &= \kappa x(t) - L^{-1}\Big(f(\kappa x(t), \kappa \mathbf{x}_t, \kappa \dot{x}(t)) - \kappa x(t)\Big) \\
&= x(-t) - L^{-1}\Big(f(x(-t), x(-t+\tau_1), \dots, x(-t+\tau_{m-1})), -\dot{x}(-t)) - x(-t)\Big) \text{ (by (3.22))} \\
&= x(-t) - L^{-1}\Big(f(x(-t), x(-(t+2\pi-\tau_1)), \dots, x(-(t+2\pi-\tau_{m-1})), -\dot{x}(-t)) - x(-t)\Big) \\
&\qquad \text{(by periodicity of } x) \\
&= x(-t) - L^{-1}\Big(f(x(-t), x(-(t+2\pi-\tau_1)), \dots, x(-(t+2\pi-\tau_{m-1})), \dot{x}(-t)) - x(-t)\Big) \\
&\qquad \text{(by (A2)(i))} \\
&= x(-t) - L^{-1}\Big(f(x(-t), x(-t-\tau_{m-1}), \dots, x(-t-\tau_1), \dot{x}(-t)) - x(-t)\Big) \text{ (by choice of } \tau_k) \\
&= x(-t) - L^{-1}\Big(f(x(-t), x(-t-\tau_1), \dots, x(-t-\tau_{m-1}), \dot{x}(-t)) - x(-t)\Big) \text{ (by (R))} \\
&= \kappa x(t) - \kappa L^{-1}\Big(f(x(t), x(t-\tau_1), \dots, x(t-\tau_m), \dot{x}(t)) - x(t)\Big) \text{ (by (3.22))} \\
&= \kappa\Big(x(t) - L^{-1}\big(f(x(t), \mathbf{x}_t, \dot{x}(t)) - x(t)\Big) \\
&= \kappa \mathscr{F}(x)(t).
\end{aligned}$$

□

Remark 3.1.2. Notice that if f satisfies (A_2)(ii), then $x(t) \equiv 0$ is a solution to equation (3.28). Also, the operator

$$\mathscr{A} := D\mathscr{F}(0) : \mathscr{E} \longrightarrow \mathscr{E} \tag{3.30}$$

is correctly defined provided that condition (A_3) is satisfied. Moreover, in this case,

$$\mathscr{A} = \mathrm{Id} - L^{-1}\Big(DN_f(0) \circ j - \mathfrak{i}\Big) : \mathscr{E} \longrightarrow \mathscr{E} \tag{3.31}$$

and $\mathscr{A}$ is a Fredholm operator of index zero; in particular, $\mathscr{A}$ is an isomorphism if and only if $0 \notin \sigma(\mathscr{A})$. Furthermore, if f satisfies (A_1)–(A_3), then the G-equivariance of $\mathscr{F}$ together with $G(0) = 0$ imply the G-equivariance of $\mathscr{A}$.

We will also need the following lemma (its proof is standard and can be found in [4]).

Lemma 3.1.3. *Under the assumptions (R), (A_1)-(A_3), suppose, in addition, that* $0 \notin \sigma(\mathscr{A})$ *(here* $\sigma(\mathscr{A})$ *stands for the spectrum of* $\mathscr{A}$*) (cf.* (3.28) *and* (3.30)-(3.31)*). Then, for a sufficiently small* $\varepsilon > 0$*, the map* $\mathscr{F}$ *is* $B_\varepsilon(0)$*-admissibly* $\mathfrak{G}$*-equivariantly homotopic to* $\mathscr{A}$*.*

3.1.3 Abstract equivariant degree based result

Assuming that conditions (η_1)–(η_6), (R) and (A_1)–(A_6) (resp. (η_1)–(η_6), (R), (A_1)–(A_5) and (A_6')) are satisfied, we are going to formulate an equivariant degree based result related to problem (3.1). To this end, one needs: (i) to construct an open bounded G-invariant domain $\Omega \subset \mathscr{E}$, $0 \in \Omega$, such that $\mathscr{F}(x)$ is Ω-admissible, and (ii) to introduce additional concepts related to *maximality* of orbit types.

Take ϕ from assumption (A_5) and K from assumption (A_6) (resp.(A_6')). With an eye towards deforming $\mathscr{F}$ by an Ω-admissible G-homotopy and to be on the safe side, take $M := M(2\phi, \eta, K+1, p, R)$ (resp. $M := M(2\phi, \alpha, K+1, p, R)$ provided by Lemma 3.0.1 (resp. Remark 3.0.2) Next, take $N > 0$ provided by Lemma 3.0.3 and put

$$\Omega := \{x \in \mathscr{E} : \forall_{t\in\mathbb{R}} \;\; x(t) \in D, \; \|\dot{x}\|_\infty < M+1, \; \|\ddot{x}\|_\infty < N+1\} \tag{3.32}$$

(see (η_6) for the definition of D). It is easy to see that Ω is an open bounded and G-invariant set. Moreover,

Lemma 3.1.4. *Under the assumptions* (η_1)*–*(η_6)*, (R) and* (A_1)*–*(A_6) *(resp.* (η_1)*–*(η_6)*, (R),* (A_1)*–*(A_5) *and* (A_6')*), the map* $\mathscr{F}$ *(given by* (3.28)*) is* Ω*-admissible.*

Proof: Suppose for contradiction, that there exists $x \in \partial\Omega$ such that $\mathscr{F}(x) = 0$. Then, there exists a sequence $\{x_n\} \subset \Omega$ such that $\|x_n - x\|_{\infty,2} \to 0$ and $x \notin \Omega$. In particular (see (3.32)),

$$\forall_{n\in\mathbb{N}}\, \forall_{t\in\mathbb{R}} \quad x_n(t) \in D \subset \overline{D}. \tag{3.33}$$

Combining (3.33) with the uniform convergence yields

$$\forall_{t\in\mathbb{R}} \quad x(t) \in \overline{D}. \tag{3.34}$$

Since $\mathfrak{F}(x) = 0$, relation (3.34) together with Lemmas 3.0.1 and 3.0.3 imply:

$$\|\dot{x}\|_\infty \le M < M+1 \qquad \text{and} \qquad \|\ddot{x}(t)\|_\infty \le N < N+1. \tag{3.35}$$

Since $x \notin \Omega$, inequalities (3.35) together with (3.32) imply that there exists $t_o \in \mathbb{R}$ such that $x(t_o) \notin D$, hence (see again (3.34)), $x(t_o) \in C := \partial D$, but this contradicts Lemma 3.0.4. $\square$

Observe that under the assumptions of Lemma 3.1.4, the $\mathfrak{G}$-equivariant degree $\mathfrak{G}\text{-Deg}(\mathscr{F}, \Omega)$ is well-defined. Also, under the assumptions of Lemma 3.1.3, $\mathfrak{G}\text{-Deg}(\mathscr{A}, B_\varepsilon(0))$ is well-defined. Put

$$\omega := \mathfrak{G}\text{-Deg}(\mathscr{F}, \Omega) - \mathfrak{G}\text{-Deg}(\mathscr{A}, B_\varepsilon(0)). \tag{3.36}$$

Using ω, we are going to present a result characterizing spatio-temporal symmetries of solutions to problem (3.1). Being of topological nature, this result allows us to completely characterize the spacial component of the symmetry in question while the temporal one can be characterized up to a folding only (in particular, the result does *not* provide an information on the *minimal period*). To be more formal, we need the following

Definition 3.1.5. (a) An orbit type (H) in the space $\mathscr{E}$ is said to be of *maximal kind* if there exists $k \ge 1$ and $u \neq 0, u \in \mathbb{V}_k$, such that $H = G_u$ and (H) is a *maximal* orbit type in $\Phi(G, \mathbb{V}_k \setminus \{0\})$.

(b) Take $x \in \mathscr{E}$ and assume that there exists $p \in \mathbb{N}$ such that $(\phi_p(G_x)) = (H)$, where (H) is of maximal kind and the homomorphism $\phi_p : O(2) \times \Gamma \times \mathbb{Z}_2 \to O(2) \times \Gamma \times \mathbb{Z}_2$ is given by

$$\phi_p(g, h, \pm 1) = (\mu_p(g), h, \pm 1), \quad g \in O(2), \quad h \in \Gamma$$

(here $\mu_p : O(2) \to O(2)/\mathbb{Z}_p \simeq O(2)$ is the natural p-folding homomorphism of $O(2)$ into itself). Then, x is said to have an *extended orbit type* (H).

We are now in a position to formulate the following abstract result.

Proposition 3.1.6. *Assume that $\eta : \mathbf{V} \to \mathbb{R}$ satisfies (η_1)–(η_6) and let $f : \mathbf{V} \times \mathbf{V}^{m-1} \times \mathbf{V} \to \mathbf{V}$ satisfy conditions (R) and (A_1)–(A_6) (resp. (A_1)–(A_5) and (A_6')). Assume, in addition, that $0 \notin \sigma(\mathscr{A})$ (cf.* (3.28), (3.30), (3.31)*). Assume, finally,*

$$\omega = n_1(H_1) + n_2(H_2) + \cdots + n_s(H_s), \quad n_j \neq 0, (H_j) \in \Phi_0(G) \tag{3.37}$$

(cf. (3.36)*). Then:*

- *(a) for every $j = 1, 2, \ldots, m$, there exists a G-orbit of 2π-periodic solutions $x \in \Omega$ to* (3.1) *such that $(G_x) \geq (H_j)$;*
- *(b) if H_j is of maximal kind, then the solution x is non-constant and has the extended orbit type $(\mathscr{H}_j)$ (cf. Definition 3.1.5).*

Proof: (a) Without loss of generality, one can chose ε so small that $B_\varepsilon(0) \subset \Omega$ (cf. conditions (η_4) and (η_6)). Put $\Omega' := \Omega \setminus \Omega_\varepsilon$. Then, by the additivity property of the equivariant degree, one has:

$$\mathfrak{G}\text{-Deg}(\mathscr{F}, \Omega') = \mathfrak{G}\text{-Deg}(\mathscr{F}, \Omega) - \mathfrak{G}\text{-Deg}(\mathscr{A}, B_\varepsilon(0)). \tag{3.38}$$

Next, combining (3.36), (3.37) and (3.38) with the existence property of the equivariant degree, implies part (a).

(b) Observe that if $x \in \mathscr{E}$ is a constant function, then $\mathfrak{G}_x \geq O(2)$. On the other hand, for any $(\mathscr{H})$ of maximal kind, the following property is satisfied: if $\mathscr{K} \in \Phi_0(\mathfrak{G}, \mathscr{E}\setminus\{0\})$ and $(\mathscr{K}) \geq (\mathscr{H})$, then there exists $p \in \mathbb{N}$ such that $(\mathscr{K}) = \Big(\phi_p^{-1}\Big(\mathscr{H}\Big)\Big)$. In particular, $(\mathscr{K})$ is of maximal kind as well. Therefore, $(\mathscr{K})$ is an orbit type of a non-constant 2π-periodic function. Combining this with (a), one obtains (b). $\square$

3.2 Computation of $\mathfrak{G}$-Deg$(\mathscr{A}, B_\varepsilon(0))$

Proposition 3.1.6 reduces the study of problem (3.1) to computing $\mathfrak{G}$-Deg$(\mathscr{A}, B_\varepsilon(0))$ and $\mathfrak{G}$-Deg$(\mathscr{F}, \Omega)$. In this section, we will develop a “workable" formula for $\mathfrak{G}$-Deg$(\mathscr{A}, B_\varepsilon(0))$.

3.2.1 Spectrum of $\mathscr{A}$

To begin with, we collect the equivariant spectral data related to $\mathscr{A}$. Since $\mathscr{A}$ is $\mathfrak{G}$-equivariant, it respects isotypic decomposition (3.23). Put $\gamma := e^{\frac{i2\pi}{m}}$ and $\mathscr{A}_k := \mathscr{A}|_{\mathbb{V}_k}$. Keeping in mind the commensurateness of delays in problem (3.1) and taking into account (3.29), one easily obtains:

$$\mathscr{A}_k = \text{Id} + \frac{1}{k^2+1}\left(\sum_{j=0}^{m-1} \gamma^{jk} A_j - \text{Id}\right), \quad k = 0, 1, 2 \dots, \tag{3.39}$$

where A_j stands for the derivative of f with respect to j-th variable (see condition (A_3)) By assumption (R), $A_j = A_{m-j}$ for $j = 1, ..., m-1$, hence (3.39) can be simplified as follows:

$$\mathscr{A}_k = \text{Id} + \frac{1}{k^2+1}\left(A_0 + \sum_{j=1}^{r} 2\cos\frac{2\pi jk}{m} A_j - \varepsilon_m A_r - \text{Id}\right), \quad k = 0, 1, 2 \dots, \; r = \left\lfloor \frac{m-1}{2} \right\rfloor, \tag{3.40}$$

where

$$\varepsilon_m = \begin{cases} 1 & \text{if } m \text{ is even;} \\ 0 & \text{otherwise.} \end{cases} \tag{3.41}$$

Since the matrices A_j are Γ-equivariant, one has $\mathscr{A}_k(V_{k,l}^-) \subset V_{k,l}^-$ ($k = 0, 1, 2, \dots$ and $l = 0, 1, 2, \dots, \mathfrak{r}$). In particular, $A_j(V_l^-) \subset V_l^-$, so put

$$A_{j,l} := A_j|_{V_l^-}, \quad l = 0, 1, 2, \dots, \mathfrak{r}.$$

To simplify the computations, we will assume that instead of (A_3) the following condition is satisfied:

(A_3') $A_{j,l} = \mu_j^l \mathrm{Id}$ for $l = 0, 1, 2, \dots, \mathfrak{r}$ and $j = 0, 1, \dots, m-1$.

Clearly, under the condition (A_3'), the matrices A_j commute with each other, therefore, condition (A_3) follows. In particular, their corresponding eigenspaces coincide: $E(\mu_j^l) = E(\mu_{j'}^l)$. This way, one obtains the following description of the spectrum of $\mathscr{A}$:

$$\sigma(\mathscr{A}) = \bigcup_{k=0}^{\infty} \sigma(\mathscr{A}_k), \tag{3.42}$$

where

$$\sigma(\mathscr{A}_k) = \left\{ 1 + \frac{1}{1+k^2} \left(\mu_0^l + \sum_{j=1}^{r} 2\cos\frac{2\pi jk}{m} \mu_j^l - \varepsilon_m \mu_r^l - 1 \right) : l = 0, 1, \dots, \mathfrak{r},\ r = \left\lfloor \frac{m-1}{2} \right\rfloor \right\}. \tag{3.43}$$

3.2.2 Reduction to basic G-degrees

For any $l = 0, 1, \dots, \mathfrak{r}$ and $k = 0, 1, ...$, put (cf. (3.43))

$$\xi_{k,l} := 1 + \frac{1}{1+k^2} \left(\mu_0^l + \sum_{j=1}^{r} 2\cos\frac{2\pi jk}{m} \mu_j^l - \varepsilon_m \mu_r^l - 1 \right), \ r = \left\lfloor \frac{m-1}{2} \right\rfloor, \tag{3.44}$$

One can show that $\xi_{k,l}$ contributes $\mathfrak{G}$-Deg$(\mathscr{A}, B(\mathscr{E}))$ only if $\xi_{k,l} < 0$. Clearly (cf. (3.44)), $\xi_{k,l}$ is negative (i.e. $\xi_{k,l} \in \sigma_-(\mathscr{A})$) if and only if

$$k^2 < -\mu_0^l - \sum_{j=1}^{r} 2\cos\frac{2\pi jk}{m} \mu_j^l + \varepsilon_m \mu_r^l, \quad l = 0, 1, \dots, \mathfrak{r},\ r = \left\lfloor \frac{m-1}{2} \right\rfloor,\ k = 0, 1, ... \tag{3.45}$$

By condition (A_3'), the $\mathcal{V}_l^-$-isotypic multiplicity of μ_j^l is independent of j and is equal to

$$m^l := \dim E(\mu_j^l)/\dim \mathcal{V}_l^- = \dim V_l^-/\dim \mathcal{V}_l^-. \tag{3.46}$$

Put (cf. (3.45)-(3.46))

$$m_{k,l} := \begin{cases} m^l & \text{if } \ k^2 < -\mu_0^l - \sum_{j=1}^r 2\cos\frac{2\pi jk}{m}\mu_j^l + \varepsilon_m \mu_r^l \\ 0 & \text{otherwise.} \end{cases} \tag{3.47}$$

Then,

$$\mathfrak{G}\text{-Deg}(\mathscr{A}, B(\mathscr{E})) = \prod_{k=0}^{\infty}\prod_{l=0}^{\mathfrak{r}} \left(\deg_{\mathcal{V}_{k,l}^-}\right)^{m_{k,l}} \tag{3.48}$$

Remark 3.2.1. (a) Notice that in the product (3.48), one has $m_{k,l} \neq 0$ for finitely many values of k and l (cf. (3.47)). Hence, for almost all the factors in (3.48), one has $(\deg_{\mathcal{V}_{k,l}})^0 = (G)$, which is the unit element in $A(G)$. Thus, formula (3.48) is well-defined.

(b) Using the relation $(\deg_{\mathcal{V}_{k,l}^-})^2 = (G)$, one can further simplify formula (3.48). Clearly, only the exponents $m_{k,l} \neq 0$ which are odd will contribute to the value of (3.48).

3.2.3 Maximal orbit types in products of basic G-degrees

In order to effectively apply Proposition 3.1.6(c), one should answer the following question: which orbit types of maximal kind (see Definition 3.1.5) appearing in the right-hand side of formula (3.48) will "survive" in the resulting product? This question has been studied in detail in [3]. Here we will present one result from [3] essentially used in what follows.

To begin with, take $\deg_{\mathcal{V}_{k,l}^-}$ appearing in (3.48) and let (H_o) be a maximal orbit type in $\mathcal{V}_{k,l}^- \setminus \{0\}$. Then,

$$\deg_{\mathcal{V}_{k,l}^-} = (G) - x_o(H_o) + a, \quad -x_o := \frac{(-1)^{\dim \mathcal{V}_{k,l}^{-H_o}} - 1}{|W(H_o)|}, \tag{3.49}$$

where $a \in A(G)$ has a zero coefficient corresponding to (H_o). Then, by (3.49), one has

$$x_o = \begin{cases} 0 & \text{if } \dim \mathcal{V}_{k,l}^{-H_o} \text{ is even} \\ 1 & \text{if } \dim \mathcal{V}_{k,l}^{-H_o} \text{ is odd and } |W(H_o)| = 2 \\ 2 & \text{if } \dim \mathcal{V}_{k,l}^{-H_o} \text{ is odd and } |W(H_o)| = 1. \end{cases} \tag{3.50}$$

We need additional notations.

Definition 3.2.2. (i) For any $(H_o) \in \Phi_0(G)$, define the function $\mathrm{coeff}^{H_o} : A(G) \to \mathbb{Z}$ assigning to any $a = \sum_{(H)} n_H(H) \in A(G)$ the coefficient n_{H_o} standing by (H_o).

(ii) Given an orbit type $(H_o) \in \Phi_0(G, \mathscr{E})$ of maximal kind (see Definition 3.1.5(a)) and $k = 0, 1, 2, \dots$, define the integer

$$\mathfrak{n}_k^{H_o} := \sum_{l=0}^{\mathfrak{r}} \mathfrak{l}_{k,l}^{H_o} \cdot m_{k,l}, \tag{3.51}$$

where $m_{k,l}$ is given by (3.47) and

$$\mathfrak{l}_{k,l}^{H_o} := \begin{cases} 1 & \text{if } \dim \mathcal{V}_{k,l}^{-H_o} \text{ is odd} \\ 0 & \text{otherwise} \end{cases} \tag{3.52}$$

(cf. formulas (3.48)–(3.50)).

The following statement was proved in [3].

Lemma 3.2.3. *Let $(H_o) \in \Phi_0(G, \mathscr{E})$ be an orbit type of maximal kind (see Definition 3.1.5(a)) and assume that for some $k \geq 0$, the number $\mathfrak{n}_k^{H_o}$ is odd (see Definition 3.2.2). Then,*

$$\mathrm{coeff}^{H_o}\Big(\mathfrak{G}\text{-}\mathrm{Deg}(\mathscr{A}, B_\varepsilon(0))\Big) = \pm x_o, \tag{3.53}$$

where x_o is given by (3.50).

3.3 Computation of $\mathfrak{G}$-Deg$(\mathscr{F}, \Omega)$

In this section, following the scheme suggested in [1], where the non-equivariant case without delays was considered, we are going to establish the following

Proposition 3.3.1. *Under the assumptions (η_1)–(η_6), (R), (A_1)–(A_2) and (A_4)–(A_6) (resp. (η_1)–(η_6), (R), (A_1)–(A_2), (A_4)–(A_5) and (A_6')), one has*

$$\mathfrak{G}\text{-Deg}(\mathscr{F}, \Omega) = \mathfrak{G}\text{-Deg}(\nu, C) \tag{3.54}$$

(here $\nu : C \to S^{n-1}$ stands for the Gauss map and $O(2)$ is assumed to act trivially on $\mathbf{V}$ identified with constant $\mathbf{V}$-valued maps).

The proof of the above proposition splits into several steps related to successive Ω-admissible $\mathfrak{G}$-equivariant homotopies.

3.3.1 Outward homotopy

To begin with, denote by $\mathfrak{n} : V \to V$ a continuous extension of the Gauss map $\nu : C \to S^{n-1}$, such that $|\mathfrak{n}(x)| \leq 1$, $\mathfrak{n}(\gamma x) = \gamma \mathfrak{n}(x)$ and $\mathfrak{n}(-x) = -\mathfrak{n}(x)$ for all $x \in \mathbf{V}$ and $\gamma \in \Gamma$. Such an extension exists due to the equivariant version of the Tietze Theorem (see, for example, [35]).

Next, for $\lambda \in [0, 2]$, define the map $f_\lambda : \mathbf{V} \times \mathbf{V}^{m-1} \times \mathbf{V} \to \mathbf{V}$ by

$$f_\lambda(x, \mathbf{y}, z) := f(x, \mathbf{y}, z) + \lambda\Big(\max\Big\{0, -\langle f(x, \mathbf{y}, z), \mathfrak{n}(x)\rangle\Big\} + \frac{1}{2} \min\Big\{1, \phi(z)\Big\}\Big)\mathfrak{n}(x), \tag{3.55}$$

where $x \in \mathbf{V}$, $\mathbf{y} \in \mathbf{V}^{m-1}$, $z \in \mathbf{V}$. One has the following

Lemma 3.3.2. *Under the assumptions of Proposition 3.3.1, the map f_λ given by* (3.55) *satisfies the following properties:*

(R^λ) *$f_\lambda(x, y^1, \cdots, , y^{m-1}, z) = f_\lambda(x, y^{m-1}, y^{m-2}, \cdots, y^2, y^1, z)$ for all $(x, y^1, \cdots, y^{m-1}, z) \in \mathbf{V}^{m+1}$,*

(A_1^λ) f_λ is Γ-equivariant;

(A_2^λ) for all $x, z \in \mathbf{V}$ and $\mathbf{y} \in \mathbf{V}^{m-1}$, one has:

(i) $f_\lambda(x, \mathbf{y}, -z) = f_\lambda(z, \mathbf{y}, z)$,

(ii) $f_\lambda(-x, -\mathbf{y}, z) = -f_\lambda(z, \mathbf{y}, z)$;

(A_4^λ) for any $x \in C$, $\mathbf{y} \in \mathbf{V}^{m-1}$ and $z \in \mathbf{V}$ such that $|\mathbf{y}| \leq R$ and $z \perp n_x$, one has

$$\langle f_\lambda(x, \mathbf{y}, z), n_x\rangle > \mathbb{I}_x(z); \tag{3.56}$$

(A_5^λ) for any $(x, \mathbf{y}, z) \in \mathbf{V} \times \mathbf{V}^{m-1} \times \mathbf{V}$ with $|x|, |\mathbf{y}| \leq R$, one has

$$|f_\lambda(x, \mathbf{y}, z)| \leq 2\phi(|z|),$$

where ϕ is from (A_5);

(A_6^λ) for any $(x, \mathbf{y}, z) \in \mathbf{V} \times \mathbf{V}^{m-1} \times \mathbf{V}$ with $|x|, |\mathbf{y}| \leq R$, one has

$$|f_\lambda(x, \mathbf{y}, z)| \leq \nabla^2\eta(x)(z, z) + \langle f_\lambda(x, \mathbf{y}, z), \nabla\eta(x)\rangle + K + 1,$$

provided that f satisfies (A_6);

(${A'}_6^\lambda$)

$$\forall_{|x|\leq R}\,\forall_{|\mathbf{y}|\leq R}\,\forall_{z\in V}\quad |f_\lambda(x, \mathbf{y}, z)| \leq \alpha(\langle x, f_\lambda(x, \mathbf{y}, z)\rangle + |z|^2) + K,$$

provided that f satisfies (A'_6).

Proof: (R$^\lambda$) For any $(x, y^1, \cdots, y^{m-1}, z) \in \mathbf{V}^{m+1}$, one has:

$$\begin{aligned}
f_\lambda(x, y^1, \cdots, y^{m-1}, z) &= f(x, y^1, \cdots, y^{m-1}, z)\\
&+ \lambda\Big(\max\big\{0, -\langle f(x, y^1, \cdots, y^{m-1}, z), \mathfrak{n}(x)\rangle\big\} + \frac{1}{2}\min\big\{1, \phi(z)\big\}\Big)\mathfrak{n}(x)\\
&= f(x, y^{m-1}, \cdots, y^1, z)\\
&+ \lambda\Big(\max\big\{0, -\langle f(x, y^{m-1}, \cdots, y^1, z), \mathfrak{n}(x)\rangle\big\} + \frac{1}{2}\min\big\{1, \phi(z)\big\}\Big)\mathfrak{n}(x)\\
&= f_\lambda(x, y^{m-1}, \cdots, y^1, z)
\end{aligned}$$

(A_1^λ) Recall that Γ acts orthogonally on $\mathbf{V}$, f and $\mathfrak{n}$ are Γ-equivariant, and ϕ is Γ-invariant. Hence, for any $\gamma \in \Gamma$ and $(x, \mathbf{y}, z) \in \mathbf{V}^{m+1}$, one has:

$$\begin{aligned}
f_\lambda(\gamma(x, \mathbf{y}, z)) &= f_\lambda(\gamma x, \gamma\mathbf{y}, \gamma z) \\
&= f(\gamma x, \gamma\mathbf{y}, \gamma z) + \lambda\Big(\max\big\{0, -\langle f(\gamma x, \gamma\mathbf{y}, \gamma z), \mathfrak{n}(\gamma x)\rangle\big\} + \frac{1}{2}\min\big\{1, \phi(\gamma z)\big\}\Big)\mathfrak{n}(\gamma x) \\
&= \gamma f(x, \mathbf{y}, z) + \lambda\Big(\max\big\{0, -\langle \gamma f(x, \mathbf{y}, z), \gamma\mathfrak{n}(x)\rangle\big\} + \frac{1}{2}\min\big\{1, \phi(z)\big\}\Big)\gamma\mathfrak{n}(x) \\
&= \gamma f_\lambda(x, \mathbf{y}, z).
\end{aligned}$$

(A_2^λ) For any $(x, \mathbf{y}, z) \in \mathbf{V}^{m+1}$, one has (by (A_2)):

$$\begin{aligned}
f_\lambda(x, \mathbf{y}, -z) &= f(x, \mathbf{y}, -z) + \lambda\Big(\max\big\{0, -\langle f(x, \mathbf{y}, -z), \mathfrak{n}(x)\rangle\big\} + \frac{1}{2}\min\big\{1, \phi(z)\big\}\Big)\mathfrak{n}(x) \\
&= f(x, \mathbf{y}, z) + \lambda\Big(\max\big\{0, -\langle f(x, \mathbf{y}, z), \mathfrak{n}(x)\rangle\big\} + \frac{1}{2}\min\big\{1, \phi(z)\big\}\Big)\mathfrak{n}(x) = f_\lambda(x, \mathbf{y}, z).
\end{aligned}$$

Also,

$$\begin{aligned}
f_\lambda(-x, -\mathbf{y}, z) &= f(-x, -\mathbf{y}, z) + \lambda\Big(\max\big\{0, -\langle f(-x, -\mathbf{y}, z), \mathfrak{n}(-x)\rangle\big\} + \frac{1}{2}\min\big\{1, \phi(z)\big\}\Big)\mathfrak{n}(-x) \\
&= -f(x, \mathbf{y}, z) + \lambda\Big(\max\big\{0, -\langle -f(x, \mathbf{y}, z), -\mathfrak{n}(x)\rangle\big\} + \frac{1}{2}\min\big\{1, \phi(z)\big\}\Big)(-\mathfrak{n}(x)) \\
&= -f_\lambda(x, \mathbf{y}, z).
\end{aligned}$$

(A_4^λ) For any $(x, \mathbf{y}, z) \in \mathbf{V}^{m+1}$, one has (by (A_4)):

$$\begin{aligned}
\langle f_\lambda(x, \mathbf{y}, z), \mathfrak{n}(x)\rangle &= \langle f(x, \mathbf{y}, z), \mathfrak{n}(x)\rangle + \lambda\Big(\max\big\{0, -\langle f(x, \mathbf{y}, z), \mathfrak{n}(x)\rangle\big\} + \frac{1}{2}\min\big\{1, \phi(z)\big\}\Big)|\mathfrak{n}(x)|^2 \\
&> \mathbb{I}_x(z) + \lambda\Big(\max\big\{0, -\langle f(x, \mathbf{y}, z), \mathfrak{n}(x)\rangle\big\} + \frac{1}{2}\min\big\{1, \phi(z)\big\}\Big) \geq \mathbb{I}_x(z).
\end{aligned}$$

Finally, to prove (A_5^λ), (A_6^λ) and (A_7^λ), one can use the same argument as in [1], p. 299. $\square$

Using (3.55), define the map $\mathscr{F}_\lambda : \mathscr{E} \to \mathscr{E}$ by

$$\mathscr{F}_\lambda(x) := x - L^{-1}\Big(N_{f_\lambda}(j(x)) - \mathfrak{i}(x)\Big), \quad x \in \mathscr{E}, \tag{3.57}$$

where $N_{f_\lambda} : C(S^1, \mathbf{V}^{m+1}) \to C(S^1, \mathbf{V})$ is the Nemytskii operator given by

$$(N_{f_\lambda}(x, \mathbf{y}, z))(t) := f_\lambda(x(t), y^1(t), \dots, y^{m-1}(t), z(t)) \quad (t \in \mathbb{R}, \;\; \lambda \in [0, 2]). \tag{3.58}$$

Combining Lemma 3.3.2 with the definition of Ω and the argument used in the proof of Lemma 3.0.4, one obtains the following

Corollary 3.3.3. *Under the assumptions of Proposition 3.3.1, formulas* (3.57)–(3.58) *define a $\mathfrak{G}$-equivariant Ω-admissible homotopy. In particular,*

$$\mathfrak{G}\text{-Deg}(\mathscr{F}, \Omega) = \mathfrak{G}\text{-Deg}(\mathscr{F}_2, \Omega). \tag{3.59}$$

Remark 3.3.4. Obviously (see (3.55) and [1], p. 300), the following inequality takes place:

$$\forall_{x \in C,\, z \in \mathbf{V}, \mathbf{y} \in \mathbf{V}^{m-1}} \qquad \langle f_2(x, \mathbf{y}, z), \mathfrak{n}(x) \rangle > 0. \tag{3.60}$$

It follows from (3.60) that for any $x \in C$, the vector

$$\Psi(x) := f_2(x, x, \cdots, x, 0) \tag{3.61}$$

is pointed outward the interior of $\overline{D}$ (giving rise to the title of this subsection). Hence, Ψ and ν are $\mathfrak{G}$-equivariantly homotopic and

$$\mathfrak{G}\text{-Deg}(\Psi, C) = \mathfrak{G}\text{-Deg}(\nu, C) \tag{3.62}$$

3.3.2 Scaling homotopy

To perform further deformations, we need the following

Lemma 3.3.5. *Under the assumptions of Proposition 3.3.1, take M provided by Lemma 3.0.1, f_2 given by* (3.55) *and $\widetilde{\lambda} \in (0, 1)$. Then, any C^2-smooth solution $x_{\widetilde{\lambda}} = x_{\widetilde{\lambda}}(t)$ to problem*

$$\begin{cases} \ddot{x}(t) = \widetilde{\lambda}^2 f_2\left(x(t), \mathbf{x}_t, \widetilde{\lambda}^{-1}\dot{x}(t)\right), \quad t \in \mathbb{R},\ x(t) \in \overline{D} \subset \mathbf{V} = \mathbb{R}^n, \\ x(t) = x(t+p), \quad \dot{x}(t) = \dot{x}(t+p) \end{cases} \tag{3.63}$$

satisfies the inequality

$$\forall_{t \in \mathbb{R}} \quad |\dot{x}_{\widetilde{\lambda}}(t)| \leq \widetilde{\lambda} M. \tag{3.64}$$

Proof: Put $u(t) := x_{\widetilde{\lambda}}(t/\widetilde{\lambda})$. Since $\ddot{u} = \widetilde{\lambda}^{-2}\ddot{x}(t/\widetilde{\lambda})$, one can easily show (cf. [1], pp. 297-298) that

$$\begin{cases} \ddot{u}(t) = f_2\left(u(t), u(t-\widetilde{\lambda}\tau_1), \cdots, u(t-\widetilde{\lambda}\tau_{m-1}), \dot{u}(t)\right), \quad t \in \mathbb{R},\ u(t) \in \overline{D} \subset \mathbf{V} = \mathbb{R}^n, \\ u(t) = u(t+p\widetilde{\lambda}), \quad \dot{u}(t) = \dot{u}(t+p\widetilde{\lambda}) \end{cases} \tag{3.65}$$

Therefore, one can use formula (3.14) (resp. (3.15)) to obtain $M_1 = M_1(2\phi, \eta, K+1, p\widetilde{\lambda}, R)$ (resp. $M_1 = M_1(2\phi, \alpha, K+1, p\widetilde{\lambda}, R)$ such that

$$\forall_{t\in\mathbb{R}} \quad |\dot{u}(t)| \le M_1. \tag{3.66}$$

Since $\widetilde{\lambda} < 1$ and Φ in (3.14) (resp. (3.15)) is increasing, formula (3.66) combined with the chain rule yields (3.64). $\square$

Remark 3.3.6. In contrast to problem (3.63), problem (3.65) is not equivariant. The reader should not be confused with that: Lemma 3.0.1 providing a priori bound for the first derivative of solution is *independent* of the symmetry conditions (A_1) and (A_2).

Given $u \in C(S^1, \mathbf{V})$, denote

$$\overline{u} := \frac{1}{2\pi}\int_0^p u(t)dt. \tag{3.67}$$

Formula (3.67) suggests two projections $Q_0, P_0 : C(S^1, \mathbf{V}) \to C(S^1, \mathbf{V})$ given by

$$Q_0 u := \overline{u} \qquad \text{and} \quad P_0 := \text{Id} - Q_0 \tag{3.68}$$

(as usual, we identify $\mathbf{V}$ with the image of Q_0 – the subspace of constant $\mathbf{V}$-valued maps $S^1 \to \mathbf{V}$). Similarly to (3.67) and (3.68), define projections $Q_2, P_2 : \mathscr{E} \to \mathscr{E}$, respectively.

For any $\widetilde{\lambda} \in (0,1)$, put $f_{2,\widetilde{\lambda}}(x, \mathbf{y}, z) := \widetilde{\lambda}^2 f_2\Big(x, \mathbf{y}, \widetilde{\lambda}^{-1} z\Big)$ (cf. (3.63)) and consider a μ-parameterized family of operators $\mathfrak{F}_{\widetilde{\lambda},\mu} : \mathscr{E} \to \mathscr{E}$ given by

$$\mathfrak{F}_{\widetilde{\lambda},\mu}(x) := x - L^{-1}\left(Q_0 N_{f_{2,\widetilde{\lambda}}}(jx) + \mu P_0 N_{f_{2,\widetilde{\lambda}}}(jx) - \mathbf{i}(x)\right), \qquad \mu \in [0,1] \tag{3.69}$$

(here the projections P_0, Q_0 are given by (3.67)–(3.68) and $N_{f_{2,\widetilde{\lambda}}}$ denotes the corresponding Nemytskii operator).

Lemma 3.3.7. *Under the assumptions of Proposition 3.3.1, there exists* $\widetilde{\lambda}_o \in (0,1]$ *such that the* μ*-parameterized family* $\mathfrak{F}_{\widetilde{\lambda}_o,\mu}$ *(see* (3.69)*) is an* Ω*-admissible* $\mathfrak{G}$*-equivariant homotopy. In particular (cf.* (3.59)*),*

$$\mathfrak{G}\text{-Deg}(\mathfrak{F},\Omega) = \mathfrak{G}\text{-Deg}(\mathfrak{F}_{\widetilde{\lambda}_o,0},\Omega), \tag{3.70}$$

where $\mathfrak{F}_{\widetilde{\lambda}_o,0}(x) = x - L^{-1}\Big(Q_0 N_{f_{2,\widetilde{\lambda}_o}}(jx) - \mathfrak{i}(x)\Big)$.

Proof: Following the same lines as in the proof of Lemma 3.3.2((R^λ), (A_1^λ) and (A_2^λ)), one can easily establish that (3.69) is $\mathfrak{G}$-equivariant for any $\widetilde{\lambda} \in (0,1)$ and $\mu \in [0,1]$. Next, keeping in mind that $\mathbb{I}_x(\cdot)$ is a quadratic form and using (A_4^λ), one obtains

$$\langle f_{2,\widetilde{\lambda}}(x,\mathbf{y},z), n_x\rangle = \langle \widetilde{\lambda}^2 f_2(x,\mathbf{y},\widetilde{\lambda}^{-1}z), n_x\rangle > \widetilde{\lambda}^2 \mathbb{I}_x(\widetilde{\lambda}^{-1}z,\widetilde{\lambda}^{-1}z) = \mathbb{I}_x(z),$$

so that $f_{2,\widetilde{\lambda}}$ satisfies the analog of (A_4^λ). Finally, arguing by contradiction, and combining the same idea as in [1], p. 300, with estimate (3.64) one arrives at the contradiction with Lemma 3.0.4, from which the existence of the required $\widetilde{\lambda}_o$ follows. □

To complete the proof of Proposition 3.3.1, it remains to establish the following

Lemma 3.3.8. *Under the assumptions of Proposition 3.3.1, one has (cf.* (3.70)*)*

$$\mathfrak{G}\text{-Deg}(\mathfrak{F}_{\widetilde{\lambda}_o,0},\Omega) = \mathfrak{G}\text{-Deg}(\nu, C). \tag{3.71}$$

Proof: One has

$$\begin{aligned}
\mathfrak{F}_{\widetilde{\lambda}_o,0}(x) &= x - L^{-1}\Big(Q_0 N_{f_{2,\widetilde{\lambda}_o}}(jx) - \mathfrak{i}(x)\Big) \\
&= Q_2x + P_2x - L^{-1}\Big(Q_0 N_{f_{2,\widetilde{\lambda}_o}}(jQ_2x + jP_2x) - \mathfrak{i}(Q_2x + P_2x)\Big) \\
&= \Big(Q_2x - L^{-1}\Big(Q_0 N_{f_{2,\widetilde{\lambda}_o}}(jQ_2x + jP_2x)\Big) + L^{-1}\mathfrak{i}Q_2x\Big) + \Big(P_2x + L^{-1}(\mathfrak{i}P_2x)\Big) \\
&= -L^{-1}\Big(Q_0 N_{f_{2,\widetilde{\lambda}_o}}(jQ_2x + jP_2x)\Big) + \Big(P_2x + L^{-1}(\mathfrak{i}P_2x)\Big)
\end{aligned}$$

(cf. (3.29)). Formula

$$\mathfrak{F}_{\widetilde{\lambda}_o,0,\delta}(x) = -L^{-1}\left(Q_0 N_{f_{2,\widetilde{\lambda}_o}}(jQ_2x + (1-\delta)jP_2x)\right) + \\ (P_2x + (1-\delta)L^{-1}(\mathfrak{i}P_2x)), \quad \delta \in [0,1], \tag{3.72}$$

defines a $\mathfrak{G}$-equivariant Ω-admissible homotopy of $\mathfrak{F}_{\widetilde{\lambda}_o,0}$ to

$$\widetilde{\mathfrak{F}}(x) := \left(-L^{-1}Q_0 N_{f_{2,\widetilde{\lambda}_o}}(jQ_2x), P_2x\right)$$

(see again(3.29)). Clearly,

$$\mathfrak{G}\text{-Deg}(\widetilde{\mathfrak{F}},\Omega) = \mathfrak{G}\text{-Deg}\left(-L^{-1}Q_0 N_{f_{2,\widetilde{\lambda}_o}}(jQ_2), D\right) \cdot \mathfrak{G}\text{-Deg}(\mathrm{Id}\,, B(P_2\mathscr{E})),$$

where $B(P_2\mathscr{E})$ stands for the unit ball in $P_2\mathscr{E}$. It remains to observe that

$$\mathfrak{G}\text{-Deg}\left(-L^{-1}Q_0 N_{f_{2,\widetilde{\lambda}_o}}(jQ_2), D\right) = \mathfrak{G}\text{-Deg}(\Psi, D)$$

(see (3.61)) and use (3.62). □

Using the same Morse Lemma argument as in the proof of Theorem 5.6 from [1], one can easily establish the following

Lemma 3.3.9. *Let $\eta : \mathbf{V} \to \mathbb{R}$ satisfy (η_1), (η_4)–(η_6), and let $f : \mathbf{V} \times \mathbf{V}^{m-1} \times \mathbf{V} \to \mathbf{V}$ be a continuous map satisfying (A_5)–(A_6). Then, D is contractible.*

Corollary 3.3.10. *Under the assumptions of Proposition 3.3.1,*

$$\mathfrak{G}\text{-Deg}(\mathfrak{F},\Omega) = (G). \tag{3.73}$$

Proof: Since $0 \in D$, Lemma 3.3.9 implies that the Gauss map ν is $\mathfrak{G}$-equivariantly homotopic to the identity map and the result follows from Proposition 3.3.1. □

3.4 Main Results and Example

3.4.1 Main result

In this section, we will present our main results and describe an illustrating example with $G = O(2) \times D_8 \times \mathbb{Z}_2$. The "non-degenerate" version of the main result is:

Theorem 3.4.1. *Assume that $\eta : \mathbf{V} \to \mathbb{R}$ satisfies (η_1)–(η_6) and let $f : \mathbf{V} \times \mathbf{V}^{m-1} \times \mathbf{V} \to \mathbf{V}$ satisfy conditions (R), (A_1)–(A_2), (A_3'), (A_4)–(A_6) (resp. (R), (A_1)–(A_2), (A_3'), (A_4)–(A_5) and (A_6')). Assume, in addition, that $0 \notin \sigma(\mathscr{A})$, where $\sigma(\mathscr{A})$ is given by* (3.42)–(3.43) *(see also* (3.41)*). Assume, finally, that there exist $k \in \mathbb{N}$ and an orbit type (H_o) in $\Phi_0(G, \mathscr{E})$ of maximal kind such that $\mathfrak{n}_k^{H_o}$ is odd (see Definitions 3.1.5(a) and 3.2.2).*

Then, system (3.1) *admits a non-constant 2π-periodic solution with the extended orbit type (H_o) (cf. Definition 3.1.5(b)).*

Proof: Formulas (3.48)–(3.50) show that $\mathfrak{G}\text{-Deg}(\mathscr{A}, B_\varepsilon(0)) = (G) + a$, where a has a zero coefficient corresponding to (G). Hence, ω given by (3.36) has a zero coefficient corresponding to (G) (cf. Corollary 3.3.10). Now, the proof follows immediately from Lemma 3.2.3 and Proposition 3.1.6(c). $\square$

Using a similar argument, one can easily establish the following degenerate counterpart of Theorem 3.4.1.

Theorem 3.4.2. *Assume that $\eta : \mathbf{V} \to \mathbb{R}$ satisfies (η_1)–(η_6) and let $f : \mathbf{V} \times \mathbf{V}^{m-1} \times \mathbf{V} \to \mathbf{V}$ satisfy conditions (R), (A_1)–(A_2), (A_3'), (A_4)–(A_6) (resp. (R), (A_1)–(A_2), (A_3'), (A_4)–(A_5) and (A_6')). Put*

$$\begin{aligned} \mathscr{C} := \Big\{ k \in \mathbb{N} \cup \{0\} : k^2 = -\mu_0^l - \textstyle\sum_{j=1}^r 2\cos\frac{2\pi jk}{m}\mu_j^l + \\ \varepsilon_m \mu_r^l, \;\; l = 0, 1, 2, \dots, \mathfrak{r}, \; r := \left\lfloor \tfrac{m-1}{2} \right\rfloor \Big\} \end{aligned} \tag{3.74}$$

and choose $s \in \mathbb{N}$ such that

$$\mathscr{C} \cap \{(2k-1)s : k \in \mathbb{N}\} = \emptyset. \tag{3.75}$$

Assume that there exist $k \in \mathbb{N}$ and an orbit type (H_o) in $\Phi_0(G, \mathscr{E})$ of maximal kind such that $\mathfrak{n}^{H_o}_{(2k-1)s}$ is odd (see Definitions 3.1.5(a) and 3.2.2).

Then, system (3.1) *admits a non-constant 2π-periodic solution with the extended orbit type (H_o) (cf. Definition 3.1.5(b)).*

3.4.2 Example

To construct an example supporting Theorem 3.4.1 with condition (A_6) being satisfied, take $\mathbf{V} := \mathbb{R}^2$ and consider the domain $D \subset \mathbf{V}$ described in polar coordinates (r, θ) as follows:

$$D := \{(r, \theta) \in \mathbb{R}^2 : 2r^4 - r^4 \cos(8\theta) - 1 < 0\}. \tag{3.76}$$

The curve $C := \partial D$ can be easily plotted (see Figure 3.2). Clearly, D is invariant under the natural action of the dihedral group $D_8 =: \Gamma$ on $\mathbf{V} \simeq \mathbb{C}$ (in particular, D is symmetric).

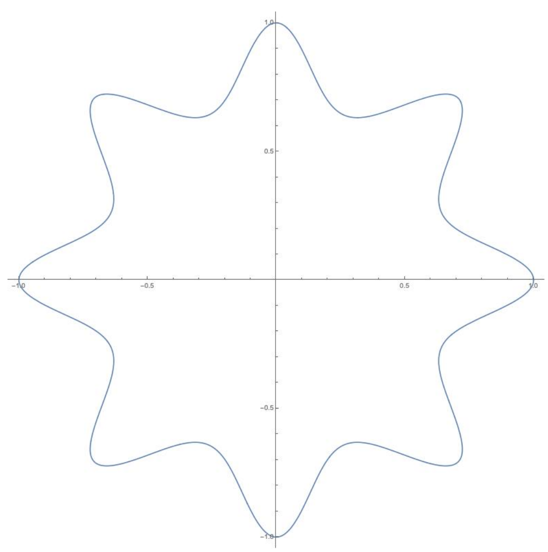

Figure 3.2. Domain D

Since D is star shaped, the Gauss curvature of C can be easily computed as a function of θ:

$$\kappa(\theta) = \frac{\sqrt{2}(-19 + 56\cos(8\theta) - 3\cos(16\theta))(2 - \cos(8\theta))^{\frac{5}{4}}}{(13 - 8\cos(8\theta) - 3\cos(16\theta))^{\frac{3}{2}}} \tag{3.77}$$

The graph of $\kappa(\theta)$ is shown on Figure 3.3.

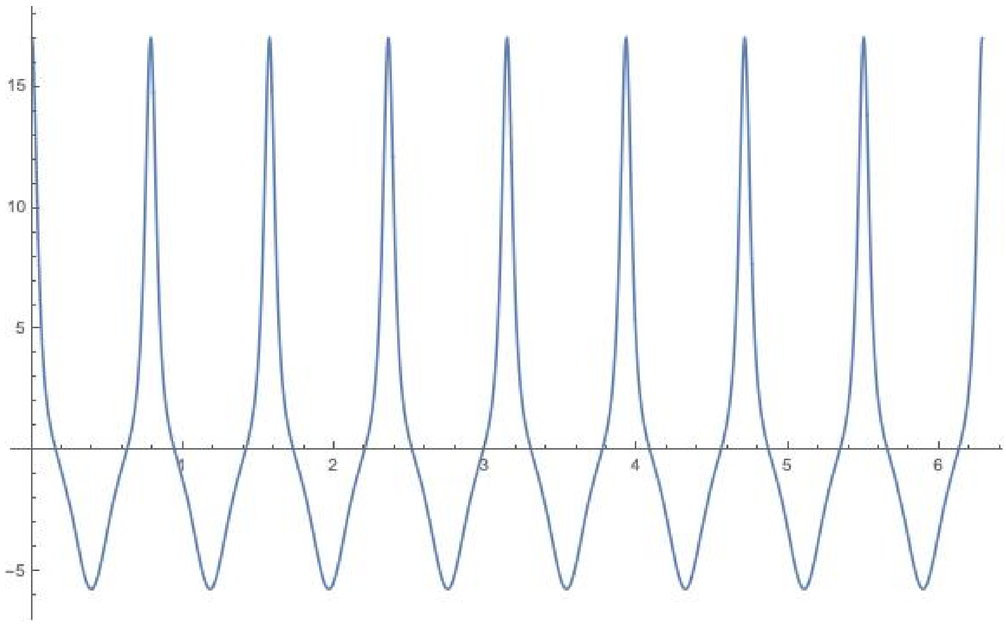

Figure 3.3. Curvature of C

Define the function $\eta : \mathbb{R}^2 \to \mathbb{R}$ given in polar coordinates as follows:

$$\eta(r, \theta) := 2r^4 - r^4 \cos(8\theta) - 1.$$

One can easily verify (directly from the formula) that η is D_8-invariant. Passing to Cartesian coordinates, one obtains:

$$\eta(x_1, x_2) = \begin{cases} 2(x_1^2 + x_2^2)^2 - \frac{x_1^8 - 28x_1^6x_2^2 + 70x_1^4x_2^4 - 28x_1^2x_2^6 + x_2^8}{(x_1^2+x_2^2)^2} - 1 & \text{if } (x_1, x_2) \neq (0,0), \\ -1 & \text{if } (x_1, x_2) \neq (0,0). \end{cases} \tag{3.78}$$

By direct verification, one has:

$$\nabla\eta(x_1, x_2) = \begin{bmatrix} \frac{8x_1(x_1^2+x_2^2)^4 - (16x^7 - 168x_1^5x_2^2 + 280x_1^3x_2^4 - 56x_1x_2^6)(x_1^2+x_2^2) + 4x_1(2x_1^8 - 28x_1^6x_2^2 + 70x_1^4x_2^4 - 28x_1^2x_2^6)}{(x_1^2+x_2^2)^3} \\ \frac{8x_2(x_1^2+x_2^2)^4 + (56x_1^6x_2 - 280x_1^4x_2^3 + 168x_1^2x_2^5)(x_1^2+x_2^2) + 4x_2(2x_1^8 - 28x_1^6x_2^2 + 70x_1^4x_2^4 - 28x_1^2x_2^6)}{(x_1^2+x_2^2)^3} \end{bmatrix}$$

for $(x_1, x_2) \neq (0,0)$ and

$$\lim_{\substack{x_1 \to 0 \\ x_2 \to 0}} \nabla\eta(x_1, x_2) = (0, 0).$$

Notice that η is of class C^2 and $\eta(x_1, x_2) = 0$, if and only if $x := (x_1, x_2) \in C$, so η satisfies conditions (η_1)–(η_6) and $\nabla\eta(0,0) = 0$. We are now in a position to define the required map $f : \mathbf{V} \times \mathbf{V}^{m-1} \times \mathbf{V} \to \mathbf{V}$ by the formula

$$f(x, y^1, y^2, \dots, y^{m-1}, z) := (|z|^2 + 1)\nabla\eta(x) + \mu_0 x + \sum_{j=1}^{m-1} \mu_j y^j \quad (x, y^j, z \in \mathbf{V}), \tag{3.79}$$

where μ_0 and μ_j are some constants. So far, f satisfies (A_1)–(A_2) and (A_3') while constants μ_0 and μ_j are a subject to satisfy the remaining conditions of Theorem 3.4.1.

To satisfy (R), we will assume $\mu_j = \mu_{m-j}$ for $j = 1, ..., m-1$. Next, to satisfy (A_4), we need to estimate $|\nabla\eta(x(\theta))|$. For this purpose, we will use again the polar coordinates and, by substituting $r = \sqrt[4]{\frac{1}{2-\cos(8\theta)}}$, one obtains:

$$|\nabla\eta(x(\theta))| = 2\frac{\sqrt{52 - 51\cos(8\theta) + 4\cos(16\theta) - \cos(24\theta)}}{(2 - \cos 8\theta)}$$

Observe also that

$$\mathbb{I}_x(z) = -\kappa(x)|z|^2, \tag{3.80}$$

where

$$\kappa(x(\theta)) = \frac{-\sqrt{2}(19 - 56\cos(8\theta) + 3\cos(16\theta)(2 - \cos(8\theta))^{\frac{5}{4}}}{(13 - 8\cos(8\theta) - 3\cos(16\theta))^{\frac{3}{2}}},$$

and the following estimates take place:

$$17 \geq \kappa(x) > -5.8, \quad 4 \leq |\nabla\eta(x)| \leq 21, \quad (x \in C). \tag{3.81}$$

We make the following assumption for (3.79):

$$\sum_{j=0}^{m-1} |\mu_j| > -4, \tag{3.82}$$

and put $R := 1$ (cf. (A_4)). Then, combining (3.79)–(3.82) with the inequality $|\nabla\eta(x)| + \kappa(x) > 1$ (see Figure 3.4, where the graph of $|\nabla\eta(x(\theta))| + \kappa((\theta))$, $x(\theta) \in C$, $\theta \in [0, 2\pi]$, is shown), one obtains:

$$
\begin{aligned}
\langle f(x,\mathbf{y},z), n_x\rangle &= (|z|^2+1)\langle \nabla\eta(x), n_x\rangle + \mu_0\langle x, n_x\rangle + \sum_{j=1}^{m-1}\langle \mu_j y^j, n_x\rangle \\
&= |z|^2|\nabla\eta(x)| + |\nabla\eta(x)| - |\mu_0||x| + \sum_{j=1}^{m-1}\langle \mu_j y^j, n_x\rangle \\
&\geq |z|^2\Big(|\nabla\eta(x)|\Big) + 4 - \sum_{j=0}^{m-1}|\mu_j||y^j| \\
&\geq |z|^2\bigg(-\kappa(x) + \Big(|\nabla\eta(x)| + \kappa(x)\Big)\bigg) + 4 - \sum_{j=0}^{m-1}|\mu_j| \\
&> -|z|^2\kappa(x) = \mathbb{I}_x(z),
\end{aligned}
$$

so that condition (A_4) is satisfied.

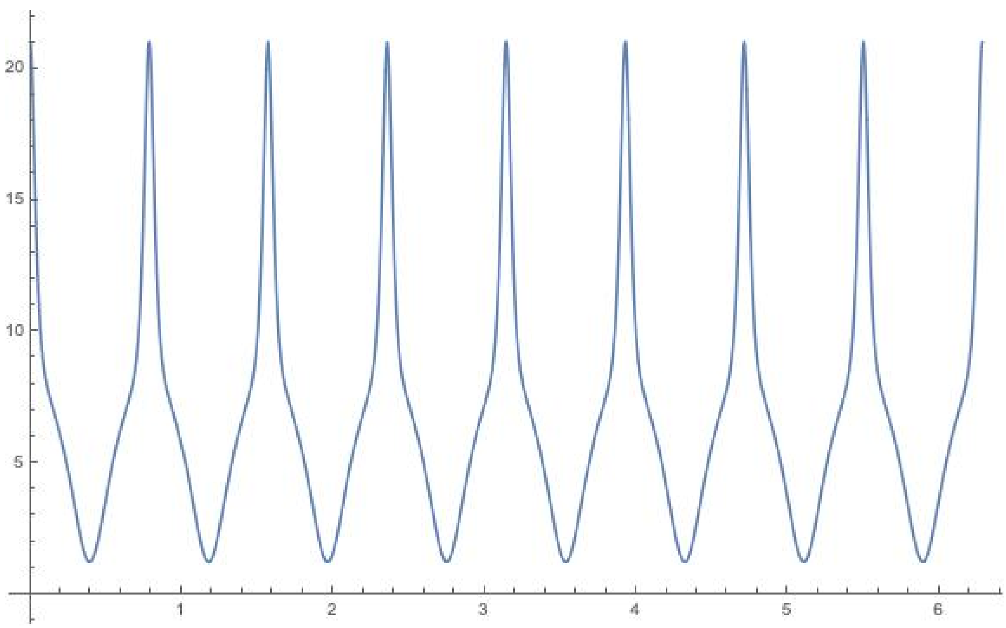

Figure 3.4. The values of $|\nabla\eta(x(\theta))| + \kappa(x)$ along the curve C. The minimal value of $|\nabla\eta(x(\theta))| + \kappa(x)$ is larger equal than 1.22522

It is easy to see that under the assumptions (3.82), the map (3.79) satisfies condition (A_5) with $A := 21$ and $B := \sum_{j=0}^{m-1}|\mu_j|$.

In order to show that condition (A_6') is satisfied, recall that $R = 1$ and one has the following relations for $x = (r\cos(\theta), r\sin(\theta))$, $r \le 1$ and $\alpha = 4\sqrt{13}$:

$$\begin{aligned}
|f(x,\mathbf{y},z)| &= \left|(|z|^2+1)\nabla\eta(x) + \mu_0 x + \sum_{j=1}^{m-1}\mu_j y^j\right| \\
&\le |z|^2|\nabla\eta(x)| + 21 + \sum_{j=0}^{m-1}|\mu_j| \\
&= |z^2|4\sqrt{r^4(4-4\cos(8\theta)+\cos^2(8t)+4r^4\sin^2(8\theta)} + C \\
&\le \alpha\Big(\Big(4r^3(2-\cos(8\theta))\Big)|z|^2 + |z|^2\Big) + C \\
&\le \alpha\Big(\Big(|z|^2\langle x, \nabla\eta(x)\rangle + |z|^2\Big) + C \\
&\le \alpha\Big(\langle x, f(x,\mathbf{y},\mathbf{z})\rangle + |z|^2\Big) + (1+\alpha)C,
\end{aligned}$$

where

$$C := 21 + \sum_{j=0}^{m-1}|\mu_j|.$$

Clearly condition (A_6') is satisfied with $K := (1+\alpha)\left(21 + \sum_{j=0}^{m-1}|\mu_j|\right)$.

We are now in a position to apply the main Theorem 3.4.1 with the group $G := O(2)\times D_8\times \mathbb{Z}_2$ and $V := \mathbb{R}^2$ being the natural D_8-representation. To this end, we need to study spectrum of the linearization at the origin (see (3.42)-(3.43)). We make the following assumption (cf. (3.41)):

$$\mu_0 + \sum_{j=1}^{r} 2\cos\frac{2\pi j}{m}\mu_j - \varepsilon_m\mu_{\frac{m}{2}} < -1, \tag{3.83}$$

where $r = \left\lfloor \frac{m-1}{2}\right\rfloor$. Then, $0 \notin \sigma(\mathcal{A})$ and

$$\sigma_-(\mathscr{A}) := \{\xi_0, \xi_1\},$$

where

$$\xi_0 = \mu_0 + \sum_{j=1}^{m}\mu_j, \quad \xi_1 := 1 + \frac{1}{2}\left(\sum_{j=1}^{r} 2\cos\frac{2\pi j}{m}\mu_j - \varepsilon_m\mu_{\frac{m}{2}}\right).$$

In this case, formulas (3.47)–(3.48) suggest:

$$\mathfrak{G}\text{-Deg}(\mathscr{A}, B(\mathscr{E})) = \deg_{\mathcal{V}_{0,1}^-} \cdot \deg_{\mathcal{V}_{1,1}^-},$$

where

$$\begin{aligned}
\deg_{\mathcal{V}_{0,1}^-} &:= (G) + (O(2) \times \mathbb{Z}_2^-) - (O(2) \times D_2^d) - (O(2) \times \widetilde{D}_2^d) \\
\deg_{\mathcal{V}_{1,1}^-} &:= (G) + 2(D_2 \times_{\mathbb{Z}_2}^{\mathbb{Z}_2^-} \widetilde{D}_2^q) + 2(D_2 \times_{\mathbb{Z}_2}^{\mathbb{Z}_2^-} D_2^q) + (D_2 {}^{D_1}\times_{\mathbb{Z}_2}^{\mathbb{Z}_2^-} \mathbb{Z}_2^q) - (D_2 {}^{D_1}\times_{\mathbb{Z}_2}^{\widetilde{D}_2^d} \widetilde{D}_2^q) \\
&\quad - (D_2 {}^{D_1}\times_{\mathbb{Z}_2}^{D_2^d} D_2^q) - 2(D_8 \times_{\mathbb{Z}_2}^{\mathbb{Z}_2^-} D_8^q).
\end{aligned}$$

Remark 3.4.3. (i) For any subgroup $S \leq D_8$, the symbol S^q stands for $S \times \mathbb{Z}_2$.

(ii) Given two subgroups $H \leq O(2)$ and $K \leq D_8^q$, we refer to Subsection 2.1.2 for the "amalgamated notation" $H\,{}^{Z}\times_L^R K$.

(iii) We refer to [9] for the explicit description of the (sub)groups $\widetilde{D}_k$, D_k^z, D_k^d, $\widetilde{D}_k^d$, and $\mathbb{Z}_2^-$.

The maximal orbit types in $\mathcal{V}_{1,1}^- \setminus \{0\}$ are:

$$(D_2 {}^{D_1}\times_{\mathbb{Z}_2}^{\widetilde{D}_2^d} \widetilde{D}_2^q), \quad (D_2 {}^{D_1}\times_{\mathbb{Z}_2}^{D_2^d} D_2^q), \quad (D_8 \times_{D_8}^{\mathbb{Z}_2^-} D_8^q). \tag{3.84}$$

We summarize our considerations in the statement following below.

Theorem 3.4.4. *Assume that D is given by* (3.76)*. Let $\Gamma = D_8$, $\mathbf{V} := \mathbb{R}^2$ be the natural D_8-representation and $f : \mathbf{V} \times \mathbf{V}^{m-1} \times \mathbf{V} \to \mathbf{V}$ be given by* (3.79)*, where the constants μ_0, μ_1, ..., μ_{m-1} satisfy conditions* (3.82) *and*

(3.83)*. Let (H_o) be one of the orbit types listed in* (3.84)*. Then:*

(i) (H_o) of maximal type (see Definition 3.1.5(a));

(ii) $\mathfrak{n}_1^{H_o} = 1$ (see Definition 3.2.2);

(iii) system (3.1) *admits a non-constant 2π-periodic solution $x(t)$ with the extended orbit type (H_o) (see Definition 3.1.5(b)).*

Actually, for our example, the equivariant invariant $\omega = (G) - \mathfrak{G}\text{-Deg}(\mathscr{A}, B(\mathscr{E}))$ can be exactly computed using the *Equideg* package in GAP system:

$$
\begin{aligned}
\omega = {} & 2(D_1 \times_{\mathbb{Z}_2}^{\mathbb{Z}_2^-} D_2^d) + 2(D_1 \times_{\mathbb{Z}_2}^{\mathbb{Z}_2^-} \widetilde{D}_2^d) + 2(D_1 \times \mathbb{Z}_2^-) - 2(D_2 \times_{\mathbb{Z}_2}^{\mathbb{Z}_2^-} \widetilde{D}_2^q) \\
& - 2(D_2 {}^{\mathbb{Z}_2}\times_{\mathbb{Z}_2^-}^{D_2^q}) - (D_2 {}^{D_1}\times_{\mathbb{Z}_2}^{\mathbb{Z}_2^-} D_2^d) - (D_1 \times D_2^d) - (D_2 {}^{D_1}\times_{\mathbb{Z}_2}^{\mathbb{Z}_2^-} \widetilde{D}_2^d) \\
& - (D_1 \times \widetilde{D}_2^d) - (D_2 {}^{D_1}\times_{\mathbb{Z}_2}^{\mathbb{Z}_2^-} \mathbb{Z}_2^q) + (D_2 {}^{D_1}\times_{\mathbb{Z}-2}^{\widetilde{D}_2^d} D_2^q) + (D_2 {}^{D_1}\times_{\mathbb{Z}_2}^{D_2^d} D_2^q) \\
& + 2(D_8 \times_{D_8}^{\mathbb{Z}_2^-} D_8^q) - (O(2) \times \mathbb{Z}_2^-) + (O(2) \times D_2^d) + (O(2) \times \widetilde{D}_2^d).
\end{aligned}
$$

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BIOGRAPHICAL SKETCH

Adrian C. Murza was born in Braşov, Romania. He studied at the Liceul de Matematică-Fizică, in Sfântu-Gheorghe, Romania. After an incursion in the med school and chemistry and the doctorate at Complutense University of Madrid, he had several postdocs in chemistry at the Tyndall Air Force Base (Florida), University of Pittsburgh and University of Wyoming.

He graduated (BS) in mathematics at the University of Porto, obtained his master's in mathematics at the University of Balearic Islands (2009) in Palma de Mallorca (under the supervision of Professors Antonio E. Teruel and Rafel Prohens), with a thesis on the dynamics of completely integrable ODEs of Lotka-Volterra type. Then, he earned his doctorate in applied mathematics (2014) from the University of Porto (under Professor Isabel Labouriau) with the thesis "Bifurcation of periodic solutions of differential equations with finite symmetry groups."

In 2015 he worked for one semester as a lecturer in the Department of Mathematics at the Transilvania University of Braşov. During this time, he started a collaboration with Professor Jaume Llibre (Autonomous University of Barcelona) on integrable systems (especially the Darboux Theory of Integrability), collaboration which continues nowadays. In 2016 he accepted a postdoc contract at the Institute of Mathematics "Simion Stoilow" of the Romanian Academy, in Bucharest. During this time, he had the chance to meet Professors Balanov and Krawcewicz and started a collaboration with their Nonlinear Analysis research group, from which he greatly benefited. This was his first tangency to the degree theory. In 2017, he became a PhD student in mathematics at UTD, under Professor Balanov's supervision.

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22. *Flow Analysis and First Integrals of a Family of 3D Lotka–Volterra Systems*, Master Thesis, University of Balearic Islands (2009).

23. *Chemical Oscillations in a Closed Sequence of Protein Folding Equilibria*, Libertas Mathematica **27** (2007), 125–130 (with I. Oprea and G. Dangelmayr).

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25. *A Stable C1b Soluble Protein and Its Regulation of Soluble Type 7 Adenylyl Cyclase: a Prototype for Soluble C1b Models*, Biohemistry **43** (2004), 15463–15471 (with J. Beeler, S. Yan, S. Bykov, S. Asher and W. Tang).

26. *Interaction of Antitumoral 9–Aminoacridine Drug with DNA and Dextran Sulfate Studied by Fluorescence and Surface–Enhanced Raman Spectroscopy*, Biopolymers **72** (2003), 174–184 (with S. Álvarez Méndez, S. Sánchez Cortés and J. García Ramos).

27. *Surface–Enhanced Raman and Steady Fluorescence Study of the Interaction between Antitumoral Drug 9–Aminoacridine and Guanidinobenzoatase: A Trypsin–like Protease Related to Metastasis Processes*, Biopolymers **62** (2001), 85–94 (with S. Sánchez Cortés and J. García Ramos).

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29. *Interaction of Antimalarial Drug Quinacrine with Nucleic Acids of Variable Sequence Studied by Spectroscopic Methods*, Journal of Biomolecular Structure and Dynamics **18** (2000), 371–383 (with L. Rivas, S. Sánchez Cortés and J. García Ramos).

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31. *Essential Role of the Concentration of Immobilized Ligands in Affinity Chromatography: Purification of Guanidinobenzoatase on an Ionized Ligand*, Journal of Chromatography B Biomedical Sciences and Applications **740** (2000), 211–218 (with R. Fernández Lafuente and J. Guisán).

32. *Affinity Chromatography of Plasma Proteins (Guanidinobenzoatase): Use of Mimetic Matrices and Mimetic Soluble Ligands to Prevent the Binding of Albumin on the Target Affinity Matrices* , Journal of Chromatography B Biomedical Sciences and Applications **732** (1999), 165–172 (with A. Robledo Aguilar, R. Fernández Lafuente and J. Guisán).

33. *Fluorescence and Surface Enhanced Raman Study of 9-Aminoacridine in Relation to Its Aggregation and Excimer Emission in Aqueous Solution and on Silver Surface*, Biospectroscopy **4** (1998), 327–339 (with S. Sánchez Cortés and J. García Ramos).

Conferences, Workshops and Invited Talks:

2018	University of Murcia, Murcia, Spain (invited talk)
2018	Complutense University of Madrid, Madrid, Spain (invited talk)
2018	University of Balearic Islands, Palma de Mallorca, Spain (invited talk)
2014	Workshop on Symmetry and Dynamics, Porto, Portugal (poster)
2011	Congress on Differential Equations and Applications, Palma de Mallorca, Spain (oral presentation)
2009	Workshop on Network Dynamics, Exeter, UK (poster)
2009	Biosim Conference, Copenhagen, Denmark (poster and oral presentation)
2009	Workshop on Mathematical Modelling, Dubrovnik, Croatia (poster and oral presentation)
2008	Biosim Conference, Budapest, Hungary (poster and oral presentation)
2008	XV Congress on Statistical Physics, Salamanca, Spain (poster)
2007	Biosim Conference, Potsdam, Germany (poster and oral presentation)
2007	Biosim Conference, Basel, Switzerland (poster and oral presentation)
2007	Workshop on Coherent Behavior In Neuronal Networks, Palma de Mallorca, Spain (poster)
1998	XV National Annual Meeting on Spectroscopy, Sevilla, Spain (poster)