PRICING, UPGRADE, AND CONTRACT MANAGEMENT

by

Xiao Zhang

APPROVED BY SUPERVISORY COMMITTEE:

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Özalp Özer, Chair

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Metín Çakanyıldırım

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Serdar Şimşek

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Shengqi Ye

Dedicated to my beloved wife, Jing Li.

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XIAO ZHANG, BS, MS

DISSERTATION

Presented to the Faculty of

The University of Texas at Dallas

in Partial Fulfillment

of the Requirements

for the Degree of

DOCTOR OF PHILOSOPHY IN

MANAGEMENT SCIENCE

THE UNIVERSITY OF TEXAS AT DALLAS

August 2018

ACKNOWLEDGMENTS

I am grateful to my committee members for their advice, encouragement, and continual guidance. First, I want to express my deepest gratitude to my advisor, Dr. Özalp Özer. I started working with Dr. Özer right after joining the PhD program. It is Dr. Özer who introduced me to an eye-opening internship opportunity, who encouraged me to take up challenging research questions, who set a great example of a responsible researcher by demonstrating academic rigor and integrity, and who led me to where I am today. I am also indebted to his funding support during my PhD study. Secondly, I would like to express my sincere gratitude to Dr. Metin Çakanyıldırım for his unreserved support. It is fortunate for me to have Dr. Çakanyıldırım who has always been prompt in giving feedback, knowledgeable to provide insightful advice, patient when listening to my problems and explaining things, and most importantly, encouraging whenever I had difficulties. My further gratitude goes to Dr. Serdar Şimşek and Dr. Shengqi Ye for their timely and valuable comments that greatly improved the presentation and contents of this dissertation. This work could not have been accomplished without all their continual help.

I want to acknowledge the unconditional support I received during my job search from faculty members in the Department of Operations Management: Drs. Alain Bensoussan, Qi Chen, Milind Dawande, Carol Flannery, Dorothée Honhon, Ganesh Janakiraman, Elena Katok, Anyan Qi, Suresh Sethi, Kathryn Stecke, and Shouqiang Wang. My gratitude is also extended to Amanda Besch, Jennifer de Leon, Ashley Desouza, Jamie Selander and Lindsay Wilson for their excellent administrative support.

My thanks go to my dear friends at UT Dallas: Yang Bo, Ying Cao, Bahriye Cesaret, Jiayu Chen, Shaokuan Chen, Wei Chen, Emre Ertan, Zhichao Feng, Blair Flicker, Harish Guda, Shivam Gupta, Varun Gupta, Bharadwaj Kadiyala, Ismail Kirci, Chungseung Lee, Jingyun Li, Meng Li, Lantian Liang, Ting Luo, Liying Mu, Cheng Nie, Sandun Perera, Xi Shan, Sina Shokoohyar, Yulia Vorotyntseva, and many others. I cherish our friendship and will always remember all the unforgettable moments that we spent together. This last word of acknowledgment I have saved for my parents, sister and wife for their unconditional love, understanding, and dedication.

June 2018

PRICING, UPGRADE, AND CONTRACT MANAGEMENT

Xiao Zhang, PhD
The University of Texas at Dallas, 2018

Supervising Professor: Özalp Özer, Chair

The theory and practice of pricing have advanced significantly over the past decades. Pricing management, the process of integrating all perspectives and information necessary to consistently reach optimal pricing decisions, is a critical key to business success. This dissertation explores the roles of pricing management in revenue maximization and supply chain coordination. In particular, we study how firms price dynamic upgrades to improve revenue and how supply chain members share inventory risk through promised lead time pricing contracts. Adopting a methodology that combines theoretical modeling and numerical analysis, we provide detailed pricing guidance and business insights to practitioners.

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CHAPTER 1

INTRODUCTION

This dissertation consists of two essays that focus on recent innovative pricing management strategies. In particular, we study how firms price dynamic upgrades to improve revenue and how supply chain members share inventory risk through promised lead time pricing contracts.

Upgrading – offering a premium product to recall and replace a reserved but unused regular product – has been widely adopted in practice by travel companies. Traditionally, upgrades are static in time, since they are offered once to customers either at the check-in time (e.g., car rental industry) or at the booking time (e.g., hotel industry). In Chapter 2, we study the upgrades that are dynamically offered if profitable between and including the booking time and the check-in time. They are more flexible in terms of offer quantity and timing. We remark that the traditional static upgrades are special cases of the dynamic one considered in Chapter 2.

The research question in Chapter 2 is motivated by a Dallas-based hotels upgrading process. The hotel hosts several conferences every year and offers two types of rooms: the premium room and the regular room. Room prices are predetermined for each conference, and the price of the premium room is naturally higher. The revenue manager uses an email upgrading method. If the number of leftover premium rooms is high, the manager sends emails to regular room purchasers several days before the check-in date and asks whether they want to upgrade to premium rooms for a small upgrade fee. Currently, the hotel's upgrade process is manually done. The hotel is exploring the possibility of using an automated system to jointly optimize the timing, quantity, and pricing of upgrades. Firms that are similar to the Dallas-based hotel can also benefit from upgrades offered via emails. In fact, such upgrade capabilities have recently been incorporated into some existing automated revenue management systems (e.g., Rentalcar.com and Malaysia Airlines). These recent industry practices further attest to the timeliness and importance of studying dynamic upgrades offered between the booking and the check-in times.

The trade-offs in the dynamic upgrade process are subtle. When the number of leftover premium products is large, it is possible that some of them remain unsold by the check-in time. Moreover, when the leftover regular products are few in numbers, they might be fully booked, and the firm loses the chance to capture regular customers with low reservation prices. These two outcomes can both be mitigated by upgrading. The firm also needs to maintain enough premium capacity for forthcoming premium customers. When selling a premium product through an upgrade, the firm loses the opportunity of selling it at the full price but frees up a regular product capacity for a possible future sale. The firm needs to analyze the trade-offs between the upgrade fee and the opportunity gain/loss of replacing a leftover premium product by a regular one.

In Chapter 2, we focus on a firm selling two types of products, premium and regular, over a sales season that naturally ends after the check-in date. No replenishment of products is considered. Customers for each product arrive over time. Depending on the leftover capacities of both products at a particular time, the firm can incite regular product purchasers to buy premium products by sending them upgrade notifications that contain upgrade links. After receiving an upgrade link, a regular product purchaser becomes an upgradeable customer. After clicking the link, the upgradeable customer is directed to the firm's upgrade website where she sees an upgrade fee. The upgrade process generates another customer arrival stream in addition to the direct arrivals for premium and regular products. This additional stream of demand depends on previous regular customer arrivals as well as how the firm manages the upgrade process. The firm has the ability to deactivate some or all upgrade links, if upgrading is no longer profitable. Deactivation of the link received by an upgradeable customer cuts off the connection to the upgrade website and shuts down the upgrade demand potentially coming from this customer. Upgrades are time-limited, and the firm controls their durations by sending and deactivating upgrade links. The upgrade notifications sent to customers contains no pricing information. At the arrival of an upgradeable customer to the upgrade website (when an upgradeable customer clicks the link), an upgrade fee is instantaneously generated. The upgrade fee depends on the cost-benefit analysis of replacing a premium product by a regular one, which is in turn driven by current leftover capacities as well as the future demand expectations. Hence, postponing the upgrade fee until the arrival of an upgradeable customer gives the firm the most up-to-date information in determining an optimal fee.

Our main contributions include the description of the dynamic upgrade process and the optimization of this process to provide insights. The firm's revenue maximization problem is formulated as a dynamic program, and we show that the optimal upgrade policy is of a pulsing type; the firm either maintains zero or the maximum number of upgrade links. Both the optimal number of active links and the optimal upgrade fee are monotone with respect to the leftover capacities. When there are more regular (resp., premium) products leftover, it is optimal for the firm to maintain fewer (resp., more) active upgrade links and to offer a higher (resp., lower) upgrade fee. To obtain these results analytically, we introduce new properties of DH-modularity and DV-modularity for a function and prove these properties for the optimal expected revenue function. Using a systematic numerical study, we compare the industry-standard check-in fixed-price upgrade and the dynamic upgrade and quantify the potential revenue improvement by switching from the static strategy to the dynamic one. We determine when the revenue improvement is significant and how different market environment parameters affect it. For example, we show that a firm can improve its revenue by as much as 49% (in a market with a high premium product capacity level, a low regular product capacity level, a low premium demand, a high regular demand, a high premium price, a low regular price, and a high click rate) when it switches from a static check-in upgrade policy to a dynamic upgrade policy. Finally, our model takes notification spamming into consideration and leads to a detailed guidance on industry implementations. We remark that this chapter is written as a paper submitted for a publication. We request that the interested reader to reference Çakanyıldırım et al. (2018).

Chapter 3 focuses on the promised lead time contract (PLTC). In a two-level supply chain where stochastic customer demand is fulfilled only by downstream retailers, a PLTC designed by the upstream manufacturer specifies the delivery lead times between the supply chain members. The manufacturer guarantees shipment of each order on time and in full to a retailer after the lead time. When the lead time is zero, the retailer can receive shipment immediately to fulfill end customer demand and does not need to hold any inventory, while the manufacturer needs to have ample inventory to satisfy the retailer's order. When the lead time is long enough, the manufacturer can start production after receiving the retailer's order and does not need to hold any inventory, while the retailer needs to place advance orders so that it can receive shipments timely to fulfill end customer demand. Supply chain members share inventory cost under a PLTC. A cost-benefit analysis of the shared inventory cost determines the pricing of a certain promised lead time – the retailer pays a premium for a short lead time or gets compensated for a long one.

The research question in Chapter 3 is inspired by the PLTCs designed by Herman Miller, a major American manufacturer which produces customizable office furniture. We investigate why Herman Miller designs different contracts for its dealers. In our model, we specifically study the PLTCs between a manufacturer and two retailers with different inventory costs and end customer demands. The manufacturer's optimal contract design depends on whether retailers are in the same or different markets; the manufacturer has to treat same-market retailers fair, while it can discriminate retailers from different markets. We characterize the optimal PLTCs in both the same-market and different-market settings, and compare these decentralized results to those in the centralized control setting. The PLTC is efficient in the different-market setting but not in the same-market setting. The manufacturer applies process standardization to produce multiple products; the initial steps in production are standardized, and products are not differentiated until later customization steps. The standardized production process delays the differentiation point, increases flexibility of handling fluctuating multiple-product orders from the retailers, decreases inventory costs and further affects the optimal PLTCs. We quantify the impact of differentiation postponement on the optimal promised lead times in all three settings and identify the conditions under which the manufacturer shifts its production mode from make-to-order to make-to-stock. Chapter 3 explains why manufacturer offers different lead times to its retailers. Our analysis of postponement illustrates and quantifies its indirect impact on supply chain PLTCs. For further discussion and insights, we also refer the reader to Lutze et al. (2018).

CHAPTER 2

DYNAMIC PRICING AND TIMING OF UPGRADES

2.1 Introduction

Upgrading – offering a premium product to recall and replace a reserved but unused regular product – has been widely adopted in practice by travel companies. Traditionally, upgrading is free of monetary charges and incorporated as a key feature in loyalty programs to improve customer relationships. Most traditional upgrades are offered at the check-in time depending on the availability of the premium product. Take the "500-mile upgrades" program from American Airlines as an example. If better seats (than booked) are available at the check-in time, high-tier loyalty program members get complimentary upgrades, and low-tier loyalty program members may need to redeem 500 miles for the upgrades. Some other traditional upgrades are offered at the booking time. In "systemwide upgrades" from American Airlines, the highest tier loyalty program members receive 4 complimentary one-way upgrades to business class annually when they reserve economy seats.

Recently emerged upgrades require monetary charges in the form of upgrade fees and bring extra revenues to firms. One basic type of revenue-generating upgrade is carried out at the check-in time. Examples include the upgrades offered by the front desk personnel at a hotel or the check-in kiosk software at an airport. Such upgrades are especially popular in the car rental industry, in which check-in agents ask customers if they want to upgrade the currently-reserved car to a premium one. According to the Consumer Federation of America, an average car rental company makes 10% of its revenue from such upgrades[1]. The other type of revenue-generating upgrade is offered at the booking time. Upon completion of a regular product reservation, customers may see an upgrade fee menu on the confirmation webpage, based on which they decide whether to accept the upgrades or not. This type of

[1] https://www.allianztravelinsurance.com/travel/rental-cars/rental-car-upsells.htm

booking-time upgrade is popular in the hotel industry. It is designed by nor1, a hospitality merchandising technology company, under the name of *eStandby Upgrade*[2]. Both types of upgrades are static in time, since they are offered once to customers either at the check-in time or the booking time. In this chapter, we study the upgrades that are dynamically offered if profitable after the booking time until the check-in time. They are more flexible in terms of offer quantity and timing. We remark that the static check-in upgrade is a special case of the dynamic one considered in this chapter.

This chapter is partially motivated by a Dallas-based medium-sized hotel's upgrading process. The hotel hosts several conferences every year and offers two types of rooms: the premium room and the regular room. Room prices are predetermined for each conference, and the price of the premium room is naturally higher. The hotel revenue manager uses a simple upgrading method. If the number of leftover premium rooms for a particular check-in date is high, the manager sends emails to regular room purchasers several days before that check-in date and asks whether they want to upgrade to premium rooms for a small upgrade fee. The manager prefers in-advance email upgrading to check-in front-desk upgrading, because the former expedites the check-in process, gives more in-advance visibility for cleaning the rooms and helps to realize the revenue earlier. Currently, the hotel's upgrade process is manually done. The hotel is exploring the possibility of using an automated system to jointly optimize the timing, quantity and pricing of upgrades.

Firms that are similar to the Dallas-based hotel can benefit from upgrades offered via emails. In fact, such upgrade capabilities have recently been incorporated into some existing automated revenue management systems. Figure 2.1 shows screenshots of upgrade emails from Rentalcar.com and Malaysia Airlines. These recent industry practices further attest to the timeliness and importance of studying upgrades offered between the booking and the check-in times. The medium of upgrade notifications is not restricted to email. *eXpress*

[2] http://www.nor1.com/estandby-upgrade/

Upgrade[3], newly developed by nor1, is an upgrade product adopting in-app push notifications to engage customers (e.g., to deliver upgrade notifications). Recent statistics show that the engagement rate of push notifications in the travel industry is about 30%[4].

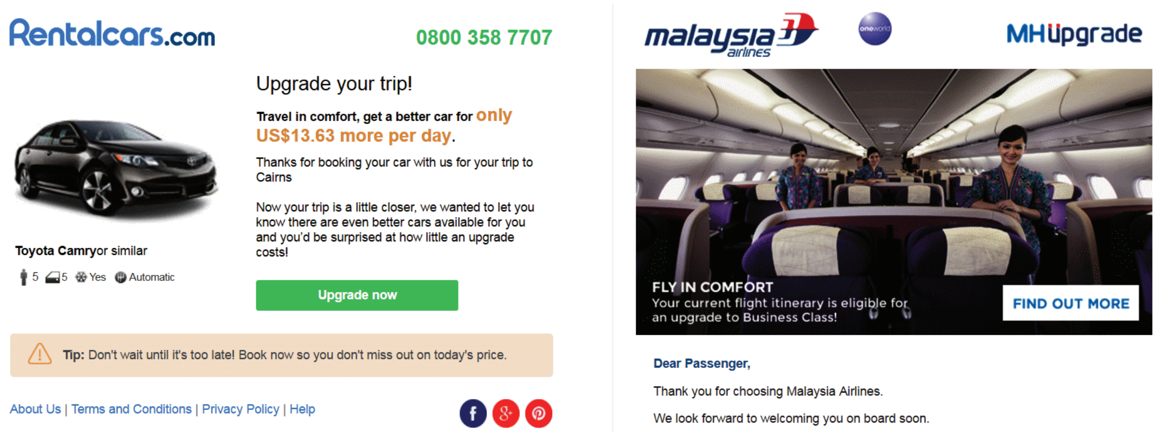

Figure 2.1. Upgrade email examples from Rentalcars.com and Malaysia Airlines

The trade-offs in the dynamic upgrade process are subtle. When the number of leftover premium products (e.g., rooms, cars, or seats) is large, it is possible that some of them remain unsold by the check-in time. Moreover, when the leftover regular products are few in numbers, they might be fully booked and the firm loses the opportunity to capture regular product customers with low reservation prices. These two outcomes can both be mitigated by upgrading. The firm also needs to maintain enough premium capacity for forthcoming premium customers. When selling a premium product through an upgrade, the firm loses the opportunity of selling it at the full price but frees up a regular product capacity for a possible future sale. The firm needs to analyze the trade-offs between the upgrade fee and the opportunity gain/loss of replacing a leftover premium product by a regular one.

In this chapter, we focus on a firm (e.g., a hotel, a car rental company, or a cruise operator) selling two types of products, *premium* and *regular*, over a sales season that naturally

[3]http://www.nor1.com/express-upgrade/

[4]http://andrewchen.co/new-data-on-push-notification-ctrs-shows-the-best-apps-perform-4x-better-than-the-worst-heres-why-guest-post/

ends after the check-in day of these products. No replenishment of products is considered. Customers for each product arrive over time. Depending on the leftover capacities of both products at a particular time, the firm can incite *regular product purchasers* to buy premium products by sending them upgrade notifications that contain upgrade links. After receiving an upgrade link, a regular product purchaser becomes an *upgradeable customer*. After clicking the link, the upgradeable customer is directed to the firm's upgrade website where she sees an upgrade fee.

Firms often prefer to send upgrade notifications to a subset of regular product purchasers. Reasons for limited upgrade notifications include the following. The duration between the booking and the check-in times often allows firms to have several time epochs, each of which provides an opportunity to send upgrade notifications. Some regular purchasers may be contacted later, if they are not contacted at the present time. Discretely distributing upgrade offers over time and sending notifications to a small set of regular purchasers at a time epoch can reduce forgetting and inactivity of these purchasers. Moreover, sending many notifications at once is riskier than sending a few at multiple times. If upgrades are made available at once to all regular purchasers and are accepted, then all of the regular sales can become premium sales. Consequently, the firm may be left with too few premium products to sell at the original high price in the future. This outcome goes against the spirit of upgrading, which is to smoothly balance the leftover capacities against the future demands.

The upgrade process generates another customer arrival stream in addition to the arrivals for premium and regular products. This additional stream of demand depends on previous regular customer arrivals as well as how the firm manages the upgrade process. The notifications emphasize the time-limitedness of the upgrade offers to create a sense of urgency (similar to the Rentalcar.com email in Figure 2.1) but do not specify the deadlines of the upgrade offers (similar to the Malaysia Airlines email in Figure 2.1). The firm has the ability to deactivate some or all upgrade links, if upgrading is no longer profitable. Deactivation of the link received by an upgradeable customer cuts off the connection to the upgrade website and shuts down the upgrade demand potentially coming from this customer. When upgrading becomes profitable again, the firm sends out new upgrade notifications with active links. Therefore, upgrades are time-limited, and the firm controls their durations by sending and deactivating upgrade links. Throughout the chapter, the number of upgradeable customers and the number of active upgrade links are used interchangeably. The number of active links maintained by the firm at any time affects the upgrade demand. Hence, the upgrade process formulated in this chapter is a dynamic demand shaping strategy.

The upgrade notifications sent to customers include little pricing information (e.g., the Rentalcars.com email in Figure 2.1) or no pricing information (e.g., the Malaysia Airlines email in Figure 2.1). At the arrival of an upgradeable customer to the firm's upgrade website (when an upgradeable customer clicks the link), an upgrade fee is instantaneously generated. The upgrade fee depends on the cost-benefit analysis of replacing a premium product by a regular one, which is in turn driven by current leftover capacities as well as the future demand expectations. Hence, postponing the determination of the upgrade fee until the arrival of an upgradeable customer gives the firm the most up-to-date information in determining the optimal fee. A lower upgrade fee increases the chance of a sale but decreases the marginal revenue of the sale. Thus, the trade-off in the specification of the upgrade fee is similar to that in classic dynamic pricing models.

The upgrade demand stream is created to mitigate a leftover capacity imbalance, i.e., when there are too many premium products and/or too few regular products. Another method of such mitigation is dynamic pricing of the premium and regular products throughout the season. Despite the presence of some online retailers opting for dynamic pricing, several firms including small- or medium-sized hotels and car rental companies do not dynamically optimize product prices. Keeping product prices constant during the season is simpler to implement, fosters credibility of the firm and the trust between the firm and most customers, eliminates cannibalization due to too-closely set dynamic prices, and avoids strategic waiting of customers for lower prices. Because of these reasons and also to focus on the problem of dynamic upgrades, we keep prices of both products constant during the season. This setting also fits to our motivating example of the Dallas-based hotel. The firm in this chapter dynamically decides on the timing, quantity and pricing of upgrades.

Some of our contributions include the description of the dynamic upgrade process and the optimization of this process as well as its variations to provide insights. We develop three model variations to study the dynamic upgrade pricing and timing problem. In the base model, the firm chooses the upgrade fee from an interval. In the second model, the firm is subject to fee restrictions and chooses upgrade fees from a subset of the interval (e.g., a subinterval or a discrete set). In the third model, we incorporate upward substitution into the base model; i.e., when the firm runs out of the regular product, it can consider selling the premium product below its original price to an incoming regular customer. The selling price can be interpreted as the regular product price plus a substitution fee. The firm's revenue maximization problems in all three models are formulated as dynamic programs. In all models, we show that the optimal upgrade policies are of a pulsing type; the firm either maintains zero or the maximum number of upgrade links. In the base and the restricted fee model, both the optimal number of active links and the optimal upgrade fee are monotone with respect to the leftover capacities. When there are more regular (resp., premium) products leftover, it is optimal for the firm to maintain fewer (resp., more) active upgrade links and to offer a higher (resp., lower) upgrade fee. In order to obtain these results analytically, we introduce new properties of DH-modularity and DV-modularity for a function and prove these properties for the optimal expected revenue functions. By comparing optimal policies across models, we show, for example, when restricting upgrade fees increases the optimal number of upgrade notifications. We also show that the optimal substitution fee is always smaller than the optimal upgrade fee.

In addition, our model takes notification spamming into consideration and leads to a detailed guidance on implementation. Using a systematic numerical study, we compare the industry-standard check-in fixed-price upgrades and the dynamic upgrades and quantify the potential revenue improvement by switching from the static strategy to a dynamic one. We determine when the revenue improvement is significant and how the revenue improvement is affected by the firm's upgrade fee restriction and upward substitution. For example, we show that a firm can improve its revenue by as much as 49% (in a market with a high premium product capacity level, a low regular product capacity level, a low premium demand, a high regular demand, a high premium price, a low regular price, and a high click rate) when it switches from a static check-in upgrade policy to a dynamic upgrade policy.

The remainder of the chapter is organized as follows. In §2.2, we review the related literature. In §2.3, we model the firm's dynamic upgrade problem as a base model and characterize the structure of the optimal upgrade policy. In §2.4, we analyze two variations of the base model. In §2.5, we illustrate the implementation of the dynamic upgrade policy. In §2.6, we quantify the benefits of dynamic upgrades through a numerical study. In §2.7, we conclude the chapter. Proofs, counterexamples and additional arguments are relegated to the appendices.

2.2 Literature Review

Chapter 2 is mainly related to three streams of literature on (i) dynamic pricing and advertising, (ii) firm-driven upgrades, and (iii) email and push notification management.

Dynamic pricing and advertising: The literature on dynamic pricing of limited capacity was pioneered by Gallego and van Ryzin (1994) and Bitran and Mondschein (1997). We adopt the modeling framework by Bitran and Mondschein (1997) in which the time period is small enough so that at most one customer shows up[5]. This literature has been developed

[5]Dividing time into short intervals is common in the revenue management literature and allows for a formulation with an arbitrary sequence of customer arrivals. An alternative formulation is to assume that

subsequently to incorporate multi-product pricing, incomplete demand information, strategic customer behavior and competition. We refer the reader to Talluri and van Ryzin (2004), Bitran and Caldentey (2003), Özer and Phillips (2012) and Chen and Chen (2015) for a detailed review. Papers in this literature optimize prices of actual products, while we focus on the optimal pricing of upgrades.

The dynamic upgrade notifications have an advertising effect as they inform the customers about the upgrades. Thus, Chapter 2 is also related to the extensive marketing literature on dynamic advertising; see Feichtinger et al. (1994) for a comprehensive review of earlier papers. The most relevant papers within this literature focus on the pulsing policy, in which a firm alternates between zero and a high level of advertising. One of the goals of these papers is to identify models and conditions under which the pulsing policy is superior to the policy of a constant level of advertising (e.g., Simon 1982, Mahajan and Muller 1986, Mesak 1992, and Aravindakshan and Naik 2015). The optimal upgrade policy in our model is also a pulsing type: the firm either maintains zero or the maximum number of active upgrade links. There is also a limited literature on joint dynamic pricing and advertising. This literature focuses on a single product setting (e.g., MacDonald and Rasmussen 2010, Ye et al. 2015, and Schlosser 2016). The key difference in our model is that the firm sells two products and controls the upgradeable customer arrival rate by sending out upgrade notifications.

Firm-driven upgrades: Gallego and Stefanescu (2012) and Chen and Chen (2015) refer to upgrading as replacing a customer's regular product with a premium one for free. They use upselling to denote a situation in which such a replacement comes with an extra charge. In this chapter, we use the term upgrade to represent both the free and non-free replacement. The literature studying firm-driven upgrades can be broadly classified into three categories.

arrivals are according to a Poisson process. Instead of discrete time intervals, the decision controls can be embedded at customer arrival epochs (see Puterman 1994, Chapter 11). Our results continue to hold for this case as well.

The first category studies upgrades for dynamic capacity management. In these studies, a firm decides the initial capacities and allocates multiple classes of products to demands from the same number of customer classes. Shumsky and Zhang (2009) find the optimal allocation policy within the class of single-level free upgrade policies, in which a customer whose preferred product has been depleted can be upgraded by at most one level. Extending this work, Yu et al. (2015) study the optimal multi-level free upgrade policy. Unlike these two papers primarily focusing on the capacity allocation problem, we concentrate on the timing, quantity and pricing of upgrades.

The second category studies upgrades for revenue management, in which the number of customer classes can be greater than the number of capacity classes. Gallego and Stefanescu (2009) study capacity holder's and reseller's upgrade problems with both an independent demand model and a multinomial logit model. They formulate the stochastic optimal control problems and analyze the corresponding fluid models. Steinhardt and Gönsch (2012) analyze the dynamic program with independent demands from Gallego and Stefanescu (2009) and propose new structural results. They also propose two different dynamic programming decomposition approaches to get tight upper bounds on the value of the original dynamic program. Recently, McCaffrey and Walczak (2016) solve an airline-specific upgrade problem with two classes of capacities (business and economy seats). The upgrades in these three papers are offered either at the booking time or postponed until the check-in time. In contrast, Chapter 2 endogenizes the timing and pricing of upgrades as decisions.

The third category focuses on the recently emerged conditional upgrade pricing problem. In the conditional upgrade, a customer upon completion of her booking of a regular product may see an upgrade menu on the confirmation webpage or receive an email that leads to an upgrade menu. The customer accepts or rejects the upgrade based on the corresponding upgrade fee. The upgrade is fulfilled at the check-in time, and the customer pays the upgrade fee only if the premium product is available. Cui et al. (2016) and Yılmaz et al. (2016) analyze the interaction between the firm and customers with game theoretic models and evaluate the benefit of conditional upgrades in the presence of strategic customer behavior. Biyalogorsky et al. (2005) study upgrades as probabilistic goods (payments of upgrades are required at the booking time) and identify the situation in which upgrading is more profitable than advance selling. The upgrades considered in this category of papers are offered at the booking time. The upgrades, in Chapter 2, do not have probabilistic features and are dynamically priced and offered from the booking time until the check-in time.

Email and push notification management: Emails and push notifications, the vehicles of upgrade notifications in Chapter 2, are widely-adopted marketing tools in industry. However, the dynamic management of such tools has received limited attention in marketing and operations literature. Neslin et al. (2013) point out that a customer's response to marketing and purchase probability depend on her "recency", which is the length of time since the customer's previous purchase. They suggest a recency-based customer targeting strategy through emails. Investigating the impact of the number of emails sent by a firm on its profitability, Zhang et al. (2017) provide a guidance for email marketing campaigns. The email marketing policies in both papers are driven by customer response behavior. In contrast, our optimal upgrade notification policy is driven by capacity imbalance. Recently, Wang et al. (2017) formulate a dynamic push notification campaign as a large-scale resource-allocation problem and analyze the problem in an asymptotic regime.

2.3 The Base Model

Here we introduce the base model for the dynamic upgrade pricing and timing problem. In §2.3.1, we describe the dynamic upgrade process and formulate the firm's revenue maximization problem as a dynamic program. In §2.3.2, we show the structure of the optimal upgrade policy and the corresponding monotonicity properties.

2.3.1 Dynamic Upgrade Process and Formulation

Consider a firm that sells two types of products, premium and regular, at predetermined prices over a finite sales season. No replenishment is allowed, and leftover products have zero value after the end of the season (e.g., after the check-in date or the departure time). Customers arrive over time and reserve the products whose consumption takes place at the end of the season. Each new customer tries to reserve a premium or a regular product. If the preferred product (premium or regular) is available, the customer pays the corresponding price and becomes a (premium or regular) purchaser. Otherwise, the customer leaves without purchasing. Cancellation is not allowed. As a consequence of random arrivals, leftover capacities may be imbalanced compared to the future demands.

Dynamic upgrades are used to balance the leftover capacities until the end of the season. When there are many leftover premium products and/or few leftover regular products, the firm can balance the leftover capacities by sending upgrade notifications via emails or phone push notifications to regular purchasers. An upgrade notification contains a time-limited upgrade link, and its purpose is to incite a regular purchaser to upgrade to a premium product at a discounted price. If upgrades later cease to be beneficial to the firm, the links can be deactivated. A regular purchaser who has an active upgrade link is called an upgradeable customer. After clicking the link, the upgradeable customer is directed to the firm's upgrade website and sees an upgrade fee. Based on the upgrade fee, the upgradeable customer decides whether to accept the upgrade to a premium product or not.

The sales season is divided into N periods. Period 1 is the starting period of the season and period $N + 1$ is the final consumption period after the sales season. Throughout the chapter, we use the following notation for brevity. For any nonnegative integers a and b with $a < b$, $[a : b]$ is defined as the set containing all integers between and including a and b; $[a : b] := \{a, a + 1, ..., b\}$. All notations are summarized in Appendix A.1.

The firm tracks the leftover capacities as state variables. The premium product price is higher than that of the regular product. We use (h_n, l_n) to denote the pair of leftover capacities of premium and regular products in any period $n \in [1 : N + 1]$. In period 1, the initial capacities of premium and regular products are H and L. In period n, the firm has $L - l_n$ regular purchasers.

The firm has two decisions to make in any period n. The first decision is how many upgrade links to send and to deactivate. Sending and deactivating upgrade links is equivalent to deciding on the number u_n of active upgrade links to maintain. With a greater u_n, it is more likely that one of the upgradeable customers clicks the upgrade link. The firm uses u_n to control the clicking (or the arrival) process of the upgradeable customers. The firm considers upgrades only when the leftover regular capacity l_n is less than or equal to a prespecified threshold M, which is the *upgrade triggering level*. This threshold is set by the management to ensure that the firm sells enough regular products (or accumulates enough regular product purchasers) before considering upgrades. When $l_n > M$, the firm has an ample leftover capacity of regular products, so an upgrade to release a unit of regular capacity is likely to be unnecessary. This upgrade triggering mechanism is similar to *eXpress Upgrade* by nor1. Both upgrades are initiated closer to the end of the sales season. *eXpress Upgrade* is triggered by a preset triggering date, while our upgrade process is triggered by the upgrade triggering threshold M. The maximum number of active upgrade links the firm can maintain is C, which measures the firm's *maximum upgrade capability*. A Firm often sets such a limit on the total active upgrade links at any given time to better manage the spamming issue and its brand image. The ratio of C and $L - M$ is set below 1, which ensures that the firm has enough regular purchasers to notify once the upgrade process is triggered. We return to the discussion of M and C in §2.5.

The second decision is the upgrade fee p_n to charge in period n when an upgradeable customer clicks. We use p^h and p^l with $p^h > p^l$ to represent the prices of the premium and the regular products, respectively. An upgradeable customer, who already paid p^l for a regular product, has an upgrade reservation price (willingness-to-pay for the upgrade) no greater than $p^h - p^l$. We assume that the reservation price is identical and independent across upgradeable customers and over time periods. The upgrade reservation price is modeled as a random variable, whose distribution has a support of $[0, p^h - p^l]$ and a tail probability of $\alpha(\cdot)$. When an upgrade fee $p_n \in [0, p^h - p^l]$ is charged in period n, an upgradeable customer accepts the upgrade with probability $\alpha(p_n)$ and rejects it with probability $1 - \alpha(p_n)$. The acceptance probability $\alpha(p_n)$ is decreasing in p_n[6]. The firm uses p_n to control the upgrade acceptance rate. An upgradeable customer who rejects an upgrade may receive upgrades again in the future. The assumption of independence over time implies that an upgradeable customer's acceptance decision is not affected by her previous upgrade rejections, if any. The assumption of identical reservation price over time allows us to simplify the notation by using $\alpha(\cdot)$ for all periods. If this assumption fails, the analysis in the chapter still holds by appropriately replacing $\alpha(\cdot)$ with $\alpha_n(\cdot)$.

The sequence of events is as follows: (1) At the beginning of period n, the firm observes the leftover capacities (h_n, l_n). If $h_n > 0$ and $l_n \leq M$ (the premium product is available and the leftover regular capacity falls below the triggering level), the firm decides on (i) how many active upgrade links $u_n \in [0 : C]$ to maintain and (ii) what upgrade fee $p_n \in [0, p^h - p^l]$ to charge if upgrades are offered. (2) During period n, a premium customer arrives with probability λ_n^h, a regular customer arrives with probability λ_n^l, an upgradeable customer arrives with probability $u_n \lambda_n$, where λ_n is the clicking probability if there is only one active upgrade link. Alternatively, no customer arrives with probability $1 - \lambda_n^h - \lambda_n^l - u_n \lambda_n$. (3) The arriving premium/regular customer buys her preferred product if it is available. The upgradeable customer clicking the link sees the upgrade fee p_n, who then accepts the upgrade with probability $\alpha(p_n)$ or rejects it with probability $1 - \alpha(p_n)$. The arrival process with at

[6]Throughout the dissertation, we use increasing and decreasing in the weak sense.

most one customer per period is widely used in revenue management literature (see, e.g., Talluri and van Ryzin 2005), and is related to merging and splitting of Poisson arrival processes.

We now formulate the firm's revenue maximizing dynamic upgrade problem as a dynamic program. In the sequel, we drop the time index n of the state variables, decision variables and all probability parameters when their meanings are clear from the context. Let $V_n(h,l)$ denote the firm's optimal expected revenue at the beginning of period $n \in [1:N+1]$ with h units of premium product and l units of regular product. The dynamic programming formulation is given by

$$
\begin{aligned}
V_n(h,l) &= \max_{\substack{u\in[0:C]\\ p\in[0,p^h-p^l]}} \Big\{(1-\lambda^h-\lambda^l-u\lambda)V_{n+1}(h,l) \\
&\qquad +\lambda^h[p^h+V_{n+1}(h-1,l)]+\lambda^l[p^l\mathbb{1}_{l\geq 1}+V_{n+1}(h,l-\mathbb{1}_{l\geq 1})] \\
&\qquad +u\lambda\alpha(p)[p+V_{n+1}(h-1,l+1)]+u\lambda[1-\alpha(p)]V_{n+1}(h,l)\Big\} \\
&\quad \text{for } h\in[1:H], l\in[0:M] \text{ and } n\in[1:N], \qquad (2.1) \\
V_n(h,l) &= (1-\lambda^h-\lambda^l)V_{n+1}(h,l)+\lambda^h[p^h+V_{n+1}(h-1,l)]+\lambda^l[p^l+V_{n+1}(h,l-1)] \\
&\quad \text{for } h\in[1:H], l\in[M+1:L] \text{ and } n\in[1:N], \qquad (2.2) \\
V_n(0,l) &= (1-\lambda^h-\lambda^l)V_{n+1}(0,l)+\lambda^h V_{n+1}(0,l)+\lambda^l[p^l\mathbb{1}_{l\geq 1}+V_{n+1}(0,l-\mathbb{1}_{l\geq 1})] \\
&= (1-\lambda^l)V_{n+1}(0,l)+\lambda^l[p^l\mathbb{1}_{l\geq 1}+V_{n+1}(0,l-\mathbb{1}_{l\geq 1})] \\
&\quad \text{for } l\in[0:L] \text{ and } n\in[1:N], \qquad (2.3) \\
V_{N+1}(h,l) &= 0 \text{ for } h\in[0:H] \text{ and } l\in[0:L], \qquad (2.4)
\end{aligned}
$$

where $\mathbb{1}_{\mathcal{A}} \in \{0,1\}$ is an indicator function taking the value of 1 only when $\mathcal{A}$ is true. The first term inside Equation (2.1) is the firm's expected revenue-to-go in the event that no customer shows up. The second and third terms correspond to a premium and a regular customer arrival, respectively. The forth and fifth terms correspond to the upgrade acceptance and rejection respectively by an upgradeable customer clicking the link. Notice that the firm has one fewer premium product and one more regular product in the acceptance case. The region $[1:H] \times [0:M]$ in Equation (2.1) contains all possible states (h, l) in which the firm may offer upgrades, and is referred to as *the potential upgrade region.* Equation (2.2) is for the region above the upgrade triggering level M. In this region, the firm does not consider offering upgrades. Equation (2.3) is for the vertical boundary, where no premium product is left and the firm cannot offer upgrades. Equation (2.4) is for the terminal condition, which indicates that the salvage values of both products are zero.

2.3.2 Optimal Policy Structure

Equation (2.1) can be simplified and equivalently written as

$$\begin{aligned}
V_n(h,l) &= (1-\lambda^h-\lambda^l)V_{n+1}(h,l) + \lambda^h[p^h + V_{n+1}(h-1,l)] + \lambda^l[p^l \mathbb{1}_{l\geq 1} + V_{n+1}(h, l-\mathbb{1}_{l\geq 1})] \\
&\quad + \max_{\substack{u\in[0:C] \\ p\in[0,p^h-p^l]}} u\lambda\alpha(p)[p+\Delta_{n+1}(h,l)] \\
&= (1-\lambda^h-\lambda^l)V_{n+1}(h,l) + \lambda^h[p^h + V_{n+1}(h-1,l)] + \lambda^l[p^l \mathbb{1}_{l\geq 1} + V_{n+1}(h, l-\mathbb{1}_{l\geq 1})] \\
&\quad + \max_{u\in[0:C]} \left\{ u\lambda \max_{p\in[0,p^h-p^l]} \alpha(p)[p+\Delta_{n+1}(h,l)] \right\}, \qquad (2.5)
\end{aligned}$$

where $\Delta_{n+1}(h,l) := V_{n+1}(h-1,l+1) - V_{n+1}(h,l)$ is the change in expected revenue after replacing one unit of premium product by one unit of regular product in period $n+1$ when there are h units of premium product and l units of regular product leftover. Equation (2.5) indicates that the firm's joint optimization problem over (u, p) in period n can be solved sequentially by finding the optimal upgrade fee first and then the optimal number of active upgrade links.

We first solve the first-stage optimal upgrade pricing problem. For $(h,l) \in [1:H] \times [0:M]$, we define

$$\delta_n(p,h,l) := \alpha(p)[p+\Delta_{n+1}(h,l)] \quad \text{and} \quad \delta_n^*(h,l) := \max_{p\in[0,p^h-p^l]} \delta_n(p,h,l), \qquad (2.6)$$

where $\delta_n(p,h,l)$ is the expected revenue from an arriving upgradeable customer in period n when the upgrade fee is set to p, and $\delta_n^*(h,l)$ is the corresponding optimal expected upgrade revenue at the optimal upgrade fee.

Revenue maximization problem in the form of $\alpha(p)(p+\Delta)$ similar to (2.6) is commonly seen in dynamic pricing literature (e.g., Bitran and Mondschein 1997 and Ye et al. 2015). Δ usually has a negative value, and is perceived as the opportunity cost. $\Delta_{n+1}(h,l)$ in our model can be either positive or negative, and we refer to it as the *upgrade opportunity value*. We make the following regularity assumption: The upgrade reservation price distribution with tail probability $\alpha(\cdot)$ is smooth enough, such that $\left\{p \in [0, p^h - p^l] : \delta_n(p,h,l) = \delta_n^*(h,l)\right\}$ is a closed set. The assumption is satisfied by distributions with increasing failure rate, which makes the set of maximizer(s) a singleton. It is also satisfied by certain distributions with non-increasing failure rate, such as beta distribution with shape parameters of 1/2. The assumption ensures that the set of maximizer(s) of the optimization problem in (2.6) is closed, and that we can pick the largest maximizer as the optimal upgrade fee in period n at state (h,l):

$$p_n^*(h,l) := \max\{p \in [0, p^h - p^l] : \delta_n(p,h,l) = \delta_n^*(h,l)\}.$$

The solution of the second-stage optimization problem in (2.5) is the optimal number of active upgrade links in period n at state (h,l):

$$u_n^*(h,l) = \arg\max_{u \in [0:C]} u\lambda\delta_n^*(h,l).$$

We have the following result.

Proposition 1. *The optimal upgrade policy is of a pulsing type:* $u_n^*(h,l) = \mathbb{1}_{\delta_n^*(h,l)>0} C$.

First note that when $\delta_n^*(h,l) > 0$, upgrading is profitable. Thus, the firm should maintain as many active links as possible. When $\delta_n^*(h,l) \leq 0$, upgrading is not profitable and the firm should deactivate all of the upgrade links.

To better understand the optimal dynamic upgrade policy, we investigate properties of $V_n(h,l)$ over the region of $[0:H] \times [0:M+1]$. This region is the union of the potential upgrade region $[1:H] \times [0:M]$, its left boundary $\{0\} \times [0:M]$ and its upper boundary $[0:H] \times \{M+1\}$. We have the following results.

Proposition 2. *For $n \in [1:N+1]$, $V_n(h,l)$ satisfies the following properties:*

a) *Submodularity: $V_n(h,l+1) - V_n(h,l) \geq V_n(h+1,l+1) - V_n(h+1,l)$ for $(h,l) \in [0:H-1] \times [0:M]$,*

b) *DH-modularity: $V_n(h-1,l+1) - V_n(h,l) \leq V_n(h,l+1) - V_n(h+1,l)$ for $(h,l) \in [1:H-1] \times [0:M]$,*

c) *DV-modularity: $V_n(h-1,l+1) - V_n(h,l) \geq V_n(h-1,l+2) - V_n(h,l+1)$ for $(h,l) \in [1:H] \times [0:M-1]$,*

d) *H-concavity: $V_n(h+1,l) - V_n(h,l) \leq V_n(h,l) - V_n(h-1,l) \leq p^h$ for $(h,l) \in [1:H-1] \times [0:M+1]$,*

e) *V-concavity: $V_n(h,l+1) - V_n(h,l) \leq V_n(h,l) - V_n(h,l-1) \leq p^l$ for $(h,l) \in [0:H] \times [1:M]$.*

Property a) of submodularity is based on horizontally comparing vertical differences of V_n: $V_n(h,l+1) - V_n(h,l) \geq V_n(h+1,l+1) - V_n(h+1,l)$ illustrated in Figure 2.2, or vertically comparing horizontal differences of V_n: $V_n(h+1,l) - V_n(h,l) \geq V_n(h+1,l+1) - V_n(h,l+1)$. Interestingly, these two comparisons are equivalent, and there is no need to differentiate them. Property b) DH-modularity and property c) DV-modularity are more refined properties compared to submodularity. They can be written as $\Delta_n(h,l) \leq \Delta_n(h+1,l)$ and $\Delta_n(h,l) \geq \Delta_n(h,l+1)$, respectively. These two inequalities involve the *horizontal* and *vertical* comparisons of *diagonal* differences of $V_n(h,l)$, hence named as DH- and DV-modularity; see Figure 2.2. DH- and DV-modularity are not equivalent to each other and require separate notations to differentiate between each other. Any four properties out of five in Proposition 2 do not imply the leftover one. In Figure 2.2, we give a counterexample by failing only submodularity. More counterexamples can be found in Appendix A.1.

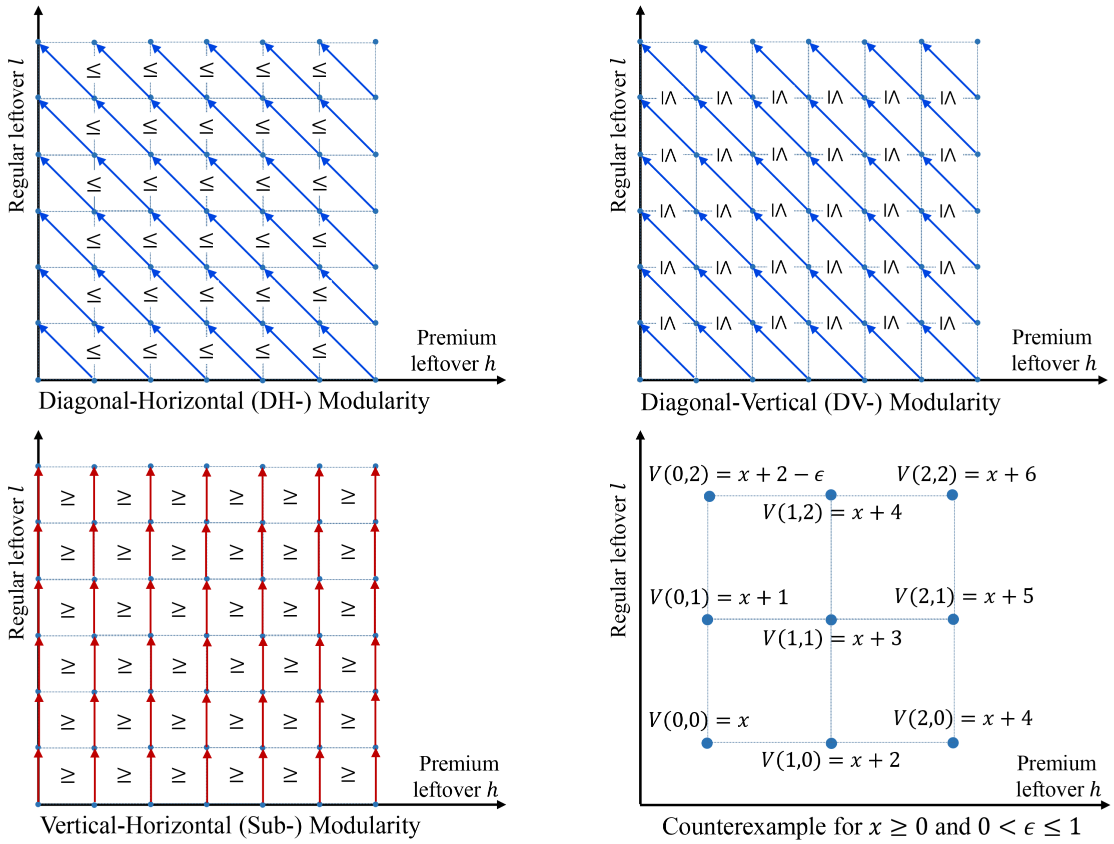

Figure 2.2. Top-left panel: Diagonal difference on the right is larger. Top-right panel: Diagonal difference above is smaller. Bottom-left panel: Vertical difference on the left is larger. Bottom-right panel: Submodularity is not implied by DH- and DV-modularity, H- and V-concavity.

Property a) states that the premium product and the regular product have a substitution effect on each other, i.e., the premium product can be used to capture regular product demand (see Chapter 1 and 2 in Topkis 1998 for more about submodularity and substitutability). Property b) of Proposition 2 implies that the expected revenue difference after replacing one unit of premium product by one unit of regular product is larger when there are more premium products. Property c) implies that the same expected revenue difference is smaller when there are more regular products. Properties d) and e) imply that the marginal value of either product is larger when there are fewer of them and that each product's marginal value is always smaller than its price.

Properties a), d) and e) of Proposition 2 directly lead to the following proposition, which implies that the marginal value of one premium product as well as one regular product is decreasing when there are more leftover capacities.

Proposition 3. *For $n \in [1 : N+1]$, we have the following properties of $V_n(h,l)$*

a) $V_n(h+1,l+1) - V_n(h,l) \geq V_n(h+2,l+1) - V_n(h+1,l)$ *for* $(h,l) \in [0 : H-2] \times [0 : M]$,

b) $V_n(h+1,l+1) - V_n(h,l) \geq V_n(h+1,l+2) - V_n(h,l+1)$ *for* $(h,l) \in [0 : H-1] \times [0 : M-1]$.

With properties b) and c) of Proposition 2, we can derive the following result.

Proposition 4. *The optimal number of active upgrade links $u_n^*(h,l)$ is increasing in h and decreasing in l. The optimal upgrade fee $p_n^*(h,l)$ is decreasing in h and increasing in l.*

Proposition 4 characterizes the monotonicity properties of the optimal policy. When there are more premium products, they are less likely to be sold out before the end of season. Therefore, it is optimal to sell premium products through a large number of upgrade links at a lower upgrade fee. When there are more regular products, they are less likely to be sold out before the end of the season. The firm is less concerned about losing regular customer demand and freeing up regular product capacity. Therefore, it is optimal to maintain upgrade links for fewer regular purchasers at a higher price. Due to the monotonicities of the optimal number of active upgrade links, the potential upgrade region $[1 : H] \times [0 : M]$ can be divided into two subregions. The firm offers upgrades only in the lower right subregion (e.g., see Figure 2.3), which can be referred to as the *upgrade region* and has high premium capacity and low regular capacity.

Next, we consider the monotonicity of optimal upgrade links and fees over time. Classic single product dynamic pricing literature shows that the optimal price decreases over time for a given capacity level if the customer reservation price distribution is stationary (e.g., Gallego and van Ryzin 1994, Bitran and Mondschein 1997, and Zhao and Zheng 2000). The reason for this classical time-monotonicity property is that the product's opportunity cost is decreasing over time. In contrast, the time monotonicity is not true in the dynamic upgrade pricing setting, even under the stationarity of the reservation price and the arrival processes. The reason is that the upgrade opportunity value $\Delta_{n+1}(h, l)$, which depends on both capacity levels and future demands, is not monotone with respect to time. Hence, it is not optimal to reduce or increase upgrade fees over time. Since the upgrade fee is not decreasing over time, strategically-waiting customers are not guaranteed an upgrade offer with a lower fee. Hence, the firm need not be overly concerned about strategic customer behavior and is more willing to implement the dynamic upgrade policy.

The optimal dynamic upgrade fees are robust with respect to proportionally changing prices. In a setting where p^h, p^l, and the upgrade reservation price increase or decrease by the same proportion and the arrival rate parameters are kept the same, the firm can get new optimal fees by simply increasing or decreasing the previous optimal upgrade fees by the same proportion. However, in a setting where p^h and p^l increase or decrease by the same amount and the upgrade reservation price stays the same, the firm needs to find the new optimal policy by solving a new dynamic program. The reason for the difference is that $\Delta_{n+1}(h, l)$ is affected by p^h and p^l, instead of $p^h - p^l$ alone.

Our model can be used to optimize the prices of both the premium and regular products at the beginning of the sales season: $\max_{0 \leq p^l \leq p^h} V_1(H, L|p^h, p^l)$. Likewise, incorporated with the cost $c(H, L)$ of acquiring H and L units of premium and regular capacities, our model can also be used to optimize the initial capacity levels: $\max_{H,L \geq 0} V_1(H, L) - c(H, L)$. These price and capacity optimization problems are static, but their input V_1 is obtained from the dynamic programming recursion of the upgrade pricing and timing problem.

2.4 Restricted Upgrade Fee and Upward Substitution

In this section, we build two model variations of the base model. The first considers a firm choosing upgrade fees from a restricted subset. The second incorporates upward substitution, in which the firm can sell a premium product to an arriving regular customer if the regular product stocks out.

2.4.1 Dynamic Upgrade with Restricted Upgrade Fee

Firms in industry may have varying levels of flexibility in adjusting prices; that is, they may not be able to optimize the upgrade fee over the interval $[0, p^h - p^l]$. In one scenario, firms may deliberately restrict the upgrade fee choice set. One example is that firms want to protect their brand images and avoid selling many premium products at low upgrade fees. Another example is that firms want to avert high upgrade fees and make upgrades more acceptable and effective. Therefore, the upgrade fee choice set may be $[\underline{p}, \overline{p}] \subseteq [0, p^h - p^l]$. In another scenario, firms want to avoid pricing at an arbitrary decimal (e.g., \$13.76) and changing prices dramatically. As another example, firms may prefer using a predetermined discrete set of prices, such as a set of \$4.99, \$9.99, \$14.99 and \$19.99, from which they pick the optimal one. When firms want to discretize the upgrade fee choice set, the set would be $\{p_1, ..., p_m\} \subseteq [0, p^h - p^l]$. An extreme example is that the upgrade fee choice set is a singleton; firms charge a fixed upgrade fee over the sales season and only control dynamic upgrade availability.

Define $V_n^r(h, l)$ as the optimal expected revenue of the restricted fee model at state (h, l) in period n. This revenue can be obtained by replacing $[0, p^h - p^l]$ with $[\underline{p}, \overline{p}]$ or $\{p_1, ..., p_m\}$ in Equation (2.1). $V_n^r(h, l)$ still satisfies the five properties in Proposition 2, and the analytical results from the base model can be extended to the restricted fee model. The structure of its optimal policy is stated in the following corollary.

Corollary 1. *For a restricted fee model, we have the similar results in Proposition 1 and 4:*

a) *The optimal upgrade policy is of a pulsing type.*

b) *The optimal number of active upgrade links is increasing in h and decreasing in l, and the optimal upgrade fee is decreasing in h and increasing in l.*

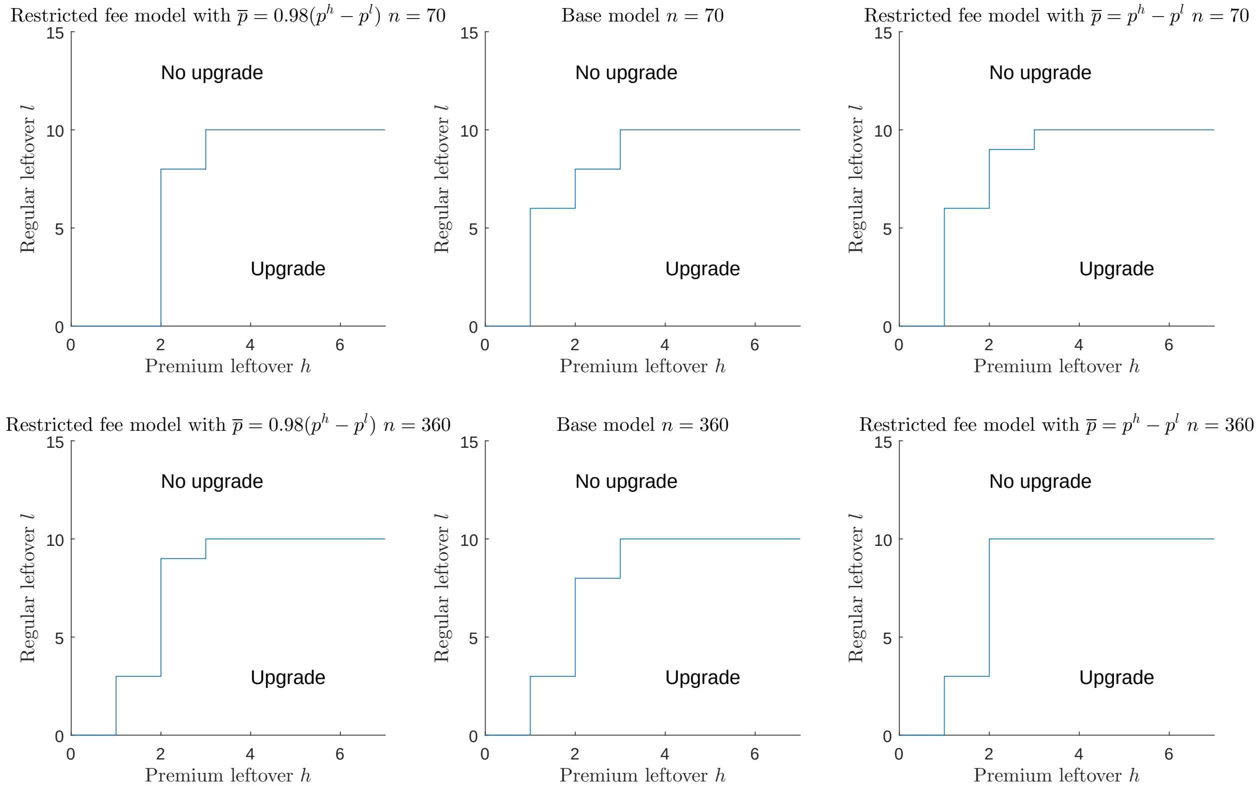

Figure 2.3. Upgrade region comparisons across three models and two time periods. Top to bottom, rows have time period 70 and 360. Left to right, columns have restricted fee model with $\overline{p} = 0.98(p^h - p^l)$, base model, and restricted fee model with $\overline{p} = (p^h - p^l)$.

A common question of interest is how the price restriction affects the optimal expected revenue and the optimal policy. Here, we answer the question by mainly focusing on the comparison between the restricted fee model with choice set $[\underline{p}, \overline{p}]$ and the base model with $[0, p^h - p^l]$. First, we compare expected revenues. The upgrade fee restriction decreases the pricing flexibility and leads to a lower expected revenue; the restricted fee model always provides a revenue lower bound for the base model. Second, we compare the optimal upgrade regions. Unlike the expected revenue comparison, the upgrade region comparison is subtle and depends on $\overline{p}$. We provide six examples in Figure 2.3, which depict the upgrade region comparisons between the base and restricted fee models. When $\overline{p} < p^h - p^l$, there is no clear answer. Note from Figure 2.3 that the upgrade region of the base model can be either larger or smaller than the restricted fee model. However, when $\overline{p} = p^h - p^l$, we numerically find that the upgrade region of the restricted fee model contains that of the base model. This finding of containment between upgrade regions indicates that if the base model optimally offers upgrades at state (h, l) in period n, it is also optimal for the restricted fee model to offer upgrades at the same state in the same period. We theoretically prove this numerical finding of containment for the left boundary of the potential upgrade region by Proposition 5 and Proposition 6. We define $\Delta_n^r(h, l) := V_n^r(h-1, l+1) - V_n^r(h, l)$ as the upgrade opportunity value and $u_n^{r,*}(h, l)$ as the optimal number of upgrade links in the restricted fee model at state (h, l) in period n.

Proposition 5. *If $\overline{p} = p^h - p^l$, then for $n \in [1 : N+1]$, we have the following across-model comparisons:*

a) *Diagonal difference:* $\Delta_n(1, l) \le \Delta_n^r(1, l)$ *for* $l \in [0 : M]$*,*

b) *Horizontal difference:* $V_n(1, l) - V_n(0, l) \ge V_n^r(1, l) - V_n^r(0, l)$ *for* $l \in [0 : M]$*,*

c) *Vertical difference:* $V_n(0, l+1) - V_n(0, l) = V_n^r(0, l+1) - V_n^r(0, l)$ *for* $l \in [0 : L-1]$*.*

Property a) states that the upgrade opportunity value is smaller in the base model. Property b) implies that the expected revenue of a premium product (horizontal difference) is larger in the base model. Finally, property c) says that the upgrade fee restriction does not affect the marginal value of the regular product (vertical difference) when there is no premium product leftover. Property a) in Proposition 5 leads to the containment result on the left boundary of the potential upgrade region in Proposition 6. In particular, if the base model offers upgrades at a state in a period, it is also optimal for a restricted fee model to offer upgrades at the same state in the same period.

Proposition 6. *If $\overline{p} = p^h - p^l$, $u_n^*(1, l) > 0$ implies $u_n^{r,*}(1, l) > 0$ for $l \in [0 : M]$.*

Despite its numerical illustration and analytical proof, the containment property is counter-intuitive. At first glance, one would say that the firm is more inclined to offer upgrades when it has more control (less restriction) in choosing the upgrade fees. This general intuition has traces in dynamic pricing literature. For example, Aydin and Ziya (2008), in an upselling context, find that a firm is more inclined to offer upsells with discounts when there is more flexibility of choosing the discount level. After a careful inspection, however, the containment property makes sense in our upgrade context. Whether to offer upgrades or not depends on whether upgrades can generate a positive revenue. A larger fee p helps bring in positive revenues $p + \Delta_{n+1}(h, l)$ and $p + \Delta^r_{n+1}(h, l)$ at more states of leftover capacities (h, l), but lowers the upgrade acceptance probability $\alpha(p)$. The base model can charge lower fees to make upgrades more acceptable, while the restricted fee model has less pricing flexibility, which makes upgrades less acceptable. To compensate for the lower acceptance rate, the restricted fee model offers upgrades at more states and its upgrade region contains that of the base model.

2.4.2 Dynamic Upgrade with Upward Substitution

When the regular product is out of stock, the firm can offer the premium product to an incoming regular customer. This practice can be termed as (stockout-based) upward substitution and be incorporated to extend the base model. Note that the premium product is offered directly to a regular customer in the case of upward substitution, whereas it is offered to an upgradeable customer through a notification in the case of upgrade. When the premium product is available but the regular product is not, the firm in the substitution model offers a premium product at the price $p^l + f^s \in [p^l, p^h]$ to an incoming regular customer. $f^s \in [0, p^h - p^l]$ is the substitution fee. The tail probability of the substitution reservation price distribution is captured by $\alpha^s(\cdot)$, which satisfies $\alpha^s(0) = 1$ and $\alpha^s(p^h - p^l) = 0$. A regular customer accepts a substitution offer at fee f^s with probability $\alpha^s(f^s)$. Both tail probabilities $\alpha(\cdot)$ associated with an upgrade and $\alpha^s(\cdot)$ associated with a substitution have the same domain of $[0, p^h - p^l]$.

Let $V_n^s(h,l)$ denote the optimal expected revenue with substitution when starting in period n with h units of premium product and l units of regular product. The dynamic programming equations for $V_n^s(h,l)$ are the same as Equation (2.2), (2.3) and (2.4). However, Equation (2.1) applies only for $h \in [1:H]$, $l \in [1:M]$ and $n \in [1:N]$ and the next equation applies for $h \in [1:H]$, $l = 0$ and $n \in [1:N]$.

$$
\begin{aligned}
V_n^s(h,0) \;=\; \max_{\substack{u^s \in [0:C] \\ f^s,\, p^s \in [0, p^h - p^l]}} & \Big\{ (1 - \lambda^h - \lambda^l - u^s\lambda) V_{n+1}^s(h,0) + \lambda^h [p^h + V_{n+1}^s(h-1,0)] \\
& + \lambda^l \alpha^s(f^s)[p^l + f^s + V_{n+1}^s(h-1,0)] + \lambda^l[1 - \alpha^s(f^s)] V_{n+1}^s(h,0) \\
& + u^s \lambda \alpha(p^s)[p^s + V_{n+1}^s(h-1,1)] + u^s\lambda[1 - \alpha(p^s)] V_{n+1}^s(h,0) \Big\},
\end{aligned}
$$

where u^s and p^s are respectively the number of upgrade links and the upgrade fee. Similar to the base model, we can define $\Delta_{n+1}^s(h,l) := V_{n+1}^s(h-1,l+1) - V_{n+1}^s(h,l)$, $\delta_n^s(p^s,h,l) := \alpha(p^s)[p^s + \Delta_{n+1}^s(h,l)]$ and $\delta_n^{s,*}(h,l) := \max\{\delta_n^s(p^s,h,l) : p^s \in [0, p^h - p^l]\}$. Using these to rewrite the DP equations over the lower boundary and the interior of the potential upgrade region, we arrive at the following equations analogous to Equation (2.5) for $h \in [1:H]$ and $n \in [1:N]$ and their consequence stated as the following corollary.

$$
\begin{aligned}
V_n^s(h,0) &= (1 - \lambda^h) V_{n+1}^s(h,0) + \lambda^h[p^h + V_{n+1}^s(h-1,0)] \\
&\quad + \max_{f^s \in [0, p^h - p^l]} \Big\{ \lambda^l \alpha^s(f^s)[p^l + f^s + V_{n+1}^s(h-1,0) - V_{n+1}^s(h,0)] \Big\} \\
&\quad + \max_{u^s \in [0:C]} \Big\{ u^s \lambda \delta_n^{s,*}(h,0) \Big\},
\end{aligned}
$$

$$
\begin{aligned}
V_n^s(h,l) &= (1 - \lambda^h - \lambda^l) V_{n+1}^s(h,l) + \lambda^h[p^h + V_{n+1}^s(h-1,l)] \\
&\quad + \lambda^l[p^l + V_{n+1}^s(h,l-1)] + \max_{u^s \in [0:C]} \Big\{ u^s \lambda \delta_n^{s,*}(h,l) \Big\} \quad \text{for } l \in [1:M].
\end{aligned}
$$

Corollary 2. *The optimal upgrade policy is a pulsing type.*

Unlike the base model, the optimal upgrade policy in the substitution model does not have a monotone property with respect to the leftover capacities. The reason is that Proposition

2 is no longer true for the upward substitution model; specifically, $V_n^s(h,l)$ may fail to satisfy the DV-modular property. The DV-modularity requires $\Delta_n^s(h,l) \geq \Delta_n^s(h,l+1)$. To illustrate the failing of DV-modularity, we provide an example with the opposite inequality $\Delta_n^s(h,0) < \Delta_n^s(h,1)$ in Appendix A.2. The possibility of $\Delta_n^s(h,l)$ increasing in l leads to the possibility of the optimal number of upgrade links increasing in l. That is, the firm may avoid offering upgrades when the regular leftover capacity is very low, but upgrades may be offered when the regular leftover capacity is high. In contrast, the firm in the base model always tends to offer upgrades with the purpose of increasing the regular leftover capacity when it is low. The intuition of the finding in the upward substitution model is as follows: Since the upgrade opportunity value $\Delta_n^s(h,0)$ might be smaller than $\Delta_n^s(h,1)$, upgrades are less beneficial when the regular product stocks out. The firm may prefer the regular product stockout and get a higher revenue through upward substitution. A further examination of $\Delta_n^s(h,l)$ reveals that it may be neither increasing nor decreasing in l; the firm may avoid upgrades when the regular capacity is either low or high but offers upgrades when the regular capacity is medium.

The firm needs to make two pricing decisions when the regular product is out of stock and the premium product is still available: the optimal substitution fee $f_n^{s,*}(h)$ and the optimal upgrade fee $p_n^{s,*}(h,0)$. The following proposition implies that the optimal substitution fee is no greater than the optimal upgrade fee, when the substitution reservation price and the upgrade reservation price are identically distributed.

Proposition 7. *$f_n^{s,*}(h) \leq p_n^{s,*}(h,0)$, if $\alpha^s(p) = \alpha(p)$ for $p \in [0, p^h - p^l]$.*

An accepted substitution consumes a premium product and at least generates p^l revenue, while an accepted upgrade consumes a premium product and frees a regular product capacity whose value is at most p^l. Since the opportunity value of a substitution $p^l + V_{n+1}^s(h-1,0) - V_{n+1}^s(h,0)$ is higher than that of an upgrade $V_{n+1}^s(h-1,1) - V_{n+1}^s(h,0)$, the firm charges a higher upgrade fee to compensate for the lower upgrade opportunity value.

2.5 Implementation of the Dynamic Upgrade Policy

A dynamic upgrade policy implementation requires firms to send notifications and engage in after-sales interaction with regular purchasers. These notifications, if too many, can cause inadvertent spamming indiscriminately and repetitively sending notifications to a regular purchaser. Most firms consider upgrades as promotional events[7]. A large number of upgrades sell the premium product below its original selling price and may devalue the product in consumers' minds. Hence, in order to protect their brand images (e.g., to appear less promotional and reduce possibility of being perceived as spammers), firms often monitor the number of notifications and the resulting upgrade sales. The extent of spamming and the volume of upgrade sales are related to the values of the upgrade triggering level M and the maximum upgrade capability C introduced in §2.3.1. An implementation also requires firms to maintain customer lists (such as the list of upgradeable customers). We elaborate on these issues next.

The upgrade triggering level $M < L$ ensures that a firm sells enough regular products before considering upgrades. In other words, when $l_n \leq M$, the firm has a limited number of leftover regular products and can use upgrades to free regular products for future demand. When upgrading is considered, the number of regular products sold or the number of regular purchasers, $L - l_n$, is at least $L - M$. The firm can choose to offer upgrades to all regular purchasers $L - l_n$. However, the firm may want to maintain a small number of active upgrade links, that is C – the maximum number of active upgrade links at any given point in time. The firm sets C based on the consideration of spamming and premium product devaluation. Broadly speaking, a larger C results in a larger number of notifications, more spamming and more upgrade sales. Without a limit on the maximum active upgrade links (i.e., $C = \infty$), the pulsing policy of Proposition 1 remains optimal. This optimal policy requires the firm

[7]http://europe.etbtravelnews.global/99459/rail-europe-announces-swiss-pass-free-upgrade-promotion/

to send upgrade links to all regular purchasers or to deactivate all links. Such a policy may cause the firm to send many upgrade notifications, leading to spamming. Kumar et al. (2015), in a permission-based marketing context, empirically show that sending emails too often speeds up customer opt-outs, reduces the number of receivers and eventually leads to response reduction. To mitigate these concerns, our model employs the maximum upgrade capability C where $C \leq L - M$. The firm, when needed, can draw C customers out of $L - l_n$ regular purchasers and maintains active upgrade links only for them. When the ratio of C to $L - M$ is small, the firm can identify and notify C regular purchasers who have not been notified recently. Therefore, the firm can reduce spamming and limit upgrade sales.

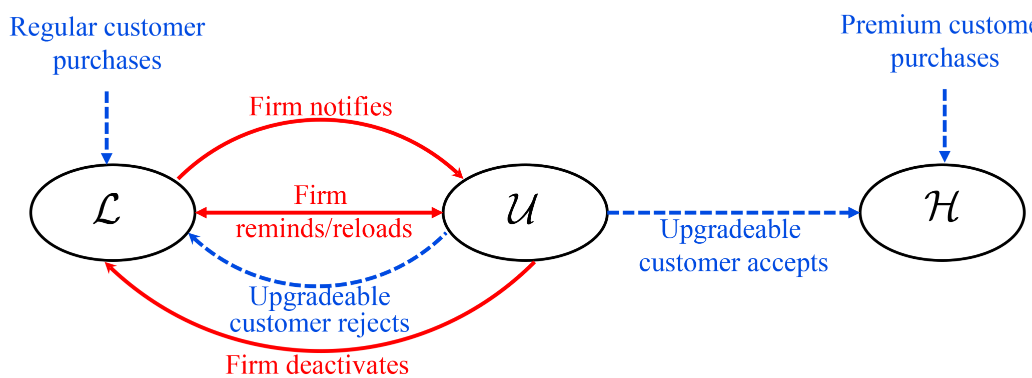

Figure 2.4. Movement of purchasers among three lists. Upward substitution is not considered here. Solid lines denote moves by the firm; broken lines denote moves by the customers.

The implementation of dynamic upgrades requires the firm to track three lists. $\mathcal{H}$ is the list of purchasers with premium products, who are either premium purchasers or upgraded regular purchasers. $\mathcal{U}$ is the list of upgradeable customers. $\mathcal{L}$ is the list of regular purchasers who are neither upgradeable nor upgraded. The regular purchasers in $\mathcal{L}$ are ordered based on their upgrade notification recency; the purchasers who received an upgrade notification most recently are put at the bottom of the list. $\mathcal{H}, \mathcal{U}$ and $\mathcal{L}$ are mutually exclusive. $\mathcal{H} \cup \mathcal{U} \cup \mathcal{L}$ is the set of all purchasers, and $\mathcal{U} \cup \mathcal{L}$ is the set of all regular purchasers who are not upgraded yet. When a premium (resp., regular) customer purchases a premium (resp., regular) product, she enters $\mathcal{H}$ (resp., $\mathcal{L}$), see Figure 2.4. The firm updates the three lists dynamically during the sales season.

The implementation of dynamic upgrades requires the action of *purchaser movement* among $\mathcal{H}$, $\mathcal{L}$ and $\mathcal{U}$. When upgrades turn profitable, the firm sends notifications to the top C regular purchasers in $\mathcal{L}$, and moves them to $\mathcal{U}$. When an upgradeable customer accepts the upgrade, she is moved from $\mathcal{U}$ to $\mathcal{H}$. When an upgradeable customer rejects the upgrade, she is moved from $\mathcal{U}$ to the bottom of $\mathcal{L}$. When upgrades turn unprofitable, the firm deactivates all the upgrade links, and moves all upgradeable customers from $\mathcal{U}$ to the bottom of $\mathcal{L}$.

We use a small-sized hotel as an example to show the purchaser movement among lists $\mathcal{H}$, $\mathcal{L}$ and $\mathcal{U}$. The hotel has 15 premium rooms and 20 regular rooms, i.e., $(H, L) = (15, 20)$, to sell over 500 periods. If each period is an hour, the sales season is approximately 20 days. In the example, we use $M = 10$ and $C = 5$. Suppose that upgrading becomes optimal for the first time in period 250 when the leftover capacity is $(h_n, l_n) = (8, 5)$. The hotel has sold 15 regular rooms and keeps all 15 regular purchasers in the ordered list $\mathcal{L}$ at the end of period 249. After the hotel sends active upgrade links to regular purchasers $1:5$ and moves them from $\mathcal{L}$ to $\mathcal{U}$, $\mathcal{L}$ contains regular purchasers $6:15$. Suppose that the upgradeable customer 1 in $\mathcal{U}$ clicks the link but rejects the upgrade, the hotel moves her back to the bottom of list $\mathcal{L}$, which becomes $6:15, 1$. Then $\mathcal{U}$ contains four upgradeable customers $2:5$, who have not clicked the upgrade links. In period 251, the leftover capacity stays at $(h_n, l_n) = (8, 5)$. Suppose the optimal decision is still to offer upgrades. The hotel then moves purchaser 6 from $\mathcal{L}$ to $\mathcal{U}$ and sends her an upgrade link, and $\mathcal{U}$ contains upgradeable customers $2:6$. Suppose upgradeable customer 3 clicks the link and accepts the upgrade, she is moved to $\mathcal{H}$. $\mathcal{U}$ now only contains upgradeable customers $2, 4, 5, 6$. The hotel then has one fewer premium room and one more regular room. In period 252, $(h_n, l_n) = (7, 6)$. If it is optimal to stop upgrading, the hotel deactivates all four active links and puts upgradeable customers $2, 4, 5, 6$ back to the bottom of $\mathcal{L}$. Suppose a regular customer arrives, the hotel numbers her as purchaser 16, adds her into $\mathcal{L}$ and has one fewer regular room.

Regular purchasers who reject upgrades (i.e., customer 1 in the example above) are put back into $\mathcal{L}$. Such purchasers may receive upgrade notifications again in the future. We assume in §2.3.1 that upgrade reservation price distributions are independent over time. This assumption helps simplify the dynamic upgrade pricing problem; the firm does not need to keep a record of customers rejecting upgrade offers and the corresponding upgrade fees. When the ratio of C to $L - M$ is relatively small, the firm can only notify a limited number of regular purchasers each time it sends out new upgrade links. Since the list of $\mathcal{L}$ is long (at least contains $L - M$ regular purchasers), it will take a long time for a regular purchaser to receive an upgrade link again. During this time, the purchaser may forget her previous upgrade fee(s) and change her reservation price, which justifies the independence of reservation prices over time.

An optional action of dynamic upgrade implementation is *purchaser reminding/reloading*. This action is tied to the upgradeable customer clicking behavior. We assume in §2.3.1 that the upgrade link clicking probability is not affected by the length of the time duration after an upgradeable customer receiving a notification. The customer clicking probability may decay over time. However, the relaxation of this assumption requires the firm to track when each upgradeable customer received her upgrade notification, which enlarges the state space and makes the dynamic upgrade pricing problem intractable. To remedy this assumption and to make dynamic upgrade implementation effective, the firm can adopt purchaser reminding/reloading. If an upgradeable customer does not respond to the offer for a certain amount of time, the firm can either send a reminder to her or move her back to $\mathcal{L}$ and reload $\mathcal{U}$ with another regular purchaser from $\mathcal{L}$.

To summarize, the dynamic upgrade model features parameters M and C. A small M gives a long list of regular purchasers to upgrade from. A small C ensures that each regular purchaser receives the upgrade notification infrequently. Together, they help the firm limit the upgrade sales and protect its brand image. A small ratio of C to $L-M$ and the ordering of purchasers in $\mathcal{L}$ help the firm avoid repetitively and indiscriminately sending upgrade notifications, hence reduce spamming. The dynamic upgrade implementation involves purchaser movement and purchaser reminding/reloading among three lists. We provide an algorithm in Table A.4 in Appendix A.3 to show a particular implementation.

2.6 Quantifying the Values of Dynamic Upgrades

In this section, we conduct a systematic numerical study to a) compare different upgrade strategies and quantify the benefits of dynamic upgrades, b) quantify the impact of different operating factors on the benefits of dynamic upgrades, and c) explain the trade-off between revenue maximization and brand image protection.

2.6.1 Benefits of Dynamic Upgrades

Numerical Study Setup. We generate 2,187 $= 3^7$ different problem instances, for which we solve dynamic programs to optimality. The parameter values used in our study are summarized in Table 2.1. Parameters of instances come from the Cartesian product of these sets. The number N of periods is 500; if each time period is an hour long, the entire sales season lasts approximately 20 days. The value sets of initial capacity parameters (H, L) apply to small to medium sized hotels with fewer premium rooms than regular rooms. We use $(\mu^h, \mu^l) = (N\lambda^h, N\lambda^l)$ to represent the total expected demands over the entire sales season. Possible combinations of (H, L, μ^h, μ^l) contain the instances where the capacity and demand are balanced (e.g., $(5, 15, 5, 15)$ and $(15, 25, 15, 25)$) and imbalanced (e.g., $(5, 15, 15, 15)$ and $(15, 15, 10, 25)$). The click rate parameter λ takes three values of low, medium and high. We use $\mu = N\lambda$ to represent the expected number of total clicks over the entire sales season if only one upgrade link is active. Since the sales season is approximately of 20 days, $\mu = 1$ (5 or 10) implies that an upgradeable customer on average clicks the upgrade link once every 20 (4 or 2) days. (M, C) is fixed at $(10, 5)$, which works for all instances because of $C \leq L - M$ for all possible L. We use the uniform distribution for the upgrade reservation price; i.e., the upgrade acceptance probability for a fee $p \in [0, p^h - p^l]$ is $\alpha(p) = (p^h - p^l - p)/(p^h - p^l)$.

Table 2.1. Parameter value sets

Parameter	Value set	Parameter	Value set	Parameter	Value set	Parameter	Value set
N	$\{500\}$	H	$\{5, 10, 15\}$	L	$\{15, 20, 25\}$	$\mu = N\lambda$	$\{1, 5, 10\}$
M	$\{10\}$	p^h	$\{1.2, 1.5, 1.8\}$	p^l	$\{0.4, 0.7, 1\}$		
C	$\{5\}$	$\mu^h = N\lambda^h$	$\{5, 10, 15\}$	$\mu^l = N\lambda^l$	$\{15, 20, 25\}$		

Models of Interest. We consider four different dynamic upgrade strategies: dynamic upgrade with a fixed fee (DF), dynamic upgrade with a set of discrete fees (DD), dynamic upgrade with an interval of fees (DI), and dynamic upgrade with an interval of fees and upward substitution (DIUS). DI is in our base model. We compare these four strategies with the check-in upgrade with fixed fee (CF). In CF, the firm needs to decide only on the check-in upgrade fee. If the firm has h units of premium product and l units of regular product leftover at the check-in time, it can offer at most $\min\{h, L - l\}$ upgrades. The optimal expected revenue of CF strategy in period n and at state (h, l) is given by

$$
\begin{aligned}
V_n^c(h,l) &= (1 - \lambda^h - \lambda^l)V_{n+1}^c(h,l) + \lambda^h[p^h \mathbb{I}_{h\geq 1} + V_{n+1}^c(h - \mathbb{I}_{h\geq 1}, l)] \\
&\quad + \lambda^l[p^l \mathbb{I}_{l\geq 1} + V_{n+1}^c(h, l - \mathbb{I}_{l\geq 1})] \text{ for } n \in [1:N], \\
V_{N+1}^c(h,l) &= \min\{h, L-l\} \max_{p\in[0,p^h-p^l]} \alpha(p)p.
\end{aligned}
$$

With the assumption of uniform upgrade reservation price distribution, the optimal check-in upgrade fee is $(p^h - p^l)/2$, and the corresponding accepting probability is $\alpha((p^h - p^l)/2) = 1/2$. The theoretical analysis in §2.3 and §2.4 does not incorporate check-in upgrades and assumes $V_{N+1}(h,l) = V_{N+1}^r(h,l) = V_{N+1}^s(h,l) = 0$. To make a fair comparison, we replace $V_{N+1}(h,l) = V_{N+1}^r(h,l) = V_{N+1}^s(h,l)$ by

$$
V_{N+1}^c(h,l) = \min\{h, L-l\} \max_{p\in[0,p^h-p^l]} \alpha(p)p = \min\{h, L-l\}\frac{p^h - p^l}{4}
$$

and incorporate the check-in upgrades into all four dynamic upgrade strategies. The new terminal condition $V^c_{N+1}(h,l)$ does not affect the optimality of the pulsing solution in any one of the four dynamic upgrade strategies. It also satisfies the five properties in Proposition 2, so the monotonicity structure of the pulsing solution in DF, DD and DI still holds. In DF, we assume that the upgrade fee is fixed at $(p^h - p^l)/2$, which is equal to the optimal check-in upgrade fee. In DD, we assume upgrade fees are dynamically picked from the set of $\{0, (p^h - p^l)/4, (p^h - p^l)/2, 3(p^h - p^l)/4, (p^h - p^l)\}$. Finally, in DIUS, we assume that the upward substitution is accepted at fee f^s with probability $\alpha(f^s)$; the substitution reservation price distribution and the upgrade reservation price distribution are the same.

Benefits of Dynamic Upgrades. Despite the clear advantage of dynamic upgrade strategy, they are not always preferred for multiple reasons. Some firms simply do not have the technical capabilities required for the implementation of a dynamic strategy, whereas some others intending to adopt the dynamic upgrade strategy prefer to avoid changing upgrade fee frequently and dramatically. Therefore, it is of interest to understand the potential benefits that would be gained through dynamic upgrade availability, upgrade fee pricing flexibility and upward substitution. The five strategies in our numerical study can be ordered as CF, DF, DD, DI, and DIUS based on their sophistication levels. For the 2,187 instances generated, we compute the optimal expected revenue under each policy and the percentage improvements obtained by switching from less sophisticated strategies to more sophisticated ones.

Table 2.2. Percentage improvements in expected revenue

	CF → DF	CF → DD	CF → DI	CF → DIUS	DI → DIUS
Max	38.25	46.08	47.39	49.35	17.84
Min	0.00	0.00	0.00	0.00	0.00
Average	2.86	3.49	3.64	4.31	0.61

Table 2.2 provides the summary statistics such as maximum, minimum and average improvements over the 2,187 instances. It demonstrates the value of dynamic upgrade strategies; even the simplest dynamic strategy DF improves revenue by 2.86% on average over

the static CF, and DIUS achieves an average 4.31% improvement. For a highly imbalanced instance with premium product overcapacity and regular product undercapacity, such as $(H, L, \mu^h, \mu^l) = (15, 15, 5, 25)$, DIUS can generate a significant 49.35% revenue improvement over CF.

For a firm currently using CF and considering switching to a dynamic strategy, the amount of pricing flexibility to adopt is crucial. More flexibility brings in a higher revenue, but often at the expense of a higher technological and transactional investment. The average revenue improvements in DD and DI are 3.49% and 3.64%, respectively, which are not significantly different from each other. For an average firm with tight budget constraint, DD or even DF can be a good option. It is also important to check the extra benefit of upward substitution on top of dynamic pricing and timing of upgrades. The average revenue improvement gained by switching from DI to DIUS is 0.61%, whereas the average improvement gained by switching from CF to DI is 3.64%. Therefore, although the upward substitution brings in some extra revenue, the adoption of the dynamic pricing and timing of upgrades has a much more significant impact.

2.6.2 Impact of Environment on Benefits of Dynamic Upgrades

We quantify the impact of various operating factors on the dynamic upgrade strategies. In particular, we investigate how these factors affect the revenue improvements achieved by advancing from CF to DF, DD, DI and DIUS . We use the following parameters in the base instance: $N = 500$, $H = 15$, $L = 20$, $\mu^h = 10$, $\mu^l = 20$, $\mu = 5$, $p^h = 1.5$, $p^l = 0.7$, $M = 10$, and $C = 5$. For this instance, $(H, L, \mu^h, \mu^l) = (15, 20, 10, 20)$ means that the firm's premium capacity is slightly over its expected demand while the regular capacity and its demand match. We choose this base instance, since most travel firms (e.g., airlines, cruise lines and hotels) purposefully build extra premium capacity. Below we change one parameter at a time while keeping the others constant.

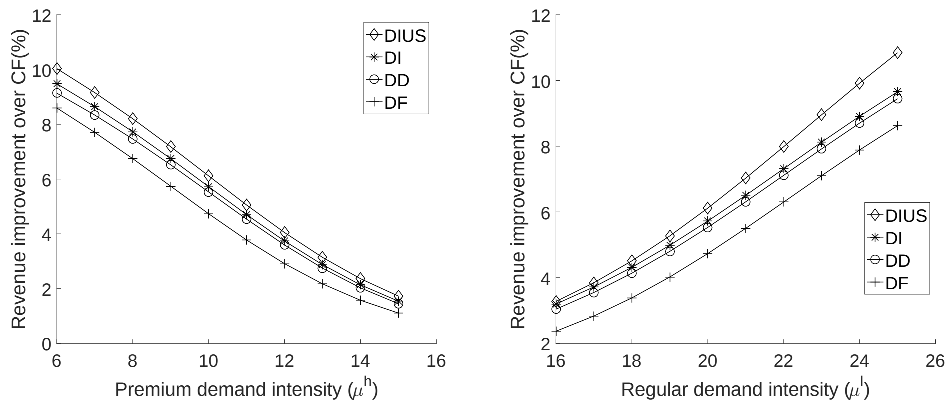

Figure 2.5. Impact of demand intensities μ^h and μ^l

Impact of Demand Intensities. We test the impact of demand intensities by varying $\mu^h \in [6:15]$ and $\mu^l \in [16:25]$. The results are illustrated in Figure 2.5. We observe that the percentage revenue improvements from CF to four dynamic strategies decrease with the premium demand intensity while they increase with the regular demand intensity. Intuitively, when the premium demand intensity is high, most of the premium capacity is going to be sold at its original price. The firm has less incentive to offer upgrades, and the value of dynamic upgrades shrinks. When the regular demand intensity is high, the regular capacity depletion is more likely to happen, after which the firm loses the opportunity to capture the regular demand. Dynamic upgrades, which allow the firm to free regular capacity, bring in a higher revenue improvement when this capacity depletion happens faster. In the right panel of Figure 2.5, we can also observe that the revenue improvement from DI to DIUS increases with the regular demand intensity. The upward substitution, as an extra means to capture regular demand, provides a larger value when the regular demand is higher.

Impact of Initial Capacity Levels. Figure 2.6 illustrates the impacts of initial capacity levels as measured by H and L. A higher initial premium capacity amplifies the firm's ability to extract extra revenue from regular purchasers through dynamic upgrades. Hence, the revenue improvements increase with H. In contrast, a higher regular initial capacity reduces the firm's need to free regular capacity through upgrades. Consequently, the revenue improvements decrease with L. We can also observe that the revenue improvement from DI to DIUS is shrinking with a higher L; with a higher regular initial capacity, the probability of regular product stockout is low, and the firm is less likely to collect revenue through upward substitution.

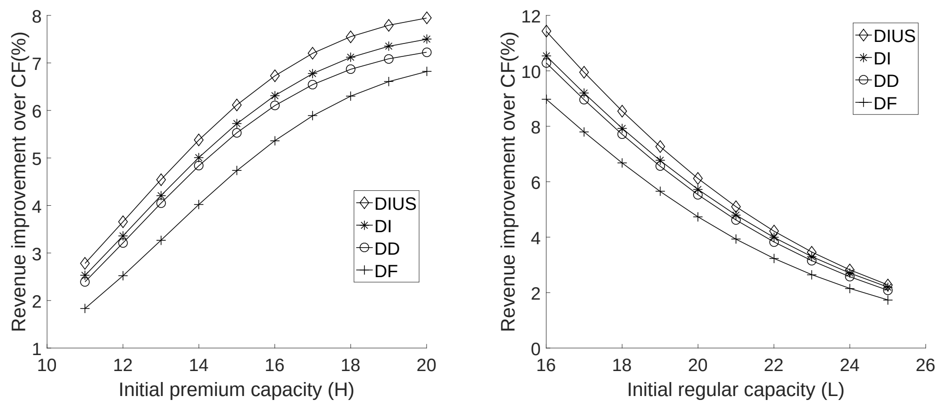

Figure 2.6. Impact of initial capacities H and L

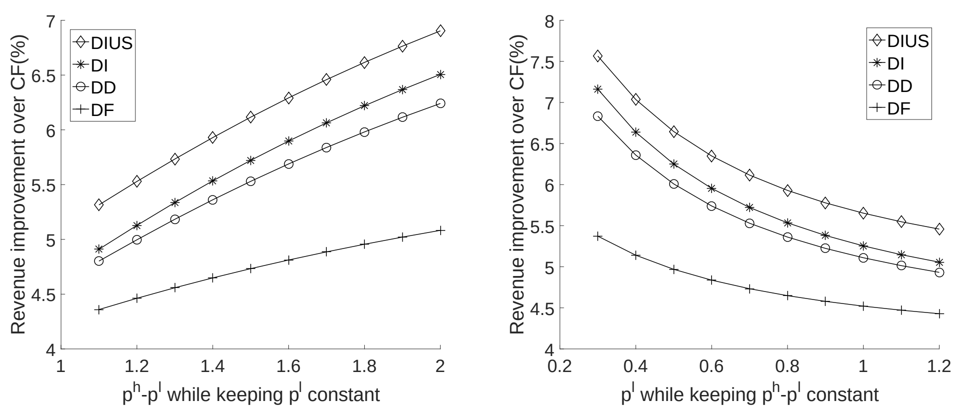

Figure 2.7. Impact of price differential $p^h - p^l$ and regular price p^l

Impact of Product Prices. Companies in travel industry usually use state-of-the-art price optimization software to set the regular product price, on top of which they add a differential and get the premium product price (Yılmaz et al. 2016). The impact of the price differential and the regular product price is shown in Figure 2.7. In the left panel of Figure

2.7, a larger price differential gives the firm an opportunity to charge a higher upgrade fee and further boosts the revenue improvements. Hence, the revenue improvements increase as the price differential increases. In the right panel, since the price differential stays constant, the revenue from upgrades stays about the same. Hence, a higher regular product price leads to smaller percentage revenue improvements.

Impact of Click Rate. The click rate μ is a measure of the dynamic upgrade effectiveness. The percentage revenue improvements increase with the click rate, as shown in Figure 2.8. A high click rate and a high pricing flexibility are complementary. Even with a high click rate, DF strategy with its fixed upgrade fee is not as capable as DD or DI strategy in converting an incoming upgradeable customer into an upgrade purchaser. As a consequence, the revenue improvement from DF to a flexible strategy (e.g., DD or DI) also increases with the click rate. Both dynamic upgrade and upward substitution are tools to balance the leftover capacities. When the click rate is high, the balancing effect from dynamic upgrade dominates the one from substitution, and the revenue improvement gap between DI and DIUS strategies decreases with the click rate.

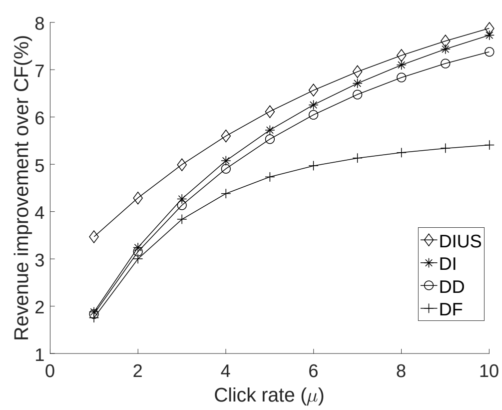

Figure 2.8. Impact of click rate μ

These numerical results help identify that dynamic pricing and timing of upgrades yield a high revenue compared to check-in fixed-fee upgrades, in particular, when (i) the premium product demand is low, (ii) the regular demand is high, (iii) the premium product capacity level is high, (iv) the regular capacity level is low, (v) the price differential is high, (vi) the regular product price is low, and (vii) the click rate is high.

2.6.3 Impact of Environment on Revenues

In this section, we illustrate the impact of operating environment on optimal expected revenues. In general, the revenues are increasing with higher initial capacities, higher prices, higher demand intensities and a higher click rate. We now investigate how the upgrade triggering level M, the maximum upgrade capability C and customers' upgrade reservation price distribution affect the revenues.

Revenue Maximization versus Brand Image Protection. In the numerical study above, we assume fixed $M = 10$ and $C = 5$. Now we focus on the base instance with $N = 500$, $H = 15$, $L = 20$, $\mu^h = 10$, $\mu^l = 20$, $\mu = 5$, $p^h = 1.5$ and $p^l = 0.7$ and check the impact of M and C. A larger M allows the firm to activate dynamic upgrades earlier, and a larger C allows for notifying more customers. In the left panel of Figure 2.9, the highest optimal expected revenue 30.87 is achieved at $(M, C) = (10, 10)$. Although pursuing the highest expected revenue is crucial, a firm deciding on (M, C) also needs to consider its brand image. In the middle panel of Figure 2.9, we show the expected number of upgrade notifications sent during the sales season, which measures the amount of spamming. The highest expected number of notifications 24.62 occurs at $(M, C) = (11, 9)$. In the right panel of Figure 2.9, we show the expected volume of upgrade sales during the sales season, which is related to premium product devaluation. The highest volume 4.74 occurs at $(M, C) = (12, 8)$. As we can see from Figure 2.9, a slightly increased revenue comes with a much larger number of upgrade notifications and a larger volume of upgrade sales. A firm deciding on M and C can use our model and graphs similar to Figure 2.9 to trade off a larger expected revenue against a higher level of spamming and premium product devaluation.

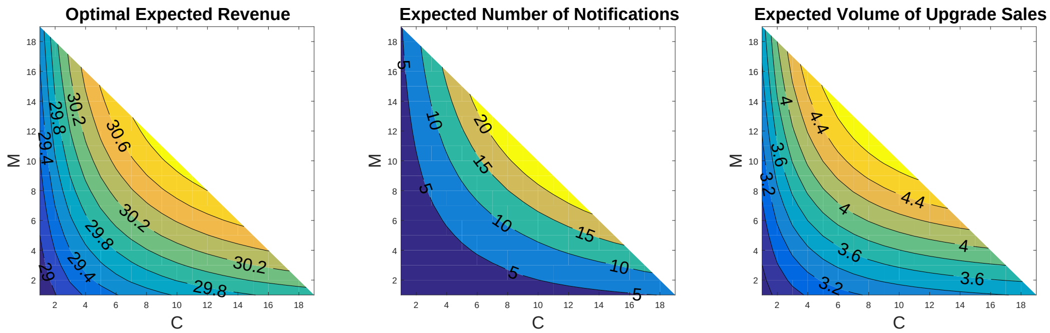

Figure 2.9. Impact of M and C

Impact of Upgrade Reservation Price Distribution. So far, we assume that the upgrade reservation price distribution is uniform. Another question of interest is how different upgrade reservation price distributions affect the firm's expected revenue, especially, whether a larger variance has a positive or negative impact. A larger variance means that the firm has less information about the reservation price distribution. Without a careful inspection, one may say that a larger variance hurts the firm. The actual finding, in our dynamic upgrade pricing context, reveals that a larger variance may either increase or decrease the firm's expected revenue depending on the instance.

For the purpose of illustrating the benefit of a large variance, we construct the following example for DI strategy. $\lambda^h = \lambda^l = \lambda = 0.1$, $p^h = 2$, $p^l = 1$, and $C = 4$. We consider two reservation price distributions. Both distributions have the same mean 0.5. The base distribution is one concentrated at the single point of 0.5; all upgradeable customers have the same reservation price and at most want to pay 0.5 for the upgrades. The other one is a two-point distribution with equal mass of 0.5 on each point; half of the upgradeable customers at most want to pay 0.1 and the other half at most want to pay 0.9 for the upgrades. The sales season has two periods. We now calculate the expected revenue V_n^1 (single-point distribution) and V_n^2 (two-point distribution) for period 2 and 1. As we can see in Table 2.3, $V_2^1(h,l) \geq V_2^2(h,l)$ is true for all possible states. However, $V_1^1(1,l) < V_1^2(1,l)$ for $l \in [0 : M]$.

Table 2.3. Expected revenue comparison

State (h,l)	Less variability		More variability
$(0,0)$	$V_N^1(0,0) = 0.00$	$\geq$	$V_N^2(0,0) = 0.00$
$(h,0)$ with $h \in [1:H]$	$V_N^1(h,0) = 0.40$	$\geq$	$V_N^2(h,0) = 0.38$
$(0,l)$ with $l \in [1:M]$	$V_N^1(0,l) = 0.10$	$\geq$	$V_N^2(0,l) = 0.10$
$(h,l) \in [1:H] \times [1:M]$	$V_N^1(h,l) = 0.50$	$\geq$	$V_N^2(h,l) = 0.48$
$(0,0)$	$V_{N-1}^1(0,0) = 0.00$	$\geq$	$V_{N-1}^2(0,0) = 0$
$(1,0)$	$V_{N-1}^1(1,0) = 0.64$	$<$	$V_{N-1}^2(1,0) = 0.67$
$(h,0)$ with $h \in [2:H]$	$V_{N-1}^1(h,0) = 0.84$	$\geq$	$V_{N-1}^2(h,0) = 0.78$
$(0,1)$	$V_{N-1}^1(0,1) = 0.19$	$\geq$	$V_{N-1}^2(0,1) = 0.19$
$(0,l)$ with $l \in [2:M]$	$V_{N-1}^1(0,l) = 0.20$	$\geq$	$V_{N-1}^2(0,l) = 0.20$
$(1,1)$	$V_{N-1}^1(1,1) = 0.79$	$<$	$V_{N-1}^2(1,1) = 0.84$
$(h,0)$ with $h \in [2:H]$	$V_{N-1}^1(h,1) = 0.99$	$\geq$	$V_{N-1}^2(h,1) = 0.95$
$(1,l)$ with $l \in [2:M]$	$V_{N-1}^1(1,l) = 0.80$	$<$	$V_{N-1}^2(1,l) = 0.85$
$(h,l) \in [2:H] \times [2:M]$	$V_{N-1}^1(h,l) = 1.00$	$\geq$	$V_{N-1}^2(h,l) = 0.96$

This finding can be explained as follows. When the mean of the reservation price stays unchanged, a larger variance implies the existence of more customers with lower reservation prices. With a certain fee, it is less likely to make an upgrade offer acceptable. However, a larger variance also means the existence of more customers with higher reservation prices, which allows the firm to charge a higher fee and still to capture the demand from a customer with high reservation price. These are the reasons why a larger variance may help or hurt the firm. From a mathematical point view, this finding is due to the lack of convexity of the revenue function in the customer reservation price. Jensen's inequality cannot be applied; a larger variance does not necessarily imply less revenue.

2.7 Conclusion

Upgrading has become a common operational tactic for companies to boost revenue. In Chapter 2, we describe a dynamic upgrade process for a firm that sells two types of products (premium and regular) at fixed prices and offers upgrades from the booking time until the check-in time. The firm sends regular purchasers upgrade notifications (via email or push notification) that contain upgrade links. The links lead customers to the firm's upgrade website, who then accept or reject the upgrades based on the upgrade fees determined also dynamically by the firm. An optimal policy specifies the timing (availability) and pricing of the dynamic upgrades. Such decisions involve analyzing subtle trade-offs. The pricing decision requires the firm to analyze the loss and gain of replacing a unit of premium capacity by a regular one. The availability decision at a particular time depends on the profitability of upgrades at that time. Based on a dynamic programming model that incorporates the above trade-offs, we characterize the optimal upgrade policy as a pulsing policy. The firm either maintains zero or the maximum number of active upgrade links. Both the optimal availability and pricing decisions exhibit monotonicity properties with respect to the capacity levels. We also consider two extensions of our base model. The first one restricts the upgrade fee. We identify a condition under which the upgrade region of the restricted fee model counterintuitively contains that of the base model. This optimal policy comparison further reveals managerial insights related to upgrading. The second extension incorporates upward substitution to capture the regular customer demand when the regular product stocks out. We show that the substitution fee should be no greater than the upgrade fee.

Our comprehensive and systematic numerical study helps managers identify suitable contexts to adopt the dynamic upgrade strategy and understand the effects of upgrade fee pricing flexibility as well as upward substitution on the benefit of dynamic upgrades. Specifically, the revenue improvement from the current industry-standard check-in fixed-fee upgrades to dynamic upgrades is significant (e.g., up to 49% improvement) when the premium product capacity level is high, the regular capacity level is low, the premium product demand is low, the regular demand is high, the regular product price is low, the price differential is high, and the click rate is high. A high pricing flexibility magnifies the benefit of a high click rate. A low regular demand and a high regular capacity weaken the contribution of the upward substitution. Emails and push notifications are widely used marketing tools to attract customers. This chapter is the first one to study how to use such tools to engage existing purchasers for additional revenue through upgrades while managing spamming and brand image related issues. The implementation of the upgrade models is easy, and we provide an algorithm.

Given the novelty of dynamic upgrades, it has many potential applications in the practice of selling services/products to maximize revenues. Potential application-oriented research includes the incorporation of dynamic upgrades into existing dynamic pricing system, multi-product dynamic upgrades, and demand learning through dynamic upgrade pricing. The comparison of consumer behaviors towards dynamic upgrades and other existing upgrade strategies would also be exciting.

CHAPTER 3
IMPACT OF POSTPONEMENT MANUFACTURING STRATEGY ON CONTRACT DESIGN

3.1 Introduction

Mass customization has been adopted by many industries in response to diverse customer demands, even though it comes with higher production and inventory costs. To cope with these disadvantages, firms apply process standardization to the initial steps in production, so that products are not differentiated until later customization steps. Delaying the product differentiation increases firms' flexibility of handling fluctuating multi-product demands. The strategy of postponement, or delayed product differentiation, was first introduced by Alderson (1950). Since then, there has been an extensive literature studying this topic. We refer the reader to Anand and Mendelson (1998), Swaminathan and Lee (2003), Yang et al. (2004), Forza et al. (2008) and Cheng et al. (2010) for a detailed review. A stream of research within this vast literature develops mathematical models to evaluate the benefits of postponement. For example, Lee (1996) show that postponement always leads to inventory reduction under stationary demand assumption; such reduction is greater when the end product demands are negatively correlated. Lee and Whang (1998) extend Lee (1996) by modeling the demand as a random walk process. They quantify the benefit of postponement as the value of uncertainty resolutions and the value of forecast improvement. Aviv and Federgruen (2001) develop a multi-period model with demand learning in a Bayesian framework and illustrate the incremental benefits of postponement from the learning effect. Hu et al. (2016) evaluate the value of postponement for a two-product newsvendor under social influence, where customers arrive sequentially and their purchase decisions can be influenced by earlier purchases. These papers focus on inventory systems of an individual manufacturer, and study the benefits of postponement through inventory cost reduction. However, as a member of a supply chain, the manufacturer's postponement strategy may also affect other supply chain members as well as the contracts between them. In this chapter, our goal is to evaluate the impact of postponement on supply chain contracts.

We consider a two-stage supply chain with a manufacturer and two retailers. The manufacturer first produces a batch of common intermediate products and customizes them into different end products. This multi-product manufacturing system can also be interpreted as a distribution system (see, e.g., Eppen and Schrage 1981, Federgruen and Zipkin 1984a, Schwarz 1989, Erkip et al. 1990, Güllü 1997, Özer 2003, and Gürbüz et al. 2007), which consists of a central depot and several warehouses. Demands are observed and satisfied at the warehouses. The depot does not hold inventory; it only places orders and allocates them to the warehouses. Both retailers order the customized end products from the manufacturer and meet their stochastic end customer demands respectively. Unsatisfied demands are backlogged. Examples of this type of supply chain structure are common in traditional manufacturing industry. Take Herman Miller, a major American manufacturer of office furniture, as an example. It manufactures and sells customized office desks and chairs to corporate customers through its dealerships (retailers).

Supply chain members face replenishment lead times. The manufacturer has a production lead time and prefers retailers placing orders in advance of their requirement. However, retailers face order fulfillment lead times and prefer that the manufacturer fully fulfills orders quickly. Since the supply chain members in either stage want to avoid the demand uncertainty during lead times, there are incentive conflicts between the two stages. The supply chain incentive can be aligned by a promised lead time contract, which was discussed by Hariharan and Zipkin (1995) and Lutze and Özer (2008). Under such a contract, the retailer places advance orders with the manufacturer. The manufacturer guarantees shipment of each order on time and in full after a promised lead time. The promised lead time contract eliminates the retailer's risk from uncertain supply and decreases the manufacturer's risk from uncertain demand. A cost benefit analysis of this interaction and the resulting inventory costs determine who pays for the promised lead time contract. Notice that the promised lead time is not the retailer's replenishment lead time; his total replenishment lead time is the promised lead time plus a transportation lead time from the manufacturer to the retailer. When the promised lead time is zero, the manufacturer holds ample inventory and satisfies the retailer's order instantaneously. When the promised lead time is equal to manufacturer's total production lead time, the retailer places his order well in advance. The manufacturer then holds zero inventory and starts production after receiving the order. The longest promised lead time is the manufacturer's total production lead time, since any longer promised lead time does not further eliminate the manufacturer demand uncertainty but hurts the retailer.

The contract design is affected by the supply chain setting. We study the promised lead time contract in the two-stage supply chain under three settings. To create a benchmark, we first establish optimal promised lead times for a centralized setting. When system control is not centralized, we study the different market setting and the same market setting. In the different market setting, the retailers are geographically dispersed and cannot observe each other's contract terms. The manufacturer can fully discriminate the retailers by offering them different contracts. In the same market setting, contracts offered by the manufacturer are public information to both retailers. Perfect discrimination is not feasible and the retailers self-select their contracts from a menu designed by the manufacturer. In all three settings, we show that an optimal promised lead time is either zero or equal to the manufacturer's total production lead time; it is never optimal to split the inventory between the manufacturer and an individual retailer. The optimal promised lead times in the centralized setting and the different market setting are the same. However, the same market setting has different contracts, in which the retailer with a higher inventory cost always gets a shorter promised lead time.

Two papers studying promised lead times in a two-stage supply chain are closely related to ours. Barnes-Schuster et al. (2006) study a centralized supply chain under normally distributed demand. Retailers have identical holding and penalty costs. They show that the manufacturer optimally offers the longest promised lead time to retailers with large standard deviations while giving the shortest promised lead time to retailers with small standard deviations. Lutze and Özer (2008) analyze a supply chain facing a finite planning horizon, where the retailer may have private information about his shortage cost. To minimize her own inventory cost while ensuring the retailer's participation, the manufacturer designs a contract which specifies a promised lead time and a lump-sum payment. They derive the optimal contract under both full and asymmetric information. Chapter 3 is similar to the above two in that we study how a manufacturer shares the demand uncertainty with downstream retailers by specifying promised lead times. However, the manufacturer in Chapter 3 produces multiple products.

This chapter characterizes the impact of postponement on supply chain contract design (promised lead time contracts in particular) and the manufacturer production mode selection in three different market settings. When the promised lead times equal manufacturer's total production lead time (resp., zero), the manufacturer is applying a make-to-order (resp., make-to-stock) production mode. Postponement, under certain conditions, shifts the manufacturer's production mode from make-to-order to make-to-stock. Gupta and Benjaafar (2004) and Su et al. (2010) are among the first to study postponement and production mode. Both papers adopt queuing models to study capacitated manufacturing systems, where the production modes (make-to-order and/or make-to-stock) are fixed. Gupta and Benjaafar (2004) consider the production stage of the common intermediate product as a make-to-stock system and the stage of the customized end products as a make-to-order system. They specifically evaluate the benefits of postponement when delivery lead times are load dependent and induced by the capacity constraint. Su et al. (2010) focus on a make-to-order system and identify when and why postponement is beneficial using inventory cost and waiting time as performance metrics. In contrast, the production modes in this chapter are endogenous and driven by the optimal promised lead time contracts, and we study how postponement affects the contract design and further changes the manufacturer's production mode selection. For a comprehensive review of the broad literature on make-to-order versus make-to-stock, see Soman et al. (2004).

The remainder of this chapter is organized as follows. In §3.2, we first treat the promised lead times as given and study the production and inventory problems of the supply chain members. In §3.3, we characterize the optimal promised lead time contracts in three settings based on the results from §3.2. We analyze the impact of postponement on the optimal promised lead time contracts in §3.4 and quantify the impact using numerical examples in §3.5. In §3.6, we conclude the chapter. Proofs and additional arguments are relegated to the appendices.

3.2 Two-stage Supply Chain

We study a two-stage supply chain consisting of a multi-product manufacturer and two retailers, as shown in Figure 3.1, over an infinite planning horizon. The manufacturer requires L periods to produce a common intermediate product and allocates this product among customization sequences for J different end products. Each customization sequence requires l periods to complete. Retailer $k \in \{1, 2\}$ orders J end products, and the manufacturer ships out the full order $s_k \geq 0$ periods later, i.e., the promised lead time. Retailer k receives the order after a transportation lead time. Without loss of generality, we assume that the transportation lead times are zero for both retailers; both retailers receive deliveries of full orders immediately after the manufacturer's shipments. All our results are still true when the transportation lead times are positive. Stochastic end customer demands are satisfied through retailers' on hand inventory. Otherwise, they are backlogged. Demand in period t for end product $j \in \{1, 2, ..., J\}$ at retailer $k \in \{1, 2\}$, denoted by D^t_{kj}, is modeled as a sequence of stationary, independent and normally distributed random variables with finite mean μ_{kj} and standard deviation σ_{kj}. The manufacturer and the retailers know the demand distributions. We use $\phi(\cdot)$ and $\Phi(\cdot)$ to represent the *pdf* and *cdf* of the standard normal distribution.

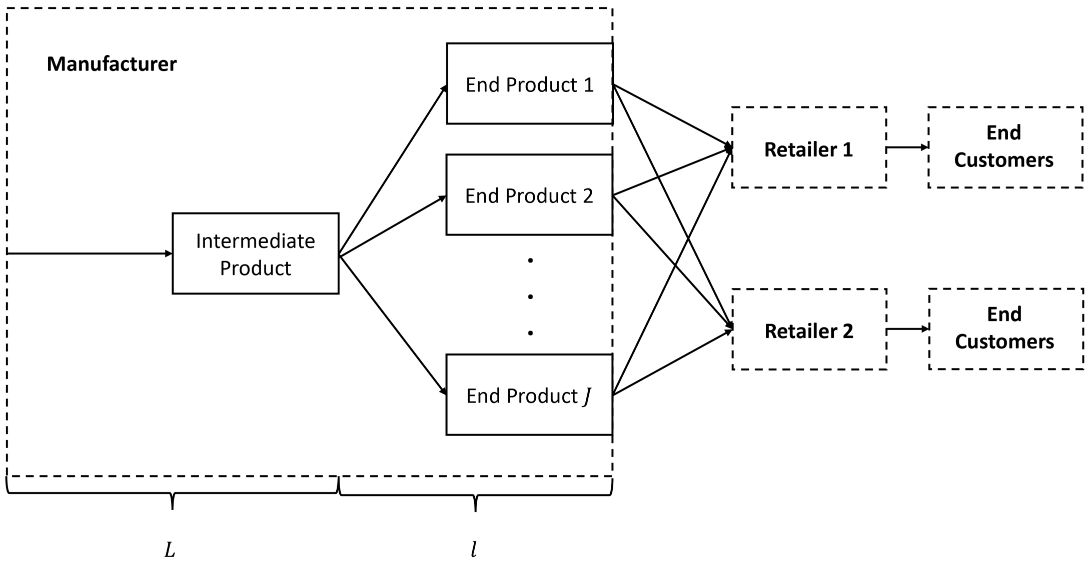

Figure 3.1. Supply Chain Structure

The inventory levels at both stages are periodically reviewed. The sequence of events is shown in Figure 3.2. At the beginning of each period t, the manufacturer receives both finished intermediate products and customized end products. She then produces a new batch of intermediate products. The manufacturer does not hold inventory for the intermediate product and allocates them immediately to the J customization sequences. During period t, the manufacturer fully fulfills retailer k's orders $d_{k1}^{t-s_k,t}$, ... , $d_{kJ}^{t-s_k,t}$, which were placed s_k periods ago and are due for delivery. She also receives orders $d_{k1}^{t,t+s_k}$, ... , $d_{kJ}^{t,t+s_k}$ from each retailer k to be delivered in period $t + s_k$. The manufacturer incurs the same unit holding cost h_m for any remaining inventory of J end products. The manufacturer's unit penalty cost p_m represents the cost of borrowing a unit of any product j from an emergency source when her on-hand inventory is not enough to satisfy retailers' orders. The emergency source must be returned, and the manufacturer incurs the penalty cost until doing so. To our knowledge, the usage of an emergency source of this nature first appeared in Lee et al. (2000) and Graves and Willems (2000). We remark that $s_1 = s_2 = 0$ corresponds to the classical postponement inventory problem and formulation in Lee (1996).

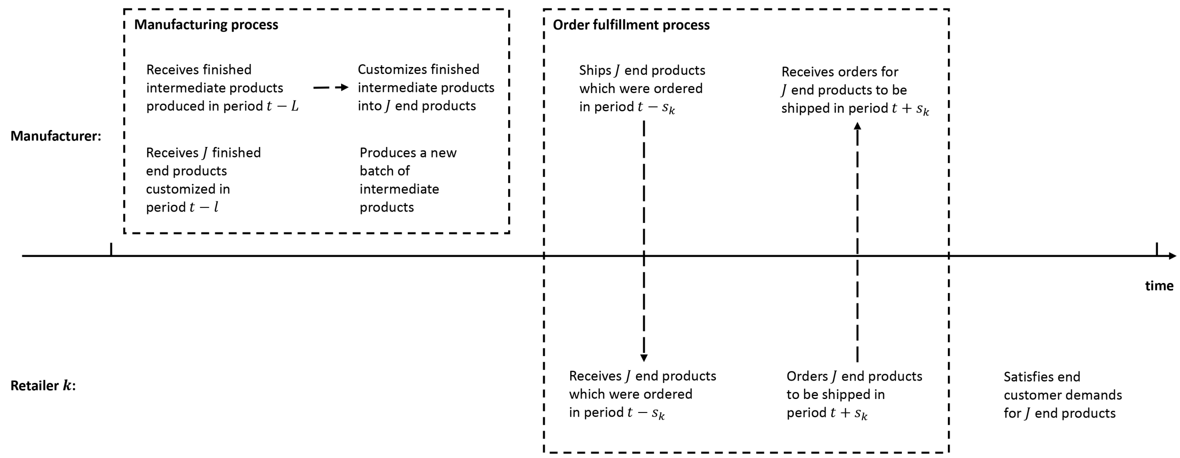

Figure 3.2. Sequence of events in period t

During period t, retailer k receives the orders of $d_{k1}^{t-s_k,t}$, ... , $d_{kJ}^{t-s_k,t}$, and places a new batch of orders $d_{k1}^{t,t+s_k}$, ... , $d_{kJ}^{t,t+s_k}$. At the end of period t, the end customer demand D_{kj}^t for every end product j at retailer k is realized. Retailer k either satisfies customer demands through on-hand inventory or backlogs. He incurs holding cost for the leftover inventory or penalty cost for backlog across J end products. We allow retailers to have different inventory costs. In particular, retailer k has unit holding cost $c_k h_r$ and unit penalty cost $c_k p_r$. A retailer having a higher value of c_k can be interpreted as one located in a neighborhood where storage space is more expensive and the cost of customer impatience is higher. This proportional inventory cost structure was introduced first by Federgruen and Zipkin (1984b).

We assume that unit production costs, wholesale prices, and retail prices are exogenously fixed constants across all J end products, respectively. The retailers capture all end customer demands, since unsatisfied demands are backlogged. Thus, for the retailers, maximizing profit is equivalent to minimizing inventory cost. Similarly, the manufacturer fully satisfies orders from the retailers due to the existence of the emergency source. Her goal of profit maximization is also equivalent to inventory cost minimization. Hence, we focus on the inventory cost minimization problems.

The promised lead times affect the distribution of demand uncertainty among two retailers and the manufacturer. For retailer k, the promised lead time requires him to place an order s_k periods in advance. A longer promised lead time increases retailer k's demand uncertainty, since he carries inventory to protect against end customer demand uncertainty over the promised lead time. The promised lead times generate advance orders and reduce the manufacturer's demand uncertainty. Recall that the manufacturer's total production time is $L + l$ and she plans production based on the retailers' orders. Note that when $s_1 = s_2 = L + l + 1$, the manufacturer produces to order and carries zero inventory. Hence, s_k greater than $L + l + 1$ does not further eliminate the manufacturer's demand uncertainty but hurts retailer k. Thus, it is never optimal to set s_k greater than $L + l + 1$, i.e., $s_k \in \{0, 1, \ldots, L + l + 1\}$. Also note that when $s_1 = s_2 = 0$, the manufacturer employs make-to-stock production mode, and retailers get instantaneous order deliveries.

3.2.1 Retailer's Problem

For a given promised lead time s_k, retailer k minimizes his inventory cost over an infinite horizon. Due to the manufacturer's emergency source that decouples the two-stage supply chain, each retailer k is guaranteed on-time delivery of all orders and his optimal order quantity is not affected by the availability of inventory at the manufacturer. Also note that end customer demands for different end products are independent. Hence, for each product j, retailer k independently solves a stationary, single location, periodic review inventory control problem with a fixed lead time s_k and no upstream supply restriction. This problem, shown by Arrow et al. (1958), can be rewritten as a problem with zero lead time by modifying the demand distribution to incorporate all demands during the lead time. A myopic base-stock policy is optimal for this problem under either the expected discounted cost criterion or the long-run average cost criterion (e.g., Veinott 1965, Iglehart 1961, and Lovejoy 1990). Retailer k can obtain the optimal base-stock level for end product j by solving the following problem:

$$\max_{y_{kj}} E[c_k h_r(y_{kj} - \sum_{n=0}^{s_k} D_{kj}^{t+n})^+ + c_k p_r(y_{kj} - \sum_{n=0}^{s_k} D_{kj}^{t+n})^-]. \tag{3.1}$$

y_{kj} is the base-stock level, and the expectation is taken over $\sum_{n=0}^{s_k} D_{kj}^{t+n}$, which is the total demand during lead time s_k. Since the end customer demand arrives after retailer k receives the delivery, he still needs to hold inventory against one-period demand uncertainty even if $s_k = 0$. Retailer k's optimal inventory cost over an infinite horizon can be characterized by the expected average cost per period under the optimal base-stock policy. We provide closed-form expressions for retailer k's optimal base-stock levels and the resulting expected average cost per period as follows.

Proposition 8. *Given a promised lead time s_k, retailer k's optimal base-stock level $y_{kj}(s_k)$ for product j, and the corresponding optimal expected cost per period $G_k(s_k)$ are*

$$\begin{aligned} y_{kj}(s_k) &= \mu_{kj}(s_k+1) + \Phi^{-1}(\frac{p_r}{p_r+h_r})\sigma_{kj}\sqrt{s_k+1}, \text{ and} \\ G_k(s_k) &= c_k(h_r+p_r)\phi(\Phi^{-1}(\frac{p_r}{p_r+h_r}))\psi_k(s_k), \end{aligned}$$

where $\psi_k(s_k) = (\sum_{j=1}^{J}\sigma_{kj})\sqrt{s_k+1}$ is concave increasing in s_k.

The first term of $y_{kj}(s_k)$ is the expected demand for product j over retailer k's promised lead time. The second term is his safety stock. The function $\psi_k(s_k)$ in Proposition 8 represents the effective standard deviation of demands for all J products over the promised lead time at retailer k. $\psi_k(s_k)$ and retailer k's inventory cost are increasing in s_k. Therefore, the retailer prefers a short promised lead time. The concavity property of $\psi_k(s_k)$ implies that retailer k is more sensitive to an increase in his promised lead time (or, equivalently, benefits more from a decrease in his promised lead time) when the promised lead time is short.

3.2.2 Manufacturer's Problem

The manufacturer's problem can be modeled as a two-echelon inventory problem over an infinite horizon. It requires L periods to produce a common intermediate product and another l periods to customize the intermediate product into J end products. The manufacturer's demand depends on the retailers' ordering policy. From Proposition 8, we know that retailer k optimally follows a stationary base-stock policy. As a result, he orders in each period to recover the units demanded by the end consumers from the previous period. The manufacturer, therefore, observes the same end consumer demand stream after a single period delay. That is, $d_{kj}^{t,t+s_k} = D_{kj}^{t-1} \sim N(\mu_{kj}, \sigma_{kj}^2)$. At the beginning of each period t, the demand for end product j due for delivery in period $t+n$ is the sum of the *observed* portion

$$o_j^{t,t+n} := d_{1j}^{t+n-s_1,t+n}\mathbb{I}_{\{s_1>n\}} + d_{2j}^{t+n-s_2,t+n}\mathbb{I}_{\{s_1>n\}}$$

and the *unobserved* portion

$$u_j^{t,t+n} := d_{1j}^{t+n-s_1,t+n}\mathbb{I}_{\{s_1\le n\}} + d_{2j}^{t+n-s_2,t+n}\mathbb{I}_{\{s_2\le n\}},$$

where $\mathbb{I}_{\mathcal{A}} \in \{0,1\}$ is an indicator function taking the value of 1 only when $\mathcal{A}$ is true. The manufacturer may incur unit holding cost h_m for leftover inventory of J end products or unit penalty cost p_m for borrowing products from an emergency source.

Establishing the manufacturer's optimal policy is computationally intensive, even in the absence of retailers' advance demand information. Our goal here is to obtain a good approximation to the problem, so that we can quantify manufacturer's inventory cost in a closed form. We use one of the best methods known to date to solve the manufacturer's complex production problem. The method is based on restricting the policy space to a set of base-stock policies with myopic allocation and invoking the *Allocation Assumption* from Eppen and Schrage (1981). Several other researchers (e.g., Federgruen and Zipkin 1984a, Erkip et al. 1990, Aviv and Federgruen 2001, Özer 2003, and Alptekinoglu and Tang 2005) have used this method since then. Under the policy restriction, the manufacturer produces a batch of intermediate products in period t and brings the system inventory to a base-stock level. She then distributes the finished intermediate products in period $t+L$ among J customization sequences by following a myopic allocation rule. The myopic allocation minimizes the expected costs in period $t+L+l$ when the allocation actually takes effect, while ignores costs in all subsequent periods. Under the *Allocation Assumption*, the manufacturer always receives sufficient intermediate products in period $t+L$, so that each customization sequence can be allocated sufficient intermediate products to ensure an equal probability of stockout (or the same service level) across all J end products in period $t+L+l$. With the policy restriction and the *Allocation Assumption*, we can find the optimal base stock level and the corresponding expected inventory cost per period in closed forms.

Now, we develop the manufacturer's cost function. Due to the base-stock policy, a batch of intermediate product is produced to bring the total system stock to y^t at the beginning of period t. The system stock y^t includes the on-hand and in-transit inventory of both the intermediate product and J end products. It protects the system from unobserved demand variation over $L+l+1$ periods. We define the total demand over the intermediate product production lead time L as

$$V^t := \sum_{n=0}^{L-1} \sum_{j=1}^{J} (o_j^{t,t+n} + u_j^{t,t+n})$$

and the total demand for end product $j \in \{1, 2, ..., J\}$ during periods $t+L$ through $t+L+l$ as

$$W_j^{t+L} := \sum_{n=L}^{L+l} (o_j^{t,t+n} + u_j^{t,t+n}).$$

At the beginning of period $t+L$, the amount of all end products to be finished by period $t+L+l$ is $y^t - V^t$. To minimize the inventory cost at the end of period $t+L+l$, the manufacturer solves

$$\begin{aligned}
\min_{y^t, y_1^{t+L}, ..., y_J^{t+L}} \quad & \sum_{j=1}^{J} G_j(y_j^{t+L}) \\
\text{s. t.} \quad & \sum_{j=1}^{J} y_j^{t+L} = y^t - V^t \\
& y_j^{t+L} \geq y_j^{t+L-1} - (o_j^{t,t+L-1} + u_j^{t,t+L-1}) \text{ for } j \in \{1, 2, ..., J\},
\end{aligned}$$

where y_j^{t+L} represents the total amount of on-hand and in-transit inventory of end product j in period $t+L$ after the myopic allocation, and the inventory cost of product j at the end of period $t+L+l$ is represented by

$$G_j(y_j^{t+L}) := E[h_m(y_j^{t+L} - W_j^{t+L})^+ + p_m(y_j^{t+L} - W_j^{t+L})^-],$$

where the expectation is taken over product j's unobserved demand from period $t+L$ to period $t+L+l$. The equality constraint exists because the manufacturer does not hold inventory of intermediate product; all available intermediate products must be allocated to J end products. The inequality constraints ensure that the allocation of intermediate products to each end product is non-negative. Given the *Allocation Assumption*, the inequality constraints are always satisfied, and the myopic allocation problem can be simplified into

$$\min_{y^t, y_1^{t+L}, ..., y_J^{t+L}} \sum_{j=1}^{J} G_j(y_j^{t+L}) \text{ s. t.} \sum_{j=1}^{J} y_j^{t+L} = y^t - V^t. \tag{3.2}$$

Since the end customer demand is independent and stationary, the expected inventory cost charged to period $t+L+l$ would be the same as any other period during the infinite time horizon. Hence, we characterize the manufacturer's inventory cost over infinite horizon by the expected average cost per period solved from (3.2). Next, we provide the multi-product manufacturer's optimal inventory policy and the resulting expected inventory cost in closed forms.

Proposition 9. *Given lead times (L, l, s_1, s_2), the manufacturer's optimal base-stock level in each period t and the resulting expected inventory cost per period are*

$$
\begin{aligned}
& y^t(L, l, s_1, s_2) \\
= & \sum_{j=1}^{J} \sum_{n=0}^{L+l} o_j^{t,t+n} + \sum_{j=1}^{J} \sum_{k=1}^{2} (L + l + 1 - s_k)\mu_{kj} + \Phi^{-1}\left(\frac{p_m}{h_m + p_m}\right) \psi_m(L, l, s_1, s_2)
\end{aligned}
$$

and

$$
G_m(L, l, s_1, s_2) = (h_m + p_m)\phi\left[\Phi^{-1}\left(\frac{p_m}{h_m + p_m}\right)\right] \psi_m(L, l, s_1, s_2),
$$

where

$$
\begin{aligned}
& \psi_m(L, l, s_1, s_2) \\
= & \sqrt{\sum_{j=1}^{J} \sum_{k=1}^{2} (L - s_k)\sigma_{kj}^2 \mathbb{1}_{\{s_k < L\}} + \left(\sum_{j=1}^{J} \sqrt{\begin{array}{l} \sum_{k=1}^{2} (l+1)\sigma_{kj}^2 \mathbb{1}_{\{s_k < L\}} \\ + \sum_{k=1}^{2} (L + l + 1 - s_k)\sigma_{kj}^2 \mathbb{1}_{\{L \le s_k \le L+l+1\}} \end{array}} \right)^2 }.
\end{aligned}
$$

The first term of $y^t(L, l, s_1, s_2)$ is the observed demand during the production lead time, the second is the expected value of the unobserved demand, and the third is the safety stock. The function $\psi_m(L, l, s_1, s_2)$ represents the effective standard deviation of the total demand during the production lead time for the multi-product manufacturer. The first term measures the demand variation during the first L periods and the second term measures the demand variation during the l periods of end product customization. Proposition 9 extends the multi-product inventory control literature to account for advance demand information from retailers with possibly different promised lead times. Proposition 9 can also be easily extended to consider more general supply chain settings with more than two retailers.

Proposition 10. *$\psi_m(L, l, s_1, s_2)$ satisfies the following properties:*

a) $\psi_m(L, l, s_1, s_2) \geq \psi_m(L, l, s_1 + 1, s_2)$,

b) $\psi_m(L, l, s_1, s_2) \geq \psi_m(L, l, s_1, s_2 + 1)$,

c) $\psi_m(L, l, s_1, s_2) - \psi_m(L, l, s_1 - 1, s_2) \geq \psi_m(L, l, s_1 + 1, s_2) - \psi_m(L, l, s_1, s_2)$,

d) $\psi_m(L, l, s_1, s_2) - \psi_m(L, l, s_1, s_2 - 1) \geq \psi_m(L, l, s_1, s_2 + 1) - \psi_m(L, l, s_1, s_2)$.

Proposition 10 implies that the manufacturer's effective inventory cost and total inventory position are concave decreasing in s_1 and s_2 respectively. Therefore, she prefers longer promised lead times. For an extreme case in which $s_1 = s_2 = L+l+1$, the manufacturer has perfect information about future demand, employs make-to-order production and carries no inventory. The concavity property implies that the marginal benefit of promised lead times increases as they get longer. In other words, a reduction in promised lead times hurts the manufacturer more when the promised lead times are long.

So far, we provide an approximation of the manufacturer's inventory cost with the help of policy restriction and *Allocation Assumption*. The accuracy of this approximation depends on whether *Allocation Assumption* actually holds. Eppen and Schrage (1981) and Erkip et al. (1990) show that this assumption holds with high probability. Even if *Allocation Assumption* does not hold, Federgruen and Zipkin (1984a) show that the resulting production and allocation decisions yield expected costs very close to optimal for multi-product inventory systems with a low demand coefficient of variation. We remark that prior applications of *Allocation Assumption* do not consider the possibility of promised lead times, i.e., $s_k > 0$. We examine the impact of promised lead times in the following proposition.

Proposition 11. *Allocation Assumption holds with certainty when $s_k \geq L+1$ for $k \in \{1, 2\}$.*

When the promised lead times are shorter than $L+1$, the necessary condition of *Allocation Assumption* is similar to that from Eppen and Schrage (1981), making it equally possible to restore the system to an equal probability of stockout across all J end products. When the promised lead times are equal to or longer than $L+1$, *Allocation Assumption* holds with certainty. When $s_k \geq L+1$, the manufacturer at period t fully observes all the demands from period t to period $t+L$. She knows in advance how the multi-product inventory system will shift away from the equal probability of stockout, so that she can produce just enough intermediate products and allocate them exactly to restore the equal probability of stockout.

3.3 Optimal Lead Times

Our goal is to arrive at optimal promised lead times in the two-stage supply chain. Proposition 8 and 10 highlight the incentive conflicts between the manufacturer and the retailers. The concavity properties of their costs imply potential structural results of the optimal promised lead times. As a benchmark, we first determine the promised lead times that minimize the expected per period inventory cost of a supply chain under centralized control. Next, we study two decentralized systems from the manufacturer's perspective, whose goal is to design promised lead time contracts for the retailers which minimize her own inventory cost.

We assume, without loss of generality, that the retailers' inventory cost and risk parameters satisfy the following inequality:

$$c_1 \sum_{j=1}^{J} \sigma_{1j} > c_2 \sum_{j=1}^{J} \sigma_{2j}. \tag{3.3}$$

This assumption guarantees $G_1(s) > G_2(s)$ for any $s \in \{0, ..., L+l+1\}$. It means that retailer 1 faces higher holding and penalty costs and/or a greater uncertainty in her end customer demands. We introduce the following notations, which will be used in the derivation of optimal lead times.

$$\begin{aligned}
\alpha_r &:= (h_r + p_r)\phi\left(\Phi^{-1}\left(\frac{p_r}{p_r + h_r}\right)\right), \quad \alpha_m := (h_m + p_m)\phi\left(\Phi^{-1}\left(\frac{p_m}{p_m + h_m}\right)\right), \\
a &:= c_1\left[\psi_1(L+l+1) - \psi_1(0)\right], \quad b := c_2\left[\psi_2(L+l+1) - \psi_2(0)\right], \\
x &:= \psi_m(L, l, 0, 0), \quad y := \psi_m(L, l, 0, L+l+1), \quad z := \psi_m(L, l, L+l+1, 0).
\end{aligned}$$

α_r and α_m measure the costs of demand uncertainty. We can rewrite the manufacturer's and retailer k's inventory costs as $G_m(L, l, s_1, s_2) = \alpha_m \psi_m(L, l, s_1, s_2)$ and $G_k(s_k) = c_k \alpha_r \psi_k(s_k)$. We have $a > b > 0$ because of inequality (3.3). $\alpha_r a$ and $\alpha_r b$ measure the additional inventory costs when retailer 1 and 2 respectively shifts his promised lead time from 0 to $L + l + 1$. $\alpha_m x$, $\alpha_m y$, $\alpha_m z$ represent the manufacturer's inventory costs under different combinations of promised lead times. Also notice that $x > y > 0$ and $x > z > 0$ are always true from Proposition 10.

3.3.1 Centralized Supply Chain

When the supply chain is centralized, we denote the promised lead times that minimize the total supply chain inventory cost $G_m(L, l, s_1, s_2) + G_1(s_1) + G_2(s_2)$ as $s_1^C(L, l)$ and $s_2^C(L, l)$.

Proposition 12. *The optimal promised lead times that minimize the expected inventory cost of the centralized supply chain are characterized as follows:*

$$\left(s_1^C(L,l), s_2^C(L,l)\right) = \begin{cases} (L+l+1, L+l+1), & \text{if } \frac{\alpha_r}{\alpha m} \in (0, \underline{M}] \\ (L+l+1, 0), & \text{if } \frac{\alpha_r}{\alpha m} \in (\underline{M}, \overline{M}] \cap (0, M] \\ (0, L+l+1), & \text{if } \frac{\alpha_r}{\alpha m} \in (\underline{M}, \overline{M}] \cap (M, +\infty) \\ (0, 0), & \text{if } \frac{\alpha_r}{\alpha m} \in (\overline{M}, +\infty) \end{cases},$$

where $\underline{M} := \min\left\{\frac{x}{a+b}, \frac{y}{a}, \frac{z}{b}\right\}$, $\overline{M} := \max\left\{\frac{x}{a+b}, \frac{x-z}{a}, \frac{x-y}{b}\right\}$, *and* $M := \frac{y-z}{a-b}$.

Since the supply chain total cost is concave in s_1 and s_2 respectively from Proposition 8 and 10, its minimum must occur at one of the extreme points. Sharing responsibility for inventory uncertainty between the manufacturer and a retailer is not optimal for the supply chain. The ratio of α_r/α_m measures retailers' cost of demand uncertainty relative to the manufacturer's. When the ratio of α_r/α_m is lower than $\underline{M}$, the manufacturer's cost of uncertainty is sufficiently higher than the retailers'. It is cheaper to store inventory at the retailers' end. The optimal promised lead times are $L + l + 1$, and the manufacturer *makes to order* for both retailers. When the ratio of α_r/α_m lies between $\underline{M}$ and $\overline{M}$, the manufacturer's cost of uncertainty is close to the retailers', and the retailers have different promised lead times. Whether retailer 1 or 2 gets 0 promised lead time depends on the magnitude of supply chain cost reduction by shifting the inventory from either retailer to the manufacturer. For example, when α_r/α_m is smaller than M, retailer 2 is an expensive location to store inventory. It is optimal for the manufacturer to hold inventory for retailer 2 under 0 promised lead time. When the ratio of α_r/α_m is higher than $\overline{M}$, the manufacturer's cost of uncertainty is sufficiently lower than the retailers'. It is cheaper to store inventory at the manufacturer end. The optimal promised lead times are 0, and the manufacturer *makes to stock* for both retailers.

Corollary 3. *When $c_1 = c_2$ and $\frac{\sigma_{1j}}{\sigma_{2j}} = c$ for all $j \in \{1, ..., J\}$, $\underline{M} = \overline{M} = \frac{x}{a+b}$ and*

$$\left(s_1^C(L,l), s_2^C(L,l)\right) = \begin{cases} (L+l+1, L+l+1), & \text{if } \frac{\alpha_r}{\alpha m} \in (0, \frac{x}{a+b}] \\ (0,0), & \text{if } \frac{\alpha_r}{\alpha m} \in (\frac{x}{a+b}, +\infty) \end{cases}.$$

Note that $c_1 = c_2$ implies both retailers have the same holding and penalty costs. $\frac{\sigma_{1j}}{\sigma_{2j}} = c$ implies that the ratio between the standard deviations of two retailers are identical across J products. Corollary 3 gives a condition under which different promised lead times are never optimal. We remark that Proposition 12 and Corollary 3 also generalize the result from the single-product supply chain in Barnes-Schuster et al. (2006) to a multi-product supply chain.

3.3.2 Decentralized Supply Chain

The previous section establishes the optimal promised lead-times for a centrally controlled supply chain. Next we focus on a decentralized supply chain in which the manufacturer's goal is to minimize her inventory cost by designing promised lead time contracts for the retailers. Proposition 8 and 10 imply that the manufacturer prefers a longer promised lead time while retailers prefer shorter promised lead times. The concavity properties of the cost functions suggest that the manufacturer may need to compensate/charge retailers with a non-linear pricing scheme, so that the retailers are willing to accept different promised lead times.

A promised lead time contract has two parameters: promised lead time s_k and corresponding per-period payment π_k. The analysis of inventory costs under a certain s_k determines π_k. When π_k is positive, we interpret the monetary transaction as a payment from the manufacturer to retailer k. When designing promised lead time contracts, the manufacturer should take two important factors into consideration. First, each retailer k's inventory cost under the contract should not exceed the market protection level, that is, the maximum acceptable inventory cost U_k. Market protection levels prevent the manufacturer from over-exploiting the retailers and create sufficiently profitable margins for them. Only if retailer k's total expected cost under promised lead time s_k plus the corresponding payment π_k is below U_k, he is willing to accept the manufacturer's contract. Second, the manufacturer should also consider whether the retailers are in different markets (the manufacturer may offer different contract terms) or in the same market (the manufacturer needs to offer similar terms and let the retailers self select). In the following two subsections, we solve for the optimal lead times for both market settings.

3.3.3 Different Market Setting

When the manufacturer sells to retailers in separate (e.g., geographically dispersed) markets, retailer k only observes his own promised lead time contract. The manufacturer can design one contract for each retailer individually, while making sure each retailer accepts the contract. The U_1 and U_2 may be different due to the different market setting. The manufacturer's inventory minimization problem is as follows:

$$
\begin{aligned}
\underset{(s_1,\pi_1),(s_2,\pi_2)}{\text{minimize}} \quad & G_m(L,l,s_1,s_2)+\pi_1+\pi_2 \\
\text{subject to} \quad & G_1(s_1)-\pi_1 \le U_1 \\
& G_2(s_2)-\pi_2 \le U_2 \\
& s_1,s_2 \in \{0,...,L+l+1\}
\end{aligned}
\tag{3.4}
$$

We denote the optimal solutions to (3.4) as $s_1^D(L,l)$, $s_2^D(L,l)$, $\pi_1^D(L,l)$, and $\pi_2^D(L,l)$. The manufacturer's objective function requires her to consider the total cost in both markets while offering separate contracts in each. Even when the retailers operate in different markets, the manufacturer's problem above is not separable; the contract terms that the manufacturer offers in one market depend on the contract terms offered in the other market. This occurs because the manufacturer's two-stage production process makes multiple customized end products, each potentially serving to replenish inventory in both markets.

Proposition 13. $s_k^D(L,l)=s_k^C(L,l)$ *and* $\pi_k^D(L,l)=G_k\left(s_k^D(L,l)\right)-U_k$ *for* $k\in\{1,2\}$.

When the multi-product manufacturer faces two retailers in different markets, she optimally offers the promised lead times that minimize not only her own expected cost, but also the total supply chain expected inventory cost. In other words, the promised lead time contracts coordinate the supply chain when retailers operate in different markets. Also notice that the manufacturer may offer different terms to retailers from Proposition 12 and 13. Retailers always incur their maximum acceptable inventory costs regardless of the promised lead times.

3.3.4 Same Market Setting

When the retailers are in the same markets, the manufacturer may not be able to offer different terms to different retailers in the same market, since offering retailer 1 a shorter promised lead time than that of retailer 2 may appear to be preferential treatment. Since passing the Sherman Antitrust Act in 1890, the United States Congress has repeatedly demonstrated the interest of the federal government in protecting normal marketplace competition in interstate commerce. The Federal Trade Commission enforces fair and nondiscriminatory business practices according to statutes such as the Sherman Act (1890), Clayton Act (1914), and Robinson-Patman Act (1936). Yet even if such service discrimination were legal, retailers may demand fair treatment, an equal service policy. The manufacturer can avoid this complication and maximize her profit by offering both retailers a menu of (s_1, π_1) and (s_2, π_2) and allowing them to pick the contract of their choice. The retailers then segment themselves according to their self-selections. The manufacturer's optimization problem is as follows:

$$\begin{array}{lll} \underset{(s_1,\pi_1),(s_2,\pi_2)}{\text{minimize}} & G_m(L, l, s_1, s_2) + \pi_1 + \pi_2 & \\ \text{subject to} & G_k(s_k) - \pi_k \leq U & \text{for } k \in \{1,2\} \\ & G_k(s_k) - \pi_k \leq G_k(s_q) - \pi_q & \text{for } k, q \in \{1,2\} \text{ and } k \neq q \\ & s_k \in \{0, \ldots, L+l+1\} & \text{for } k \in \{1,2\} \end{array} \tag{3.5}$$

Problem (3.5) is a nonlinear program over s_1, s_2, π_1, and π_2. We denote its optimal solutions as $s_1^S(L,l)$, $s_2^S(L,l)$, $\pi_1^S(L,l)$, and $\pi_2^S(L,l)$. The first set of participation constraints in problem (3.5) ensure that the total expected cost $G_k(s_k) - \pi_k$ for each retailer k will not exceed his maximum acceptable inventory cost U. Due to the same market setting, we assume without loss of generality (and to keep notation and discussion simple) the market protection levels are the same for both retailers, i.e., $U_1 = U_2 = U$. The second set of self-selection constraints ensure that each retailer k prefers the promised lead time contract the manufacturer designs for him over the other.

Proposition 14.

$$\begin{aligned}
\left(s_1^S(L,l), s_2^S(L,l)\right) &= \begin{cases} (L+l+1, L+l+1) & if\ \frac{\alpha_r}{\alpha m} \in (0, \underline{N}] \\ (0, L+l+1) & if\ \frac{\alpha_r}{\alpha m} \in (\underline{N}, \overline{N}] \\ (0,0) & if\ \frac{\alpha_r}{\alpha m} \in (\overline{N}, +\infty) \end{cases}, \\
\pi_1^S(L,l) &= G_1(s_1^S(L,l)) - U, \\
\pi_2^S(L,l) &= [G_1(s_1^S(L,l)) - G_2(s_1^S(L,l))] + G_2(s_2^D(L,l)) - U,
\end{aligned}$$

where $\underline{N} := \min\{\frac{x}{2\alpha}, \frac{y}{2\alpha-b}\}$, $\overline{N} := \max\{\frac{x}{2\alpha}, \frac{x-y}{b}\}$, *and* $\pi_1^D(L,l) \leq \pi_2^D(L,l)$.

According to Proposition 14, the optimal promised lead times fall into the set of $\{(L+l+1, L+l+1), (0, L+l+1), (0,0)\}$. When the ratio of α_r/α_m is lower than $\underline{N}$ (resp., higher than $\overline{N}$), the manufacturer's cost of uncertainty is sufficiently higher (resp., lower) than the retailers'. It is cheaper to store inventory at the retailers' (resp., manufacturer's) end. The optimal promised lead times are $L+l+1$ (resp., 0), and the manufacturer *makes to order* (resp., *makes to stock*) for both retailers. Recall that in both the centralized setting and the different market setting, either retailer may get 0 promised lead time when the manufacturer's cost of uncertainty is close to the retailers' (i.e., $\underline{M} \leq \alpha_r/\alpha_m \leq \overline{M}$). In contrast, when the ratio of α_r/α_m lies between $\underline{N}$ and $\overline{N}$ in the same market setting, the manufacturer designs a shorter promised lead time for retailer 1 who has a higher inventory cost, and compensate him less or charge him more for his acceptance of the shorter lead time. The difference is due to the self-selection constraints in problem (3.5). The manufacturer knows that retailer 1, compared to retailer 2, has a higher inventory cost and is more sensitive to an increase in promised lead time. To induce retailer 1 to select the contract intended for him, the manufacturer need to design retailer 1's contract with a shorter lead time. By charging a higher premium for the shorter lead time, the manufacturer also prevents retailer 2 from selecting retailer 1's contract.

3.4 Postponement Effect

A postponement strategy corresponds to increasing the production lead time for the common intermediate product L, while keeping the total production time $L+l$ fixed. By increasing L for a fixed $L+l$, the manufacturer can delay the point of product differentiation and thereby be more responsive to retailers' orders. We explore the properties of $\psi_m(L, l, s_1, s_2)$ that are necessary in determining the impact of postponement on the optimal lead time contract.

Proposition 15. *The manufacturer's effective standard deviation of production lead time demand $\psi_m(L, l, s_1, s_2)$ is concave decreasing in postponement.*

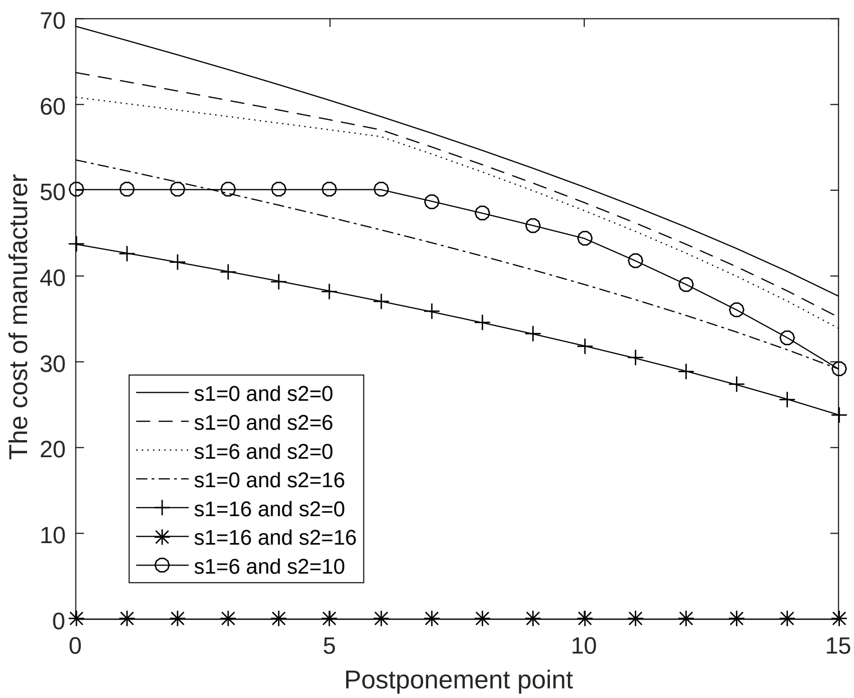

Figure 3.3. Impact of postponement $p_m = 3,\ h_m = 2,\ L+l = 15,\ J = 4,\ \sigma_{1j}^2 = 3,\ \sigma_{j2}^2 = 2$

Postponement affects the manufacturer's effective standard deviation and therefore affects both the optimal base stock level and the resulting expected per-period inventory cost. An example is given in Figure 3.3 to illustrate Proposition 15. We fix $L+l = 15$ and let L, or the postponement point, change from 0 to 15. If we also fix s_1 and s_2, the manufacturer's inventory cost G_m is a function of postponement. In Figure 3.3, G_m is concave decreasing in postponement. When $\min\{s_1, s_2\} > L$, the manufacturer knows all necessary demand information before allocating intermediate product for customization. Further delaying the production differentiation point does not bring in any extra benefit. Thus G_m is unaffected by the postponement.

So far, we have extended existing literature about the impact of postponement on inventory cost by incorporating promised lead times. Now we want to take supply chain contract design into consideration. From Proposition 12, 13 and 14, optimal lead times in three different market settings are specified by critical thresholds $\underline{M}$, $\overline{M}$, $\underline{N}$ and $\overline{N}$. The changing tendency of these thresholds with respect to postponement reveals the impact of postponement on the promised lead time contracts.

Proposition 16. a) *$\underline{M}$ and $\underline{N}$ are always decreasing in postponement,*

b) *$\overline{M}$ is decreasing in postponement if*

$$\frac{\sum_{j=1}^{J} \sigma_{1j}^2}{\left[\sum_{j=1}^{J} \sigma_{1j}\right]^2} \geqslant \frac{\sum_{j=1}^{J} (\sigma_{1j}^2 + \sigma_{2j}^2)}{\left[\sum_{j=1}^{J} \sqrt{\sigma_{1j}^2 + \sigma_{2j}^2}\right]^2} \text{ and } \frac{\sum_{j=1}^{J} \sigma_{2j}^2}{\left[\sum_{j=1}^{J} \sigma_{2j}\right]^2} \geqslant \frac{\sum_{j=1}^{J} (\sigma_{1j}^2 + \sigma_{2j}^2)}{\left[\sum_{j=1}^{J} \sqrt{\sigma_{1j}^2 + \sigma_{2j}^2}\right]^2},$$

c) *$\overline{N}$ is decreasing in postponement if*

$$\frac{\sum_{j=1}^{J} \sigma_{1j}^2}{\left[\sum_{j=1}^{J} \sigma_{1j}\right]^2} \geqslant \frac{\sum_{j=1}^{J} (\sigma_{1j}^2 + \sigma_{2j}^2)}{\left[\sum_{j=1}^{J} \sqrt{\sigma_{1j}^2 + \sigma_{2j}^2}\right]^2}.$$

Proposition 16 illustrates the condition under which $\underline{M}$, $\overline{M}$, $\underline{N}$ and $\overline{N}$ are decreasing in postponement. When the promised lead times are long (resp., short), the manufacturer adopts a make-to-order (resp., make-to-stock) production mode. A postponement strategy increases the likelihood that the manufacturer should optimally offer short promised lead times and make to stock. In other words, a postponement strategy enables the supply chain to carry more inventory at the manufacturer end and be more responsive to the retailers. Proposition 16 also shows that considering only the manufacturer's inventory cost while overlooking the impact on promised lead time contracts under estimates the benefits of postponement strategy.

Corollary 4. *When $J = 2$ or $\frac{\sigma_{1j}}{\sigma_{2j}} = c$ for all $j \in \{1, ..., J\}$, $\underline{M}$, $\overline{M}$, $\underline{N}$ and $\overline{N}$ are always decreasing in postponement.*

If there are only two end products or the ratio between the standard deviations of retailers' demands are identical across J products, postponement always leads to shorter promised lead times.

3.5 Quantifying the Value of Postponement

To illustrate the impact of postponement on the optimal promised lead time contracts and supply chain members' inventory costs, we present a simple example in which the manufacturer produces two products in $L + l = 4$ periods. We set $p_m = p_r = 3$ and $h_m = h_r = 2$ for both the manufacturer and retailers. Retailer 1 has cost parameter $c_1 = 0.8$ and demand variances $\sigma^2_{1,1} = \sigma^2_{1,2} = 7$, while retailer 2 has cost parameter $c_2 = 1.5$ and demand variances $\sigma^2_{2,1} = \sigma^2_{2,2} = 1.5$. Compared to retailer 1, retailer 2 suffers less demand variation but more holding and backlogging costs. The market protection levels are $U_1 = U_2 = 10$ for both the retailers in both the different market setting and the same market setting.

Table 3.1. Postponement Impact in Centrailized Setting

Postponement		Total Cost	M's Cost	R's Costs		Optimal Lead Times	
L	l	$G_m + G_1 + G_2$	G_m	G_1	G_2	s_1^C	s_2^C
0	4	37.42	0	20.03	17.39	5	5
1	3	37.17	10.04	20.03	7.10	5	0
2	2	36.59	9.46	20.03	7.10	5	0
3	1	35.98	8.85	20.03	7.10	5	0
4	0	34.78	19.51	8.18	7.10	0	0

In Table 3.1, we display the optimal promised lead time contract as a function of postponement (L, l). From the first row, we can see that the optimal lead times are long, since postponement effect is not strong enough to have an impact. It is optimal to store inventory at the retailers' end. Retailer 2 has much larger cost parameters than the manufacturer does. When the manufacturer redesigns its production process and delay production differentiation to a later point in the manufacturing process (i.e., for large L), storing inventory at the manufacturer becomes less costly than at retailer 2. Thus, the manufacturer, in the 2nd, 3rd and 4th rows, begins offering instantaneous delivery for retailer 2. As postponement goes even further, manufacturer's cost gets low enough to store inventory for the entire supply chain. The promised lead times become zero in row 5. It is also easy to notice that the supply chain total cost is decreasing with postponement. In Table 3.2, the total costs and optimal lead times are the same as those in Table 3.1. The market protection level balance the total cost among the manufacturer and the retailers. Both retailers always suffer their largest cost, which is 10. In Table 3.3, the first interesting finding is that the total cost is not decreasing in postponement. Compared to the different market case, the same market setting has a higher manufacturer's cost and lower retailers' costs and different promised lead time contracts.

Table 3.2. Postponement Impact in Different Market Setting

Postponement		Total Cost	M's Cost	R's Cost		Optimal Lead Times		Optimal Payments	
L	l	$G_m + G_1 + G_2$	G_m	G_1	G_2	s_1^D	s_2^D	π_1^D	π_2^D
0	4	37.42	17.42	10	10	5	5	10.03	7.39
1	3	37.17	17.17	10	10	5	0	10.03	-2.90
2	2	36.59	16.59	10	10	5	0	10.03	-2.90
3	1	35.98	15.98	10	10	5	0	10.03	-2.90
4	0	34.78	14.78	10	10	0	0	-1.82	-2.90

Table 3.3. Postponement Impact in Same Market Setting

Postponement		Total Cost	M's Cost	R's Cost		Optimal Lead Times		Optimal Payments	
L	l	$G_m + G_1 + G_2$	G_m	G_1	G_2	s_1^S	s_2^S	π_1^S	π_2^S
0	4	37.42	20.06	10	7.36	5	5	10.03	10.03
1	3	37.42	20.06	10	7.36	5	5	10.03	10.03
2	2	37.80	18.88	10	8.92	0	0	-1.82	-1.82
3	1	36.34	17.43	10	8.92	0	0	-1.82	-1.82
4	0	34.78	15.86	10	8.92	0	0	-1.82	-1.82

3.6 Conclusion

In Chapter 3, we extend existing supply chain inventory cost models to a multi-product, multi-retailer model with advance orders and delayed production differentiation. Additionally, we allow retailers to differ in the products and service levels they provide to end consumers. We discover the significant influence that postponement has on optimal promised lead time contracts between a manufacturer and two retailers.

Due to supply chain coordination issues between the multi-product manufacturer and the retailers, the manufacturer may need to compensate retailers with a non-linear pricing scheme to accept different promised lead times. The ensuing portfolio of promised lead time agreements is influenced by the extent to which the manufacturer postpones customization. In a centrally controlled system, the manufacturer optimally offers either make-to-stock or make-to-order service to each retailer.

When system control is not centralized, even when the retailers are geographically dispersed or in different markets, the manufacturer's problem of determining optimal promised lead times is not separable. That is, the optimal choice of promised lead times is not same as the optimal promised lead time for a single retailer. Hence, our results provide a valuable guide to determining the optimal promised lead time agreement. Retailers incur their maximum acceptable inventory cost, regardless of promised lead times or postponement.

When retailers are in the same market, the manufacturer provides a shorter promised lead time for the retailer with a higher expected inventory cost, but the retailer pays more for it. We examine the behavior of the manufacturer's non-linear pricing scheme with respect to promised lead times. For the retailer having lower inventory costs, corresponding payment is concave increasing in the promised lead times for both retailers. For the retailer having higher inventory costs, the result differs. The corresponding payments to such a retailer is concave increasing in his own promised lead time but not affected by the low cost retailer's promised lead time.

Our main contribution is the analysis of the impact of postponement on promised lead time contracts. Postponement increases the likelihood that the manufacturer should optimally make to stock (immediate delivery), and the retailers optimally do not carry inventory. Our numerical example illustrate the potential requirement of supply chain contract reoptimization after postponement being adopted on the manufacturer end.

CHAPTER 4

CONCLUSION

This dissertation explores how firms price dynamic upgrades to improve revenue and how supply chain members share inventory risk through promised lead time pricing contracts.

Upgrading is a travel industry practice used to mitigate supply-demand mismatches among products of different quality levels. Such upgrades are usually implemented either at the booking time or at the check-in time. In Chapter 2, we consider dynamically-offered upgrades between the booking and the check-in times by a firm that sells two types of products (premium and regular). The firm decides on the timing and quantity of upgrades. Customers who purchased the regular product may be offered upgrades via notifications containing a link to an upgrade website. A regular product purchaser either accepts or rejects the upgrade offer after clicking the link and observing the upgrade fee (price) dynamically determined by the firm. The upgrade is time limited. When the upgrade process is not profitable, the firm can stop it by deactivating the upgrade links. Formulating the firm's revenue maximization problem as a dynamic program, we show that the optimal upgrade policy is of a pulsing type. The firm either maintains zero or the maximum number of active links. Both the optimal number of active links and the optimal upgrade fee are monotone with respect to the leftover capacities. We then propose and analyze two model variations: one with a restricted upgrade fee choice set and one with upward stockout substitution, in which the firm can sell a premium product to an arriving regular customer at a discount if the regular product stocks out. Finally, through a systematic numerical study, we quantify the revenue improvement from industry-standard check-in fixed-price upgrades to dynamic pricing and timing of upgrades. We also identify the market environment, in which the revenue improvement is significant across various models.

Postponement, or delayed product differentiation, is a common strategy for mass customization. It directly affects the inventory policy and cost of a multi-product manufacturer, and may also have an impact on the interactions among supply chain members. In Chapter 3, we focus on a two-stage supply chain consisting of a multi-product manufacturer and two retailers. We specifically study the impact of postponement on promised lead time contracts, under which the manufacturer guarantees on-time shipments of complete orders to the retailers within the promised lead times. The optimal contracts designed by the manufacturer depend on whether the retailers are in different markets (both retailers cannot observe each other's contract terms, and the manufacturer may discriminate between them) or in the same market (contract terms are public information, and both retailers self-select their contracts from a menu designed by the manufacturer). We characterize the optimal promised lead times in both settings and compare them to those in a base setting where the supply chain is under a centralized control. In all three settings, the optimal promised lead time for each retailer is equal to either the total length of the manufacturer's production time (the longest) or zero (the shortest). In contrast to the centralized and different market settings, the retailer with a lower inventory cost always gets a shorter promised lead time in the same market setting. We then analyze the impact of postponement on the optimal promised lead times in all three settings and characterize the conditions under which the manufacturer shifts its production mode from make-to-order (when the promised lead times are the longest) to make-to-stock (when the promised lead times are the shortest). Finally, through numerical examples, we quantify the impact of postponement on the promised lead time contracts and the inventory costs of supply chain members.

APPENDIX A

SUPPLEMENTAL MATERIALS FOR CHAPTER 2

A.1 Notations and Proofs

Table A.1. Notations for Chapter 2

Firm	
H :	premium product initial capacity
L :	regular product initial capacity
p^h :	premium product price
p^l :	regular product price
M :	upgrade triggering level
C :	maximum upgrade capability (i.e., maximum number of active links the firm can maintain)
h :	premium leftover capacity
l :	regular leftover capacity
u :	number of active upgrade links
p :	upgrade fee
n :	time period
N :	total number of time periods in the sales season
$V_n(h,l)$:	expected optimal revenue function over periods $[n : N+1]$
$\Delta_n(h,l)$:	upgrade opportunity value (i.e., diagonal difference of expected optimal revenue function)
$\delta_n(p,h,l)$:	expected upgrade revenue from an upgradeable customer at upgrade fee p
$u_n^*(h,l)$:	the optimal number of active upgrade links
$p_n^*(h,l)$:	the optimal upgrade fee
$\delta_n^*(h,l)$:	the optimal expected upgrade revenue from an upgradeable customer
$V_n^r(h,l)$:	expected optimal revenue function of the restricted upgrade fee model
$\Delta_n^r(h,l)$:	upgrade opportunity value of the restricted upgrade fee model
u^r :	number of active upgrade links in the restricted upgrade fee model
$u_n^{r,*}(h,l)$:	the optimal number of active upgrade links in the restricted upgrade fee model
$V_n^s(h,l)$:	expected optimal revenue function of the upward substitution model
$\Delta_n^s(h,l)$:	upgrade opportunity value of the upward substitution model
f^s :	upward substitution fee
$f_n^{s,*}(h)$:	the optimal upward substitution fee
p^s :	upgrade fee in the upward substitution model
$p_n^{s,*}(h,l)$:	the optimal upgrade fee in the upward substitution model
u^s :	number of active upgrade links in the upward substitution model
Customer	
λ^h :	premium customer arrival probability
λ^l :	regular customer arrival probability
λ :	upgradeable customer arrival probability when there is only one active link
$\alpha(p)$:	upgrade accepting probability at upgrade fee p
$\alpha^s(f^s)$:	upward substitution accepting probability at substitution fee f^s

We define the revenue function $\pi(p,\Delta) = \alpha(p)(p+\Delta)$ for a sale between a seller and a buyer. $\Delta > 0$ can be interpreted as a third-party subsidy paid to the seller after each sale, whereas $\Delta \le 0$ is a cost suffered by the seller. $\alpha(p)$ defined on $[a,b]$ is the probability of a sale when the price is p, which can be derived from the tail probability of the reservation price distribution. Let the maximum revenue be $\Pi(\Delta) = \max_{p\in[a,b]} \alpha(p)(p+\Delta)$. The set of optimal prices is $\{p \in [a,b] : \pi(p,\Delta) = \Pi(\Delta)\}$, which is assumed to be closed. This assumption is satisfied by distributions with an increasing failure rate, which makes the set of optimal prices a singleton. Distributions with a non-increasing failure rate, such as beta distribution with shape parameters as 1/2, also satisfy this assumption. The maximal optimal price is defined as $p^*(\Delta) = \max\{p \in [a,b] : \pi(p,\Delta) = \Pi(\Delta)\}$.

Lemma 1. *For $\Delta_1 \geq \Delta_2$, we have $\Pi(\Delta_1) \geq \Pi(\Delta_2)$ and $p^*(\Delta_1) \leq p^*(\Delta_2)$.*

Proof of Lemma 1: The revenue function inequality follows from $\Pi(\Delta_2) = \alpha(p^*(\Delta_2))(\Delta_2 + p^*(\Delta_2)) \leq \alpha(p^*(\Delta_2))(p^*(\Delta_2)+\Delta_1) \leq \alpha(p^*(\Delta_1))(p^*(\Delta_1)+\Delta_1) = \Pi(\Delta_1)$. A sufficient condition for the optimal price inequality is $\pi(p,\Delta_2) \leq \pi(p^*(\Delta_1),\Delta_2)$ for $p \leq p^*(\Delta_1)$. Since $\pi(p^*(\Delta_1),\Delta_1) - \pi(p,\Delta_1) \geq 0$, the sufficient condition holds if $\pi(p^*(\Delta_1),\Delta_2) - \pi(p,\Delta_2) \geq \pi(p^*(\Delta_1),\Delta_1) - \pi(p,\Delta_1)$ for $p \leq p^*(\Delta_1)$. The last inequality reduces to $\alpha(p)(\Delta_1 - \Delta_2) \geq \alpha(p^*(\Delta_1))(\Delta_1 - \Delta_2)$, which is true because $\Delta_1 \geq \Delta_2$ and $\alpha(p) \geq \alpha(p^*(\Delta_1))$ for $p \leq p^*(\Delta_1)$. $\square$

From Lemma 1, the optimal revenue increases while the optimal price decreases in Δ. In classic dynamic pricing literature, Δ usually represents the opportunity cost of one unit of product, which is always nonpositive. In Chapter 2, Δ can be either positive, zero or negative. It represents the upgrade opportunity value $\Delta_{n+1}(h,l) = V_{n+1}(h-1,l+1) - V_{n+1}(h,l)$.

Proof of Proposition 1: For $(h,l) \in [1:H] \times [0:M]$, the optimal expected revenue is:

$$
\begin{aligned}
V_n(h,l) &= (1-\lambda^h-\lambda^l)V_{n+1}(h,l) + \lambda^h[p^h + V_{n+1}(h-1,l)] + \lambda^l[p^l \mathbb{1}_{l\geq 1} + V_{n+1}(h, l-\mathbb{1}_{l\geq 1})] \\
&\quad + \max_{u\in[0:C]} \left\{ u\lambda \max_{p\in[0,p^h-p^l]} \alpha(p)[p + \Delta_{n+1}(h,l)] \right\} \\
&= (1-\lambda^h-\lambda^l)V_{n+1}(h,l) + \lambda^h[p^h + V_{n+1}(h-1,l)] + \lambda^l[p^l \mathbb{1}_{l\geq 1} + V_{n+1}(h, l-\mathbb{1}_{l\geq 1})] \\
&\quad + \max_{u\in[0:C]} \left\{ u\lambda\delta_n^*(h,l) \right\}.
\end{aligned}
$$

Notice that $V_n(h,l)$ is linear in u, whose coefficient is $\lambda\delta_n^*(h,l)$. If $\delta_n^*(h,l) > 0$, $u_n^*(h,l) = C$. If $\delta_n^*(h,l) \le 0$, $u_n^*(h,l) = 0$. Thus, the optimal number of upgrade links in period n at state (h,l) is: $u_n^*(h,l) = \mathbb{1}_{\delta_n^*(h,l)>0}C$. $\square$

To better understand the optimal dynamic upgrade policy, we study the properties of the value function $V_n(h,l)$. The following Lemma 2 is part of property d) and e) in Proposition 2 and can be proved separately. It implies that the expected value of one unit of product is never greater than its price.

Lemma 2. *$V_n(1,0) - V_n(0,0) \le p^h$ and $V_n(0,1) - V_n(0,0) \le p^l$ for $n \in [1 : N+1]$.*

Proof of Lemma 2: For brevity, we use $p_{h,l}^*$ to represent $p_n^*(h,l)$ during this proof. Because the optimal upgrade fee is picked from $[0, p^h - p^l]$, we have $0 \le p_{h,l}^* \le p^h - p^l$. Since $V_n(0,0) = 0$, Lemma 2 is equivalent to $V_n(0,1) \le p^l$ and $V_n(1,0) \le p^h$. The proof is by induction. In period $N+1$, $V_{N+1}(0,1) = V_{N+1}(1,0) = 0 \le \min\{p^l, p^h\}$. Assuming $V_{n+1}(0,1) \le p^l$ and $V_{n+1}(1,0) \le p^h$, we want to validate $V_n(0,1) \le p^l$ and $V_n(1,0) \le p^h$. We have $V_n(0,1) \le p^l$, since $V_n(0,1) = (1-\lambda^l)V_{n+1}(0,1) + \lambda^l p^l$. For the inequality of $V_n(1,0)$, we use Equation (2.5). When $\delta_n^*(1,0) \le 0$, $V_n(1,0) = (1-\lambda^h)V_{n+1}(1,0) + \lambda^h p^h \le p^h$. When $\delta_n^*(1,0) > 0$,

$$\begin{aligned}
V_n(1,0) &= [1-\lambda^h - C\lambda]V_{n+1}(1,0) + \lambda^h p^h + C\lambda\alpha(p_{1,0}^*)[p_{1,0}^* + V_{n+1}(0,1)] \\
&\quad + C\lambda[1-\alpha(p_{1,0}^*)]V_{n+1}(1,0) \\
&\le [1-\lambda^h - C\lambda]p^h + \lambda^h p^h + C\lambda\alpha(p_{1,0}^*)(p_{1,0}^* + p^l) + C\lambda[1-\alpha(p_{1,0}^*)]p^h \\
&\le [1-\lambda^h - C\lambda]p^h + \lambda^h p^h + C\lambda\alpha(p_{1,0}^*)p^h + C\lambda[1-\alpha(p_{1,0}^*)]p^h \\
&= p^h.
\end{aligned}$$

So $V_n(0,1) \le p^l$ and $V_n(1,0) \le p^h$, which complete the induction step. $\square$

We introduce three new notations to shorten our following proofs:

$$\begin{aligned}
\Psi[V_{n+1}(h,l)] &= (1-\lambda^h-\lambda^l)V_{n+1}(h,l)+\lambda^h[p^h\mathbb{1}_{h\geq 1}+V_{n+1}(h-\mathbb{1}_{h\geq 1},l)] \\
&\quad +\lambda^l[p^l\mathbb{1}_{l\geq 1}+V_{n+1}(h,l-\mathbb{1}_{l\geq 1})], \\
\Delta_n^h(h,l) &= V_n(h,l)-V_n(h-1,l), \\
\Delta_n^l(h,l) &= V_n(h,l)-V_n(h,l-1).
\end{aligned}$$

Functional Ψ in $\Psi[V_{n+1}(h,l)]$ is a short-hand notation to capture the expected revenue at state (h,l) over periods $[n:N+1]$ if upgrades are not offered in period n. Thus, the expected optimal revenue at $(h,l)\in[1:H]\times[0:M]$ can be written as

$$V_n(h,l) = \Psi[V_{n+1}(h,l)]+\max_{u\in[0:C] \text{ and } p\in[0,p^h-p^l]} u\lambda\alpha(p)[p+\Delta_{n+1}(h,l)].$$

The horizontal difference $\Delta_n^h(h,l)$ is defined to represent the marginal value of the hth unit of premium product when the leftover capacities are (h,l) in period n. The vertical difference $\Delta_n^l(h,l)$ is defined to represent the marginal value of the lth unit of regular product when the leftover capacities are (h,l) in period n.

Before the proof of Proposition 2, we provide two counterexamples in Figure A.1 to show that neither DH-modularity nor DV-modularity follows from the other four properties. Similar to these and the counterexample in the main body of Chapter 2 showing that submodularity does not follow from the other four properties, we can also construct counterexamples showing that neither H-concavity nor V-concavity follows from the other four properties. Thus, each property needs to be individually proved.

Proof of Proposition 2: We prove all five properties by induction. They are true in period $N+1$, since $V_{N+1}(h,l)=0$. As the induction hypothesis, we assume all five properties are true in period $n+1$, and validate them one by one in period n. DP formulations are different on the corner point $(0,0)$, two boundaries $(0,l)$ for $l>0$ and $(h,0)$ for $h>0$, and

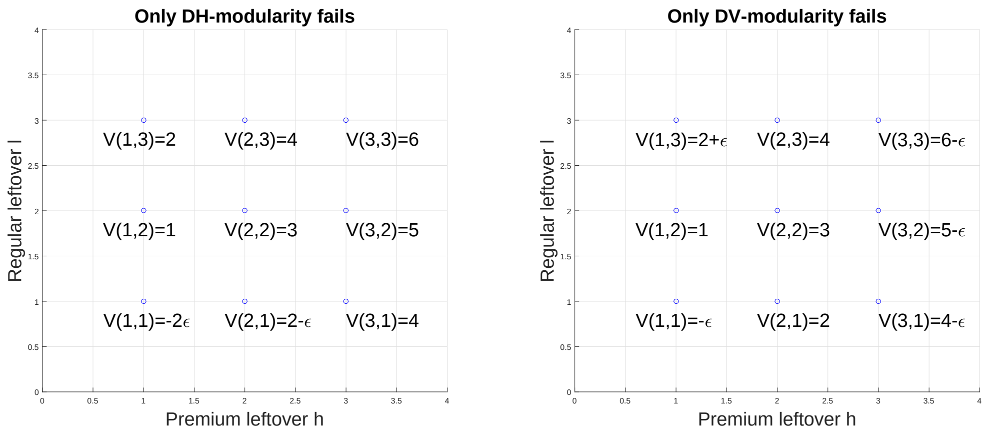

Figure A.1. DH-modularity and DV-modularity counterexamples

in the interior region (h,l) for $h,l>0$. The proof of each property consists of four parts corresponding to these four regions. For brevity, we use $p^*_{h,l}$ to represent $p^*_n(h,l)$.

When properties DH- and DV-modularity are true in period $n+1$, $\delta_n(p,h,l) = \alpha(p)[p + \Delta_{n+1}(h,l)]$, $\delta^*_n(h,l) = \max_{p\in[0,p^h-p^l]} \delta_n(p,h,l)$ and Lemma 1 imply

$$\delta^*_n(h,l+1) \le \delta^*_n(h,l) \le \delta^*_n(h+1,l) \text{ and } p^*_{h,l+1} \ge p^*_{h,l} \ge p^*_{h+1,l}. \tag{A.1}$$

Inequalities in (A.1) are used repeatedly to determine if upgrades should be offered at a certain state (h,l) in period n. Because of inequalities in (A.1), the proof for each property in each region contains multiple cases listed in Table A.2. DH-, DV-, sub-modularity are two-dimensional properties, whose proofs are similar. H- and V-concavities are one-dimensional properties, whose proofs are similar. We introduce the following three notations to shorten our proofs for the interior region:

$$\begin{aligned}
\Psi[V_{n+1}(h,l)] &= (1-\lambda^h-\lambda^l)V_{n+1}(h,l) + \lambda^h[p^h \mathbb{1}_{h\ge1} + V_{n+1}(h-\mathbb{1}_{h\ge1},l)] \\
&\quad + \lambda^l[p^l \mathbb{1}_{l\ge1} + V_{n+1}(h,l-\mathbb{1}_{l\ge1})], \\
\Delta^h_n(h,l) &= V_n(h,l) - V_n(h-1,l), \\
\Delta^l_n(h,l) &= V_n(h,l) - V_n(h,l-1).
\end{aligned}$$

Table A.2. Number of cases and the required properties in period $n+1$ for each property and region pair

	Regions			
Property in period n	Interior	Horizontal boundary	Vertical boundary	Corner
a) Submodularity	6 cases	6 cases	3 cases	3 cases
Proof of a) requires	a), b), c), d) ,e)	a), b), c), d), e)	a), b), c)	a), b), c)
b) DH-modularity	5 cases	5 cases	4 cases	4 cases
Proof of b) requires	b), c)	a), b), c)	a), b), c), Lemma 2	a), b), c), Lemma 2
c) DV-modularity	5 cases	5 cases	3 cases	3 cases
Proof of c) requires	b), c)	a), b), c), Lemma 2	a), b), c)	a), b), c), Lemma 2
d) H-concavity	4 cases	4 cases	3 cases	3 cases
Proof of d) requires	a), b), c), d)	a), b), c), d)	a), b), c), d), Lemma 2	a), b), c), d), Lemma 2
e) V-concavity	4 cases	4 cases	1 case	1 case
Proof of e) requires	a), b), c), e)	a), b), c), e), Lemma 2	e)	e), Lemma 2

Proof of property a): $[0:H-1]\times[0:M]$ is partitioned into the interior $[1:H-1]\times[1:M]$, the horizontal boundary $[1:H-1]\times\{0\}$, the vertical boundary $\{0\}\times[1:M]$ and the corner $(0,0)$. The submodularity property at state $(h,l)\in[0:H-1]\times[0:M]$ in period n can be equivalently expressed as

$$\begin{aligned}
V_n(h,l+1)-V_n(h,l) &\geq V_n(h+1,l+1)-V_n(h+1,l),\\
\Delta_n^l(h,l+1) &\geq \Delta_n^l(h+1,l+1),\\
V_n(h+1,l)-V_n(h,l) &\geq V_n(h+1,l+1)-V_n(h,l+1),\\
\Delta_n^h(h+1,l) &\geq \Delta_n^h(h+1,l+1).
\end{aligned}$$

For most cases, we obtain the last two inequality. The inequalities involve states $(h,l+1)$, (h,l), $(h+1,l+1)$ and $(h+1,l)$. Specializing Inequalities (A.1) for these states, we obtain $\delta_n^*(h,l+1)\leq\delta_n^*(h,l)\leq\delta_n^*(h+1,l)$ and $\delta_n^*(h,l+1)\leq\delta_n^*(h+1,l+1)\leq\delta_n^*(h+1,l)$. Notice that not all of δ^* can be ordered. In particular, $\delta_n^*(h,l)$ can be either larger or smaller than $\delta_n^*(h+1,l+1)$.

Interior $[1:H-1]\times[1:M]$. There are 6 possible cases.

Case 1: $\delta_n^*(h, l+1) \leq \{\delta_n^*(h,l), \delta_n^*(h+1,l+1)\} \leq \delta_n^*(h+1,l) \leq 0$.

$$\begin{aligned}
& V_n(h+1,l) \\
= \quad & (1-\lambda^h-\lambda^l)V_{n+1}(h+1,l) + \lambda^h[p^h + V_{n+1}(h,l)] + \lambda^l[p^l + V_{n+1}(h+1,l-1)], \\
& V_n(h,l) \\
= \quad & (1-\lambda^h-\lambda^l)V_{n+1}(h,l) + \lambda^h[p^h + V_{n+1}(h-1,l)] + \lambda^l[p^l + V_{n+1}(h,l-1)], \\
& V_n(h+1,l+1) \\
= \quad & (1-\lambda^h-\lambda^l)V_{n+1}(h+1,l+1) + \lambda^h[p^h + V_{n+1}(h,l+1)] + \lambda^l[p^l + V_{n+1}(h+1,l)], \\
& V_n(h,l+1) \\
= \quad & (1-\lambda^h-\lambda^l)V_{n+1}(h,l+1) + \lambda^h[p^h + V_{n+1}(h-1,l+1)] + \lambda^l[p^l + V_{n+1}(h,l)].
\end{aligned}$$

From the submodularity in period $n+1$, we have $\Delta_n^h(h+1,l) \geq \Delta_n^h(h+1,l+1)$ through term-by-term comparisons.

Case 2: $\delta_n^*(h, l+1) \leq \{\delta_n^*(h,l), \delta_n^*(h+1,l+1)\} < \delta_n^*(h+1,l)$.

$$\begin{aligned}
V_n(h+1,l) &= \Psi[V_{n+1}(h+1,l)] + C\lambda\delta_n^*(h+1,l) \geq \Psi[V_{n+1}(h+1,l)], \\
V_n(h,l) &= \Psi[V_{n+1}(h,l)], \\
V_n(h+1,l+1) &= \Psi[V_{n+1}(h+1,l+1)], \\
V_n(h,l+1) &= \Psi[V_{n+1}(h,l+1)].
\end{aligned}$$

The inequality is from $\delta_n^*(h+1,l) \geq 0$. Due to the submodularity in period $n+1$, we have $\Delta_n^h(h+1,l) \geq \Delta_n^h(h+1,l+1)$.

Case 3: $\delta_n^*(h,l+1) \leq \delta_n^*(h+1,l+1) \leq 0 < \delta_n^*(h,l) \leq \delta_n^*(h+1,l)$.

$$\begin{aligned}
V_n(h+1,l) &= \Psi[V_{n+1}(h+1,l)] + C\lambda\delta_n^*(h+1,l), \\
V_n(h,l) &= \Psi[V_{n+1}(h,l)] + C\lambda\delta_n^*(h,l), \\
V_n(h+1,l+1) &= \Psi[V_{n+1}(h+1,l+1)], \\
V_n(h,l+1) &= \Psi[V_{n+1}(h,l+1)].
\end{aligned}$$

$$
\begin{aligned}
\Delta_n^h(h+1,l) &= (1-\lambda^h-\lambda^l)\Delta_{n+1}^h(h+1,l)+\lambda^h\Delta_{n+1}^h(h,l)+\lambda^l\Delta_{n+1}^h(h+1,l-1) \\
&\quad +C\lambda[\delta_n^*(h+1,l)-\delta_n^*(h,l)] \\
&\geq (1-\lambda^h-\lambda^l)\Delta_{n+1}^h(h+1,l)+\lambda^h\Delta_{n+1}^h(h,l)+\lambda^l\Delta_{n+1}^h(h+1,l-1), \\
\Delta_n^h(h+1,l+1) &= (1-\lambda^h-\lambda^l)\Delta_{n+1}^h(h+1,l+1)+\lambda^h\Delta_{n+1}^h(h,l+1)+\lambda^l\Delta_{n+1}^h(h+1,l).
\end{aligned}
$$

The inequality is from $\delta_n^*(h+1,l) \geq \delta_n^*(h,l)$. Due to the submodularity in period $n+1$, we have $\Delta_n^h(h+1,l) \geq \Delta_n^h(h+1,l+1)$ through term-by-term comparisons.

Case 4: $\delta_n^*(h,l+1) \leq \delta_n^*(h,l) \leq 0 < \delta_n^*(h+1,l+1) \leq \delta_n^*(h+1,l)$.

$$
\begin{aligned}
V_n(h+1,l) &= \Psi[V_{n+1}(h+1,l)]+C\lambda\delta_n^*(h+1,l), \\
V_n(h,l) &= \Psi[V_{n+1}(h,l)], \\
V_n(h+1,l+1) &= \Psi[V_{n+1}(h+1,l+1)]+C\lambda\delta_n^*(h+1,l+1), \\
V_n(h,l+1) &= \Psi[V_{n+1}(h,l+1)].
\end{aligned}
$$

$$
\begin{aligned}
\Delta_n^h(h+1,l) &= (1-\lambda^h-\lambda^l)\Delta_{n+1}^h(h+1,l)+\lambda^h\Delta_{n+1}^h(h,l)+\lambda^l\Delta_{n+1}^h(h+1,l-1) \\
&\quad +C\lambda\delta_n^*(h+1,l), \\
\Delta_n^h(h+1,l+1) &= (1-\lambda^h-\lambda^l)\Delta_{n+1}^h(h+1,l+1)+\lambda^h\Delta_{n+1}^h(h,l+1)+\lambda^l\Delta_{n+1}^h(h+1,l) \\
&\quad +C\lambda\delta_n^*(h+1,l+1).
\end{aligned}
$$

Due to the submodularity in period $n+1$ and $\delta_n^*(h+1,l) \geq \delta_n^*(h+1,l+1)$, we have $\Delta_n^h(h+1,l) \geq \Delta_n^h(h+1,l+1)$ through term-by-term comparisons.

Case 5: $\delta_n^*(h,l+1) \leq 0 < \{\delta_n^*(h+1,l+1), \delta_n^*(h,l)\} \leq \delta_n^*(h+1,l)$.

Since $\delta_n^*(h+1,l+1)$ can be larger or smaller than $\delta_n^*(h,l)$, this is the most complicated case in the entire proof. We consider two subcases: Subcase A is $\Delta_{n+1}(h,l) \leq \Delta_{n+1}(h+1,l+1)$ and Subcase B is $\Delta_{n+1}(h,l) \geq \Delta_{n+1}(h+1,l+1)$. We obtain different but equivalent submodularity inequalities for each case, in particular $\Delta_n^l(h,l+1) \geq \Delta_n^l(h+1,l+1)$ for Subcase A and $\Delta_n^h(h+1,l) \geq \Delta_n^h(h+1,l+1)$ for Subcase B.

Subcase A: $\Delta_{n+1}(h,l) \leq \Delta_{n+1}(h+1,l+1)$. By Lemma 1, $\Delta_{n+1}(h,l) \leq \Delta_{n+1}(h+1,l+1)$ implies $\delta_n^*(h,l) \leq \delta_n^*(h+1,l+1)$, $p_{h,l}^* \geq p_{h+1,l+1}^*$ and $\alpha(p_{h,l}^*) \leq \alpha(p_{h+1,l+1}^*)$.

$$\begin{aligned}
V_n(h,l+1) &= \Psi[V_{n+1}(h,l+1)] \\
&\geq \Psi[V_{n+1}(h,l+1)] + C\lambda\delta_n^*(h,l+1) \\
&\geq \Psi[V_{n+1}(h,l+1)] + C\lambda\alpha(p_{h,l}^*)[p_{h,l}^* + V_{n+1}(h-1,l+2) - V_{n+1}(h,l+1)], \\
V_n(h,l) &= \Psi[V_{n+1}(h,l)] + C\lambda\alpha(p_{h,l}^*)[p_{h,l}^* + V_{n+1}(h-1,l+1) - V_{n+1}(h,l)], \\
V_n(h+1,l+1) &= \Psi[V_{n+1}(h+1,l+1)] \\
&\quad + C\lambda\alpha(p_{h+1,l+1}^*)[p_{h+1,l+1}^* + V_{n+1}(h,l+2) - V_{n+1}(h+1,l+1)], \\
V_n(h+1,l) &= \Psi[V_{n+1}(h+1,l)] + C\lambda\alpha(p_{h+1,l}^*)[p_{h+1,l}^* + V_{n+1}(h,l+1) - V_{n+1}(h+1,l)] \\
&\geq \Psi[V_{n+1}(h+1,l)] \\
&\quad + C\lambda\alpha(p_{h+1,l+1}^*)[p_{h+1,l+1}^* + V_{n+1}(h,l+1) - V_{n+1}(h+1,l)].
\end{aligned}$$

Above, the first inequality follows from $\delta_n^*(h,l+1) \leq 0$. The second and third follow from the facts that $p_{h,l}^*$ and $p_{h+1,l+1}^*$ are not the optimal fees for $\delta_n(p,h,l+1)$ and $\delta_n(p,h+1,l)$, respectively.

$$\begin{aligned}
&\Delta_n^l(h,l+1) \\
\geq\ & (1-\lambda^h-\lambda^l)\Delta_{n+1}^l(h,l+1) + \lambda^h\Delta_{n+1}^l(h-1,l+1) + \lambda^l\Delta_{n+1}^l(h,l) \\
& + C\lambda\alpha(p_{h,l}^*)[V_{n+1}(h-1,l+2) - V_{n+1}(h-1,l+1) + V_{n+1}(h,l) - V_{n+1}(h,l+1)] \\
=\ & [1-\lambda^h-\lambda^l - C\lambda\alpha(p_{h+1,l+1}^*)]\Delta_{n+1}^l(h,l+1) + \lambda^h\Delta_{n+1}^l(h-1,l+1) + \lambda^l\Delta_{n+1}^l(h,l) \\
& + C\lambda\alpha(p_{h,l}^*)\Delta_{n+1}^l(h-1,l+2) + C\lambda[\alpha(p_{h+1,l+1}^*) - \alpha(p_{h,l}^*)]\Delta_{n+1}^l(h,l+1) \\
\geq\ & [1-\lambda^h-\lambda^l - C\lambda\alpha(p_{h+1,l+1}^*)]\Delta_{n+1}^l(h,l+1) + \lambda^h\Delta_{n+1}^l(h-1,l+1) + \lambda^l\Delta_{n+1}^l(h,l) \\
& + C\lambda\alpha(p_{h+1,l+1}^*)\Delta_{n+1}^l(h,l+2)
\end{aligned}$$

The second inequality is from $\alpha(p^*_{h+1,l+1})-\alpha(p^*_{h,l}) \geq 0$, the submodularity $\Delta^l_{n+1}(h-1,l+2) \geq \Delta^l_{n+1}(h,l+2)$ and the V-concavity $\Delta^l_{n+1}(h,l+1) \geq \Delta^l_{n+1}(h,l+2)$.

$$\begin{aligned}
& \Delta^l_n(h+1,l+1) \\
\leq\ & (1-\lambda^h-\lambda^l)\Delta^l_{n+1}(h+1,l+1) + \lambda^h\Delta^l_{n+1}(h,l+1) + \lambda^l\Delta^l_{n+1}(h+1,l) \\
& +C\lambda\alpha(p^*_{h+1,l+1})[V_{n+1}(h,l+2) - V_{n+1}(h,l+1) + V_{n+1}(h+1,l) - V_{n+1}(h+1,l+1)] \\
=\ & [1-\lambda^h-\lambda^l - C\lambda\alpha(p^*_{h+1,l+1})]\Delta^l_{n+1}(h+1,l+1) + \lambda^h\Delta^l_{n+1}(h,l+1) + \lambda^l\Delta^l_{n+1}(h+1,l) \\
& +C\lambda\alpha(p^*_{h+1,l+1})\Delta^l_{n+1}(h,l+2)
\end{aligned}$$

The submodularity in period $n+1$ then implies the submodularity $\Delta^l_n(h,l+1) \geq \Delta^l_n(h+1,l+1)$ in period n through term-by-term comparisons.

Subcase B: $\Delta_{n+1}(h,l) \geq \Delta_{n+1}(h+1,l+1)$. By Lemma 1, $\Delta_{n+1}(h,l) \geq \Delta_{n+1}(h+1,l+1)$ implies $\delta^*_n(h,l) \geq \delta^*_n(h+1,l+1)$, $p^*_{h,l} \leq p^*_{h+1,l+1}$ and $\alpha(p^*_{h,l}) \geq \alpha(p^*_{h+1,l+1})$.

$$\begin{aligned}
V_n(h+1,l) &= \Psi[V_{n+1}(h+1,l)] + C\lambda\alpha(p^*_{h+1,l})[p^*_{h+1,l} + V_{n+1}(h,l+1) - V_{n+1}(h+1,l)] \\
&\geq \Psi[V_{n+1}(h+1,l)] + C\lambda\alpha(p^*_{h,l})[p^*_{h,l} + V_{n+1}(h,l+1) - V_{n+1}(h+1,l)], \\
V_n(h,l) &= \Psi[V_{n+1}(h,l)] + C\lambda\alpha(p^*_{h,l})[p^*_{h,l} + V_{n+1}(h-1,l+1) - V_{n+1}(h,l)], \\
V_n(h+1,l+1) &= \Psi[V_{n+1}(h+1,l+1)] \\
&\quad +C\lambda\alpha(p^*_{h+1,l+1})[p^*_{h+1,l+1} + V_{n+1}(h,l+2) - V_{n+1}(h+1,l+1)], \\
V_n(h,l+1) &= \Psi[V_{n+1}(h,l+1)] \\
&\geq \Psi[V_{n+1}(h,l+1)] + C\lambda\delta^*_n(h,l+1) \\
&\geq \Psi[V_{n+1}(h,l+1)] \\
&\quad +C\lambda\alpha(p^*_{h+1,l+1})[p^*_{h+1,l+1} + V_{n+1}(h-1,l+2) - V_{n+1}(h,l+1)].
\end{aligned}$$

Above, the first inequality follows from the fact that $p^*_{h,l}$ is not the optimal fee for $\delta_n(p,h+1,l)$. The second inequality is from $\delta^*_n(h,l+1) \leq 0$. The third inequality follows from the fact that $p^*_{h+1,l+1}$ is not the optimal fee for $\delta_n(p,h,l+1)$.

$$
\begin{aligned}
& \Delta_n^h(h+1,l) \\
\geq \quad & (1-\lambda^h-\lambda^l)\Delta_{n+1}^h(h+1,l)+\lambda^h\Delta_{n+1}^h(h,l)+\lambda^l\Delta_{n+1}^h(h+1,l-1) \\
& +C\lambda\alpha(p^*_{h,l})[V_{n+1}(h,l+1)-V_{n+1}(h-1,l+1)+V_{n+1}(h,l)-V_{n+1}(h+1,l)] \\
= \quad & [1-\lambda^h-\lambda^l-C\lambda\alpha(p^*_{h,l})]\Delta_{n+1}^h(h+1,l)+\lambda^h\Delta_{n+1}^h(h,l)+\lambda^l\Delta_{n+1}^h(h+1,l-1) \\
& +C\lambda\alpha(p^*_{h,l})\Delta_{n+1}^h(h,l+1). \\
& \Delta_n^h(h+1,l+1) \\
\leq \quad & (1-\lambda^h-\lambda^l)\Delta_{n+1}^h(h+1,l+1)+\lambda^h\Delta_{n+1}^h(h,l+1)+\lambda^l\Delta_{n+1}^h(h+1,l) \\
& +C\lambda\alpha(p^*_{h+1,l+1})[V_{n+1}(h,l+2)-V_{n+1}(h-1,l+2)+V_{n+1}(h,l+1)-V_{n+1}(h+1,l+1)] \\
= \quad & [1-\lambda^h-\lambda^l-C\lambda\alpha(p^*_{h,l})]\Delta_{n+1}^h(h+1,l+1)+\lambda^h\Delta_{n+1}^h(h,l+1)+\lambda^l\Delta_{n+1}^h(h+1,l) \\
& +C\lambda\alpha(p^*_{h+1,l+1})\Delta_{n+1}^h(h,l+2)+C\lambda[\alpha(p^*_{h,l})-\alpha(p^*_{h+1,l+1})]\Delta_{n+1}^h(h+1,l+1) \\
\leq \quad & [1-\lambda^h-\lambda^l-C\lambda\alpha(p^*_{h,l})]\Delta_{n+1}^h(h+1,l+1)+\lambda^h\Delta_{n+1}^h(h,l+1)+\lambda^l\Delta_{n+1}^h(h+1,l) \\
& +C\lambda\alpha(p^*_{h,l})\Delta_{n+1}^h(h,l+1).
\end{aligned}
$$

The third inequality follows from the submodularity $\Delta_{n+1}^h(h,l+2) \leq \Delta_{n+1}^h(h,l+1)$ and the H-concavity $\Delta_{n+1}^h(h+1,l+1) \leq \Delta_{n+1}^h(h,l+1)$. The submodularity in period $n+1$ then implies the submodularity $\Delta_n^h(h+1,l) \geq \Delta_n^h(h+1,l+1)$ in period n through term-by-term comparisons.

Case 6: $0 < \delta_n^*(h,l+1) \leq \{\delta_n^*(h+1,l+1), \delta_n^*(h,l)\} \leq \delta_n^*(h+1,l)$. The proof follows similar arguments as in Case 5.

Subcase A: $\Delta_{n+1}(h,l) \leq \Delta_{n+1}(h+1,l+1)$. By Lemma 1, $\Delta_{n+1}(h,l) \leq \Delta_{n+1}(h+1,l+1)$ implies $\delta_n^*(h,l) \leq \delta_n^*(h+1,l+1)$, $p^*_{h,l} \geq p^*_{h+1,l+1}$ and $\alpha(p^*_{h,l}) \leq \alpha(p^*_{h+1,l+1})$.

$$
\begin{aligned}
& V_n(h,l+1) \\
= \;& \Psi[V_{n+1}(h,l+1)] + C\lambda\delta_n^*(h,l+1) \\
\geq \;& \Psi[V_{n+1}(h,l+1)] + C\lambda\alpha(p^*_{h,l})[p^*_{h,l} + V_{n+1}(h-1,l+2) - V_{n+1}(h,l+1)], \\
& V_n(h,l) \\
= \;& \Psi[V_{n+1}(h,l)] + C\lambda\alpha(p^*_{h,l})[p^*_{h,l} + V_{n+1}(h-1,l+1) - V_{n+1}(h,l)], \\
& V_n(h+1,l+1) \\
= \;& \Psi[V_{n+1}(h+1,l+1] + C\lambda\alpha(p^*_{h+1,l+1})[p^*_{h+1,l+1} + V_{n+1}(h,l+2) - V_{n+1}(h+1,l+1)], \\
& V_n(h+1,l) \\
= \;& \Psi[V_{n+1}(h+1,l) + C\lambda\alpha(p^*_{h+1,l})[p^*_{h+1,l} + V_{n+1}(h,l+1) - V_{n+1}(h+1,l)] \\
\geq \;& \Psi[V_{n+1}(h+1,l)] + C\lambda\alpha(p^*_{h+1,l+1})[p^*_{h+1,l+1} + V_{n+1}(h,l+1) - V_{n+1}(h+1,l)].
\end{aligned}
$$

Above, the first and second inequalities follow from the facts that $p^*_{h,l}$ and $p^*_{h+1,l+1}$ are not the optimal fees for $\delta_n(p,h,l+1)$ and $\delta_n(p,h+1,l)$, respectively.

$$
\begin{aligned}
& \Delta_n^l(h,l+1) \\
\geq \;& (1-\lambda^h-\lambda^l)\Delta^l_{n+1}(h,l+1) + \lambda^h\Delta^l_{n+1}(h-1,l+1) + \lambda^l\Delta^l_{n+1}(h,l) \\
& +C\lambda\alpha(p^*_{h,l})[V_{n+1}(h-1,l+2) - V_{n+1}(h-1,l+1) + V_{n+1}(h,l) - V_{n+1}(h,l+1)] \\
= \;& [1-\lambda^h-\lambda^l - C\lambda\alpha(p^*_{h+1,l+1})]\Delta^l_{n+1}(h,l+1) + \lambda^h\Delta^l_{n+1}(h-1,l+1) + \lambda^l\Delta^l_{n+1}(h,l) \\
& +C\lambda\alpha(p^*_{h,l})\Delta^l_{n+1}(h-1,l+2) + C\lambda[\alpha(p^*_{h+1,l+1}) - \alpha(p^*_{h,l})]\Delta^l_{n+1}(h,l+1) \\
\geq \;& [1-\lambda^h-\lambda^l - C\lambda\alpha(p^*_{h+1,l+1})]\Delta^l_{n+1}(h,l+1) + \lambda^h\Delta^l_{n+1}(h-1,l+1) + \lambda^l\Delta^l_{n+1}(h,l) \\
& +C\lambda\alpha(p^*_{h+1,l+1})\Delta^l_{n+1}(h,l+2).
\end{aligned}
$$

The second inequality is from $\alpha(p^*_{h+1,l+1})-\alpha(p^*_{h,l}) \geq 0$, the submodularity $\Delta^l_{n+1}(h-1,l+2) \geq \Delta^l_{n+1}(h,l+2)$ and the V-concavity $\Delta^l_{n+1}(h,l+1) \geq \Delta^l_{n+1}(h,l+2)$.

$$\begin{aligned}
& \Delta^l_n(h+1,l+1) \\
\leq\ & (1-\lambda^h-\lambda^l)\Delta^l_{n+1}(h+1,l+1)+\lambda^h\Delta^l_{n+1}(h,l+1)+\lambda^l\Delta^l_{n+1}(h+1,l) \\
& +C\lambda\alpha(p^*_{h+1,l+1})[V_{n+1}(h,l+2)-V_{n+1}(h,l+1)+V_{n+1}(h+1,l)-V_{n+1}(h+1,l+1)] \\
=\ & [1-\lambda^h-\lambda^l-C\lambda\alpha(p^*_{h+1,l+1})]\Delta^l_{n+1}(h+1,l+1)+\lambda^h\Delta^l_{n+1}(h,l+1)+\lambda^l\Delta^l_{n+1}(h+1,l) \\
& +C\lambda\alpha(p^*_{h+1,l+1})\Delta^l_{n+1}(h,l+2).
\end{aligned}$$

The submodularity in period $n+1$ then implies the submodularity $\Delta^l_n(h,l+1) \geq \Delta^l_n(h+1,l+1)$ in period n through term-by-term comparisons.

Subcase B: $\Delta_{n+1}(h,l) \geq \Delta_{n+1}(h+1,l+1)$. By Lemma 1, $\Delta_{n+1}(h,l) \geq \Delta_{n+1}(h+1,l+1)$ implies $\delta^*_n(h,l) \geq \delta^*_n(h+1,l+1)$, $p^*_{h,l} \leq p^*_{h+1,l+1}$ and $\alpha(p^*_{h,l}) \geq \alpha(p^*_{h+1,l+1})$.

$$\begin{aligned}
V_n(h+1,l) &= \Psi[V_{n+1}(h+1,l)]+C\lambda\alpha(p^*_{h+1,l})[p^*_{h+1,l}+V_{n+1}(h,l+1)-V_{n+1}(h+1,l)] \\
&\geq \Psi[V_{n+1}(h+1,l)]+C\lambda\alpha(p^*_{h,l})[p^*_{h,l}+V_{n+1}(h,l+1)-V_{n+1}(h+1,l)], \\
V_n(h,l) &= \Psi[V_{n+1}(h,l)]+C\lambda\alpha(p^*_{h,l})[p^*_{h,l}+V_{n+1}(h-1,l+1)-V_{n+1}(h,l)], \\
V_n(h+1,l+1) &= \Psi[V_{n+1}(h+1,l+1)] \\
&\quad +C\lambda\alpha(p^*_{h+1,l+1})[p^*_{h+1,l+1}+V_{n+1}(h,l+2)-V_{n+1}(h+1,l+1)], \\
V_n(h,l+1) &= \Psi[V_{n+1}(h,l+1)+C\lambda\delta^*_n(h,l+1) \\
&\geq \Psi[V_{n+1}(h,l+1) \\
&\quad +C\lambda\alpha(p^*_{h+1,l+1})[p^*_{h+1,l+1}+V_{n+1}(h-1,l+2)-V_{n+1}(h,l+1)].
\end{aligned}$$

Above, the first and second inequalities follow from the facts that $p^*_{h,l}$ and $p^*_{h+1,l+1}$ are not the optimal fees for $\delta_n(p, h+1, l)$ and $\delta_n(p, h, l+1)$, respectively.

$$
\begin{aligned}
& \Delta^h_n(h+1, l) \\
\geq\ & (1-\lambda^h-\lambda^l)\Delta^h_{n+1}(h+1,l) + \lambda^h\Delta^h_{n+1}(h,l) + \lambda^l\Delta^h_{n+1}(h+1,l-1) \\
& +C\lambda\alpha(p^*_{h,l})[V_{n+1}(h,l+1) - V_{n+1}(h-1,l+1) + V_{n+1}(h,l) - V_{n+1}(h+1,l)] \\
=\ & [1-\lambda^h-\lambda^l - C\lambda\alpha(p^*_{h,l})]\Delta^h_{n+1}(h+1,l) + \lambda^h\Delta^h_{n+1}(h,l) + \lambda^l\Delta^h_{n+1}(h+1,l-1) \\
& +C\lambda\alpha(p^*_{h,l})\Delta^h_{n+1}(h,l+1). \\
& \Delta^h_n(h+1,l+1) \\
\leq\ & (1-\lambda^h-\lambda^l)\Delta^h_{n+1}(h+1,l+1) + \lambda^h\Delta^h_{n+1}(h,l+1) + \lambda^l\Delta^h_{n+1}(h+1,l) \\
& +C\lambda\alpha(p^*_{h+1,l+1})[V_{n+1}(h,l+2) - V_{n+1}(h-1,l+2) + V_{n+1}(h,l+1) - V_{n+1}(h+1,l+1)] \\
=\ & [1-\lambda^h-\lambda^l - C\lambda\alpha(p^*_{h,l})]\Delta^h_{n+1}(h+1,l+1) + \lambda^h\Delta^h_{n+1}(h,l+1) + \lambda^l\Delta^h_{n+1}(h+1,l) \\
& +C\lambda\alpha(p^*_{h+1,l+1})\Delta^h_{n+1}(h,l+2) + C\lambda[\alpha(p^*_{h,l}) - \alpha(p^*_{h+1,l+1})]\Delta^h_{n+1}(h+1,l+1) \\
\leq\ & [1-\lambda^h-\lambda^l - C\lambda\alpha(p^*_{h,l})]\Delta^h_{n+1}(h+1,l+1) + \lambda^h\Delta^h_{n+1}(h,l+1) + \lambda^l\Delta^h_{n+1}(h+1,l) \\
& +C\lambda\alpha(p^*_{h,l})\Delta^h_{n+1}(h,l+1).
\end{aligned}
$$

The second inequality follows from the submodularity $\Delta^h_{n+1}(h,l+2) \leq \Delta^h_{n+1}(h,l+1)$, the H-concavity $\Delta^h_{n+1}(h+1,l+1) \leq \Delta^h_{n+1}(h,l+1)$ and $\alpha(p^*_{h,l}) - \alpha(p^*_{h+1,l+1}) \geq 0$. The submodularity in period $n+1$ then implies the submodularity $\Delta^h_n(h+1,l) \geq \Delta^h_n(h+1,l+1)$ in period n through term-by-term comparisons.

<u>Horizontal boundary $[1 : H-1] \times \{0\}$</u>. There are 6 possible cases similar to the interior region.

Case 1: $\delta_n^*(h,1) \leq \{\delta_n^*(h,0), \delta_n^*(h+1,1)\} \leq \delta_n^*(h+1,0) \leq 0$.

$$\begin{aligned}
V_n(h+1,0) &= (1-\lambda^h)V_{n+1}(h+1,0) + \lambda^h[p^h + V_{n+1}(h,0)], \\
V_n(h,0) &= (1-\lambda^h)V_{n+1}(h,0) + \lambda^h[p^h + V_{n+1}(h-1,0)], \\
V_n(h+1,1) &= (1-\lambda^h-\lambda^l)V_{n+1}(h+1,1) + \lambda^h[p^h + V_{n+1}(h,1)] + \lambda^l[p^l + V_{n+1}(h+1,0)], \\
V_n(h,1) &= (1-\lambda^h-\lambda^l)V_{n+1}(h,1) + \lambda^h[p^h + V_{n+1}(h-1,1)] + \lambda^l[p^l + V_{n+1}(h,0)].
\end{aligned}$$

$$\begin{aligned}
\Delta_n^h(h+1,0) &= (1-\lambda^h)\Delta_{n+1}^h(h+1,0) + \lambda^h\Delta_{n+1}^h(h,0) \\
&= (1-\lambda^h-\lambda^l)\Delta_{n+1}^h(h+1,0) + \lambda^h\Delta_{n+1}^h(h,0) + \lambda^l\Delta_{n+1}^h(h+1,0) \\
\Delta_n^h(h+1,1) &= (1-\lambda^h-\lambda^l)\Delta_{n+1}^h(h+1,1) + \lambda^h\Delta_{n+1}^h(h,1) + \lambda^l\Delta_{n+1}^h(h+1,0).
\end{aligned}$$

Due to the submodularity in period $n+1$, we have $(1-\lambda^h-\lambda^l)\Delta_{n+1}^h(h+1,0)+\lambda^h\Delta_{n+1}^h(h,0) \geq (1-\lambda^h-\lambda^l)\Delta_{n+1}^h(h+1,1) + \lambda^h\Delta_{n+1}^h(h,1)$. Thus $\Delta_n^h(h+1,0) \geq \Delta_n^h(h+1,1)$.

Case 2: $\delta_n^*(h,1) \leq \{\delta_n^*(h,0), \delta_n^*(h+1,1)\} \leq 0 < \delta_n^*(h+1,0)$.

$$\begin{aligned}
V_n(h+1,0) &= (1-\lambda^h)V_{n+1}(h+1,0) + \lambda^h[p^h + V_{n+1}(h,0)] + C\lambda\delta_n^*(h+1,0) \\
&\geq (1-\lambda^h)V_{n+1}(h+1,0) + \lambda^h[p^h + V_{n+1}(h,0)], \\
V_n(h,0) &= (1-\lambda^h)V_{n+1}(h,0) + \lambda^h[p^h + V_{n+1}(h-1,0)], \\
V_n(h+1,1) &= (1-\lambda^h-\lambda^l)V_{n+1}(h+1,1) + \lambda^h[p^h + V_{n+1}(h,1)] + \lambda^l[p^l + V_{n+1}(h+1,0)], \\
V_n(h,1) &= (1-\lambda^h-\lambda^l)V_{n+1}(h,1) + \lambda^h[p^h + V_{n+1}(h-1,1)] + \lambda^l[p^l + V_{n+1}(h,0)].
\end{aligned}$$

The inequality is from $\delta_n^*(h+1,0) \geq 0$. Then, the proof is similar to Case 1.

Case 3: $\delta_n^*(h,1) \leq \delta_n^*(h+1,1) \leq 0 < \delta_n^*(h,0) \leq \delta_n^*(h+1,0)$.

$$\begin{aligned}
V_n(h+1,0) &= (1-\lambda^h)V_{n+1}(h+1,0) + \lambda^h[p^h + V_{n+1}(h,0)] + C\lambda\delta_n^*(h+1,0), \\
V_n(h,0) &= (1-\lambda^h)V_{n+1}(h,0) + \lambda^h[p^h + V_{n+1}(h-1,0)] + C\lambda\delta_n^*(h,0), \\
V_n(h+1,1) &= (1-\lambda^h-\lambda^l)V_{n+1}(h+1,1) + \lambda^h[p^h + V_{n+1}(h,1)] + \lambda^l[p^l + V_{n+1}(h+1,0)], \\
V_n(h,1) &= (1-\lambda^h-\lambda^l)V_{n+1}(h,1) + \lambda^h[p^h + V_{n+1}(h-1,1)] + \lambda^l[p^l + V_{n+1}(h,0)].
\end{aligned}$$

$$\begin{aligned}
\Delta_n^h(h+1,0) &= (1-\lambda^h)\Delta_{n+1}^h(h+1,0) + \lambda^h \Delta_{n+1}^h(h,0) + C\lambda[\delta_n^*(h+1,0) - \delta_n^*(h,0)] \\
&\geq (1-\lambda^h)\Delta_{n+1}^h(h+1,0) + \lambda^h \Delta_{n+1}^h(h,0), \\
\Delta_n^h(h+1,1) &= (1-\lambda^h-\lambda^l)\Delta_{n+1}^h(h+1,1) + \lambda^h \Delta_{n+1}^h(h,1) + \lambda^l \Delta_{n+1}^h(h+1,0).
\end{aligned}$$

The inequality is from $\delta_n^*(h+1,0) \geq \delta_n^*(h,0)$. Then, the proof is similar to Case 1.

Case 4: $\delta_n^*(h,1) \leq \delta_n^*(h,0) \leq 0 < \delta_n^*(h+1,1) \leq \delta_n^*(h+1,0)$.

$$\begin{aligned}
V_n(h+1,0) &= (1-\lambda^h)V_{n+1}(h+1,0) + \lambda^h[p^h + V_{n+1}(h,0)] + C\lambda\delta_n^*(h+1,0), \\
V_n(h,0) &= (1-\lambda^h)V_{n+1}(h,0) + \lambda^h[p^h + V_{n+1}(h-1,0)], \\
V_n(h+1,1) &= (1-\lambda^h-\lambda^l)V_{n+1}(h+1,1) + \lambda^h[p^h + V_{n+1}(h,1)] + \lambda^l[p^l + V_{n+1}(h+1,0)] \\
&\quad + C\lambda\delta_n^*(h+1,1), \\
V_n(h,1) &= (1-\lambda^h-\lambda^l)V_{n+1}(h,1) + \lambda^h[p^h + V_{n+1}(h-1,1)] + \lambda^l[p^l + V_{n+1}(h,0)]. \\
\Delta_n^h(h+1,0) &= (1-\lambda^h)\Delta_{n+1}^h(h+1,0) + \lambda^h \Delta_{n+1}^h(h,0) + C\lambda\delta_n^*(h+1,0), \\
\Delta_n^h(h+1,1) &= (1-\lambda^h-\lambda^l)\Delta_{n+1}^h(h+1,1) + \lambda^h \Delta_{n+1}^h(h,1) + \lambda^l \Delta_{n+1}^h(h+1,0) \\
&\quad + C\lambda\delta_n^*(h+1,1).
\end{aligned}$$

We have $\delta_n^*(h+1,0) \geq \delta_n^*(h+1,1)$. Then, the proof is similar to Case 1.

Case 5: $\delta_n^*(h,1) \leq 0 < \{\delta_n^*(h,0), \delta_n^*(h+1,1)\} \leq \delta_n^*(h+1,0)$.

Subcase A: $\Delta_{n+1}(h,0) \leq \Delta_{n+1}(h+1,1)$. By Lemma 1, $\Delta_{n+1}(h,0) \leq \Delta_{n+1}(h+1,1)$ implies $\delta_n^*(h,0) \leq \delta_n^*(h+1,1)$, $p_{h,0}^* \geq p_{h+1,1}^*$ and $\alpha(p_{h,0}^*) \leq \alpha(p_{h+1,1}^*)$.

$$\begin{aligned}
V_n(h,1) &\geq (1-\lambda^h-\lambda^l)V_{n+1}(h,1) + \lambda^h[p^h + V_{n+1}(h-1,1)] + \lambda^l[p^l + V_{n+1}(h,0)] \\
&\quad + C\lambda\delta_n^*(h,1) \\
&\geq (1-\lambda^h-\lambda^l)V_{n+1}(h,1) + \lambda^h[p^h + V_{n+1}(h-1,1)] + \lambda^l[p^l + V_{n+1}(h,0)] \\
&\quad + C\lambda\alpha(p_{h,0}^*)[p_{h,0}^* + V_{n+1}(h-1,2) - V_{n+1}(h,1)], \\
V_n(h,0) &= (1-\lambda^h-\lambda^l)V_{n+1}(h,0) + \lambda^h[p^h + V_{n+1}(h-1,0)] + \lambda^l V_{n+1}(h,0) \\
&\quad + C\lambda\alpha(p_{h,0}^*)[p_{h,0}^* + V_{n+1}(h-1,1) - V_{n+1}(h,0)], \\
V_n(h+1,1) &= (1-\lambda^h-\lambda^l)V_{n+1}(h+1,1) + \lambda^h[p^h + V_{n+1}(h,1)] + \lambda^l[p^l + V_{n+1}(h+1,0)] \\
&\quad + C\lambda\alpha(p_{h+1,1}^*)[p_{h+1,1}^* + V_{n+1}(h,2) - V_{n+1}(h+1,1)], \\
V_n(h+1,0) &= (1-\lambda^h-\lambda^l)V_{n+1}(h+1,0) + \lambda^h[p^h + V_{n+1}(h,0)] + \lambda^l V_{n+1}(h+1,0) \\
&\quad + C\lambda\alpha(p_{h+1,0}^*)[p_{h+1,0}^* + V_{n+1}(h,1) - V_{n+1}(h+1,0)]
\end{aligned}$$

$$\begin{aligned}
&\geq (1-\lambda^h-\lambda^l)V_{n+1}(h+1,0) + \lambda^h[p^h + V_{n+1}(h,0)] + \lambda^l V_{n+1}(h+1,0) \\
&\quad + C\lambda\alpha(p_{h+1,1}^*)[p_{h+1,1}^* + V_{n+1}(h,1) - V_{n+1}(h+1,0)].
\end{aligned}$$

Above the first inequality follows from $\delta_n^*(h,1) \leq 0$; the second and third respectively follow from the facts that $p_{h,0}^*$ and $p_{h+1,1}^*$ are not the optimal fee for $\delta_n(p,h,1)$ and $\delta_n(p,h+1,0)$.

$$\begin{aligned}
\Delta_n^l(h,1) &\geq (1-\lambda^h-\lambda^l)\Delta_{n+1}^l(h,1) + \lambda^h\Delta_{n+1}^l(h-1,1) + \lambda^l p^l \\
&\quad + C\lambda\alpha(p_{h,0}^*)[V_{n+1}(h-1,2) - V_{n+1}(h-1,1) + V_{n+1}(h,0) - V_{n+1}(h,1)] \\
&= [1-\lambda^h-\lambda^l - C\lambda\alpha(p_{h+1,1}^*)]\Delta_{n+1}^l(h,1) + \lambda^h\Delta_{n+1}^l(h-1,1) + \lambda^l p^l \\
&\quad + C\lambda\alpha(p_{h,0}^*)\Delta_{n+1}^l(h-1,2) + C\lambda[\alpha(p_{h+1,1}^*) - \alpha(p_{h,0}^*)]\Delta_{n+1}^l(h,1) \\
&\geq [1-\lambda^h-\lambda^l - C\lambda\alpha(p_{h+1,1}^*)]\Delta_{n+1}^l(h,1) + \lambda^h\Delta_{n+1}^l(h-1,1) + \lambda^l p^l \\
&\quad + C\lambda\alpha(p_{h+1,1}^*)\Delta_{n+1}^l(h,2).
\end{aligned}$$

The second inequality is from $\alpha(p^*_{h+1,1}) - \alpha(p^*_{h,0}) \geq 0$, the submodularity $\Delta^l_{n+1}(h-1,2) \geq \Delta^l_{n+1}(h,2)$ and the V-concavity $\Delta^l_{n+1}(h,1) \geq \Delta^l_{n+1}(h,2)$.

$$\begin{aligned}
\Delta^l_n(h+1,1) &\leq (1-\lambda^h-\lambda^l)\Delta^l_{n+1}(h+1,1) + \lambda^h \Delta^l_{n+1}(h,1) + \lambda^l p^l \\
&\quad + C\lambda\alpha(p^*_{h+1,1})[V_{n+1}(h,2) - V_{n+1}(h,1) + V_{n+1}(h+1,0) - V_{n+1}(h+1,1)] \\
&= [1-\lambda^h-\lambda^l - C\lambda\alpha(p^*_{h+1,1})]\Delta^l_{n+1}(h+1,1) + \lambda^h \Delta^l_{n+1}(h,1) + \lambda^l p^l \\
&\quad + C\lambda\alpha(p^*_{h+1,1})\Delta^l_{n+1}(h,2).
\end{aligned}$$

The submodularity in period $n+1$ then implies the submodularity $\Delta^l_n(h,1) \geq \Delta^l_n(h+1,1)$ in period n through term-by-term comparisons.

Subcase B: $\Delta_{n+1}(h,0) \geq \Delta_{n+1}(h+1,1)$. By Lemma 1, $\Delta_{n+1}(h,0) \geq \Delta_{n+1}(h+1,1)$ implies $\delta^*_n(h,0) \geq \delta^*_n(h+1,1)$, $p^*_{h,0} \leq p^*_{h+1,1}$ and $\alpha(p^*_{h,0}) \geq \alpha(p^*_{h+1,1})$.

$$\begin{aligned}
V_n(h+1,0) &= (1-\lambda^h-\lambda^l)V_{n+1}(h+1,0) + \lambda^h[p^h + V_{n+1}(h,0)] + \lambda^l V_{n+1}(h+1,0) \\
&\quad + C\lambda\alpha(p^*_{h+1,0})[p^*_{h+1,0} + V_{n+1}(h,1) - V_{n+1}(h+1,0)] \\
&\geq (1-\lambda^h-\lambda^l)V_{n+1}(h+1,0) + \lambda^h[p^h + V_{n+1}(h,0)] + \lambda^l V_{n+1}(h+1,0) \\
&\quad + C\lambda\alpha(p^*_{h,0})[p^*_{h,0} + V_{n+1}(h,1) - V_{n+1}(h+1,0)], \\
V_n(h,0) &= (1-\lambda^h-\lambda^l)V_{n+1}(h,0) + \lambda^h[p^h + V_{n+1}(h-1,0)] + \lambda^l V_{n+1}(h,0) \\
&\quad + C\lambda\alpha(p^*_{h,0})[p^*_{h,0} + V_{n+1}(h-1,1) - V_{n+1}(h,0)], \\
V_n(h+1,1) &= (1-\lambda^h-\lambda^l)V_{n+1}(h+1,1) + \lambda^h[p^h + V_{n+1}(h,1)] + \lambda^l[p^l + V_{n+1}(h+1,0)] \\
&\quad + C\lambda\alpha(p^*_{h+1,1})[p^*_{h+1,1} + V_{n+1}(h,2) - V_{n+1}(h+1,1)], \\
V_n(h,1) &\geq (1-\lambda^h-\lambda^l)V_{n+1}(h,1) + \lambda^h[p^h + V_{n+1}(h-1,1)] + \lambda^l[p^l + V_{n+1}(h,0)] \\
&\quad + C\lambda\delta^*_n(h,1) \\
&\geq (1-\lambda^h-\lambda^l)V_{n+1}(h,1) + \lambda^h[p^h + V_{n+1}(h-1,1)] + \lambda^l[p^l + V_{n+1}(h,0)] \\
&\quad + C\lambda\alpha(p^*_{h+1,1})[p^*_{h+1,1} + V_{n+1}(h-1,2) - V_{n+1}(h,1)].
\end{aligned}$$

Above the second inequality is from $\delta_n^*(h,1) \leq 0$; the first and third respectively follow from the facts that $p_{h,0}^*$ and $p_{h+1,1}^*$ are not the optimal fee for $\delta_n(p,h+1,0)$ and $\delta_n(p,h,1)$.

$$\begin{aligned}
& \Delta_n^h(h+1,0) \\
\geq\ & (1-\lambda^h-\lambda^l)\Delta_{n+1}^h(h+1,0)+\lambda^h\Delta_{n+1}^h(h,0)+\lambda^l\Delta_{n+1}^h(h+1,0) \\
& +C\lambda\alpha(p_{h,0}^*)[V_{n+1}(h,1)-V_{n+1}(h-1,1)+V_{n+1}(h,0)-V_{n+1}(h+1,0)] \\
=\ & [1-\lambda^h-\lambda^l-C\lambda\alpha(p_{h,0}^*)]\Delta_{n+1}^h(h+1,0)+\lambda^h\Delta_{n+1}^h(h,0)+\lambda^l\Delta_{n+1}^h(h+1,0) \\
& +C\lambda\alpha(p_{h,0}^*)\Delta_{n+1}^h(h,1),
\end{aligned}$$

$$\begin{aligned}
& \Delta_n^h(h+1,1) \\
\leq\ & (1-\lambda^h-\lambda^l)\Delta_{n+1}^h(h+1,1)+\lambda^h\Delta_{n+1}^h(h,1)+\lambda^l\Delta_{n+1}^h(h+1,0) \\
& +C\lambda\alpha(p_{h+1,1}^*)[V_{n+1}(h,2)-V_{n+1}(h-1,2)+V_{n+1}(h,1)-V_{n+1}(h+1,1)]
\end{aligned}$$

$$\begin{aligned}
=\ & [1-\lambda^h-\lambda^l-C\lambda\alpha(p_{h,0}^*)]\Delta_{n+1}^h(h+1,1)+\lambda^h\Delta_{n+1}^h(h,1)+\lambda^l\Delta_{n+1}^h(h+1,0) \\
& +C\lambda\alpha(p_{h+1,1}^*)\Delta_{n+1}^h(h,2)+C\lambda[\alpha(p_{h,0}^*)-\alpha(p_{h+1,1}^*)][\Delta_{n+1}^h(h+1,1)] \\
\leq\ & [1-\lambda^h-\lambda^l-C\lambda\alpha(p_{h,0}^*)]\Delta_{n+1}^h(h+1,1)+\lambda^h\Delta_{n+1}^h(h,1)+\lambda^l\Delta_{n+1}^h(h+1,0) \\
& +C\lambda\alpha(p_{h,0}^*)\Delta_{n+1}^h(h,1).
\end{aligned}$$

The second inequality follows from the submodularity $\Delta_{n+1}^h(h,2) \leq \Delta_{n+1}^h(h,1)$, the H-concavity $\Delta_{n+1}^h(h+1,1) \leq \Delta_{n+1}^h(h,1)$ and $\alpha(p_{h,0}^*)-\alpha(p_{h+1,1}^*) \geq 0$. The submodularity in period $n+1$ then implies the submodularity $\Delta_n^h(h+1,0) \geq \Delta_n^h(h+1,1)$ in period n through term-by-term comparisons.

Case 6: $0 < \delta_n^*(h,1) \leq \{\delta_n^*(h,0), \delta_n^*(h+1,1)\} \leq \delta_n^*(h+1,0)$.

Subcase A: $\Delta_{n+1}(h,0) \leq \Delta_{n+1}(h+1,1)$. By Lemma 1, $\Delta_{n+1}(h,0) \leq \Delta_{n+1}(h+1,1)$ implies $\delta_n^*(h,0) \leq \delta_n^*(h+1,1)$, $p^*_{h,0} \geq p^*_{h+1,1}$ and $\alpha(p^*_{h,0}) \leq \alpha(p^*_{h+1,1})$.

$$
\begin{aligned}
V_n(h,1) &= (1-\lambda^h-\lambda^l)V_{n+1}(h,1) + \lambda^h[p^h + V_{n+1}(h-1,1)] + \lambda^l[p^l + V_{n+1}(h,0)] \\
&\quad + C\lambda\delta_n^*(h,1) \\
&\geq (1-\lambda^h-\lambda^l)V_{n+1}(h,1) + \lambda^h[p^h + V_{n+1}(h-1,1)] + \lambda^l[p^l + V_{n+1}(h,0)] \\
&\quad + C\lambda\alpha(p^*_{h,0})[p^*_{h,0} + V_{n+1}(h-1,2) - V_{n+1}(h,1)], \\
V_n(h,0) &= (1-\lambda^h-\lambda^l)V_{n+1}(h,0) + \lambda^h[p^h + V_{n+1}(h-1,0)] + \lambda^l V_{n+1}(h,0) \\
&\quad + C\lambda\alpha(p^*_{h,0})[p^*_{h,0} + V_{n+1}(h-1,1) - V_{n+1}(h,0)], \\
V_n(h+1,1) &= (1-\lambda^h-\lambda^l)V_{n+1}(h+1,1) + \lambda^h[p^h + V_{n+1}(h,1)] + \lambda^l[p^l + V_{n+1}(h+1,0)] \\
&\quad + C\lambda\alpha(p^*_{h+1,1})[p^*_{h+1,1} + V_{n+1}(h,2) - V_{n+1}(h+1,1)], \\
V_n(h+1,0) &= (1-\lambda^h-\lambda^l)V_{n+1}(h+1,0) + \lambda^h[p^h + V_{n+1}(h,0)] + \lambda^l V_{n+1}(h+1,0) \\
&\quad + C\lambda\alpha(p^*_{h+1,0})[p^*_{h+1,0} + V_{n+1}(h,1) - V_{n+1}(h+1,0)] \\
&\geq (1-\lambda^h-\lambda^l)V_{n+1}(h+1,0) + \lambda^h[p^h + V_{n+1}(h,0)] + \lambda^l V_{n+1}(h+1,0) \\
&\quad + C\lambda\alpha(p^*_{h+1,1})[p^*_{h+1,1} + V_{n+1}(h,1) - V_{n+1}(h+1,0)].
\end{aligned}
$$

Then the proof is identical to that of Case 5 Subcase A.

Subcase B: $\Delta_{n+1}(h,0) \geq \Delta_{n+1}(h+1,1)$. By Lemma 1, $\Delta_{n+1}(h,0) \geq \Delta_{n+1}(h+1,1)$ implies $\delta_n^*(h,0) \geq \delta_n^*(h+1,1)$, $p_{h,0}^* \leq p_{h+1,1}^*$ and $\alpha(p_{h,0}^*) \geq \alpha(p_{h+1,1}^*)$.

$$\begin{aligned}
V_n(h+1,0) &= (1-\lambda^h-\lambda^l)V_{n+1}(h+1,0) + \lambda^h[p^h + V_{n+1}(h,0)] + \lambda^l V_{n+1}(h+1,0) \\
&\quad +C\lambda\alpha(p_{h+1,0}^*)[p_{h+1,0}^* + V_{n+1}(h,1) - V_{n+1}(h+1,0)] \\
&\geq (1-\lambda^h-\lambda^l)V_{n+1}(h+1,0) + \lambda^h[p^h + V_{n+1}(h,0)] + \lambda^l V_{n+1}(h+1,0) \\
&\quad +C\lambda\alpha(p_{h,0}^*)[p_{h,0}^* + V_{n+1}(h,1) - V_{n+1}(h+1,0)], \\
V_n(h,0) &= (1-\lambda^h-\lambda^l)V_{n+1}(h,0) + \lambda^h[p^h + V_{n+1}(h-1,0)] + \lambda^l V_{n+1}(h,0) \\
&\quad +C\lambda\alpha(p_{h,0}^*)[p_{h,0}^* + V_{n+1}(h-1,1) - V_{n+1}(h,0)], \\
V_n(h+1,1) &= (1-\lambda^h-\lambda^l)V_{n+1}(h+1,1) + \lambda^h[p^h + V_{n+1}(h,1)] + \lambda^l[p^l + V_{n+1}(h+1,0)] \\
&\quad +C\lambda\alpha(p_{h+1,1}^*)[p_{h+1,1}^* + V_{n+1}(h,2) - V_{n+1}(h+1,1)], \\
V_n(h,1) &= (1-\lambda^h-\lambda^l)V_{n+1}(h,1) + \lambda^h[p^h + V_{n+1}(h-1,1)] + \lambda^l[p^l + V_{n+1}(h,0)] \\
&\quad +C\lambda\delta_n^*(h,1) \\
&\geq (1-\lambda^h-\lambda^l)V_{n+1}(h,1) + \lambda^h[p^h + V_{n+1}(h-1,1)] + \lambda^l[p^l + V_{n+1}(h,0)] \\
&\quad +C\lambda\alpha(p_{h+1,1}^*)[p_{h+1,1}^* + V_{n+1}(h-1,2) - V_{n+1}(h,1)].
\end{aligned}$$

Then the proof is identical to that of Case 5 Subcase B.

<u>Vertical boundary $\{0\} \times [1:M]$</u>. With $h=0$, $\delta_n^*(0,l)$ and $\delta_n^*(0,l+1)$ are not defined as no upgrades can be offered with no premium product. We have $\delta_n^*(1,l+1) \leq \delta_n^*(1,l)$ and there are 3 possible cases similar to Cases 1, 2 and 4 in the interior.

Case 1: $\delta_n^*(1,l+1) \leq \delta_n^*(1,l) \leq 0$.

$$\begin{aligned}
V_n(0,l+1) &= (1-\lambda^l)V_{n+1}(0,l+1) + \lambda^l[p^l + V_{n+1}(0,l)], \\
V_n(0,l) &= (1-\lambda^l)V_{n+1}(0,l) + \lambda^l[p^l + V_{n+1}(0,l-1)], \\
V_n(1,l+1) &= (1-\lambda^h-\lambda^l)V_{n+1}(1,l+1) + \lambda^h[p^h + V_{n+1}(0,l+1)] + \lambda^l[p^l + V_{n+1}(1,l)], \\
V_n(1,l) &= (1-\lambda^h-\lambda^l)V_{n+1}(1,l) + \lambda^h[p^h + V_{n+1}(0,l)] + \lambda^l[p^l + V_{n+1}(1,l-1)].
\end{aligned}$$

$$\begin{aligned}\Delta_n^l(0,l+1) &= (1-\lambda^l)\Delta_{n+1}^l(0,l+1)+\lambda^l\Delta_{n+1}^l(0,l)\\ &= (1-\lambda^h-\lambda^l)\Delta_{n+1}^l(0,l+1)+\lambda^l\Delta_{n+1}^l(0,l)+\lambda^h\Delta_{n+1}^l(0,l+1),\\ \Delta_n^l(1,l+1) &= (1-\lambda^h-\lambda^l)\Delta_{n+1}^l(1,l+1)+\lambda^l\Delta_{n+1}^l(1,l)+\lambda^h\Delta_{n+1}^l(0,l+1).\end{aligned}$$

Due to the submodularity in period $n+1$, we have $(1-\lambda^h-\lambda^l)\Delta_{n+1}^l(0,l+1)+\lambda^h\Delta_{n+1}^l(0,l) \geq (1-\lambda^h-\lambda^l)\Delta_{n+1}^l(1,l+1)+\lambda^h\Delta_{n+1}^l(1,l)$. Thus $\Delta_n^l(0,l+1) \geq \Delta_n^l(1,l+1)$.

Case 2: $\delta_n^*(1,l+1) \leq 0 < \delta_n^*(1,l)$.

$$\begin{aligned}V_n(0,l+1) &= (1-\lambda^l)V_{n+1}(0,l+1)+\lambda^l[p^l+V_{n+1}(0,l)],\\ V_n(0,l) &= (1-\lambda^l)V_{n+1}(0,l)+\lambda^l[p^l+V_{n+1}(0,l-1)],\\ V_n(1,l+1) &= (1-\lambda^h-\lambda^l)V_{n+1}(1,l+1)+\lambda^h[p^h+V_{n+1}(0,l+1)]+\lambda^l[p^l+V_{n+1}(1,l)],\\ V_n(1,l) &= (1-\lambda^h-\lambda^l)V_{n+1}(1,l)+\lambda^h[p^h+V_{n+1}(0,l)]+\lambda^l[p^l+V_{n+1}(1,l-1)]\\ &\quad +C\lambda\delta_n^*(1,l)\\ &\geq (1-\lambda^h-\lambda^l)V_{n+1}(1,l)+\lambda^h[p^h+V_{n+1}(0,l)]+\lambda^l[p^l+V_{n+1}(1,l-1)],\end{aligned}$$

where the last inequality is due to $\delta_n^*(1,l) > 0$.

$$\begin{aligned}\Delta_n^l(0,l+1) &= (1-\lambda^l)\Delta_{n+1}^l(0,l+1)+\lambda^l\Delta_{n+1}^l(0,l)\\ &= (1-\lambda^h-\lambda^l)\Delta_{n+1}^l(0,l+1)+\lambda^l\Delta_{n+1}^l(0,l)+\lambda^h\Delta_{n+1}^l(0,l+1),\\ \Delta_n^l(1,l+1) &\leq (1-\lambda^h-\lambda^l)\Delta_{n+1}^l(1,l+1)+\lambda^l\Delta_{n+1}^l(1,l)+\lambda^h\Delta_{n+1}^l(0,l+1).\end{aligned}$$

Due to the submodularity in period $n+1$, we have $(1-\lambda^h-\lambda^l)\Delta_{n+1}^l(0,l+1)+\lambda^h\Delta_{n+1}^l(0,l) \geq (1-\lambda^h-\lambda^l)\Delta_{n+1}^l(1,l+1)+\lambda^h\Delta_{n+1}^l(1,l)$. Thus $\Delta_n^l(0,l+1) \geq \Delta_n^l(1,l+1)$.

Case 3: $0 < \delta_n^*(1,l+1) \leq \delta_n^*(1,l)$.

$$\begin{aligned}V_n(0,l+1) &= (1-\lambda^l)V_{n+1}(0,l+1)+\lambda^l[p^l+V_{n+1}(0,l)],\\ V_n(0,l) &= (1-\lambda^l)V_{n+1}(0,l)+\lambda^l[p^l+V_{n+1}(0,l-1)],\\ V_n(1,l+1) &= (1-\lambda^h-\lambda^l)V_{n+1}(1,l+1)+\lambda^h[p^h+V_{n+1}(0,l+1)]+\lambda^l[p^l+V_{n+1}(1,l)]\\ &\quad +C\lambda\delta_n^*(1,l+1),\end{aligned}$$

$$
\begin{aligned}
V_n(1,l) &= (1-\lambda^h-\lambda^l)V_{n+1}(1,l)+\lambda^h[p^h+V_{n+1}(0,l)]+\lambda^l[p^l+V_{n+1}(1,l-1)] \\
&\quad +C\lambda\delta_n^*(1,l). \\
\Delta_n^l(0,l+1) &= (1-\lambda^l)\Delta_{n+1}^l(0,l+1)+\lambda^l\Delta_{n+1}^l(0,l) \\
&= (1-\lambda^h-\lambda^l)\Delta_{n+1}^l(0,l+1)+\lambda^l\Delta_{n+1}^l(0,l)+\lambda^h\Delta_{n+1}^l(0,l+1), \\
\Delta_n^l(1,l+1) &= (1-\lambda^h-\lambda^l)\Delta_{n+1}^l(1,l+1)+\lambda^l\Delta_{n+1}^l(1,l)+\lambda^h\Delta_{n+1}^l(0,l+1) \\
&\quad +C\lambda[\delta_n^*(1,l+1)-\delta_n^*(1,l)] \\
&\leq (1-\lambda^h-\lambda^l)\Delta_{n+1}^l(1,l+1)+\lambda^l\Delta_{n+1}^l(1,l)+\lambda^h\Delta_{n+1}^l(0,l+1).
\end{aligned}
$$

The last inequality is from $\delta_n^*(1,l+1)-\delta_n^*(1,l)\leq 0$. Due to the submodularity in period $n+1$, we have $(1-\lambda^h-\lambda^l)\Delta_{n+1}^l(0,l+1)+\lambda^h\Delta_{n+1}^l(0,l)\geq(1-\lambda^h-\lambda^l)\Delta_{n+1}^l(1,l+1)+\lambda^h\Delta_{n+1}^l(1,l)$. Thus $\Delta_n^l(0,l+1)\geq\Delta_n^l(1,l+1)$.

<u>Corner $(0,0)$</u>. There are 3 possible cases similar to the vertical boundary.

Case 1: $\delta_n^*(1,1)\leq\delta_n^*(1,0)\leq 0$.

$$
\begin{aligned}
V_n(0,1) &= (1-\lambda^h-\lambda^l)V_{n+1}(0,1)+\lambda^h[0+V_{n+1}(0,1)]+\lambda^l[p^l+V_{n+1}(0,0)], \\
V_n(0,0) &= (1-\lambda^h-\lambda^l)V_{n+1}(0,0)+\lambda^h[0+V_{n+1}(0,0)]+\lambda^l[0+V_{n+1}(0,0)], \\
V_n(1,1) &= (1-\lambda^h-\lambda^l)V_{n+1}(1,1)+\lambda^h[p^h+V_{n+1}(0,1)]+\lambda^l[p^l+V_{n+1}(1,0)], \\
V_n(1,0) &= (1-\lambda^h-\lambda^l)V_{n+1}(1,0)+\lambda^h[p^h+V_{n+1}(0,0)]+\lambda^l[0+V_{n+1}(1,0)]. \\
\Delta_n^l(0,1) &= (1-\lambda^h-\lambda^l)\Delta_{n+1}^l(0,1)+\lambda^h\Delta_{n+1}^l(0,1)+\lambda^lp^l, \\
\Delta_n^l(1,1) &= (1-\lambda^h-\lambda^l)\Delta_{n+1}^l(1,1)+\lambda^h\Delta_{n+1}^l(0,1)+\lambda^lp^l.
\end{aligned}
$$

Due to the submodularity in period $n+1$, we have $\Delta_n^l(0,1)\geq\Delta_n^l(1,1)$ through term-by-term comparisons.

Case 2: $\delta_n^*(1,1)\leq 0<\delta_n^*(1,0)$.

$$
\begin{aligned}
V_n(0,1) &= (1-\lambda^h-\lambda^l)V_{n+1}(0,1)+\lambda^h[0+V_{n+1}(0,1)]+\lambda^l[p^l+V_{n+1}(0,0)], \\
V_n(0,0) &= (1-\lambda^h-\lambda^l)V_{n+1}(0,0)+\lambda^h[0+V_{n+1}(0,0)]+\lambda^l[0+V_{n+1}(0,0)], \\
V_n(1,1) &= (1-\lambda^h-\lambda^l)V_{n+1}(1,1)+\lambda^h[p^h+V_{n+1}(0,1)]+\lambda^l[p^l+V_{n+1}(1,0)],
\end{aligned}
$$

$$
\begin{aligned}
V_n(1,0) &= (1-\lambda^h-\lambda^l)V_{n+1}(1,0)+\lambda^h[p^h+V_{n+1}(0,0)]+\lambda^l[0+V_{n+1}(1,0)] \\
&\quad +C\lambda\delta_n^*(1,0). \\
\Delta_n^l(0,1) &= (1-\lambda^h-\lambda^l)\Delta_{n+1}^l(0,1)+\lambda^h\Delta_{n+1}^l(0,1)+\lambda^l p^l, \\
\Delta_n^l(1,1) &= (1-\lambda^h-\lambda^l)\Delta_{n+1}^l(1,1)+\lambda^h\Delta_{n+1}^l(0,1)+\lambda^l p^l-C\lambda\delta_n^*(1,0) \\
&\le (1-\lambda^h-\lambda^l)\Delta_{n+1}^l(1,1)+\lambda^h\Delta_{n+1}^l(0,1)+\lambda^l p^l.
\end{aligned}
$$

The last inequality is from $\delta_n^*(1,0)>0$. Due to the submodularity in period $n+1$, we have $\Delta_n^l(0,1)\ge\Delta_n^l(1,1)$ through term-by-term comparisons.

Case 3: $0<\delta_n^*(1,1)\le\delta_n^*(1,0)$.

$$
\begin{aligned}
V_n(0,1) &= (1-\lambda^h-\lambda^l)V_{n+1}(0,1)+\lambda^h[0+V_{n+1}(0,1)]+\lambda^l[p^l+V_{n+1}(0,0)], \\
V_n(0,0) &= (1-\lambda^h-\lambda^l)V_{n+1}(0,0)+\lambda^h[0+V_{n+1}(0,0)]+\lambda^l[0+V_{n+1}(0,0)], \\
V_n(1,1) &= (1-\lambda^h-\lambda^l)V_{n+1}(1,1)+\lambda^h[p^h+V_{n+1}(0,1)]+\lambda^l[p^l+V_{n+1}(1,0)]+C\lambda\delta_n^*(1,1), \\
V_n(1,0) &= (1-\lambda^h-\lambda^l)V_{n+1}(1,0)+\lambda^h[p^h+V_{n+1}(0,0)]+\lambda^l[0+V_{n+1}(1,0)]+C\lambda\delta_n^*(1,0). \\
\Delta_n^l(0,1) &= (1-\lambda^h-\lambda^l)\Delta_{n+1}^l(0,1)+\lambda^h\Delta_{n+1}^l(0,1)+\lambda^l p^l, \\
\Delta_n^l(1,1) &= (1-\lambda^h-\lambda^l)\Delta_{n+1}^l(1,1)+\lambda^h\Delta_{n+1}^l(0,1)+\lambda^l p^l+C\lambda[\delta_n^*(1,1)-\delta_n^*(1,0)] \\
&\le (1-\lambda^h-\lambda^l)\Delta_{n+1}^l(1,1)+\lambda^h\Delta_{n+1}^l(0,1)+\lambda^l p^l.
\end{aligned}
$$

The last inequality is from $\delta_n^*(1,1)\le\delta_n^*(1,0)$. Due to the submodularity in period $n+1$, we have $\Delta_n^l(0,1)\ge\Delta_n^l(1,1)$ through term-by-term comparisons.

Proof of property b): $[1:H-1]\times[0:M]$ is partitioned into the interior $[2:H-1]\times[1:M]$, the horizontal boundary $[2:H-1]\times\{0\}$, the vertical boundary $\{1\}\times[1:M]$ and the corner $(1,0)$. The proof is customized for each region. The DH-modularity at state $(h,l)\in[1:H-1]\times[0:M]$ in period n can be equivalently expressed as

$$
V_n(h-1,l+1)-V_n(h,l) \;\le\; V_n(h,l+1)-V_n(h+1,l),
$$

$$
\begin{aligned}
\Delta_n(h,l) &\leq \Delta_n(h+1,l), \\
V_n(h+1,l) - V_n(h,l) &\leq V_n(h,l+1) - V_n(h-1,l+1), \\
\Delta_n^h(h+1,l) &\leq \Delta_n^h(h,l+1).
\end{aligned}
$$

We focus on the last two expressions since they are more convenient to prove. These inequalities involve states $(h+1,l)$, (h,l), $(h,l+1)$ and $(h-1,l+1)$. Specializing inequalities (A.1) for these states, we obtain

$$
p^*_{h-1,l+1} \geq p^*_{h,l+1} \geq p^*_{h,l} \geq p^*_{h+1,l} \;\text{ and }\; \delta^*_n(h-1,l+1) \leq \delta^*_n(h,l+1) \leq \delta^*_n(h,l) \leq \delta^*_n(h+1,l).
$$

Shifting 0 from the right-hand side of the last inequality to its left-hand side, we construct cases for each region.

<u>Interior $[2:H-1]\times[1:M]$</u>. There are 5 possible cases.

Case 1: $\delta^*_n(h-1,l+1) \leq \delta^*_n(h,l+1) \leq \delta^*_n(h,l) \leq \delta^*_n(h+1,l) \leq 0$.

$$
\begin{aligned}
& V_n(h-1,l+1) \\
= \;& (1-\lambda^h-\lambda^l)V_{n+1}(h-1,l+1) + \lambda^h[p^h + V_{n+1}(h-2,l+1)] + \lambda^l[p^l + V_{n+1}(h-1,l)], \\
& V_n(h,l) \\
= \;& (1-\lambda^h-\lambda^l)V_{n+1}(h,l) + \lambda^h[p^h + V_{n+1}(h-1,l)] + \lambda^l[p^l + V_{n+1}(h,l-1)], \\
& V_n(h,l+1) \\
= \;& (1-\lambda^h-\lambda^l)V_{n+1}(h,l+1) + \lambda^h[p^h + V_{n+1}(h-1,l+1)] + \lambda^l[p^l + V_{n+1}(h,l)], \\
& V_n(h+1,l) \\
= \;& (1-\lambda^h-\lambda^l)V_{n+1}(h+1,l) + \lambda^h[p^h + V_{n+1}(h,l)] + \lambda^l[p^l + V_{n+1}(h+1,l-1)].
\end{aligned}
$$

Due to the DH-modularity in period $n+1$, we have $V_n(h+1,l) - V_n(h,l) \leq V_n(h,l+1) - V_n(h-1,l+1)$ or $\Delta_n^h(h+1,l) \leq \Delta_n^h(h,l+1)$ through term-by-term comparisons.

Case 2: $\delta_n^*(h-1,l+1) \le \delta_n^*(h,l+1) \le \delta_n^*(h,l) \le 0 < \delta_n^*(h+1,l)$.

$$\begin{aligned} V_n(h-1,l+1) &= \Psi[V_{n+1}(h-1,l+1)], \\ V_n(h,l) &= \Psi[V_{n+1}(h,l)] \ge \Psi[V_{n+1}(h,l)] + C\lambda\delta_n^*(h,l) \\ &\ge \Psi[V_{n+1}(h,l)] + C\lambda\alpha(p^*_{h+1,l})[p^*_{h+1,l} + V_{n+1}(h-1,l+1) - V_{n+1}(h,l)], \end{aligned}$$

where the first inequality is due to $\delta_n^*(h,l) \le 0$ and the second inequality follows from the fact that $p^*_{h+1,l}$ is not the optimal fee for $\delta_n(p,h,l)$.

$$\begin{aligned} V_n(h,l+1) &= \Psi[V_{n+1}(h,l+1)], \\ V_n(h+1,l) &= \Psi[V_{n+1}(h+1,l)] + C\lambda\delta_n^*(h+1,l) \\ &= \Psi[V_{n+1}(h+1,l)] + C\lambda\alpha(p^*_{h+1,l})[p^*_{h+1,l} + V_{n+1}(h,l+1) - V_{n+1}(h+1,l)]. \end{aligned}$$

Using the three equalities and one inequality from above,

$$\begin{aligned} &\Delta_n^h(h+1,l) \\ \le\ & (1-\lambda^h-\lambda^l)\Delta_{n+1}^h(h+1,l) + \lambda^h\Delta_{n+1}^h(h,l) + \lambda^l\Delta_{n+1}^h(h+1,l-1) \\ & + C\lambda\alpha(p^*_{h+1,l})[V_{n+1}(h,l+1) - V_{n+1}(h-1,l+1) + V_{n+1}(h,l) - V_{n+1}(h+1,l)] \\ =\ & [1-\lambda^h-\lambda^l - C\lambda\alpha(p^*_{h+1,l})]\Delta_{n+1}^h(h+1,l) + \lambda^h\Delta_{n+1}^h(h,l) + \lambda^l\Delta_{n+1}^h(h+1,l-1) \\ & + C\lambda\alpha(p^*_{h+1,l})\Delta_{n+1}^h(h,l+1). \\ &\Delta_n^h(h,l+1) \\ =\ & (1-\lambda^h-\lambda^l)\Delta_{n+1}^h(h,l+1) + \lambda^h\Delta_{n+1}^h(h-1,l+1) + \lambda^l\Delta_{n+1}^h(h,l). \end{aligned}$$

Application of the DH-modularity in period $n+1$ shows $\lambda^h\Delta_{n+1}^h(h,l) \le \lambda^h\Delta_{n+1}^h(h-1,l+1)$ and $\lambda^l\Delta_{n+1}^h(h+1,l-1) \le \lambda^l\Delta_{n+1}^h(h,l)$. Hence, for $\Delta_n^h(h+1,l) \le \Delta_{n+1}^h(h,l+1)$, it is sufficient to prove

$$\begin{aligned} & [1-\lambda^h-\lambda^l - C\lambda\alpha(p^*_{h+1,l})]\Delta_{n+1}^h(h+1,l) + C\lambda\alpha(p^*_{h+1,l})\Delta_{n+1}^h(h,l+1) \\ \le\ & (1-\lambda^h-\lambda^l)\Delta_{n+1}^h(h,l+1). \end{aligned}$$

This also follows from the DH-modularity in period $n+1$: $[1-\lambda^h-\lambda^l-C\lambda\alpha(p^*_{h+1,l})]\Delta^h_{n+1}(h+1,l) \le (1-\lambda^h-\lambda^l-C\lambda\alpha(p^*_{h+1,l}))\Delta^h_{n+1}(h,l+1)$. Hence, $\Delta^h_n(h+1,l) \le \Delta^h_{n+1}(h,l+1)$ and the proof for this case is complete.

Case 3: $\delta^*_n(h-1,l+1) \le \delta^*_n(h,l+1) \le 0 < \delta^*_n(h,l) \le \delta^*_n(h+1,l)$.

$$
\begin{aligned}
V_n(h-1,l+1) &= \Psi[V_{n+1}(h-1,l+1)],\\
V_n(h,l) &= \Psi[V_{n+1}(h,l)] + C\lambda\delta^*_n(h,l)\\
&\ge \Psi[V_{n+1}(h,l)] + C\lambda\alpha(p^*_{h+1,l})[p^*_{h+1,l} + V_{n+1}(h-1,l+1) - V_{n+1}(h,l)],\\
V_n(h,l+1) &= \Psi[V_{n+1}(h,l+1)],\\
V_n(h+1,l) &= \Psi[V_{n+1}(h+1,l) + C\lambda\alpha(p^*_{h+1,l})[p^*_{h+1,l} + V_{n+1}(h,l+1) - V_{n+1}(h+1,l)].
\end{aligned}
$$

We further have

$$
\begin{aligned}
&\Delta^h_n(h+1,l)\\
\le\ & (1-\lambda^h-\lambda^l)\Delta^h_{n+1}(h+1,l) + \lambda^h\Delta^h_{n+1}(h,l) + \lambda^l\Delta^h_{n+1}(h+1,l-1)\\
&+C\lambda\alpha(p^*_{h+1,l})[V_{n+1}(h,l+1) - V_{n+1}(h-1,l+1) + V_{n+1}(h,l) - V_{n+1}(h+1,l)]\\
=\ & [1-\lambda^h-\lambda^l-C\lambda\alpha(p^*_{h+1,l})]\Delta^h_{n+1}(h+1,l) + \lambda^h\Delta^h_{n+1}(h,l) + \lambda^l\Delta^h_{n+1}(h+1,l-1)\\
&+C\lambda\alpha(p^*_{h+1,l})\Delta^h_{n+1}(h,l+1),\\
&\Delta^h_n(h,l+1)\\
=\ & (1-\lambda^h-\lambda^l)\Delta^h_{n+1}(h,l+1) + \lambda^h\Delta^h_{n+1}(h-1,l+1) + \lambda^l\Delta^h_{n+1}(h,l).
\end{aligned}
$$

Then the proof of $\Delta^h_n(h+1,l) \le \Delta^h_{n+1}(h,l+1)$ is identical to that of Case 2.

Case 4: $\delta^*_n(h-1,l+1) \le 0 < \delta^*_n(h,l+1) \le \delta^*_n(h,l) \le \delta^*_n(h+1,l)$.

$$
\begin{aligned}
V_n(h-1,l+1) &= \Psi[V_{n+1}(h-1,l+1)],\\
V_n(h,l) &= \Psi[V_{n+1}(h,l)] + C\lambda\delta^*_n(h,l)\\
&\ge \Psi[V_{n+1}(h,l)] + C\lambda\alpha(p^*_{h+1,l})[p^*_{h+1,l} + V_{n+1}(h-1,l+1) - V_{n+1}(h,l)],
\end{aligned}
$$

$$
\begin{aligned}
V_n(h,l+1) &= \Psi[V_{n+1}(h,l+1)] + C\lambda\delta_n^*(h,l+1) \\
&\geq \Psi[V_{n+1}(h,l+1)], \\
V_n(h+1,l) &= \Psi[V_{n+1}(h+1,l)] + C\lambda\alpha(p_{h+1,l}^*)[p_{h+1,l}^* + V_{n+1}(h,l+1) - V_{n+1}(h+1,l)].
\end{aligned}
$$

The second inequality is from $\delta_n^*(h,l+1) > 0$. Then,

$$
\begin{aligned}
\Delta_n^h(h+1,l) &\leq [1-\lambda^h-\lambda^l-C\lambda\alpha(p_{h+1,l}^*)]\Delta_{n+1}^h(h+1,l) + \lambda^h\Delta_{n+1}^h(h,l) \\
&\quad +\lambda^l\Delta_{n+1}^h(h+1,l-1) + C\lambda\alpha(p_{h+1,l}^*)\Delta_{n+1}^h(h,l+1), \\
\Delta_n^h(h,l+1) &\geq (1-\lambda^h-\lambda^l)\Delta_{n+1}^h(h,l+1) + \lambda^h\Delta_{n+1}^h(h-1,l+1) + \lambda^l\Delta_{n+1}^h(h,l).
\end{aligned}
$$

Then the proof of $\Delta_n^h(h+1,l) \leq \Delta_{n+1}^h(h,l+1)$ is identical to that of Case 2.

Case 5: $0 < \delta_n^*(h-1,l+1) \leq \delta_n^*(h,l+1) \leq \delta_n^*(h,l) \leq \delta_n^*(h+1,l)$.

$$
\begin{aligned}
V_n(h-1,l+1) &= \Psi[V_{n+1}(h-1,l+1)] + C\lambda\delta_n^*(h-1,l+1), \\
V_n(h,l) &= \Psi[V_{n+1}(h,l)] + C\lambda\delta_n^*(h,l) \\
&\geq \Psi[V_{n+1}(h,l)] + C\lambda\alpha(p_{h+1,l}^*)[p_{h+1,l}^* + V_{n+1}(h-1,l+1) - V_{n+1}(h,l)], \\
V_n(h,l+1) &= \Psi[V_{n+1}(h,l+1)] + C\lambda\delta_n^*(h,l+1), \\
V_n(h+1,l) &= \Psi[V_{n+1}(h+1,l)] + C\lambda\alpha(p_{h+1,l}^*)[p_{h+1,l}^* + V_{n+1}(h,l+1) - V_{n+1}(h+1,l)].
\end{aligned}
$$

$$
\begin{aligned}
\Delta_n^h(h+1,l) &\leq [1-\lambda^h-\lambda^l-C\lambda\alpha(p_{h+1,l}^*)]\Delta_{n+1}^h(h+1,l) + \lambda^h\Delta_{n+1}^h(h,l) \\
&\quad +\lambda^l\Delta_{n+1}^h(h+1,l-1) + C\lambda\alpha(p_{h+1,l}^*)\Delta_{n+1}^h(h,l+1), \\
\Delta_n^h(h,l+1) &= (1-\lambda^h-\lambda^l)\Delta_{n+1}^h(h,l+1) + \lambda^h\Delta_{n+1}^h(h-1,l+1) + \lambda^l\Delta_{n+1}^h(h,l) \\
&\quad +C\lambda[\delta_n^*(h,l+1) - \delta_n^*(h-1,l+1)] \\
&\geq (1-\lambda^h-\lambda^l)\Delta_{n+1}^h(h,l+1) + \lambda^h\Delta_{n+1}^h(h-1,l+1) + \lambda^l\Delta_{n+1}^h(h,l).
\end{aligned}
$$

The last inequality is from $\delta_n^*(h,l+1) \geq \delta_n^*(h-1,l+1)$. Then the proof of $\Delta_n^h(h+1,l) \leq \Delta_{n+1}^h(h,l+1)$ is identical to that of Case 2.

Horizontal boundary $[2 : H-1] \times \{0\}$. Exactly the same 5 cases of the interior are used.

Case 1: $\delta_n^*(h-1,1) \leq \delta_n^*(h,1) \leq \delta_n^*(h,0) \leq \delta_n^*(h+1,0) \leq 0$.

$$\begin{aligned}
V_n(h-1,1) &= (1-\lambda^h-\lambda^l)V_{n+1}(h-1,1)+\lambda^h[p^h+V_{n+1}(h-2,1)] \\
&\quad +\lambda^l[p^l+V_{n+1}(h-1,0)], \\
V_n(h,0) &= (1-\lambda^h)V_{n+1}(h,0)+\lambda^h[p^h+V_{n+1}(h-1,0)], \\
V_n(h,1) &= (1-\lambda^h-\lambda^l)V_{n+1}(h,1)+\lambda^h[p^h+V_{n+1}(h-1,1)]+\lambda^l[p^l+V_{n+1}(h,0)], \\
V_n(h+1,0) &= (1-\lambda^h)V_{n+1}(h+1,0)+\lambda^h[p^h+V_{n+1}(h,0)].
\end{aligned}$$

$$\begin{aligned}
\Delta_n^h(h+1,0) &= (1-\lambda^h)\Delta_{n+1}^h(h+1,0)+\lambda^h\Delta_{n+1}^h(h,0), \\
\Delta_n^h(h,1) &= (1-\lambda^h-\lambda^l)\Delta_{n+1}^h(h,1)+\lambda^h\Delta_{n+1}^h(h-1,1)+\lambda^l\Delta_{n+1}^h(h,0) \\
&= (1-\lambda^h)\Delta_{n+1}^h(h,1)+\lambda^h\Delta_{n+1}^h(h-1,1)+\lambda^l[\Delta_{n+1}^h(h,0)-\Delta_{n+1}^h(h,1)] \\
&\geq (1-\lambda^h)\Delta_{n+1}^h(h,1)+\lambda^h\Delta_{n+1}^h(h-1,1).
\end{aligned}$$

The inequality is from the submodularity in period $n+1$: $\Delta_{n+1}^h(h,0) \geq \Delta_{n+1}^h(h,1)$. Due to DH-modularity in period $n+1$, we have $\Delta_n^h(h+1,0) \leq \Delta_n^h(h,1)$ through term-by-term comparisons.

Case 2: $\delta_n^*(h-1,1) \leq \delta_n^*(h,1) \leq \delta_n^*(h,0) \leq 0 < \delta_n^*(h+1,0)$.

$$\begin{aligned}
V_n(h-1,1) &= (1-\lambda^h-\lambda^l)V_{n+1}(h-1,1)+\lambda^h[p^h+V_{n+1}(h-2,1)] \\
&\quad +\lambda^l[p^l+V_{n+1}(h-1,0)], \\
V_n(h,0) &= (1-\lambda^h)V_{n+1}(h,0)+\lambda^h[p^h+V_{n+1}(h-1,0)] \\
&\geq (1-\lambda^h)V_{n+1}(h,0)+\lambda^h[p^h+V_{n+1}(h-1,0)]+C\lambda\delta_n^*(h,0) \\
&\geq (1-\lambda^h)V_{n+1}(h,0)+\lambda^h[p^h+V_{n+1}(h-1,0)] \\
&\quad +C\lambda\alpha(p_{h+1,0}^*)[p_{h+1,0}^*+V_{n+1}(h-1,1)-V_{n+1}(h,0)],
\end{aligned}$$

where the first inequality is due to $\delta_n^*(h,0) \leq 0$ and the second inequality follows from the fact that $p_{h+1,0}^*$ is not the optimal fee for $\delta_n(p,h,0)$.

$$\begin{aligned}
V_n(h,1) &= (1-\lambda^h-\lambda^l)V_{n+1}(h,1) + \lambda^h[p^h + V_{n+1}(h-1,1)] + \lambda^l[p^l + V_{n+1}(h,0)], \\
V_n(h+1,0) &= (1-\lambda^h)V_{n+1}(h+1,0) + \lambda^h[p^h + V_{n+1}(h,0)] + C\lambda\delta_n^*(h+1,0) \\
&= (1-\lambda^h)V_{n+1}(h+1,0) + \lambda^h[p^h + V_{n+1}(h,0)] \\
&\quad +C\lambda\alpha(p_{h+1,0}^*)[p_{h+1,0}^* + V_{n+1}(h,1) - V_{n+1}(h+1,0)],
\end{aligned}$$

Then

$$\begin{aligned}
\Delta_n^h(h+1,0) &\leq (1-\lambda^h)\Delta_{n+1}^h(h+1,0) + \lambda^h\Delta_{n+1}^h(h,0) \\
&\quad +C\lambda\alpha(p_{h+1,0}^*)[V_{n+1}(h,1) - V_{n+1}(h-1,1) + V_{n+1}(h,0) - V_{n+1}(h+1,0)] \\
&= [1-\lambda^h - C\lambda\alpha(p_{h+1,0}^*)]\Delta_{n+1}^h(h+1,0) + \lambda^h\Delta_{n+1}^h(h,0) \\
&\quad +C\lambda\alpha(p_{h+1,0}^*)[\Delta_{n+1}^h(h,1), \\
\Delta_n^h(h,1) &= (1-\lambda^h-\lambda^l)\Delta_{n+1}^h(h,1) + \lambda^h\Delta_{n+1}^h(h-1,1) + \lambda^l\Delta_{n+1}^h(h,0) \\
&= (1-\lambda^h)\Delta_{n+1}^h(h,1) + \lambda^h\Delta_{n+1}^h(h-1,1) + \lambda^l[\Delta_{n+1}^h(h,0) - \Delta_{n+1}^h(h,1)] \\
&\geq (1-\lambda^h)\Delta_{n+1}^h(h,1) + \lambda^h\Delta_{n+1}^h(h-1,1).
\end{aligned}$$

The second inequality is from the submodularity in period $n+1$: $\Delta_{n+1}^h(h,0) \geq \Delta_{n+1}^h(h,1)$. Application of the DH-modularity in period $n+1$ yields $\lambda^h\Delta_{n+1}^h(h,0) \leq \lambda^h\Delta_{n+1}^h(h-1,1)$. Hence, it is sufficient to prove

$$[1-\lambda^h - C\lambda\alpha(p_{h+1,0}^*)]\Delta_{n+1}^h(h+1,0) + C\lambda\alpha(p_{h+1,0}^*)\Delta_{n+1}^h(h,1) \leq (1-\lambda^h)\Delta_{n+1}^h(h,1).$$

This also follows from the DH-modularity in period $n+1$: $[1-\lambda^h - C\lambda\alpha(p_{h+1,0}^*)]\Delta_{n+1}^h(h+1,0) \leq (1-\lambda^h - C\lambda\alpha(p_{h+1,0}^*))\Delta_{n+1}^h(h,1)$. Hence, $\Delta_n^h(h+1,0) \leq \Delta_n^h(h,1)$ and the proof for this case is complete.

Case 3: $\delta_n^*(h-1,1) \le \delta_n^*(h,1) \le 0 < \delta_n^*(h,0) \le \delta_n^*(h+1,0)$.

$$\begin{aligned} V_n(h-1,1) &= (1-\lambda^h-\lambda^l)V_{n+1}(h-1,1) + \lambda^h[p^h + V_{n+1}(h-2,1)] \\ &\quad +\lambda^l[p^l + V_{n+1}(h-1,0)], \end{aligned}$$

$$\begin{aligned} V_n(h,0) &= (1-\lambda^h)V_{n+1}(h,0) + \lambda^h[p^h + V_{n+1}(h-1,0)] + C\lambda\delta_n^*(h,0) \\ &\ge (1-\lambda^h)V_{n+1}(h,0) + \lambda^h[p^h + V_{n+1}(h-1,0)] \\ &\quad +C\lambda\alpha(p^*_{h+1,0})[p^*_{h+1,0} + V_{n+1}(h-1,1) - V_{n+1}(h,0)], \\ V_n(h,1) &= (1-\lambda^h-\lambda^l)V_{n+1}(h,1) + \lambda^h[p^h + V_{n+1}(h-1,1)] + \lambda^l[p^l + V_{n+1}(h,0)], \\ V_n(h+1,0) &= (1-\lambda^h)V_{n+1}(h+1,0) + \lambda^h[p^h + V_{n+1}(h,0)] + C\lambda\delta_n^*(h+1,0) \\ &= (1-\lambda^h)V_{n+1}(h+1,0) + \lambda^h[p^h + V_{n+1}(h,0)] \\ &\quad +C\lambda\alpha(p^*_{h+1,0})[p^*_{h+1,0} + V_{n+1}(h,1) - V_{n+1}(h+1,0)]. \end{aligned}$$

Then the proof of $\Delta_n^h(h+1,0) \le \Delta_n^h(h,1)$ is identical to that of Case 2.

Case 4: $\delta_n^*(h-1,1) \le 0 < \delta_n^*(h,1) \le \delta_n^*(h,0) \le \delta_n^*(h+1,0)$.

$$\begin{aligned} V_n(h-1,1) &= (1-\lambda^h-\lambda^l)V_{n+1}(h-1,1) + \lambda^h[p^h + V_{n+1}(h-2,1)] \\ &\quad +\lambda^l[p^l + V_{n+1}(h-1,0)], \\ V_n(h,0) &= (1-\lambda^h)V_{n+1}(h,0) + \lambda^h[p^h + V_{n+1}(h-1,0)] + C\lambda\delta_n^*(h,0) \\ &\ge (1-\lambda^h)V_{n+1}(h,0) + \lambda^h[p^h + V_{n+1}(h-1,0)] \\ &\quad +C\lambda\alpha(p^*_{h+1,0})[p^*_{h+1,0} + V_{n+1}(h-1,1) - V_{n+1}(h,0)], \\ V_n(h,1) &= (1-\lambda^h-\lambda^l)V_{n+1}(h,1) + \lambda^h[p^h + V_{n+1}(h-1,1)] + \lambda^l[p^l + V_{n+1}(h,0)] \\ &\quad +C\lambda\delta_n^*(h,1) \\ &\ge (1-\lambda^h-\lambda^l)V_{n+1}(h,1) + \lambda^h[p^h + V_{n+1}(h-1,1)] + \lambda^l[p^l + V_{n+1}(h,0)], \\ V_n(h+1,0) &= (1-\lambda^h)V_{n+1}(h+1,0) + \lambda^h[p^h + V_{n+1}(h,0)] + C\lambda\delta_n^*(h+1,0) \\ &= (1-\lambda^h)V_{n+1}(h+1,0) + \lambda^h[p^h + V_{n+1}(h,0)] \\ &\quad +C\lambda\alpha(p^*_{h+1,0})[p^*_{h+1,0} + V_{n+1}(h,1) - V_{n+1}(h+1,0)]. \end{aligned}$$

The second inequality is due to $\delta_n^*(h,1) > 0$. Then the proof of $\Delta_n^h(h+1,0) \le \Delta_n^h(h,1)$ is identical to that of Case 2 .

Case 5: $0 < \delta_n^*(h-1,1) \le \delta_n^*(h,1) \le \delta_n^*(h,0) \le \delta_n^*(h+1,0)$.

$$
\begin{aligned}
V_n(h-1,1) &= (1-\lambda^h-\lambda^l)V_{n+1}(h-1,1) + \lambda^h[p^h + V_{n+1}(h-2,1)] \\
&\quad +\lambda^l[p^l + V_{n+1}(h-1,0)] + C\lambda\delta_n^*(h-1,1), \\
V_n(h,0) &= (1-\lambda^h)V_{n+1}(h,0) + \lambda^h[p^h + V_{n+1}(h-1,0)] + C\lambda\delta_n^*(h,0) \\
&\geq (1-\lambda^h)V_{n+1}(h,0) + \lambda^h[p^h + V_{n+1}(h-1,0)] \\
&\quad +C\lambda\alpha(p_{h+1,0}^*)[p_{h+1,0}^* + V_{n+1}(h-1,1) - V_{n+1}(h,0)], \\
V_n(h,1) &= (1-\lambda^h-\lambda^l)V_{n+1}(h,1) + \lambda^h[p^h + V_{n+1}(h-1,1)] + \lambda^l[p^l + V_{n+1}(h,0)] \\
&\quad +C\lambda\delta_n^*(h,1), \\
V_n(h+1,0) &= (1-\lambda^h)V_{n+1}(h+1,0) + \lambda^h[p^h + V_{n+1}(h,0)] + C\lambda\delta_n^*(h+1,0) \\
&= (1-\lambda^h)V_{n+1}(h+1,0) + \lambda^h[p^h + V_{n+1}(h,0)] \\
&\quad +C\lambda\alpha(p_{h+1,0}^*)[p_{h+1,0}^* + V_{n+1}(h,1) - V_{n+1}(h+1,0)].
\end{aligned}
$$

$$
\begin{aligned}
\Delta_n^h(h+1,0) &\leq (1-\lambda^h)\Delta_{n+1}^h(h+1,0) + \lambda^h\Delta_{n+1}^h(h,0) \\
&\quad +C\lambda\alpha(p_{h+1,0}^*)[V_{n+1}(h,1) - V_{n+1}(h-1,1) + V_{n+1}(h,0) - V_{n+1}(h+1,0)] \\
&= [1-\lambda^h - C\lambda\alpha(p_{h+1,0}^*)]\Delta_{n+1}^h(h+1,0) + \lambda^h\Delta_{n+1}^h(h,0) \\
&\quad +C\lambda\alpha(p_{h+1,0}^*)[\Delta_{n+1}^h(h,1), \\
\Delta_n^h(h,1) &= (1-\lambda^h-\lambda^l)\Delta_{n+1}^h(h,1) + \lambda^h\Delta_{n+1}^h(h-1,1) + \lambda^l\Delta_{n+1}^h(h,0) \\
&\quad +C\lambda[\delta_n^*(h,1) - \delta_n^*(h-1,1)] \\
&= (1-\lambda^h)\Delta_{n+1}^h(h,1) + \lambda^h\Delta_{n+1}^h(h-1,1) + \lambda^l[\Delta_{n+1}^h(h,0) - \Delta_{n+1}^h(h,1)] \\
&\quad +C\lambda[\delta_n^*(h,1) - \delta_n^*(h-1,1)] \\
&\geq (1-\lambda^h)\Delta_{n+1}^h(h,1) + \lambda^h\Delta_{n+1}^h(h-1,1).
\end{aligned}
$$

The last inequality is from the submodularity $\Delta^h_{n+1}(h,0) \geq \Delta^h_{n+1}(h,1)]$ and $\delta^*_n(h,1) - \delta^*_n(h-1,1) \geq 0$. Then the proof of $\Delta^h_n(h+1,0) \leq \Delta^h_n(h,1)$ is identical to that of Case 2.

<u>Vertical boundary $\{1\} \times [1:M]$</u>. When $h-1=0$, $\delta^*_n(h-1,l+1)$ is not defined as there is no premium products to offer upgrades. There are 4 possible cases similar to the first 4 cases in the interior. In addition to the DH-modularity and submodularity properties, the arguments also need Lemma 2. Lemma 2 and the submodularity property together imply $\Delta^h_{n+1}(1,l) \leq p^h$.

Case 1: $\delta^*_n(1,l+1) \leq \delta^*_n(1,l) \leq \delta^*_n(2,l) \leq 0$.

$$\begin{aligned}
V_n(0,l+1) &= (1-\lambda^l)V_{n+1}(0,l+1) + \lambda^l[p^l + V_{n+1}(0,l)] \\
&= (1-\lambda^h-\lambda^l)V_{n+1}(0,l+1) + \lambda^h V_{n+1}(0,l+1) + \lambda^l[p^l + V_{n+1}(0,l)], \\
V_n(1,l) &= (1-\lambda^h-\lambda^l)V_{n+1}(1,l) + \lambda^h[p^h + V_{n+1}(0,l)] + \lambda^l[p^l + V_{n+1}(1,l-1)], \\
V_n(1,l+1) &= (1-\lambda^h-\lambda^l)V_{n+1}(1,l+1) + \lambda^h[p^h + V_{n+1}(0,l+1)] + \lambda^l[p^l + V_{n+1}(1,l)], \\
V_n(2,l) &= (1-\lambda^h-\lambda^l)V_{n+1}(2,l) + \lambda^h[p^h + V_{n+1}(1,l)] + \lambda^l[p^l + V_{n+1}(2,l-1)].
\end{aligned}$$

$$\begin{aligned}
\Delta^h_n(2,l) &= (1-\lambda^h-\lambda^l)\Delta^h_{n+1}(2,l) + \lambda^h\Delta^h_{n+1}(1,l) + \lambda^l\Delta^h_{n+1}(2,l-1), \\
\Delta^h_n(1,l+1) &= (1-\lambda^h-\lambda^l)\Delta^h_{n+1}(1,l+1) + \lambda^h p^h + \lambda^l\Delta^h_{n+1}(1,l).
\end{aligned}$$

From the DH-modularity in period $n+1$ and $\Delta^h_{n+1}(1,l) \leq p^h$, we have $\Delta^h_n(2,l) \leq \Delta^h_n(1,l+1)$ through term-by-term comparisons.

Case 2: $\delta^*_n(1,l+1) \le \delta^*_n(1,l) \le 0 < \delta^*_n(2,l)$.

$$
\begin{aligned}
V_n(0,l+1) &= (1-\lambda^l)V_{n+1}(0,l+1) + \lambda^l[p^l + V_{n+1}(0,l)] \\
&= (1-\lambda^h-\lambda^l)V_{n+1}(0,l+1) + \lambda^h V_{n+1}(0,l+1) + \lambda^l[p^l + V_{n+1}(0,l)], \\
V_n(1,l) &= \Psi[V_{n+1}(1,l)] \\
&\ge \Psi[V_{n+1}(1,l)] + C\lambda\delta^*_n(1,l) \\
&\ge \Psi[V_{n+1}(1,l)] + C\lambda\alpha(p^*_{2,l})[p^*_{2,l} + V_{n+1}(0,l+1) - V_{n+1}(1,l)], \\
V_n(1,l+1) &= (1-\lambda^h-\lambda^l)V_{n+1}(1,l+1) + \lambda^h[p^h + V_{n+1}(0,l+1)] + \lambda^l[p^l + V_{n+1}(1,l)], \\
V_n(2,l) &= \Psi[V_{n+1}(2,l)] + C\lambda\delta^*_n(2,l) \\
&= \Psi[V_{n+1}(2,l)] + C\lambda\alpha(p^*_{2,l})[p^*_{2,l} + V_{n+1}(1,l+1) - V_{n+1}(2,l)].
\end{aligned}
$$

The first inequality is due to $\delta^*_n(1,l) \le 0$ and the second inequality follows from the fact that $p^*_{2,l}$ is not the optimal fee for $\delta_n(p,1,l)$.

$$
\begin{aligned}
\Delta^h_n(2,l) &\le (1-\lambda^h-\lambda^l)\Delta^h_{n+1}(2,l) + \lambda^h\Delta^h_{n+1}(1,l) + \lambda^l\Delta^h_{n+1}(2,l-1) \\
&\quad + C\lambda\alpha(p^*_{2,l})[V_{n+1}(1,l+1) - V_{n+1}(0,l+1) + V_{n+1}(1,l)] - V_{n+1}(2,l)] \\
&= [1-\lambda^h-\lambda^l - C\lambda\alpha(p^*_{2,1})]\Delta^h_{n+1}(2,l) + \lambda^h\Delta^h_{n+1}(1,l) + \lambda^l\Delta^h_{n+1}(2,l-1) \\
&\quad + C\lambda\alpha(p^*_{2,l})\Delta^h_{n+1}(1,l+1), \\
\Delta^h_n(1,l+1) &= (1-\lambda^h-\lambda^l)\Delta^h_{n+1}(1,l+1) + \lambda^h p^h + \lambda^l\Delta^h_{n+1}(1,l).
\end{aligned}
$$

Application of the DH-modularity in period $n+1$ to the multiplier of λ^l and $\Delta^h_{n+1}(1,l) \le p^h$ show that it is sufficient to prove

$$
\begin{aligned}
&[1-\lambda^h-\lambda^l - C\lambda\alpha(p^*_{2,1})]\Delta^h_{n+1}(2,l) + C\lambda\alpha(p^*_{2,l})\Delta^h_{n+1}(1,l+1) \\
&\le (1-\lambda^h-\lambda^l)\Delta^h_{n+1}(1,l+1).
\end{aligned}
$$

This also follows from the DH-modularity in period $n+1$:

$$
[1-\lambda^h-\lambda^l - C\lambda\alpha(p^*_{2,1})]\Delta^h_{n+1}(2,l) \le [1-\lambda^h-\lambda^l - C\lambda\alpha(p^*_{2,1})]\Delta^h_{n+1}(1,l+1).
$$

Hence, $\Delta_n^h(2,l) \leq \Delta_n^h(1,l+1)$ and the proof for this case is complete.

Case 3: $\delta_n^*(1,l+1) \leq 0 < \delta_n^*(1,l) \leq \delta_n^*(2,l)$.

$$\begin{aligned}
V_n(0,l+1) &= (1-\lambda^l)V_{n+1}(0,l+1) + \lambda^l[p^l + V_{n+1}(0,l)] \\
&= (1-\lambda^h-\lambda^l)V_{n+1}(0,l+1) + \lambda^h V_{n+1}(0,l+1) + \lambda^l[p^l + V_{n+1}(0,l)], \\
V_n(1,l) &= \Psi[V_{n+1}(1,l)] + C\lambda\delta_n^*(1,l) \\
&\geq \Psi[V_{n+1}(1,l)] + C\lambda\alpha(p_{2,l}^*)[p_{2,l}^* + V_{n+1}(0,l+1) - V_{n+1}(1,l)], \\
V_n(1,l+1) &= (1-\lambda^h-\lambda^l)V_{n+1}(1,l+1) + \lambda^h[p^h + V_{n+1}(0,l+1)] + \lambda^l[p^l + V_{n+1}(1,l)], \\
V_n(2,l) &= \Psi[V_{n+1}(2,l)] + C\lambda\delta_n^*(2,l) \\
&= \Psi[V_{n+1}(2,l)] + C\lambda\alpha(p_{2,l}^*)[p_{2,l}^* + V_{n+1}(1,l+1) - V_{n+1}(2,l)].
\end{aligned}$$

Then the proof of $\Delta_n^h(2,l) \leq \Delta_n^h(1,l+1)$ is identical to that of Case 2.

Case 4: $0 < \delta_n^*(1,l+1) \leq \delta_n^*(1,l) \leq \delta_n^*(2,l)$.

$$\begin{aligned}
V_n(0,l+1) &= (1-\lambda^l)V_{n+1}(0,l+1) + \lambda^l[p^l + V_{n+1}(0,l)] \\
&= (1-\lambda^h-\lambda^l)V_{n+1}(0,l+1) + \lambda^h V_{n+1}(0,l+1) + \lambda^l[p^l + V_{n+1}(0,l)], \\
V_n(1,l) &= \Psi[V_{n+1}(1,l)] + C\lambda\delta_n^*(1,l) \\
&\geq \Psi[V_{n+1}(1,l)] + C\lambda\alpha(p_{2,l}^*)[p_{2,l}^* + V_{n+1}(0,l+1) - V_{n+1}(1,l)],
\end{aligned}$$

$$\begin{aligned}
V_n(1,l+1) &= (1-\lambda^h-\lambda^l)V_{n+1}(1,l+1) + \lambda^h[p^h + V_{n+1}(0,l+1)] + \lambda^l[p^l + V_{n+1}(1,l)] \\
&\quad + C\lambda\delta_n^*(1,l+1) \\
&\geq (1-\lambda^h-\lambda^l)V_{n+1}(1,l+1) + \lambda^h[p^h + V_{n+1}(0,l+1)] + \lambda^l[p^l + V_{n+1}(1,l)], \\
V_n(2,l) &= \Psi[V_{n+1}(2,l)] + C\lambda\delta_n^*(2,l) \\
&= \Psi[V_{n+1}(2,l)] + C\lambda\alpha(p_{2,l}^*)[p_{2,l}^* + V_{n+1}(1,l+1) - V_{n+1}(2,l)].
\end{aligned}$$

Then the proof of $\Delta_n^h(2,l) \leq \Delta_n^h(1,l+1)$ is identical to that of Case 2.

Corner $(1,0)$. There are 4 possible cases similar to the 4 cases in the vertical boundary. $\delta_n^*(0,1)$ is not defined, since upgrades cannot be offered when there is no premium product. In addition to the DH-modularity and submodularity, the arguments also need Lemma 2.

Case 1: $\delta_n^*(1,1) \leq \delta_n^*(1,0) \leq \delta_n^*(2,0) \leq 0$.

$$
\begin{aligned}
V_n(0,1) &= (1-\lambda^l)V_{n+1}(0,1) + \lambda^l[p^l + V_{n+1}(0,0)] \\
&= (1-\lambda^h)V_{n+1}(0,1) + \lambda^h V_{n+1}(0,1) + \lambda^l[p^l - V_{n+1}(0,1) + V_{n+1}(0,0)], \\
V_n(1,0) &= (1-\lambda^h)V_{n+1}(1,0) + \lambda^h[p^h + V_{n+1}(0,0)], \\
V_n(1,1) &= (1-\lambda^h-\lambda^l)V_{n+1}(1,1) + \lambda^h[p^h + V_{n+1}(0,1)] + \lambda^l[p^l + V_{n+1}(1,0)] \\
&= (1-\lambda^h)V_{n+1}(1,1) + \lambda^h[p^h + V_{n+1}(0,1)] + \lambda^l[p^l - V_{n+1}(1,1) + V_{n+1}(1,0)], \\
V_n(2,0) &= (1-\lambda^h)V_{n+1}(2,0) + \lambda^h[p^h + V_{n+1}(1,0)].
\end{aligned}
$$

$$
\begin{aligned}
\Delta_n^h(2,0) &= (1-\lambda^h)\Delta_{n+1}^h(2,0) + \lambda^h\Delta_{n+1}^h(1,0), \\
\Delta_n^h(1,1) &= (1-\lambda^h)\Delta_{n+1}^h(1,1) + \lambda^h p^h + \lambda^l[\Delta_{n+1}^l(0,1) - \Delta_{n+1}^l(1,1)] \\
&\geq (1-\lambda^h)\Delta_{n+1}^h(1,1) + \lambda^h p^h,
\end{aligned}
$$

where the last inequality is from the submodularity $\Delta_{n+1}^l(0,1) \geq \Delta_{n+1}^l(1,1)$. Then, $\Delta_n^h(2,0) \leq \Delta_n^h(1,1)$ is implied by $\Delta_{n+1}^h(1,0) \leq p^h$ through term-by-term comparisons.

Case 2: $\delta_n^*(1,1) \le \delta_n^*(1,0) \le 0 < \delta_n^*(2,0)$.

$$\begin{aligned}
V_n(0,1) &= (1-\lambda^h)V_{n+1}(0,1) + \lambda^h V_{n+1}(0,1) + \lambda^l[p^l - V_{n+1}(0,1) + V_{n+1}(0,0)], \\
V_n(1,0) &= (1-\lambda^h)V_{n+1}(1,0) + \lambda^h[p^h + V_{n+1}(0,0)] \\
&\ge (1-\lambda^h)V_{n+1}(1,0) + \lambda^h[p^h + V_{n+1}(0,0)] + C\lambda\delta_n^*(1,0) \\
&\ge (1-\lambda^h)V_{n+1}(1,0) + \lambda^h[p^h + V_{n+1}(0,0)] \\
&\quad + C\lambda\alpha(p_{2,0}^*)[p_{2,0}^* + V_{n+1}(0,1) - V_{n+1}(1,0)], \\
V_n(1,1) &= (1-\lambda^h)V_{n+1}(1,1) + \lambda^h[p^h + V_{n+1}(0,1)] + \lambda^l[p^l - V_{n+1}(1,1) + V_{n+1}(1,0)], \\
V_n(2,0) &= (1-\lambda^h)V_{n+1}(2,0) + \lambda^h[p^h + V_{n+1}(1,0)] + C\lambda\delta_n^*(2,0) \\
&= (1-\lambda^h)V_{n+1}(2,0) + \lambda^h[p^h + V_{n+1}(1,0)] \\
&\quad + C\lambda\alpha(p_{2,0}^*)[p_{2,0}^* + V_{n+1}(1,1) - V_{n+1}(2,0)].
\end{aligned}$$

The first inequality is due to $\delta_n^*(1,0) \le 0$ and the second inequality follows from the fact that $p_{2,0}^*$ is not the optimal fee for $\delta_n(p,1,0)$.

$$\begin{aligned}
\Delta_n^h(2,0) &\le (1-\lambda^h)\Delta_{n+1}^h(2,0) + \lambda^h\Delta_{n+1}^h(1,0) \\
&\quad + C\lambda\alpha(p_{2,0}^*)[V_{n+1}(1,1) - V_{n+1}(0,1) + V_{n+1}(1,0) - V_{n+1}(2,0)] \\
&= [1-\lambda^h - C\lambda\alpha(p_{2,0}^*)]\Delta_{n+1}^h(2,0) + \lambda^h\Delta_{n+1}^h(1,0) + C\lambda\alpha(p_{2,0}^*)\Delta_{n+1}^h(1,1), \\
\Delta_n^h(1,1) &= (1-\lambda^h)\Delta_{n+1}^h(1,1) + \lambda^h p^h + \lambda^l[\Delta_{n+1}^l(0,1) - \Delta_{n+1}^l(1,1)] \\
&\ge (1-\lambda^h)\Delta_{n+1}^h(1,1) + \lambda^h p^h,
\end{aligned}$$

where the last inequality is from the submodularity $\Delta_{n+1}^l(0,1) \ge \Delta_{n+1}^l(1,1)$. $\Delta_{n+1}^h(1,0) \le p^h$ shows that it is sufficient to prove

$$[1-\lambda^h - C\lambda\alpha(p_{2,0}^*)]\Delta_{n+1}^h(2,0) + C\lambda\alpha(p_{2,0}^*)\Delta_{n+1}^h(1,1) \le (1-\lambda^h)\Delta_{n+1}^h(1,1).$$

This follows from the DH-modularity in period $n+1$:

$$[1-\lambda^h - C\lambda\alpha(p_{2,0}^*)]\Delta_{n+1}^h(2,0) \le [1-\lambda^h - C\lambda\alpha(p_{2,0}^*)]\Delta_{n+1}^h(1,1)..$$

Hence, $\Delta_n^h(2,0) \le \Delta_n^h(1,1)$ and the proof for this case is complete.

Case 3: $\delta_n^*(1,1) \le 0 < \delta_n^*(1,0) \le \delta_n^*(2,0)$.

$$\begin{aligned}
V_n(0,1) &= (1-\lambda^h)V_{n+1}(0,1) + \lambda^h V_{n+1}(0,1) + \lambda^l[p^l - V_{n+1}(0,1) + V_{n+1}(0,0)], \\
V_n(1,0) &= (1-\lambda^h)V_{n+1}(1,0) + \lambda^h[p^h + V_{n+1}(0,0)] + C\lambda\delta_n^*(1,0) \\
&\ge (1-\lambda^h)V_{n+1}(1,0) + \lambda^h[p^h + V_{n+1}(0,0)] \\
&\quad + C\lambda\alpha(p_{2,0}^*)[p_{2,0}^* + V_{n+1}(0,1) - V_{n+1}(1,0)], \\
V_n(1,1) &= (1-\lambda^h)V_{n+1}(1,1) + \lambda^h[p^h + V_{n+1}(0,1)] + \lambda^l[p^l - V_{n+1}(1,1) + V_{n+1}(1,0)], \\
V_n(2,0) &= (1-\lambda^h)V_{n+1}(2,0) + \lambda^h[p^h + V_{n+1}(1,0)] \\
&\quad + C\lambda\alpha(p_{2,0}^*)[p_{2,0}^* + V_{n+1}(1,1) - V_{n+1}(2,0)].
\end{aligned}$$

Then the proof of $\Delta_n^h(2,0) \le \Delta_n^h(1,1)$ is identical to that of Case 2.

Case 4: $0 < \delta_n^*(1,1) \le \delta_n^*(1,0) \le \delta_n^*(2,0)$.

$$\begin{aligned}
V_n(0,1) &= (1-\lambda^h)V_{n+1}(0,1) + \lambda^h V_{n+1}(0,1) + \lambda^l[p^l - V_{n+1}(0,1) + V_{n+1}(0,0)], \\
V_n(1,0) &\ge (1-\lambda^h)V_{n+1}(1,0) + \lambda^h[p^h + V_{n+1}(0,0)] \\
&\quad + C\lambda\alpha(p_{2,0}^*)[p_{2,0}^* + V_{n+1}(0,1) - V_{n+1}(1,0)], \\
V_n(1,1) &= (1-\lambda^h-\lambda^l)V_{n+1}(1,1) + \lambda^h[p^h + V_{n+1}(0,1)] + \lambda^l[p^l + V_{n+1}(1,0)] + C\lambda\delta_n^*(1,1) \\
&\ge (1-\lambda^h)V_{n+1}(1,1) + \lambda^h[p^h + V_{n+1}(0,1)] + \lambda^l[p^l - V_{n+1}(1,1) + V_{n+1}(1,0)], \\
V_n(2,0) &= (1-\lambda^h)V_{n+1}(2,0) + \lambda^h[p^h + V_{n+1}(1,0)] \\
&\quad + C\lambda\alpha(p_{2,0}^*)[p_{2,0}^* + V_{n+1}(1,1) - V_{n+1}(2,0)].
\end{aligned}$$

Then the proof of $\Delta_n^h(2,0) \le \Delta_n^h(1,1)$ is identical to that of Case 2.

Proof of property c): $[1:H]\times[0:M-1]$ is partitioned into the interior $[2:H]\times[1:M-1]$, the horizontal boundary $[2:H]\times\{0\}$, the vertical boundary $\{1\}\times[1:M-1]$ and the corner

$(1, 0)$. The DV-modularity at state $(h, l) \in [1 : H] \times [0 : M-1]$ in period n can be equivalently expressed as

$$\begin{aligned}
V_n(h-1, l+1) - V_n(h, l) &\geq V_n(h-1, l+2) - V_n(h, l+1), \\
\Delta_n(h, l) &\geq \Delta_n(h, l+1), \\
V_n(h, l+1) - V_n(h, l) &\geq V_n(h-1, l+2) - V_n(h-1, l+1), \\
\Delta_n^l(h, l+1) &\geq \Delta_n^l(h-1, l+2).
\end{aligned}$$

We focus on the last two expressions since they are more convenient to prove. These inequalities involve states $(h, l), (h, l+1), (h-1, l+1)$ and $(h-1, l+2)$. Specializing inequalities (A.1) for these states, we obtain

$$\begin{aligned}
& p^*_{h-1,l+2} \geq p^*_{h-1,l+1} \geq p^*_{h,l+1} \geq p^*_{h,l}, \\
& \delta_n^*(h-1, l+2) \leq \delta_n^*(h-1, l+1) \leq \delta_n^*(h, l+1) \leq \delta_n^*(h, l).
\end{aligned}$$

Cases are constructed by shifting 0 from the right-hand side of the last inequality to its left-hand side.

<u>Interior $[2 : H] \times [1 : M-1]$</u>. There are 5 cases.

Case 1: $\delta_n^*(h-1, l+2) \leq \delta_n^*(h-1, l+1) \leq \delta_n^*(h, l+1) \leq \delta_n^*(h, l) \leq 0$.

$$\begin{aligned}
V_n(h, l) &= (1 - \lambda^h - \lambda^l)V_{n+1}(h, l) + \lambda^h[p^h + V_{n+1}(h-1, l)] + \lambda^l[p^l + V_{n+1}(h, l-1)], \\
V_n(h, l+1) &= (1 - \lambda^h - \lambda^l)V_{n+1}(h, l+1) + \lambda^h[p^h + V_{n+1}(h-1, l+1)] \\
&\quad + \lambda^l[p^l + V_{n+1}(h, l)], \\
V_n(h-1, l+1) &= (1 - \lambda^h - \lambda^l)V_{n+1}(h-1, l+1) + \lambda^h[p^h + V_{n+1}(h-2, l+1)] \\
&\quad + \lambda^l[p^l + V_{n+1}(h-1, l)],
\end{aligned}$$

$$\begin{aligned}
V_n(h-1, l+2) &= (1 - \lambda^h - \lambda^l)V_{n+1}(h-1, l+2) + \lambda^h[p^h + V_{n+1}(h-2, l+2)] \\
&\quad + \lambda^l[p^l + V_{n+1}(h-1, l+1)].
\end{aligned}$$

From DV-modularity in period $n+1$ and through term-by-term comparisons, we have $V_n(h,l+1) - V_n(h,l) \geq V_n(h-1,l+2) - V_n(h-1,l+1)$ or $\Delta_n^l(h,l+1) \geq \Delta_n^l(h-1,l+2)$.

Case 2: $\delta_n^*(h-1,l+2) \leq \delta_n^*(h-1,l+1) \leq \delta_n^*(h,l+1) \leq 0 < \delta_n^*(h,l)$.

$$\begin{aligned}
V_n(h,l) &= \Psi[V_{n+1}(h,l)] + C\lambda\delta_n^*(h,l) \\
&= \Psi[V_{n+1}(h,l)] + C\lambda\alpha(p_{h,l}^*)[p_{h,l}^* + V_{n+1}(h-1,l+1) - V_{n+1}(h,l)], \\
V_n(h,l+1) &= \Psi[V_{n+1}(h,l+1)] \geq \Psi[V_{n+1}(h,l+1)] + C\lambda\delta_n^*(h,l+1) \\
&\geq \Psi[V_{n+1}(h,l+1)] + C\lambda\alpha(p_{h,l}^*)[p_{h,l}^* + V_{n+1}(h-1,l+2) - V_{n+1}(h,l+1)], \\
V_n(h-1,l+1) &= \Psi[V_{n+1}(h-1,l+1)], \\
V_n(h-1,l+2) &= \Psi[V_{n+1}(h-1,l+2)],
\end{aligned}$$

where the first inequality is due to $\delta_n^*(h,l+1) \leq 0$ and the second inequality follows from the fact that $p_{h,l}^*$ is not the optimal fee for $\delta_n(p,h,l+1)$. Then,

$$\begin{aligned}
& \Delta_n^l(h,l+1) \\
\geq\ & (1-\lambda^h-\lambda^l)\Delta_{n+1}^l(h,l+1) + \lambda^h\Delta_{n+1}^l(h-1,l+1) + \lambda^l\Delta_{n+1}^l(h,l) \\
& + C\lambda\alpha(p_{h,l}^*)[V_{n+1}(h-1,l+2) - V_{n+1}(h-1,l+1) + V_{n+1}(h,l) - V_{n+1}(h,l+1)] \\
=\ & [1-\lambda^h-\lambda^l - C\lambda\alpha(p_{h,l}^*)]\Delta_{n+1}^l(h,l+1) + \lambda^h\Delta_{n+1}^l(h-1,l+1) + \lambda^l\Delta_{n+1}^l(h,l) \\
& + C\lambda\alpha(p_{h,l}^*)\Delta_{n+1}^l(h-1,l+2), \\
& \Delta_n^l(h-1,l+2) \\
=\ & (1-\lambda^h-\lambda^l)\Delta_{n+1}^l(h-1,l+2) + \lambda^h\Delta_n^l(h-2,l+2) + \lambda^l\Delta_{n+1}^l(h-1,l+1).
\end{aligned}$$

Application of the DV-modularity in period $n+1$ to the terms multiplied by λ^h and λ^l shows that it is sufficient to prove

$$\begin{aligned}
& [1-\lambda^h-\lambda^l - C\lambda\alpha(p_{h,l}^*)]\Delta_{n+1}^l(h,l+1) + C\lambda\alpha(p_{h,l}^*)\Delta_{n+1}^l(h-1,l+2) \\
\geq\ & (1-\lambda^h-\lambda^l)\Delta_{n+1}^l(h-1,l+2).
\end{aligned}$$

This also follows from the DV-modularity in period $n+1$: $[1-\lambda^h-\lambda^l-C\lambda\alpha(p^*_{h,l})]\Delta^l_{n+1}(h,l+1) \geq [1-\lambda^h-\lambda^l-C\lambda\alpha(p^*_{h,l})]\Delta^l_{n+1}(h-1,l+2)$. Hence, $\Delta^l_n(h,l+1) \geq \Delta^l_n(h-1,l+2)$ and the proof for this case is complete.

Case 3: $\delta^*_n(h-1,l+2) \leq \delta^*_n(h-1,l+1) \leq 0 < \delta^*_n(h,l+1) \leq \delta^*_n(h,l)$.

$$
\begin{aligned}
V_n(h,l) &= \Psi[V_{n+1}(h,l)] + C\lambda\delta^*_n(h,l) \\
&= \Psi[V_{n+1}(h,l)] + C\lambda\alpha(p^*_{h,l})[p^*_{h,l} + V_{n+1}(h-1,l+1) - V_{n+1}(h,l)], \\
V_n(h,l+1) &= \Psi[V_{n+1}(h,l+1)] + C\lambda\delta^*_n(h,l+1) \\
&\geq \Psi[V_{n+1}(h,l+1] + C\lambda\alpha(p^*_{h,l})[p^*_{h,l} + V_{n+1}(h-1,l+2) - V_{n+1}(h,l+1)], \\
V_n(h-1,l+1) &= \Psi[V_{n+1}(h-1,l+1)], \\
V_n(h-1,l+2) &= \Psi[V_{n+1}(h-1,l+2)].
\end{aligned}
$$

Then the proof of $\Delta^l_n(h,l+1) \geq \Delta^l_n(h-1,l+2)$ is identical to that of Case 2.

Case 4: $\delta^*_n(h-1,l+2) \leq 0 < \delta^*_n(h-1,l+1) \leq \delta^*_n(h,l+1) \leq \delta^*_n(h,l)$.

$$
\begin{aligned}
V_n(h,l) &= \Psi[V_{n+1}(h,l)] + C\lambda\alpha(p^*_{h,l})[p^*_{h,l} + V_{n+1}(h-1,l+1) - V_{n+1}(h,l)], \\
V_n(h,l+1) &= \Psi[V_{n+1}(h,l+1)] + C\lambda\delta^*_n(h,l+1) \\
&\geq \Psi[V_{n+1}(h,l+1)] + C\lambda\alpha(p^*_{h,l})[p^*_{h,l} + V_{n+1}(h-1,l+2) - V_{n+1}(h,l+1)], \\
V_n(h-1,l+1) &= \Psi[V_{n+1}(h-1,l+1) + C\lambda\delta^*_n(h-1,l+1) \\
&\geq \Psi[V_{n+1}(h-1,l+1)], \\
V_n(h-1,l+2) &= \Psi[V_{n+1}(h-1,l+2)].
\end{aligned}
$$

The second inequality is from $\delta^*_n(h-1,l+1) > 0$. Then the proof of $\Delta^l_n(h,l+1) \geq \Delta^l_n(h-1,l+2)$ is identical to that of Case 2.

Case 5: $0 < \delta^*_n(h-1,l+2) \leq \delta^*_n(h-1,l+1) \leq \delta^*_n(h,l+1) \leq \delta^*_n(h,l)$.

$$
\begin{aligned}
V_n(h,l) &= \Psi[V_{n+1}(h,l)] + C\lambda\alpha(p^*_{h,l})[p^*_{h,l} + V_{n+1}(h-1,l+1) - V_{n+1}(h,l)], \\
V_n(h,l+1) &= \Psi[V_{n+1}(h,l+1)] + C\lambda\delta^*_n(h,l+1) \\
&\geq \Psi[V_{n+1}(h,l+1)] + C\lambda\alpha(p^*_{h,l})[p^*_{h,l} + V_{n+1}(h-1,l+2) - V_{n+1}(h,l+1)],
\end{aligned}
$$

$$
\begin{aligned}
V_n(h-1,l+1) &= \Psi[V_{n+1}(h-1,l+1)] + C\lambda\delta_n^*(h-1,l+1), \\
V_n(h-1,l+2) &= \Psi[V_{n+1}(h-1,l+2)] + C\lambda\delta_n^*(h-1,l+2).
\end{aligned}
$$

$$
\begin{aligned}
\Delta_n^l(h,l+1) &\geq [1-\lambda^h-\lambda^l-C\lambda\alpha(p_{h,l}^*)]\Delta_{n+1}^l(h,l+1) + \lambda^h\Delta_{n+1}^l(h-1,l+1) \\
&\quad +\lambda^l\Delta_{n+1}^l(h,l) + C\lambda\alpha(p_{h,l}^*)\Delta_{n+1}^l(h-1,l+2), \\
\Delta_n^l(h-1,l+2) &= (1-\lambda^h-\lambda^l)\Delta_{n+1}^l(h-1,l+2) + \lambda^h\Delta_{n+1}^l(h-2,l+2) \\
&\quad +\lambda^l\Delta_{n+1}^l(h-1,l+1) + C\lambda[\delta_n^*(h-1,l+2)-\delta_n^*(h-1,l+1)] \\
&\leq (1-\lambda^h-\lambda^l)\Delta_{n+1}^l(h-1,l+2) + \lambda^h\Delta_n^l(h-2,l+2) \\
&\quad +\lambda^l\Delta_{n+1}^l(h-1,l+1).
\end{aligned}
$$

The last inequality is from $\delta_n^*(h-1,l+2) - \delta_n^*(h-1,l+1) \leq 0$ in Inequalities (A.1). Then the proof of $\Delta_n^l(h,l+1) \geq \Delta_n^l(h-1,l+2)$ is identical to that of Case 2.

Horizontal boundary $[2:H]\times\{0\}$. Exactly the same 5 cases of the interior are used. In addition to DV-modularity in period $n+1$, the arguments also need $\Delta_{n+1}^l(h-1,1) \leq p^l$, which can be derived from the submodularity property in c) and Lemma 2.

Case 1: $\delta_n^*(h-1,2) \leq \delta_n^*(h-1,1) \leq \delta_n^*(h,1) \leq \delta_n^*(h,0) \leq 0$.

$$
\begin{aligned}
V_n(h,0) &= (1-\lambda^h)V_{n+1}(h,0) + \lambda^h[p^h + V_{n+1}(h-1,0)] \\
&= (1-\lambda^h-\lambda^l)V_{n+1}(h,0) + \lambda^h[p^h + V_{n+1}(h-1,0)] + \lambda^l V_{n+1}(h,0), \\
V_n(h,1) &= (1-\lambda^h-\lambda^l)V_{n+1}(h,1) + \lambda^h[p^h + V_{n+1}(h-1,1)] + \lambda^l[p^l + V_{n+1}(h,0)], \\
V_n(h-1,1) &= (1-\lambda^h-\lambda^l)V_{n+1}(h-1,1) + \lambda^h[p^h + V_{n+1}(h-2,1)] \\
&\quad +\lambda^l[p^l + V_{n+1}(h-1,0)], \\
V_n(h-1,2) &= (1-\lambda^h-\lambda^l)V_{n+1}(h-1,2) + \lambda^h[p^h + V_{n+1}(h-2,2)] \\
&\quad +\lambda^l[p^l + V_{n+1}(h-1,1)].
\end{aligned}
$$

$$
\begin{aligned}
\Delta_n^l(h,1) &= (1-\lambda^h-\lambda^l)\Delta_{n+1}^l(h,1) + \lambda^h\Delta_{n+1}^l(h-1,1) + \lambda^l p^l, \\
\Delta_n^l(h-1,2) &= (1-\lambda^h-\lambda^l)\Delta_{n+1}^l(h-1,2) + \lambda^h\Delta_{n+1}^l(h-2,1) + \lambda^l\Delta_{n+1}^l(h-1,1).
\end{aligned}
$$

From DV-modularity in period $n+1$ and $p^l \geq \Delta^l_{n+1}(h-1,1)$, we have $\Delta^l_n(h,1) \geq \Delta^l_n(h-1,2)$ through term-by-term comparisons.

Case 2: $\delta^*_n(h-1,2) \leq \delta^*_n(h-1,1) \leq \delta^*_n(h,1) \leq 0 < \delta^*_n(h,0)$.

$$\begin{aligned}
V_n(h,0) &= (1-\lambda^h)V_{n+1}(h,0) + \lambda^h[p^h + V_{n+1}(h-1,0)] + C\lambda\delta^*_n(h,0) \\
&= (1-\lambda^h-\lambda^l)V_{n+1}(h,0) + \lambda^h[p^h + V_{n+1}(h-1,0)] + \lambda^l V_{n+1}(h,0) \\
&\quad + C\lambda\alpha(p^*_{h,0})[p^*_{h,0} + V_{n+1}(h-1,1) - V_{n+1}(h,0)], \\
V_n(h,1) &= (1-\lambda^h-\lambda^l)V_{n+1}(h,1) + \lambda^h[p^h + V_{n+1}(h-1,1)] + \lambda^l[p^l + V_{n+1}(h,0)] \\
&\geq (1-\lambda^h-\lambda^l)V_{n+1}(h,1) + \lambda^h[p^h + V_{n+1}(h-1,1)] + \lambda^l[p^l + V_{n+1}(h,0)] \\
&\quad + C\lambda\delta^*_n(h,1) \\
&\geq (1-\lambda^h-\lambda^l)V_{n+1}(h,1) + \lambda^h[p^h + V_{n+1}(h-1,1)] + \lambda^l[p^l + V_{n+1}(h,0)] \\
&\quad + C\lambda\alpha(p^*_{h,0})[p^*_{h,0} + V_{n+1}(h-1,2) - V_{n+1}(h,1)], \\
V_n(h-1,1) &= (1-\lambda^h-\lambda^l)V_{n+1}(h-1,1) + \lambda^h[p^h + V_{n+1}(h-2,1)] \\
&\quad + \lambda^l[p^l + V_{n+1}(h-1,0)], \\
V_n(h-1,2) &= (1-\lambda^h-\lambda^l)V_{n+1}(h-1,2) + \lambda^h[p^h + V_{n+1}(h-2,2)] \\
&\quad + \lambda^l[p^l + V_{n+1}(h-1,1)],
\end{aligned}$$

where the first inequality is due to $\delta^*_n(h,1) \leq 0$ and the second inequality follows from the fact that $p^*_{h,0}$ is not the optimal fee for $\delta_n(p,h,1)$. Then,

$$\begin{aligned}
\Delta^l_n(h,1) &= (1-\lambda^h-\lambda^l)\Delta^l_{n+1}(h,1) + \lambda^h\Delta^l_{n+1}(h-1,1) + \lambda^l p^l \\
&\quad + C\lambda\alpha(p^*_{h,0})[V_{n+1}(h-1,2) - V_{n+1}(h-1,1) + V_{n+1}(h,0) - V_{n+1}(h,1)] \\
&= [1-\lambda^h-\lambda^l - C\lambda\alpha(p^*_{h,0})]\Delta^l_{n+1}(h,1) + \lambda^h\Delta^l_{n+1}(h-1,1) + \lambda^l p^l \\
&\quad + C\lambda\alpha(p^*_{h,0})\Delta^l_{n+1}(h-1,2), \\
\Delta^l_n(h-1,2) &= (1-\lambda^h-\lambda^l)\Delta^l_{n+1}(h-1,2) + \lambda^h\Delta^l_{n+1}(h-2,2) + \lambda^l\Delta^l_{n+1}(h-1,1).
\end{aligned}$$

Application of the DV-modularity in period $n+1$ to the term multiplied by λ^h gives $\lambda^h\Delta^l_{n+1}(h-1,1) \geq \lambda^h\Delta^l_{n+1}(h-2,2)$. We also have $p^l \geq \Delta^l_{n+1}(h-1,1)$. Then, we only need to prove

$$\begin{aligned} & [1-\lambda^h-\lambda^l-C\lambda\alpha(p^*_{h,0})]\Delta^l_{n+1}(h,1)+C\lambda\alpha(p^*_{h,0})\Delta^l_{n+1}(h-1,2) \\ \geq \; & (1-\lambda^h-\lambda^l)\Delta^l_{n+1}(h-1,2). \end{aligned}$$

This also follows from the DV-modularity in period $n+1$: $[1-\lambda^h-\lambda^l-C\lambda\alpha(p^*_{h,0})]\Delta^l_{n+1}(h,1) \geq [1-\lambda^h-\lambda^l-C\lambda\alpha(p^*_{h,0})]\Delta^l_{n+1}(h-1,2)$. Hence, $\Delta^l_n(h,1) \geq \Delta^l_n(h-1,2)$ and the proof for this case is complete.

Case 3: $\delta^*_n(h-1,2) \leq \delta^*_n(h-1,1) \leq 0 < \delta^*_n(h,1) \leq \delta^*_n(h,0)$.

$$\begin{aligned} V_n(h,0) &= (1-\lambda^h)V_{n+1}(h,0)+\lambda^h[p^h+V_{n+1}(h-1,0)]+C\lambda\delta^*_n(h,0) \\ &= (1-\lambda^h-\lambda^l)V_{n+1}(h,0)+\lambda^h[p^h+V_{n+1}(h-1,0)]+\lambda^l V_{n+1}(h,0) \\ &\quad +C\lambda\alpha(p^*_{h,0})[p^*_{h,0}+V_{n+1}(h-1,1)-V_{n+1}(h,0)], \\ V_n(h,1) &= (1-\lambda^h-\lambda^l)V_{n+1}(h,1)+\lambda^h[p^h+V_{n+1}(h-1,1)]+\lambda^l[p^l+V_{n+1}(h,0)] \\ &\quad +C\lambda\delta^*_n(h,1) \\ &\geq (1-\lambda^h-\lambda^l)V_{n+1}(h,1)+\lambda^h[p^h+V_{n+1}(h-1,1)]+\lambda^l[p^l+V_{n+1}(h,0)] \\ &\quad +C\lambda\alpha(p^*_{h,0})[p^*_{h,0}+V_{n+1}(h-1,2)-V_{n+1}(h,1)], \\ V_n(h-1,1) &= (1-\lambda^h-\lambda^l)V_{n+1}(h-1,1)+\lambda^h[p^h+V_{n+1}(h-2,1)] \\ &\quad +\lambda^l[p^l+V_{n+1}(h-1,0)], \\ V_n(h-1,2) &= (1-\lambda^h-\lambda^l)V_{n+1}(h-1,2)+\lambda^h[p^h+V_{n+1}(h-2,2)] \\ &\quad +\lambda^l[p^l+V_{n+1}(h-1,1)]. \end{aligned}$$

Then the proof of $\Delta^l_n(h,1) \geq \Delta^l_n(h-1,2)$ is identical to that of Case 2.

Case 4: $\delta_n^*(h-1,2) < 0 \leq \delta_n^*(h-1,1) \leq \delta_n^*(h,1) \leq \delta_n^*(h,0)$.

$$
\begin{aligned}
V_n(h,0) &= (1-\lambda^h)V_{n+1}(h,0) + \lambda^h[p^h + V_{n+1}(h-1,0)] + C\lambda\delta_n^*(h,0) \\
&= (1-\lambda^h-\lambda^l)V_{n+1}(h,0) + \lambda^h[p^h + V_{n+1}(h-1,0)] + \lambda^l V_{n+1}(h,0) \\
&\quad +C\lambda\alpha(p_{h,0}^*)[p_{h,0}^* + V_{n+1}(h-1,1) - V_{n+1}(h,0)], \\
V_n(h,1) &= (1-\lambda^h-\lambda^l)V_{n+1}(h,1) + \lambda^h[p^h + V_{n+1}(h-1,1)] + \lambda^l[p^l + V_{n+1}(h,0)] \\
&\quad +C\lambda\delta_n^*(h,1) \\
&\geq (1-\lambda^h-\lambda^l)V_{n+1}(h,1) + \lambda^h[p^h + V_{n+1}(h-1,1)] + \lambda^l[p^l + V_{n+1}(h,0)] \\
&\quad +C\lambda\alpha(p_{h,0}^*)[p_{h,0}^* + V_{n+1}(h-1,2) - V_{n+1}(h,1)], \\
V_n(h-1,1) &= (1-\lambda^h-\lambda^l)V_{n+1}(h-1,1) + \lambda^h[p^h + V_{n+1}(h-2,1)] \\
&\quad +\lambda^l[p^l + V_{n+1}(h-1,0)] + C\lambda\delta_n^*(h-1,1) \\
&\geq (1-\lambda^h-\lambda^l)V_{n+1}(h-1,1) + \lambda^h[p^h + V_{n+1}(h-2,1)] \\
&\quad +\lambda^l[p^l + V_{n+1}(h-1,0)], \\
V_n(h-1,2) &= (1-\lambda^h-\lambda^l)V_{n+1}(h-1,2) + \lambda^h[p^h + V_{n+1}(h-2,2)] \\
&\quad +\lambda^l[p^l + V_{n+1}(h-1,1)].
\end{aligned}
$$

The second inequality is from $\delta_n^*(h-1,1) > 0$. Then the proof of $\Delta_n^l(h,1) \geq \Delta_n^l(h-1,2)$ is identical to that of Case 2.

Case 5: $0 < \delta_n^*(h-1,2) \leq \delta_n^*(h-1,1) \leq \delta_n^*(h,1) \leq \delta_n^*(h,0)$.

$$
\begin{aligned}
V_n(h,0) &= (1-\lambda^h)V_{n+1}(h,0) + \lambda^h[p^h + V_{n+1}(h-1,0)] + C\lambda\delta_n^*(h,0) \\
&= (1-\lambda^h-\lambda^l)V_{n+1}(h,0) + \lambda^h[p^h + V_{n+1}(h-1,0)] + \lambda^l V_{n+1}(h,0) \\
&\quad +C\lambda\alpha(p_{h,0}^*)[p_{h,0}^* + V_{n+1}(h-1,1) - V_{n+1}(h,0)], \\
V_n(h,1) &= (1-\lambda^h-\lambda^l)V_{n+1}(h,1) + \lambda^h[p^h + V_{n+1}(h-1,1)] + \lambda^l[p^l + V_{n+1}(h,0)] \\
&\quad +C\lambda\delta_n^*(h,1) \\
&\geq (1-\lambda^h-\lambda^l)V_{n+1}(h,1) + \lambda^h[p^h + V_{n+1}(h-1,1)] + \lambda^l[p^l + V_{n+1}(h,0)] \\
&\quad +C\lambda\alpha(p_{h,0}^*)[p_{h,0}^* + V_{n+1}(h-1,2) - V_{n+1}(h,1)],
\end{aligned}
$$

$$
\begin{aligned}
V_n(h-1,1) &= (1-\lambda^h-\lambda^l)V_{n+1}(h-1,1)+\lambda^h[p^h+V_{n+1}(h-2,1)] \\
&\quad +\lambda^l[p^l+V_{n+1}(h-1,0)]+C\lambda\delta_n^*(h-1,1), \\
V_n(h-1,2) &= (1-\lambda^h-\lambda^l)V_{n+1}(h-1,2)+\lambda^h[p^h+V_{n+1}(h-2,2)] \\
&\quad +\lambda^l[p^l+V_{n+1}(h-1,1)]+C\lambda\delta_n^*(h-1,2),
\end{aligned}
$$

where the first inequality follows from the fact that $p_{h,0}^*$ is not the optimal fee for $\delta_n(p,h,1)$. Then,

$$
\begin{aligned}
\Delta_n^l(h,1) &\geq (1-\lambda^h-\lambda^l)\Delta_{n+1}^l(h,1)+\lambda^h\Delta_{n+1}^l(h-1,1)+\lambda^l p^l \\
&\quad +C\lambda\alpha(p_{h,0}^*)[V_{n+1}(h-1,2)-V_{n+1}(h-1,1)+V_{n+1}(h,0)-V_{n+1}(h,1)] \\
&= [1-\lambda^h-\lambda^l-C\lambda\alpha(p_{h,0}^*)]\Delta_{n+1}^l(h,1)+\lambda^h\Delta_{n+1}^l(h-1,1) \\
&\quad +\lambda^l p^l+C\lambda\alpha(p_{h,0}^*)\Delta_{n+1}^l(h-1,2), \\
\Delta_n^l(h-1,2) &= (1-\lambda^h-\lambda^l)\Delta_{n+1}^l(h-1,2)+\lambda^h\Delta_{n+1}^l(h-2,2)+\lambda^l\Delta_{n+1}^l(h-1,1) \\
&\quad +C\lambda[\delta_n^*(h-1,2)-\delta_n^*(h-1,1)] \\
&\leq (1-\lambda^h-\lambda^l)\Delta_{n+1}^l(h-1,2)+\lambda^h\Delta_{n+1}^l(h-2,2)+\lambda^l\Delta_{n+1}^l(h-1,1).
\end{aligned}
$$

The last inequality is from $\delta_n^*(h-1,2)-\delta_n^*(h-1,1)\leq 0$ in Inequalities (A.1). Then the proof of $\Delta_n^l(h,1)\geq\Delta_n^l(h-1,2)$ is identical to that of Case 2.

<u>Vertical boundary $\{1\}\times[1:M-1]$.</u> With $h-1=0$, neither $\delta_n^*(h-1,2)$ nor $\delta_n^*(h-1,1)$ is defined as there is no premium product to offer upgrades. Remaining two δ_n^* values satisfy $\delta_n^*(h,l+1)\leq\delta_n^*(h,l)$. Hence, we have 3 cases that are similar to the first 3 cases in the interior. Arguments need the DV-modularity and the submodularity in period $n+1$.

Case 1: $\delta_n^*(h, l+1) \leq \delta_n^*(h, l) \leq 0$.

$$
\begin{aligned}
V_n(1,l) &= (1-\lambda^h-\lambda^l)V_{n+1}(1,l) + \lambda^h[p^h + V_{n+1}(0,l)] + \lambda^l[p^l + V_{n+1}(1,l-1)] \\
&= (1-\lambda^l)V_{n+1}(1,l) + \lambda^l[p^l + V_{n+1}(1,l-1)] + \lambda^h[p^h + V_{n+1}(0,l) - V_{n+1}(1,l)], \\
V_n(1,l+1) &= (1-\lambda^h-\lambda^l)V_{n+1}(1,l+1) + \lambda^h[p^h + V_{n+1}(0,l+1)] + \lambda^l[p^l + V_{n+1}(1,l)] \\
&= (1-\lambda^l)V_{n+1}(1,l+1) + \lambda^l[p^l + V_{n+1}(1,l)] \\
&\quad + \lambda^h[p^h + V_{n+1}(0,l+1) - V_{n+1}(1,l+1)], \\
V_n(0,l+1) &= (1-\lambda^l)V_{n+1}(0,l+1) + \lambda^l[p^l + V_{n+1}(0,l)], \\
V_n(0,l+2) &= (1-\lambda^l)V_{n+1}(0,l+2) + \lambda^l[p^l + V_{n+1}(0,l+1)].
\end{aligned}
$$

$$
\begin{aligned}
\Delta_n^l(1,l+1) &= (1-\lambda^l)\Delta_{n+1}^l(1,l+1) + \lambda^l\Delta_{n+1}^l(1,l) + \lambda^h[\Delta_{n+1}^h(1,l) - \Delta_{n+1}^h(1,l+1)] \\
&\geq (1-\lambda^l)\Delta_{n+1}^l(1,l+1) + \lambda^l\Delta_{n+1}^l(1,l), \\
\Delta_n^l(0,l+2) &= (1-\lambda^l)\Delta_{n+1}^l(0,l+2) + \lambda^l\Delta_{n+1}^l(0,l+1).
\end{aligned}
$$

The inequality is from the submodularity property $\Delta_{n+1}^h(1,l) \geq \Delta_{n+1}^h(1,l+1)$. Due to DV-modularity in period $n+1$, we have $\Delta_n^l(1,l+1) \geq \Delta_n^l(0,l+2)$ through term-by-term comparisons.

Case 2: $\delta_n^*(h, l+1) \le 0 < \delta_n^*(h, l)$.

$$
\begin{aligned}
V_n(1,l) &= (1-\lambda^h-\lambda^l)V_{n+1}(1,l) + \lambda^h[p^h + V_{n+1}(0,l)] + \lambda^l[p^l + V_{n+1}(1,l-1)] \\
&\quad + C\lambda\delta_n^*(1,l) \\
&= (1-\lambda^l)V_{n+1}(1,l) + \lambda^l[p^l + V_{n+1}(1,l-1)] + \lambda^h[p^h + V_{n+1}(0,l) - V_{n+1}(1,l)] \\
&\quad + C\lambda\alpha(p_{1,l}^*)[p_{1,l}^* + V_{n+1}(0,l+1) - V_{n+1}(1,l)], \\
V_n(1,l+1) &= (1-\lambda^h-\lambda^l)V_{n+1}(1,l+1) + \lambda^h[p^h + V_{n+1}(0,l+1)] + \lambda^l[p^l + V_{n+1}(1,l)] \\
&\ge (1-\lambda^l)V_{n+1}(1,l+1) + \lambda^l[p^l + V_{n+1}(1,l)] \\
&\quad + \lambda^h[p^h + V_{n+1}(0,l+1) - V_{n+1}(1,l+1)] + C\lambda\delta_n^*(1,l+1) \\
&\ge (1-\lambda^l)V_{n+1}(1,l+1) + \lambda^l[p^l + V_{n+1}(1,l)] \\
&\quad + \lambda^h[p^h + V_{n+1}(0,l+1) - V_{n+1}(1,l+1)] \\
&\quad + C\lambda\alpha(p_{1,l}^*)[p_{1,l}^* + V_{n+1}(0,l+2) - V_{n+1}(1,l+1)],
\end{aligned}
$$

$$
\begin{aligned}
V_n(0,l+1) &= (1-\lambda^l)V_{n+1}(0,l+1) + \lambda^l[p^l + V_{n+1}(0,l)], \\
V_n(0,l+2) &= (1-\lambda^l)V_{n+1}(0,l+2) + \lambda^l[p^l + V_{n+1}(0,l+1)],
\end{aligned}
$$

where the first inequality is due to $\delta_n^*(1, l+1) \le 0$ and the second inequality follows from the fact that $p_{1,l}^*$ is not the optimal fee for $\delta_n(p, 1, l+1)$. Then,

$$
\begin{aligned}
\Delta_n^l(1,l+1) &\ge (1-\lambda^l)\Delta_{n+1}^l(1,l+1) + \lambda^l\Delta_{n+1}^l(1,l) + \lambda^h[\Delta_{n+1}^h(1,l) - \Delta_{n+1}^h(1,l+1)] \\
&\quad + C\lambda\alpha(p_{1,l}^*)[V_{n+1}(0,l+2) - V_{n+1}(0,l+1) + V_{n+1}(1,l) - V_{n+1}(1,l+1)] \\
&\ge [1-\lambda^l - C\lambda\alpha(p_{1,l}^*)]\Delta_{n+1}^l(1,l+1) + \lambda^l\Delta_{n+1}^l(1,l) + C\lambda\alpha(p_{1,l}^*)\Delta_{n+1}^l(0,l+2), \\
\Delta_n^l(0,l+2) &= (1-\lambda^l)\Delta_{n+1}^l(0,l+2) + \lambda^l\Delta_{n+1}^l(0,l+1).
\end{aligned}
$$

The second inequality is from the submodularity property $\Delta_{n+1}^h(1,l) \ge \Delta_{n+1}^h(1,l+1)$. Application of the DV-modularity in period $n+1$ to the term multiplied by λ^l gives $\lambda^l\Delta_{n+1}^l(1,l) \ge \lambda^l\Delta_{n+1}^l(0,l+1)$. Then, we only need to prove

$$[1-\lambda^l - C\lambda\alpha(p_{1,l}^*)]\Delta_{n+1}^l(1,l+1) + C\lambda\alpha(p_{1,l}^*)\Delta_{n+1}^l(0,l+2) \ge (1-\lambda^l)\Delta_{n+1}^l(0,l+2).$$

This also follows from the DV-modularity in period $n+1$: $[1-\lambda^l-C\lambda\alpha(p^*_{1,l})]\Delta^l_{n+1}(1,l+1) \geq [1-\lambda^l-C\lambda\alpha(p^*_{1,l})]\Delta^l_{n+1}(0,l+2)$. Hence, $\Delta^l_n(1,l+1) \geq \Delta^l_n(0,l+2)$ and the proof for this case is complete.

Case 3: $0 < \delta^*_n(h,l+1) \leq \delta^*_n(h,l)$.

$$\begin{aligned}
V_n(1,l) &= (1-\lambda^h-\lambda^l)V_{n+1}(1,l)+\lambda^h[p^h+V_{n+1}(0,l)]+\lambda^l[p^l+V_{n+1}(1,l-1)] \\
&\quad +C\lambda\delta^*_n(1,l) \\
&= (1-\lambda^l)V_{n+1}(1,l)+\lambda^l[p^l+V_{n+1}(1,l-1)]+\lambda^h[p^h+V_{n+1}(0,l)-V_{n+1}(1,l)] \\
&\quad +C\lambda\alpha(p^*_{1,l})[p^*_{1,l}+V_{n+1}(0,l+1)-V_{n+1}(1,l)],
\end{aligned}$$

$$\begin{aligned}
V_n(1,l+1) &= (1-\lambda^l)V_{n+1}(1,l+1)+\lambda^l[p^l+V_{n+1}(1,l)] \\
&\quad +\lambda^h[p^h+V_{n+1}(0,l+1)-V_{n+1}(1,l+1)]+C\lambda\delta^*_n(1,l+1) \\
&\geq (1-\lambda^l)V_{n+1}(1,l+1)+\lambda^l[p^l+V_{n+1}(1,l)] \\
&\quad +\lambda^h[p^h+V_{n+1}(0,l+1)-V_{n+1}(1,l+1)] \\
&\quad +C\lambda\alpha(p^*_{1,l})[p^*_{1,l}+V_{n+1}(0,l+2)-V_{n+1}(1,l+1)], \\
V_n(0,l+1) &= (1-\lambda^l)V_{n+1}(0,l+1)+\lambda^l[p^l+V_{n+1}(0,l)], \\
V_n(0,l+2) &= (1-\lambda^l)V_{n+1}(0,l+2)+\lambda^l[p^l+V_{n+1}(0,l+1)].
\end{aligned}$$

Then the proof of $\Delta^l_n(1,l+1) \geq \Delta^l_n(0,l+2)$ is identical to that of Case 2.

Corner $(1,0)$. There are 3 cases, which are the same as those in the vertical boundary. The proof uses Lemma 2, the DV-modularity and submodularity.

Case 1: $\delta_n^*(1,1) \leq \delta_n^*(1,0) \leq 0$.

$$
\begin{aligned}
V_n(1,0) &= (1-\lambda^h)V_{n+1}(1,0) + \lambda^h[p^h + V_{n+1}(0,0)] \\
&= (1-\lambda^l)V_{n+1}(1,0) + \lambda^h[p^h + V_{n+1}(0,0) - V_{n+1}(1,0)] + \lambda^l V_{n+1}(1,0), \\
V_n(1,1) &= (1-\lambda^h-\lambda^l)V_{n+1}(1,1) + \lambda^h[p^h + V_{n+1}(0,1)] + \lambda^l[p^l + V_{n+1}(1,0)] \\
&= (1-\lambda^l)V_{n+1}(1,1) + \lambda^h[p^h + V_{n+1}(0,1) - V_{n+1}(1,1)] + \lambda^l[p^l + V_{n+1}(1,0)], \\
V_n(0,1) &= (1-\lambda^l)V_{n+1}(0,1) + \lambda^l[p^l + V_{n+1}(0,0)], \\
V_n(0,2) &= (1-\lambda^l)V_{n+1}(0,2) + \lambda^l[p^l + V_{n+1}(0,1)]. \\
\Delta_n^l(1,1) &= (1-\lambda^l)\Delta_{n+1}^l(1,1) + \lambda^l p^l + \lambda^h[\Delta_{n+1}^h(1,0) - \Delta_{n+1}^h(1,1)] \\
&\geq (1-\lambda^l)\Delta_{n+1}^l(1,1) + \lambda^l p^l, \\
\Delta_n^l(0,2) &= (1-\lambda^l)\Delta_{n+1}^l(0,2) + \lambda^l \Delta_{n+1}^l(0,1).
\end{aligned}
$$

The inequality follows from submodularity property c) $\Delta_{n+1}^h(1,0) \geq \Delta_{n+1}^h(1,1)$. Due to DV-modularity in period $n+1$ and Lemma 2, we have $\Delta_n^l(1,1) \geq \Delta_n^l(0,2)$ through term-by-term comparisons.

Case 2: $\delta_n^*(1,1) \leq 0 < \delta_n^*(1,0)$.

$$
\begin{aligned}
V_n(1,0) &= (1-\lambda^h)V_{n+1}(1,0) + \lambda^h[p^h + V_{n+1}(0,0)] + C\lambda\delta_n^*(1,0) \\
&= (1-\lambda^l)V_{n+1}(1,0) + \lambda^h[p^h + V_{n+1}(0,0) - V_{n+1}(1,0)] + \lambda^l V_{n+1}(1,0) \\
&\quad + C\lambda\alpha(p_{1,0}^*)[p_{1,0}^* + V_{n+1}(0,1) - V_{n+1}(1,0)], \\
V_n(1,1) &= (1-\lambda^h-\lambda^l)V_{n+1}(1,1) + \lambda^h[p^h + V_{n+1}(0,1)] + \lambda^l[p^l + V_{n+1}(1,0)] \\
&\geq (1-\lambda^h-\lambda^l)V_{n+1}(1,1) + \lambda^h[p^h + V_{n+1}(0,1)] + \lambda^l[p^l + V_{n+1}(1,0)] + C\lambda\delta_n^*(1,1) \\
&\geq (1-\lambda^l)V_{n+1}(1,1) + \lambda^h[p^h + V_{n+1}(0,1) - V_{n+1}(1,1)] + \lambda^l[p^l + V_{n+1}(1,0)] \\
&\quad + C\lambda\alpha(p_{1,0}^*)[p_{1,0}^* + V_{n+1}(0,2) - V_{n+1}(1,1)], \\
V_n(0,1) &= (1-\lambda^l)V_{n+1}(0,1) + \lambda^l[p^l + V_{n+1}(0,0)], \\
V_n(0,2) &= (1-\lambda^l)V_{n+1}(0,2) + \lambda^l[p^l + V_{n+1}(0,1)],
\end{aligned}
$$

where the first inequality is due to $\delta_n^*(1,1) \leq 0$ and the second inequality follows from the fact that $p_{1,0}^*$ is not the optimal fee for $\delta_n(p,1,1)$. Then,

$$\begin{aligned}
\Delta_n^l(1,1) &\geq (1-\lambda^l)\Delta_{n+1}^l(1,1) + \lambda^l p^l + \lambda^h[\Delta_{n+1}^h(1,0) - \Delta_{n+1}^h(1,1)] \\
&\quad + C\lambda\alpha(p_{1,0}^*)[V_{n+1}(0,2) - V_{n+1}(0,1) + V_{n+1}(1,0) - V_{n+1}(1,1)] \\
&\geq [1-\lambda^l - C\lambda\alpha(p_{1,0}^*)]\Delta_{n+1}^l(1,1) + \lambda^l p^l + C\lambda\alpha(p_{1,0}^*)\Delta_{n+1}^l(0,2) \\
\Delta_n^l(0,2) &= (1-\lambda^l)\Delta_{n+1}^l(0,2) + \lambda^l\Delta_{n+1}^l(0,1).
\end{aligned}$$

The inequality is from the submodularity property $\Delta_{n+1}^h(1,0) \geq \Delta_{n+1}^h(1,1)$. Due to Lemma 2, we also have $\lambda^l p^l \geq \lambda^l \Delta_{n+1}^l(0,1)$. Then, we only need to prove

$$[1-\lambda^l - C\lambda\alpha(p_{1,0}^*)]\Delta_{n+1}^l(1,1) + C\lambda\alpha(p_{1,0}^*)\Delta_{n+1}^l(0,2) \geq (1-\lambda^l)\Delta_{n+1}^l(0,2).$$

This also follows from the DV-modularity in period $n+1$: $[1-\lambda^l - C\lambda\alpha(p_{1,0}^*)]\Delta_{n+1}^l(1,1) \geq [1-\lambda^l - C\lambda\alpha(p_{1,0}^*)]\Delta_{n+1}^l(0,2)$. Hence, $\Delta_n^l(1,1) \geq \Delta_n^l(0,2)$ and the proof for this case is complete.

Case 3: $0 < \delta_n^*(1,1) \leq \delta_n^*(1,0)$.

$$\begin{aligned}
V_n(1,0) &= (1-\lambda^h)V_{n+1}(1,0) + \lambda^h[p^h + V_{n+1}(0,0)] + C\lambda\delta_n^*(1,0) \\
&= (1-\lambda^l)V_{n+1}(1,0) + \lambda^h[p^h + V_{n+1}(0,0) - V_{n+1}(1,0)] + \lambda^l V_{n+1}(1,0) \\
&\quad + C\lambda\alpha(p_{1,0}^*)[p_{1,0}^* + V_{n+1}(0,1) - V_{n+1}(1,0)], \\
V_n(1,1) &= (1-\lambda^h-\lambda^l)V_{n+1}(1,1) + \lambda^h[p^h + V_{n+1}(0,1)] + \lambda^l[p^l + V_{n+1}(1,0)] + C\lambda\delta_n^*(1,1) \\
&\geq (1-\lambda^l)V_{n+1}(1,1) + \lambda^h[p^h + V_{n+1}(0,1) - V_{n+1}(1,1)] + \lambda^l[p^l + V_{n+1}(1,0)] \\
&\quad + C\lambda\alpha(p_{1,0}^*)[p_{1,0}^* + V_{n+1}(0,2) - V_{n+1}(1,1)], \\
V_n(0,1) &= (1-\lambda^l)V_{n+1}(0,1) + \lambda^l[p^l + V_{n+1}(0,0)], \\
V_n(0,2) &= (1-\lambda^l)V_{n+1}(0,2) + \lambda^l[p^l + V_{n+1}(0,1)].
\end{aligned}$$

Then the proof of $\Delta_n^l(1,1) \geq \Delta_n^l(0,2)$ is identical to that of Case 2.

Proof of property d): $[1:H]\times[0:M+1]$ is partitioned into the interior $[2:H]\times[1:M+1]$, the horizontal boundary $[2:H]\times\{0\}$, the vertical boundary $\{1\}\times[1:M+1]$ and the corner point $(1,0)$. H-concavity in period n is $V_n(h+1,l)-V_n(h,l)\le V_n(h,l)-V_n(h-1,l)$ or $\Delta_n^h(h+1,l)\le\Delta_n^h(h,l)$. This inequality involves states $(h+1,l),(h,l)$ and $(h-1,l)$. Specializing Inequalities (A.1) for these states, we obtain $p^*_{h+1,l}\le p^*_{h,l}\le p^*_{h-1,l}$ and $\delta_n^*(h-1,l)\le\delta_n^*(h,l)\le\delta_n^*(h+1,l)$. Cases are constructed by shifting 0 from the right-hand side of the second inequality to its left-hand side.

Interior $[2:H]\times[1:M+1]$. There are 4 possible cases.

Case 1: $\delta_n^*(h-1,l)\le\delta_n^*(h,l)\le\delta_n^*(h+1,l)\le 0$.

$$\begin{aligned}
V_n(h+1,l) &= (1-\lambda^h-\lambda^l)V_{n+1}(h+1,l)+\lambda^h[p^h+V_{n+1}(h,l)]\\
&\quad +\lambda^l[p^l+V_{n+1}(h+1,l-1)],\\
V_n(h,l) &= (1-\lambda^h-\lambda^l)V_{n+1}(h,l)+\lambda^h[p^h+V_{n+1}(h-1,l)]+\lambda^l[p^l+V_{n+1}(h,l-1)],\\
V_n(h-1,l) &= (1-\lambda^h-\lambda^l)V_{n+1}(h-1,l)+\lambda^h[p^h+V_{n+1}(h-2,l)]\\
&\quad +\lambda^l[p^l+V_{n+1}(h-1,l-1)].
\end{aligned}$$

The H-concavity in period $n+1$ and term-by-term comparisons of the value functions above yield $\Delta_n^h(h+1,l)\le\Delta_n^h(h,l)$.

Case 2: $\delta_n^*(h-1,l)\le\delta_n^*(h,l)\le 0<\delta_n^*(h+1,l)$.

$$\begin{aligned}
V_n(h+1,l) &= \Psi[V_{n+1}(h+1,l)]+C\lambda\delta_n^*(h+1,l)\\
&= \Psi[V_{n+1}(h+1,l)]+C\lambda\alpha(p^*_{h+1,l})[p^*_{h+1,l}+V_{n+1}(h,l+1)-V_{n+1}(h+1,l)],\\
V_n(h,l) &= \Psi[V_{n+1}(h,l)]\\
&\ge \Psi[V_{n+1}(h,l)]+C\lambda\delta_n^*(h,l)\\
&\ge \Psi[V_{n+1}(h,l)]+C\lambda\alpha(p^*_{h+1,l})[p^*_{h+1,l}+V_{n+1}(h-1,l+1)-V_{n+1}(h,l)],\\
V_n(h-1,l) &= \Psi[V_{n+1}(h-1,l)],
\end{aligned}$$

where the first inequality is from $\delta_n^*(h,l) \le 0$ and the second is from the non-optimality of $p_{h+1,l}^*$ for $\delta_n(p,h,l)$. Then,

$$\begin{aligned}
& \Delta_n^h(h+1,l) \\
\le\ & (1-\lambda^h-\lambda^l)\Delta_{n+1}^h(h+1,l) + \lambda^h\Delta_{n+1}^h(h,l) + \lambda^l\Delta_{n+1}^h(h+1,l-1) \\
& + C\lambda\alpha(p_{h+1,l}^*)[V_{n+1}(h,l+1) - V_{n+1}(h-1,l+1) + V_{n+1}(h,l) - V_{n+1}(h+1,l)] \\
=\ & [1-\lambda^h-\lambda^l - C\lambda\alpha(p_{h+1,l}^*)]\Delta_{n+1}^h(h+1,l) + \lambda^h\Delta_{n+1}^h(h,l) + \lambda^l\Delta_{n+1}^h(h+1,l-1) \\
& + C\lambda\alpha(p_{h+1,l}^*)\Delta_{n+1}^h(h,l+1), \\
& \Delta_n^h(h,l) \\
=\ & [1-\lambda^h-\lambda^l - C\lambda\alpha(p_{h+1,l}^*)]\Delta_{n+1}^h(h,l) + \lambda^h\Delta_{n+1}^h(h-1,l) + \lambda^l\Delta_{n+1}^h(h,l-1) \\
& + C\lambda\alpha(p_{h+1,l}^*)\Delta_{n+1}^h(h,l).
\end{aligned}$$

Using submodularity in period $n+1$ on the terms multiplied by $C\lambda\alpha(p_{h+1,l}^*)$ and the H-concavity in period $n+1$ on the other terms, we obtain $\Delta_n^h(h+1,l) \le \Delta_n^h(h,l)$.

Case 3: $\delta_n^*(h-1,l) \le 0 < \delta_n^*(h,l) \le \delta_n^*(h+1,l)$.

$$V_n(h+1,l) = \Psi[V_{n+1}(h+1,l)] + C\lambda\alpha(p_{h+1,l}^*)[p_{h+1,l}^* + V_{n+1}(h,l+1) - V_{n+1}(h+1,l)],$$

$$\begin{aligned}
V_n(h,l) &= \Psi[V_{n+1}(h,l) + C\lambda\delta_n^*(h,l) \\
&\ge \Psi[V_{n+1}(h,l)] + C\lambda\alpha(p_{h+1,l}^*)[p_{h+1,l}^* + V_{n+1}(h-1,l+1) - V_{n+1}(h,l)], \\
V_n(h,l) &\ge \Psi[V_{n+1}(h,l)], \\
V_n(h-1,l) &= \Psi[V_{n+1}(h-1,l)].
\end{aligned}$$

Note that two lower bounds are given for $V_n(h,l)$; the first is used to find an upper bound for $\Delta_n^h(h+1,l)$ and the second for a lower bound for $\Delta_n^h(h,l)$. By following the arguments of Case 2, we can show that the upper bound of $\Delta_n^h(h+1,l)$ is not more than the lower bound of $\Delta_n^h(h,l)$, so $\Delta_n^h(h+1,l) \le \Delta_n^h(h,l)$.

Case 4: $0 < \delta_n^*(h-1,l) \le \delta_n^*(h,l) \le \delta_n^*(h+1,l)$.

$$\begin{aligned}
V_n(h+1,l) &= \Psi[V_{n+1}(h+1,l)] + C\lambda\alpha(p^*_{h+1,l})[p^*_{h+1,l} + V_{n+1}(h,l+1) - V_{n+1}(h+1,l)],\\
V_n(h,l) &= \Psi[V_{n+1}(h,l)] + C\lambda\delta_n^*(h,l)\\
&\ge \Psi[V_{n+1}(h,l)] + C\lambda\alpha(p^*_{h+1,l})[p^*_{h+1,l} + V_{n+1}(h-1,l+1) - V_{n+1}(h,l)],\\
V_n(h-1,l) &= \Psi[V_{n+1}(h-1,l)] + C\lambda\delta_n^*(h-1,l).
\end{aligned}$$

$$\begin{aligned}
& \Delta_n^h(h+1,l)\\
\le\ & (1-\lambda^h-\lambda^l)\Delta_{n+1}^h(h+1,l) + \lambda^h\Delta_{n+1}^h(h,l) + \lambda^l\Delta_{n+1}^h(h+1,l-1)\\
& +C\lambda\alpha(p^*_{h+1,l})[V_{n+1}(h,l+1) - V_{n+1}(h-1,l+1) + V_{n+1}(h,l) - V_{n+1}(h+1,l)]\\
=\ & [1-\lambda^h-\lambda^l - C\lambda\alpha(p^*_{h+1,l})]\Delta_{n+1}^h(h+1,l) + \lambda^h\Delta_{n+1}^h(h,l)\\
& +\lambda^l\Delta_{n+1}^h(h+1,l-1) + C\lambda\alpha(p^*_{h+1,l})\Delta_{n+1}^h(h,l+1),\\
& \Delta_n^h(h,l)\\
=\ & (1-\lambda^h-\lambda^l)\Delta_{n+1}^h(h,l) + \lambda^h\Delta_{n+1}^h(h-1,l) + \lambda^l\Delta_{n+1}^h(h,l-1)\\
& +C\lambda[\delta_n^*(h,l) - \delta_n^*(h-1,l)]\\
\ge\ & (1-\lambda^h-\lambda^l)\Delta_{n+1}^h(h,l) + \lambda^h\Delta_{n+1}^h(h-1,l) + \lambda^l\Delta_{n+1}^h(h,l-1)\\
=\ & [1-\lambda^h-\lambda^l - C\lambda\alpha(p^*_{h+1,l})]\Delta_{n+1}^h(h,l) + \lambda^h\Delta_{n+1}^h(h-1,l)\\
& +\lambda^l\Delta_{n+1}^h(h,l-1) + C\lambda\alpha(p^*_{h+1,l})\Delta_{n+1}^h(h,l),
\end{aligned}$$

where the last inequality is from $\delta_n^*(h,l) - \delta_n^*(h-1,l) \ge 0$. Then the proof of $\Delta_n^h(h+1,l) \le \Delta_n^h(h,l)$ is finished as in Case 2.

Horizontal boundary $[2:H]\times\{0\}$. There are 4 cases.

Case 1: $\delta_n^*(h-1,0) \le \delta_n^*(h,0) \le \delta_n^*(h+1,0) \le 0$.

$$\begin{aligned}
V_n(h+1,0) &= (1-\lambda^h)V_{n+1}(h+1,0) + \lambda^h[p^h + V_{n+1}(h,0)],\\
V_n(h,0) &= (1-\lambda^h)V_{n+1}(h,0) + \lambda^h[p^h + V_{n+1}(h-1,0)],\\
V_n(h-1,0) &= (1-\lambda^h)V_{n+1}(h-1,0) + \lambda^h[p^h + V_{n+1}(h-2,0)].
\end{aligned}$$

The H-concavity in period n is inherited from that in period $n+1$.

Case 2: $\delta_n^*(h-1,0) \le \delta_n^*(h,0) \le 0 < \delta_n^*(h+1,0)$.

$$
\begin{aligned}
V_n(h+1,0) &= (1-\lambda^h)V_{n+1}(h+1,0) + \lambda^h[p^h + V_{n+1}(h,0)] + C\lambda\delta_n^*(h+1,0) \\
&= (1-\lambda^h)V_{n+1}(h+1,0) + \lambda^h[p^h + V_{n+1}(h,0)] \\
&\quad + C\lambda\alpha(p^*_{h+1,0})[p^*_{h+1,0} + V_{n+1}(h,1) - V_{n+1}(h+1,0)], \\
V_n(h,0) &= (1-\lambda^h)V_{n+1}(h,0) + \lambda^h[p^h + V_{n+1}(h-1,0)] \\
&\ge (1-\lambda^h)V_{n+1}(h,0) + \lambda^h[p^h + V_{n+1}(h-1,0)] + C\lambda\delta_n^*(h,0) \\
&\ge (1-\lambda^h)V_{n+1}(h,0) + \lambda^h[p^h + V_{n+1}(h-1,0)] \\
&\quad + C\lambda\alpha(p^*_{h+1,0})[p^*_{h+1,0} + V_{n+1}(h-1,1) - V_{n+1}(h,0)], \\
V_n(h-1,0) &= (1-\lambda^h)V_{n+1}(h-1,0) + \lambda^h[p^h + V_{n+1}(h-2,0)],
\end{aligned}
$$

where the first inequality is from $\delta_n^*(h,0) \le 0$ and the second is from the non-optimality of $p^*_{h+1,0}$ for $\delta_n(p,h,0)$. Then,

$$
\begin{aligned}
\Delta_n^h(h+1,0) &\le (1-\lambda^h)\Delta_{n+1}^h(h+1,0) + \lambda^h\Delta_{n+1}^h(h,l) \\
&\quad + C\lambda\alpha(p^*_{h+1,0})[V_{n+1}(h,1) - V_{n+1}(h-1,1) + V_{n+1}(h,0) - V_{n+1}(h+1,0)] \\
&= [1-\lambda^h - C\lambda\alpha(p^*_{h+1,0})]\Delta_{n+1}^h(h+1,0) + \lambda^h\Delta_{n+1}^h(h,0) \\
&\quad + C\lambda\alpha(p^*_{h+1,0})\Delta_{n+1}^h(h,1),
\end{aligned}
$$

$$
\begin{aligned}
\Delta_n^h(h,0) &= [1-\lambda^h - C\lambda\alpha(p^*_{h+1,0})]\Delta_{n+1}^h(h,0) + \lambda^h\Delta_{n+1}^h(h-1,0) \\
&\quad + C\lambda\alpha(p^*_{h+1,0})\Delta_{n+1}^h(h,0).
\end{aligned}
$$

Using submodularity in period $n+1$ on the terms multiplied by $C\lambda\alpha(p^*_{h+1,0})$ and the H-concavity in period $n+1$ on the other terms, we obtain $\Delta_n^h(h+1,0) \le \Delta_n^h(h,0)$.

Case 3: $\delta_n^*(h-1,0) \le 0 < \delta_n^*(h,0) \le \delta_n^*(h+1,0)$.

$$\begin{aligned}
V_n(h+1,0) &= (1-\lambda^h)V_{n+1}(h+1,0) + \lambda^h[p^h + V_{n+1}(h,0)] + C\lambda\delta_n^*(h+1,0) \\
&= (1-\lambda^h)V_{n+1}(h+1,0) + \lambda^h[p^h + V_{n+1}(h,0)] \\
&\quad +C\lambda\alpha(p^*_{h+1,0})[p^*_{h+1,0} + V_{n+1}(h,1) - V_{n+1}(h+1,0)], \\
V_n(h,0) &= (1-\lambda^h)V_{n+1}(h,0) + \lambda^h[p^h + V_{n+1}(h-1,0)] + C\lambda\delta_n^*(h,0) \\
&\ge (1-\lambda^h)V_{n+1}(h,0) + \lambda^h[p^h + V_{n+1}(h-1,0)] \\
&\quad +C\lambda\alpha(p^*_{h+1,0})[p^*_{h+1,0} + V_{n+1}(h-1,1) - V_{n+1}(h,0)], \\
V_n(h,0) &\ge (1-\lambda^h)V_{n+1}(h,0) + \lambda^h[p^h + V_{n+1}(h-1,0)], \\
V_n(h-1,0) &= (1-\lambda^h)V_{n+1}(h-1,0) + \lambda^h[p^h + V_{n+1}(h-2,0)].
\end{aligned}$$

Note that two lower bounds are given for $V_n(h,0)$; the first is used to find an upper bound for $\Delta_n^h(h+1,0)$ and the second for a lower bound for $\Delta_n^h(h,0)$. By following the arguments of Case 2, we can show that the upper bound of $\Delta_n^h(h+1,0)$ is not more than the lower bound of $\Delta_n^h(h,0)$, so $\Delta_n^h(h+1,0) \le \Delta_n^h(h,0)$.

Case 4: $0 < \delta_n^*(h-1,0) \le \delta_n^*(h,0) \le \delta_n^*(h+1,0)$.

$$\begin{aligned}
V_n(h+1,0) &= (1-\lambda^h)V_{n+1}(h+1,0) + \lambda^h[p^h + V_{n+1}(h,0)] + C\lambda\delta_n^*(h+1,0) \\
&= (1-\lambda^h)V_{n+1}(h+1,0) + \lambda^h[p^h + V_{n+1}(h,0)] \\
&\quad +C\lambda\alpha(p^*_{h+1,0})[p^*_{h+1,0} + V_{n+1}(h,1) - V_{n+1}(h+1,0)], \\
V_n(h,0) &= (1-\lambda^h)V_{n+1}(h,0) + \lambda^h[p^h + V_{n+1}(h-1,0)] + C\lambda\delta_n^*(h,0) \\
&\ge (1-\lambda^h)V_{n+1}(h,0) + \lambda^h[p^h + V_{n+1}(h-1,0)] \\
&\quad +C\lambda\alpha(p^*_{h+1,0})[p^*_{h+1,0} + V_{n+1}(h-1,1) - V_{n+1}(h,0)], \\
V_n(h-1,0) &= (1-\lambda^h)V_{n+1}(h-1,0) + \lambda^h[p^h + V_{n+1}(h-2,0)] + C\lambda\delta_n^*(h-1,0).
\end{aligned}$$

$$
\begin{aligned}
& \Delta_n^h(h+1,0) \\
\leq\ & (1-\lambda^h)\Delta_{n+1}^h(h+1,0)+\lambda^h\Delta_{n+1}^h(h,l) \\
& +C\lambda\alpha(p^*_{h+1,0})[V_{n+1}(h,1)-V_{n+1}(h-1,1)+V_{n+1}(h,0)-V_{n+1}(h+1,0)] \\
=\ & [1-\lambda^h-C\lambda\alpha(p^*_{h+1,0})]\Delta_{n+1}^h(h+1,0)+\lambda^h\Delta_{n+1}^h(h,0) \\
& +C\lambda\alpha(p^*_{h+1,0})\Delta_{n+1}^h(h,1), \\
& \Delta_n^h(h,0) \\
=\ & (1-\lambda^h)\Delta_{n+1}^h(h,0)++\lambda^h\Delta_{n+1}^h(h-1,0)+C\lambda[\delta_n^*(h,0)-\delta_n^*(h-1,0)] \\
\geq\ & [1-\lambda^h-C\lambda\alpha(p^*_{h+1,0})]\Delta_{n+1}^h(h,0)+\lambda^h\Delta_{n+1}^h(h-1,0)+C\lambda\alpha(p^*_{h+1,0})\Delta_{n+1}^h(h,0),
\end{aligned}
$$

where the last inequality is from $\delta_n^*(h,0)-\delta_n^*(h-1,0)\geq 0$. Then the proof of $\Delta_n^h(h+1,0)\leq\Delta_n^h(h,0)$ is finished as in Case 2.

Vertical boundary $\{1\}\times[1:M+1]$. Since $h-1=0$, $\delta_n^*(h-1,l)$ is not defined. We have $\delta_n^*(1,l)\leq\delta_n^*(2,l)$ and 3 cases similar to the first three in the interior. Proofs require submodularity, H-concavity in period $n+1$ and $\Delta_{n+1}^h(1,l)\leq p^h$, which is from Lemma 2 and submodularity. We prove the H-concavity in Case 1.

Case 1: $\delta_n^*(1,l)\leq\delta_n^*(2,l)\leq 0$.

$$
\begin{aligned}
V_n(2,l) &= (1-\lambda^h-\lambda^l)V_{n+1}(2,l)+\lambda^h[p^h+V_{n+1}(1,l)]+\lambda^l[p^l+V_{n+1}(2,l-1)], \\
V_n(1,l) &= (1-\lambda^h-\lambda^l)V_{n+1}(1,l)+\lambda^h[p^h+V_{n+1}(0,l)]+\lambda^l[p^l+V_{n+1}(1,l-1)], \\
V_n(0,l) &= (1-\lambda^h-\lambda^l)V_{n+1}(0,l)+\lambda^hV_{n+1}(0,l)+\lambda^l[p^l+V_{n+1}(0,l-1)]. \\
\Delta_n^h(2,l) &= (1-\lambda^h-\lambda^l)\Delta_{n+1}^h(2,l)+\lambda^h\Delta_{n+1}^h(1,l)+\lambda^l\Delta_{n+1}^h(2,l-1), \\
\Delta_n^h(1,l) &= (1-\lambda^h-\lambda^l)\Delta_{n+1}^h(1,l)+\lambda^hp^h+\lambda^l\Delta_{n+1}^h(1,l-1),
\end{aligned}
$$

H-concavity in period $n+1$ and $\Delta_{n+1}^h(1,l)\leq p^h$ imply $\Delta_n^h(2,l)\leq\Delta_n^h(1,l)$.

Case 2: $\delta_n^*(1,l) \le 0 < \delta_n^*(2,l)$.

$$\begin{aligned}
V_n(2,l) &= (1-\lambda^h-\lambda^l)V_{n+1}(2,l) + \lambda^h[p^h + V_{n+1}(1,l)] + \lambda^l[p^l + V_{n+1}(2,l-1)] \\
&\quad + C\lambda\delta_n^*(2,l) \\
&= (1-\lambda^h-\lambda^l)V_{n+1}(2,l) + \lambda^h[p^h + V_{n+1}(1,l)] + \lambda^l[p^l + V_{n+1}(2,l-1)] \\
&\quad + C\lambda\alpha(p_{2,l}^*)[p_{2,l}^* + V_{n+1}(1,l+1) - V_{n+1}(2,l)], \\
V_n(1,l) &= (1-\lambda^h-\lambda^l)V_{n+1}(1,l) + \lambda^h[p^h + V_{n+1}(0,l)] + \lambda^l[p^l + V_{n+1}(1,l-1)] \\
&\ge (1-\lambda^h-\lambda^l)V_{n+1}(1,l) + \lambda^h[p^h + V_{n+1}(0,l)] + \lambda^l[p^l + V_{n+1}(1,l-1)] \\
&\quad + C\lambda\delta_n^*(1,l) \\
&\ge (1-\lambda^h-\lambda^l)V_{n+1}(1,l) + \lambda^h[p^h + V_{n+1}(0,l)] + \lambda^l[p^l + V_{n+1}(1,l-1)] \\
&\quad + C\lambda\alpha(p_{2,l}^*)[p_{2,l}^* + V_{n+1}(0,l+1) - V_{n+1}(1,l)], \\
V_n(0,l) &= (1-\lambda^h-\lambda^l)V_{n+1}(0,l) + \lambda^h V_{n+1}(0,l) + \lambda^l[p^l + V_{n+1}(0,l-1)],
\end{aligned}$$

where the first inequality is from $\delta_n^*(1,l) \le 0$ and the second is from the non-optimality of $p_{2,l}^*$ for $\delta_n(p,1,l)$. Then,

$$\begin{aligned}
\Delta_n^h(2,l) &\le (1-\lambda^h-\lambda^l)\Delta_{n+1}^h(2,l) + \lambda^h\Delta_{n+1}^h(1,l) + \lambda^l\Delta_{n+1}^h(2,l-1) \\
&\quad + C\lambda\alpha(p_{2,l}^*)[V_{n+1}(1,l+1) - V_{n+1}(0,l+1) + V_{n+1}(1,l) - V_{n+1}(2,l)] \\
&= [1-\lambda^h-\lambda^l - C\lambda\alpha(p_{2,l}^*)]\Delta_{n+1}^h(2,l) + \lambda^h\Delta_{n+1}^h(1,l) + \lambda^l\Delta_{n+1}^h(2,l-1) \\
&\quad + C\lambda\alpha(p_{2,l}^*)\Delta_{n+1}^h(1,l+1),
\end{aligned}$$

$$\begin{aligned}
\Delta_n^h(1,l) &= [1-\lambda^h-\lambda^l - C\lambda\alpha(p_{2,l}^*)]\Delta_{n+1}^h(1,l) + \lambda^h p^h + \lambda^l\Delta_{n+1}^h(1,l-1) \\
&\quad + C\lambda\alpha(p_{2,l}^*)\Delta_{n+1}^h(1,l).
\end{aligned}$$

Using submodularity in period $n+1$ on the terms multiplied by $C\lambda\alpha(p_{2,l}^*)$, Lemma 2 and submodularity in period $n+1$ on the terms multiplied by λ^h, and the H-concavity in period $n+1$ on the other terms, we obtain $\Delta_n^h(2,l) \le \Delta_n^h(1,l)$.

Case 3: $0 < \delta_n^*(1,l) \le \delta_n^*(2,l)$.

$$\begin{aligned}
V_n(2,l) &= (1-\lambda^h-\lambda^l)V_{n+1}(2,l) + \lambda^h[p^h + V_{n+1}(1,l)] + \lambda^l[p^l + V_{n+1}(2,l-1)] \\
&\quad + C\lambda\delta_n^*(2,l) \\
&= (1-\lambda^h-\lambda^l)V_{n+1}(2,l) + \lambda^h[p^h + V_{n+1}(1,l)] + \lambda^l[p^l + V_{n+1}(2,l-1)] \\
&\quad + C\lambda\alpha(p_{2,l}^*)[p_{2,l}^* + V_{n+1}(1,l+1) - V_{n+1}(2,l)], \\
V_n(1,l) &= (1-\lambda^h-\lambda^l)V_{n+1}(1,l) + \lambda^h[p^h + V_{n+1}(0,l)] + \lambda^l[p^l + V_{n+1}(1,l-1)] \\
&\quad + C\lambda\delta_n^*(1,l) \\
&\ge (1-\lambda^h-\lambda^l)V_{n+1}(1,l) + \lambda^h[p^h + V_{n+1}(0,l)] + \lambda^l[p^l + V_{n+1}(1,l-1)] \\
&\quad + C\lambda\alpha(p_{2,l}^*)[p_{2,l}^* + V_{n+1}(0,l+1) - V_{n+1}(1,l)], \\
V_n(1,l) &\ge (1-\lambda^h-\lambda^l)V_{n+1}(1,l) + \lambda^h[p^h + V_{n+1}(0,l)] + \lambda^l[p^l + V_{n+1}(1,l-1)], \\
V_n(0,l) &= (1-\lambda^h-\lambda^l)V_{n+1}(0,l) + \lambda^h V_{n+1}(0,l) + \lambda^l[p^l + V_{n+1}(0,l-1)].
\end{aligned}$$

Note that two lower bounds are given for $V_n(1,l)$; the first is used to find an upper bound for $\Delta_n^h(2,l)$ and the second for a lower bound for $\Delta_n^h(1,l)$. By following the arguments of Case 2, we can show that the upper bound of $\Delta_n^h(2,l)$ is not more than the lower bound of $\Delta_n^h(1,l)$, so $\Delta_n^h(2,l) \le \Delta_n^h(1,l)$.

Corner $(1,0)$. There are 3 cases and proofs require submodularity, $\Delta_{n+1}^h(1,l) \le p^h$ and H-concavity as in the vertical boundary. We prove Case 1.

Case 1: $\delta_n^*(1,0) \le \delta_n^*(2,0) \le 0$.

$$\begin{aligned}
V_n(2,0) &= (1-\lambda^h)V_{n+1}(2,0) + \lambda^h[p^h + V_{n+1}(1,0)], \\
V_n(1,0) &= (1-\lambda^h)V_{n+1}(1,0) + \lambda^h[p^h + V_{n+1}(0,0)], \\
V_n(0,0) &= (1-\lambda^h)V_{n+1}(0,0) + \lambda^h V_{n+1}(0,0).
\end{aligned}$$

$$\begin{aligned}
\Delta_n^h(2,0) &= (1-\lambda^h)\Delta_{n+1}^h(2,0) + \lambda^h\Delta_{n+1}^h(1,0), \\
\Delta_n^h(1,0) &= (1-\lambda^h)\Delta_{n+1}^h(1,0) + \lambda^h p^h.
\end{aligned}$$

H-concavity in period $n+1$ and Lemma 2 yield $\Delta_n^h(2,0) \leq \Delta_n^h(1,0)$.

Case 2: $\delta_n^*(1,0) \leq 0 < \delta_n^*(2,0)$.

$$\begin{aligned}
V_n(2,0) &= (1-\lambda^h)V_{n+1}(2,0) + \lambda^h[p^h + V_{n+1}(1,0)] + C\lambda\delta_n^*(2,0) \\
&= (1-\lambda^h)V_{n+1}(2,0) + \lambda^h[p^h + V_{n+1}(1,0)] \\
&\quad +C\lambda\alpha(p_{2,0}^*)[p_{2,0}^* + V_{n+1}(1,1) - V_{n+1}(2,0)], \\
V_n(1,0) &= (1-\lambda^h)V_{n+1}(1,0) + \lambda^h[p^h + V_{n+1}(0,0)] \\
&\geq (1-\lambda^h)V_{n+1}(1,0) + \lambda^h[p^h + V_{n+1}(0,0)] + C\lambda\delta_n^*(1,0) \\
&\geq (1-\lambda^h)V_{n+1}(1,0) + \lambda^h[p^h + V_{n+1}(0,0)] \\
&\quad +C\lambda\alpha(p_{2,0}^*)[p_{2,0}^* + V_{n+1}(0,1) - V_{n+1}(1,0)], \\
V_n(0,0) &= (1-\lambda^h)V_{n+1}(0,0) + \lambda^h V_{n+1}(0,0),
\end{aligned}$$

where the first inequality is from $\delta_n^*(1,0) \leq 0$ and the second is from the non-optimality of $p_{2,0}^*$ for $\delta_n(p,1,0)$. Then,

$$\begin{aligned}
\Delta_n^h(2,0) &= (1-\lambda^h)\Delta_{n+1}^h(2,0) + \lambda^h\Delta_{n+1}^h(1,0) \\
&\quad +C\lambda\alpha(p_{2,0}^*)[V_{n+1}(1,1) - V_{n+1}(0,1) + V_{n+1}(1,0) - V_{n+1}(2,0)] \\
&= [1-\lambda^h-\lambda^l - C\lambda\alpha(p_{2,0}^*)]\Delta_{n+1}^h(2,0) + \lambda^h\Delta_{n+1}^h(1,0) + C\lambda\alpha(p_{2,0}^*)\Delta_{n+1}^h(1,1), \\
\Delta_n^h(1,0) &= [1-\lambda^h - C\lambda\alpha(p_{2,0}^*)]\Delta_{n+1}^h(1,0) + \lambda^h p^h + C\lambda\alpha(p_{2,0}^*)\Delta_{n+1}^h(1,0).
\end{aligned}$$

Using submodularity in period $n+1$ on the terms multiplied by $C\lambda\alpha(p_{2,0}^*)$, Lemma 2 in period $n+1$ on the terms multiplied by λ^h, and the H-concavity in period $n+1$ on the other terms, we obtain $\Delta_n^h(2,0) \leq \Delta_n^h(1,0)$.

Case 3: $0 < \delta_n^*(1,0) \leq \delta_n^*(2,0)$.

$$
\begin{aligned}
V_n(2,0) &= (1-\lambda^h)V_{n+1}(2,0) + \lambda^h[p^h + V_{n+1}(1,0)] + C\lambda\delta_n^*(2,0) \\
&= (1-\lambda^h)V_{n+1}(2,0) + \lambda^h[p^h + V_{n+1}(1,0)] \\
&\quad + C\lambda\alpha(p_{2,0}^*)[p_{2,0}^* + V_{n+1}(1,1) - V_{n+1}(2,0)], \\
V_n(1,0) &= (1-\lambda^h)V_{n+1}(1,0) + \lambda^h[p^h + V_{n+1}(0,0)] + C\lambda\delta_n^*(1,0) \\
&\geq (1-\lambda^h)V_{n+1}(1,0) + \lambda^h[p^h + V_{n+1}(0,0)] \\
&\quad + C\lambda\alpha(p_{2,0}^*)[p_{2,0}^* + V_{n+1}(0,1) - V_{n+1}(1,0)], \\
V_n(1,0) &\geq (1-\lambda^h)V_{n+1}(1,0) + \lambda^h[p^h + V_{n+1}(0,0)], \\
V_n(0,0) &= (1-\lambda^h)V_{n+1}(0,0) + \lambda^h V_{n+1}(0,0).
\end{aligned}
$$

Note that two lower bounds are given for $V_n(1,0)$; the first is used to find an upper bound for $\Delta_n^h(2,0)$ and the second for a lower bound for $\Delta_n^h(1,0)$. By following the arguments of Case 2, we can show that the upper bound of $\Delta_n^h(2,0)$ is not more than the lower bound of $\Delta_n^h(1,0)$, so $\Delta_n^h(2,0) \leq \Delta_n^h(1,0)$.

Proof of property e): $[0 : H] \times [1 : M]$ is partitioned into the interior $[1 : H] \times [2 : M]$, the horizontal boundary $[1 : H] \times \{1\}$, the vertical boundary $\{0\} \times [2 : M]$ and the corner $(0,1)$. The V-concavity in period n is $V_n(h,l) - V_n(h,l-1) \geq V_n(h,l+1) - V_n(h,l)$ or $\Delta_n^l(h,l) \geq \Delta_n^l(h,l+1)$. This inequality involves states $(h,l-1)$, (h,l) and $(h,l+1)$. Specializing Inequalities (A.1) for these states, we have $p_{h,l-1}^* \leq p_{h,l}^* \leq p_{h,l+1}^*$ and $\delta_n^*(h,l+1) \leq \delta_n^*(h,l) \leq \delta_n^*(h,l-1)$, which is used to construct the cases below.

<u>Interior $[0 : H] \times [1 : M]$</u>. There are 4 possible cases.

Case 1: $\delta_n^*(h,l+1) \le \delta_n^*(h,l) \le \delta_n^*(h,l-1) \le 0$.

$$\begin{aligned}
V_n(h,l-1) &= (1-\lambda^h-\lambda^l)V_{n+1}(h,l-1) + \lambda^h[p^h + V_{n+1}(h-1,l-1)] \\
&\quad +\lambda^l[p^l + V_{n+1}(h,l-2)], \\
V_n(h,l) &= (1-\lambda^h-\lambda^l)V_{n+1}(h,l) + \lambda^h[p^h + V_{n+1}(h-1,l)] + \lambda^l[p^l + V_{n+1}(h,l-1)], \\
V_n(h,l+1) &= (1-\lambda^h-\lambda^l)V_{n+1}(h,l+1) + \lambda^h[p^h + V_{n+1}(h-1,l+1)] \\
&\quad +\lambda^l[p^l + V_{n+1}(h,l)].
\end{aligned}$$

V-concavity in period $n+1$ and term-by-term comparisons of the value functions above yield $\Delta_n^l(h,l) \ge \Delta_n^l(h,l+1)$.

Case 2: $\delta_n^*(h,l+1) \le \delta_n^*(h,l) \le 0 < \delta_n^*(h,l-1)$.

$$\begin{aligned}
V_n(h,l-1) &= \Psi[V_{n+1}(h,l-1)] + C\lambda\delta_n^*(h,l-1) \\
&= \Psi[V_{n+1}(h,l-1)] + C\lambda\alpha(p^*_{h,l-1})[p^*_{h,l-1} + V_{n+1}(h-1,l) - V_{n+1}(h,l-1)], \\
V_n(h,l) &= \Psi[V_{n+1}(h,l)] \\
&\ge \Psi[V_{n+1}(h,l)] + C\lambda\delta_n^*(h,l) \\
&\ge \Psi[V_{n+1}(h,l)] + C\lambda\alpha(p^*_{h,l-1})[p^*_{h,l-1} + V_{n+1}(h-1,l+1) - V_{n+1}(h,l)], \\
V_n(h,l+1) &= \Psi[V_{n+1}(h,l+1)],
\end{aligned}$$

where the first inequality is from $\delta_n^*(h,l) \le 0$ and the second is from the non-optimality of $p^*_{h,l-1}$ for $\delta_n(p,h,l)$. Then,

$$\begin{aligned}
\Delta_n^l(h,l) &\ge (1-\lambda^h-\lambda^l)\Delta_{n+1}^l(h,l) + \lambda^h\Delta_{n+1}^l(h-1,l) + \lambda^l\Delta_{n+1}^l(h,l-1) \\
&\quad +C\lambda\alpha(p^*_{h,l-1})[V_{n+1}(h-1,l+1) - V_{n+1}(h-1,l) + V_{n+1}(h,l-1) - V_{n+1}(h,l)] \\
&= (1-\lambda^h-\lambda^l - C\lambda\alpha(p^*_{h,l-1}))\Delta_{n+1}^l(h,l) + \lambda^h\Delta_{n+1}^l(h-1,l) + \lambda^l\Delta_{n+1}^l(h,l-1) \\
&\quad +C\lambda\alpha(p^*_{h,l-1})\Delta_{n+1}^l(h-1,l+1),
\end{aligned}$$

$$
\begin{aligned}
\Delta_n^l(h,l+1) &= (1-\lambda^h-\lambda^l)\Delta_{n+1}^l(h,l+1)+\lambda^h\Delta_{n+1}^l(h-1,l+1)+\lambda^l\Delta_{n+1}^l(h,l) \\
&= (1-\lambda^h-\lambda^l-C\lambda\alpha(p_{h,l-1}^*))\Delta_{n+1}^l(h,l+1)+\lambda^h\Delta_{n+1}^l(h-1,l+1) \\
&\quad +\lambda^l\Delta_{n+1}^l(h,l)+C\lambda\alpha(p_{h,l-1}^*)\Delta_{n+1}^l(h,l+1).
\end{aligned}
$$

Using submodularity in period $n+1$ on the terms multiplied by $C\lambda\alpha(p_{h,l-1}^*)$ and V-concavity in period $n+1$ on the other terms, we obtain $\Delta_n^l(h,l) \geq \Delta_n^l(h,l+1)$.

Case 3: $\delta_n^*(h,l+1) \leq 0 < \delta_n^*(h,l) \leq \delta_n^*(h,l-1)$.

$$
\begin{aligned}
V_n(h,l-1) &= \Psi[V_{n+1}(h,l-1)]+C\lambda\alpha(p_{h,l-1}^*)[p_{h,l-1}^*+V_{n+1}(h-1,l)-V_{n+1}(h,l-1)], \\
V_n(h,l) &= \Psi[V_{n+1}(h,l)]+C\lambda\delta_n^*(h,l) \\
&\geq \Psi[V_{n+1}(h,l)]+C\lambda\alpha(p_{h,l-1}^*)[p_{h,l-1}^*+V_{n+1}(h-1,l+1)-V_{n+1}(h,l)], \\
V_n(h,l) &\geq \Psi[V_{n+1}(h,l)], \\
V_n(h,l+1) &= \Psi[V_{n+1}(h,l+1].
\end{aligned}
$$

Note that two lower bounds are given for $V_n(h,l)$; the first is used to find a lower bound for $\Delta_n^l(h,l)$ and the second for an upper bound for $\Delta_n^l(h,l+1)$. By following the arguments of Case 2, we can show that the upper bound of $\Delta_n^l(h,l+1)$ is not more than the lower bound of $\Delta_n^l(h,l)$, so $\Delta_n^l(h,l+1) \leq \Delta_n^l(h,l)$.

Case 4: $0 < \delta_n^*(h,l+1) \leq \delta_n^*(h,l) \leq \delta_n^*(h,l-1)$.

$$
\begin{aligned}
V_n(h,l-1) &= \Psi[V_{n+1}(h,l-1)]+C\lambda\alpha(p_{h,l-1}^*)[p_{h,l-1}^*+V_{n+1}(h-1,l)-V_{n+1}(h,l-1)], \\
V_n(h,l) &= \Psi[V_{n+1}(h,l)]+C\lambda\delta_n^*(h,l) \\
&\geq \Psi[V_{n+1}(h,l)]+C\lambda\alpha(p_{h,l-1}^*)[p_{h,l-1}^*+V_{n+1}(h-1,l+1)-V_{n+1}(h,l)], \\
V_n(h,l+1) &= \Psi[V_{n+1}(h,l+1)]+C\lambda\delta_n^*(h,l+1).
\end{aligned}
$$

$$
\begin{aligned}
& \Delta_n^l(h,l) \\
\geq\ & (1-\lambda^h-\lambda^l)\Delta_{n+1}^l(h,l)+\lambda^h\Delta_{n+1}^l(h-1,l)+\lambda^l\Delta_{n+1}^l(h,l-1) \\
& +C\lambda\alpha(p_{h,l-1}^*)[V_{n+1}(h-1,l+1)-V_{n+1}(h-1,l)+V_{n+1}(h,l-1)-V_{n+1}(h,l)] \\
=\ & (1-\lambda^h-\lambda^l-C\lambda\alpha(p_{h,l-1}^*))\Delta_{n+1}^l(h,l)+\lambda^h\Delta_{n+1}^l(h-1,l)+\lambda^l\Delta_{n+1}^l(h,l-1) \\
& +C\lambda\alpha(p_{h,l-1}^*)\Delta_{n+1}^l(h-1,l+1), \\
& \Delta_n^l(h,l+1) \\
=\ & (1-\lambda^h-\lambda^l)\Delta_{n+1}^l(h,l+1)+\lambda^h\Delta_{n+1}^l(h-1,l+1)+\lambda^l\Delta_{n+1}^l(h,l) \\
& +C\lambda[\delta_n^*(h,l+1)-\delta_n^*(h,l)], \\
\leq\ & (1-\lambda^h-\lambda^l)\Delta_{n+1}^l(h,l+1)+\lambda^h\Delta_{n+1}^l(h-1,l+1)+\lambda^l\Delta_{n+1}^l(h,l) \\
=\ & (1-\lambda^h-\lambda^l-C\lambda\alpha(p_{h,l-1}^*))\Delta_{n+1}^l(h,l+1)+\lambda^h\Delta_{n+1}^l(h-1,l+1)+\lambda^l\Delta_{n+1}^l(h,l) \\
& +C\lambda\alpha(p_{h,l-1}^*)\Delta_{n+1}^l(h,l+1),
\end{aligned}
$$

where the last inequality is from $\delta_n^*(h,l+1)-\delta_n^*(h,l)\leq 0$. Then the proof of $\Delta_n^l(h,l)\geq\Delta_n^l(h,l+1)$ is finished as in Case 2.

Horizontal boundary $[1:H]\times\{1\}$. There are 4 cases similar to the interior. Proofs require submodularity, V-concavity in period $n+1$ and $V_{n+1}(h,1)-V_{n+1}(h,0)\leq p^l$, which can be derived from Lemma 2 and the submodularity.

Case 1: $\delta_n^*(h,2)\leq\delta_n^*(h,1)\leq\delta_n^*(h,0)\leq 0$.

$$
\begin{aligned}
V_n(h,0) &= (1-\lambda^h-\lambda^l)V_{n+1}(h,0)+\lambda^h[p^h+V_{n+1}(h-1,0)]+\lambda^lV_{n+1}(h,0), \\
V_n(h,1) &= (1-\lambda^h-\lambda^l)V_{n+1}(h,1)+\lambda^h[p^h+V_{n+1}(h-1,1)]+\lambda^l[p^l+V_{n+1}(h,0)], \\
V_n(h,2) &= (1-\lambda^h-\lambda^l)V_{n+1}(h,2)+\lambda^h[p^h+V_{n+1}(h-1,2)]+\lambda^l[p^l+V_{n+1}(h,1)]. \\
\Delta_n^l(h,1) &= (1-\lambda^h-\lambda^l)\Delta_{n+1}^l(h,1)+\lambda^h\Delta_{n+1}^l(h-1,1)+\lambda^lp^l, \\
\Delta_n^l(h,2) &= (1-\lambda^h-\lambda^l)\Delta_{n+1}^l(h,2)+\lambda^h\Delta_{n+1}^l(h-1,2)+\lambda^l\Delta_{n+1}^l(h,1).
\end{aligned}
$$

V-concavity in period $n+1$ and $p^l\geq\Delta_{n+1}^l(h,1)$ imply $\Delta_n^l(h,1)\geq\Delta_n^l(h,2)$.

Case 2: $\delta_n^*(h,2) \le \delta_n^*(h,1) \le 0 < \delta_n^*(h,0)$.

$$
\begin{aligned}
V_n(h,0) &= (1-\lambda^h-\lambda^l)V_{n+1}(h,0)+\lambda^h[p^h+V_{n+1}(h-1,0)]+\lambda^l V_{n+1}(h,0) \\
&\quad +C\lambda\alpha(p^*_{h,0})[p^*_{h,0}+V_{n+1}(h-1,1)-V_{n+1}(h,0)], \\
V_n(h,1) &= (1-\lambda^h-\lambda^l)V_{n+1}(h,1)+\lambda^h[p^h+V_{n+1}(h-1,1)]+\lambda^l[p^l+V_{n+1}(h,0)] \\
&\ge (1-\lambda^h-\lambda^l)V_{n+1}(h,1)+\lambda^h[p^h+V_{n+1}(h-1,1)]+\lambda^l[p^l+V_{n+1}(h,0)] \\
&\quad +C\lambda\delta_n^*(h,1) \\
&\ge (1-\lambda^h-\lambda^l)V_{n+1}(h,1)+\lambda^h[p^h+V_{n+1}(h-1,1)]+\lambda^l[p^l+V_{n+1}(h,0)] \\
&\quad +C\lambda\alpha(p^*_{h,0})[p^*_{h,0}+V_{n+1}(h-1,2)-V_{n+1}(h,1)], \\
V_n(h,2) &= (1-\lambda^h-\lambda^l)V_{n+1}(h,2)+\lambda^h[p^h+V_{n+1}(h-1,2)]+\lambda^l[p^l+V_{n+1}(h,1)].
\end{aligned}
$$

$$
\begin{aligned}
\Delta_n^l(h,1) &\ge (1-\lambda^h-\lambda^l)\Delta_{n+1}^l(h,1)+\lambda^h\Delta_{n+1}^l(h-1,1)+\lambda^l p^l \\
&\quad +C\lambda\alpha(p^*_{h,0})[V_{n+1}(h-1,2)-V_{n+1}(h-1,1)+V_{n+1}(h,0)-V_{n+1}(h,1)] \\
&= [1-\lambda^h-\lambda^l-C\lambda\alpha(p^*_{h,0})]\Delta_{n+1}^l(h,1)+\lambda^h\Delta_{n+1}^l(h-1,1)+\lambda^l p^l \\
&\quad +C\lambda\alpha(p^*_{h,0})\Delta_{n+1}^l(h-1,2), \\
\Delta_n^l(h,2) &= [1-\lambda^h-\lambda^l-C\lambda\alpha(p^*_{h,0})]\Delta_{n+1}^l(h,2)+\lambda^h\Delta_{n+1}^l(h-1,2)+\lambda^l\Delta_{n+1}^l(h,1) \\
&\quad +C\lambda\alpha(p^*_{h,0})\Delta_{n+1}^l(h,2).
\end{aligned}
$$

Using submodularity in period $n+1$ on the terms multiplied by $C\lambda\alpha(p^*_{h,0})$, $p^l \ge \Delta_{n+1}^l(h,1)$ on the terms multiplied by λ^l and V-concavity in period $n+1$ on the other terms, we obtain $\Delta_n^l(h,1) \ge \Delta_n^l(h,2)$.

Case 3: $\delta_n^*(h,2) \le 0 < \delta_n^*(h,1) \le \delta_n^*(h,0)$.

$$
\begin{aligned}
V_n(h,0) &= (1-\lambda^h-\lambda^l)V_{n+1}(h,0)+\lambda^h[p^h+V_{n+1}(h-1,0)]+\lambda^l V_{n+1}(h,0) \\
&\quad +C\lambda\alpha(p^*_{h,0})[p^*_{h,0}+V_{n+1}(h-1,1)-V_{n+1}(h,0)],
\end{aligned}
$$

$$
\begin{aligned}
V_n(h,1) &= (1-\lambda^h-\lambda^l)V_{n+1}(h,1)+\lambda^h[p^h+V_{n+1}(h-1,1)]+\lambda^l[p^l+V_{n+1}(h,0)] \\
&\quad +C\lambda\delta_n^*(h,1) \\
&\geq (1-\lambda^h-\lambda^l)V_{n+1}(h,1)+\lambda^h[p^h+V_{n+1}(h-1,1)]+\lambda^l[p^l+V_{n+1}(h,0)] \\
&\quad +C\lambda\alpha(p_{h,0}^*)[p_{h,0}^*+V_{n+1}(h-1,2)-V_{n+1}(h,1)], \\
V_n(h,1) &\geq (1-\lambda^h-\lambda^l)V_{n+1}(h,1)+\lambda^h[p^h+V_{n+1}(h-1,1)]+\lambda^l[p^l+V_{n+1}(h,0)] \\
V_n(h,2) &= (1-\lambda^h-\lambda^l)V_{n+1}(h,2)+\lambda^h[p^h+V_{n+1}(h-1,2)]+\lambda^l[p^l+V_{n+1}(h,1)].
\end{aligned}
$$

Note that two lower bounds are given for $V_n(h,1)$; the first is used to find a lower bound for $\Delta_n^l(h,1)$ and the second for an upper bound for $\Delta_n^l(h,2)$. By following the arguments of Case 2, we can show that the upper bound of $\Delta_n^l(h,2)$ is not more than the lower bound of $\Delta_n^l(h,1)$, so $\Delta_n^l(h,2) \leq \Delta_n^l(h,1)$.

Case 4: $0 < \delta_n^*(h,2) \leq \delta_n^*(h,1) \leq \delta_n^*(h,0)$.

$$
\begin{aligned}
V_n(h,0) &= (1-\lambda^h-\lambda^l)V_{n+1}(h,0)+\lambda^h[p^h+V_{n+1}(h-1,0)]+\lambda^l V_{n+1}(h,0) \\
&\quad +C\lambda\alpha(p_{h,0}^*)[p_{h,0}^*+V_{n+1}(h-1,1)-V_{n+1}(h,0)], \\
V_n(h,1) &= (1-\lambda^h-\lambda^l)V_{n+1}(h,1)+\lambda^h[p^h+V_{n+1}(h-1,1)]+\lambda^l[p^l+V_{n+1}(h,0)] \\
&\quad +C\lambda\delta_n^*(h,1) \\
&\geq (1-\lambda^h-\lambda^l)V_{n+1}(h,1)+\lambda^h[p^h+V_{n+1}(h-1,1)]+\lambda^l[p^l+V_{n+1}(h,0)] \\
&\quad +C\lambda\alpha(p_{h,0}^*)[p_{h,0}^*+V_{n+1}(h-1,2)-V_{n+1}(h,1)], \\
V_n(h,2) &= (1-\lambda^h-\lambda^l)V_{n+1}(h,2)+\lambda^h[p^h+V_{n+1}(h-1,2)]+\lambda^l[p^l+V_{n+1}(h,1)] \\
&\quad +C\lambda\delta_n^*(h,2). \\
\Delta_n^l(h,1) &\geq (1-\lambda^h-\lambda^l)\Delta_{n+1}^l(h,1)+\lambda^h\Delta_{n+1}^l(h-1,1)+\lambda^l p^l \\
&\quad +C\lambda\alpha(p_{h,0}^*)[V_{n+1}(h-1,2)-V_{n+1}(h-1,1)+V_{n+1}(h,0)-V_{n+1}(h,1)] \\
&= [1-\lambda^h-\lambda^l-C\lambda\alpha(p_{h,0}^*)]\Delta_{n+1}^l(h,1)+\lambda^h\Delta_{n+1}^l(h-1,1)+\lambda^l p^l \\
&\quad +C\lambda\alpha(p_{h,0}^*)\Delta_{n+1}^l(h-1,2),
\end{aligned}
$$

$$
\begin{aligned}
\Delta_n^l(h,2) &= (1-\lambda^h-\lambda^l)\Delta_{n+1}^l(h,2)+\lambda^h\Delta_{n+1}^l(h-1,2)+\lambda^l\Delta_{n+1}^l(h,1) \\
&\quad +C\lambda[\delta_n^*(h,2)-\delta_n^*(h,1)] \\
&\leq (1-\lambda^h-\lambda^l)\Delta_{n+1}^l(h,2)+\lambda^h\Delta_{n+1}^l(h-1,2)+\lambda^l\Delta_{n+1}^l(h,1) \\
&= [1-\lambda^h-\lambda^l-C\lambda\alpha(p_{h,0}^*)]\Delta_{n+1}^l(h,2)+\lambda^h\Delta_{n+1}^l(h-1,2)+\lambda^l\Delta_{n+1}^l(h,1) \\
&\quad +C\lambda\alpha(p_{h,0}^*)\Delta_{n+1}^l(h,2),
\end{aligned}
$$

where the last inequality follows from $\delta_n^*(h,2)-\delta_n^*(h,1)\leq 0$. Then the proof of $\Delta_n^l(h,1)\geq\Delta_n^l(h,2)$ is finished as in Case 2.

Vertical boundary $0\times[2:M]$. Without a premium product, no upgrades can be offered.

$$
\begin{aligned}
V_n(0,l-1) &= (1-\lambda^l)V_{n+1}(0,l-1)+\lambda^l[p^l+V_{n+1}(0,l-2)], \\
V_n(0,l) &= (1-\lambda^l)V_{n+1}(0,l)+\lambda^l[p^l+V_{n+1}(0,l-1)], \\
V_n(0,l+1) &= (1-\lambda^l)V_{n+1}(0,l+1)+\lambda^l[p^l+V_{n+1}(0,l)].
\end{aligned}
$$

V-concavity in period $n+1$ implies $V_n(0,l)-V_n(0,l-1)\geq V_n(0,l+1)-V_n(0,l)$.

Corner $(0,1)$. No upgrades can be offered as in the vertical boundary.

$$
\begin{aligned}
V_n(0,0) &= (1-\lambda^l)V_{n+1}(0,0)+\lambda^lV_{n+1}(0,0), \\
V_n(0,1) &= (1-\lambda^l)V_{n+1}(0,1)+\lambda^l[p^l+V_{n+1}(0,0)], \\
V_n(0,2) &= (1-\lambda^l)V_{n+1}(0,2)+\lambda^l[p^l+V_{n+1}(0,1)]. \\
\Delta_n^l(0,1) &= (1-\lambda^l)\Delta_{n+1}^l(0,1)+\lambda^lp^l, \\
\Delta_n^l(0,2) &= (1-\lambda^l)\Delta_{n+1}^l(0,2)+\lambda^l\Delta_{n+1}^l(0,1).
\end{aligned}
$$

V-concavity in period $n+1$ and $p^l\geq\Delta_{n+1}^l(0,1)$ together imply $\Delta_n^l(0,1)\geq\Delta_n^l(0,2)$. $\square$

Proof of Proposition 3: From property a) of Proposition 2, we have $V_n(h+1,l+1)-V_n(h+1,l)\geq V_n(h+2,l+1)-V_n(h+2,l)$. From property d) of Proposition 2, we have

$V_n(h+1,l) - V_n(h,l) \geq V_n(h+2,l) - V_n(h+1,l)$. Thus, $V_n(h+1,l+1) - V_n(h,l) \geq V_n(h+2,l+1) - V_n(h+1,l)$ and property a) of Proposition 3 is true. From property a) of Proposition 2, we have $V_n(h+1,l) - V_n(h,l) \geq V_n(h+1,l+1) - V_n(h,l+1)$. From property e) of Proposition 2, we have $V_n(h+1,l+1) - V_n(h+1,l) \geq V_n(h+1,l+2) - V_n(h+1,l+1)$. Thus, $V_n(h+1,l+1) - V_n(h,l) \geq V_n(h+1,l+2) - V_n(h,l+1)$ and property b) of Proposition 3 is true. □

Proof of Proposition 4: From property b) of Proposition 2, we have $\Delta_n(h+1,l) \geq \Delta_n(h,l)$. Then Lemma 1 implies that $\delta_n^*(h+1,l) \geq \delta_n^*(h,l)$ and $p_n^*(h+1,l) \leq p_n^*(h,l)$. $\delta_n^*(h+1,l) \geq \delta_n^*(h,l)$ further leads to $u_n^*(h+1,l) \geq u_n^*(h,l)$. From property c) of Proposition 2, we have $\Delta_n(h,l) \geq \Delta_n(h,l+1)$. Then Lemma 1 implies that $\delta_n^*(h,l) \geq \delta_n^*(h,l+1)$ and $p_n^*(h,l) \leq p_n^*(h,l+1)$. $\delta_n^*(h,l) \geq \delta_n^*(h,l+1)$ further leads to $u_n^*(h,l) \geq u_n^*(h,l+1)$. Thus, the optimal number of upgrade links is increasing in h and decreasing in l, while the optimal upgrade fee is decreasing in h and increasing l. □

Proof of Corollary 1 The proof is identical to those of Lemma 2, Proposition 1, 2, and 4. □

Proof of Proposition 5: Property c) can be proved individually, and we prove it first.

Proof of property c): The proof is by induction. In period $N+1$, $V_{N+1}(0,l) = V_{N+1}^r(0,l) = 0$ for $l \in [0:L]$. Thus $V_{N+1}(0,l+1) - V_{N+1}(0,l) = V_{N+1}^r(0,l+1) - V_{N+1}^r(0,l)$ for $l \in [0:L-1]$. We assume $V_{n+1}(0,l+1) - V_{n+1}(0,l) = V_{n+1}^r(0,l+1) - V_{n+1}^r(0,l)$ is true, we want to validate $V_n(0,l+1) - V_n(0,l) = V_n^r(0,l+1) - V_n^r(0,l)$. From the dynamic program formulation on the vertical boundary $(0,l)$ with $l \in [1:L-1]$, we have

$$
\begin{aligned}
V_n(0,l+1) - V_n(0,l) &= (1-\lambda^l)[V_{n+1}(0,l+1) - V_{n+1}(0,l)] + \lambda^l[V_{n+1}(0,l) - V_{n+1}(0,l-1)], \\
V_n^r(0,l+1) - V_n^r(0,l) &= (1-\lambda^l)[V_{n+1}^r(0,l+1) - V_{n+1}^r(0,l)] + \lambda^l[V_{n+1}^r(0,l) - V_{n+1}^r(0,l-1)].
\end{aligned}
$$

From the dynamic program formulation on the corner point $(0,0)$, we have

$$\begin{aligned} V_n(0,1) - V_n(0,0) &= (1-\lambda^l)[V_{n+1}(0,1) - V_{n+1}(0,0)] + \lambda^l p^l, \\ V_n^r(0,1) - V_n^r(0,0) &= (1-\lambda^l)[V_{n+1}^r(0,1) - V_{n+1}^r(0,0)] + \lambda^l p^l. \end{aligned}$$

So $V_n(0,l+1) - V_n(0,l) = V_n^r(0,l+1) - V_n^r(0,l)$ for $l \in [0:L-1]$, which completes the induction step.

We prove the other two properties together by induction. They are true in period $N+1$, since $V_{N+1}(h,l) = V_{N+1}^r(h,l) = 0$. As the induction hypothesis, we assume property a) and b) are true in period $n+1$, and validate them one by one in period n. DP formulations are different on the corner point $(0,0)$ and the vertical boundary $(0,l)$ for $l > 0$. The proof of each property consists of two parts corresponding to these two regions.

We define the optimal upgrade revenue per customer and optimal upgrade fee in the restricted fee model as follows:

$$\begin{aligned} \delta_n^{r,*}(h,l) &= \max_{p \in [\underline{p},\overline{p}]} \delta_n^r(p,h,l) = \max_{p \in [\underline{p},\overline{p}]} \alpha(p)[p + V_{n+1}^r(h-1,l+1) - V_{n+1}^r(h,l)], \\ p_n^{r,*}(h,l) &= \max\{p \in [\underline{p},\overline{p}] : \delta_n^r(p,h,l) = \delta_n^{r,*}(h,l)\}. \end{aligned}$$

For simplicity, we use $p^{r,*}$ to represent $p_n^{r,*}(h,l)$ when the time period and the state are clear. Since property a) is true in period $n+1$ and $\overline{p} = p^h - p^l$, $\delta_n^*(1,l) > 0$ implies $\delta_n^{r,*}(1,l) > 0$. If upgrade is offered in the base model in period n at state $(1,l)$, it should also be offered in the restricted fee model. This argument is used repeatedly to determine if upgrades should be offered at certain states in two models. Also because of this argument, the proof for each property in each region contains multiple cases listed in Table A.3.

Proof of property a): $\{1\} \times [0:M]$ is partitioned into the vertical boundary $\{1\} \times [1:M]$ and the corner $(1,0)$. The proof is customized for each region. Property a) in period n can be expressed as in the statement of a) or alternatively as

$$V_n(1,l) - V_n^r(1,l) \;\geq\; V_n(0,l+1) - V_n^r(0,l+1) \text{ for } l \in [0:M].$$

Table A.3. Number of cases and the required properties for each property and region pair.

	Regions	
Property in period n	Vertical Boundary	Corner
a) Diagonal difference	3 cases	3 cases
Proof requires from period $n+1$	a), c)	a), b), c)
b) Horizontal difference	3 cases	3 cases
Proof requires from period $n+1$	a), b), c)	a), b), c)

We focus on the alternative expression for most of the cases for convenience. The inequality involve states $(1,l)$ and $(0,l+1)$. From property a) in period $n+1$, we know that $\delta_n^*(1,l) > 0$ implies $\delta_n^{r,*}(1,l) > 0$. Cases are constructed by examining whether $\delta_n^*(1,l)$ and $\delta_n^{r,*}(1,l)$ are positive or not.

Vertical boundary $\{1\} \times [1:M]$. There are 3 possible cases.

Case 1: $\delta_n^*(1,l) \leq 0$ and $\delta_n^{r,*}(1,l) \leq 0$.

$$\begin{aligned}
V_n(0,l+1) &= (1-\lambda^h-\lambda^l)V_{n+1}(0,l+1) + \lambda^h V_{n+1}(0,l+1) + \lambda^l[p^l + V_{n+1}(0,l)], \\
V_n^r(0,l+1) &= (1-\lambda^h-\lambda^l)V_{n+1}^r(0,l+1) + \lambda^h V_{n+1}^r(0,l+1) + \lambda^l[p^l + V_{n+1}^r(0,l)], \\
V_n(1,l) &= (1-\lambda^h-\lambda^l)V_{n+1}(1,l) + \lambda^h[p^h + V_{n+1}(0,l)] + \lambda^l[p^l + V_{n+1}(1,l-1)], \\
V_n^r(1,l) &= (1-\lambda^h-\lambda^l)V_{n+1}^r(1,l) + \lambda^h[p^h + V_{n+1}^r(0,l)] + \lambda^l[p^l + V_{n+1}^r(1,l-1)].
\end{aligned}$$

Due to property a) and c) in period $n+1$, we have $V_n(1,l) - V_n^r(1,l) \geq V_n(0,l+1) - V_n^r(0,l+1)$ through term-by-term comparisons.

Case 2: $\delta_n^*(1,l) \leq 0$ and $0 < \delta_n^{r,*}(1,l)$.

$$\begin{aligned}
V_n(1,l) &= (1-\lambda^h-\lambda^l)V_{n+1}(1,l) + \lambda^h[p^h + V_{n+1}(0,l)] + \lambda^l[p^l + V_{n+1}(1,l-1)] \\
&\geq (1-\lambda^h-\lambda^l)V_{n+1}(1,l) + \lambda^h[p^h + V_{n+1}(0,l)] + \lambda^l[p^l + V_{n+1}(1,l-1)] \\
&\quad + C\lambda\delta_n^*(1,l) \\
&\geq (1-\lambda^h-\lambda^l)V_{n+1}(1,l) + \lambda^h[p^h + V_{n+1}(0,l)] + \lambda^l[p^l + V_{n+1}(1,l-1)] \\
&\quad + C\lambda\alpha(p^{r,*})[p^{r,*} + V_{n+1}(0,l+1) - V_{n+1}(1,l)],
\end{aligned}$$

$$
\begin{aligned}
V_n^r(1,l) &= (1-\lambda^h-\lambda^l)V_{n+1}^r(1,l)+\lambda^h[p^h+V_{n+1}^r(0,l)]+\lambda^l[p^l+V_{n+1}^r(1,l-1)] \\
&\quad +C\lambda\delta_n^{r,*}(1,l) \\
&= (1-\lambda^h-\lambda^l)V_{n+1}^r(1,l)+\lambda^h[p^h+V_{n+1}^r(0,l)]+\lambda^l[p^l+V_{n+1}^r(1,l-1)] \\
&\quad +C\lambda\alpha(p^{r,*})[p^{r,*}+V_{n+1}^r(0,l+1)-V_{n+1}^r(1,l)], \\
V_n(0,l+1) &= (1-\lambda^h-\lambda^l)V_{n+1}(0,l+1)+\lambda^h V_{n+1}(0,l+1)+\lambda^l[p^l+V_{n+1}(0,l)], \\
V_n^r(0,l+1) &= (1-\lambda^h-\lambda^l)V_{n+1}^r(0,l+1)+\lambda^h V_{n+1}^r(0,l+1)+\lambda^l[p^l+V_{n+1}^r(0,l)],
\end{aligned}
$$

where the first inequality is due to $\delta_n^*(1,l)\le 0$ and the second inequality follows from the fact that $p^{r,*}$ may not be the optimal fee for $\delta_n(p,1,l)$. Using the three equalities and one inequality from above, we have

$$
\begin{aligned}
& V_n(0,l+1)-V_n^r(0,l+1) \\
= {} & (1-\lambda^h-\lambda^l)[V_{n+1}(0,l+1)-V_{n+1}^r(0,l+1)] \\
& +\lambda^h[V_{n+1}(0,l+1)-V_{n+1}^r(0,l+1)]+\lambda^l[V_{n+1}(0,l)-V_{n+1}^r(0,l)], \\
& V_n(1,l)-V_n^r(1,l) \\
\ge {} & (1-\lambda^h-\lambda^l)[V_{n+1}(1,l)-V_{n+1}^r(1,l)] \\
& +\lambda^h[V_{n+1}(0,l)-V_{n+1}^r(0,l)]+\lambda^l[V_{n+1}(1,l-1)-V_{n+1}^r(1,l-1)] \\
& +C\lambda\alpha(p^{r,*})[V_{n+1}(0,l+1)-V_{n+1}^r(0,l+1)+V_{n+1}^r(1,l)-V_{n+1}(1,l)] \\
= {} & [1-\lambda^h-\lambda^l-C\lambda\alpha(p^{r,*})][V_{n+1}(1,l)-V_{n+1}^r(1,l)] \\
& +\lambda^h[V_{n+1}(0,l)-V_{n+1}^r(0,l)]+\lambda^l[V_{n+1}(1,l-1)-V_{n+1}^r(1,l-1)] \\
& +C\lambda\alpha(p^{r,*})[V_{n+1}(0,l+1)-V_{n+1}^r(0,l+1)].
\end{aligned}
$$

Application of property c) and a) in period $n+1$ shows $\lambda^h[V_{n+1}(0,l)-V_{n+1}^r(0,l)]=\lambda^h[V_{n+1}(0,l+1)-V_{n+1}^r(0,l+1)]$ and $\lambda^l[V_{n+1}(1,l-1)-V_{n+1}^r(1,l-1)]\ge\lambda^l[V_{n+1}(0,l)-V_{n+1}^r(0,l)]$. Hence, for $V_n(1,l)-V_n^r(1,l)\ge V_n(0,l+1)-V_n^r(0,l+1)$, it is sufficient to prove

$$
\begin{aligned}
& [1-\lambda^h-\lambda^l-C\lambda\alpha(p^{r,*})][V_{n+1}(1,l)-V_{n+1}^r(1,l)] \\
\ge {} & (1-\lambda^h-\lambda^l)[V_{n+1}(0,l+1)-V_{n+1}^r(0,l+1)]-C\lambda\alpha(p^{r,*})[V_{n+1}(0,l+1)-V_{n+1}^r(0,l+1)].
\end{aligned}
$$

This also follows from property a) in period $n+1$: $[1-\lambda^h-\lambda^l-C\lambda\alpha(p^{r,*})][V_{n+1}(1,l)-V^r_{n+1}(1,l)] \geq [1-\lambda^h-\lambda^l-C\lambda\alpha(p^{r,*})][V_{n+1}(0,l+1)-V^r_{n+1}(0,l+1)]$. Hence, $V_n(1,l)-V^r_n(1,l) \geq V_n(0,l+1)-V^r_n(0,l+1)$ and the proof for this case is complete.

Case 3: $0<\delta^*_n(h,l)$ and $0<\delta^{r,*}_n(h,l)$.

$$
\begin{aligned}
V_n(0,l+1) &= (1-\lambda^h-\lambda^l)V_{n+1}(0,l+1)+\lambda^h V_{n+1}(0,l+1)+\lambda^l[p^l+V_{n+1}(0,l)],\\
V^r_n(0,l+1) &= (1-\lambda^h-\lambda^l)V^r_{n+1}(0,l+1)+\lambda^h V^r_{n+1}(0,l+1)+\lambda^l[p^l+V^r_{n+1}(0,l)],\\
V_n(1,l) &= (1-\lambda^h-\lambda^l)V_{n+1}(1,l)+\lambda^h[p^h+V_{n+1}(0,l)]+\lambda^l[p^l+V_{n+1}(1,l-1)]\\
&\quad +C\lambda\delta^*_n(1,l)\\
&\geq (1-\lambda^h-\lambda^l)V_{n+1}(1,l)+\lambda^h[p^h+V_{n+1}(0,l)]+\lambda^l[p^l+V_{n+1}(1,l-1)]\\
&\quad +C\lambda\alpha(p^{r,*})[p^{r,*}+V_{n+1}(0,l+1)-V_{n+1}(1,l)],\\
V^r_n(1,l) &= (1-\lambda^h-\lambda^l)V^r_{n+1}(1,l)+\lambda^h[p^h+V^r_{n+1}(0,l)]+\lambda^l[p^l+V^r_{n+1}(0,l-1)]\\
&\quad +C\lambda\delta^{r,*}_n(1,l)\\
&= (1-\lambda^h-\lambda^l)V^r_{n+1}(1,l)+\lambda^h[p^h+V^r_{n+1}(0,l)]+\lambda^l[p^l+V^r_{n+1}(1,l-1)]\\
&\quad +C\lambda\alpha(p^{r,*})[p^{r,*}+V^r_{n+1}(0,l+1)-V^r_{n+1}(1,l)].
\end{aligned}
$$

We further have

$$
\begin{aligned}
&V_n(1,l)-V^r_n(1,l)\\
\geq\ & (1-\lambda^h-\lambda^l)[V_{n+1}(1,l)-V^r_{n+1}(1,l)]\\
&+\lambda^h[V_{n+1}(0,l)-V^r_{n+1}(0,l)]+\lambda^l[V_{n+1}(1,l-1)-V^r_{n+1}(1,l-1)]\\
&+C\lambda\alpha(p^{r,*})[V_{n+1}(0,l+1)-V^r_{n+1}(0,l+1)+V^r_{n+1}(1,l)-V_{n+1}(1,l)]\\
=\ & [1-\lambda^h-\lambda^l-C\lambda\alpha(p^{r,*})][V_{n+1}(1,l)-V^r_{n+1}(1,l)]\\
&+\lambda^h[V_{n+1}(0,l)-V^r_{n+1}(0,l)]+\lambda^l[V_{n+1}(1,l-1)-V^r_{n+1}(1,l-1)]\\
&+C\lambda\alpha(p^{r,*})[V_{n+1}(0,l+1)-V^r_{n+1}(0,l+1)],
\end{aligned}
$$

$$
\begin{aligned}
& V_n(0,l+1) - V_n^r(0,l+1) \\
= \; & (1-\lambda^h-\lambda^l)[V_{n+1}(0,l+1) - V_{n+1}^r(0,l+1)] \\
& +\lambda^h[V_{n+1}(0,l+1) - V_{n+1}^r(0,l+1)] + \lambda^l[V_{n+1}(0,l) - V_{n+1}^r(0,l)].
\end{aligned}
$$

Then the proof of $V_n(1,l) - V_n^r(1,l) \geq V_n(0,l+1) - V_n^r(0,l+1)$ is identical to that of Case 2.

Corner $(1,0)$. Upgrades cannot be offered at state $(0,1)$ due to zero premium capacity. There are 3 possible cases similar to vertical boundary.

Case 1: $\delta_n^*(1,0) \leq 0$ and $\delta_n^{r,*}(1,0) \leq 0$.

$$
\begin{aligned}
V_n(0,1) &= (1-\lambda^h-\lambda^l)V_{n+1}(0,1) + \lambda^h V_{n+1}(0,1) + \lambda^l[p^l + V_{n+1}(0,0)], \\
V_n^r(0,1) &= (1-\lambda^h-\lambda^l)V_{n+1}^r(0,1) + \lambda^h V_{n+1}^r(0,1) + \lambda^l[p^l + V_{n+1}^r(0,0)], \\
V_n(1,0) &= (1-\lambda^h-\lambda^l)V_{n+1}(1,0) + \lambda^h[p^h + V_{n+1}(0,0)] + \lambda^l V_{n+1}(1,0), \\
V_n^r(1,0) &= (1-\lambda^h-\lambda^l)V_{n+1}^r(1,0) + \lambda^h[p^h + V_{n+1}^r(0,0)] + \lambda^l V_{n+1}^r(1,0).
\end{aligned}
$$

Then we have

$$
\begin{aligned}
V_n(0,1) - V_n^r(0,1) &= (1-\lambda^h-\lambda^l)[V_{n+1}(0,1) - V_{n+1}^r(0,1)] + \lambda^h[V_{n+1}(0,1) - V_{n+1}^r(0,1)] \\
&\quad +\lambda^l[V_{n+1}(0,0) - V_{n+1}^r(0,0)], \\
V_n(1,0) - V_n^r(1,0) &= (1-\lambda^h-\lambda^l)[V_{n+1}(1,0) - V_{n+1}^r(1,0)] + \lambda^h[V_{n+1}(0,0) - V_{n+1}^r(0,0)] \\
&\quad +\lambda^l[V_{n+1}(1,0) - V_{n+1}^r(1,0)].
\end{aligned}
$$

Due to property a) in period $n+1$, we have $V_{n+1}(0,1) - V_{n+1}^r(0,1) \leq V_{n+1}(1,0) - V_{n+1}^r(1,0)$. Due to property c) in period $n+1$, we have $V_{n+1}(0,1) - V_{n+1}^r(0,1) = V_{n+1}(0,0) - V_{n+1}^r(0,0)$. Due to property b) in period $n+1$, we have $V_{n+1}(0,0) - V_{n+1}^r(0,0) \leq V_{n+1}(1,0) - V_{n+1}^r(1,0)$. Thus we have $V_n(0,1) - V_n^r(0,1) \leq V_n(1,0) - V_n^r(1,0)$ through term-by-term comparisons.

Case 2: $\delta_n^*(1,0) \leq 0$ and $\delta_n^{r,*}(1,0) > 0$.

$$
\begin{aligned}
V_n(0,1) &= (1-\lambda^h-\lambda^l)V_{n+1}(0,1) + \lambda^h V_{n+1}(0,1) + \lambda^l[p^l + V_{n+1}(0,0)], \\
V_n^r(0,1) &= (1-\lambda^h-\lambda^l)V_{n+1}^r(0,1) + \lambda^h V_{n+1}^r(0,1) + \lambda^l[p^l + V_{n+1}^r(0,0)],
\end{aligned}
$$

$$\begin{aligned}
V_n(1,0) &= (1-\lambda^h-\lambda^l)V_{n+1}(1,0)+\lambda^h[p^h+V_{n+1}(0,0)]+\lambda^l V_{n+1}(1,0) \\
&\geq (1-\lambda^h-\lambda^l)V_{n+1}(1,0)+\lambda^h[p^h+V_{n+1}(0,0)]+\lambda^l V_{n+1}(1,0)+C\lambda\delta_n^*(1,0) \\
&\geq (1-\lambda^h-\lambda^l)V_{n+1}(1,0)+\lambda^h[p^h+V_{n+1}(0,0)]+\lambda^l V_{n+1}(1,0) \\
&\quad +C\lambda\alpha(p^{r,*})[p^{r,*}+V_{n+1}(0,1)-V_{n+1}(1,0)], \\
V_n^r(1,0) &= (1-\lambda^h-\lambda^l)V_{n+1}^r(1,0)+\lambda^h[p^h+V_{n+1}^r(0,0)]+\lambda^l V_{n+1}^r(1,0)+C\lambda\delta_n^{r,*}(1,0) \\
&= (1-\lambda^h-\lambda^l)V_{n+1}^r(1,0)+\lambda^h[p^h+V_{n+1}^r(0,0)]+\lambda^l V_{n+1}^r(1,0) \\
&\quad +C\lambda\alpha(p^{r,*})[p^{r,*}+V_{n+1}^r(0,1)-V_{n+1}^r(1,0)],
\end{aligned}$$

where the first inequality is due to $\delta_n^*(1,0) \leq 0$ and the second inequality follows from the fact that $p^{r,*}$ may not be the optimal fee for $\delta_n(p,1,0)$. Using the three equalities and one inequality from above, we have

$$\begin{aligned}
V_n(1,0)-V_n^r(1,0) &\geq (1-\lambda^h-\lambda^l)[V_{n+1}(1,0)-V_{n+1}^r(1,0)] \\
&\quad +\lambda^h[V_{n+1}(0,0)-V_{n+1}^r(0,0)]+\lambda^l[V_{n+1}(1,0)-V_{n+1}^r(1,0)] \\
&\quad +C\lambda\alpha(p^{r,*})[V_{n+1}(0,1)-V_{n+1}^r(0,1)+V_{n+1}^r(1,0)-V_{n+1}(1,0)] \\
&= [1-\lambda^h-\lambda^l-C\lambda\alpha(p^{r,*})][V_{n+1}(1,0)-V_{n+1}^r(1,0)] \\
&\quad +\lambda^h[V_{n+1}(0,0)-V_{n+1}^r(0,0)] \\
&\quad +\lambda^l[V_{n+1}(1,0)-V_{n+1}^r(1,0)]+C\lambda\alpha(p^{r,*})[V_{n+1}(0,1)-V_{n+1}^r(0,1)], \\
V_n(0,1)-V_n^r(0,1) &= (1-\lambda^h-\lambda^l)[V_{n+1}(0,1)-V_{n+1}^r(0,1)]+\lambda^h[V_{n+1}(0,1)-V_{n+1}^r(0,1)] \\
&\quad +\lambda^l[V_{n+1}(0,0)-V_{n+1}^r(0,0)],
\end{aligned}$$

Application of property c) and b) in period $n+1$ shows $\lambda^h[V_{n+1}(0,0)-V_{n+1}^r(0,0)] = \lambda^h[V_{n+1}(0,1)-V_{n+1}^r(0,1)]$ and $\lambda^l[V_{n+1}(1,0)-V_{n+1}^r(1,0)] \geq \lambda^l[V_{n+1}(0,0)-V_{n+1}^r(0,0)]$. Hence, for $V_n(1,0)-V_n^r(1,0) \geq V_n(0,1)-V_n^r(0,1)$, it is sufficient to prove

$$\begin{aligned}
&[1-\lambda^h-\lambda^l-C\lambda\alpha(p^{r,*})][V_{n+1}(1,0)-V_{n+1}^r(1,0)]+C\lambda\alpha(p^{r,*})[V_{n+1}(0,1)-V_{n+1}^r(0,1)] \\
\geq\ &(1-\lambda^h-\lambda^l)[V_{n+1}(0,1)-V_{n+1}^r(0,1)].
\end{aligned}$$

This also follows from property a) in period $n+1$: $[1-\lambda^h-\lambda^l-C\lambda\alpha(p^{r,*})][V_{n+1}(1,0)-V^r_{n+1}(1,0)] \geq [1-\lambda^h-\lambda^l-C\lambda\alpha(p^{r,*})][V_{n+1}(0,1)-V^r_{n+1}(0,1)]$. Hence, $V_n(1,0)-V^r_n(1,0) \geq V_n(0,1)-V^r_n(0,1)$ and the proof for this case is complete.

Case 3: $\delta^*_n(1,0) > 0$ and $\delta^{r,*}_n(1,0) > 0$.

$$\begin{aligned}
V_n(0,1) &= (1-\lambda^h-\lambda^l)V_{n+1}(0,1)+\lambda^h V_{n+1}(0,1)+\lambda^l[p^l+V_{n+1}(0,0)],\\
V^r_n(0,1) &= (1-\lambda^h-\lambda^l)V^r_{n+1}(0,1)+\lambda^h V^r_{n+1}(0,1)+\lambda^l[p^l+V^r_{n+1}(0,0)],\\
V_n(1,0) &= (1-\lambda^h-\lambda^l)V_{n+1}(1,0)+\lambda^h[p^h+V_{n+1}(0,0)]+\lambda^l V_{n+1}(1,0)+C\lambda\delta^*_n(1,0)\\
&\geq (1-\lambda^h-\lambda^l)V_{n+1}(1,0)+\lambda^h[p^h+V_{n+1}(0,0)]+\lambda^l V_{n+1}(1,0)\\
&\quad +C\lambda\alpha(p^{r,*})[p^{r,*}+V_{n+1}(0,1)-V_{n+1}(1,0)],\\
V^r_n(1,0) &= (1-\lambda^h-\lambda^l)V^r_{n+1}(1,0)+\lambda^h[p^h+V^r_{n+1}(0,0)]+\lambda^l V^r_{n+1}(1,0)\\
&\quad +C\lambda\alpha(p^{r,*})[p^{r,*}+V^r_{n+1}(0,1)-V^r_{n+1}(1,0)].
\end{aligned}$$

Then the proof of $V_n(1,0)-V^r_n(1,0) \geq V_n(0,1)-V^r_n(0,1)$ is identical to that of Case 2.

Proof of property b): $0 \times [0:M]$ is partitioned into the vertical boundary $\{0\} \times [1:M]$ and the corner $(0,0)$. The proof is customized for each region. Property b) in period n can be expressed as in the statement of b) or alternatively as $V_n(1,l)-V^r_n(1,l) \geq V_n(0,l)-V^r_n(0,l)$ for $l \in [0:M]$. We focus on the alternative expression for most of the cases, since it is more convenient to prove. The inequality involve states $(1,l)$ and $(0,l)$. From property a) in period $n+1$, $\delta^*_n(1,l) > 0$ implies $\delta^{r,*}_n(1,l) > 0$. Cases are constructed by examining whether $\delta^*_n(1,l)$ and $\delta^{r,*}_n(1,l)$ are positive or not.

<u>Vertical boundary $\{1\} \times [1:M]$</u>. There are 3 possible cases.

Case 1: $\delta^*_n(1,l) \leq 0$ and $\delta^{r,*}_n(1,l) \leq 0$.

$$\begin{aligned}
V_n(1,l) &= (1-\lambda^h-\lambda^l)V_{n+1}(1,l)+\lambda^h[p^h+V_{n+1}(0,l)]+\lambda^l[p^l+V_{n+1}(1,l-1)],\\
V^r_n(1,l) &= (1-\lambda^h-\lambda^l)V^r_{n+1}(1,l)+\lambda^h[p^h+V^r_{n+1}(0,l)]+\lambda^l[p^l+V^r_{n+1}(1,l-1)],\\
V_n(0,l) &= (1-\lambda^h-\lambda^l)V_{n+1}(0,l)+\lambda^h V_{n+1}(0,l)+\lambda^l[p^l+V_{n+1}(0,l-1)],\\
V^r_n(0,l) &= (1-\lambda^h-\lambda^l)V^r_{n+1}(0,l)+\lambda^h V^r_{n+1}(0,l)+\lambda^l[p^l+V^r_{n+1}(0,l-1)].
\end{aligned}$$

Due to property b) in period $n+1$, we have $V_n(1,l)-V_n^r(1,l) \geq V_n(0,l)-V_n^r(0,l)$ through term-by-term comparisons.

Case 2: $\delta_n^*(1,l) \leq 0$ and $0 < \delta_n^{r,*}(1,l)$.

$$\begin{aligned}
V_n(1,l) &= (1-\lambda^h-\lambda^l)V_{n+1}(1,l) + \lambda^h[p^h + V_{n+1}(0,l)] + \lambda^l[p^l + V_{n+1}(1,l-1)] \\
&\geq (1-\lambda^h-\lambda^l)V_{n+1}(1,l) + \lambda^h[p^h + V_{n+1}(0,l)] + \lambda^l[p^l + V_{n+1}(1,l-1)] + C\lambda\delta_n^*(1,l) \\
&\geq (1-\lambda^h-\lambda^l)V_{n+1}(1,l) + \lambda^h[p^h + V_{n+1}(0,l)] + \lambda^l[p^l + V_{n+1}(1,l-1)] \\
&\quad +C\lambda\alpha(p^{r,*})[p^{r,*} + V_{n+1}(0,l+1) - V_{n+1}(1,l)], \\
V_n^r(1,l) &= (1-\lambda^h-\lambda^l)V_{n+1}^r(1,l) + \lambda^h[p^h + V_{n+1}^r(0,l)] + \lambda^l[p^l + V_{n+1}^r(1,l-1)] \\
&\quad +C\lambda\alpha(p^{r,*})[p^{r,*} + V_{n+1}^r(0,l+1) - V_{n+1}^r(1,l)], \\
V_n(0,l) &= (1-\lambda^h-\lambda^l)V_{n+1}(0,l) + \lambda^h V_{n+1}(0,l) + \lambda^l[p^l + V_{n+1}(0,l-1)], \\
V_n^r(0,l) &= (1-\lambda^h-\lambda^l)V_{n+1}^r(0,l) + \lambda^h V_{n+1}^r(0,l) + \lambda^l[p^l + V_{n+1}^r(0,l-1)],
\end{aligned}$$

where the first inequality is due to $\delta_n^*(1,l) \leq 0$ and the second inequality follows from the fact that $p^{r,*}$ may not be the optimal fee for $\delta_n(p,1,l)$. Using the three equalities and one inequality from above, we have

$$\begin{aligned}
&V_n(1,l) - V_n^r(1,l) \\
\geq\ & (1-\lambda^h-\lambda^l)[V_{n+1}(1,l) - V_{n+1}^r(1,l)] \\
&+\lambda^h[V_{n+1}(0,l) - V_{n+1}^r(0,l)] + \lambda^l[V_{n+1}(1,l-1) - V_{n+1}^r(1,l-1)] \\
&+C\lambda\alpha(p^{r,*})[V_{n+1}(0,l+1) - V_{n+1}^r(0,l+1) + V_{n+1}^r(1,l) - V_{n+1}(1,l)] \\
=\ & [1-\lambda^h-\lambda^l - C\lambda\alpha(p^{r,*})][V_{n+1}(1,l) - V_{n+1}^r(1,l)] + \lambda^h[V_{n+1}(0,l) - V_{n+1}^r(0,l)] \\
&+\lambda^l[V_{n+1}(1,l-1) - V_{n+1}^r(1,l-1)] + C\lambda\alpha(p^{r,*})[V_{n+1}(0,l+1) - V_{n+1}^r(0,l+1)], \\
&V_n(0,l) - V_n^r(0,l) \\
=\ & (1-\lambda^h-\lambda^l)[V_{n+1}(0,l) - V_{n+1}^r(0,l)] + \lambda^h[V_{n+1}(0,l) - V_{n+1}^r(0,l)] \\
&+\lambda^l[V_{n+1}(0,l-1) - V_{n+1}^r(0,l-1)].
\end{aligned}$$

Application of property b) in period $n+1$ shows $\lambda^l[V_{n+1}(1,l-1) - V^r_{n+1}(1,l-1)] \geq \lambda^l[V_{n+1}(0,l-1) - V^r_{n+1}(0,l-1)]$. Hence, for $V_n(1,l) - V^r_n(1,l) \geq V_n(0,l) - V^r_n(0,l)$, it is sufficient to prove

$$
\begin{aligned}
&[1-\lambda^h-\lambda^l-C\lambda\alpha(p^{r,*})][V_{n+1}(1,l)-V^r_{n+1}(1,l)] \\
&\geq (1-\lambda^h-\lambda^l)[V_{n+1}(0,l)-V^r_{n+1}(0,l)] - C\lambda\alpha(p^{r,*})[V_{n+1}(0,l+1)-V^r_{n+1}(0,l+1)].
\end{aligned}
$$

This also follows from property b) in period $n+1$: $V_{n+1}(1,l) - V^r_{n+1}(1,l) \geq V_{n+1}(0,l) - V^r_{n+1}(0,l)$ and property c) in period $n+1$: $V_{n+1}(0,l+1) - V^r_{n+1}(0,l+1) = V_{n+1}(0,l) - V^r_{n+1}(0,l)$. Hence, $V_n(1,l) - V^r_n(1,l) \geq V_n(0,l) - V^r_n(0,l)$ and the proof for this case is complete.

Case 3: $0 < \delta^*_n(1,l)$ and $0 < \delta^{r,*}_n(1,l)$.

$$
\begin{aligned}
V_n(1,l) &= (1-\lambda^h-\lambda^l)V_{n+1}(1,l) + \lambda^h[p^h + V_{n+1}(0,l)] + \lambda^l[p^l + V_{n+1}(1,l-1)] \\
&\quad + C\lambda\delta^*_n(1,l) \\
&\geq (1-\lambda^h-\lambda^l)V_{n+1}(1,l) + \lambda^h[p^h + V_{n+1}(0,l)] + \lambda^l[p^l + V_{n+1}(1,l-1)] \\
&\quad + C\lambda\alpha(p^{r,*})[p^{r,*} + V_{n+1}(0,l+1) - V_{n+1}(1,l)], \\
V^r_n(1,l) &= (1-\lambda^h-\lambda^l)V^r_{n+1}(1,l) + \lambda^h[p^h + V^r_{n+1}(0,l)] + \lambda^l[p^l + V^r_{n+1}(1,l-1)] \\
&\quad + C\lambda\delta^{r,*}_n(1,l) \\
&= (1-\lambda^h-\lambda^l)V^r_{n+1}(1,l) + \lambda^h[p^h + V^r_{n+1}(0,l)] + \lambda^l[p^l + V^r_{n+1}(1,l-1)] \\
&\quad + C\lambda\alpha(p^{r,*})[p^{r,*} + V^r_{n+1}(0,l+1) - V^r_{n+1}(1,l)], \\
V_n(0,l) &= (1-\lambda^h-\lambda^l)V_{n+1}(0,l) + \lambda^h V_{n+1}(0,l) + \lambda^l[p^l + V_{n+1}(0,l-1)], \\
V^r_n(0,l) &= (1-\lambda^h-\lambda^l)V^r_{n+1}(0,l) + \lambda^h V^r_{n+1}(0,l) + \lambda^l[p^l + V^r_{n+1}(0,l-1)].
\end{aligned}
$$

Then the proof of $V_n(1,l) - V^r_n(1,l) \geq V_n(0,l) - V^r_n(0,l)$ is identical to that of Case 2.

<u>Corner point $\{0\} \times \{0\}$</u>. Upgrades cannot be offered at state $(0,1)$ due to zero premium capacity. There are 3 possible cases similar to the vertical boundary.

Case 1: $\delta_n^*(1,0) \le 0$ and $\delta_n^{r,*}(1,0) \le 0$.

$$
\begin{aligned}
V_n(1,0) &= (1-\lambda^h-\lambda^l)V_{n+1}(1,0) + \lambda^h[p^h + V_{n+1}(0,0)] + \lambda^l V_{n+1}(1,0), \\
V_n^r(1,0) &= (1-\lambda^h-\lambda^l)V_{n+1}^r(1,0) + \lambda^h[p^h + V_{n+1}^r(0,0)] + \lambda^l V_{n+1}^r(1,0), \\
V_n(0,0) &= (1-\lambda^h-\lambda^l)V_{n+1}(0,0) + \lambda^h V_{n+1}(0,0) + \lambda^l V_{n+1}(0,0), \\
V_n^r(0,0) &= (1-\lambda^h-\lambda^l)V_{n+1}^r(0,0) + \lambda^h V_{n+1}^r(0,0) + \lambda^l V_{n+1}^r(0,0).
\end{aligned}
$$

Due to property b) in period $n+1$, we have $V_n(1,0) - V_n^r(1,0) \ge V_n(0,0) - V_n^r(0,0)$ through term-by-term comparisons.

Case 2: $\delta_n^*(1,0) \le 0$ and $\delta_n^{r,*}(1,0) > 0$.

$$
\begin{aligned}
V_n(1,0) &= (1-\lambda^h-\lambda^l)V_{n+1}(1,0) + \lambda^h[p^h + V_{n+1}(0,0)] + \lambda^l V_{n+1}(1,0) \\
&\ge (1-\lambda^h-\lambda^l)V_{n+1}(1,0) + \lambda^h[p^h + V_{n+1}(0,0)] + \lambda^l V_{n+1}(1,0) + C\lambda\delta_n^*(1,0) \\
&\ge (1-\lambda^h-\lambda^l)V_{n+1}(1,0) + \lambda^h[p^h + V_{n+1}(0,0)] + \lambda^l V_{n+1}(1,0) \\
&\quad + C\lambda\alpha(p^{r,*})[p^{r,*} + V_{n+1}(0,1) - V_{n+1}(1,0)], \\
V_n^r(1,0) &= (1-\lambda^h-\lambda^l)V_{n+1}^r(1,0) + \lambda^h[p^h + V_{n+1}^r(0,0)] + \lambda^l V_{n+1}^r(1,0) \\
&\quad + C\lambda\alpha(p^{r,*})[p^{r,*} + V_{n+1}^r(0,1) - V_{n+1}^r(1,0)], \\
V_n(0,0) &= (1-\lambda^h-\lambda^l)V_{n+1}(0,0) + \lambda^h V_{n+1}(0,0) + \lambda^l V_{n+1}(0,0), \\
V_n^r(0,0) &= (1-\lambda^h-\lambda^l)V_{n+1}^r(0,0) + \lambda^h V_{n+1}^r(0,0) + \lambda^l V_{n+1}^r(0,0),
\end{aligned}
$$

where the first inequality is due to $\delta_n^*(1,0) \le 0$ and the second inequality follows from the fact that $p^{r,*}$ may not be the optimal fee for $\delta_n(p,1,0)$. Using the three equalities and one inequality from above, we have

$$
\begin{aligned}
V_n(0,0) - V_n^r(0,0) &= (1-\lambda^h-\lambda^l)[V_{n+1}(0,0) - V_{n+1}^r(0,0)] + \lambda^h[V_{n+1}(0,0) - V_{n+1}^r(0,0)] \\
&\quad + \lambda^l[V_{n+1}(0,0) - V_{n+1}^r(0,0)],
\end{aligned}
$$

$$\begin{aligned}
V_n(1,0) - V_n^r(1,0) &\geq (1-\lambda^h-\lambda^l)[V_{n+1}(1,0) - V_{n+1}^r(1,0)] \\
&\quad +\lambda^h[V_{n+1}(0,0) - V_{n+1}^r(0,0)] + \lambda^l[V_{n+1}(1,0) - V_{n+1}^r(1,0)] \\
&\quad +C\lambda\alpha(p^{r,*})[V_{n+1}(0,1) - V_{n+1}^r(0,1) + V_{n+1}^r(1,0) - V_{n+1}(1,0)] \\
&= [1-\lambda^h-\lambda^l - C\lambda\alpha(p^{r,*})][V_{n+1}(1,0) - V_{n+1}^r(1,0)] \\
&\quad +\lambda^h[V_{n+1}(0,0) - V_{n+1}^r(0,0)] \\
&\quad +\lambda^l[V_{n+1}(1,0) - V_{n+1}^r(1,0)] + C\lambda\alpha(p^{r,*})[V_{n+1}(0,1) - V_{n+1}^r(0,1)].
\end{aligned}$$

Application of property b) in period $n+1$ shows $\lambda^l[V_{n+1}(1,0) - V_{n+1}^r(1,0)] \geq \lambda^l[V_{n+1}(0,0) - V_{n+1}^r(0,0)]$. Hence, for $V_n(1,0) - V_n^r(1,0) \geq V_n(0,0) - V_n^r(0,0)$, it is sufficient to prove

$$\begin{aligned}
&[1-\lambda^h-\lambda^l - C\lambda\alpha(p^{r,*})][V_{n+1}(1,0) - V_{n+1}^r(1,0)] + C\lambda\alpha(p^{r,*})[V_{n+1}(0,1) - V_{n+1}^r(0,1)] \\
&\geq (1-\lambda^h-\lambda^l)[V_{n+1}(0,0) - V_{n+1}^r(0,0)].
\end{aligned}$$

This also follows from property b) in period $n+1$: $V_{n+1}(1,0) - V_{n+1}^r(1,0) \geq V_{n+1}(0,0) - V_{n+1}^r(0,0)$ and property c) in period $n+1$: $V_{n+1}(0,1) - V_{n+1}^r(0,1) = V_{n+1}(0,0) - V_{n+1}^r(0,0)$. Hence, $V_n(1,0) - V_n^r(1,0) \geq V_n(0,0) - V_n^r(0,0)$ and the proof for this case is complete.

Case 3: $\delta_n^*(1,0) > 0$ and $\delta_n^{r,*}(1,0) > 0$.

$$\begin{aligned}
V_n(1,0) &= (1-\lambda^h-\lambda^l)V_{n+1}(1,0) + \lambda^h[p^h + V_{n+1}(0,0)] + \lambda^l V_{n+1}(1,0) + C\lambda\delta_n^*(1,0) \\
&\geq (1-\lambda^h-\lambda^l)V_{n+1}(1,0) + \lambda^h[p^h + V_{n+1}(0,0)] + \lambda^l V_{n+1}(1,0) \\
&\quad +C\lambda\alpha(p^{r,*})[p^{r,*} + V_{n+1}(0,1) - V_{n+1}(1,0)], \\
V_n^r(1,0) &= (1-\lambda^h-\lambda^l)V_{n+1}^r(1,0) + \lambda^h[p^h + V_{n+1}^r(0,0)] + \lambda^l V_{n+1}^r(1,0) \\
&\quad +C\lambda\alpha(p^{r,*})[p^{r,*} + V_{n+1}^r(0,1) - V_{n+1}^r(1,0)], \\
V_n(0,0) &= (1-\lambda^h-\lambda^l)V_{n+1}(0,0) + \lambda^h V_{n+1}(0,0) + \lambda^l V_{n+1}(0,0), \\
V_n^r(0,0) &= (1-\lambda^h-\lambda^l)V_{n+1}^r(0,0) + \lambda^h V_{n+1}^r(0,0) + \lambda^l V_{n+1}^r(0,0),
\end{aligned}$$

Then the proof of $V_n(1,0) - V_n^r(1,0) \geq V_n(0,0) - V_n^r(0,0)$ is identical to that of Case 2. $\square$

Proof of Proposition 6: $u_n^*(1,l) > 0$ implies that an optimal fee $p_n^*(1,l) \in [0, p^h - p^l]$ makes $\delta_n^*(1,l) = \alpha(p_n^*(1,l))[p_n^*(1,l) + \Delta_n(1,l)] > 0$. From property a) of Proposition 5, we have $\Delta_n(1,l) \leq \Delta_n^r(1,l)$. Since $0 \leq \underline{p} \leq \overline{p} = p^h - p^l$, there must exist a $p \in [\underline{p}, \overline{p}]$ which makes $\alpha(p)[p + \Delta_n^r(1,l)] > 0$. Then, $\delta_n^{r,*}(1,l) = \alpha(p_n^{r,*}(1,l))[p_n^{r,*}(1,l) + \Delta_n^r(1,l)] > 0$. Thus, $u_n^*(1,l) > 0$ implies $u_n^{r,*}(1,l) > 0$. $\square$

Proof of Corollary 2 The proof is identical to that of Proposition 1. $\square$

Proof of Proposition 7:

The comparison of the optimal substitution fee and the optimal upgrade fee requires us to examine the optimization of the following two revenue functions:

$$\alpha(f^s)[f^s + p^l + V_{n+1}^s(h-1,0) - V_{n+1}^s(h,0)] \text{ and } \alpha(p^s)[p^s + V_{n+1}^s(h-1,1) - V_{n+1}^s(h,0)].$$

From Lemma 2, we know that $f_n^{s,*}(h) \leq p_n^{s,*}(h,0)$ if $p^l + V_{n+1}^s(h-1,0) - V_{n+1}^s(h,0) \geq V_{n+1}^s(h-1,1) - V_{n+1}^s(h,0)$. The second inequality is equivalent to $V_{n+1}^s(h-1,1) - V_{n+1}^s(h-1,0) \leq p^l$; a unit of regular capacity cannot bring more revenue than its market price. A formal induction proof is provided as follows.

$V_{N+1}^s(h,l+1) - V_{N+1}^s(h,l) \leq p^l$ is true in period $N+1$, since $V_{N+1}(h,l) = 0$. As the induction hypothesis, we assume the inequality is true in period $n+1$ and validate it in period n. DP formulations are different on the corner point $(0,0)$, two boundaries $(0,l)$ for $l > 0$ and $(h,0)$ for $h > 0$, and in the interior region (h,l) for $h,l > 0$. The proof consists of four parts corresponding to these four regions. For brevity, we use $p_{h,l}^{s,*}$ to represent $p_n^{s,*}(h,l)$ and $f_h^{s,*}$ to represent $f_n^{s,*}(h)$ when the time period is clear.

Interior $[1:H] \times [1:L]$. There are 4 possible cases.

Case 1: $\delta_n^{s,*}(h, l+1) \leq 0$ and $\delta_n^{s,*}(h, l) \leq 0$.

$$\begin{aligned}
V_n^s(h, l+1) &= (1-\lambda^h-\lambda^l)V_{n+1}^s(h, l+1) + \lambda^h[p^h + V_{n+1}^s(h-1, l+1)] \\
&\quad +\lambda^l[p^l + V_{n+1}^s(h, l)], \\
V_n^s(h, l) &= (1-\lambda^h-\lambda^l)V_{n+1}^s(h, l) + \lambda^h[p^h + V_{n+1}^s(h-1, l)] \\
&\quad +\lambda^l[p^l + V_{n+1}^s(h, l-1)].
\end{aligned}$$

$V_{n+1}^s(h, l+1) - V_{n+1}^s(h, l) \leq p^l$ and term-by-term comparisons of the value functions above yield $V_n^s(h, l+1) - V_n^s(h, l) \leq p^l$.

Case 2: $\delta_n^{s,*}(h, l+1) > 0$ and $\delta_n^{s,*}(h, l) \leq 0$.

$$\begin{aligned}
V_n^s(h, l+1) &= (1-\lambda^h-\lambda^l)V_{n+1}^s(h, l+1) + \lambda^h[p^h + V_{n+1}^s(h-1, l+1)] \\
&\quad +\lambda^l[p^l + V_{n+1}^s(h, l)] + C\lambda\delta_n^{s,*}(h, l+1) \\
&= (1-\lambda^h-\lambda^l)V_{n+1}^s(h, l+1) + \lambda^h[p^h + V_{n+1}^s(h-1, l+1)] \\
&\quad +\lambda^l[p^l + V_{n+1}^s(h, l)] \\
&\quad +C\lambda\alpha(p_{h,l+1}^{s,*})[p_{h,l+1}^{s,*} + V_{n+1}^s(h-1, 1+2) - V_{n+1}^s(h, l+1)], \\
V_n^s(h, l) &= (1-\lambda^h-\lambda^l)V_{n+1}^s(h, l) + \lambda^h[p^h + V_{n+1}^s(h-1, l)] + \lambda^l[p^l + V_{n+1}^s(h, l-1)] \\
&\geq (1-\lambda^h-\lambda^l)V_{n+1}^s(h, l) + \lambda^h[p^h + V_{n+1}^s(h-1, l)] + \lambda^l[p^l + V_{n+1}^s(h, l-1)] \\
&\quad +C\lambda\delta_n^{s,*}(h, l) \\
&\geq (1-\lambda^h-\lambda^l)V_{n+1}^s(h, l) + \lambda^h[p^h + V_{n+1}^s(h-1, l)] + \lambda^l[p^l + V_{n+1}^s(h, l-1)] \\
&\quad +C\lambda\alpha(p_{h,l+1}^{s,*})[p_{h,l+1}^{s,*} + V_{n+1}^s(h-1, 1+1) - V_{n+1}^s(h, l)],
\end{aligned}$$

where the first inequality is from $\delta_n^{s,*}(h, l) \leq 0$ and the second inequality is due to the non-optimality of $p_{h,l+1}^{s,*}$ for $\alpha(p^s)[p^s + V_{n+1}^s(h-1, l+1) - V_{n+1}^s(h, l)]$. Then, $V_{n+1}^s(h, l+1) - V_{n+1}^s(h, l) \leq p^l$ and term-by-term comparisons of the value functions above yield $V_n^s(h, l+1) - V_n^s(h, l) \leq p^l$.

Case 3: $\delta_n^{s,*}(h,l+1) \le 0$ and $\delta_n^{s,*}(h,l) > 0$.

$$
\begin{aligned}
V_n^s(h,l+1) &= (1-\lambda^h-\lambda^l)V_{n+1}^s(h,l+1) + \lambda^h[p^h + V_{n+1}^s(h-1,l+1)] \\
&\quad + \lambda^l[p^l + V_{n+1}^s(h,l)], \\
V_n^s(h,l) &= (1-\lambda^h-\lambda^l)V_{n+1}^s(h,l) + \lambda^h[p^h + V_{n+1}^s(h-1,l)] + \lambda^l[p^l + V_{n+1}^s(h,l-1)] \\
&\quad + C\lambda\delta_n^{s,*}(h,l) \\
&> (1-\lambda^h-\lambda^l)V_{n+1}^s(h,l) + \lambda^h[p^h + V_{n+1}^s(h-1,l)] + \lambda^l[p^l + V_{n+1}^s(h,l-1)],
\end{aligned}
$$

where the inequality is from $\delta_n^{s,*}(h,l) > 0$. Then, $V_{n+1}^s(h,l+1) - V_{n+1}^s(h,l) \le p^l$ and term-by-term comparisons of the value functions above yield $V_n^s(h,l+1) - V_n^s(h,l) \le p^l$.

Case 4: $\delta_n^{s,*}(h,l+1) > 0$ and $\delta_n^{s,*}(h,l) > 0$.

$$
\begin{aligned}
V_n^s(h,l+1) &= (1-\lambda^h-\lambda^l)V_{n+1}^s(h,l+1) + \lambda^h[p^h + V_{n+1}^s(h-1,l+1)] \\
&\quad + \lambda^l[p^l + V_{n+1}^s(h,l)] + C\lambda\delta_n^{s,*}(h,l+1) \\
&= (1-\lambda^h-\lambda^l)V_{n+1}^s(h,l+1) + \lambda^h[p^h + V_{n+1}^s(h-1,l+1)] \\
&\quad + \lambda^l[p^l + V_{n+1}^s(h,l)] \\
&\quad + C\lambda\alpha(p_{h,l+1}^{s,*})[p_{h,l+1}^{s,*} + V_{n+1}^s(h-1,1+2) - V_{n+1}^s(h,l+1)], \\
V_n^s(h,l) &= (1-\lambda^h-\lambda^l)V_{n+1}^s(h,l) + \lambda^h[p^h + V_{n+1}^s(h-1,l)] \\
&\quad + \lambda^l[p^l + V_{n+1}^s(h,l-1)] + C\lambda\delta_n^{s,*}(h,l) \\
&\ge (1-\lambda^h-\lambda^l)V_{n+1}^s(h,l) + \lambda^h[p^h + V_{n+1}^s(h-1,l)] + \lambda^l[p^l + V_{n+1}^s(h,l-1)] \\
&\quad + C\lambda\alpha(p_{h,l+1}^{s,*})[p_{h,l+1}^{s,*} + V_{n+1}^s(h-1,1+1) - V_{n+1}^s(h,l)],
\end{aligned}
$$

where the inequality is due to the non-optimality of $p_{h,l+1}^{s,*}$ for $\alpha(p^s)[p^s + V_{n+1}^s(h-1,l+1) - V_{n+1}^s(h,l)]$. Then, $V_{n+1}^s(h,l+1) - V_{n+1}^s(h,l) \le p^l$ and term-by-term comparisons of the value functions above yield $V_n^s(h,l+1) - V_n^s(h,l) \le p^l$.

<u>Horizontal boundary $[1:H]\times\{1\}$</u>. There are 4 cases similar to the interior.

Case 1: $\delta_n^{s,*}(h,1) \leq 0$ and $\delta_n^{s,*}(h,0) \leq 0$.

$$\begin{aligned} V_n^s(h,1) &= (1-\lambda^h-\lambda^l)V_{n+1}^s(h,1) + \lambda^h[p^h + V_{n+1}^s(h-1,1)] + \lambda^l[p^l + V_{n+1}^s(h,0)], \\ V_n^s(h,0) &\geq (1-\lambda^h-\lambda^l)V_{n+1}^s(h,0) + \lambda^h[p^h + V_{n+1}^s(h-1,0)] + \lambda^l V_{n+1}^s(h,0). \end{aligned}$$

The inequality is true because we eliminate the potential nonnegative revenue from substitution. $V_{n+1}^s(h,l+1) - V_{n+1}^s(h,l) \leq p^l$ and term-by-term comparisons of the value functions above yield $V_n^s(h,1) - V_n^s(h,0) \leq p^l$.

Case 2: $\delta_n^{s,*}(h,1) > 0$ and $\delta_n^{s,*}(h,0) \leq 0$.

$$\begin{aligned} V_n^s(h,1) &= (1-\lambda^h-\lambda^l)V_{n+1}^s(h,1) + \lambda^h[p^h + V_{n+1}^s(h-1,1)] + \lambda^l[p^l + V_{n+1}^s(h,0)] \\ &\quad + C\lambda\delta_n^{s,*}(h,1) \\ &= (1-\lambda^h-\lambda^l)V_{n+1}^s(h,1) + \lambda^h[p^h + V_{n+1}^s(h-1,1)] + \lambda^l[p^l + V_{n+1}^s(h,0)] \\ &\quad + C\lambda\alpha(p_{h,1}^{s,*})[p_{h,1}^{s,*} + V_{n+1}^s(h-1,2) - V_{n+1}^s(h,1)], \\ V_n^s(h,0) &\geq (1-\lambda^h-\lambda^l)V_{n+1}^s(h,0) + \lambda^h[p^h + V_{n+1}^s(h-1,0)] + \lambda^l V_{n+1}^s(h,0) \\ &\geq (1-\lambda^h-\lambda^l)V_{n+1}^s(h,0) + \lambda^h[p^h + V_{n+1}^s(h-1,0)] + \lambda^l V_{n+1}^s(h,0) \\ &\quad + C\lambda\delta_n^{s,*}(h,0) \\ &\geq (1-\lambda^h-\lambda^l)V_{n+1}^s(h,0) + \lambda^h[p^h + V_{n+1}^s(h-1,0)] + \lambda^l V_{n+1}^s(h,0) \\ &\quad + C\lambda\alpha(p_{h,1}^{s,*})[p_{h,1}^{s,*} + V_{n+1}^s(h-1,1) - V_{n+1}^s(h,0)], \end{aligned}$$

where the first inequality is because of the potential substitution revenue elimination, and the second inequality is from $\delta_n^{s,*}(h,0) \leq 0$ and the third inequality is due to the non-optimality of $p_{h,1}^{s,*}$ for $\alpha(p^s)[p^s + V_{n+1}^s(h-1,1) - V_{n+1}^s(h,0)]$. Then, $V_{n+1}^s(h,l+1) - V_{n+1}^s(h,l) \leq p^l$ and term-by-term comparisons of the value functions above yield $V_n^s(h,1) - V_n^s(h,0) \leq p^l$.

Case 3: $\delta_n^{s,*}(h,1) \le 0$ and $\delta_n^{s,*}(h,0) > 0$.

$$\begin{aligned}
V_n^s(h,1) &= (1-\lambda^h-\lambda^l)V_{n+1}^s(h,1) + \lambda^h[p^h + V_{n+1}^s(h-1,1)] + \lambda^l[p^l + V_{n+1}^s(h,0)], \\
V_n^s(h,0) &\ge (1-\lambda^h-\lambda^l)V_{n+1}^s(h,0) + \lambda^h[p^h + V_{n+1}^s(h-1,0)] + \lambda^l V_{n+1}^s(h,0) \\
&\quad + C\lambda\delta_n^{s,*}(h,0) \\
&> (1-\lambda^h-\lambda^l)V_{n+1}^s(h,0) + \lambda^h[p^h + V_{n+1}^s(h-1,0)] + \lambda^l V_{n+1}^s(h,0),
\end{aligned}$$

where the first inequality is because of the potential substitution revenue elimination and the second inequality is from $\delta_n^{s,*}(h,0) > 0$. Then, $V_{n+1}^s(h,l+1) - V_{n+1}^s(h,l) \le p^l$ and term-by-term comparisons of the value functions above yield $V_n^s(h,1) - V_n^s(h,0) \le p^l$.

Case 4: $\delta_n^{s,*}(h,1) > 0$ and $\delta_n^{s,*}(h,0) > 0$.

$$\begin{aligned}
V_n^s(h,1) &= (1-\lambda^h-\lambda^l)V_{n+1}^s(h,1) + \lambda^h[p^h + V_{n+1}^s(h-1,1)] + \lambda^l[p^l + V_{n+1}^s(h,0)] \\
&\quad + C\lambda\delta_n^{s,*}(h,1) \\
&= (1-\lambda^h-\lambda^l)V_{n+1}^s(h,1) + \lambda^h[p^h + V_{n+1}^s(h-1,1)] + \lambda^l[p^l + V_{n+1}^s(h,0)] \\
&\quad + C\lambda\alpha(p_{h,1}^{s,*})[p_{h,1}^{s,*} + V_{n+1}^s(h-1,2) - V_{n+1}^s(h,1)], \\
V_n^s(h,0) &\ge (1-\lambda^h-\lambda^l)V_{n+1}^s(h,0) + \lambda^h[p^h + V_{n+1}^s(h-1,0)] + \lambda^l V_{n+1}^s(h,0) \\
&\quad + C\lambda\delta_n^{s,*}(h,0) \\
&\ge (1-\lambda^h-\lambda^l)V_{n+1}^s(h,0) + \lambda^h[p^h + V_{n+1}^s(h-1,0)] + \lambda^l V_{n+1}^s(h,0) \\
&\quad + C\lambda\alpha(p_{h,1}^{s,*})[p_{h,1}^{s,*} + V_{n+1}^s(h-1,1) - V_{n+1}^s(h,0)],
\end{aligned}$$

where the first inequality is because of the potential substitution revenue elimination and the second inequality is from the non-optimality of $p_{h,1}^{s,*}$ for $\alpha(p^s)[p^s + V_{n+1}^s(h-1,1) - V_{n+1}^s(h,0)]$. Then, $V_{n+1}^s(h,l+1) - V_{n+1}^s(h,l) \le p^l$ and term-by-term comparisons of the value functions above yield $V_n^s(h,1) - V_n^s(h,0) \le p^l$.

Vertical boundary $0 \times [1 : L-1]$. Without a premium product, no upgrades can be offered.

$$\begin{aligned}
V_n(0,l+1) &= (1-\lambda^l)V_{n+1}(0,l+1) + \lambda^l[p^l + V_{n+1}(0,l)], \\
V_n(0,l) &= (1-\lambda^l)V_{n+1}(0,l) + \lambda^l[p^l + V_{n+1}(0,l-1)].
\end{aligned}$$

Term-by-term comparisons of the value functions above yield $V_n^s(0,l+1) - V_n^s(0,l) \leq p^l$.

Corner $(0,1)$. No upgrades can be offered as in the vertical boundary.

$$\begin{aligned} V_n(0,1) &= (1-\lambda^l)V_{n+1}(0,1) + \lambda^l[p^l + V_{n+1}(0,0)] \\ V_n(0,0) &= (1-\lambda^l)V_{n+1}(0,0) + \lambda^l V_{n+1}(0,0). \end{aligned}$$

$V_{n+1}^s(0,1) - V_{n+1}^s(0,0) \leq p^l$ and term-by-term comparisons of the value functions above yield $V_n^s(0,1) - V_n^s(0,0) \leq p^l$. □

A.2 A Counterexample for Failing DV-Modularity in the Substitution Model

Consider a single period problem with $N = 1$. The substitution reservation price distribution is assumed to be a single-point distribution; all regular customers are willing to accept the substitution if the corresponding fee is smaller or equal to $0.5(p^h - p^l)$. Hence, the substitution acceptance probability is given by

$$\alpha^s(f^s) = \begin{cases} 1 & f^s \leq 0.5(p^h - p^l) \\ 0 & f^s > 0.5(p^h - p^l) \end{cases}.$$

From the definition of the terminal value function, we have $V_2^s(h,l) = 0$, $V_2^s(h-1,0) - V_2^s(h,0) = 0$ and $\Delta_2^s(h,l) = 0$. Since the substitution opportunity cost and upgrade opportunity value are both zero, any substitution fee and any upgrade fee bring in non-negative revenues. Thus, it is always optimal to offer substitutions and upgrades if possible in period 1. We can also argue that the optimal substitution fee is $f_1^{s,*} = 0.5(p^h - p^l)$. The optimal upgrade fee in period 1 is independent of (h,l) and denoted by $p_1^{s,*}$. The value of $V_1^s(h,l)$ is:

$$\begin{aligned} V_1^s(0,0) &= 0, \\ V_1^s(0,l) &= \lambda^l p^l \quad \text{ for } l \in [1:L] \\ V_1^s(h,0) &= \lambda^h p^h + \lambda^l[p^l + 0.5(p^h - p^l)] + C\lambda\alpha(p_1^{s,*})p_1^{s,*} \quad \text{ for } h \in [1:H], \\ V_1^s(h,l) &= \lambda^h p^h + \lambda^l p^l + C\lambda\alpha(p_1^{s,*})p_1^{s,*} \quad \text{ for } h \in [1:H] \text{ and } l \in [1:M], \end{aligned}$$

$$V_1^s(h,l) = \lambda^h p^h + \lambda^l p^l \quad \text{for } h \in [1:H] \text{ and } l \in [M+1:L].$$

Then the value of $\Delta_N^s(h,l)$ is:

$$\begin{aligned}
\Delta_1^s(1,0) &= \lambda^l p^l - \lambda^l[p^l + 0.5(p^h - p^l)] - \lambda^h p^h - C\lambda\alpha(p_1^{s,*})p_1^{s,*}, \\
\Delta_1^s(1,l) &= -\lambda^h p^h - C\lambda\alpha(p_1^{s,*})p_1^{s,*} \quad \text{for } l \in [1:M-1], \\
\Delta_1^s(h,0) &= \lambda^l p^l - \lambda^l[p^l + 0.5(p^h - p^l)] = -\lambda^l 0.5(p^h - p^l) \quad \text{for } h \in [2:H], \\
\Delta_1^s(h,l) &= 0 \quad \text{for } h \in [2:H] \text{ and } l \in [1:M-1], \\
\Delta_1^s(h,M) &= -C\lambda\alpha(p_1^{s,*})p_1^{s,*} \quad \text{for } h \in [2:H].
\end{aligned}$$

We have $\Delta_1^s(h,0) < \Delta_1^s(h,1)$ since $\lambda^l 0.5(p^h - p^l) > 0$.

A.3 A Detailed Upgrade Pseudocode

Table A.4. Pseudocode for upgrade implementation algorithm for given H, L, N, M, and C. The extent of reminding/reloading unresponsive customers is controlled by R.

```
/* c is the customer index. */
/* r(c) is the most recent period in which customer c receives an upgrade notification (upgrade recency). */
/* H contains customer index c, and L and U contain customer index c and upgrade recency r(c). */
Initialize: n = 1, h = H, l = L, H = ∅, L = ∅, U = ∅;
While n ≤ N
  /* Not in the potential upgrade region. */
  If h > 0 and l > M
    If there is a customer c, then
      If customer c is for a premium product, then
        h = h − 1 and H := H ∪ {c};
      ElseIf customer c is for a regular product, then
        l = l − 1, r(c) = 0 and L := L ∪ {[c, r(c)]};      /* customer c has not received a upgrade notification. */
      EndIf
    EndIf
  /* In the potential upgrade region. */
  ElseIf h > 0 and l ≤ M
    /* Do not upgrade */
    If δ*_n(h, l) ≤ 0, then
      L = L ∪ U and U = ∅;
      If there is a customer c, then
        If customer c is for a premium product, then
          h = h − 1 and H = H ∪ {c};
        ElseIf customer c is for a regular product, then
          l = l − 1, r(c) = 0 and L = L ∪ {[c, r(c)]};
        EndIf
      EndIf
    /* Upgrade */
    ElseIf δ*_n(h, l) > 0, then
      /* Moving or reminding unresponsive upgradeable customers in U. */
      For c ∈ U
        If n − r(c) ≥ R, then
          either move customer c: U = U \ {[c, r(c)]} and L = L ∪ {[c, r(c)]}
          or remind customer c by a new notification: r(c) = n;
        EndIf
      EndFor
      /* Loading U with new upgradeable customers. */
      While |U| < C
        Pick a customer c with the smallest r(c) in L and send her an upgrade notification:
        L = L \ {[c, r(c)]}, r(c) = n and U = U ∪ {[c, r(c)]};
      EndWhile
      If there is a customer c, then
        If customer c is for a premium product, then
          h = h − 1 and H = H ∪ {c};
        ElseIf customer c is for a regular product, then
          l = l − 1, r(c) = 0 and L = L ∪ {[c, r(c)]};
        ElseIf customer c is for an upgrade, then
          If customer c accepts the upgrade, then
            h = h − 1, l = l + 1, U = U \ {[c, r(c)]} and H = H ∪ {c};
          ElseIf customer c rejects the upgrade, then
            U = U \ {[c, r(c)]} and L = L ∪ {[c, r(c)]};
          EndIf
        EndIf
      EndIf
    EndIf
  EndIf
  n = n + 1;
EndWhile.
```

APPENDIX B

SUPPLEMENTAL MATERIALS FOR CHAPTER 3

B.1 Notations and Proofs

Table B.1. Notations for Chapter 3

Manufacturer	
L :	number of periods to produce common intermediate product
l :	number of periods for each customization sequence
J :	number of customization sequences
j :	customized end product index
p_m :	unit shortage cost
h_m :	unit holding cost
y^t	system-wide total inventory level after ordering in period t
y_j^t :	total inventory level of end product j in period t after the myopic allocation
$d_{kj}^{t,t+s_k}$:	demand from retailer k for product j placed in period t for period $t+s_k$ delivery
$o_j^{t,t+n}$:	demand for end product j observed in period t due for $t+n$
$u_j^{t,t+n}$:	demand for end product j unobserved in period t due for $t+n$
V^t :	total demand from period t to period $t+L-1$
W_j^{t+L} :	total demand for end product j from period $t+L$ to period $t+L+l$
$G_j(y_j^{t+L})$:	expected inventory cost for end product j at the end of period $t+L+l$
$y^t(L,l,s_1,s_2)$:	manufacturer's optimal base-stock level
$G_m(L,l,s_1,s_2)$:	manufacturer's optimal expected inventory cost per period
$\psi_m(L,l,s_1,s_2)$:	manufacturer's effective demand standard deviation during production lead time
Retailers	
c_k :	retailer k's cost coefficient
$c_k p_r$:	retailer k's unit shortage cost
$c_k h_r$:	retailer k's unit holding cost
D_{kj}^t	retailer k's end customer demand for product j in period t
$y_{kj}(s_k)$:	retailer k's optimal base-stock level for product j
$G_k(s_k)$:	retailer k's optimal expected inventory cost per period
$\psi_k(s_s)$:	retailer k's effective demand standard deviation during promised lead time
Supply chain	
s_k :	retailer k's promised lead time
α_m :	manufacturer's cost of demand uncertainty
$c_k\alpha_r$:	retailer k's cost of demand uncertainty
$s_k^C(L,l)$:	optimal promised lead time for retailer k in the centralized supply chain
U_k :	retailer k's maximum acceptable inventory cost per period in the different market setting
$s_k^D(L,l)$:	optimal promised lead time for retailer k in the different market setting
$\pi_k^D(L,l)$:	optimal payment for retailer k in the different market setting
U :	retailers' maximum acceptable inventory cost per period in the same market setting
$s_k^S(L,l)$:	optimal promised lead time for retailer k in the same market setting
$\pi_k^S(L,l)$:	optimal payment for retailer k in the same market setting
$\underline{M}, M, \overline{M}$:	thresholds that determine optimal promised lead times for the centralized supply chain
$\underline{N}, \overline{N}$:	thresholds that determine optimal promised lead times for the same market setting

Proof of Proposition 8: Since retailer k is operating over infinite-time horizon, he minimizes either the discounted cost or the long-run average cost. Veinott (1965) and Iglehart (1963) show the optimality of the (s, S) policy under either criterion respectively. Proposition 8 is a special case of their optimality results.

Because $\psi_k(s_k) = \left(\sum_{j=1}^{J} \sigma_{kj}\right) \sqrt{s_k + 1}$, we find its first and second derivatives as follows.

$$\begin{aligned} \frac{d\psi_k(s_k)}{ds_k} &= \frac{1}{2} \left(\sum_{j=1}^{J} \sigma_{kj} \right) (s_k + 1)^{-\frac{1}{2}} > 0 \\ \frac{d^2\psi_k(s_k)}{ds_k^2} &= -\frac{1}{4} \left(\sum_{j=1}^{J} \sigma_{kj} \right) (s_k + 1)^{-\frac{3}{2}} < 0 \end{aligned}$$

Hence, $\psi_k(s_k)$ is concave increasing in s_k. □

Proof of Proposition 9:

The proof of Proposition 9 is similar to the proofs in Eppen and Schrage(1981), Erkip et al. (1990), and Özer (2003). In the main body of Chapter 3, we have made four assumptions:

1. Unit holding cost h_m and penalty cost p_m are the same across J end products.

2. Independent demand $d_{kj}^{t,t+s_k}$ follows a normal distribution with mean μ_{kj} and variance σ_{kj}^2 for $k \in \{1, 2\}$ and $j \in \{1, ..., J\}$.

3. *Allocation Assumption* holds: the manufacturer always receives sufficient intermediate products in period $t + L$, so that each customization sequence can be allocated sufficient intermediate products to ensure the same service level (an equal fractile of the demand distribution) across all J end products in period $t + L + l$.

4. We restrict the policy space to the class of base-stock policies with myopic allocation.

We will see later, under assumption 1, 2 and 3, the myopic allocation in the optimal policy insures that an equal fractile of the demand distribution must be restored for each end product j.

At the beginning of period t, a batch of intermediate product is produced by the manufacturer and brings the total system stock to y^t. The system stock y^t includes the on-hand and in-transit inventory of both the intermediate product and the end products. It will protect the system from the variation of demand over $L + l + 1$ periods. $o_j^{t,t+n} := d_{1j}^{t+n-s_1,t+n}\mathbb{I}_{\{s_1>n\}} + d_{2j}^{t+n-s_2,t+n}\mathbb{I}_{\{s_2>n\}}$ is the observed demand of product j at period t due for delivery in period $t + n$. $u_j^{t,t+n} := d_{1j}^{t+n-s_1,t+n}\mathbb{I}_{\{s_1\le n\}} + d_{2j}^{t+n-s_2,t+n}\mathbb{I}_{\{s_2\le n\}}$ is the unobserved demand of product j at period t due for delivery in period $t + n$. The summation of $o_j^{t,t+n}$ and $u_j^{t,t+n}$ is the actual demand for product j due for delivery in period $t + n$. $V^t := \sum_{n=0}^{L-1}\sum_{j=1}^{J}(o_j^{t,t+n} + u_j^{t,t+n})$ represents the total demand over the intermediate product lead time L. We define

$$W_j^{t+L} := \sum_{n=L}^{L+l}(o_j^{t,t+n} + u_j^{t,t+n}) \text{ for } j \in \{1, 2, ..., J\},$$

which represents the total demand of end product j from periods $t + L$ to $t + L + l$. From the definitions of $d_{kj}^{t,t+s_k}$, $o_j^{t,t+n}$, and $u_j^{t,t+n}$, we have

$$\begin{aligned}
E(V^t) &= \sum_{n=0}^{L-1}\sum_{j=1}^{J} o_j^{t,t+n} + \sum_{j=1}^{J}\sum_{k=1}^{2}(L - s_k)\mu_{kj}\mathbb{I}_{\{s_k<L\}}, \\
Var(V^t) &= \sum_{j=1}^{J}\sum_{k=1}^{2}(L - s_k)\sigma_{kj}^2\mathbb{I}_{\{s_k<L\}}, \\
E(W_j^{t+L}) &= \sum_{n=L}^{L+l} o_j^{t,t+n} + \sum_{k=1}^{2}(l + 1)\mu_{kj}\mathbb{I}_{\{s_k<L\}} + \sum_{k=1}^{2}(L + l + 1 - s_k)\mu_{kj}\mathbb{I}_{\{L\le s_k\le L+l+1\}}, \\
Var(W_j^{t+L}) &= \sum_{k=1}^{2}(l + 1)\sigma_{kj}^2\mathbb{I}_{\{s_k<L\}} + \sum_{k=1}^{2}(L + l + 1 - s_k)\sigma_{kj}^2\mathbb{I}_{\{L\le s_k\le L+l+1\}}.
\end{aligned}$$

At the beginning of period $t+L$, the amount of end products to be finished by period $t+L+l$ is $y^t - V^t$. We first treat the system base-stock y^t as given and focus on the allocation decisions $y_1^{t+L}, ..., y_J^{t+L}$. To minimize the cost of myopic allocation under *Allocation Assumption*, we solve the problem in (3.2) while assuming y^t is an exogenous constant:

$$\min_{y_1^{t+L},...,y_J^{t+L}} \sum_{j=1}^{J} G_j(y_j^{t+L}) \text{ s.t. } \sum_{j=1}^{J} y_j^{t+L} = y^t - V^t.$$

y_j^{t+L} represents the total amount of on-hand and in-transit inventory of end product j. $G_j(y_j^{t+L} := E[h_m(y_j^{t+L} - W_j^{t+L})^+ + p_m(y_j^{t+L} - W_j^{t+L})^-]$ is the expected inventory cost of product j at the end of period $t + L + l$. After introducing the Lagrange multiplier λ, we minimize the Lagrangian function. For each end product, the optimal allocation for each product is

$$y_j^{t+L}(L, l, s_1, s_2) = E(W_j^{t+L}) + \sqrt{Var(W_j^{t+L})}\Phi^{-1}(\frac{p_m + \lambda}{p_m + h_m}),$$

where Φ is the cdf of standard normal random variable. We sum $y_j^{t+L}(L, l, s_1, s_2)$ across all end products, equate it to $y^t - V^t$ and solve for $\Phi^{-1}[(p_m + \lambda)/(p_m + h_m)]$. By substituting $\Phi^{-1}[(p_m + \lambda)/(p_m + h_m)]$ into $y_j^{t+L}(L, l, s_1, s_2)$, we have

$$y_j^{t+L}(L, l, s_1, s_2) = E(W_j^{t+L}) + [y^t - V^t - \sum_{j=1}^{J} E(W_j^{t+L})]\frac{\sqrt{Var(W_j^{t+L})}}{\sum_{j=1}^{J}\sqrt{Var(W_j^{t+L})}}.$$

Now we focus on end product j. The level of on-hand inventory of product j at the end of period $t + L + l$ is given by

$$\begin{aligned}
& y_j^{t+L}(L, l, s_1, s_2) - W_j^{t+L} \\
= \;& \left\{E(W_j^{t+L}) + [y^t - \sum_{j=1}^{J} E(W_j^{t+L})]\frac{\sqrt{Var(W_j^{t+L})}}{\sum_{j=1}^{J}\sqrt{Var(W_j^{t+L})}}\right\} \\
& - \left\{W_j^{t+L} + V^t\frac{\sqrt{Var(W_j^{t+L})}}{\sum_{j=1}^{J}\sqrt{Var(W_j^{t+L})}}\right\} \\
= \;& s_j^{t+L} - \xi_j^{t+L},
\end{aligned}$$

where the deterministic component s_j^{t+L} is defined as

$$s_j^{t+L} = E(W_j^{t+L}) + [y^t - \sum_{j=1}^{J} E(W_j^{t+L})]\frac{\sqrt{Var(W_j^{t+L})}}{\sum_{j=1}^{J}\sqrt{Var(W_j^{t+L})}},$$

and the stochastic component ξ_j^{t+L} is defined as

$$\xi_j^{t+L} = W_j^{t+L} + V^t \frac{\sqrt{Var(W_j^{t+L})}}{\sum_{j=1}^J \sqrt{Var(W_j^{t+L})}}.$$

The corresponding expected inventory cost is

$$\begin{aligned} G_j(y_j^{t+L}(L,l,s_1,s_2)) &= E[h_m(y_j^{t+L}(L,l,s_1,s_2) - W_j^{t+L})^+ + p_m(y_j^{t+L}(L,l,s_1,s_2) - W_j^{t+L})^-] \\ &= E[h_m(s_j^{t+L} - \xi_j^{t+L})^+ + p_m(s_j^{t+L} - \xi_j^{t+L})^-], \end{aligned}$$

where the second expectation is taken over ξ_j^{t+L}. Notice that $G_j(y_j^{t+L}(L,l,s_1,s_2))$ is a function of y^t, since s_j^{t+L} contains y^t.

Now we optimize $G_j(y_j^{t+L}(L,l,s_1,s_2))$ over y^t and search for the optimal base stock level $y^t(L,l,s_1,s_2)$. The optimizer of the function $G_j(y_j^{t+L}(L,l,s_1,s_2))$ is achieved by choosing y^t such that $P(\xi_j^{t+L} \leq s_j^{t+L}) = p_m/(p_m + h_m)$. From this, we have

$$\Phi^{-1}(\frac{p_m}{p_m + h_m}) = \frac{s_j^{t+L} - E(\xi_j^{t+L})}{\sqrt{Var(\xi_j^{t+L})}}.$$

Then the optimal base stock level is

$$y^t(L,l,s_1,s_2) = E(V^t) + \sum_{j=1}^J E(W_j^{t+L}) + \Phi^{-1}(\frac{p_m}{p_m + h_m})\sqrt{Var(V^t) + \left[\sum_{j=1}^J \sqrt{Var(W_j^{t+L})}\right]^2}.$$

We can observe that $y^t(L,l,s_1,s_2)$ is independent of j, which means that it is the optimizer of all $G_j(y_j^{t+L}(L,l,s_1,s_2))$ and $\sum_{j=1}^J G_j G_j(y_j^{t+L}(L,l,s_1,s_2))$. By plugging $E(V^t)$, $Var(V^t)$, $E(W_j^{t+L})$, and $Var(W_j^{t+L})$ into $y^t(L,l,s_1,s_2)$, we have

$$\begin{aligned} y^t(L,l,s_1,s_2) &= \sum_{n=0}^{L+l} \sum_{j=1}^J o_j^{t,t+n} + \sum_{j=1}^J \sum_{k=1}^2 (L+l+1-s_k)\mu_{kj} \\ &\quad + \Phi^{-1}\left(\frac{p_m}{h_m + p_m}\right)\psi_m(L,l,s_1,s_2), \end{aligned}$$

where

$$
\begin{aligned}
&\psi_m(L,l,s_1,s_2)\\
= &\sqrt{Var(V^t) + \left[\sum_{j=1}^{J}\sqrt{Var(W_j^{t+L})}\right]^2}\\
= &\sqrt{\sum_{j=1}^{J}\sum_{k=1}^{2}(L-s_k)\sigma_{kj}^2\mathbb{I}_{\{s_k<L\}} + \left(\sum_{j=1}^{J}\sqrt{\begin{array}{l}\sum_{k=1}^{2}(l+1)\sigma_{kj}^2\mathbb{I}_{\{s_k<L\}}\\ +\sum_{k=1}^{2}(L+l+1-s_k)\sigma_{kj}^2\mathbb{I}_{\{L\le s_k\le L+l+1\}}\end{array}}\right)^2}.
\end{aligned}
$$

Henc, the manufacturer's minimum expected inventory cost per period is

$$
\begin{aligned}
G_m(L,l,s_1,s_2) &= \sum_{j=1}^{J} G_j(y_j^{t+L}(L,l,s_1,s_2))|_{y^t=y^t(L,l,s_1,s_2)}\\
&= (h_m+p_m)\phi\left(\Phi^{-1}\left(\frac{p_m}{p_m+h_m}\right)\right)\sum_{j=1}^{J}\sqrt{Var(\xi_j^{t+L})}\\
&= (h_m+p_m)\phi\left(\Phi^{-1}\left(\frac{p_m}{p_m+h_m}\right)\right)\psi_m(L,l,s_1,s_2).
\end{aligned}
$$

□

Proof of Proposition 10: Here we only prove property a) and c) for s_1. By similar arguments, property b) and d) also hold for s_2.

We first prove property a). Since $\psi_m(L,l,s_1,s_2)\ge 0$, $\psi_m^2(L,l,s_1+1,s_2)-\psi_m^2(L,l,s_1,s_2)\le 0$ is sufficient to validate the decreasing property of $\psi_m(L,l,s_1,s_2)$ in s_1. We examine two cases: $s_1<L$ and $L\le s_2$. First, if $s_1<L$, then

$$\psi_m^2(L,l,s_1+1,s_2)-\psi_m^2(L,l,s_1,s_2) = -\sum_{j=1}^{J}\sigma_{1j}^2 \le 0.$$

Second, if $L \leq s_1$, then

$$\begin{aligned}
&\psi_m^2(L, l, s_1+1, s_2) - \psi_m^2(L, l, s_1, s_2) \\
= &\left[\sum_{j=1}^{J} \sqrt{(L+1-s_1)\sigma_{1j}^2 + (l+1)\sigma_{2j}^2 \mathbb{I}_{\{s_2<L\}} + (L+l+1-s_2)\sigma_{2j}^2 \mathbb{I}_{\{L\leq s_2\}}}\right]^2 \\
&- \left[\sum_{j=1}^{J} \sqrt{(L+1+1-s_1)\sigma_{1j}^2 + (l+1)\sigma_{2j}^2 \mathbb{I}_{\{s_2<L\}} + (L+l+1-s_2)\sigma_{2j}^2 \mathbb{I}_{\{L\leq s_2\}}}\right]^2.
\end{aligned}$$

Since $(L+1-s_1)\sigma_{1j}^2 \leq (L+1+1-s_1)\sigma_{1j}^2$, we have $\psi_m^2(L, l, s_1+1, s_2) - \psi_m^2(L, l, s_1, s_2) \leq 0$. Thus $\psi_m(L, l, s_1, s_2)$ is decreasing in s_1.

Now we prove property c). Because the proof is the same for all (L, l, s_2), we suppress these variables in the function names that follow. To aid in the proof, we define two new functions, namely,

$$\begin{aligned}
\gamma_j(s_1) &:= \sqrt{\begin{array}{l} (l+1)\left[\sigma_{1j}^2 \mathbb{I}_{\{s_1<L\}} + \sigma_{2j}^2 \mathbb{I}_{\{s_2<L\}}\right] + (L+l+1-s_1)\sigma_{1j}^2 \mathbb{I}_{\{L\leq s_1\}} \\ +(L+l+1-s_2)\sigma_{2j}^2 \mathbb{I}_{\{L\leq s_2\}} \end{array}}, \\
\Psi(s_1) &:= [\psi_m(s_1)]^2 \\
&= \left[(L-s_1)\mathbb{I}_{\{s_1<L\}} \sum_{j=1}^{J} \sigma_{1j}^2 + (L-s_2)\mathbb{I}_{\{s_2<L\}} \sum_{j=1}^{J} \sigma_{2j}^2\right] + \left[\sum_{j=1}^{J} \gamma_j(s_1)\right]^2.
\end{aligned}$$

From now on in this proof, we treat $\psi_m(s_1)$ as a continuous function. Because property c) in the continuous setting is equivalent to

$$\frac{d^2\psi_m(s_1)}{ds_1^2} \leq 0$$

and because

$$\begin{aligned}
\frac{d\psi_m(s_1)}{ds_1} &= \frac{1}{2}[\psi_m(s_1)]^{-1}\left(\frac{d\Psi(s_1)}{ds_1}\right) \text{ and} \\
\frac{d^2\psi_m(s_1)}{ds_1^2} &= \frac{1}{2}\left\{-\frac{1}{2}[\psi_m(s_1)]^{-\frac{3}{2}}\left[\frac{d\Psi(s_1)}{ds_1}\right]^2 + [\psi_m(s_1)]^{-1}\left[\frac{d^2\Psi(s_1)}{ds_1^2}\right]\right\},
\end{aligned}$$

property c) or the concavity property is proved if we can show

$$2\left[\frac{d^2\Psi(s_1)}{ds_1^2}\right]\Psi(s_1) \leq \left[\frac{d\Psi(s_1)}{ds_1}\right]^2. \tag{B.1}$$

When $s_1 < L$, we have $\frac{d\Psi(s_1)}{ds_1} = -\sum_{j=1}^{J} \sigma_{1j}^2$ and $\frac{d^2\Psi(s_1)}{ds_1^2} = 0$, so (B.1) holds. When $L \leq s_1 \leq L + l + 1$, we have

$$\begin{aligned} \frac{d\gamma_j(s_1)}{ds_1} &= -\frac{1}{2}\sigma_{1j}^2 \left[\gamma_j(s_1)\right]^{-1} \text{ and} \\ \frac{d^2\gamma_j(s_1)}{ds_1^2} &= -\frac{1}{4}\sigma_{1j}^4 \left[\gamma_j(s_1)\right]^{-3}. \end{aligned}$$

We determine the first and second derivatives of $\Psi(s_1)$ as follows:

$$\begin{aligned} \frac{d\Psi(s_1)}{ds_1} &= 2\left[\sum_{j=1}^{J}\gamma_j(s_1)\right](-\frac{1}{2})\left[\sum_{j=1}^{J}\frac{d\gamma_j(s_1)}{ds_1}\right] = -\left[\sum_{j=1}^{J}\gamma_j(s_1)\right]\left[\sum_{j=1}^{J}\frac{\sigma_{1j}^2}{\gamma_j(s_1)}\right] \\ \frac{d^2\Psi(s_1)}{ds_1^2} &= \frac{1}{2}\left\{\left[\sum_{j=1}^{J}\frac{\sigma_{1j}^2}{\gamma_j(s_1)}\right]^2 - \left[\sum_{j=1}^{J}\gamma_j(s_1)\right]\left[\sum_{j=1}^{J}\frac{\sigma_{1j}^4}{\left[\gamma_j(s_1)\right]^3}\right]\right\} \end{aligned}$$

Substituting these expressions into (B.1) yields

$$\left[(L - s_2)\sum_{j=1}^{J}\sigma_{2j}^2 \mathbb{I}_{\{s_2 < L\}}\right]\left[\begin{array}{l} \left[\sum_{j=1}^{J}\frac{\sigma_{1j}^2}{\gamma_j(s_1)}\right]^2 \\ -\left[\sum_{j=1}^{J}\gamma_j(s_1)\right]\left[\sum_{j=1}^{J}\frac{\sigma_{1j}^4}{\left[\gamma_j(s_1)\right]^3}\right] \end{array}\right]$$

$$\leq \left[\sum_{j=1}^{J}\gamma_j(s_1)\right]^3\left[\sum_{j=1}^{J}\frac{\sigma_{1j}^4}{\left[\gamma_j(s_1)\right]^3}\right]. \tag{B.2}$$

It is easy to see that the sufficient condition of (B.1) is

$$\left[\sum_{j=1}^{J}\frac{\sigma_{1j}^2}{\gamma_j(s_1)}\right]\left[\sum_{j=1}^{J}\frac{\sigma_{1j}^2}{\gamma_j(s_1)}\right] \leq \left[\sum_{j=1}^{J}\gamma_j(s_1)\right]\left[\sum_{j=1}^{J}\frac{\sigma_{1j}^4}{\left[\gamma_j(s_1)\right]^3}\right]. \tag{B.3}$$

To proceed we adopt a term-wise comparison for items i and j. For $i = j$, we note that

$$\left[\frac{\sigma_{1j}^2}{\gamma_j(s_1)}\right]^2 = \gamma_j(s_1)\frac{\sigma_{1j}^4}{\left[\gamma_j(s_1)\right]^3}.$$

For $i \neq j$, we compare

$$\frac{\sigma_{1j}^2}{\gamma_j(s_1)}\frac{\sigma_{1i}^2}{\gamma_i(s_1)} + \frac{\sigma_{1i}^2}{\gamma_i(s_1)}\frac{\sigma_{1j}^2}{\gamma_j(s_1)} \tag{B.4}$$

and

$$\gamma_j(s_1)\frac{\sigma_{1i}^4}{[\gamma_i(s_1)]^3}+\gamma_i(s_1)\frac{\sigma_{1j}^4}{[\gamma_j(s_1)]^3}. \tag{B.5}$$

If (B.4) is less than (B.5) for all i and j pairs, then (B.3) holds. Multiplying both (B.4) and (B.5) by $\gamma_i(s_1)\gamma_j(s_1)$ implies that (B.3) holds whenever

$$2\sigma_{1j}^2\sigma_{2j}^2 \leq \sigma_{1j}^4\left(\frac{\gamma_i(s_1)}{\gamma_j(s_1)}\right)^2+\sigma_{1i}^4\left(\frac{\gamma_j(s_1)}{\gamma_i(s_1)}\right)^2.$$

Bringing all terms to the right side and factoring shows

$$0 \leq \left[\sigma_{1j}^2\left(\frac{\gamma_i(s_1)}{\gamma_j(s_1)}\right)-\sigma_{1i}^2\left(\frac{\gamma_j(s_1)}{\gamma_i(s_1)}\right)\right]^2,$$

which is certainly true. Hence (B.3) holds and condition (B.2) is met. We can also show the left hand derivative of $\psi_m(s_1)$ at L is larger than the right hand derivative. Thus, $\psi_m(s_1)$ is concave in s_1. □

Proof of Proposition 11: To analyze the impact of promised lead time on *Allocation Assumption* and compare the result with that from Eppen and Schrage (1981), we focus on a supply chain with only one retailer. The proof can be generalized to the two-retailer supply chain studied in the main body of Chapter 3. The optimal base stock level is given by

$$y^t(L,l,s) \;=\; \mathbb{1}_{\{s>1\}}\sum_{n=0}^{s-1}\sum_{j=1}^{J}o_j^{t,t+n}+(L+l+1-s)\sum_{j=1}^{J}\mu_j+\Phi^{-1}\left(\frac{p}{p+h}\right)\psi(L,l,s),$$

where

$$\psi(L,l,s) = \sqrt{(L-s)\mathbb{1}_{\{s<L\}}\sum_{j=1}^{J}\sigma_j^2+\left[\sum_{j=1}^{J}\sqrt{(l+1)\mathbb{1}_{\{s<L\}}\sigma_j^2+(L+l+1-s)\mathbb{1}_{\{L\leq s\leq L+l+1\}}\sigma_j^2}\right]^2}.$$

$\psi(L, l, s)$ represents the effective standard deviation of lead time demand for the manufacturer. $s \in \{0, 1, ..., L+l+1\}$ is the single retailer's promised lead time. $o_j^{t,t+n}$ is the demand for end product j in period $t+n$ observed in period t.

Allocation Assumption: In each allocation period $t+L$, the manufacturer receives sufficient intermediate products, which were ordered L periods ago, so that each end product can be allocated intermediate products in sufficient quantity to ensure that probability of stock out in period $t+L+l$ is the same for all end products.

When $s = 0$, Lemma 1 in Eppen and Schrage (1981) shows a necessary condition of *Allocation Assumption* being true

$$\sum_{j=1}^{J} d_j^{t-1} \geq \max_{j=1,..,J} \left\{ \sum_{\substack{i=1 \\ i \neq j}}^{J} d_j^{t+L-1} + \left(1 - \frac{\sum_{j=1}^{J} \sigma_j}{\sigma_j}\right) d_j^{t-L-1} \right\},$$

where d_j^t is the demand for product j in period t. We want to find the necessary condition of *Allocation Assumption* being true when the promised lead time $s > 0$.

When $s \in \{1, ..., L\}$, the proof is similar to Lemma 1 in Eppen and Schrage (1981), since the manufacturer at the beginning of period t does not observe demand from period $t+L$ to $t+L+l$. The only difference is that when the promised lead time is positive, the base stock level $y^t(L, l, s)$ is not stationary and depends on $o_j^{t,t+n}$. We assume that the inventory on hand, plus on order, is equal to the same fractile position for each end product at period $t+L-1$. Then inventory positions can be represented as

$$IP_j^{t+L-1} = (l+1)\mu_j + z_1\sqrt{l+1}\sigma_j, \text{ for } j = 1, ..., J.$$

The manufacturer satisfies demand d_j^{t+L-1}, receives $\sum_{j=1}^{J} d_j^{t-1+s}$ intermediate products produced in period t, and allocates a_j units to end product j. Thus,

$$IP_j^{t+L} = (l+1)\mu_j + z_1\sqrt{l+1}\sigma_j - d_j^{t+L-1} + a_j, \text{ for } j = 1, ..., J.$$

An equal fractile position can be achieved if one can find a set of a_j's such that

$$\sum_{j=1}^{J} a_j = \sum_{j=1}^{J} d_j^{t-1+s} \text{ and } a_j \geq 0$$
$$\text{for } j = 1, ...J,$$

and

$$IP_j^{t+L} = (l+1)\mu_j + z_2\sqrt{l+1}\sigma_j, \text{ for } j = 1, ..., J.$$

Thus,

$$a_j = (z_2 - z_1)\sqrt{l+1}\sigma_j + d_j^{t+L-1}, \text{ for } j = 1, ..., J,$$

and

$$\sum_{j=1}^{J} a_j = \sum_{j=1}^{J} d_j^{t-1+s} = (z_2 - z_1)\sqrt{l+1}\sum_{j=1}^{J}\sigma_j + \sum_{j=1}^{J} d_j^{t+L-1}.$$

Solving for $(z_2 - z_1)$ and substituting yields

$$a_j = \frac{(\sum_{j=1}^{J} d_j^{t-1+s} - \sum_{j=1}^{J} d_j^{t+L-1})}{\sum_{j=1}^{J}\sigma_j}\sigma_j + d_j^{t+L-1}, \text{ for } j = 1, ..., J.$$

Note that $a_j \geq 0$ if

$$\sum_{j=1}^{J} d_j^{t-1+s} \geq \sum_{j=1}^{J} d_j^{t+L-1} - \frac{\sum_{j=1}^{J}\sigma_j}{\sigma_j} d_j^{t+L-1}, \text{ for } j = 1, ..., J.$$

All $a_j \geq 0$ if

$$\sum_{j=1}^{J} d_j^{t-1+s} \geq \max_{j=1,..,J}\left\{\sum_{\substack{i=1\\ i\neq j}}^{J} d_j^{t+L-1} + \left(1 - \frac{\sum_{j=1}^{J}\sigma_j}{\sigma_j}\right) d_j^{t+L-1}\right\}.$$

When $s \in \{L+1, ..., L+l+1\}$, the proof is slightly different. Since $s \geq L+1$, the manufacturer at period t can observe some future demand in and/or after period $t+L$. We use o_j^t to represent the observed demand in period t for product j. We still assume that the inventory on hand, plus on order, is equal to the same fractile position for each end product at period $t+L-1$. Then inventory position can now be represented as

$$IP_j^{t-1+L} = \sum_{n=0}^{s-L-1} o_j^{t-1+L+n} + (L+l+1-s)\mu_j + z_1\sqrt{L+l+1-s}\sigma_j, \text{ for } j = 1, ..., J.$$

The observed demand o_j^{t-1+L} then occurs. The order of intermediate products placed at time t which equals $\sum_{j=1}^{J} d_j^{t-1+s} = \sum_{j=1}^{J} o_j^{t-1+s}$ arrives and the manufacturer allocates a_j unites to end product j. Thus,

$$IP_j^{t+L} = \mathbb{1}_{\{s>L+1\}} \sum_{n=1}^{s-L-1} o_j^{t-1+L+n} + (L+l+1-s)\mu_j + z_1\sqrt{L+l+1-s}\sigma_j + a_j, \text{ for } j = 1, ..., J.$$

An equal fractile position can be achieved if one can find a set of a_j such that

$$\sum_{j=1}^{J} a_j = \sum_{j=1}^{J} d_j^{t-1+s} = \sum_{j=1}^{J} o_j^{t-1+s}, \; a_j \geq 0 \text{ for } j = 1, ..., J,$$

and

$$IP_j^{t+L} = \sum_{n=0}^{s-L-1} o_j^{t+L+n} + (L+l+1-s)\mu_j + z_2\sqrt{L+l+1-s}\sigma_j \text{ for } j = 1, ..., J.$$

Then,

$$\begin{aligned} a_j &= (z_2 - z_1)\sqrt{L+l-s}\sigma_j + \sum_{n=0}^{s-L-1} o_j^{t+L+n} - \mathbb{1}_{\{s>L+1\}} \sum_{n=1}^{s-L-1} o_j^{t-1+L+n} \\ &= (z_2 - z_1)\sqrt{L+l-s}\sigma_j + o_j^{t-1+s}, \text{ for } j = 1, ...J, \end{aligned}$$

and

$$\sum_{j=1}^{J} a_j = \sum_{j=1}^{J} o_j^{t-1+s} = (k_2 - k_1)\sqrt{L+l-s}\sum_{j=1}^{J} \sigma_j + \sum_{j=1}^{J} o_j^{t-1+s}.$$

Thus $k_2 = k_1$ and $a_j = o_j^{t-1+s} = d_j^{t-1+s} > 0$ for $j = 1, ..., J$. The manufacturer can always restore the system to an equal fractile position after intermediate product allocation. $\square$

Proof of Proposition 12: Since the objective function $G_m(L,l,s_1,s_2)+G_1(s_1)+G_2(s_2)$ is concave in s_k from Proposition 8 and 10, $s_k^C(L,l)\in\{0,L+l+1\}$ for $k\in\{1,2\}$.

We describe the total supply chain expected inventory cost as

$$\Gamma(L,l,s_1,s_2)=a_m\psi_m(L,l,s_1,s_2)+a_r\left[c_1\psi_1(s_1)+c_2\psi_2(s_2)\right].$$

Using operator $\Diamond\in\{<,\leqslant,=,\geqslant,>\}$, we discover the following:

$$\begin{aligned}
\Gamma(L,l,0,0)\Diamond\Gamma(L,l,0,L+l+1) &\quad\Leftrightarrow\quad a_m(x-y)\Diamond a_r b,\\
\Gamma(L,l,0,0)\Diamond\Gamma(L,l,L+l+1,0) &\quad\Leftrightarrow\quad a_m(x-z)\Diamond a_r a,\\
\Gamma(L,l,0,0)\Diamond\Gamma(L,l,L+l+1,L+l+1) &\quad\Leftrightarrow\quad a_m x\Diamond a_r(a+b),\\
\Gamma(L,l,0,L+l+1)\Diamond\Gamma(L,l,L+l+1,L+l+1) &\quad\Leftrightarrow\quad a_m y\Diamond a_r a,\\
\Gamma(L,l,L+l+1,0)\Diamond\Gamma(L,l,L+l+1,L+l+1) &\quad\Leftrightarrow\quad a_m z\Diamond a_r b.
\end{aligned}$$

We assume the manufacturer offers $s_k^c=L+l+1$ when $s_k=0$ and $s_k=L+l+1$ yield the same minimum inventory cost. We find that $(s_1^C(L,l),s_2^C(L,l))=(0,0)$ when $\Gamma(L,l,0,0)<\min\{\Gamma(L,l,0,L+l+1),\Gamma(L,l,L+l+1,0),\Gamma(L,l,L+l+1,L+l+1)\}$. It is equivalent to $\frac{\alpha_r}{\alpha_m}>\max\left\{\frac{x}{a+b},\frac{x-z}{a},\frac{x-y}{b}\right\}=\overline{M}$. Similarly, $(s_1^C(L,l),s_2^C(L,l))=(L+l+1,L+l+1)$ when $\Gamma(L,l,L+l+1,L+l+1)<\min\{\Gamma(L,l,0,L+l+1),\Gamma(L,l,L+l+1,0),\Gamma(L,l,0,0)\}$, which is equivalent to $\frac{\alpha_r}{\alpha_m}\leqslant\underline{M}=\min\{\frac{x}{a+b},\frac{y}{a},\frac{z}{b}\}$. Suppose $\underline{M}<\frac{\alpha_r}{\alpha m}\leqslant\overline{M}$. Then $(s_1^C(L,l),s_2^C(L,l))=\{(0,L+l+1),(L+l+1,0)\}$. We have $(s_1^C(L,l),s_2^C(L,l))=(0,L+l+1)$ when $\Gamma(L,l,0,L+l+1)<\Gamma(L,l,L+l+1,0)$, which occurs when $a_m(y-z)<a_r a-a_r b$, or equivalently,

$$\frac{\alpha_r}{\alpha_m}>\frac{y-z}{a-b}=M.$$

$\square$

Proof of Corollary 3 We can show

$$\begin{aligned}
\Gamma(L,l,L+l+1,0)<\Gamma(L,l,L+l+1,L+l+1) &\quad\Rightarrow\quad \Gamma(L,l,0,0)<\Gamma(L,l,L+l+1,0),\\
\Gamma(L,l,0,L+l+1)<\Gamma(L,l,L+l+1,L+l+1) &\quad\Rightarrow\quad \Gamma(L,l,0,0)<\Gamma(L,l,0,L+l+1).
\end{aligned}$$

Thus, $(L+l+1,0)$ and $(0,L+l+1)$ can never be optimal. $\square$

Proof of Proposition 13: We can tell that two constraints in the problem (3.4) must be tight. So problem (3.4) is equivalent to finding s_1 and s_2 to minimize $G_m(L,l,s_1,s_2) + G_1(s_1) - U_1 + G_2(s_2) - U_2$. Then the optimal solutions are $s_1^C(L,l)$ and $s_2^C(L,l)$. $\square$

Before solving problem (3.5) and proving Proposition 12, we provide two lemmas to simplify problem (3.5).

Lemma 3. *For problem (3.5), any feasible solution must satisfy $s_1 \leq s_2$ and $\pi_1 \leq \pi_2$.*

Proof of Lemma 3: For the first result, we add the constraints $G_1(s_1) - \pi_1 \leqslant G_1(s_2) - \pi_2$ and $G_2(s_2) - \pi_2 \leqslant G_2(s_1) - \pi_1$ to obtain $G_1(s_1) + G_2(s_2) \leqslant G_1(s_2) + G_2(s_1)$. Rearranging the terms of this inequality yields $G_1(s_1) - G_1(s_2) \leqslant G_2(s_1) - G_2(s_2)$, or equivalently $c_1 \sum_{j=1}^{J} \sigma_{1j} \left[\sqrt{s_1} - \sqrt{s_2}\right] \leqslant c_2 \sum_{j=1}^{J} \sigma_{2j} \left[\sqrt{s_1} - \sqrt{s_2}\right]$. Because of $c_1 \sum_{j=1}^{J} \sigma_{1j} > c_2 \sum_{j=1}^{J} \sigma_{2j}$ by assumption, the previous inequality implies $s_1 \leqslant s_2$ at any feasible solution.

We prove the second result by contradiction. Assuming $\pi_1 > \pi_2$, we have $G_2(s_2) - \pi_2 \geqslant G_2(s_1) - \pi_2 > G_2(s_1) - \pi_1$ because of $s_1 \leqslant s_2$. This inequality violates a constraint of (3.5). Thus, $\pi_1 \leqslant \pi_2$ is true for any feasible solution. $\square$

If we fix s_1 and s_2, problem (3.5) degenerates into the following linear program over π_1 and π_2.

$$
\begin{aligned}
\underset{\pi_1,\pi_2}{\text{minimize}} \quad & \pi_1 + \pi_2 \\
\text{subject to} \quad & G_1(s_1) - \pi_1 \leq U \\
& G_2(s_2) - \pi_2 \leq U \\
& G_1(s_1) - \pi_1 \leq G_1(s_2) - \pi_2 \\
& G_2(s_2) - \pi_2 \leq G_2(s_1) - \pi_1
\end{aligned}
\tag{B.6}
$$

To solve the nonlinear program (3.5), we first solve for the optimal $\pi_1^D(L, l, s_1, s_2)$ and $\pi_2^D(L, l, s_1, s_2)$ for all possible combination of (s_1, s_2) from problem (B.6). Lemma 3 indicates that problem (3.5) is infeasible when $s_1 > s_2$. Thus, we only need to focus on problem (B.6) with $s_1 \leq s_2$. We then insert $\pi_1^D(L, l, s_1, s_2)$ and $\pi_2^D(L, l, s_1, s_2)$ back into the original objective function in problem (3.5) and find the optimal $s_1^D(L, l)$ and $s_2^D(L, l)$ in the set of $\{0, ..., L + l + 1\}$. The optimal solutions of problem (B.6) are given by the following proposition.

Lemma 4. *Given (s_1, s_2) with $s_1 \leq s_2$, the optimal payments in problem (B.6) are*

$$\begin{aligned}
\pi_1^D(L, l, s_1, s_2) &= G_1(s_1) - U \\
\pi_2^D(L, l, s_1, s_2) &= [G_1(s_1) - G_2(s_1)] + G_2(s_2) - U.
\end{aligned}$$

Proof of Lemma 4: Problem (3.5) is a classic linear program, whose optimal solutions can only happen at $(G_1(s_1) - U, G_1(s_1) - G_2(s_1)] + G_2(s_2) - U)$ $\square$

Lemma 4 gives the functional form of $\pi_1^D(L, l, s_1, s_2)$ and $\pi_2^D(L, l, s_1, s_2)$. We can further analyze their properties with respect to s_1 and s_2. $\pi_1^D(L, l, s_1, s_2)$ is concave increasing in s_1 and unaffected by s_2, while $\pi_2^D(L, l, s_1, s_2)$ is concave increasing in s_1 and s_2 respectively. Plugging $\pi_1^D(L, l, s_1, s_2)$ and $\pi_2^D(L, l, s_1, s_2)$ into problem (3.5), we have

$$\begin{aligned}
\underset{(s_1, s_2)}{\text{minimize}} \quad & G_m(L, l, s_1, s_2) + [2G_1(s_1) - G_2(s_1)] + G_2(s_2) - 2U \\
\text{subject to} \quad & s_1 \leqslant s_2 \\
& s_1, s_2 \in \{0, ..., L + l + 1\}
\end{aligned} \tag{B.7}$$

Proof of Proposition 14 We describe the objective function of problem (B.7) as

$$\Gamma(L, l, s_1, s_2) = G_m(L, l, s_1, s_2) + [2G_1(s_1) - G_2(s_1)] + G_2(s_2) - 2U,$$

where $(s_1^*(L,l), s_2^*(L,l))$ minimizes $\Gamma(L,l,s_1,s_2)$ for given (L,l). Since the objective function $\Gamma(L,l,s_1,s_2)$ is concave in both s_1 and s_2, $(s_1^*(L,l), s_2^*(L,l)) \in \{(0,0), (0, L+l+1), (L+l+1, L+l+1)\}$. We assume the manufacturer offers $s_k = L+l+1$ when $s_k = 0$ and $s_k = L+l+1$ yield the same result. $(s_1^*(L,l), s_2^*(L,l)) = (0,0)$ when $\Gamma(L,l,0,0) < \min\{\Gamma(L,l,0,L+l+1), \Gamma(L,l,L+l+1,L+l+1)\}$. It is equivalent to

$$\frac{\alpha_r}{\alpha m} > \max\left\{\frac{x}{2a}, \frac{x-y}{b}\right\} = \overline{N}.$$

Similarly, $(s_1^*(L,l), s_2^*(L,l)) = (L+l+1, L+l+1)$ when $\Gamma(L,l,L+l+1,L+l+1) < \min\{\Gamma(L,l,0,L+l+1), \Gamma(L,l,0,0)\}$, which is equivalent to $\frac{\alpha_r}{\alpha m} \leqslant \underline{N} = \min\left\{\frac{x}{2a}, \frac{y}{2a-b}\right\}$. When $\underline{N} < \frac{\alpha_r}{\alpha m} \leqslant \overline{N}$. Then $(s_1^*(L,l), s_2^*(L,l)) = (0, L+l+1)$. $\square$

Proof of Proposition 15 To check the property of $\psi_m(L,l,s_1,s_2)$ with respect to postponement, we fix the total production time at a constant level TT. From $TT = L + l$, $\psi_m(L, TT-L, s_1, s_2)$ is a function of L, s_1, and s_2, where L is the postponement variable. For any combination of s_1 and s_2, we analyze $\psi_m(L, TT-L, s_1, s_2)$'s counterpart $\psi_m^c(L, TT-L, s_1, s_2)$ in the continuous interval of $[0, TT]$. If $\psi_m^c(L, TT-L, s_1, s_2)$ is concave decreasing in L in the continuous interval, $\psi_m(L, TT-L, s_1, s_2)$ is also concave decreasing on the discrete domain $\{0, 1, ..., TT\}$. Since the actual functional form of $\psi_m^c(L, TT-L, s_1, s_2)$ depends on the relationship of L, s_1, and s_2, we prove the concave decreasing property case by case.

Case 1: $s_1 = s_2 = s$

$$\psi_m^c(L, TT-L, s_1, s_2) = \begin{cases} \sqrt{(TT+1-s)\left(\sum_{j=1}^J \sqrt{(\sigma_{1j}^2 + \sigma_{2j}^2)}\right)^2} & \text{if } L \leqslant s \\ \sqrt{(L-s)\sum_{j=1}^J (\sigma_{1j}^2 + \sigma_{2j}^2) + (TT+1-L)\left(\sum_{j=1}^J \sqrt{(\sigma_{1j}^2 + \sigma_{2j}^2)}\right)^2} & \text{if } L > s \end{cases}$$

First of all, $\psi_m^c(L, TT-L, s_1, s_2)$ is continuous in $[0, TT]$. When $L \leqslant s$, $\psi_m^c(L, TT - L, s_1, s_2)$ dose not depend on L. Thus manufacturer's effective standard deviation is unaffected by postponement. When $L > s$,

$$\psi_m^c(L, TT-L, s_1, s_2) = \sqrt{a - bL},$$

where

$$\begin{aligned} a &= (TT+1)\left(\sum_{j=1}^{J}\sqrt{(\sigma_{1j}^2+\sigma_{2j}^2)}\right)^2 - s\sum_{j=1}^{J}(\sigma_{1j}^2+\sigma_{2j}^2) > 0, \text{ and} \\ b &= \left[\left(\sum_{j=1}^{J}\sqrt{(\sigma_{1j}^2+\sigma_{2j}^2)}\right)^2 - \sum_{j=1}^{J}(\sigma_{1j}^2+\sigma_{2j}^2)\right] > 0. \end{aligned}$$

By Lemma 1, we know $\psi_m^c(L, TT-L, s_1, s_2)$ is concave decreasing in $(s, TT]$. Now we check the concavity property around s. From

$$\begin{aligned} 0 &= \frac{d_-\psi_m^c(L, TT-L, s_1, s_2)}{dL}|_{L=s} \\ &> \frac{d_+\psi_m^c(L, TT-L, s_1, s_2)}{dL}|_{L=s}, \end{aligned}$$

$\psi_m^c(L, TT-L, s_1, s_2)$ is concave decreasing in the whole interval of $[0, TT]$.

Case 2: $s_1 \neq s_2$

Without loss of generality, we assume $s_1 < s_2$. $\psi_m^c(L, TT-L, s_1, s_2)$ is also continuous in $[0, TT]$.

When $L \leqslant s_1 < s_2$,

$$\psi_m^c(L, TT-L, s_1, s_2) = \sqrt{\left(\sum_{j=1}^{J}\sqrt{(TT+1-s_1)\sigma_{1j}^2 + (TT+1-s_2)\sigma_{2j}^2}\right)^2}$$

is independent of L. Thus manufacturer's effective standard deviation is unaffected by postponement when $L \leqslant s_1 < s_2$.

When $s_1 < L \leqslant s_2$,

$$\psi_m^c(L, TT{-}L, s_1, s_2) = \sqrt{(L - s_1)\sum_{j=1}^{J}\sigma_{1j}^2 + \left(\sum_{j=1}^{J}\sqrt{(TT+1-L)\sigma_{1j}^2 + (TT+1-s_2)\sigma_{2j}^2}\right)^2}.$$

we define two new functions

$$\begin{aligned} \Psi(L) &\equiv [\psi_m^c(L, TT-L, s_1, s_2)]^2 = (L - s_1)\sum_{j=1}^{J}\sigma_{1j}^2 + \left[\sum_{j=1}^{J} r_j(L)\right]^2, \text{ and} \\ r_j(L) &\equiv \sqrt{(TT+1-L)\sigma_{1j}^2 + (TT+1-s_2)\sigma_{2j}^2}, \end{aligned} \tag{B.8}$$

to aid our proof. Since

$$\begin{aligned} \frac{\partial \psi_m^c(L, TT-L, s_1, s_2)}{\partial L} &= (\frac{1}{2})\,[\Psi(L)]^{-\frac{1}{2}}\,\frac{\partial \Psi(L)}{\partial L}, \text{ and} \\ \frac{\partial^2 \psi_m^c(L, TT-L, s_1, s_2)}{\partial L^2} &= (\frac{1}{2})\left\{-\frac{1}{2}\,[\Psi(L)]^{-\frac{3}{2}}\left[\frac{\partial \Psi(L)}{\partial L}\right]^2 + [\Psi(L)]^{-\frac{1}{2}}\,\frac{\partial^2 \Psi(L)}{\partial L^2}\right\}, \end{aligned} \tag{B.9}$$

concavity is proved if we can show

$$2\frac{\partial^2 \Psi(L)}{\partial L^2}\Psi(L) \leqslant \left[\frac{\partial \Psi(L)}{\partial L}\right]^2. \tag{B.10}$$

Because

$$\begin{aligned} \frac{\partial r_j(L)}{\partial L} &= -\frac{1}{2}\frac{\sigma_{1j}^2}{r_j(L)}, \text{ and} \\ \frac{\partial^2 r_j(L)}{\partial L^2} &= -\frac{1}{4}\frac{\sigma_{1j}^4}{r_j(L)^3}, \end{aligned}$$

we have the derivatives of $\Psi(L)$ as follows.

$$\begin{aligned} \frac{\partial \Psi(L)}{\partial L} &= \sum_{j=1}^{J}\sigma_{1j}^2 + 2\left[\sum_{j=1}^{J} r_j(L)\right]\left[\sum_{j=1}^{J}\frac{\partial r_j(L)}{\partial L}\right] \\ &= \sum_{j=1}^{J}\sigma_{1j}^2 - \left[\sum_{j=1}^{J} r_j(L)\right]\left[\sum_{j=1}^{J}\frac{\sigma_{1j}^2}{r_j(L)}\right], \text{ and} \\ \frac{\partial^2 \Psi(L)}{\partial L^2} &= 2\left[\sum_{j=1}^{J}\frac{\partial r_j(L)}{\partial L}\right]^2 + 2\left[\sum_{j=1}^{J} r_j(L)\right]\left[\sum_{j=1}^{J}\frac{\partial^2 r_j(L)}{\partial L^2}\right] \\ &= \frac{1}{2}\left\{\left[\sum_{j=1}^{J}\frac{\sigma_{1j}^2}{r_j(L)}\right]^2 - \left[\sum_{j=1}^{J} r_j(L)\right]\left[\sum_{j=1}^{J}\frac{\sigma_{1j}^4}{r_j(L)^3}\right]\right\}. \end{aligned} \tag{B.11}$$

Substituting (B.8) and (B.11) into (B.10) yields

$$\left\{\begin{array}{c}\left[\sum_{j=1}^{J}\frac{\sigma_{1j}^2}{r_j(L)}\right]^2 \\ -\left[\sum_{j=1}^{J} r_j(L)\right]\left[\sum_{j=1}^{J}\frac{\sigma_{1j}^4}{r_j(L)^3}\right]\end{array}\right\}\cdot\left\{\begin{array}{c}(L-s_1)\sum_{j=1}^{J}\sigma_{1j}^2 \\ +\left[\sum_{j=1}^{J} r_j(L)\right]^2\end{array}\right\} \tag{B.12}$$
$$\leqslant \left\{\sum_{j=1}^{J}\sigma_{1j}^2-\left[\sum_{j=1}^{J} r_j(L)\right]\left[\sum_{j=1}^{J}\frac{\sigma_{1j}^2}{r_j(L)}\right]\right\}^2.$$

From Lemma 2, the first term of (B.12) is negative. Thus (B.12) and (B.10) always hold. From (B.11) and (B.9), we also know

$$\begin{aligned}\frac{\partial\psi_m^c(L,TT-L,s_1,s_2)}{\partial L} &= (\frac{1}{2})\,[\Psi(L)]^{-\frac{1}{2}}\,\frac{\partial\Psi(L)}{\partial L}\\ &= (\frac{1}{2})\,[\Psi(L)]^{-\frac{1}{2}}\left\{\sum_{j=1}^{J}\sigma_{1j}^2-\left[\sum_{j=1}^{J} r_j(L)\right]\left[\sum_{j=1}^{J}\frac{\sigma_{1j}^2}{r_j(L)}\right]\right\}\\ &< 0.\end{aligned}$$

Thus $\psi_m^c(L)$ is concave decreasing in postponement.

When $s_1 < s_2 < L$,

$$\begin{aligned}&\psi_m^c(L,TT-L,s_1,s_2)\\ =&\sqrt{(L-s_1)\sum_{j=1}^{J}\sigma_{1j}^2+(L-s_2)\sum_{j=1}^{J}\sigma_{2j}^2+\left(\sum_{j=1}^{J}\sqrt{(TT+1-L)(\sigma_{1j}^2+\sigma_{2j}^2)}\right)^2}\\ =&\sqrt{(L-s_1)\sum_{j=1}^{J}\sigma_{1j}^2+(L-s_2)\sum_{j=1}^{J}\sigma_{2j}^2+(TT+1-L)\left(\sum_{j=1}^{J}\sqrt{(\sigma_{1j}^2+\sigma_{2j}^2)}\right)^2}\\ =&\sqrt{a-bL},\end{aligned}$$

where

$$\begin{aligned}a &= (TT+1)\left(\sum_{j=1}^{J}\sqrt{(\sigma_{1j}^2+\sigma_{2j}^2)}\right)^2-s_1\sum_{j=1}^{J}\sigma_{1j}^2-s_2\sum_{j=1}^{J}\sigma_{2j}^2>0,\text{ and}\\ b &= \left[\left(\sum_{j=1}^{J}\sqrt{(\sigma_{1j}^2+\sigma_{2j}^2)}\right)^2-\sum_{j=1}^{J}(\sigma_{1j}^2+\sigma_{2j}^2)\right]>0.\end{aligned}$$

By Lemma 1, we know $\psi_m(L, l, s_1, s_2)$ is concave decreasing in L.

Now we check the concavity property around s_1 and s_2. From

$$\begin{aligned}
\frac{d_-\psi_m^c(L, TT-L, s_1, s_2)}{dL}|_{L=s_1} &> \frac{d_+\psi_m^c(L, TT-L, s_1, s_2)}{dL}|_{L=s_1} \text{ and} \\
\frac{d_-\psi_m^c(L, TT-L, s_1, s_2)}{dL}|_{L=s_2} &> \frac{d_+\psi_m^c(L, TT-L, s_1, s_2)}{dL}|_{L=s_2},
\end{aligned}$$

$\psi_m^c(L, TT-L, s_1, s_2)$ is concave decreasing on $[0, TT]$. Thus manufacturer's effective standard deviation $\psi_m(L, TT-L, s_1, s_2)$ is concave decreasing in L on the discrete domain $\{0, 1, ..., TT\}$ for any pair of s_1 and s_2. $\square$

Proof of Proposition 16 (a) From Proposition 3, we know that x, y, and z are concave decreasing in postponement. Thus $\underline{M}$ and $\underline{N}$ are always decreasing in postponement.

(b) From Lemma 3, we know that $x - y$ is decreasing in postponement if

$$\frac{\sum_{j=1}^J \sigma_{1j}^2}{\left[\sum_{j=1}^J \sigma_{1j}\right]^2} \geqslant \frac{\sum_{j=1}^J (\sigma_{1j}^2 + \sigma_{2j}^2)}{\left[\sum_{j=1}^J \sqrt{\sigma_{1j}^2 + \sigma_{2j}^2}\right]^2},$$

and that $x - z$ is decreasing in postponement if

$$\frac{\sum_{j=1}^J \sigma_{2j}^2}{\left[\sum_{j=1}^J \sigma_{2j}\right]^2} \geqslant \frac{\sum_{j=1}^J (\sigma_{1j}^2 + \sigma_{2j}^2)}{\left[\sum_{j=1}^J \sqrt{\sigma_{1j}^2 + \sigma_{2j}^2}\right]^2}.$$

Thus $\overline{M}$ is also decreasing in postponement.

(c) From Lemma 3, we know that $x - y$ is decreasing in postponement if

$$\frac{\sum_{j=1}^J \sigma_{1j}^2}{\left[\sum_{j=1}^J \sigma_{1j}\right]^2} \geqslant \frac{\sum_{j=1}^J (\sigma_{1j}^2 + \sigma_{2j}^2)}{\left[\sum_{j=1}^J \sqrt{\sigma_{1j}^2 + \sigma_{2j}^2}\right]^2}.$$

Thus $\overline{N}$ is also decreasing in postponement. $\square$

Proof of Corollary 4 We fix the total production time at a constant level TT. In two product case,

$$\begin{aligned} x &= \sqrt{L(\sigma_{11}^2+\sigma_{21}^2+\sigma_{12}^2+\sigma_{22}^2)+(TT+1-L)\left[\sqrt{\sigma_{11}^2+\sigma_{21}^2}+\sqrt{\sigma_{12}^2+\sigma_{22}^2}\right]^2}, \\ y &= \sqrt{L(\sigma_{11}^2+\sigma_{12}^2)+(TT+1-L)(\sigma_{11}+\sigma_{12})^2}, \text{ and} \\ z &= \sqrt{L(\sigma_{21}^2+\sigma_{22}^2)+(TT+1-L)(\sigma_{21}+\sigma_{22})^2}. \end{aligned}$$

Notice that $x > 0$, $y > 0$, $z > 0$, $x > y$ and $x > z$ are always true for any $L \in \{0, \ldots, TT\}$. We want to show $x - y$ and $x - z$ are always decreasing in postponement.

Without loss of generality, we focus on the proof of $x - y$.

$$(x-y)' < 0 \Leftrightarrow \frac{c-d}{C-D} < \sqrt{f(L)},$$

where

$$\begin{aligned} f(L) &\equiv \frac{(TT+1)c-(c-d)L}{(TT+1)C-(C-D)L}, \\ c &\equiv (\sigma_{11}+\sigma_{12})^2, \\ d &\equiv (\sigma_{11}^2+\sigma_{12}^2), \\ C &\equiv \left[\sqrt{\sigma_{11}^2+\sigma_{21}^2}+\sqrt{\sigma_{12}^2+\sigma_{22}^2}\right]^2, \text{ and} \\ D &\equiv (\sigma_{11}^2+\sigma_{21}^2+\sigma_{12}^2+\sigma_{22}^2). \end{aligned}$$

Notice that $C \geqslant D$, $c \geqslant d$, $C \geqslant c$, and $D \geqslant d$ are always true. $(c-d)/(C-D) < \sqrt{f(L)}$ if and only if $(c-d)/(C-D)$ is smaller than the minimum value of $\sqrt{f(L)}$. It is easy to show that

$$f(L) \text{ is } \begin{cases} \text{increasing} & \text{if } dC > Dc \\ \text{constant} & \text{if } dC = Dc \\ \text{decreasing} & \text{if } dC < Dc \end{cases}.$$

When $dC \geqslant Dc$, the minimum value of $f(L)$ is achieved at 0.

$$\begin{aligned} & dC \geqslant Dc \\ \Leftrightarrow\ & \tfrac{c-d}{C-D} < \tfrac{c}{C} < 1 \\ \Rightarrow\ & \tfrac{c-d}{C-D} < \sqrt{\tfrac{c}{C}} \\ \Rightarrow\ & \tfrac{c-d}{C-D} < \sqrt{f(0)} \end{aligned}$$

So, $(c-d)/(C-D) < \sqrt{f(L)}$is true.

When $dC < Dc$, the minimum value of $f(L)$ is achieved at TT.

$$\begin{aligned} & \left(\tfrac{c-d}{C-D}\right)^2 < \tfrac{d}{D} \\ \Leftrightarrow\ & \tfrac{c-d}{C-D} < \sqrt{\tfrac{d}{D}} \\ \Rightarrow\ & \tfrac{c-d}{C-D} < \sqrt{f(TT+1)} \\ \Rightarrow\ & \tfrac{c-d}{C-D} < \sqrt{f(TT)} \end{aligned}$$

So, $(c-d)/(C-D) < \sqrt{f(L)}$ is true.

Hence, $x-y$ always decreases in postponement. Thus all four thresholds are always decreasing in postponement. □

Lemma 5. *$a > 0$ and $b > 0$, and $f(L) = \sqrt{a-bL}$ is defined on an interval where $a-bL > 0$. Then, $f(L)$ is concave decreasing in L.*

Proof. Since

$$\begin{aligned} \frac{\partial f}{\partial L} &= -\frac{b}{2}(a-bL)^{-\frac{1}{2}} < 0, \text{ and} \\ \frac{\partial^2 f}{\partial L^2} &= -\frac{b^2}{4}(a-bL)^{-\frac{3}{2}} < 0, \end{aligned}$$

$f(L)$ is concave decreasing in L. □

Lemma 6. *$a_j > 0, b_j > 0$ for $j \in \{1, 2, ..., J\}$, then*

$$\left(\sum_{j=1}^{J} \frac{a_j}{b_j}\right)\left(\sum_{j=1}^{J} \frac{a_j}{b_j}\right) \leqslant \left(\sum_{j=1}^{J} b_j\right)\left(\sum_{j=1}^{J} \frac{a_j^2}{b_j^3}\right). \tag{B.13}$$

Proof. We expand both sides of the inequality and adopt a term-wise comparison. When $i = j$, we note that

$$\left(\frac{a_j}{b_j}\right)^2 = b_j \left(\frac{a_j^2}{b_j^3}\right),$$

so (B.13) holds. When $i \neq j$, we compare

$$\left(\frac{a_i}{b_i}\right)\left(\frac{a_j}{b_j}\right) + \left(\frac{a_j}{b_j}\right)\left(\frac{a_i}{b_i}\right) \tag{B.14}$$

and

$$b_i \left(\frac{a_j^2}{b_j^3}\right) + b_j \left(\frac{a_i^2}{b_i^3}\right). \tag{B.15}$$

Our aim is to show (B.14) is not greater than (B.15) for any combination of $i \neq j$. Since

$$0 \leqslant \left(a_j \frac{b_i}{b_j} - a_i \frac{b_j}{b_i}\right)^2 \Leftrightarrow 2a_j a_i \leqslant a_j^2 \left(\frac{b_i}{b_j}\right)^2 + a_i^2 \left(\frac{b_j}{b_i}\right)^2,$$

dividing the later inequality by $b_i b_j$ implies that (B.14) is not greater than (B.15). Thus, (B.13) holds. $\square$

Lemma 7. *$x - y$ decreases in postponement if*

$$\frac{\sum_{j=1}^{J} \sigma_{1j}^2}{\left[\sum_{j=1}^{J} \sigma_{1j}\right]^2} \geqslant \frac{\sum_{j=1}^{J} (\sigma_{1j}^2 + \sigma_{2j}^2)}{\left[\sum_{j=1}^{J} \sqrt{\sigma_{1j}^2 + \sigma_{2j}^2}\right]^2},$$

and $x - z$ decreases in postponement if

$$\frac{\sum_{j=1}^{J} \sigma_{2j}^2}{\left[\sum_{j=1}^{J} \sigma_{2j}\right]^2} \geqslant \frac{\sum_{j=1}^{J} (\sigma_{1j}^2 + \sigma_{2j}^2)}{\left[\sum_{j=1}^{J} \sqrt{\sigma_{1j}^2 + \sigma_{2j}^2}\right]^2}.$$

Proof. Fixed the total production time at TT. Without loss of generality, we focus on the proof of the first part. Define

$$\begin{aligned}
P &\equiv \left[\sum_{j=1}^{J}\sqrt{\sigma_{1j}^2+\sigma_{2j}^2}\right]^2, \\
Q &\equiv \sum_{j=1}^{J}(\sigma_{1j}^2+\sigma_{2j}^2), \\
p &\equiv \left[\sum_{j=1}^{J}\sigma_{1j}\right]^2, \text{and} \\
q &\equiv \sum_{j=1}^{J}\sigma_{1j}^2.
\end{aligned}$$

Then,

$$\begin{aligned}
x &= \sqrt{LQ+(TT+1-L)P} \\
&= \sqrt{(TT+1)P-L(P-Q)}, \\
y &= \sqrt{Lq+(TT+1-L)p} \\
&= \sqrt{(TT+1)p-L(p-q)}.
\end{aligned}$$

Thus,

$$\begin{aligned}
&(x-y)' \\
= \; &\frac{1}{2}\left[\frac{p-q}{\sqrt{(TT+1)p-L(p-q)}}-\frac{P-Q}{\sqrt{(TT+1)P-L(P-Q)}}\right].
\end{aligned}$$

$$(x-y)'<0 \Leftrightarrow \frac{p-q}{P-Q}<\sqrt{f(L)},$$

where

$$f(L)\equiv\frac{(TT+1)p-L(p-q)}{(TT+1)P-L(P-Q)}.$$

It is easy to show that

$$f(L) \text{ is } \begin{cases} \text{increasing} & \text{if } q \cdot P > Q \cdot p \\ \text{constant} & \text{if } q \cdot P = Q \cdot p \\ \text{decreasing} & \text{if } q \cdot P < Q \cdot p \end{cases} .$$

When $q \cdot P \geqslant Q \cdot p$, the minimum value of $f(L)$ is achieved at 0.

$$\begin{aligned} & q \cdot P \geqslant Q \cdot p \\ \Leftrightarrow \quad & \tfrac{p-q}{P-Q} < \tfrac{p}{P} < 1 \\ \Rightarrow \quad & \tfrac{c-d}{C-D} < \sqrt{\tfrac{p}{P}} \\ \Rightarrow \quad & \tfrac{c-d}{C-D} < \sqrt{f(0)} \end{aligned}$$

So, $(x-y)' < 0$ when $q \cdot P \geqslant Q \cdot p$, which is equivalent to

$$\frac{\sum_{j=1}^{J} \sigma_{1j}^2}{\left[\sum_{j=1}^{J} \sigma_{1j}\right]^2} \geqslant \frac{\sum_{j=1}^{J} (\sigma_{1j}^2 + \sigma_{2j}^2)}{\left[\sum_{j=1}^{J} \sqrt{\sigma_{1j}^2 + \sigma_{2j}^2}\right]^2}.$$

$\square$

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BIOGRAPHICAL SKETCH

Xiao Zhang was born in the city of Renqiu, Hebei Province, China. He graduated from Beijing Institute of Technology with BS in Applied Mathematics in 2005. Following that, he moved to US to pursue graduate studies. He received an MS degree in Computational Mathematics from Southern Methodist University and an MS degree in Statistics from The University of Texas at Dallas (UTD). He joined the PhD program in Operations Management at UTD in 2011. His research interests lie in the intersection of pricing management and customer relationship management. At UTD, he taught the undergraduate core course Operations Management in Spring 2015.

CURRICULUM VITAE

Xiao Zhang

June 2018

Contact Information:

Department of Operations Management
The University of Texas at Dallas
800 W. Campbell Rd. SM 30
Richardson, TX 75080-3021, U.S.A.
Email: xiao.zhang@utdallas.edu

Educational History:

M.S., Statistics, The University of Texas at Dallas, 2017
M.S., Computational Mathematics, Southern Methodist University, 2011
B.S., Applied Mathematics, Beijing Institute of Technology, 2009

Pricing, Upgrade, and Contract Management
Ph.D. Dissertation
Department of Operations Management, The University of Texas at Dallas
Advisor: Dr. Özalp Özer

Solving Groundwater Flow PDEs by Finite Difference Methods
Undergraduate Thesis
School of Science, Beijing Institute of Technology
Advisor: Dr. Jiafu Lin

Professional Recognitions and Honors:

JSOM Owlie Award TA of the Year, UTD, 2015
JSOM Dean's Excellence Scholarship, UTD, 2016
JSOM Dean's Excellence Scholarship, UTD, 2017
Finalist, INFORMS IBM Service Science Best Student Paper Award Competition, 2017

Professional Memberships:

Institute for Operations Research and the Management Sciences (INFORMS), 2011–present
Production and Operations Management Society (POMS), 2015–present
Manufacturing and Service Operations Management Society (MSOM), 2016–present
Decision Sciences Institute (DSI), 2017–present