STABILITY, BIFURCATION, AND CONTINUATION THEORY FOR PERTURBED SWEEPING PROCESSES

by

Lakmi Niwanthi Wadippuli Achchige

APPROVED BY SUPERVISORY COMMITTEE:

Oleg Makarenkov, Chair

Dmitry Rachinskiy

Tomoki Ohsawa

Wieslaw Krawcewicz

To Dr. Oleg Makarenkov, Dr. Stefano Luzzatto and Dr. Sanjeewa Perera.

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LAKMI NIWANTHI WADIPPULI ACHCHIGE, BSc

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FOR PERTURBED SWEEPING PROCESSES

Lakmi Niwanthi Wadippuli Achchige, PhD
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Supervising Professor: Oleg Makarenkov, Chair

This dissertation is devoted to the development of a qualitative theory of perturbed sweeping processes, which are a combination of differential equations and a moving constraint. The differential equations involved are always assumed Lipschitz continuous. As for the moving constraint, several different situations are addressed: Lipschitz continuous in time, BV-continuous in time, state-dependent, state-independent, with convex interior, with prox-regular interior, bounded in time, periodic in time, almost periodic in time. We prove the existence of local and global solutions as well as boundedness, periodicity, almost periodicity, and asymptotic stability of solutions. Furthermore, we establish results on the occurrence of periodic solutions from a switched boundary equilibrium and on bifurcation of cycles from a regular boundary equilibrium. Concrete examples illustrate the main results of the dissertation.

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INTRODUCTION

1.1 Motivation and Historical Remarks

A perturbed Moreau sweeping process reads as

$$-\dot{x}(t) \in N_{C(t)}(x(t)) + f(t, x(t)), \qquad x \in E \tag{1.1}$$

with multivalued function $t \mapsto C(t)$ and vector valued funtion $f : \mathbb{R} \times E \to E$ (see Castaing and Monteiro Marques [19], Kunze [39], Kamenskiy-Makarenkov [31], Edmond-Thibault [26]), where $N_C(x)$ is the outward normal cone defined for nonempty closed convex set $C \subset E$ as

$$N_C(x) = \begin{cases} \{\xi \in \mathbb{R}^n : \langle \xi, c - x \rangle \leq 0, \text{ for any } c \in C\}, & \text{if } x \in C, \\ \emptyset, & \text{if } x \notin C. \end{cases} \tag{1.2}$$

Here we consider E to be a finite-dimensional vector space. The unboundedness of the right-hand-sides in (1.1) makes the classical theory of differential inclusions (see e.g., Aubin et al [6], Kamenskii et al [33]) inapplicable.

Numerous applications can found in elastoplasticity (see e.g., Adly et al [3], Bastein et al [7, 8], Gilles-Ulisse [29], Kunze [39]) as well as in problems of power converters Addi et al [2] and crowd motion Maury-Venel [48]. In sweeping processes (1.1) coming from models of parallel networks of elastoplastic springs (see e.g., Bastein et al [7, 8], Gilles-Ulisse [29], Kunze [39]), $C(t)$ represents the mechanical loading of the springs and $f(t, x)$ stands for those forces which influence the masses of nodes.

Existence and uniqueness: Fundamental results on the existence, uniqueness and dependence of solutions on the initial data are proposed in Monteiro Marques [47, Ch. 3], Monteiro Marques [49], Valadier [61, 62], Edmond-Thibault [26], Castaing and Monteiro Marques [19], Adly-Le [4], Brogliato-Thibault [15], Krejci-Roche [38], Paoli [54]. Dependence of solutions on parameters is covered in Bernicot-Venel [10] and Kamenskiy-Makarenkov [31]. Optimal control problems for sweeping process (1.1) and equivalent differential equations with hysteresis operator are addressed in Edmond-Thibault [26], Adam-Outrata [1] (which also discusses applications to game theory), Brokate-Krejci [16]. Numerical schemes to compute the solutions of (1.1) are discussed through most of the papers mentioned above.

When $t \mapsto C(t)$ is Lipschitz i.e.,

$$d_H(C(t_1), C(t_2)) \le L|t_1 - t_2|, \qquad \text{for all } t_1, t_2 \in \mathbb{R}, \text{ and for some } L > 0,$$

where the Hausdorff distance $d_H(C_1, C_2)$ between two closed sets $C_1, C_2 \subset \mathbb{R}^n$ is defined as

$$d_H(C_1, C_2) = \max\left\{\sup_{x\in C_2} \operatorname{dist}(x, C_1), \sup_{x\in C_1} \operatorname{dist}(x, C_2)\right\} \tag{1.3}$$

with $\operatorname{dist}(x, C) = \inf\{|x - c| : c \in C\}$, always leads (see e.g., Edmond-Thibault [26], [27]) to the existence and uniqueness of an absolutely continuous solution $x(t)$ for any initial condition (under natural assumptions on f). But in the case where $t \mapsto C(t)$ is a convex-valued function of bounded variation doesn't ensure solvability of (1.1) in the class of absolutely continuous functions (see Adly et al [3] and references therein for application). That is why an extended concept of the derivative (called Radon-Nikodym concept) is required in (1.1) when the map $t \mapsto C(t)$ is a function of bounded variation, in which case equation (1.1) is usually formulated in terms of differential measure dx of BV continuous function x and Lebesgue measure dt as

$$-dx \in N_{C(t)}(x) + f(t, x)dt, \quad x \in E.$$

Existence and uniqueness of solutions as well as existence of a periodic solution to above system has been established by Castaing and Monteiro Marques in [19]. The problem of existence and uniqueness of solutions in the unperturbed case ($f \equiv 0$) was addressed in Moreau [51], Monteiro Marques [49], Valadier [61]. Further state-independent extensions of the system were considered in Adly et al [3], Edmond-Thibault [27], Colombo and Monteiro Marques [20].

Existence of a solution to state-dependent sweeping processes with Lipschitz $(t, x) \mapsto C(t, x)$ and $f \equiv 0$ is proved by Kunze and Monteiro Marques ([40]) by introducing an implicit numerical scheme to (1.1).

Due to challenges from crowd motion modeling (Maury-Venel [48]), the existence and uniqueness of solutions to a sweeping process with a nonconvex set have been studied. The main problem towards weakening the convexity of the set is the lack of a unique map $x \mapsto \mathrm{proj}(x, C)$. Therefore the concept of prox-regularity of sets came to the studies of the sweeping process. For the space $\mathbb{R}^n$, the set $C(t)$ is η-prox-regular, if $C(t)$ admits an external tangent ball with radius smaller than η at each point of the boundary of $C(t)$ (see Maury-Venel [48, p. 150], Colombo and Monteiro Marques [21, p. 48]).

When the set is nonconvex, the monotonicity of $x \mapsto N_C(x)$, i.e., the property $\langle \xi - \xi', x - x' \rangle \geq 0$, $\xi \in N_C(x)$, $\xi' \in N_C(x')$, $x, x' \in C$ may not hold. Therefore the proximal normal cone, which has a relevant property of hypomonotonicity is used in sweeping processes with nonconvex sets (see Edmond-Thibault [26]).

We define the proximal normal cone $N(C, x)$ to a nonempty closed set C at x as

$$N(C, x) = \{\xi \in \mathbb{R}^n : x \in \mathrm{proj}(x + \alpha\xi, C) \text{ for some } \alpha > 0\}$$

where $\mathrm{proj}(x, C)$ is the set of closest points on C to the point x. When the set C is convex, the proximal normal cone $N(C, x)$ and the outward normal cone $N_C(x)$ coincide (see e.g., Maury-Venel [48, Remark 2.9]).

Colombo-Goncharov [20], Benabdellah [9], Colombo and Monteiro Marques [21] and Thibault [59] studied the existence and uniqueness of solutions to non-perturbed sweeping process (i.e., (1.1) with $f \equiv 0$) with uniform prox-regular sets. And perturbed sweeping process is considered in Edmond-Thibault [26], [27]. A sweeping process with uniform prox-regular set values is appear for the crowd motion in Maury-Venel [48] with numerical simulations. Cao-Mordukhovich [18] studied optimal control of a nonconvex perturbed sweeping process and applied to the planar crowd motion model given by Maury-Venel [48].

In the model of crowd motion which introduced by Maury-Venel [48] using sweeping processes, the constrained set C is a nonconvex constant set and the perturbation vector function is time independent.

When defining the model, Maury-Venel [48] considered a group of N people whose positions are given by $x = (x_1, x_2, \ldots, x_N) \in \mathbb{R}^{2N}$, where each person is identified as a disk with center $x_i \in \mathbb{R}^2$ and radius r.

Then by avoiding overlapping of people, the set of feasible configuration is defined as

$$C = \{x \in \mathbb{R}^{2N} : \|x_i - x_j\| - 2r \geq 0 \text{ for all } i < j\}. \tag{1.4}$$

And the perturbed function $U(x) = (U_1(x), U_2(x), \cdots U_N(x))$ represents the spontaneous velocity of each person at the position x, i.e., $U_i(x)$ is the velocity that i-th person would have in the absence of other people.

Stability: Much less is known about the asymptotic behavior of solutions of perturbed sweeping processes as $t \to \infty$. The known results in this direction are due to Leine-Wouw [41, 42], Brogliato [13], and Brogliato-Heemels [14]. Applied to a time-independent sweeping process (1.1) the statements of Leine-Wouw [41, Theorem 8.7] (or [42, Theorem 2]), Brogliato [13, Lemma 2], and Brogliato-Heemels [14, Theorem 4.4] imply the incremental stability and global exponential stability of an equilibrium, provided that f is strongly monotone, i.e.,

$$\begin{aligned} &\langle f(t, x_1) - f(t, x_2), x_1 - x_2 \rangle \geq \alpha \|x_1 - x_2\|^2, \\ &\qquad\qquad\qquad \text{for some fixed } \alpha > 0 \text{ and for all } t \in \mathbb{R},\ x_1, x_2 \in \mathbb{R}^n. \end{aligned} \tag{1.5}$$

In particular, the results of Leine-Wouw [41, 42], Brogliato [13], Brogliato-Heemels [14] do not impose any Lipschitz regularity on $x \mapsto f(t, x)$ and the derivative in (1.1) is a differential measure, which is capable to deal with solutions x of bounded variation.

Periodicity: The papers Kamenskiy-Makarenkov [31], Castaing and Monteiro Marques [19], Kunze [39] show the existence of T-periodic solutions for T-periodic in time (1.1). Time-periodically changing C and f are most typical in laboratory experiments for models of parallel networks of elastoplastic springs (see Kamenskii-Makarenkov [30], Al Janaideh-Krejci [5], Bastein et al [8]). However, the different nature of $t \mapsto C(t)$ and $t \mapsto f(t,x)$ makes it most reasonable to not rely on the existence of a common period when the two functions receive periodic excitations, but rather to use a theory which is capable to deal with arbitrary different periods of $t \mapsto C(t)$ and $t \mapsto f(t,x)$.

While results on the existence of almost periodic solutions for differential equations are available (e.g., in Trubnikov-Perov [60] and Zhao [64]) and used in applications, the existence of almost periodic solutions to sweeping processes is not addressed in the literature. A series of results on the existence and stability of almost periodic solutions to differential inclusions is obtained in Kloeden-Kozyakin [34] and Plotnikov [35], but still for the case of bounded right-hand-sides.

1.2 Outline of the dissertation

The dissertation is organized as follows. Chapter 2 consists of the results which we already published in Kamenskiy at al [32]. Here we investigate the global existence, periodicity, almost periodicity, global stability and response to additional perturbations of sweeping processes with the monotonicity property (1.5). First, we establish the existence of a solutions to (1.1) defined on the entire $\mathbb{R}$ under the assumption that both $t \mapsto C(t)$ and $(t,x) \mapsto f(t,x)$ are uniformly Lipschitz functions, but without any use of the monotonicity assumption (1.5). When (1.5) holds, we have (Theorem 2.2.2) the uniqueness and global exponential stability of a solution defined on the entire $\mathbb{R}$.

Under the assumption that both $t \mapsto C(t)$ and $t \mapsto f(t,x)$ are almost periodic functions and $x \mapsto f(t,x)$ is monotone in the sense of (1.5), we also show (Theorem 2.3.1) that the unique global solution defined on the entire $\mathbb{R}$ is almost periodic. Here we follow Vesely [63] to introduce the concept of almost periodicity for set-valued functions and for the respective Bochner's theorem. The results of Vesely [63] are developed for functions with values in an arbitrary complete metric space and we take advantage of the completeness of the space of convex closed non-empty sets equipped with the Hausdorff metric (see e.g., Price [55]) to apply the concept of almost-periodicity to sweeping processes.

We also consider the sweeping process (1.1) with a parameter ε under the assumption that the monotonicity condition (1.5) and almost periodicity of C and f only hold for $\varepsilon = \varepsilon_0$. The results of (Theorems 2.4.1 and 2.4.2) prove that the solutions to the perturbed sweeping process with $\varepsilon \neq \varepsilon_0$ and with an initial condition $x_\varepsilon(0) \in C(0)$ approach any given inflation of the solution x_0 (as it is termed in Kloeden-Kozyakin [34]) when the values of time become large and when ε approaches ε_0. Instructive examples of Section 2.4.4 illustrate the domains of applications of Theorems 2.4.1 and 2.4.2.

Chapter 3 is devoted to a nonconvex (prox-regular) extension of the stability result that we obtained in Chapter 2 for convex set-valued functions.

Additionally to the assumptions of Theorem 2.2.2, we need that $\|f(t,x)\| \leq M_f$, for some $M_f > 0$ and all $t \in \mathbb{R}$, $x \in \bigcup\limits_{t\in\mathbb{R}} C(t)$, in order to use the property of hypomonotonicity of the prox-normal cone. And to obtain contraction of solutions of sweeping process (1.1), a lower bound on the monotonicity constant α in (1.5) is imposed. Also in this chapter we study the periodicity of solutions when input functions are periodic in (1.1) together with prox-regular set-valued function $t \mapsto C(t)$. This is an analogue of the result of Castaing and Monteiro Marques [19, Theorem 5.3] that was obtained in [19] for convex sets. Finally, Chapter 3 discusses asymptotic stability of periodic solutions.

Chapter 4 contains the sweeping process part of the results that we published in Makarenkov-

Niwanthi [46]. In this chapter we study bifurcation of limit cycles from a boundary focus equilibrium for sweeping processes (1.1) in $E = \mathbb{R}^2$, which takes the form

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} \in -N_{C(\varepsilon)}(x, y) + \begin{pmatrix} f(x, y) \\ g(x, y) \end{pmatrix}, \quad (x, y) \in \mathbb{R}^2$$

where the constraint C is no longer moving in time but it now depends on a parameter ε. First, in Proposition (4.2.1) we derive an equation of sliding along the boundary $\partial C(\varepsilon)$ of $C(\varepsilon)$ for sweeping process and an equation for the stationary point of the equation of sliding. We use these findings in Theorem (4.2.1) to establish the bifurcation of finite-time stable limit cycles of the sweeping process from a focus equilibrium of the reduced system

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} f(x, y) \\ g(x, y) \end{pmatrix},$$

when the focus equilibrium collides with the boundary $\partial C(\varepsilon)$ under varying ε.

We conclude the chapter by an illustrative example.

Chapter 5 we build upon the knowledge obtained over the previous chapters to advance the field of sweeping processes with minimal regularity properties.

Here we study the existence of periodic solutions in the following state-dependent version of sweeping process (1.1)

$$-\dot{x} \in N_{A+a(t)+c(x)}(x) + f(t, x), \quad x \in E, \tag{1.6}$$

where a is a BV-continuous function and $c : E \mapsto E$ is a Lipschitz function.

Since solutions of (1.6) are no longer absolutely continuous when the constraint $C(t) = A + a(t) + c(t)$ is only BV-continuous in time, a new concept of derivative (Radon-Nikodym derivative) is used in Chapter 5 to define solutions to (1.6).

We prove the existence of solutions to (1.6) (Theorem 5.3.1) by introducing a new implicit catching-up algorithm (5.15)-(5.18) which allows to construct a sequence $\{x_n\}_{n\in\mathbb{N}}$ of approximations of the solution x of (1.6). We use the ideas of Kunze and Monteiro Marques [40] to prove that the implicit scheme admits a solution and the variational criterion by Castaing and Monteiro Marques [19] to show that the limit of our scheme is indeed a solution.

The existence of T-periodic solutions to (1.6) if the right-hand-sides of (1.6) are T-periodic is proved in Sections 5.4 and 5.5 by establishing that

$$d(I - P^n, Q) \neq 0 \tag{1.7}$$

for the Poincare maps P^n of the n-th approximation of the catching-up scheme and suitable $Q \in E$. Here $d(I - P^n, Q)$ is the topological degree of the map P^n with respect to an open bounded set Q (see Krasnoselskii-Zabreiko [36]). After we get the existence of a fixed point for P^n we pass to the limit as $n \to \infty$ on the respective T-periodic solutions of sweeping process (1.6) and get the existence of a T-periodic solution to (1.6) even though we don't know whether $\lim_{n\to\infty} P^n(x)$ is uniform on Q or not. We offer global and local sufficient conditions to ensure (1.7). The global sufficient condition is based on construction of such a convex set Q which contains all possible values of the set $A + a(t) + c(x(t))$ for all possible solutions of (1.6). In this way, we can show that $P^n(Q) \subset \overline{Q}$ for sufficiently large $n \in \mathbb{N}$, which ensures (1.7). To design sufficient conditions that ensure the validity of (1.7) in a desired region Q (local sufficient conditions), we are no longer allowed to enlarge Q as much as we want, so we have to seek for alternative deformations of (1.7) that stick to the given region Q. We go here by a continuation approach and replace (1.6) by a parameter dependent sweeping process

$$-dx \in N_{A+a(t,\lambda)+c(x,\lambda)}(x) + f(t, x, \lambda)dt, \quad x \in E,\ \lambda \in \mathbb{R}. \tag{1.8}$$

Accordingly, the relation (1.7) gets replaced by

$$d(I - P^{\lambda,n}, Q) \neq 0. \tag{1.9}$$

We, therefore, assume that (1.7) corresponds to (1.9) with some $\lambda = \lambda_1$ and prove the validity of (1.9) for $\lambda = \lambda_1$ building upon some good properties of $P^{\lambda,n}$ for $\lambda = 0$ combined with nondegenerate homotopy between $P^{\lambda_1,n}$ and $P^{0,n}$. As for possible good properties of $P^{0,n}$ we offer both topological (Theorem 5.5.1) and algebraic (Theorem 5.8.3) conditions.

The topological condition simply assumes that (1.9) holds for $\lambda = 0$, that leads us a standard continuation principle, (Theorem 5.5.1).

To obtain easily verifiable algebraic conditions that ensure the validity of (1.9) for $\lambda = 0$, we offer sufficient conditions for asymptotic stability of a point x_0 of the target set Q. Such an approach is based on the fact that the topological degree of a Poincare map in the neighborhood of an asymptotically stable fixed point equals 1. However, just assuming that x_0 is an asymptotically stable equilibrium of (1.8) with $\lambda = 0$ is not of interest because it leads to periodic solutions that don't interact with the boundary of the constraint of (1.8) when $\lambda > 0$. Such periodic solutions will simply be solutions of the differential equation

$$-\dot{x} = f(t, x, \lambda), \quad x \in E, \ \lambda \in \mathbb{R}.$$

That is why a non-equilibrium concept of an asymptotically stable point x_0 is required to design periodic solutions of (1.8) which are intrinsically sweeping (i.e., interact with the boundary of the constraint of (1.8). Such a concept (called switched boundary equlibrium) is introduced in Chapter 4 and the occurrence of periodic solutions near x_0 is established in Theorem (4.2.1).

CHAPTER 2

GLOBAL STABILITY OF ALMOST PERIODIC SOLUTIONS OF MONOTONE SWEEPING PROCESSES AND THEIR RESPONSE TO NON-MONOTONE PERTURBATIONS

Authors – Mikhail Kamenskii[1], Oleg Makarenkov, Lakmi Niwanthi and Paul Raynaud de Fitte[2]

Department of Mathematical Sciences, FO 35

The University of Texas at Dallas

800 West Campbell Road

Richardson, Texas 75080-3021

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[1]Department of Mathematics, Voronezh State University, Voronezh, Russia

[2]Normandie University, Laboratoire Raphael Salem, UMR CNRS 6085, Rouen, France

2.1 Introduction

In this chapter we assume $f : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ to be globally Lipschitz continuous in the sense that

$$\begin{aligned} \|f(t_1, x_1) - f(t_2, x_2)\| &\leq L_f\|t_1 - t_2\| + L_f\|x_1 - x_2\|, \\ &\text{for all } t_1, t_2 \in \mathbb{R},\ x_1, x_2 \in \mathbb{R}^n,\ \text{and for some } L_f > 0. \quad (2.1) \end{aligned}$$

A similar property

$$d_H(C(t_1), C(t_2)) \leq L_C|t_1 - t_2|, \qquad \text{for all } t_1, t_2 \in \mathbb{R},\ \text{and for some } L_C > 0, \qquad (2.2)$$

is assumed for the nonempty closed convex-valued function $t \mapsto C(t)$, where d_H is the Hausdorff distance, see (1.3).

In this chapter we prove that any sweeping processes (1.1) with almost periodic monotone right-hand-sides i.e., with (1.5) admits a globally exponentially stable almost periodic solution. And then we describe the extent to which such a globally stable solution persists under non-monotone perturbations specifically when the perturbation depend on a parameter continuously and integrally continuous.

We give the definition of an almost periodic function as follows (see e.g., Vesely [63], Levitan-Zhikov [43]).

Definition 2.1.1. Let X be a complete metric space equipped with the metric d. A continuous function $\phi : \mathbb{R} \to (X, d)$ is *almost periodic*, if for any $\varepsilon > 0$, there exists a number $p(\varepsilon) > 0$ with the property that any interval of length $p(\varepsilon) > 0$ of the real line contains at least one point s, such that

$$d(\phi(t + s), \phi(t)) < \varepsilon \quad \text{ for } t \in \mathbb{R}.$$

We will be using the following Bochner's theorem from Vesely [63] (see also Levitan-Zhikov [43, p. 4]) when proving the almost periodicity of the unique solution.

Theorem 2.1.1. Let X be a complete Banach space and $\phi : \mathbb{R} \mapsto X$ be a continuous function. Then ϕ is almost periodic if and only if from any sequence $\{\phi(t+s_n)\}_{n\in\mathbb{N}}$, where s_n are real numbers, one can extract a subsequence $\{\phi(t+r_n)\}_{n\in\mathbb{N}}$, satisfying the Cauchy uniform convergence condition on $\mathbb{R}$; i.e., for any $\varepsilon > 0$, there exists $l(\varepsilon) \in \mathbb{N}$ with the property that

$$d(\phi(t+r_i), \phi(t+r_j)) \leq \varepsilon, \quad t \in \mathbb{R} \text{ for all } i,j > l(\varepsilon),\ i,j \in \mathbb{N}.$$

Let $ck(\mathbb{R}^n)$ be the space of all closed bounded nonempty sets of $\mathbb{R}^n$ equipped with the Hausdorff metric d_H. Then the space $(ck(\mathbb{R}^n), d_H)$ is a complete metric space (see e.g., Price [55]). Therefore the definition (2.1.1) and Bochner theorem 2.1.1 for closed convex valued continuous almost periodic function C also applicable.

Following Krasnoselskii-Krein [37] and Demidovich [23, Ch. V, §3], we say that $f(t,x,\varepsilon)$ is *integrally continuous* at $\varepsilon = \varepsilon_0$, if

$$\lim_{\varepsilon\to\varepsilon_0} \int_\tau^t f(s,x,\varepsilon)ds = \int_\tau^t f(s,x,\varepsilon_0)ds, \quad \text{for all } \tau, t \in \mathbb{R},\ x \in \mathbb{R}^n. \tag{2.3}$$

The following theorem is used when we describe the stability of the attractor in the case of the perturbation depend on the parameter integrally continuous.

Theorem 2.1.2. (Krasnoselskii-Krein [37]) Assume that $F : \mathbb{R} \times \mathbb{R}^k \times \mathbb{R} \to \mathbb{R}^n$ satisfies (2.18) and that $t \mapsto F(t,u,\varepsilon_0)$ is continuous for every $u \in \mathbb{R}^k$. Consider a family of continuous functions $\{u_\varepsilon(t)\}_{\varepsilon\in\mathbb{R}}$ defined on an interval $[\tau, T]$ such that $u_\varepsilon(t) \to u_0(t)$ as $\varepsilon \to \varepsilon_0$, uniformly on $[\tau, T]$. If F verifies the integral continuity property (2.3), then

$$\lim_{\varepsilon\to\varepsilon_0} \int_\tau^t F(s,u_\varepsilon(s),\varepsilon)ds = \int_\tau^t F(s,u_0(s),\varepsilon_0)ds, \qquad \text{for all } t \in [\tau, T].$$

2.2 Existence of an unique globally exponentially stable bounded solution

Under conditions (2.1) and (2.2), for any initial condition $x(t_0) \in C(t_0)$, the sweeping process (1.1) with nonempty, closed and convex $C(t)$, $t \in \mathbb{R}$, admits (Edmond-Thibault [26, Theorem 1]) a unique absolutely continuous forward solution $x(t)$, in the sense that $x(t)$ satisfies (1.1) for almost all $t \geq t_0$.

Remark 2.2.1. If x_0 is a solution to (1.1) defined on $t \geq t_0$, then $x(t) \in C(t)$, for all $t \geq t_0$, because $N_{C(t)}(x(t))$ is undefined otherwise (the interested reader can see that Edmond-Thibault [26, pp. 352–353] obtains the solution $x(t)$ as $x(t) = y(t) - \psi(t)$, where $y(t) \in C(t) + \psi(t)$). In particular, if $\|C(t)\| \leq M$ for some $M > 0$ and all $t \in \mathbb{R}$, then

$$\text{for any solution } x \text{ of (1.1) with initial contion } x(t_0) \in C(t_0), \|x(t)\| \leq M, \text{ for } t \geq t_0. \quad (2.4)$$

Theorem 2.2.1. Let $f : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ satisfy the Lipschitz condition (2.1). Assume that, for any $t \in \mathbb{R}$, the set $C(t) \subset \mathbb{R}^n$ is nonempty, closed, convex and the map $t \mapsto C(t)$ satisfies the Lipschitz condition (2.2). If C is globally bounded, then the sweeping process (1.1) admits at least one absolutely continuous solution x_0 defined on the entire $\mathbb{R}$. The solution x_0 is globally bounded.

Proof. **Step 1:** *Construction of a candidate solution x_0 defined on the entire* $\mathbb{R}$. Let $\{\xi_m\}_{m=1}^{\infty}$ be an arbitrary sequence of elements of $\mathbb{R}^n$ such that $\xi_m \in C(-m)$, $m \in \mathbb{N}$. Let $x_m(t)$ be the solution to (1.1) with the initial condition $x_m(-m) = \xi_m$. Extend each x_m from $[-m, \infty)$ to $\mathbb{R}$ by defining $x_m(t) = x_m(-m)$ for all $t < -m$. By Edmond-Thibault [26, Theorem 1], the functions of $\{x_m(t)\}_{m=1}^{\infty}$ share same Lipschitz constant $L_k > 0$ on each interval $[-k, k]$, $k \in \mathbb{N}$. Letting $\{x_m^0\}_{m=1}^{\infty} = \{x_m\}_{m=1}^{\infty}$, for each $k \in \mathbb{N}$ we can extract a subsequence $\{x_m^k(t)\}_{m=1}^{\infty}$ of $\{x_m^{k-1}(t)\}_{m=1}^{\infty}$ which converges uniformly on $[-k, k]$. By using this family of subsequences we introduce a sequence $\{x_m^*\}_{m=1}^{\infty}$ by $x_m^*(t) = x_m^m(t)$. The sequence $\{x_m^*\}_{m=1}^{\infty}$ converges uniformly on any fixed interval $[-k, k]$, $k \in \mathbb{N}$. Define $x_0(t)$ by $x_0(t) = \lim\limits_{m\to\infty} x_m^*(t)$.

Step 2: *Proof that x_0 is indeed a solution.* Let $\tau \in \mathbb{R}$ and let v be a solution to (1.1) with $v(\tau) = x_0(\tau)$. Assume $v(t_0) \neq x_0(t_0)$ for some $t_0 > \tau$, i.e., $\lim_{m\to\infty} x_m^*(t_0) \neq v(t_0)$. Then there exists $\varepsilon_0 > 0$, such that for each $m \in \mathbb{N}$, there exists $m_k > m$ such that $\|x_{m_k}^*(t_0) - v(t_0)\| \geq \varepsilon_0$. On the other hand, by continuous dependence of solutions to (1.1) on the initial condition (see Edmond-Thibault [26, Proposition 2]), there exists $\delta > 0$ such that if $\|v(\tau) - x_m^*(\tau)\| < \delta$ then $\|v(t) - x_m^*(t)\| < \varepsilon_0$ for all $m \in \mathbb{N}$ with $-m < \tau$ (which ensures that $x_m^*(t)$ is a solution of (1.1) for $t \geq \tau$) and $t \in [\tau, t_0]$, see Fig. 2.1. But since $v(\tau) = x_0(\tau) = \lim_{m\to\infty} x_m^*(\tau)$, there exists $N \in \mathbb{N}$ such that $\|v(\tau) - x_m^*(\tau)\| < \delta$ for each $m > N$. Then $\|v(t) - x_m(t)\| < \varepsilon_0$ for all $m > N$ and $t \in [\tau, t_0]$. This contradicts $\lim_{n\to\infty} x_m^*(t_0) \neq v(t_0)$. Therefore $v(t) = x_0(t)$ for each $t \geq \tau$. Hence x_0 is a solution to (1.1).

The solution x_0 is globally bounded by Remark 2.2.1. $\square$

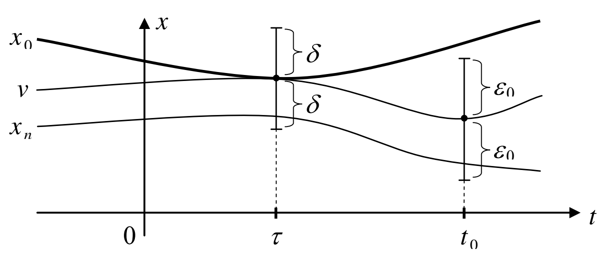

Figure 2.1. Illustration of the location of curves x_0, v, and x_m^*.

Theorem 2.2.2. Assume that the conditions of Theorem 2.2.1 hold. If f satisfies the monotonicity condition (1.5) then (1.1) admits exactly one absolutely continuous bounded solution x_0 defined on the entire $\mathbb{R}$. Moreover, x_0 is globally exponentially stable.

The following proof is known (see e.g., Leine-Wouw [41, Theorem 8.7] and Leine-Wouw [42, Lemma 2]), but we add a proof in terms of sweeping process (1.1) for completeness.

Proof. **Step 1:** Let x_1 and x_2 be solutions to (1.1) with the initial conditions $x_1(t_0), x_2(t_0) \in C(t_0)$. Assuming that $t \geq t_0$ is such that both $\dot{x}_1(t)$ and $\dot{x}_2(t)$ exist and verify (1.1), one has

$$\langle -\dot{x}_1(t) - f(t, x_1(t)), x_1(t) - x_2(t)\rangle \geq 0.$$

Therefore $\langle -f(t,x_1(t)), x_1(t)-x_2(t)\rangle \geq \langle \dot{x}_1(t), x_1(t)-x_2(t)\rangle$.

By analogy, $-\dot{x}_2(t) - f(t,x_2(t)) \in N_{C(t)}(x_2(t))$ implies

$\langle -\dot{x}_2(t), x_1(t)-x_2(t)\rangle \leq \langle f(t,x_2(t)), x_1(t)-x_2(t)\rangle$. Therefore,

$$\begin{aligned}
\frac{d}{dt}\|x_1(t)-x_2(t)\|^2 &= 2\langle \dot{x}_1(t)-\dot{x}_2(t), x_1(t)-x_2(t)\rangle \\
&= 2\langle \dot{x}_1(t), x_1(t)-x_2(t)\rangle - 2\langle \dot{x}_2(t), x_1(t)-x_2(t)\rangle \\
&\leq -2\langle f(t,x_1(t)), x_1(t)-x_2(t)\rangle + 2\langle f(t,x_2(t)), x_1(t)-x_2(t)\rangle \\
&= -2\langle f(t,x_1(t)) - f(t,x_2(t)), x_1(t)-x_2(t)\rangle \\
&\leq -2\alpha\|x_1(t)-x_2(t)\|^2
\end{aligned}$$

and by Gronwall-Bellman lemma (see Lemma 2.5.1 in the Appendix),

$$\|x_1(t)-x_2(t)\|^2 \leq e^{-2\alpha(t-t_0)}\|x_1(t_0)-x_2(t_0)\|^2, \text{ for a.e. } t \geq t_0.$$

Since both x_1 and x_2 are continuous functions,

$$\|x_1(t)-x_2(t)\|^2 \leq e^{-2\alpha(t-t_0)}\|x_1(t_0)-x_2(t_0)\|^2, \quad \text{for all } t \geq t_0. \tag{2.5}$$

Step 2. *Uniqueness of the bounded solution* x_0. Let v be another bounded solution of (1.1) defined on the entire $\mathbb{R}$. Then, given any $\tau \in \mathbb{R}$, the inequality (2.5) yields

$$\|x_0(t)-v(t)\|^2 \leq e^{-2\alpha(t-\tau)}\|x_0(\tau)-v(\tau)\|^2, \quad \text{for all } t \geq \tau.$$

Thus $\|x_0(t)-v(t)\| \leq 2Me^{-\alpha(t-\tau)}$, for all $t \geq \tau$, where M is as defined in (2.4). Now we fix $t \in \mathbb{R}$ and pass to the limit as $\tau \to -\infty$, obtaining $\|x(t)-v(t)\| \leq 0$. Thus $x(t) = v(t)$ for all $t \in \mathbb{R}$.

Step 3. *Global exponential stability of* x_0. Indeed, (2.5) implies that

$$\|x_0(t)-v(t)\| \leq e^{-\alpha(t-\tau)}\|x_0(\tau)-v(\tau)\|,$$

for any solution v of (1.1) and for any $t \geq \tau$. $\square$

Remark 2.2.2. The global boundedness of $t \mapsto C(t)$ is used in the proof of Theorem 2.2.1 just to conclude the boundedness of the global solution x_0. Accordingly, the assumption of global boundedness of $t \mapsto C(t)$ and the property of global boundedness of x_0 can be simultaneously dropped in the formulation of Theorem 2.2.1. But assuming global boundedness of $t \mapsto C(t)$ in Theorem 2.2.2 cannot be dropped as it is used in the proof in an essential way (to establish the uniqueness of x_0, not to just prove its global boundedness).

2.3 Almost periodicity of the unique bounded solution

Theorem 2.3.1. Let the conditions of Theorem 2.2.1 hold and let x_0 be the unique absolutely continuous bounded solution given by Theorem 2.2.1. If both the function $t \mapsto f(t, x)$ and the set-valued function $t \mapsto C(t)$ are almost periodic, then x_0 is almost periodic.

Proof. Let $\{h_m\}_{m=1}^{\infty} \subseteq \mathbb{R}$. We are going to prove that there exists $\{k_m(x)\}_{m=1}^{\infty} \subseteq \{h_m\}_{m=1}^{\infty}$ such that the sequence of

$$x_m(t) = x_0(t + k_m), \quad m \in N, \ t \in \mathbb{R}, \tag{2.6}$$

converges as $m \to \infty$ uniformly in $t \in \mathbb{R}$, which will imply almost periodicity of x_0 by Bochner's theorem 2.1.1.

Step 1. *The existence of* $\{l_m\}_{m=1}^{\infty} \subseteq \{h_m\}_{m=1}^{\infty}$ *such that* $f_m(t, x) = f(t + l_m, x)$ *converges as* $m \to \infty$ *uniformly.* Since $f(t, x)$ is almost periodic, then, for each $x \in \mathbb{R}^n$, Bochner's theorem (see e.g., Levitan-Zhikov [43, p. 4]) implies the existence of $\{l_m(x)\}_{m=1}^{\infty} \subseteq \{h_m\}_{m=1}^{\infty}$ such that the sequence of functions $\{f(\cdot + l_m(x), x)\}_{m=1}^{\infty}$ converges in the sup-norm. The standard diagonal method allows to construct $\{l_m(x)\}_{m=1}^{\infty}$ independent on x. Indeed, considering $\{x_m\}_{m=1}^{\infty} = \mathbb{Q}^n$, we first construct sequences $\{l_m(x_1)\}_{m=1}^{\infty} \supseteq \{l_m(x_2)\}_{m=1}^{\infty} \supseteq \ldots$, such that each individual sequence $\{f(\cdot + l_m(x_1), x_1)\}_{m=1}^{\infty}$, $\{f(\cdot + l_m(x_2), x_2)\}_{m=1}^{\infty}$, ... converges. And then define $\{l_m\}_{m=1}^{\infty} \subseteq \{h_m\}_{m=1}^{\infty}$ as $l_m = l_m(x_m)$, $m \in \mathbb{N}$. Put

$$f_m(t, x) = f(t + l_m, x), \quad \text{for all } t \in \mathbb{R}, \ x \in \mathbb{Q}^n, \ m \in \mathbb{N}. \tag{2.7}$$

So constructed, $\{f_m(\cdot,x)\}_{m=1}^{\infty}$ converges for each fixed $x \in \mathbb{Q}^n$. Let

$$\hat{f}(t,x) = \lim_{m\to\infty} f_m(t,x), \quad \text{for all } t \in \mathbb{R},\ x \in \mathbb{Q}^n. \tag{2.8}$$

By (2.1) both f_m and $\hat{f}$ are Lipschitz continuous with constant L_f on $\mathbb{R}\times\mathbb{R}^n$ and $\mathbb{R}\times\mathbb{Q}^n$ respectively. Now we extend $\hat{f}$ from $\mathbb{R}\times\mathbb{Q}^n$ to $\mathbb{R}\times\mathbb{R}^n$ by taking an arbitrary sequence $\mathbb{Q}^n \ni x_k \to x_0 \in \mathbb{R}$, as $k\to\infty$, and defining $\hat{f}(t,x_0) = \lim_{k\to\infty} \hat{f}(t,x_k)$. The limit exists because $\{\hat{f}(t,x_k)\}_{k=1}^{\infty}$ is a Cauchy sequence for each fixed $t\in\mathbb{R}$, which follows from Lipschitz continuity of $\hat{f}$ on $\mathbb{R}\times\mathbb{Q}^n$. Lipschitz continuity of $\hat{f}$ extends from $\mathbb{R}\times\mathbb{Q}^n$ to $\mathbb{R}\times\mathbb{R}^n$ by continuity. The latter property also implies that

$$\left\|\hat{f}(t,x_0) - \hat{f}(t,x_k)\right\| \le L_f\|x_0 - x_k\|, \quad \text{for all } k \in \mathbb{N}.$$

Finally, to show that

$$f_m(t,x) \to \hat{f}(t,x) \quad \text{as } m\to\infty,\ \text{uniformly in } t\in\mathbb{R},\ x\in\mathbb{R}^n, \tag{2.9}$$

we estimate $f_m(t,x) - \hat{f}(t,x)$ as

$$\left\|f_m(t,x) - \hat{f}(t,x)\right\| \le \|f_m(t,x) - f_m(t,x_*)\| + \left\|f_m(t,x_*) - \hat{f}(t,x_*)\right\| + \left\|\hat{f}(t,x_*) - \hat{f}(t,x)\right\|.$$

Given $x\in\mathbb{R}^n$ and $\varepsilon>0$, we choose $x_*\in\mathbb{Q}^n$ so close to x that $\|f_m(t,x)-f_m(t,x_*)\| < \varepsilon/3$ and $\left\|\hat{f}(t,x_*) - \hat{f}(t,x)\right\| < \varepsilon/3$, for all $m\in\mathbb{N}$, $t\in\mathbb{R}$. By (2.8) we can now select $m_0\in\mathbb{N}$ such that $\left\|f_m(t,x_*) - \hat{f}(t,x_*)\right\| < \varepsilon/3$, for all $m>m_0$ and $t\in\mathbb{R}$. Thus, (2.9) holds.

Step 2. *The existence of $\{k_m\}_{m=1}^{\infty} \subseteq \{l_m\}_{m=1}^{\infty}$, such that $C_m(t) = C(t+k_m)$ converges as $m\to\infty$ uniformly.* By Bochner's theorem for almost periodic functions in pseudo-metric spaces (see Vesely [63, Theorem 2.4]), there exists $\{k_m\}_{m=1}^{\infty} \subseteq \{l_m\}_{m=1}^{\infty}$, such that $\{C_m(t)\}_{m=1}^{\infty}$ is a Cauchy sequence in $ck(\mathbb{R}^n)$, which is uniform in $t\in\mathbb{R}$. The convergence of $\{C_m(t)\}_{m=1}^{\infty}$ for each individual $t\in\mathbb{R}$ now follows from the completeness of $ck(\mathbb{R}^n)$ (Price

[55, the theorem of §3]). The uniformity of the convergence in $t \in \mathbb{R}$ follows along the standard lines. Indeed, let

$$\hat{C}(t) = \lim_{m\to\infty} C_m(t).$$

Given $\varepsilon > 0$, fix $m_0 > 0$ such that $d_H(C_m(t), C_{m_*}(t)) < \varepsilon/2$ for all $m > m_0$, $m_* > m_0$, and $t \in \mathbb{R}$. For each $t \in \mathbb{R}$ select $m_*(t) > m_0$ such that $d_H\left(C_{m_*(t)}(t), \hat{C}(t)\right) < \varepsilon/2$. Then

$$d_H\left(C_m(t), \hat{C}(t)\right) \leq d_H\left(C_m(t), C_{m_*(t)}(t)\right) + d_H\left(C_{m_*(t)}(t), \hat{C}(t)\right) < \varepsilon/2 + \varepsilon/2 = \varepsilon,$$

for all $m > m_0,\ t \in \mathbb{R}$.

Note that (2.2) implies that $\hat{C}$ is globally Lipschitz continuous with constant L_C.

Step 3: *The uniform convergence of* $\{x_m(t)\}_{m=1}^{\infty}$. The function x_m, see (2.6), is a solution to the sweeping process

$$-\dot{x}(t) \in N_{C_m(t)}(x(t)) + f_m(t, x(t)). \tag{2.10}$$

Along with (2.10) let us consider

$$-\dot{x}(t) \in N_{\hat{C}(t)}(x(t)) + \hat{f}(t, x(t)). \tag{2.11}$$

Both $\hat{C}$ and $\hat{f}$ are globally bounded and globally Lipschitz continuous. Moreover, by using (2.7) and (2.8) one concludes that $\hat{f}$ satisfies the monotonicity property (1.5). Therefore, by Theorem 2.2.1 the sweeping process (2.11) has a unique bounded absolutely continuous solution $\hat{x}$ defined on the entire $\mathbb{R}$. Let $t \in \mathbb{R}$ be such that both $\dot{x}_m(t)$ and $\dot{\hat{x}}(t)$ exist and satisfy the respective relations (2.10) and (2.11). Define

$$v_m = \dot{x}_m(t) + f_m(t, x_m(t)), \hat{v} = \dot{\hat{x}}(t) + \hat{f}(t, \hat{x}(t)),$$

so that $v_m \in -N_{C_m(t)}(x_m(t)), \hat{v} \in -N_{\hat{C}(t)}(\hat{x}(t))$.

Furthermore, introducing $\Delta_m(t) = d_H\left(C_m(t), \hat{C}(t)\right)$ one has

$$u_m(t) \in C_m(t) \subseteq \hat{C}(t) + \bar{B}_{\Delta_m(t)}(0), \quad \hat{u}(t) \in \hat{C}(t) \subseteq C_m(t) + \bar{B}_{\Delta_m(t)}(0), \quad \text{for all } t \in \mathbb{R}.$$

Therefore, x_m and $\hat{x}$ can be decomposed as

$$x_m(t) = \hat{d}(t) + s_m(t), \quad \hat{x}(t) = d_m(t) + \hat{s}(t),$$

where $\hat{d}(t) \in \hat{C}(t),\ d_m(t) \in C_m(t),\ \|s_m(t)\| \leq \Delta_m(t),\ \|\hat{s}(t)\| \leq \Delta_m(t)$.

Let

$$w_m(t) = \|x_m(t) - \hat{x}(t)\|^2.$$

Then,

$$\begin{aligned}
\frac{1}{2}\dot{w}_m(t) &= \langle \dot{x}_m(t) - \dot{\hat{x}}(t), x_m(t) - \hat{x}(t)\rangle \\
&= \langle v_m(t) - f_m(t, x_m(t)) - \hat{v}(t) + \hat{f}(t, \hat{x}(t)), x_m(t) - \hat{x}(t)\rangle \\
&= \langle v_m(t), x_m(t) - d_m(t) - \hat{s}(t)\rangle + \langle \hat{v}(t), \hat{x}(t) - \hat{d}(t) - s_m(t)\rangle \\
&\qquad - \langle f_m(t, x_m(t)) - \hat{f}(t, \hat{x}(t)), x_m(t) - \hat{x}(t).\rangle
\end{aligned}$$

By (1.2) we have $\langle v_m(t), x_m(t) - d_m(t)\rangle \leq 0$ and $\langle \hat{v}(t), \hat{x}(t) - \hat{d}(t)\rangle \leq 0$. Therefore, for a.a. $t \in \mathbb{R}$,

$$\begin{aligned}
\frac{1}{2}\dot{w}_m(t) &\leq -\langle v_m(t), \hat{s}(t)\rangle - \langle \hat{v}(t), s_m(t)\rangle - \langle f_m(t, x_m(t)) - \hat{f}(t, \hat{x}(t)), x_m(t) - \hat{x}(t)\rangle \\
&\leq \|v_m(t)\| \cdot \|\hat{s}(t)\| + \|\hat{v}(t)\| \cdot \|s_m(t)\| \\
&\qquad - \langle f_m(t, x_m(t)) - f_m(t, \hat{x}(t)) + f_m(t, \hat{x}(t)) - \hat{f}(t, \hat{x}(t)), x_m(t) - \hat{x}(t)\rangle.
\end{aligned}$$

Given $\varepsilon > 0$ we use the conclusions of Steps 1 and 2 to spot an $m_0 > 0$ such that

$$\|\hat{s}(t)\| \leq \varepsilon_0,\ \|s_m(t)\| \leq \varepsilon_0,\ \left\| f_m(t, \hat{x}(t)) - \hat{f}(t, \hat{x}(t)) \right\| \leq \varepsilon_0, \quad \text{for all } m \geq m_0,\ t \in \mathbb{R}^n.$$

Almost periodicity in t and the Lipschitz condition (2.1) imply that the function $f(t, x)$ is uniformly bounded when $t \in \mathbb{R}$ and $\|x\| \leq M$, where M is as introduced in Remark 2.2.1. Therefore, by Edmond-Thibault [26, Theorem 1], there exists $L_0 > 0$ such that

$$\|v_m(t)\| \leq L_0,\ \|\hat{v}(t)\| \leq L_0$$

and by using (2.4) we can estimate $\dot{w}_m(t)$ further as

$$\frac{1}{2}\dot{w}_m(t) \leq 2\varepsilon L_0 - \langle f_m(t, x_m(t)) - f_m(t, \hat{x}(t)), x_m(t) - \hat{x}(t)\rangle + 2\varepsilon M,$$

for all $m \geq m_0$, *a.a.* $t \in \mathbb{R}$.

By referring to the definition (2.7) of f_m, one observes that f_m satisfies the monotonicity estimate (1.5), which implies

$$\frac{1}{2}\dot{w}_m(t) \leq 2\varepsilon(L_0 + M) - \alpha\|x_m(t) - \hat{x}(t)\|^2 = 2\varepsilon(L_0 + M) - \alpha w_m(t),$$

for all $m \geq m_0$ and a.a. $t \in \mathbb{R}$.

Gronwall-Bellman lemma (see Lemma 2.5.1 in the Appendix) now allows to conclude that

$$\begin{aligned} w_m(t) &\leq w_m(\tau)e^{-\alpha(t-\tau)} + 2\varepsilon(L_0 + M)\int_\tau^t e^{-\alpha(t-s)}ds \\ &= w_m(\tau)e^{-\alpha(t-\tau)} + \varepsilon\frac{2(L_0 + M)}{\alpha}\left(1 - e^{-\alpha(t-\tau)}\right), \quad t, \tau \in \mathbb{R}, \ m \geq m_0. \end{aligned}$$

By passing to the limit as $\tau \to -\infty$ one gets

$$w_m(t) \leq \varepsilon \cdot 2(L_0 + M)/\alpha, \quad t \in \mathbb{R}, \ m \geq m_0.$$

Therefore, $\|x_m(t) - \hat{x}(t)\| \to 0$ as $m \to \infty$ uniformly in $t \in \mathbb{R}$, and so x_0 is almost periodic by Bochner's theorem. $\square$

Remark 2.3.1. To fulfill the assumption of global boundedness of $t \mapsto C(t)$ in Theorem 2.3.1, it is sufficient to assume that $C(t)$ is bounded for each individual $t \in \mathbb{R}$. Indeed, any almost periodic set-valued map $C(t)$ with closed bounded values is globally bounded on $\mathbb{R}$, see e.g., Levitan-Zhikov [43, p. 2] or Vesely [63, Lemma 2.2].

2.4 Stability of the attractor to non-monotone perturbations

In this section, we study the sweeping process

$$-\dot{x}(t) \in N_{C(t)}(x(t)) + f(t, x(t), \varepsilon), \tag{2.12}$$

which satisfies the monotonicity condition (1.5) only when $\varepsilon = \varepsilon_0$, i.e.,

$$\langle f(t, x_1, \varepsilon_0) - f(t, x_2, \varepsilon_0), x_1 - x_2 \rangle \geq \alpha \|x_1 - x_2\|^2,$$
$$\text{for some fixed } \alpha > 0 \text{ and for all } t \in \mathbb{R},\ x_1, x_2 \in \mathbb{R}^n. \qquad (2.13)$$

2.4.1 The case where the dependence of the perturbation on the parameter is continuous

Here we assume that

$$\|f(t_1, x_1, \varepsilon) - f(t_2, x_2, \varepsilon)\| \leq L_f \|t_1 - t_2\| + L_f \|x_1 - x_2\|, \text{ for all } t_1, t_2 \in \mathbb{R},\ x_1, x_2 \in \mathbb{R}^n. \quad (2.14)$$

Theorem 2.4.1. Let $f : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ satisfy the Lipschitz condition (2.14) and the monotonicity condition (2.13). Assume that, for any $t \in \mathbb{R}$, the set $C(t) \subset \mathbb{R}^n$ is nonempty, closed, convex and the uniformly bounded map $t \mapsto C(t)$ satisfies the Lipschitz condition (2.2). Finally, assume that $f(t, x, \varepsilon)$ is continuous at $\varepsilon = \varepsilon_0$ uniformly in $t \in \mathbb{R}$, $x \in \mathbb{R}^n$. Let $x_0 : \mathbb{R} \to \mathbb{R}^n$ be the unique solution to (2.12) with $\varepsilon = \varepsilon_0$ provided by Theorem 2.2.2. Then, given any $\gamma > 0$ there exists $t_1 \in \mathbb{R}$ such that for any solution x_ε of (2.12) defined on $[0, \infty)$, one has

$$\|x_\varepsilon(t) - x_0(t)\| < \gamma, \qquad t \geq t_1, \qquad (2.15)$$

for all ε sufficiently close to ε_0.

We remind the reader that corresponding results for differential inclusions with bounded right-hand-sides are known examples from Kloeden-Kozyakin [34].
The following lemma will be used iteratively throughout the rest of the chapter.

Lemma 2.4.1. Let x_ε be a solution of (2.12) defined on $[\tau, \infty)$. Let $x_0 = x_{\varepsilon_0}$. If (2.13) holds, then, for a.a. $t \geq \tau$,

$$\|x_\varepsilon(t) - x_0(t)\|^2 \leq e^{-2\alpha(t-\tau)} \|x_\varepsilon(\tau) - x_0(\tau)\|^2$$
$$-2 \int_\tau^t e^{-2\alpha(t-s)} \langle f(s, x_\varepsilon(s), \varepsilon) - f(s, x_\varepsilon(s), \varepsilon_0), x_\varepsilon(s) - x_0(s) \rangle ds. \quad (2.16)$$

Proof. For a.a. $t \geq \tau$ and $\varepsilon \in \mathbb{R}$ we have

$$\begin{aligned}
\frac{d}{dt}\|x_\varepsilon(t) - x_0(t)\|^2 &= 2\left\langle \dot{x_\varepsilon}(t) - \dot{x_0}(t), x_\varepsilon(t) - x_0(t)\right\rangle \\
&\leq 2\left\langle -f\left(t, x_\varepsilon(t), \varepsilon\right), x_\varepsilon(t) - x_0(t)\right\rangle + 2\left\langle f(t, x_0(t), \varepsilon_0), x_\varepsilon(t) - x_0(t)\right\rangle \\
&= -2\left\langle f(t, x_\varepsilon(t), \varepsilon) - f(t, x_\varepsilon(t), \varepsilon_0), x_\varepsilon(t) - x_0(t)\right\rangle \\
&\qquad - 2\langle f(t, x_\varepsilon(t), \varepsilon_0) - f(t, x_0(t), \varepsilon_0), x_\varepsilon(t) - x_0(t)\rangle \\
&\leq -2\alpha\|x_\varepsilon(t) - x_0(t)\|^2 - 2\langle f(t, x_\varepsilon(t), \varepsilon) - f(t, x_\varepsilon(t), \varepsilon_0), x_\varepsilon(t) - x_0(t)\rangle
\end{aligned}$$

and the conclusion follows by applying the Gronwall-Bellman lemma (see Lemma 2.5.1 in the Appendix). $\square$

Proof of Theorem 2.4.1. By Lemma 2.4.1 and (2.4) one has

$$\begin{aligned}
\|x_\varepsilon(t) - x_0(t)\|^2 &\leq e^{-2\alpha t}\|x_\varepsilon(0) - x_0(0)\|^2 \\
&+ \left(\frac{1}{2\alpha} - \frac{e^{-2\alpha t}}{2\alpha}\right) \max_{s \in [0,t]} \|f(s, x_\varepsilon(s), \varepsilon) - f(s, x_\varepsilon(s), \varepsilon_0)\| \cdot M, \quad (2.17)
\end{aligned}$$

from which the conclusion follows. $\square$

Remark 2.4.1. The estimate (2.15) can be extended to the entire $\mathbb{R}$, if x_ε is defined on the entire $\mathbb{R}$ (for example if x_ε is that given by Theorem 2.2.1). Indeed, in this case (2.17) can be strengthened to

$$\begin{aligned}
\|x_\varepsilon(t) - x_0(t)\|^2 &\leq e^{-2\alpha(t-\tau)}\|x_\varepsilon(\tau) - x_0(\tau)\|^2 \\
&+ \left(\frac{1}{2\alpha} - \frac{e^{-2\alpha(t-\tau)}}{2\alpha}\right) \max_{s \in [\tau,t]} \|f(s, x_\varepsilon(s), \varepsilon) - f(s, x_\varepsilon(s), \varepsilon_0)\| \cdot M,
\end{aligned}$$

which gives

$$\|x_\varepsilon(t) - x_0(t)\|^2 \leq \frac{1}{2\alpha} \max_{s \in (-\infty,t]} \|f(s, x_\varepsilon(s), \varepsilon) - f(s, x_\varepsilon(s), \varepsilon_0)\| \cdot M,$$

by passing to the limit as $\tau \to -\infty$.

2.4.2 The case where the dependence of the perturbation on the parameter is just integrally continuous

In this section we assume that the following version of Lipschitz condition (2.1) holds:

$$\begin{aligned} &\|f(t_1,x,\varepsilon)-f(t_2,x,\varepsilon)\| \le L_\varepsilon\|t_1-t_2\|, \quad \text{for all } t_1,t_2\in\mathbb{R},\ x\in\mathbb{R}^n,\ \varepsilon\in\mathbb{R}\backslash\{\varepsilon_0\}, \\ &\|f(t,x_1,\varepsilon)-f(t,x_2,\varepsilon)\| \le L_f\|x_1-x_2\|, \quad \text{for all } t\in\mathbb{R},\ x_1,x_2\in\mathbb{R}^n,\ \varepsilon\in\mathbb{R}, \end{aligned} \tag{2.18}$$

where $L_\varepsilon > 0$ may depend on $\varepsilon \in \mathbb{R}$ and $L_f > 0$ is independent of $\varepsilon \in \mathbb{R}^n$. The central role in this section is played by a generalization of the theorem on passage to the limit in the integral due to Krasnoselskii-Krein [37] (see also Demidovich [23, Ch. V, §3]). In the statement of (2.1.2), we take $k = n$ when referring to (2.18) and (2.3) in the context of the function F.

We are now in the position to prove the main result of this section.

Theorem 2.4.2. Let $f : \mathbb{R}\times\mathbb{R}^n\times\mathbb{R} \to \mathbb{R}^n$ satisfy both the Lipschitz condition (2.18). Assume that f satisfies the monotonicity condition (2.13). Assume that, for any $t \in \mathbb{R}$, the set $C(t) \subset \mathbb{R}^n$ is nonempty, closed, convex and the uniformly bounded map $t \mapsto C(t)$ satisfies the Lipschitz condition (2.2). Finally, assume that $f(t, x, \varepsilon)$ is integrally continuous at $\varepsilon = \varepsilon_0$. Then, given any $\gamma > 0$ there exists $t_1 \ge 0$ such that for any solution x_ε to (2.12) defined on $[0,\infty)$ and for any $t_2 \ge t_1$, one has

$$\|x_\varepsilon(t) - x_0(t)\| < \gamma, \qquad t \in [t_1, t_2],$$

for all ε sufficiently close to ε_0.

Proof. Let us fix some closed interval $[t_1, t_2]$ and assume that the statement of the theorem is wrong, i.e., assume that there exists $\gamma > 0$ such that

$$\max_{t\in[t_1,t_2]} \|x_{\varepsilon_m}(t) - x_0(t)\| \ge \gamma \tag{2.19}$$

for some sequence $\varepsilon_m \to \varepsilon_0$ as $m \to \infty$. By (2.4), we can find $\tau < 0$ such that

$$e^{-2\alpha(t-\tau)} \|x_{\varepsilon_m}(\tau) - x_0(\tau)\|^2 < \frac{\gamma^2}{2}, \quad \text{for all } m \in \mathbb{N},\ t \in [t_1, t_2]. \tag{2.20}$$

In what follows, we show that the integral term of the estimate (2.16) can be made smaller that $\gamma^2/2$ on the sequence x_{ε_m} as well. Since $f(t, x, \varepsilon)$ is uniformly bounded and C satisfies the global Lipschitz condition (2.2), by Edmond-Thibault [26, Theorem 1] we have the existence of $L_0 > 0$ such that

$$\|\dot{x}_{\varepsilon_m}(t)\| \le L_0, \qquad \text{for all } m \in \mathbb{N}, \text{ and a.a. } t \in [\tau, T]$$

where $T > 0$. Since the functions of $\{x_{\varepsilon_m}(t)\}_{m\in\mathbb{N}}$ are uniformly bounded according to (2.4), the Arzela-Ascoli theorem implies that without loss of generality the sequence $\{x_{\varepsilon_m}(t)\}_{m\in\mathbb{N}}$ can be assumed convergent uniformly on $[\tau, T]$. Introduce

$$F(t, (x_1, x_2)^T, \varepsilon) = \langle f(t, x_1, \varepsilon) - f(t, x_1, \varepsilon_0), x_2 \rangle, \quad u_m(t) = \left(x_{\varepsilon_m}(t), e^{2\alpha t}\left(x_{\varepsilon_m}(t) - x_0(t)\right)\right)^T,$$

so that $F : \mathbb{R} \times \mathbb{R}^{2n} \times \mathbb{R} \to \mathbb{R}^n$. Since $f(t, x, \varepsilon)$ is integrally continuous at $\varepsilon = \varepsilon_0$, then

$$\lim_{\varepsilon\to\varepsilon_0} \int_\tau^t F\left(s, (x_1, x_2)^T, \varepsilon\right) ds = 0, \quad \text{for all } (x_1, x_2)^T \in \mathbb{R}^{2n},\ t \in [\tau, T].$$

Furthermore, the function F satisfies the same type of Lipschitz condition (2.18) as f does. The Krasnoselskii-Krein theorem (Theorem 2.1.2), therefore, implies

$$\lim_{m\to\infty} \int_\tau^t F(s, u_m(s), \varepsilon_m) ds = 0, \quad \text{for all } t \in [\tau, T]. \tag{2.21}$$

The conclusions (2.20) and (2.21) contradict (2.19) because of (2.16). The proof follows by Lemma 2.4.1. $\square$

2.4.3 A particular case: high-frequency vibrations

In this section we consider a sweeping process

$$-\dot{x}(t) \in N_{C(t)}(x(t)) + g\left(\frac{t}{\varepsilon}, x(t)\right), \tag{2.22}$$

where both $t \mapsto C(t)$ and $t \mapsto g(t, x)$ are almost periodic and we use Theorem 2.4.2 in order to estimate the location of solutions of (2.22) for large values of time and for small values of ε.

Since g is almost periodic in the first variable, the following property holds uniformly in $a \in \mathbb{R}$ (see Bohr [12, p. 44])

$$g_0(x) = \lim_{T\to\infty} \frac{1}{T} \int\limits_0^T g(\tau, x)d\tau = \lim_{T\to\infty} \frac{1}{T} \int\limits_a^{T+a} g(\tau, x)d\tau, \tag{2.23}$$

where both limits exist. Therefore,

$$\lim_{\varepsilon\to 0} \int\limits_\tau^t g\left(\frac{s}{\varepsilon}, x\right) ds = \lim_{T\to\infty} (t-\tau)\frac{1}{T} \int\limits_{\tau T/(t-\tau)}^{T+\tau T/(t-\tau)} g(s, x)ds = \int\limits_\tau^t g_0(x)ds.$$

By the other words, the function

$$f(t, x, \varepsilon) = \begin{cases} g\left(\dfrac{t}{\varepsilon}, x\right), & \text{if } \varepsilon \neq 0, \\ g_0(x), & \text{if } \varepsilon = 0, \end{cases}$$

is integrally continuous at $\varepsilon = 0$ in the sense of (2.3).

We arrive to following corollary of Theorems 2.3.1, Remark 2.3.1 and 2.4.2.

Corollary 2.4.1. Assume that, for each $t \in \mathbb{R}$, the set $C(t) \subset \mathbb{R}^n$ is nonempty, closed, convex, and bounded. Let $t \mapsto C(t)$ be an almost periodic function that satisfies the global Lipschitz condition (2.2). Assume that, for each $x \in \mathbb{R}^n$, the function $t \mapsto g(t, x)$ is almost periodic and satisfies the global Lipschitz condition

$$\|g(t_1, x_1) - g(t_2, x_2)\| \leq L_g|t_1 - t_2| + L_g\|x_1 - x_2\|, \qquad \text{for all } t_1, t_2 \in \mathbb{R},\ x_1, x_2 \in \mathbb{R}^n.$$

Finally, assume that for some $\alpha > 0$ the function g_0 given by (2.23) satisfies the monotonicity condition

$$\langle g_0(x_1) - g_0(x_2), x_1 - x_2 \rangle \geq \alpha \|x_1 - x_2\|^2, \qquad \text{for all } x_1, x_2 \in \mathbb{R}^n.$$

If x_ε is any solution of (2.22) defined on $[0, \infty)$, then uniformly on any time-interval $[t_1, t_2]$ with sufficiently large t_1, the family $\{x_\varepsilon(t)\}_{\varepsilon \in \mathbb{R}}$ converges, as $\varepsilon \to 0$, to the unique globally exponentially stable almost periodic solution $x_0(t)$ of the averaged sweeping process

$$-\dot{x}(t) \in N_{C(t)}(x(t)) + g_0(x(t)).$$

2.4.4 Instructive examples

The examples of this section illustrate how the results of this chapter are supposed to be used in applications.

Example 2.4.1. Consider a one-dimensional sweeping process

$$-\dot{x}(t) \in N_{[\sin(t), \sin(t)+1]}(x(t)) + \varepsilon x^2(t) + \left(\sin\left(\sqrt{2} \cdot t\right) + 2\right) x(t). \tag{2.24}$$

The sweeping process (2.24) satisfies the monotonicity property (1.5) when $\varepsilon = 0$. Theorems 2.3.1 and 2.4.1 imply that for any $\gamma > 0$ there exists $t_1 > 0$ such that any solution x_ε of (2.24) with $x_\varepsilon(0) \in [0, 1]$ satisfies $\|x_\varepsilon(t) - x_0(t)\| \leq \gamma$ for all $t \geq t_1$ and for all $|\varepsilon|$ sufficiently small, where x_0 is the unique globally exponentially stable almost periodic solution to

$$-\dot{x}(t) \in N_{[\sin(t), \sin(t)+1]}(x(t)) + \left(\sin\left(\sqrt{2} \cdot t\right) + 2\right) x(t).$$

Example 2.4.2. Let us now show that the monotonicity of a sweeping process gets broken by a high-frequency ingredient as follows

$$-\dot{x}(t) \in N_{[\sin(t), \sin(t)+1]}(x(t)) + \sin\left(\frac{t}{\varepsilon}\right) x^2(t) + \left(\sin\left(\sqrt{2} \cdot t\right) + 2\right) x(t). \tag{2.25}$$

The non-monotonic term $\sin\left(\dfrac{t}{\varepsilon}\right)$ no longer approaches 0 as it took place in Example 2.4.1 and Theorem 2.4.1 is inapplicable. However, $\sin\left(\dfrac{t}{\varepsilon}\right)$ approaches 0 as $\varepsilon \to 0$ integrally (i.e., in the sense of (2.3)) on any bounded time interval $[t_1, t_2]$. Therefore, Corollary 2.4.1 ensures that given any $\gamma > 0$ there exists $t_1 > 0$ such that for any $t_2 > t_1$ and for any solution x_ε of (2.25) with $x_\varepsilon(0) \in [0,1]$ one has $\|x_\varepsilon(t) - x_0(t)\| \le \gamma$ on $[t_1, t_2]$ for all $|\varepsilon|$ sufficiently small, where x_0 is the unique globally exponentially stable almost periodic solution to the averaged sweeping process

$$-\dot{x}(t) \in N_{[\sin(t), \sin(t)+1]}(x(t)) + \left(\sin\left(\sqrt{2}\cdot t\right) + 2\right) x(t).$$

To summarize, Examples 2.4.1 and 2.4.2 establish useful qualitative properties of non-monotone sweeping processes without any need of actual computing of solutions. Numerical computation of solutions of (2.24) and (2.25) (e.g., using the catch-up algorithm of Edmond-Thibault [26]) is thus outside the scope of this chapter.

2.5 Appendix

The following version of Gronwall-Bellman lemma and its proof are taken from Trubnikov-Perov [60, Lemma 1.1.1.5].

Lemma 2.5.1. (Gronwall-Bellman) Let an absolutely continuous function $a : [0, T] \to \mathbb{R}$ satisfy

$$\dot{a}(t) \le \lambda a(t) + b(t), \qquad \text{for a.a. } t \in [0, T], \tag{2.26}$$

where $b : [0, T] \to \mathbb{R}$ is an integrable function. Then

$$a(t) \le e^{\lambda t} a(0) + \int_0^t e^{\lambda(t-s)} b(s) ds, \qquad \text{for all } t \in [0, T].$$

Proof. By introducing

$$\psi(t) = e^{\lambda t} a(0) + \int_0^t e^{\lambda(t-s)} b(s) ds,$$

one has

$$\psi(t) e^{-\lambda t} - \int_0^t e^{-\lambda s} b(s) ds = a(0)$$

and so

$$\frac{d}{dt}\left[\psi(t) e^{-\lambda t} - \int_0^t e^{-\lambda s} b(s) ds\right] = 0, \qquad \text{for a.a. } t \in [0, T],$$

which implies

$$\dot{\psi}(t) - \lambda\psi(t) = b(t) \geq \dot{a}(t) - \lambda a(t).$$

If now

$$u(t) = a(t) - \psi(t),$$

then $\dot{u}(t) \leq \lambda u(t)$ and so $\frac{d}{dt}\left[u(t)e^{-\lambda t}\right] = e^{-\lambda t}(\dot{u} - \lambda u) \leq 0$, i.e., $u(t)e^{-\lambda t} \leq u(0)$. Therefore, $u(t) \leq 0$ and

$$a(t) \leq \psi(t) = e^{\lambda t} a(0) + \int_0^t e^{\lambda(t-s)} b(s) ds.$$

□

CHAPTER 3

GLOBAL STABILITY OF NON-CONVEX MONOTONE SWEEPING PROCESSES

3.1 Introduction

Let $C : \mathbb{R} \to \mathbb{R}^n$ be a set valued map which take nonempty closed values and $f : \mathbb{R}\times\mathbb{R}^n \to \mathbb{R}^n$. Now we assume our set C is nonconvex and the normal cone in sweeping process (1.1), $N_C(x)$ is given (Edmond-Thibault [26], [27], Thibault [59], Maury-Venel [48]) as

$$N_C(x) = \{\xi \in \mathbb{R}^n : x \in \text{proj}(x + \alpha\xi, C) \text{ for some } \alpha > 0\}.$$

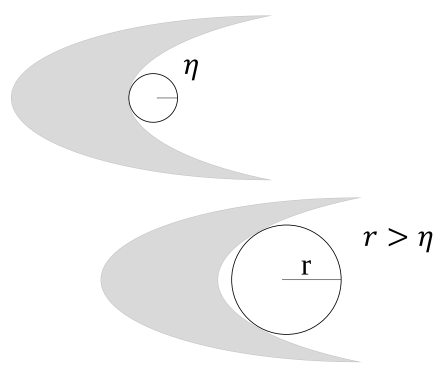

Figure 3.1. A η-prox-regular set

For the space $\mathbb{R}^n$, the set $C(t)$ is η-prox-regular, if $C(t)$ admits an external tangent ball with radius smaller than η at each $x \in \partial C(t)$ (see Maury-Venel [48, p. 150], Colombo and Monteiro Marques [21, p. 48]). A characterization of the normal cone for η-prox-regular sets is hypomonotonicity property (Edmond-Thibault [26], [27], Thibault [59]), which is given as

$$\langle \xi - \xi', x - x' \rangle \geq -\|x - x'\|^2 \text{ for } \xi \in N(C, x), \xi' \in N(C, x') \text{ such that } \|\xi\|, \|\xi'\| \leq \eta. \quad (3.1)$$

In this chapter we discuss stability of sweeping processes (1.1) with η-prox-regular set-valued function C which is Lipschitz continuous i.e.,

$$d_H(C(t_1), C(t_2)) \leq L_C|t_1 - t_2|, \qquad \text{for all } t_1, t_2 \in \mathbb{R}, \text{ and for some } L_C > 0, \tag{3.2}$$

where $d_H(C_1, C_2)$ be the Hausdorff distance between two closed sets $C_1, C_2 \subset \mathbb{R}^n$ given by (1.3).

Also we assumed the Lipschitz continuity of $f : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ with $L_f > 0$

$$\|f(t_1, x_1) - f(t_2, x_2)\| \leq L_f\|t_1 - t_2\| + L_f\|x_1 - x_2\|, \quad \text{for all } t_1, t_2 \in \mathbb{R}, \ x_1, x_2 \in \mathbb{R}^n \tag{3.3}$$

and strong monotonicity of f (1.5).

Also in this chapter we studied the periodicity of solutions when input functions are periodic in (1.1) with prox-regular set-valued function C and exponential stability of this unique periodic solution.

3.2 Existence of a unique global solution and it's stability

Theorem 3.2.1. Let $C : \mathbb{R} \to \mathbb{R}^d$ be a Lipschitz continuous function with constant L_C and $C(t)$ is nonempty, closed and η-prox-regular for each $t \in \mathbb{R}$. Let $f : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ satisfy Lipschitz condition (3.3). Then the sweeping process (1.1) has at least one global solution defined on the entire $\mathbb{R}$.

Since the proof follows the same steps as in the proof of Theorem 2.2.1 we are not giving the proof in this chapter.

Theorem 3.2.2. Let the conditions of Theorem 3.2.1 hold and $L_C \geq 0$ is as given by this theorem. Let $\|f(t, x)\| \leq M_f$, for all $t \in \mathbb{R}$, $x \in \bigcup\limits_{t\in\mathbb{R}} C(t)$ where $M_f \geq 0$ is a fixed constant. Assume (1.5) holds with

$$\alpha > \frac{L_C + M_f}{\eta}. \tag{3.4}$$

Then the sweeping process (1.1) has a unique solution x_0, defined on $\mathbb{R}$. Furthermore the global solution x_0 is exponentially stable.

Proof. We note that by Edmond-Thibault [26, Proposition 1] for a solution x of (1.1) with initial condition $x(\tau) = x_0$,

$$\|\dot{x}(t) + f(t, x(t)\| \leq \|f(t, x(t)\| + L_C, \quad \text{for } t > \tau.$$

Then with the uniform bound M_f of f we have

$$\|\dot{x}(t) + f(t, x(t)\| \leq M_f + L_C, \quad \text{for all } \ t > \tau. \tag{3.5}$$

Let x_1, x_2 be two solutions of (1.1) with initial conditions $x_1(\tau), x_2(\tau) \in C(\tau)$. Let $t \geq \tau$ such that $\dot{x}_1(t), \dot{x}_2(t)$ exist.

Since $-\dot{x}_1(t) - f(t, x_1(t)) \in N_{C(t)}(x_1(t))$ and $-\dot{x}_2(t) - f(t, x_2(t)) \in N_{C(t)}(x_2(t))$, by hypomonotonicity of the normal cone (3.1) and (3.5) we have

$$\begin{aligned}\langle\frac{-\eta}{M_f + L_C}(\dot{x}_1(t) + f(t, x_1(t))) - \frac{-\eta}{M_f + L_C}(\dot{x}_2(t) + f(t, x_2(t))),&x_1(t) - x_2(t)\rangle \\ &\geq -\|x_1(t) - x_2(t)\|^2.\end{aligned}$$

Then

$$\begin{aligned}\|x_1(t) - x_2(t)\|^2 - \frac{\eta}{M_f + L_C}&\langle f(t, x_1(t)) - f(t, x_2(t)), x_1(t) - x_2(t)\rangle \\ &\geq \frac{\eta}{M_f + L_C}\langle \dot{x}_1(t) - \dot{x}_2(t), x_1(t) - x_2(t)\rangle,\end{aligned}$$

and by (1.5)

$$\frac{\eta}{M_f + L_C}\langle \dot{x}_1(t) - \dot{x}_2(t), x_1(t) - x_2(t)\rangle \leq \|x_1(t) - x_2(t)\|^2 - \frac{\eta\alpha}{M_f + L_C}\|x_1(t) - x_2(t)\|^2.$$

Thus we have

$$\langle \dot{x}_1(t) - \dot{x}_2(t), x_1(t) - x_2(t)\rangle \leq \left(\frac{M_f + L_C}{\eta} - \alpha\right)\|x_1(t) - x_2(t)\|^2,$$

i.e.,

$$\frac{d}{dt}\|x_1(t)-x_2(t)\|^2 \leq \left(\frac{2(M_f+L_C)}{\eta}-2\alpha\right)\|x_1(t)-x_2(t)\|^2.$$

Let $\bar{\alpha}=\frac{1}{\eta}(M_f+L_C-\eta\alpha)$. Then by Gronwall Bellman lemma (see Lemma 2.5.1) for $t>\tau$

$$\|x_1(t)-x_2(t)\|^2 \leq e^{2\bar{\alpha}(t-\tau)}\|x_1(\tau)-x_2(\tau)\|^2.$$

Thus

$$\|x_1(t)-x_2(t)\| \leq e^{\bar{\alpha}(t-\tau)}\|x_1(\tau)-x_2(\tau)\|, \quad \text{for} t>\tau. \tag{3.6}$$

Let $x(t)$ be a global solution of (1.1) which exists by Theorem 3.2.1. Then (3.4) guarantees that $\bar{\alpha}<0$ and that $x(t)$ is exponentially stable. It remains to observe that $x(t)$ is the only global solution. Indeed, let $\bar{x}(t)$ be another global solution. Then, for each $t\in\mathbb{R}$ we can pass to the limit as $\tau\to\infty$ in (3.6), obtaining $\|x(t)-\bar{x}(t)\|\leq 0$, so $x=\bar{x}$. □

Now we give a theorem about periodicity of the unique global solution established in Theorem 3.2.2. The proof follows the lines of Castaing and Monteiro Marques [19, Theorem 5.3], but we include such a proof for completeness.

Theorem 3.2.3. The unique global solution x_0 which comes from Theorem 2 is T-periodic, if both maps $t\mapsto C(t)$ and $t\mapsto f(t,x)$ are T-periodic.

Proof. Note that $a\mapsto x_a(T)$ is a contraction mapping from $C(0)$ to $C(T)=C(0)$, where x_a is the solution of (1.1) on $[0,T]$ with initial condition $x_a(0)=a\in C(0)$. Indeed, by (3.6), for $a,b\in C(0)$,

$$\|x_a(T)-x_b(T)\| \leq e^{\bar{\alpha}T}\|a-b\|$$

where $\bar{\alpha}<0$.

Then, since $a\mapsto x_a(T)$ is continuous on $C(0)$ (see Edmond-Thibault [26, Proposition 2]), by the contraction mapping principle on $C(0)$ (see Rudin [57, p.220]), there exists $\bar{x}:[0,T]\to C(0)$ such that $\bar{x}(0)=\bar{x}(T)$ and satisfies (1.1) on $[0,T]$. Since both $t\mapsto C(t)$ and $t\mapsto f(t,x)$ are T-periodic, we can not extend $\bar{x}$ to a T-periodic solution defined on $\mathbb{R}$ by T-periodicity. Since the global solution x_0 given by Theorem 3.2.2 is unique, we have the result. □

3.3 Example

Let the vector field $f : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2$ be given by

$$f(t, x) := \alpha x, \quad t \in \mathbb{R},\ x \in \mathbb{R}^2, \tag{3.7}$$

where $\alpha > 0$ is a fixed constant. We define the moving set $C(t)$ using a function $b \in C^1(\mathbb{R}, \mathbb{R})$ which is bounded below by $\beta \geq 1$ and admits a global Lipschitz constant L_b, i.e.,

$$|b(t_1) - b(t_2)| \leq L_b|t_1 - t_2|, \quad \text{for all} \ \ t_1, t_2 \in \mathbb{R}. \tag{3.8}$$

Define

$$C(t) := \bar{B}_1 \bigcap S_b(t), \quad S(t) = \left\{ x \in \mathbb{R}^2 : x_1^2 + \frac{x_2^2}{b(t)^2} \geq 1 \right\}. \tag{3.9}$$

where $\bar{B}_1$ is the closed ball of radius 1 and centered at $(-1.5, 0)$.

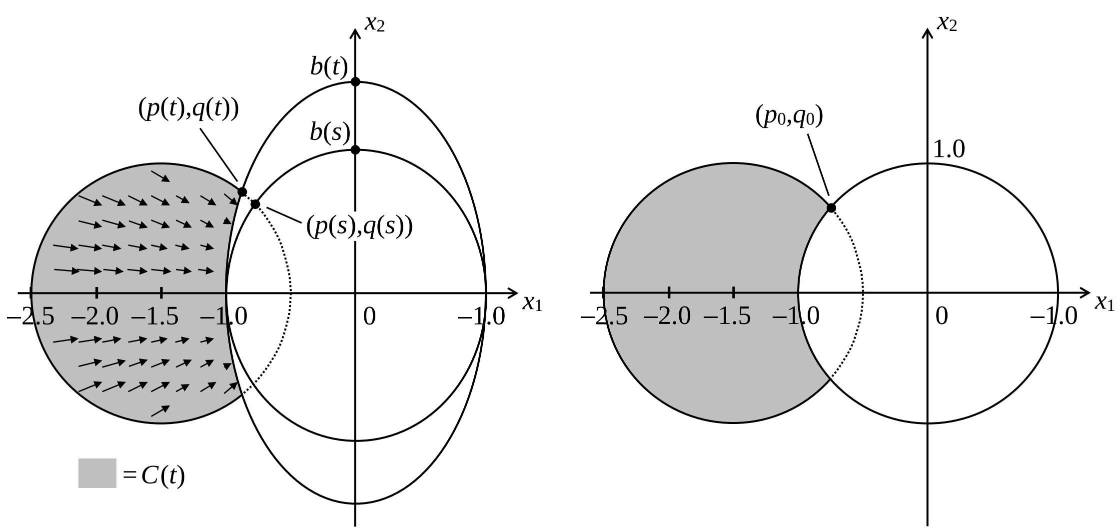

Figure 3.2. Illustrations of the notations of the example. The closed ball centered at $(-1.5, 0)$ is $\bar{B}_1$ and the white ellipses are the graphs of $S(t)$ for different values of the argument. The arrows is the vector field of $\dot{x} = -\alpha x$.

In order to apply Theorem 3.2.2, we will now analyze: *i)* strong monotonicity and uniform boundedness of $f(t, x)$, *ii)* Lipschitz continuity of $C(t)$, *iii)* prox-regularity of $C(t)$.

i) The monotonicity and boundedness of $f(t,x)$. Since $\langle f(t,x) - f(t,y), x - y\rangle = \langle \alpha x - \alpha y, x - y\rangle = \alpha\|x-y\|^2$, f is strongly monotone with constant α and bounded on $\bar{B}_1 \supset C(t)$ by $M_f = 2.5\alpha$.

ii) Lipschitz continuity of $C(t)$. The boundary $\partial\bar{B}_1$ of $\bar{B}_1$ intersects the boundary $\partial S(t)$ of $S(t)$ at a unique points $(p(t), q(t))$ with $\beta(t) \geq 0$. Since

$$d_H(C(t), C(s)) \leq \|(p(t), q(t)) - (p(s), q(s))\|$$

(see Fig. 3.2), we now aim at computing the Lipschitz constants of functions p and q. Since $b \in C^1(\mathbb{R}, [1,\infty))$, the implicit function theorem (see e.g., Zorich [65, Sec. 8.5.4 Theorem 1]) ensures that p and q are differentiable on $\mathbb{R}$. Therefore, by the mean-value theorem (see e.g., Rudin [57, Theorem 5.10]),

$$d_H(C(t), C(s)) \leq \|(p'(t_p), q'(t_q))\| \cdot |t-s|, \tag{3.10}$$

where t_p, t_q are located between t and s. To compute $(p'(t_p), q'(t_q))$, we use the formula for the derivative of the implicit function (Zorich [65, Sec. 8.5.4 Theorem 1])

$$(p'(t), q'(t))^T = -\left(F'_{(p,q)}\right)^{-1}(p(t), q(t), t) F'_t(p(t), q(t), t),$$

applied with

$$F(p,q,t) = \begin{pmatrix} (p+1.5)^2 + q^2 - 1 \\ p^2 + \dfrac{q^2}{b(t)^2} - 1 \end{pmatrix}.$$

Since

$$F'_{(p,q)}(p,q,t) = 2\begin{pmatrix} p+1.5 & q \\ p & \dfrac{q}{b(t)^2} \end{pmatrix}, \quad F'_t(p,q,t) = \begin{pmatrix} 0 \\ -2b(t)^{-3}b'(t)q^2 \end{pmatrix},$$

we get the following formula for the derivatives p' and q'

$$\begin{pmatrix} p'(t) \\ q'(t) \end{pmatrix} = -\frac{1}{\frac{1}{b(t)^2}(p(t)+1.5)q(t) - p(t)q(t)} \begin{pmatrix} q(t) \\ -(p(t)+1.5) \end{pmatrix} \frac{1}{b(t)^3} q(t)^2 b'(t).$$

Noticing that the properties $1 + p(t) > 0$ and $-p(t)b(t)^2 > 0$ imply

$$\frac{1}{b(t) \cdot (p(t) + 1.5 - p(t)b(t)^2)} \leq \frac{1}{\beta \cdot (-p(t)b(t)^2)} \leq \frac{1}{\beta^3 |p_0|},$$

we conclude

$$|p'(t)| \leq \frac{L_b}{\beta^3 |p_0|}, \qquad |q'(t)| \leq \frac{L_b}{\beta^3 |p_0|},$$

where p_0 is such that $p(t) \leq p_0$ for all $t \in \mathbb{R}$. Since $b(t) \geq 1$, we can take p_0 as the abscissa of the intersection of $\partial \bar{B}_1$ with a unit circle centered at 0, i.e.,

$$p_0 = -0.75,$$

see Fig. 3.2. Substituting these achievements to (3.10), we conclude

$$d_H(C(t), C(s)) \leq \frac{4L_b}{3\beta^3}|t - s|,$$

which gives $L_C = \dfrac{4L_b}{3\beta^3}$ for the Lipschitz constant of $t \mapsto C(t)$.

iii) The constant η in η-prox-regularity of $C(t)$. We recall that $C(t)$ is η-prox-regular if $C(t)$ admits an external tangent ball with radius smaller than η at each $x \in \partial C(t)$ (see Maury and Venel [48], Colombo and Monteiro Marques [21]). The points of $\partial C(t) \backslash \partial S(t)$ admit an external tangent ball of any radius. Therefore, to find η, which determines η-prox-regularity of $C(t)$, it is sufficient to focus on the points of $\partial C(t) \cap \partial S(t)$. That is why, for a fixed $t \in \mathbb{R}$, we can choose η as the minimum of the radius of curvature through $x \in \partial C(t) \cap \partial S_b(t)$, see e.g., Lockwood [44, p. 193].

Let us fix $t \in \mathbb{R}$ and use the parameterization $P(\phi) = (-\cos\phi, b(t)\sin\phi)$, $\phi \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, for the left-hand side of the ellipse $x^2 + \frac{y^2}{b(t)^2} = 1$. Then, the radius of curvature $R(\phi)$ of $\partial C(t) \cap \partial S(t)$ at $P(\phi)$ is (see Lockwood [44, p. xi, p. 21])

$$R(\phi) = \frac{1}{b(t)}(\sin^2\phi + b(t)^2\cos^2\phi)^{\frac{3}{2}} = \frac{1}{b(t)}\left(b(t)^2 + (1 - b(t)^2)\sin^2(\phi)\right)^{\frac{3}{2}}.$$

Observe that R decreases when $|\phi|$ increases from 0 to $\dfrac{\pi}{2}$. Indeed,

$$R'(\phi) = \frac{1}{b(t)}\frac{3}{2}\left(b(t)^2 + (1-b(t)^2)\sin^2(\phi)\right)^{\frac{1}{2}}(1-b(t)^2\sin(2\phi)$$

and $R'(\phi) = 0$ only at $\phi = 0$ on $\left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$. Then since

$$R''(0) = \frac{1}{b(t)}\frac{3}{2}(2b(t))(1-b(t)^2) < 0,$$

the function $\phi \mapsto R(\phi)$ attains the maximum on $\left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]$ at $\phi = 0$.

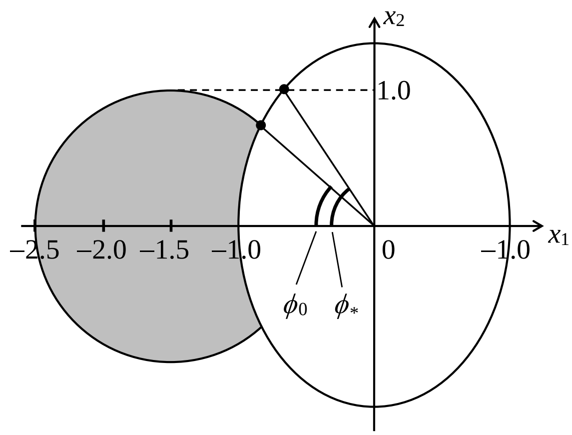

Figure 3.3. The parameters ϕ_0 and ϕ_*.

Therefore, the minimum curvature of $\partial C(t) \cap \partial S(t)$ is attained at the point $(p(t), q(t))$ as defined in ii). Let ϕ_0 be such that $P(\phi_0) = (p(t), q(t))$ and let $\phi_* > 0$ be such that the second component $P_2(\phi_*)$ of $P(\phi_*)$ equals 1, which exists because $b(t) \geq 1$ (see Fig. 3.3). Since $q(t) \leq 1$, we have $\phi_0 \leq \phi_*$, and since $\phi \mapsto R(\phi)$ decreases as $|\phi|$ increases, we have

$$R(\phi_0) \geq R(\phi_*).$$

Since $P_2(\phi_*) = 1$ implies $b(t)\sin\phi_* = 1$, we have $\sin\phi_* = \dfrac{1}{b(t)}$ and so

$$\begin{aligned} R(\phi_0) \quad &\geq \quad \frac{1}{b(t)}\left(\frac{1}{b(t)^2} + b(t)^2\left(1 - \frac{1}{b(t)^2}\right)\right)^{\frac{3}{2}} = \frac{1}{b(t)} \cdot \frac{(1+b(t)^4 - b(t)^2)^{\frac{3}{2}}}{b(t)^3} = \\ &= \left(b(t)^{-\frac{8}{3}} + b(t)^{\frac{4}{3}} - b(t)^{-\frac{2}{3}}\right)^{\frac{3}{2}} \geq \left(b(t)^{\frac{4}{3}} - b(t)^{-\frac{2}{3}}\right)^{\frac{3}{2}}. \end{aligned}$$

Noticing that the function $b \mapsto \left(b^{\frac{4}{3}} - b^{-\frac{2}{3}}\right)^{\frac{3}{2}}$ increases on $[1, \infty)$, we finally conclude

$$R(\phi_0) \geq \left(\beta^{\frac{4}{3}} - \beta^{-\frac{2}{3}}\right)^{\frac{3}{2}}.$$

Therefore, $C(t)$ is η-prox-regular with $\eta = \left(\beta^{\frac{4}{3}} - \beta^{-\frac{2}{3}}\right)^{\frac{3}{2}}$.

Substituting the values of M_f, L_C, and η into formula (3.4), we get the following statement.

Proposition 3.3.1. Let $\alpha > 0$ be an arbitrary constant and $b \in C^1(\mathbb{R}, [\beta, \infty))$ with some $\beta \geq 1$ and Lipschitz condition (3.8). If

$$\alpha > \frac{\dfrac{4L_b}{3\beta^3} + \dfrac{5}{2}\alpha}{\left(\beta^{\frac{4}{3}} - \beta^{-\frac{2}{3}}\right)^{\frac{3}{2}}},$$

then, the global solution

$$x(t) = (-1, 0), \quad t \in \mathbb{R},$$

of the sweeping process (1.1) with $C(t)$ and $f(t, x)$ given by (3.9) and (3.7), is globally asymptotically stable.

As noticed earlier, $b \mapsto \left(b^{\frac{4}{3}} - b^{-\frac{2}{3}}\right)^{\frac{3}{2}}$ increases on $[1, \infty)$, so that the condition of Proposition 3.3.1 is a lower bound on β.

3.4 Appendix

Here we explain the inapplicability of Theorem 3.2.2 in the crowd motion model given by Maury-Venel in [48].

According to the brief introduction about this model which we gave in our Chapter 1, sweeping process can be written as

$$\begin{cases} -\dot{x} \in N(C, x) - U(x) \\ x(0) = x_0 \in C. \end{cases} \tag{3.11}$$

Let's consider the situation where there are only two people. Then by Maury-Venel [48, Proposition 2.15], the set C in (1.4) is η-prox regular with $\eta = r\sqrt{2}$. Let's take $U(x) = -x$. Viewing (3.11) as (1.1), we get $\alpha = 1$ in (1.5).

Then the condition (3.4) of Theorem 3.2.2 takes the form $\sqrt{2}r > L_C + M_f$, where $L_C = 0$ (because C in (3.11) doesn't depend on t) and M_f satisfies $\|f(t,x)\| = \|x\| \leq M_f$ for each $x \in C$. Therefore (3.4) implies $M_f < \sqrt{2}r$.

On the other hand, since $\|(0,-r) - (0,r)\| = 2r$, we have $(0,-r,0,r) \in C$ and so M_f must verify $M_f \geq \|(0,-r,0,r)\| = \sqrt{2}r$.

Therefore Theorem 3.2.2 does not apply.

CHAPTER 4

BIFURCATIONS OF FINITE-TIME STABLE LIMIT CYCLES FROM FOCUS BOUNDARY EQUILIBRIA IN SWEEPING PROCESSES

Authors – Oleg Makarenkov, Lakmi Niwanthi

Department of Mathematical Sciences, FO 35

The University of Texas at Dallas

800 West Campbell Road

Richardson, Texas 75080-3021

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4.1 Introduction

In this chapter we consider $E = \mathbb{R}^2$ and the perturbed sweeping process (1.1) in the form of

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} \in -N_{C(\varepsilon)}(x,y) + \begin{pmatrix} f(x,y) \\ g(x,y) \end{pmatrix}, \tag{4.1}$$

where

$$C(\varepsilon) = \left\{(x,y) \in \mathbb{R}^2 : H(x,y,\varepsilon) \le 0\right\}, \quad H \in C^0,$$

is a nonempty closed time-independent μ-prox-regular set with fixed $\mu > 0$, for all $\varepsilon \ge 0$. Here we establish a theorem on bifurcation of a finite-time stable limit cycle as $\partial C(\varepsilon)$ collides with a focus equilibrium of the vector field

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} f(x,y) \\ g(x,y) \end{pmatrix}. \tag{4.2}$$

The only fact about $C(\varepsilon)$ that we will effectively use smoothness of H in the neighborhood of is that $0 \in \partial C(0)$, see Fig. 4.1.

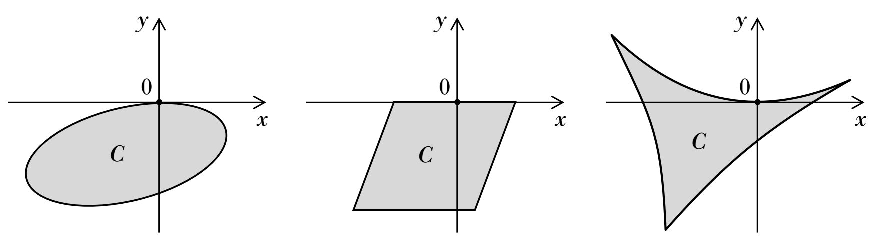

Figure 4.1. Examples of sets that can be used in sweeping process (4.1).

We will assume that f and g are C^1 globally Lipschitz functions, so that for any initial condition $(x_0, y_0) \in C(\varepsilon)$, the sweeping process (4.1) admits a unique forward solution $(x(t), y(t)) \in C(\varepsilon)$ with $(x(0), y(0)) = (x_0, y_0)$.

4.2 Existence of finite-time stable limit cycles

We will be using the following proposition in the proof of the existence of finite-time stable limit cycles.

Proposition 4.2.1. Consider $f, g \in C^2$ and assume that H is twice continuously differentiable in the neighborhood of the origin. Let the origin be an equilibrium of the subsystem (4.2) and $H(0) = 0$. Let the coordinates be rotated so that $H'_x(0) = 0$ and $H'_y(0) \neq 0$. Assume that

$$f'_x(0) \neq 0, \quad g'_x(0) \neq 0. \tag{4.3}$$

Then, there exist $r > 0$ and $\varepsilon_0 > 0$ such that for any $0 \leq \varepsilon \leq \varepsilon_0$ there exists a unique point $(A(\varepsilon), B(\varepsilon)) \in [-r, r] \times [-r, r]$ which satisfies the property

$$H_{(x,y)}(A(\varepsilon), B(\varepsilon), \varepsilon) \begin{pmatrix} f(A(\varepsilon), B(\varepsilon)) \\ g(A(\varepsilon), B(\varepsilon)) \end{pmatrix} = 0, \quad \varepsilon > 0. \tag{4.4}$$

1) The point $(A(\varepsilon), B(\varepsilon))$ satisfies

$$(A'(0), B'(0)) = \frac{H'_\varepsilon(0)}{H'_y(0)} \left(\frac{g'_y(0)}{g'_x(0)}, -1 \right). \tag{4.5}$$

2) The point $(A(\varepsilon), B(\varepsilon))$ splits

$$L = \left\{ (x, y) \in \mathbb{R}^2 : \ x, y \in [-r, r], \ H(x, y, \varepsilon) = 0 \right\}$$

into two parts

$$L_{sliding} = \left\{ (x, y) \in \mathbb{R}^2 : \ x, y \in [-r, r], \ H(x, y, \varepsilon) = 0 \right\} \cap \\ \cap \left\{ (x, y) : H_{(x,y)}(x, y, \varepsilon) \begin{pmatrix} f(x, y) \\ g(x, y) \end{pmatrix} > 0 \right\}$$

and

$$L_{crossing} = \left\{(x,y) \in \mathbb{R}^2 : \ x, y \in [-r, r], \ H(x,y,\varepsilon) = 0\right\} \cap$$
$$\cap \left\{ (x,y) : H_{(x,y)}(x,y,\varepsilon) \begin{pmatrix} f(x,y) \\ g(x,y) \end{pmatrix} < 0 \right\}.$$

3) a) Any solution $(x(t), y(t))$ of sweeping process (4.1) with the initial condition $(x(0), y(0)) \in L_{sliding}$ can escape from $L_{sliding}$ through the endpoints of $L_{sliding}$ only (i.e., through the two points of $\overline{L_{sliding}} \backslash L_{sliding}$).

b) While in $L_{sliding}$, the solution $(x(t), y(t))$ is governed by the following equation of sliding motion

$$\begin{pmatrix} \dot{x}(t) \\ \dot{y}(t) \end{pmatrix} = \alpha(x(t), y(t), \varepsilon) \begin{pmatrix} -H'_y(x(t), y(t), \varepsilon) \\ H'_x(x(t), y(t), \varepsilon) \end{pmatrix}, \tag{4.6}$$

where

$$\alpha(x,y,\varepsilon) = \frac{\left(-H'_y(x,y,\varepsilon), H'_x(x,y,\varepsilon)\right) \begin{pmatrix} f(x,y) \\ g(x,y) \end{pmatrix}}{\|H_{(x,y)}(x,y,\varepsilon)\|^2}.$$

c) The equation

$$\begin{aligned} f(a,b) + \lambda H'_x(a,b,\varepsilon) &= 0, \\ g(a,b) + \lambda H'_y(a,b,\varepsilon) &= 0 \end{aligned} \tag{4.7}$$

for the equilibrium of (4.6) possesses a unique solution $(a(\varepsilon), b(\varepsilon), \lambda(\varepsilon))$ on L with $(a'(0), b'(0), \lambda'(0))$ given by

$$\frac{H'_\varepsilon(0)}{H'_y(0)} \left(\frac{f'_y(0)}{f'_x(0)}, -1, \frac{1}{H'_y(0) f'_x(0)} \det \begin{vmatrix} f'_x(0) & f'_y(0) \\ g'_x(0) & g'_y(0) \end{vmatrix} \right). \tag{4.8}$$

4) If

$$f_{(x,y)}(0) \begin{pmatrix} A'(0) \\ B'(0) \end{pmatrix} (A'(0) - a'(0))\lambda'(0) < 0, \tag{4.9}$$

then the vector $\begin{pmatrix} f(A(\varepsilon), B(\varepsilon)) \\ g(A(\varepsilon), B(\varepsilon)) \end{pmatrix}$ (tangent to L by definition) points outwards $L_{sliding}$.

5) If condition (4.9) holds, then any solution $(x(t), y(t))$ of (4.1) with the initial condition $(x(0), y(0))$ from the $((a(\varepsilon), b(\varepsilon)), (A(\varepsilon), B(\varepsilon)))$-segment of $L_{sliding}$, escapes from $L_{sliding}$ in finite time through the point $(A(\varepsilon), B(\varepsilon))$.

6) The solution $(x(t), y(t))$ of (4.1) with the initial condition $(x(0), y(0)) = (A(\varepsilon), B(\varepsilon))$ leaves L towards

$$L^- = \left\{(x, y) \in \mathbb{R}^2 : H(x, y, \varepsilon) < 0\right\}$$

immediately under condition (4.9), in the sense that there exists Δt such that $t \mapsto (x(t), y(t))$ verifies (4.1) and $(x(t), y(t)) \in L^-$, for all $t \in (0, \Delta t]$.

Proof. The existence, uniqueness, and continuous differentiability of $(A(\varepsilon), B(\varepsilon))$ satisfying (4.4) follow by applying the Implicit Function Theorem (see e.g., Zorich [65, Sec. 8.5.4 Theorem]) to the function

$$F(A, B, \varepsilon) = \begin{pmatrix} H_{(x,y)}(A, B, \varepsilon) \begin{pmatrix} f(A, B) \\ g(A, B) \end{pmatrix} \\ H(A, B, \varepsilon) \end{pmatrix},$$

where we use that $F(0) = 0$ and $\det\left(F_{(A,B)}(0)\right) \neq 0$ by the second of the assumptions of (4.3).

1) Formula (4.5) follows by computing the derivative of $F(A(\varepsilon), B(\varepsilon), \varepsilon) = 0$ at $\varepsilon = 0$.

2) Follows from the uniqueness of $(A(\varepsilon), B(\varepsilon))$.

3) a) Fix $\varepsilon > 0$. Let $t_{escape} \geq 0$ be the time when $(x(t), y(t))$ escapes from $L_{sliding}$, specifically

$$\begin{aligned} t_{escape} &= \max\{t_0 \geq 0 : x(t) \in [-r, r],\ y(t) \in [-r, r], \\ &\qquad H(x(t), y(t), \varepsilon) = 0,\ t \in [0, t_0]\}. \end{aligned}$$

Assuming that neither $|x(t_{escape})| = r$, nor $|y(t_{escape})| = r$, we now show that

$$H_{(x,y)}(x(t_{escape}), y(t_{escape}), \varepsilon) \begin{pmatrix} f(x(t_{escape}), y(t_{escape})) \\ g(x(t_{escape}), y(t_{escape})) \end{pmatrix} \leq 0, \tag{4.10}$$

which coincides with the Statement 3a.

By the definition of t_{escape}, for any $\delta > 0$ there exist $t_\delta \in [t_{escape}, t_{escape} + \delta]$ such that $H(x(t), y(t), \varepsilon) < 0$ for each $t \in (t_{escape}, t_\delta]$. Since, the solution $(x(t), y(t))$ satisfies (4.2) on $(t_{escape}, t_\delta]$, by Mean-Value Theorem

$$\begin{aligned} H(x(t_\delta), y(t_\delta)), \varepsilon) &\quad -H(x(t_{escape}), y(t_{escape})), \varepsilon) = \\ &\quad H_{(x,y)}(x(t^*_\delta), y(t^*_\delta), \varepsilon) \begin{pmatrix} f(x(t^*_\delta), y(t^*_\delta)) \\ g(x(t^*_\delta), y(t^*_\delta)) \end{pmatrix} (t_\delta - t_{escape}), \end{aligned}$$

for some $t^*_\delta \in (t_{escape}, t_\delta)$.

This yields (4.10) as $\delta \to 0$ since $H(x(t_\delta), y(t_\delta)), \varepsilon) < 0$ for each $\delta > 0$ and $H(x(t_{escape}), y(t_{escape})), \varepsilon) = 0$.

b) Consider some $t_0 > 0$ such that $(x(t), y(t)) \in L_{sliding}$ for all $t \in [0, t_0]$. From the definition of $L_{sliding}$ we conclude that

$$H_{(x,y)}(x(t), y(t), \varepsilon) \begin{pmatrix} \dot{x}(t) \\ \dot{y}(t) \end{pmatrix} = 0, \quad \text{for a.e. } t \in [0, t_0],$$

where we use that the derivatives of solutions of (4.1) are defined for a.e. t only.

Equation (4.6) now comes by projecting (4.1) on the vector $\begin{pmatrix} -H'_y(x(t), y(t), \varepsilon) \\ H'_x(x(t), y(t), \varepsilon) \end{pmatrix}$, i.e.,

$$\begin{aligned}
&\left(-H'_y(x(t), y(t), \varepsilon), H'_x(x(t), y(t), \varepsilon)\right) \begin{pmatrix} f(t, x(t), y(t)) \\ g(t, x(t), y(t)) \end{pmatrix} \\
&= \left(-H'_y(x(t), y(t), \varepsilon), H'_x(x(t), y(t), \varepsilon)\right) \begin{pmatrix} \dot{x}(t) \\ \dot{y}(t) \end{pmatrix},
\end{aligned}$$

for a.e. $t \in [0, t_0]$. Extension of (4.6) from a.e. $t \in [0, t_0]$ to all $t \in [0, t_0]$ follows from the smoothness of (4.6).

c) To prove the existence and uniqueness of $(a(\varepsilon), b(\varepsilon))$, we apply the Implicit Function Theorem to the function

$$G(a, b, \lambda, \varepsilon) = \begin{pmatrix} f(a, b) + \lambda H'_x(a, b, \varepsilon) \\ g(a, b) + \lambda H'_y(a, b, \varepsilon) \\ H(a, b, \varepsilon) \end{pmatrix}.$$

The determinant

$$\det\left(G_{(a,b,\lambda)}(0)\right) = \det \begin{pmatrix} f'_x(0) & f'_y(0) & 0 \\ g'_x(0) & g'_y(0) & H'_y(0) \\ 0 & H'_y(0) & 0 \end{pmatrix} = -H'_y(0)^2 f'_x(0)$$

doesn't vanish by the first assumption of (4.3) and the formula

$$(a'(0), b'(0), \lambda'(0))^T = -G_{(a,b,\lambda)}(0)^{-1} G'_\varepsilon(0) \tag{4.11}$$

for the derivative of the implicit function yields (4.8).

4) Case I: $\lambda'(0) < 0$, which combined with (4.9) gives

$$f_{(x,y)}(0) \begin{pmatrix} A'(0) \\ B'(0) \end{pmatrix} (A'(0) - a'(0)) > 0. \tag{4.12}$$

Furthermore, $\lambda'(0) < 0$ implies that $(a(\varepsilon), b(\varepsilon)) \in L_{sliding}$ for all $\varepsilon > 0$ sufficiently small. Sub-case 1: $A'(0) < a'(0)$ (i.e., $(A(\varepsilon), B(\varepsilon))$ is the left endpoint of $L_{sliding}$). In this case (4.12) yields $f(A(\varepsilon), B(\varepsilon)) < 0$, i.e., the vector $\begin{pmatrix} f(A(\varepsilon), B(\varepsilon)) \\ g(A(\varepsilon), B(\varepsilon)) \end{pmatrix}$ points to the left.

Sub-case 2: By analogy, when $A'(0) > a'(0)$, the assumption (4.12) implies $f(A(\varepsilon), B(\varepsilon)) > 0$.

Case II: $\lambda'(0) > 0$. Can be considered by analogy taking into account that $\lambda'(0) > 0$ implies that $(a(\varepsilon), b(\varepsilon)) \in L_{crossing}$ for all $\varepsilon > 0$ sufficiently small.

5) The dynamics of $(x(t), y(t))$ is described by one-dimensional smooth equation of sliding motion as long as $(x(t), y(t)) \in L_{sliding}$. Part 4) implies that the vector field of the equation of sliding motion on $L_{sliding}$ points towards the endpoint $(A(\varepsilon), B(\varepsilon))$ at all the points of $L_{sliding}$ close to $(A(\varepsilon), B(\varepsilon))$. Therefore, if we assume, by contradiction, that the solution $(x(t), y(t))$ doesn't reach $(A(\varepsilon), B(\varepsilon))$ in finite-time, then the sliding vector field must possess an equilibrium on the $((a(\varepsilon), b(\varepsilon)), (A(\varepsilon), B(\varepsilon)))$-segment of $L_{sliding}$, which contradicts the uniqueness of equilibrium $(a(\varepsilon), b(\varepsilon))$.

6) Let $(x(t), y(t))$ be the solution of (4.2) with the initial condition $(x(0), y(0)) = (A(\varepsilon), B(\varepsilon))$. By the definition of $(A(\varepsilon), B(\varepsilon))$, there exists $\Delta t > 0$ such that $H(x(t), y(t), \varepsilon) < 0$ for all $t \in (0, \Delta t]$. Therefore, $(x(t), y(t))$ is the solution of (4.1) on $(0, \Delta t]$. Therefore, $(x(t), y(t))$ is the solution of (4.1) on $[0, \Delta t]$, because the definition of the solution (4.1) requires the validity of (4.1) for $(x(t), y(t))$ in a.e. time instances t only.

□

Now for $\varepsilon > 0$, we consider the following change of variables

$$\begin{pmatrix} u(t) \\ v(t) \end{pmatrix} = \frac{1}{\varepsilon} \begin{pmatrix} x \\ y \end{pmatrix}$$

to brings (4.2) to the form

$$\begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} = \frac{1}{\varepsilon}\begin{pmatrix} f(\varepsilon u, \varepsilon v) \\ g(\varepsilon u, \varepsilon v) \end{pmatrix}, \quad H(\varepsilon u, \varepsilon v, \varepsilon) < 0. \tag{4.13}$$

Along with system (4.13) we consider the following reduced system

$$\begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} = \begin{pmatrix} f_x(0) & f_y(0) \\ g_x(0) & g_y(0) \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}, \qquad \text{if } H_y(0)v + H_\varepsilon(0) < 0,. \tag{4.14}$$

and we arrive to the following result about the limit cycles of sweeping process (4.1).

Theorem 4.2.1. Consider $f, g \in C^2$ and assume that H is twice continuously differentiable in the neighborhood of the origin. Let the origin be an equilibrium of the subsystem of (4.2) and $H(0) = 0$. Assume that the coordinates are rotated so that $H'_x(0) = 0$ and $H'_y(0) \neq 0$. Let the assumption (4.9) of Proposition 4.2.1 hold with $(A'(0), B'(0))$ and $(a'(0), b'(0), \lambda'(0))$ given by (4.5) and (4.8) respectively. Let the assumption (4.3) of Proposition 4.2.1 holds. Finally, assume $(u_0(t), v_0(t))$ the solution of reduced system (4.14) with the initial condition $(u_0(0), v_0(0)) = (A'(0), B'(0))$ meets $v = B'(0)$ at time T_0. If

$$\begin{aligned} &u_0(T_0) \in (\min\{a'(0), A'(0)\}, \max\{a'(0), A'(0)\}) \quad \text{in the case when } \lambda'(0) < 0, \\ &u_0(T_0) \neq A'(0) \quad \text{in the case when } \lambda'(0) > 0, \end{aligned} \tag{4.15}$$

then for all $\varepsilon > 0$ sufficiently small, the sweeping process (4.1) admits a finite-time stable stick-slip limit cycle $(x_\varepsilon(t), y_\varepsilon(t)) \to 0$ as $\varepsilon \to 0$.

Proof. **Step 1.** Let $t \mapsto \begin{pmatrix} U(t, u, v, \varepsilon) \\ V(t, u, v, \varepsilon) \end{pmatrix}$ be the general solution of system (4.13). Introduce

$$F(T, \varepsilon) = \frac{1}{\varepsilon} H\left(\varepsilon U\left(T, \frac{A(\varepsilon)}{\varepsilon}, \frac{B(\varepsilon)}{\varepsilon}, \varepsilon\right), \varepsilon V\left(T, \frac{A(\varepsilon)}{\varepsilon}, \frac{B(\varepsilon)}{\varepsilon}, \varepsilon\right), \varepsilon\right).$$

Computing $F(T, 0)$ we get

$$F(T, 0) = H'_y(0)V(T, A'(0), B'(0)) + H'_\varepsilon(0)$$

where

$$
\begin{aligned}
&(U(T,A'(0),B'(0)),V(T,A'(0),B'(0))) \\
&\qquad = \left(\lim_{\varepsilon\to 0} U\left(T,\frac{A(\varepsilon)}{\varepsilon},\frac{B(\varepsilon)}{\varepsilon},\varepsilon\right),\lim_{\varepsilon\to 0} V\left(T,\frac{A(\varepsilon)}{\varepsilon},\frac{B(\varepsilon)}{\varepsilon},\varepsilon\right)\right)
\end{aligned}
$$

and

$$
\begin{aligned}
V_t'(T,A'(0),B'(0)) &= \lim_{\varepsilon\to 0} V_t'\left(T,\frac{A(\varepsilon)}{\varepsilon},\frac{B(\varepsilon)}{\varepsilon},\varepsilon\right) \\
&= \lim_{\varepsilon\to 0}\frac{1}{\varepsilon} g\left(\varepsilon U\left(T,\frac{A(\varepsilon)}{\varepsilon},\frac{B(\varepsilon)}{\varepsilon},\varepsilon\right),\varepsilon V\left(T,\frac{A(\varepsilon)}{\varepsilon},\frac{B(\varepsilon)}{\varepsilon},\varepsilon\right)\right) \\
&= (g_x'(0),g_y'(0))\begin{pmatrix} U(T_0,A'(0),B'(0)) \\ V(T_0,A'(0),B'(0)) \end{pmatrix}.
\end{aligned}
$$

Similarly

$$
U_t'(T,A'(0),B'(0)) = (f_x'(0),f_y'(0))\begin{pmatrix} U(T_0,A'(0),B'(0)) \\ V(T_0,A'(0),B'(0)) \end{pmatrix}.
$$

Thus $(U(T,A'(0),B'(0)),V(T,A'(0),B'(0)))$ satisfies (4.14). Therefore, $F(T_0,0)=0$ and since

$$
\begin{aligned}
F_t'(T_0,0) &= H_y'(0)V_t'(T_0,A'(0),B'(0)) = \\
& = H_y'(0)(g_x'(0),g_y'(0))\begin{pmatrix} U(T_0,A'(0),B'(0)) \\ V(T_0,A'(0),B'(0)) \end{pmatrix} \\
& = H_y'(0)(g_x'(0),g_y'(0))\begin{pmatrix} U(T_0,A'(0),B'(0)) \\ B'(0) \end{pmatrix},
\end{aligned}
$$

we have $F_t'(T_0,0)\neq 0$ from (4.5) and (4.15).

Therefore the existence of T_ε such that $F(T_\varepsilon,\varepsilon)=0$ follows by applying the Implicit Function Theorem, which in turn implies that $(x_\varepsilon(t),y_\varepsilon(t))$, solution of (4.2) with initial condition $(A(\varepsilon),B(\varepsilon))$, meets $H(x,y,\varepsilon)$ at $t=T_\varepsilon$. i.e., $H(x_\varepsilon(T_\varepsilon),y_\varepsilon(T_\varepsilon),\varepsilon)=0$.

Step 2. Condition (4.15) implies that $(x_\varepsilon(T_\varepsilon), y_\varepsilon(T_\varepsilon))$ belongs to the segment of $L_{sliding}$. Therefore by Proposition (4.2.1), $(x_\varepsilon(t), y_\varepsilon(t))$ reaches $(A(\varepsilon), B(\varepsilon))$ along $L_{sliding}$ in a finite time.

Note that $(x_\varepsilon(T_\varepsilon), y_\varepsilon(T_\varepsilon)) \in ((a(\varepsilon), b(\varepsilon)), (A(\varepsilon), B(\varepsilon)))$ when $\lambda'(0) < 0$ ensures, the solution $(x_\varepsilon(t), y_\varepsilon(t))$ reaches to $(A(\varepsilon), B(\varepsilon))$.

Step 3. Stability follows from convergence of $(A(\varepsilon), B(\varepsilon))$ and $(a(\varepsilon), b(\varepsilon))$ to the origin as $\varepsilon \to 0$. $\square$

4.3 Example

We illustrate the theorem considering the following sweeping process

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} \in -N_{C(\varepsilon)}(x, y) + \begin{pmatrix} ax - by \\ bx + ay \end{pmatrix} + M(x, y), \tag{4.16}$$

where $a, b > 0$, M is any C^2-functions such that $M(0) = M'(0) = 0$ and $C(\varepsilon) = C - \varepsilon \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ with nonempty μ-prox-regular set C satisfying

$$\partial C = \left\{(x, y) \in \mathbb{R}^2 : H(x, y) = 0\right\}, \qquad H \in C^0,$$

with such a function H which is continuously differentiable in the neighborhood of the origin, $H(0) = 0$ and $H_{(x,y)}(0, 0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

In order to check the assumptions of Theorem 4.2.1, we calculate $(A'(0), B'(0))$ and $(a'(0), b'(0), \lambda'(0))$ from (4.5) and (4.8). Then we have

$$\begin{aligned} (A'(0), B'(0)) &= \left(-\frac{a}{b}, 1\right) \\ (a'(0), b'(0), \lambda'(0)) &= \left(\frac{b}{a}, 1, -\frac{a^2 + b^2}{a}\right). \end{aligned} \tag{4.17}$$

This gives

$$\frac{-a^2-b^2}{b}\cdot\frac{-a^2-b^2}{ab}\cdot\frac{-a^2-b^2}{a}$$

for the left-hand-side of (4.9). Therefore, assumption (4.9) always holds.

To prove the existence of $(u_0(t), v_0(t))$, the solution of reduced system (4.14) with the initial condition $(u_0(0), v_0(0)) = (A'(0), B'(0))$ such that $(u_0(t), v_0(t))$ meets $v = B'(0)$ in a finite time T_0 and to check the condition (4.15) we have to compute $r = P(A'(0)) = P\left(-\frac{a}{b}\right)$ where P is the Poincaré map of linear system (4.14) induced by the cross-section $v = -\frac{H_\varepsilon(0)}{H_y(0)} = B'(0) = 1$. The linear system (4.14) corresponding to (4.16) is

$$\begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} = \begin{pmatrix} au - bv \\ bu + av \end{pmatrix}. \tag{4.18}$$

Using that a solution of (4.18) is given by

$$u(t) = e^{at}\cos(bt), \quad v(t) = e^{at}\sin(bt), \tag{4.19}$$

we build the following solution of (4.18)

$$u_0(t) = \frac{e^{a(t-t_0)}\cos(bt)}{\sin(bt_0)}, \quad v_0(t) = \frac{e^{a(t-t_0)}\sin(bt)}{\sin(bt_0)}, \quad bt_0 = \operatorname{arccot}\left(-\frac{a}{b}\right),$$

which verifies the property $(u_0(t_0), v_0(t_0)) = (A'(0), B'(0))$.

It is impossible to find the intersection of solution $(u_0(t), v_0(t))$ with $v = 1$ explicitly, so we propose an explicit approach that relies on the observation that an intersection of any solution of (4.18) with $u = 0$ is computable explicitly.

Since $\operatorname{arccot}\left(-\frac{a}{b}\right) \in \left(\frac{\pi}{2}, \pi\right)$, the first intersection of this solution with $u = 0$ occurs at $bt = \frac{\pi}{2} + \pi$, which gives

$$y_* = v_0\left(\frac{1}{b}\cdot\frac{3\pi}{2}\right) = -\exp\left(a\left(\frac{1}{b}\cdot\frac{3\pi}{2} - t_0\right)\right)\frac{1}{\sin(bt_0)}.$$

Now we assume that the intersection of $(u_0(t), v_0(t))$ with $v = 1$ occurs at some point $u = r$ and use (4.19) to compute y_* in terms of r.

Specifically, using (4.19) we build a solution

$$u^0(t) = \frac{e^{a(t-t^0)}\cos(bt)}{\sin(bt^0)},\ v^0(t) = \frac{e^{a(t-t^0)}\sin(bt)}{\sin(bt^0)},\ bt^0 = \operatorname{arccot}(r),$$

which verifies $(u^0(t^0), v^0(t^0)) = (r, 1)$.

Since $\operatorname{arccot}(r) \in (0, \pi)$, the intersection of $(u^0(t), v^0(t))$ with $u = 0,\ v < 0$, must had occurred earlier at time $bt = \frac{\pi}{2} - \pi$, which gives

$$y^* = v^0\left(\frac{1}{b}\cdot\left(-\frac{\pi}{2}\right)\right) = -\exp\left(a\left(\frac{1}{b}\left(-\frac{\pi}{2}\right) - t^0\right)\right)\frac{1}{\sin(bt^0)}$$

for the respective point of intersection with $u = 0$. Now equaling y_* and y^*, observing that $\dfrac{1}{\sin(\operatorname{arccot}\alpha)} = \sqrt{\alpha^2+1}$, and taking the natural logarithm of both sides of the equality, one gets the following implicit formula for r :

$$\begin{aligned}\frac{a}{b}\cdot\frac{3\pi}{2} - \frac{a}{b}\operatorname{arccot}\left(-\frac{a}{b}\right) + \frac{1}{2}\ln\left(1+\frac{a^2}{b^2}\right) = \\ = \frac{a}{b}\left(-\frac{\pi}{2}\right) - \frac{a}{b}\operatorname{arccot}(r) + \frac{1}{2}\ln(1+r^2).\end{aligned} \tag{4.20}$$

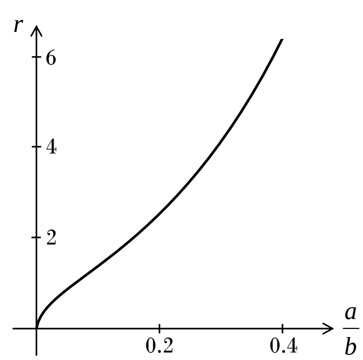

Figure 4.2. The solution of (4.20).

The graph of the implicit equation (4.20) is given in Fig. 4.2, from which we conclude that the solution $(u_0(t), v_0(t))$ returns back to the cross-section $v = 1$ at the value $r\left(\frac{a}{b}\right) = (u_0(T_0), v_0(T_0))$ which increases monotonically with $\frac{a}{b}$.

Our goal now is to establish (4.15). Based on (4.17), $\lambda'(0) < 0$.

In this case assumption (4.15) takes the form

$$r = u_0(T_0) < \frac{b}{a}.$$

Since $r \mapsto -\frac{a}{b}\text{arccot}(r) + \frac{1}{2}\ln(1+r^2)$ is a monotonically increasing function, we can combine the later inequality with (4.20) to obtain

$$\begin{aligned} \frac{a}{b} \cdot \frac{3\pi}{2} - \frac{a}{b}\text{arccot}\left(-\frac{a}{b}\right) + \frac{1}{2}\ln\left(1 + \frac{a^2}{b^2}\right) < \\ < \frac{a}{b}\left(-\frac{\pi}{2}\right) - \frac{a}{b}\text{arccot}\left(\frac{b}{a}\right) + \frac{1}{2}\ln\left(1 + \frac{b}{a}\right)^2. \end{aligned} \tag{4.21}$$

Now we arrive to the following corollary of Theorem 4.2.1.

Corollary 4.3.1. If $\frac{a}{b}$ satisfies

$$\frac{a}{b}\left(4\text{arctan}\frac{a}{b} - 3\pi\right) > 2\ln\frac{a}{b} \qquad \text{(which gives approximately } \frac{a}{b} < 0.29\text{)},$$

then for all $\varepsilon > 0$ sufficiently small, the sweeping process (4.16) admits a finite-time stable stick-slip limit cycle $(x_\varepsilon(t), y_\varepsilon(t))$ that shrinks to the origin as $\varepsilon \to 0$.

CHAPTER 5

A CONTINUATION PRINCIPLE FOR PERIODIC BV-CONTINUOUS STATE-DEPENDENT SWEEPING PROCESSES

5.1 Introduction

Here we investigates the initial-value and periodic problems to the following state-dependent version of (1.1)

$$-\,dx \in N_{A+a(t)+c(x)}(x) + f(t,x)dt, \quad x \in E, \tag{5.1}$$

where a is a BV-continuous function and $c : E \mapsto E$ is a Lipschitz function.

In order to prove existence of periodic solutions, we will be also using a continuation approach and replace (5.1) by a parameter dependent sweeping process

$$-\,dx \in N_{A+a(t,\lambda)+c(x,\lambda)}(x) + f(t,x,\lambda)dt, \quad x \in E,\ \lambda \in \mathbb{R}. \tag{5.2}$$

5.2 Definition of solution

In what follows, $\mathcal{B}([0,T])$ is the family of Borel subsets of $[0,T]$. A *Borel vector measure* on $[0,T]$ is a map $\mu : \mathcal{B}([0,T]) \to E$ such that $\mu\left(\cup_{n=1}^{\infty} B_n\right) = \Sigma_{n=1}^{\infty}\mu(B_n)$ for any sequence $\{B_n\}_{n=1}^{\infty}$ of mutually disjoint elements of $\mathcal{B}([0,T])$, see Recupero [56, §2.4] or Dinculeanu [25, Definition 1, §III.14.4, p. 297].

According to Dinculeanu [25, Theorem 1, §III.17.2, p. 358] (in our case it was according to Recupero [56] who phrased it clearly), any BV-continuous function $x : [0,T] \to E$ admits a unique vector measure of bounded variation $dx : \mathcal{B}([0,T]) \to E$ (called *Stieltjes measure* in Dinculeanu [25]) such that for every $0 < t_1 < t_2 < T$ we have

$$\begin{aligned} dx((t_1,t_2)) &= x(t_2^-) - x(t_1^+), \quad dx([t_1,t_2]) = x(t_2^+) - x(t_1^-), \\ dx([t_1,t_2)) &= x(t_2^-) - x(t_1^-), \quad dx((t_1,t_2]) = x(t_2^+) - x(t_1^+), \end{aligned} \tag{5.3}$$

where

$$x(t^-) = \lim_{\tau \to t^-} x(\tau), \quad x(t^+) = \lim_{\tau \to t^+} x(\tau), \quad 0 < t < T.$$

A vector Borel measure $d\mu$ is called continuous with respect to a scalar Borel measure $d\nu$ (or simply $d\nu$-continuous), if $\lim_{\nu(D)\to 0} \mu(D) = 0$, see Diestel-Uhl [24, p. 11]. If a vector measure $d\mu$ is $d\nu$-continuous then, according to Radon-Nikodym Theorem [24, p. 59] there is a $d\nu$-integrable function $g : [0,T] \mapsto E$ such that

$$d\mu(D) = \int_D g \, d\nu, \quad \text{for all } D \in \mathcal{B}([0,T]).$$

In this case, the function g is called Radon-Nikodym derivative of $d\mu$ with respect to $d\nu$ (or density) and is denoted by $\dfrac{d\mu}{d\nu}$. Furthermore, according to Moreau-Valadier [53, Proposition 1] (see also Valadier [61, Theorem 3]), the Radon-Nikodym derivative $\dfrac{d\mu}{d\nu}$ can be computed as

$$\frac{d\mu}{d\nu}(t) = \lim_{\varepsilon \to 0,\, \varepsilon > 0} \frac{d\mu([t, t+\varepsilon])}{d\nu([t, t+\varepsilon])}, \quad d\nu - a.e.\ on\ [0,T]. \tag{5.4}$$

We will use the following definition of the solution of (5.1) (Castaing and Monteiro Marques [19, §1]).

Definition 5.2.1. A BV continuous function x is called a solution of (5.1), if there exists a finite measure $d\nu$ for which both differential measure dx and Lebesgue measure dt are $d\nu$-continuous, and such that

$$-\frac{dx}{d\nu}(t) \in N_{A+a(t)+c(x(t))}(x(t)) + f(t, x(t))\frac{dt}{d\nu}(t), \quad d\nu - a.e.\ on\ [0,T].$$

5.3 Existence of solutions

It is customary (see Kunze and Monteiro Marques [40, Theorem 6]) to assume that the initial condition q of sweeping process (5.1) satisfies

$$q \in A + a(0) + c(q). \tag{5.5}$$

However, it will be convenient for our analysis to define solutions of (5.1) for any initial condition $q \in E$, that we will term a *generalized initial condition.* We take advantage of the fact, that for contracting map c, the equation

$$v = \mathrm{proj}(q, A + a(0) + c(v))$$

always has a solution $v = V(q)$ (see Lemma 5.6.2) and $V \in C^0(E, E)$.

The main result of this chapter is the following Theorem 5.3.1 on the existence of solutions to (5.1). As itself, the theorem won't lose anything by dropping the generalized initial condition concept. However, considering the generalized initial conditions will be convenient for applications of Theorem 5.3.1 to the problem of the occurrence of periodic solutions from a boundary equilibrium, that we consider in this chapter later (Theorem 5.8.3).

Theorem 5.3.1. Assume that $A \subset E$ is a nonempty closed convex bounded set, $a : [0, T] \to E$ is BV-continuous on $[0, T]$, $x \mapsto c(x)$ is globally Lipschitz with Lipschitz constant $0 < L_2 < 1$, and $(t, x) \mapsto f(t, x)$ is Caratheodory in (t, x) with respect to Lebesgue measure and globally Lipschitz in x. Then, for any generalized initial condition $q \in E$, the sweeping process (5.1) admits a solution, defined on $[0, T]$, with the initial condition $x(0) = V(q)$. In particular, sweeping process (5.1) admits a solution on $[0, T]$, for any initial condition $x(0) = q$, where q satisfies (5.5).

5.4 Global existence of periodic solutions

In this section we offer a result saying that, under the conditions of Theorem 5.3.1, sweeping process (5.1) always has a periodic solution, if the right-hand-sides are T-periodic.

Theorem 5.4.1. Assume that conditions of theorem 5.3.1 hold and let $L_2 \in (0, 1)$ be the Lipschitz constant of c as introduced in theorem 5.3.1. Denoting by $\xi \in E$ the unique solution of $c(\xi) = \xi$, consider the set

$$\Omega = \bigcup_{t \in [0,T]} \Omega_t, \quad \Omega_t = \bigcup_{b \in A(t)} \left\{ x : \|x - \xi\| < \frac{\|b\|}{1 - L} \right\}.$$

Then sweeping process (5.1) admits a solution $t \mapsto x(t)$ such that

$$x(T) = x(0) \in \overline{\Omega}. \tag{5.6}$$

In particular, $t \mapsto x(t)$ is a T-periodic solution of (5.1), if both $t \mapsto a(t)$ and $t \mapsto f(t,x)$ are T-periodic.

Remark 5.4.1. Throughout this chapter we prefer to work with functions defined on $[0,T]$ only. When saying $t \mapsto x(t)$ is a T-periodic solution of (5.1), we mean that $t \mapsto x(t)$ becomes a T-periodic solution after all functions are extended to $\mathbb{R}$ by T-periodicity.

5.5 Continuation of periodic solutions

This section considers a λ-dependent sweeping process (5.2) for measures dx and dt, and discovers how the existence of periodic solutions for $\lambda > 0$ can be concluded from an appropriate knowledge about (5.2) at $\lambda = 0$.

We will assume that BV-continuity of a of Theorem 5.3.1 holds uniformly with respect to λ, i.e.,

$$\begin{aligned} &\mathrm{var}(a(\cdot,\lambda),[s,t]) \le \mathrm{var}(\bar{a},[s,t]), \quad \lambda \in [0,1], \\ &\text{where } \bar{a} : [0,T] \to \mathbb{R} \text{ is a BV continuous function.} \end{aligned} \tag{5.7}$$

The map V^λ for (5.2) now depends on the parameter λ and is defined as the unique solution (according to Lemma 5.6.2) of the equation

$$v = \mathrm{proj}(q, A + a(0,\lambda) + c(v,\lambda)).$$

We will call sweeping process (5.2) T-periodic, if

$$a(t+T,\lambda) \equiv a(t,\lambda), \quad f(t+T,x,\lambda) \equiv f(t,x,\lambda).$$

In what follows, $d(I - \bar{P}, Q)$ is the topological degree of the vector field $I - \bar{P}$ on an open bounded set $Q \subset E$, see e.g., Krasnoselskii-Zabreiko [36].

Theorem 5.5.1. Assume that T-periodic sweeping process (5.2) possesses the following regularity:

I) The set $A \subset E$ is nonempty, convex, closed, and bounded. The function a satisfies (5.7). The function $x \mapsto c(x, \lambda)$ is globally Lipschitz with Lipschitz constant $0 < L_2 < 1$. The function $(t, x) \mapsto f(t, x, \lambda)$ is Caratheodory in (t, x) with respect to Lebesgue measure and globally Lipschitz in x, and both the Lipschitz constants are independent of $\lambda \in [0, 1]$. Furthermore, a, c, and f are continuous in $\lambda \in [0, 1]$ uniformly with respect to $t \in [0, T]$ and $x \in E$.

Assume, that the existence of a T-periodic solution for $\lambda = 0$ is given in the following extended way:

II) There exists an open bounded $Q \subset E$ such that, when $\lambda = 0$, the solution of (5.2) is unique for any initial condition $x(0) \in V^0(\overline{Q})$, none of the elements of $V^0(\partial Q)$ are initial conditions of T-periodic solutions of (5.2) with $\lambda = 0$, and for the Poincare map P^0 of (5.2) with $\lambda = 0$ one has

$$d(I - P^0 \circ V^0, Q) \neq 0.$$

Finally, assume the following homotopy through $\lambda \in [0, \lambda_1]$:

III) There exists $\lambda_1 \in (0, 1]$ such that sweeping process (5.2) doesn't have periodic solutions x with initial condition $x(0) \in V^\lambda(\partial Q)$, $\lambda \in [0, \lambda_1]$.

Then, for any $\lambda \in [0, \lambda_1]$, sweeping process (5.2) admits a T-periodic solution x with the initial condition $x(0) \in V^\lambda(Q)$.

Note, for $\lambda > 0$, we don't know whether or not the solutions of the sweeping process (5.2) are uniquely defined by the initial condition or depend continuously on λ. That is why the statement of the theorem is not a direct consequence of II) as it usually happens in topological degree based existence results. In particular, we cannot establish any type of continuity of solutions as $\lambda \to 0$. That is why the next theorem is not a direct consequence of Theorem 5.5.1.

Theorem 5.5.2. Assume that sweeping process (5.2) is T-periodic. Assume that conditions I) and II) of Theorem 5.5.1 hold. Then, there exists $\lambda_1 > 0$ such that condition III) of Theorem 5.5.1 holds, and, therefore, for any $\lambda \in [0, \lambda_1]$, sweeping process (5.2) admits a T-periodic solution x with the initial condition $x(0) \in V^\lambda(Q)$.

5.6 The catching-up algorithm and proofs of the abstract existence results

This section contains proofs of Theorems 5.3.1-5.5.2. The proof of the existence of solutions is based on the implicit catching-up scheme (5.15)-(5.18) which newly introduce in (section 5.6.3), which in turn relies on the following two ideas: **(i)** Castaing and Monteiro Marques change of the variables [19, Theorem 4.1] that converts (section 5.6.1) the perturbed sweeping process (5.2) with differential measure dx into a non-perturbed sweeping process (5.9) for the derivative $\frac{du}{|du|}$ with respect to the variation measure $|du|$ of du; **(ii)** Kunze and Monteiro Marques lemma ([39, Lemma 7]) to resolve (Lemma 5.6.2) the implicit catching-up scheme (5.15)-(5.18) with respect to the implicit variable. Furthermore, our Lemma 5.6.2 extends Kunze and Monteiro Marques [39, Lemma 7] by proving continuous dependence of scheme (5.15)-(5.18) on initial condition, that gave us continuity of Poincare maps $P^{\lambda,n}$ (section 5.6.4). The convergence of the scheme (5.15)-(5.18) is established in section 5.6.5 where we prove (Lemma 5.6.4) convergence of the approximations $\{u_n\}_{n\in\mathbb{N}}$ of solution u of (5.9) and then prove (Lemma 5.6.5) convergence of the respective approximations $\{x_n\}_{n\in\mathbb{N}}$ of solution x of sweeping process (1.8). In other words, Lemma 5.6.5 states that the change of the variables of Castaing and Monteiro Marques [19, Theorem 4.1] is continuous with respect to time-discretization. Finally, a result by Monteiro Marques [49, p. 15-16] (which is also Proposition 6 in Valadier [61]) is used to prove (Theorem 5.6.1 of section 5.6.6) that the limit of catching-up scheme (5.15)-(5.18) is a solution of (1.8).

5.6.1 An equivalent non-perturbed formulation of the perturbed sweeping process

Recall, that for a BV-continuous function $u : [0,T] \to E$, the *variation measure* $|du|$ (also called *modulus measure*) is defined, for any $D \in \mathcal{B}([0,T])$, as (see Diestel-Uhl [24, Definition 4, p. 2], Recupero [56, §2.4])

$$|du|(D) = \\ = \sup\left\{\sum_{n=1}^{\infty} \|u(D_n)\| : D = \bigcup_{n=1}^{\infty} D_n,\ D_n \in \mathcal{B}([0,T]),\ D_i \cap D_j = \emptyset \text{ if } i \neq j\right\}.$$

For a BV-continuous function $u : [0,T] \to \mathbb{R}$, the differential measure du is always $|du|$-continuous (it follows, for example, from Diestel-Uhl [24, Theorem 1]), i.e., a $|du|$-integrable density $\dfrac{du}{|du|}$ is well defined. Moreover, according to Castaing and Monteiro Marques [19, Theorem 4.1], if x is a solution of the perturbed sweeping process (5.2), then the BV continuous function u defined by

$$u(t) = x(t) + \int_0^t f(\tau, x(\tau))d\tau \tag{5.8}$$

is a solution to the non-perturbed sweeping process

$$-\frac{du}{|du|}(t) \in N_{A+a(t,\lambda)+c(x(t),\lambda)+\int_0^t f(\tau,x(\tau),\lambda)d\tau}(u(t)), \quad |du| - a.e. \ on \ [0,T]. \tag{5.9}$$

Lemma 5.6.1. Assume that $(t, x, \lambda) \mapsto f(t, x, \lambda)$ is Caratheodory in (t, x) with respect to Lebesgue measure and is globally Lipschitz in x with Lipschitz constant independent of $t \in [0,T]$ and $\lambda \in [0,1]$. Then, for any BV continuous $u : [0,T] \to E$, the equation (5.8) admits a unique BV continuous solution $x : [0,T] \to E$.

Lemma 5.6.1 is a direct consequence of Lemma 5.6.5 that we prove below.

Combining Castaing and Monteiro Marques [19, Theorem 4.1] and Lemma 5.6.1, we can formulate the following equivalent definition of the solution of (5.2).

Definition 5.6.1. A BV continuous function x is called a solution of perturbed sweeping process (5.2), if the function u given by (5.8) is a solution of the non-perturbed sweeping process (5.9).

5.6.2 An example

To illustrate the concept of sweeping process for measures, we consider a simple example

$$-\,dx \in N_{[0,1]+a(t)}(x), \quad t \in [0,1], \tag{5.10}$$

where $a(t)$ is a non-decreasing BV-continuous function.

Since $a(t)$ is BV-continuous on $[0,1]$, the variation measure $|da|$ is a possible measure on $[0,1]$, then, by the uniqueness of vector measure (see Section 5.2),

$$da = |da|. \tag{5.11}$$

Case 1: $t \mapsto a(t)$ is absolutely continuous. When the function $a(t)$ is absolutely continuous, the sweeping process (5.10) admits the solution $x(t) = a(t)$ with respect to Lebesgue measure dt. Indeed, according to properties (5.3) and (5.4) of the differential measure da, the Radon-Nykodim derivative $\dfrac{da}{dt}(t)$ computes as

$$\frac{da}{dt}(t) = \lim_{\varepsilon \to 0^+} \frac{da([t+\varepsilon, t])}{dt([t+\varepsilon, t])} = \lim_{\varepsilon \to 0^+} \frac{a(t+\varepsilon) - a(t)}{\varepsilon} = \dot{a}(t), \quad dt - \text{a.e. on } [0,1], \tag{5.12}$$

(in other words, Radon-Nikodym derivative of an absolutely continuous function is the regular derivative). And since $a(t)$ is a non-decreasing function, we conclude that $\dot{a}(t) \geq 0$ $dt-$a.e. on $[0,1]$. To prove that

$$-\frac{da}{dt}(t) \in N_{[0,1]+a(t)}(a(t)), \quad dt - \text{a.e. on } [0,1],$$

(i.e., to prove that the function $a(t)$ and the measure dt form a solution of (5.10) as per Definition (5.2.1)), it remains to observe that

$$N_{[0,1]+a(t)}(a(t)) = (-\infty, 0].$$

Note, when $t \mapsto a(t)$ is absolutely continuous, the differential measure da admits a Radon-Nykodim derivative $\dfrac{da}{|da|}(t)$ and, by (5.11),

$$\frac{da}{|da|}(t) = \lim_{\varepsilon \to 0^+} \frac{da([t+\varepsilon, t])}{|da|([t+\varepsilon, t])} = 1, \quad t \in [0,1]. \tag{5.13}$$

Since $-1 \in (-\infty, 0]$, the function $t \mapsto a(t)$ is a solution of (5.10) also with respect to the measure $|da|$.

Case 2: $t \mapsto a(t)$ is not absolutely continuous. An absolutely continuous function $t \mapsto a(t)$ admitted a Radon-Nykodim derivative with respect to dt because, for absolutely continuous $a(t)$, da is dt-continuous (see Proposition (5.6.1) in the end of this example).

When $t \mapsto a(t)$ is not absolutely continuous, the differential measure da is not necessary dt-continuous. Indeed, consider for example the triadic Cantor function $a(t)$ defined over the triadic Cantor set C as follows (see e.g., Stein-Shakarchi [58, p. 8, 38]). We remove the open interval $I_1 = (1/3, 2/3)$ from $[0,1]$ and then again we remove middle third intervals I_2 from remaining set $[1, 1/3] \cup [2/3, 1]$. By removing middle third from remaining set repeatedly, we get the Cantor set C. Then

$$[0,1] = C \cup \bigcup_{i=1}^{\infty} I_i.$$

The Cantor function is defined continuously and as it takes a constant value on the interval $\bar{I}_i$. By countable additivity of $|da|$,

$$1 = |da|([0,1]) = |da|(C) + \sum_{k=1}^{\infty} |da|(I_k),$$

and, since a is constant on each I_k, $1 = |da|(C) = da(C)$, see (5.11). But, $dt(C) = 0$ (see e.g., Stein-Shakarchi [58, p. 17]), so that da is not dt-continuous (see Diestel-Uhl [24, Theorem 1, p. 10]).

At the same time, the differential measure da of the triadic Cantor function $t \mapsto a(t)$ is $|da|$-continuous by (5.11) and the respective Radon-Nykodim derivative is given by (5.13), so that $t \mapsto a(t)$ is still a solution of (5.10) with respect to the variation measure $|da|$ by the same argument as in case 1.

The next fact is known. It can be obtained from the inverse of the Radon-Nikodym theorem (see Moreau [52, p. 53]). We, however, include a direct proof for completeness.

Proposition 5.6.1. If $a : [0, T] \to \mathbb{R}$ is an absolutely continuous function then da is dt-continuous.

Proof. Fix $\varepsilon > 0$. We have to show that there exists $\delta > 0$ such that

$$dt(D) < \delta \Rightarrow |da(D)| < \varepsilon \text{ for each } D \in \mathcal{B}([0, T]).$$

By absolute continuity of a, we can choose $\delta > 0$ such that for any finite sequence $\{(s_i, t_i)\}_{i=1}^n \subset [0, T]$ of pairwise disjoint sub-intervals, we have

$$\sum_{i=1}^n |t_i - s_i| < 2\delta \Rightarrow \sum_{i=1}^n |a(t_i) - a(s_i)| < \varepsilon.$$

Let now $D \in \mathcal{B}([0, T])$ be such that $dt(D) < \delta$.

Approximate D by a finite sequence $\{(s_i, t_i)\}_{i=1}^n \subset [0, T]$ of pairwise disjoint intervals, so that

$$D \subset \bigcup_{i=1}^n (s_i, t_i) \quad \text{and} \quad dt\left(\bigcup_{i=1}^n (s_i, t_i)\right) < 2\delta.$$

Then we have

$$|da(D)| \leq |da\left(\bigcup_{i=1}^n (s_i, t_i)\right)| \leq \sum_{i=1}^n |a(t_i) - a(s_i)| < \varepsilon.$$

$\square$

5.6.3 The catching-up algorithm

For each fixed $n \in \mathbb{N}$, we partition $[0,T]$ into smaller intervals by the points $\{t_0, t_1, ..., t_n\} \subset [0,T]$ defined by

$$t_0 = 0,\ t_n = T,\ t_{i+1} - t_i = \frac{T}{n}, \quad i \in \overline{1,n}.$$

In what follows, we fix some initial condition

$$x(0) = u(0) = q,$$

where q satisfies

$$q \in A + a(0,\lambda) + c(q,\lambda), \tag{5.14}$$

and use the ideas of Definition 5.6.1 in order to construct pieceiwise-linear functions u_n and x_n (linear on each $[t_i, t_{i+1}]$) that serve as approximations of the solutions u and x of Definition 5.6.1. The construction will be implemented iteratively through the intervals $[t_i, t_{i+1}]$ starting from $i = 0$, and moving towards $i = n-1$.

Denoting

$$u_n(0) = q,\ x_n(0) = q,\ u_i^n = u_n(t_i),\ x_i^n = x_n(t_i), \quad i \in \overline{0,n},$$

we apply the implicit iterative scheme

$$\begin{aligned} u_{i+1}^n &= \operatorname{proj}\left[u_i^n, A + a(t_{i+1},\lambda) + c\left(u_{i+1}^n - \int_0^{t_i} f(\tau, x_n(\tau), \lambda)d\tau, \lambda\right)\right. \\ &\quad \left. + \int_0^{t_i} f(\tau, x_n(\tau),\lambda)d\tau\right], \end{aligned} \tag{5.15}$$

$$x_{i+1}^n = u_{i+1}^n - \int_0^{t_i} f(\tau, x_n(\tau), \lambda)d\tau, \tag{5.16}$$

$$u_n(t) = u_i^n + \frac{t - t_i}{t_{i+1} - t_i}(u_{i+1}^n - u_i^n), \quad t \in [t_i, t_{i+1}], \tag{5.17}$$

$$x_n(t) = x_i^n + \frac{t - t_i}{t_{i+1} - t_i}(x_{i+1}^n - x_i^n), \quad t \in [t_i, t_{i+1}], \tag{5.18}$$

successively from $i = 0$ to $i = n-1$. Next lemma uses the idea of the implicit scheme of Kunze and Monteiro Marques ([40, Lemma 7]) and it proves that for each $i \in \overline{0, n-1}$ we can extend the definition of u_n and x_n from $[0, t_i]$ to $[0, t_{i+1}]$ according to (5.15)-(5.18).

Lemma 5.6.2. Consider a set-valued function

$$C(s_1, s_2, u, \xi) = A + \tilde{a}(s_1, \xi) + \tilde{c}(s_2, u, \xi), \quad s_1, s_2 \in [0, T],\ u \in E,\ \xi \in W,$$

where $A \subset E$ is a nonempty closed convex bounded set, $\tilde{a} : \mathbb{R} \times W \to E$, $\tilde{c} : \mathbb{R} \times E \times W \to E$, and W is a finite dimensional Euclidean space. Assume that

$$\begin{aligned} &\mathrm{var}(\tilde{a}(\cdot, \xi), [s,t]) \le \mathrm{var}(\bar{a}, [s,t]), \quad \xi \in W, \\ &\text{where } \bar{a} : [0,T] \to \mathbb{R} \text{ is a BV continuous function,} \end{aligned}$$

and $(s, \xi) \to \tilde{a}(s, \xi)$ is continuous in $\xi \in W$ uniformly in $s \in [0, T]$. Assume that $(s, u, \xi) \mapsto \tilde{c}(s, u, \xi)$ is continuous in $\xi \in W$ uniformly in $(s, u) \in [0, T] \times E$ and satisfies the Lipschitz condition

$$\begin{aligned} \|\tilde{c}(s, u, \xi) - \tilde{c}(t, v, \xi)\| &\le L_1 |s - t| + L_2 \|u - v\|, \\ &\text{for any } s, t \in [0, T],\ u, v \in E,\ \xi \in W, \end{aligned}$$

with $L_1 > 0$ and $L_2 \in (0, 1)$. Then, for any $\tau_1, \tau_2, s_1, s_2 \in [0, T]$ and any $u \in E$ there exists an unique $v = v(\tau_1, \tau_2, s_1, s_2, u, \xi)$ such that

$$v \in C(\tau_1, \tau_2, v, \xi) \quad \text{and} \quad v = \mathrm{proj}(u, C(\tau_1, \tau_2, v, \xi)). \tag{5.19}$$

Moreover, $v \in C^0([0, T] \times [0, T] \times [0, T] \times [0, T] \times E \times W, E)$. If, in addition,

$$u \in C(s_1, s_2, u, \xi),$$

then

$$\|v - u\| \le \frac{\mathrm{var}(\bar{a}, [s_1, \tau_1]) + L_1 |\tau_2 - s_2|}{1 - L_2}. \tag{5.20}$$

Lemma 5.6.3. Let C be a convex set of E. Then, for any vectors $u, c \in E$,

$$\|\mathrm{proj}(u, C) - \mathrm{proj}(u, C + c)\| \leq \|c\|.$$

Proof. From the definition of projections $v_1 = \mathrm{proj}(u, C)$ and $v_2 = \mathrm{proj}(u, C + c)$ we have (see e.g., Kunze and Monteiro Marques [40, §2])

$$u - v_1 \in N_C(v_1) \quad \text{and} \quad u - v_2 \in N_{C+c}(v_2). \tag{5.21}$$

Since $v_2 - c \in C$ and $v_1 + c \in C + c$, we conclude from (5.21) that

$$\langle u - v_1, v_2 - c - v_1 \rangle \leqslant 0 \quad \text{and} \quad \langle u - v_2, v_1 + c - v_2 \rangle \leqslant 0,$$

or, rearranging the terms,

$$\langle v_1 - u, v_1 - v_2 \rangle \leqslant \langle u - v_1, c \rangle \quad \text{and} \quad \langle u - v_2, v_1 - v_2 \rangle \leqslant \langle v_2 - u, c \rangle .$$

Finally, we add both inequalities together and get

$$\langle v_1 - v_2, v_1 - v_2 \rangle \leqslant \langle v_2 - v_1, c \rangle \leqslant \|v_1 - v_2\| \cdot \|c\|,$$

which implies the statement. $\square$

Proof of Lemma 5.6.2. Step 1. *The existence of* $v(\tau_1, \tau_2, s_1, s_2, u, \xi)$. Define $F \in C^0(E, E)$ as $F(v) = \mathrm{proj}(u, C(\tau_1, \tau_2, v, \xi))$. Using Lemma 5.6.3, we have

$$\begin{aligned}
&\|F(v_1) - F(v_2)\| = \\
&= \|\mathrm{proj}(u, A + \tilde{a}(\tau_1, \xi) + \tilde{c}(\tau_2, v_1, \xi)) - \mathrm{proj}(u, A + \tilde{a}(\tau_1, \xi) + \tilde{c}(\tau_2, v_2, \xi))\| \leq \\
&\leq \|\tilde{c}(\tau_2, v_1, \xi) - \tilde{c}(\tau_2, v_2, \xi)\| \leq L_2 \|v_1 - v_2\|,
\end{aligned} \tag{5.22}$$

so the existence of $v = v(\tau_1, \tau_2, s_1, s_2, u, \xi)$ with the required property (5.19) follows by applying the contraction mapping theorem (see e.g., Rudin [57, Theorem 9.23]).

Step 2. *Continuity of* $v(\tau_1, \tau_2, s_1, s_2, u, \xi)$.

To prove the continuity of v, let $v = v(\tau_1, \tau_2, s_1, s_2, u, \xi)$ and $\bar{v} = v(\bar{\tau}_1, \bar{\tau}_2, \bar{s}_1, \bar{s}_2, \bar{u}, \bar{\xi})$ where $s_1, s_2, \bar{s}_1, \bar{s}_2 \in [0, T]$, $\tau_1, \tau_2, \bar{\tau}_1, \bar{\tau}_2 \in [0, T]$, $\xi, \bar{\xi} \in W$ and $u, \bar{u} \in E$.

First observe that

$$\begin{aligned}
&\|\bar{v} - v\| = \\
&\quad = \|\mathrm{proj}(\bar{u}, A + \tilde{a}(\bar{\tau}_1, \bar{\xi}) + \tilde{c}(\bar{\tau}_2, \bar{v}, \bar{\xi})) - \mathrm{proj}(u, A + \tilde{a}(\tau_1, \xi) + \tilde{c}(\tau_2, v, \xi))\| \\
&\quad \leq \|\mathrm{proj}(\bar{u}, A + \tilde{a}(\bar{\tau}_1, \bar{\xi}) + \tilde{c}(\bar{\tau}_2, \bar{v}, \bar{\xi})) - \mathrm{proj}(u, A + \tilde{a}(\bar{\tau}_1, \bar{\xi}) + \tilde{c}(\bar{\tau}_2, \bar{v}, \bar{\xi}))\| \\
&\quad + \|\mathrm{proj}(u, A + \tilde{a}(\bar{\tau}_1, \bar{\xi}) + \tilde{c}(\bar{\tau}_2, \bar{v}, \bar{\xi})) - \mathrm{proj}(u, A + \tilde{a}(\tau_1, \xi) + \tilde{c}(\tau_2, v, \xi))\|.
\end{aligned}$$

Since for any nonempty, closed, convex set $C \subset E$ and any vectors $\bar{u}, u \in E$, we have (see e.g., Mordukhovich-Nam [50, Proposition 1.79])

$$\|\mathrm{proj}(\bar{u}, C) - \mathrm{proj}(u, C)\| \leq \|\bar{u} - u\|, \tag{5.23}$$

then, using also Lemma 5.6.3, we conclude that

$$\begin{aligned}
\|\bar{v} - v\| &\leq \|\bar{u} - u\| + \left\|\tilde{a}(\bar{\tau}_1, \bar{\xi}) + \tilde{c}(\bar{\tau}_2, \bar{v}, \bar{\xi}) - \tilde{a}(\tau_1, \xi) - \tilde{c}(\tau_2, v, \xi)\right\| \leq \\
&\leq \|\bar{u} - u\| + \left\|\tilde{a}(\bar{\tau}_1, \bar{\xi}) - \tilde{a}(\bar{\tau}_1, \xi)\right\| + \mathrm{var}(\bar{a}, [\tau_1, \overline{\tau}_1]) + \\
&\quad + \|\tilde{c}(\bar{\tau}_2, v, \bar{\xi}) - \tilde{c}(\tau_2, v, \xi)\| + L_1|\bar{\tau}_2 - \tau_2| + L_2\|\bar{v} - v\|,
\end{aligned} \tag{5.24}$$

so that the required continuity of $v(\tau_1, \tau_2, s_1, s_2, u, \xi)$ follows from $0 \leq L_2 < 1$.

Step 3. *Proof of the estimate (5.20).*

Assuming that $u \in C(s_1, s_2, u, \xi)$, we have follow the lines of (5.24) to get

$$\|v - u\| = \|\mathrm{proj}(u, C(\tau_1, \tau_2, v, \xi)) - u\| = \min_{\bar{v} \in C(\tau_1, \tau_2, v, \xi)} \|u - \bar{v}\|.$$

But $C(s_1, s_2, u, \xi) = A + \tilde{a}(s_1, \xi) + \tilde{c}(s_2, u, \xi)$ and $C(\tau_1, \tau_2, v, \xi) = A + \tilde{a}(\tau_1, \xi) + \tilde{c}(\tau_2, v, \xi)$. Therefore,

$$\begin{aligned}
\min_{\bar{v} \in C(\tau_1, \tau_2, v, \xi)} \|u - \bar{v}\| &\leq \|\tilde{a}(s_1, \xi) + \tilde{c}(s_2, u, \xi) - \tilde{a}(\tau_1, \xi) - \tilde{c}(\tau_2, v, \xi)\| \leq \\
&\leq \mathrm{var}(\bar{a}, [s_1, \tau_1]) + L_1|\tau_2 - s_2| + L_2\|u - v\|,
\end{aligned} \tag{5.25}$$

which implies (5.20). The proof of the lemma is complete. $\square$

Remark 5.6.1. *On the validity of Lemma 5.6.2 when $A + c(t, \xi)$ is replaced by a more general term $A(t, \xi)$.*

One can observe that estimate (5.22) holds also in the case where $A + \tilde{a}(t, \xi)$ takes a more general form $A(t, \xi)$. Furthermore, if $d_H(A_1, A_2)$ is the Hausdorff distance between nonempty closed sets $A_1, A_2 \subset E$ and $A(t, \xi)$ satisfies

$$d_H(A(s), A(t)) \leq \mathrm{var}(\bar{a}, [s, t]), \tag{5.26}$$

then (5.25) holds as well since

$$\min_{\bar{v} \in C(\tau_1, \tau_2, v, \xi)} \|u - \bar{v}\| \leq d_H(A(s_1, \xi) + +\tilde{c}(s_2, u, \xi), A(\tau_1, \xi) + \tilde{c}(\tau_2, v, \xi)).$$

To summarize, the existence of $v(\tau_1, \tau_2, s_1, s_2, u, \xi)$ (Step 1) and the estimate (5.20) (Step 3) still hold, if $A + \tilde{a}(t, \xi)$ is replaced by $A(t, \xi)$ satisfying (5.26).
On the other hand Monteiro Marques [47, Proposition 4.7, p. 26] implies that

$$\|\mathrm{proj}(u, C) - \mathrm{proj}(u, D)\| \leq \sqrt{2(\mathrm{dist}(u, C) + \mathrm{dist}(u, D))} \cdot \sqrt{d_H(C, D)}, \tag{5.27}$$

which could potentially help to obtain other versions of Lemma 5.6.2, that we don't pursue here.

Corollary 5.6.1. Assume that condition I) of Theorem 5.5.1 holds. Then, for any (q, λ) satisfying (5.14) the implicit scheme (5.15)-(5.18) is solvable iteratively from $i = 0$ to $i = n-1$ and the respective iterations $x_i^n = x_i^n(q, \lambda)$ and $u_i^n = u_i^n(q, \lambda)$ are continuous in (q, λ) on $E \times [0, 1]$. Moreover,

$$\|u_{i+1}^n(q, \lambda) - u_i^n(q, \lambda)\| \leq \frac{\mathrm{var}(\bar{a}, [t_i, t_{i+1}]) + L_1 T/n}{1 - L_2}, \quad i \in \overline{0, n-1},$$

where $L_1 > 0$ and $L_2 \in (0, 1)$.

Proof. Let $\xi = ((\xi_1, \xi_2, \cdots, \xi_{n+1}), \xi_{n+2}) \in E^{n+1} \times \mathbb{R}$ be defined as

$$\xi_i = x_{i-1}^n, \quad i \in \overline{1, n+1} \quad , \quad \xi_{n+2} = \lambda.$$

Therefore, the rule (5.18) defines a function $\Psi : E^{n+1} \times \mathbb{R} \to C^0([0,T], E)$ that relates $\xi \in E^{n+1} \times \mathbb{R}$ to a piecewise linear function $x_n(t)$ defined on $[0,T]$. The statement of the Corollary 1 now follows by applying Lemma 5.6.2 with

$$\tilde{c}(s,u,\xi) = \left(u - \int_0^s f(\tau, \Psi(\xi)(\tau), \xi_{n+2}) d\tau, \xi_{n+2} \right) + \int_0^s f(\tau, \Psi(\xi)(\tau), \xi_{n+2}) d\tau,$$
$$\tilde{a}(s,\xi) = a(s, \xi_{n+2}).$$

The proof of the corollary is complete. $\square$

5.6.4 The Poincare map associated to the catching-up algorithm

Even though we cannot ensure the existence of a Poincare map for sweeping process (5.2), we can associate the following Poincare map

$$P^{\lambda,n}(q) = x_n(T)$$

to the approximations x_n of the catching-up algorithm (5.15)-(5.18). Corollary 5.6.1 allows to formulate the following property of the map $P^{\lambda,n}$.

Corollary 5.6.2. Assume that condition I) of Theorem 5.5.1 holds. Consider an open bounded set $Q \subset E$. Then, for each fixed $\lambda \in [0,1]$ and $n \in \mathbb{N}$, the Poincare map $q \mapsto P^{\lambda,n}(q)$ is continuous on $\overline{Q}$.

5.6.5 Convergence of the catching-up algorithm

Let $(u_n(t,q,\lambda), x_n(t,q,\lambda))$ be the solution $(u_n(t), x_n(t))$ of the catching-up algorithm (5.15)-(5.18) with the parameter $\lambda \in [0,1]$ and the initial condition $u_n(0) = x_n(0) = q$.

Lemma 5.6.4. Assume that condition I) of Theorem 5.5.1 holds. Consider a sequence $(\lambda_n, q_n) \to (\lambda_0, q_0)$ as $n \to \infty$ of $[0,1] \times E$ satisfying (5.14) for each $n \in \mathbb{N}$. Then, there exists a subsequence $\{n_k\}_{k\in\mathbb{N}}$ such that $\{u_{n_k}(t, q_{n_k}, \lambda_{n_k})\}_{k\in\mathbb{N}}$ converges as $k \to \infty$ uniformly in $t \in [0, T]$.

Proof. Step 1. *Boundedness of* $\{u_n(t, q_n, \lambda_n)\}_{n\in\mathbb{N}}$. Let u_i^n, $i \in \overline{0, n}$, be the approximations given by (5.15)-(5.18) with $q = q_n$ and $\lambda = \lambda_n$. By Corollary 5.6.1,

$$\|u_n(t, q_n, \lambda_n)\| \;\; \le \;\; \|q_n\| + \frac{1}{(1 - L_2)} \left(\mathrm{var}(\bar{a}, [0, T]) + L_1 T\right),$$

so the sequence $\{u_n(t, q_n, \lambda_n)\}_{n\in\mathbb{N}}$ is bounded uniformly on $[0, T]$.

Step 2. *Equicontinuity of* $\{u_n(t, q_n, \lambda_n)\}_{n\in\mathbb{N}}$. Fix $\varepsilon > 0$. Since $\mathrm{var}(\bar{a}, [s, t]) \to 0$ as $|s - t| \to 0$ (see e.g., Lojasiewicz [45, Theorem 1.3.4]), we can choose $\delta_1 > 0$ such that

$$\frac{\mathrm{var}(\bar{a}, [s, t]) + L_1(t - s)}{1 - L_2} < \frac{\varepsilon}{3}, \quad \text{for all } 0 \le s \le t \le T \text{ with } t - s < \delta_1. \tag{5.28}$$

Fix some $0 \le s \le t \le T$ satisfying $t - s < \delta_1$ and denote by $i_s, i_t \in \overline{0, n-1}$ such indexes that

$$s \in [t_{i_s}, t_{i_s+1}], \quad t \in [t_{i_t}, t_{i_t+1}].$$

Then we can estimate $\|u_n(t) - u_n(s)\|$ as follows:

$$\begin{aligned}
&\|u_n(t) - u_n(s)\| \le \\
&\quad \le \|u_n(s) - u_n(t_{i_s+1})\| + \|u_n(t_{i_s+1}) - u_n(t_{i_t})\| + \|u_n(t_{i_t}) - u_n(t)\| \le \\
&\quad \le \mathrm{var}(u_n, [t_{i_s}, t_{i_s+1}]) + \mathrm{var}(u_n, [t_{i_s+1}, t_{i_t}]) + \mathrm{var}(u_n, [t_{i_t}, t_{i_t+1}]).
\end{aligned}$$

The second term is smaller than $\varepsilon/3$ by (5.28) right away. Assuming that $n \ge T/\delta_1$, the property (5.28) ensures that first and third terms are each smaller than $\varepsilon/3$ as well. So we proved that

$$\|u_n(t) - u_n(s)\| < \varepsilon, \quad \text{for all } 0 \le s \le t \le T \text{ with } t - s < \delta_1, \text{ and } n \ge T/\delta_1.$$

Since there is only a finite number of $n \in \mathbb{N}$ with $n < T/\delta_1$, we can find $\delta_2 > 0$ such that

$$\|u_n(t) - u_n(s)\| < \varepsilon, \quad \text{for all } 0 \le s \le t \le T \text{ with } t - s < \delta_2, \text{ and } n < T/\delta_1.$$

Letting $\delta = \min\{\delta_1, \delta_2\}$, we finally obtain

$$\|u_n(t) - u_n(s)\| < \varepsilon, \quad \text{for all } 0 \le s \le t \le T \text{ with } t - s < \delta, \text{ and } n \in \mathbb{N}.$$

The conclusion of the Lemma now follows by applying the Arzela-Ascoli theorem (see e.g., Rudin [57, Theorem 7.25]). □

Remark 5.6.2. Establishing the existence of a converging subsequence $\{x_{n_k}(t, q_{n_k}, \lambda_{n_k})\}_{k\in\mathbb{N}}$ needs more work compared to what we did in the proof of Lemma 5.6.4 because the direct corollary of (5.16)

$$x_{i+1}^n - x_i^n = u_{i+1}^n - u_i^n + \int_{t_{i-1}}^{t_i} f(\tau, x_n(\tau), \lambda) d\tau$$

doesn't imply uniform boundedness of $x_n(t, q_n, \lambda_n)$, $n \in \mathbb{N}$, directly.

To prove the convergence of $\{x_{n_k}(t, q_{n_k}, \lambda_{n_k})\}_{k\in\mathbb{N}}$ we will now extend the discrete map (5.17) to such an operator $F_n : C([0,T], E) \to C([0,T], E)$ whose fixed point is exactly $t \mapsto x_n(t, q_n, \lambda_n)$. The convergence of x_{n_k} will then follow from the continuity of F_n in n at $n = \infty$.

Let us define $P_n : C([0,T], E) \to E^{n+1}$, $l^- : E^{n+1} \to E^{n+1}$ and $Q_n : E^{n+1} \to C([0,T], E^{n+1})$ as

$$P_n(x) = \left(x(0), x\left(\frac{T}{n}\right), ..., x\left((n-1)\frac{T}{n}\right), x(T)\right), \quad x \in C([0,T], E),$$

$$[l^-(y)]_1 = 0 \,, \qquad [l^-(y)]_i = y_{i-1} \,, \quad i \in \overline{2, n+1} \,, \; y \in E^{n+1} \,,$$

$$Q_n(y)(t) = \frac{t - t_{i-1}}{1/n} y_{i+1} + \frac{t_i - t}{1/n} y_i \,, \quad y \in E^{n+1} \,, \; t \in [t_{i-1}, t_i), \; i \in \overline{1, n},$$

$$Q_n(y)(t_n) = y_{n+1} \,, \quad y \in E^{n+1}, \text{ since } t_n = T.$$

For a fixed $\lambda \in [0,1]$ and a continuous function $u : [0,T] \to E$, we introduce a continuous extension of (5.17) as

$$\begin{aligned} (F_n x)(t) &= (Q_n P_n u)(t) - (Q_n l^- P_n J)(t), \quad t \in [0,T], \\ & \text{where } J(t) = \int_0^t f(\tau, x(\tau), \lambda) d\tau. \end{aligned} \tag{5.29}$$

Then, for $x \in C([0,T],E)$ satisfying $x = F_n x$, one has

$$\begin{aligned} x(0) &= (Q_n P_n u)(0) - \big(Q_n l^- P_n J\big)(0) = [P_n u]_1 - [l^- P_n J]_1 = u(0) - 0, \\ x(t_1) &= [P_n u]_2 - [l^- P_n J]_2 = u(t_1) - [P_n J]_1 = u(t_1) - J(0) = u(t_1), \\ x(t_2) &= u(t_2) - J(t_1), \\ & \dots \\ x(t_n) &= u(t_n) - J(t_{n-1}). \end{aligned}$$

Therefore, if u_n and x_n are given by (5.15)-(5.18), then, letting $u = u_n$ in (5.29), the fixed point x of F_n verifies $x(t_i) = x_n(t_i)$, $i \in \overline{0,n}$. And, since the function $t \mapsto (F_n x)(t)$ is linear on $[t_i, t_{i+1}]$, $i \in \overline{0, n-1}$, we conclude $x_n = x$. In other words, if u in (5.29) is given by $u = u_n$, then x_n is the unique fixed point of F_n.

Lemma 5.6.5. Assume that the conditions of Lemma 5.6.1 hold. Then, there exists $\alpha > 0$ and $L \in (0,1)$ such that

$$\|F_n(x_1) - F_n(x_2)\|^* \le L\|x_1 - x_2\|^*, \quad n \in \mathbb{N},$$

for any $x_1, x_2, u \in C([0,T],E)$, $\lambda \in [0,1]$, and

$$\|x\|^* = \max_{t \in [0,T]} e^{-\alpha t}\|x(t)\|.$$

Moreover, for each $x, u \in C([0,T],E)$, and $\lambda \in [0,1]$, one has

$$\lim_{n\to\infty} \|F_n(x) - F(x)\| = 0, \quad \text{where } F(x)(t) = u(t) - \int_0^t f(\tau, x(\tau), \lambda) d\tau,$$

where $\|\cdot\|$ is the max-norm on $[0,T]$ and F is a contraction in the norm $\|\cdot\|^*$.

Proof. Step 1. Using the definition of Q_n, l^-, and P_n, we have

$$(Q_n l^- P_n J)(t_{i-1}) = [l^- P^n J]_i = [P_n J]_{i-1} = J(t_{i-2}), \quad i \in \overline{2, n+1}.$$

So that

$$(F_n x)(t_i) = u(t_i) - J(t_{i-1}).$$

Fix $i \in \overline{1, n-1}$ and choose any $t \in [t_i, t_{i+1}]$. Then,

$$\begin{aligned}
&\|F_n(x_1)(t) - F_n(x_2)(t)\| \leq \\
&\quad \leq \max\left\{\|F_n(x_1)(t_i) - F_n(x_2)(t_i)\|, \|F_n(x_1)(t_{i+1}) - F_n(x_2)(t_{i+1})\|\right\} = \\
&\quad = \max\left\{\left\|\int_0^{t_{i-1}} f(\tau, x_1(\tau), \lambda)d\tau - \int_0^{t_{i-1}} f(\tau, x_2(\tau), \lambda)d\tau\right\|,\right. \\
&\qquad\qquad \left.\left\|\int_0^{t_i} f(\tau, x_1(\tau), \lambda)d\tau - \int_0^{t_i} f(\tau, x_2(\tau), \lambda)d\tau\right\|\right\} \leq \\
&\quad \leq \bar{L}\int_0^{t_i} \|x_1(\tau) - x_2(\tau)\|d\tau \leq \bar{L}\int_0^{t_i} e^{\alpha\tau}\|x_1 - x_2\|^* d\tau,
\end{aligned}$$

where $\bar{L} > 0$ is the global Lipschitz constant of $x \mapsto f(t, x, \lambda)$ and $\alpha > 0$ is an arbitrary constant. Therefore,

$$e^{-\alpha t}\|F_n(x_1)(t) - F_n(x_2)(t)\| \leq \frac{\bar{L}}{\alpha}\left(e^{\alpha(t_i - t)} - e^{-\alpha t}\right)\|x_1 - x_2\|^* \leq \frac{\bar{L}}{\alpha}\|x_1 - x_2\|^*,$$

which holds for any $t \in [0, T]$. The case $t \in [0, t_1]$ can be considered along the same lines. This proves the contraction part of the lemma.

Step 2. To prove the convergence part, fix $i \in \overline{1, n-1}$ again and consider $t \in [t_i, t_{i+1}]$. Since $(Q_n P_n u)(t_i) = u(t_i)$, we have

$$\begin{aligned}
\|(Q_n P_n u)(t) - u(t)\| &\leq \|(Q_n P_n u)(t) - (Q_n P_n u)(t_i)\| + \|u(t) - u(t_i)\| \leq \\
&\leq \|u(t_{i+1}) - u(t_i)\| + \|u(t) - u(t_i)\|,
\end{aligned}$$

so that the convergence of $(Q_n P_n u)(t)$ to $u(t)$ as $n \to \infty$ follows from continuity of u. The convergence of $(Q_n l^- P_n J)(t)$ follows same lines.

Indeed, since $(Q_n l^- P_n J)(t_{i+1}) = J(t_i)$, one has

$$\begin{aligned}
&\|(Q_n l^- P_n J)(t) - J(t)\| \leq \\
&\quad \leq \|(Q_n l^- P_n J)(t) - (Q_n l^- P_n J)(t_{i+1})\| + \|J(t) - J(t_i)\| \leq \\
&\quad \leq \|J(t_{i-1}) - J(t_i)\| + \|J(t) - J(t_i)\|
\end{aligned}$$

and the convergence of $(Q_n l^- P_n J)(t)$ to $J(t)$ as $n \to \infty$ follows from continuity of $J(t)$. The proof of the lemma is complete. □

Corollary 5.6.3. Assume that condition I) of Theorem 5.5.1 holds. Let $\{n_k\}_{k\in\mathbb{N}}$ be the subsequence given by Lemma 5.6.4 (which ensures the convergence of $\{u_{n_k}(t, q_{n_k}, \lambda_{n_k})\}_{k\in\mathbb{N}}$). Consider the limit

$$u(t) = \lim_{k\to\infty} u_{n_k}(t, q_{n_k}, \lambda_{n_k}).$$

Let $x(t)$ be the solution of the respective equation (5.8) (which exists according to Lemma 5.6.1). Then $\{x_{n_k}(t, q_{n_k}, \lambda_{n_k})\}_{k\in\mathbb{N}}$ converges uniformly in $t \in [0, T]$, and

$$x(t) = \lim_{k\to\infty} x_{n_k}(t, q_{n_k}, \lambda_{n_k}). \tag{5.30}$$

Proof. The conclusion follows from the inequality

$$\begin{aligned}
\|x - x_n\|^* &= \|F(x) - F_n(x_n)\|^* \leq \|F(x) - F_n(x)\|^* + \|F_n(x) - F_n(x_n)\|^* \leq \\
&\leq \|F(x) - F_n(x)\|^* + L\|x - x_n\|^*,
\end{aligned}$$

where $L \in (0, 1)$ is given by Lemma 5.6.5. □

5.6.6 Verifying that the limit of the catching-up algorithm is indeed a solution

Theorem 5.6.1. Let the conditions of Corollary 5.6.3 hold and let $u(t)$ and $x(t)$ be as given by this corollary. Then, $u(t)$ is a solution of sweeping process (5.9) with the parameters $x(t)$, $\lambda = \lim_{k\to\infty} \lambda_{n_k}$, and the initial condition $u(0) = \lim_{k\to\infty} q_{n_k}$. Accordingly, by Definition 5.6.1, $x(t)$ is a solution of perturbed sweeping process (5.2).

Proof. Let $\phi(t)$, $t \in [0,T]$, be an arbitrary continuous selector of the moving set of (5.9), i.e.,

$$\phi(t) \in A + a(t,\lambda) + c(x(t),\lambda) + \int_0^t f(\tau, x(\tau), \lambda)d\tau, \quad t \in [0,T].$$

According to Monteiro Marques [49, p. 15-16] (see also Valadier [61, Proposition 6]) it is sufficient to prove that

$$\int_s^t \langle \phi(\tau), du(\tau)\rangle \geq \frac{1}{2}\left(\|u(t)\|^2 - \|u(s)\|^2\right), \quad 0 \leq s \leq t \leq T, \tag{5.31}$$

which we now establish using the ideas of Kunze and Monteiro Marques [40].

Without loss of generality we will assume that $\{n_k\}_{k\in\mathbb{N}} = \mathbb{N}$, and replace n_k, $k \in \mathbb{N}$, by n, $n \in \mathbb{N}$ in the formulation of the theorem. Fix $t > 0$ and select $i \in \overline{0, n-1}$ such that $t \in [t_i, t_{i+1}]$. Introduce $\hat{c}_n(t)$ as

$$\hat{c}_n(t) = \mathrm{proj}\left(\phi(t), A + a(t_{i+1}, \lambda_n) + c(x_{i+1}^n, \lambda_n) + \int_0^{t_i} f(\tau, x_n(\tau), \lambda_n)d\tau\right).$$

Then, by (5.15) and by convexity of A, we have (see e.g., Kunze and Monteiro Marques [40, formula (4)])

$$\langle u_n(t_{i+1}) - u_n(t_i), u_n(t_{i+1}) - \hat{c}_n(t)\rangle \leq 0, \quad t \in [t_i, t_{i+1}],$$

from where

$$\langle u_n(t_{i+1}) - u_n(t_i), u_n(t) - \hat{c}_n(t)\rangle \leq$$
$$\leq \langle u_n(t_{i+1}) - u_n(t_i), u_n(t) - u_n(t_{i+1})\rangle \leq \|u_n(t_{i+1}) - u_n(t)\|^2,$$

or

$$\langle u_n(t_{i+1}) - u_n(t_i), \hat{c}_n(t)\rangle \geq -\|u_n(t_{i+1}) - u_n(t_i)\|^2 + \langle u_n(t_{i+1}) - u_n(t_i), u_n(t)\rangle,$$

for any $t \in [t_i, t_{i+1}]$. Using the linearity of u_n on $[t_i, t_{i+1}]$, we conclude

$$\langle \hat{c}_n(t), u_n(\bar{t}_{i+1}) - u_n(\bar{t}_i)\rangle \geq$$
$$\geq \langle u_n(t), u_n(\bar{t}_{i+1}) - u_n(\bar{t}_i)\rangle - \langle u_n(\bar{t}_{i+1}) - u_n(\bar{t}_i)), (u_n(t_{i+1}) - u_n(t_i))\rangle,$$

for any $t_i \le \bar{t}_i \le t \le \bar{t}_{i+1} \le t_{i+1}$. Therefore, denoting $\tau_{j,k} = \bar{t}_i + \left(j + \frac{1}{2}\right) \frac{\bar{t}_{i+1} - \bar{t}_i}{k}$ for $j \in \{0, 1, \cdots, k-1\}$, one has (same approach is used in part (ii) of the proof of Monteiro Marques [47, Theorem 2.1, p. 12, second formula from below])

$$\begin{aligned}
&\int_{\bar{t}_i}^{\bar{t}_{i+1}} \langle \hat{c}_n(\tau), du_n(d\tau)\rangle = \\
&= \lim_{k\to\infty} \sum_{j=0}^{k-1} \left\langle \hat{c}_n\left(\tau_{j,k}\right), u_n\left(\bar{t}_i + (j+1)\frac{\bar{t}_{i+1} - \bar{t}_i}{k}\right) - u_n\left(\bar{t}_i + j\frac{\bar{t}_{i+1} - \bar{t}_i}{k}\right)\right\rangle \ge \\
&\ge \lim_{k\to\infty} \sum_{j=0}^{k-1} \left\langle u_n\left(\tau_{j,k}\right), u_n\left(\bar{t}_i + (j+1)\frac{\bar{t}_{i+1} - \bar{t}_i}{k}\right) - u_n\left(\bar{t}_i + j\frac{\bar{t}_{i+1} - \bar{t}_i}{k}\right)\right\rangle + R_n,
\end{aligned}$$

where the reminder R_n is given by

$$\begin{aligned}
R_n = -\lim_{k\to\infty} \sum_{j=0}^{k-1} \Bigg\langle u_n\left(\bar{t}_i + (j+1)\frac{\bar{t}_{i+1} - \bar{t}_i}{k}\right) - \\
- u_n\left(\bar{t}_i + j\frac{\bar{t}_{i+1} - \bar{t}_i}{k}\right), u_n(t_{i+1}) - u_n(t_i)\Bigg\rangle = \\
= -\left\langle u_n(\bar{t}_{i+1}) - u_n(\bar{t}_i), u_n(t_{i+1}) - u_n(t_i)\right\rangle.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\int_{\bar{t}_i}^{\bar{t}_{i+1}} \langle \hat{c}_n(\tau), du_n(d\tau)\rangle = \\
&= \lim_{k\to\infty} \sum_{j=0}^{k-1} \left\langle u_n(\tau_{j,k}), u_n'(\tau_{j,k})\frac{\bar{t}_{i+1} - \bar{t}_i}{k}\right\rangle + R_n = \\
&= \frac{1}{2}\lim_{k\to\infty} \sum_{j=0}^{k-1} \left(\frac{d}{d\tau}\|u_n(\tau)\|^2\right)\Bigg|_{\tau=\tau_{j,k}} \cdot \frac{\bar{t}_{i+1} - \bar{t}_i}{k} + R_n = \\
&= \frac{1}{2}\int_{\bar{t}_i}^{\bar{t}_{i+1}} \frac{d}{d\tau}\|u_n(\tau)\|^2 d\tau + R_n = \frac{1}{2}\left(\|u_n(\bar{t}_{i+1})\|^2 - \|u_n(\bar{t}_i)\|^2\right) + R_n.
\end{aligned}$$

This result can now be used to estimate the required integral (5.31) as follows

$$\int_s^t \langle \hat{c}_n(\tau), du_n(d\tau)\rangle \ge \frac{1}{2}\left(\|u_n(t)\|^2 - \|u_n(s)\|^2\right) + R, \tag{5.32}$$

where

$$|R| \le \mathrm{var}(u_n, [s,t]) \cdot \max_{i \in \overline{0,n-1}} \|u_n(t_{i+1}) - u_n(t_i)\|.$$

But according to Corollary 5.6.1,

$$\operatorname{var}(u_n, [s,t]) \le \frac{\operatorname{var}(\bar{a}, [s,t])}{1-L_2} + \frac{L_1|t-s|}{1-L_2}.$$

Therefore, the desired statement (5.31) follows from (5.32) by passing to the limit as $n \to \infty$ (the passage to the limit is valid e.g., by Monteiro Marques [47, Theorem 2.1(ii)-(iii)] combined with formula (26) of p. 7 of the same book).

The proof of the theorem is complete. □

5.6.7 Proof of Theorem 5.3.1 (sweeping process without a parameter)

Theorem 5.3.1 is a direct consequence of Theorem 5.6.1. One just view sweeping process (5.1) as sweeping process (5.2) with $\lambda = 0$.

Remark 5.6.3. Using Remark 5.6.1, Theorem 5.3.1 can be directly extended to sweeping processes of the form

$$-\,dx \in N_{A(t,\lambda)+c(x,\lambda)}(x) + f(t,x,\lambda)dt, \quad x \in E,\ \lambda \in \mathbb{R}, \tag{5.33}$$

where A is a set-valued function with nonempty closed convex bounded values that satisfies the property

$$\begin{aligned} & d_H(A(s,\lambda), A(t,\lambda)) \le \operatorname{var}(\bar{a},[s,t]), \quad \lambda \in [0,1], \\ & \text{where } \bar{a} : [0,T] \to \mathbb{R} \text{ is a BV continuous function.} \end{aligned} \tag{5.34}$$

5.6.8 Proofs of Theorems 5.5.1 and 5.5.2 (sweeping process with a parameter)

Proof of Theorem 5.5.1. Step 1. First we prove that there exists $N > 0$ such that $d(I - P^{\lambda,n} \circ V^{\lambda,n}, Q)$ is defined for $n \ge N$ and $\lambda \in [0, \lambda_1]$. Assuming the contrary, we get a sequence $n_k \to \infty$, $\lambda_k \to \lambda_0 \in [0, \lambda_1]$, and a converging sequence $\{q_k\}_{k\in\mathbb{N}} \subset \partial Q$ such that

$$P^{\lambda_k,n_k} \circ V^{\lambda_k,n_k}(q_k) = q_k, \quad k \in \mathbb{N}. \tag{5.35}$$

Applying Lemma 5.6.4, Corollary 5.6.3, and Theorem 5.6.1 we conclude that $q_0 = \lim_{k\to\infty} q_k \in \partial Q$ is the initial condition of the T-periodic solution (5.30) of sweeping process (5.2) with $\lambda = \lambda_0$, which contradicts conditions III) of Theorem 5.5.1.

The conclusion of Step 1, in particular, implies that

$$d(I - P^{\lambda,n} \circ V^{\lambda,n}, Q) = d(I - P^{0,n} \circ V^{0,n}, Q), \quad n \geq N, \ \lambda \in [0, \lambda_1].$$

Step 2. Here we use assumption II (uniqueness) of Theorem 5.5.1 to conclude that

$$P^{0,n} \circ V^{0,n}(q) \to P^0 \circ V^0(q), \quad \text{as } n \to \infty,$$

uniformly with respect to $q \in \overline{Q}$. Thus, we can diminish $N > 0$ in such a way that $d(I - P^{0,n} \circ V^{0,n}, Q) = d(I - P^0 \circ V^0, Q)$, $n \geq N$, which gives

$$d(I - P^{\lambda,n} \circ V^{\lambda,n}, Q) \neq 0, \quad n \geq N, \ \lambda \in [0, \lambda_1].$$

Therefore, for each $\lambda \in [0, \lambda_1]$ there exists $q_n \in Q$ such that the approximations $\{x_n(\cdot, q_n, \lambda)\}_{n\geq N}$ are T-periodic, so this sequence has a convergent subsequence which converges to a T-periodic solution of (5.2) with initial condition $q = \lim_{n_k\to\infty} q_{n_k}$ as $n \to \infty$ according to Corollary 5.6.3.

The proof of the theorem is complete. □

The proof of Theorem 5.5.2 follows the lines of the proof of Theorem 5.5.1. The only difference is in the beginning of Step 1, which now proves the existence of both $N > 0$ and $\lambda_1 \in (0, 1]$ such that $d(I - P^{\lambda,n} \circ V^{\lambda,n}, Q)$ is defined for $n \geq N$ and $\lambda \in [0, \lambda_1]$. Assuming the contrary, we get a sequence $n_k \to \infty$, $\lambda_k \to 0 \in [0, 1]$, and a converging sequence $\{q_k\}_{k\in\mathbb{N}} \subset \partial Q$ such that (5.35) holds, that leads to the existence of a T-periodic solution to sweeping process (5.2) with $\lambda = 0$, contradicting condition II) of Theorem 5.5.1. The rest of the proof of Theorem 5.5.2 follows the proof of Theorem 5.5.1 just literally.

5.7 Proof of the theorem on the global existence of periodic solutions

To prove Theorem 5.4.1 we will use the following well-known result (see e.g., Krasnoselskii-Zabreiko [36, Theorem 6.2]):

Theorem 5.7.1. Let $\bar{P} : E \to E$ be a continuous map and let $Q \subset E$ be an open bounded convex set. If $\bar{P}(Q) \subset \overline{Q}$ and if $\bar{P}$ doesn't have fixed points on ∂Q, then

$$d(I - \bar{P}, Q) = 1.$$

Proof of Theorem 5.4.1. Let Ω_1 be the 1-neighborhood of Ω. Since Ω is convex, then Ω_1 is convex as well. We will view sweeping process (5.1) as sweeping process (5.2) with $\lambda = 0$. So we consider the map

$$\overline{P}^{0,n}(x) = P^{0,n}(V(x)),$$

where $P^{0,n}$ is as introduced in Section 5.6.4 and V is as introduced in Section 5.3. We claim that

$$\overline{P}^{0,n}(\Omega_1) \subset \Omega, \quad \text{for all } n \in \mathbb{N}. \tag{5.36}$$

We have $V(x) \in \Omega$ by the definition of the map V. Then, according to the catching-up scheme (5.15)-(5.18), we have that

$$x_{i+1}^n \in A + a(t_{i+1}, 0) + c(x_{i+1}^n), \quad \text{i.e., } x_{i+1}^n \in \Omega_{t_{i+1}},\ i \in \overline{0, n-1},$$

and so $x_n(T) \in \Omega_T$, which implies (5.36).

Using the continuity of $P^{0,n}$ (Corollary 5.6.2) and V (Lemma 5.6.2) along with Theorem 5.7.1, we get the existence of $q_n \in \Omega$ such that $\overline{P}^{0,n}(q_n) = q_n$, which implies

$$P^{0,n}(q_n) = q_n, \quad n \in \mathbb{N},$$

because $V(q_n) \in \Omega$. In other words, we have $x_n(T, q_n, 0) = x_n(0, q_n, 0)$ for all $n \in \mathbb{N}$. Now, Theorem 5.6.1 applied with $\lambda_n = 0$, implies the existence of a convergent subsequence $\{x_{n_k}(t, q_{n_k}, 0)\}$ whose limit $x(t)$ is solution of (5.1) with the required T-periodicity property (5.6). The proof is complete. □

5.8 Existence of periodic solutions in the neighborhood of a boundary equilibrium (the theorem and its proof)

This section is devoted to establishing conditions for the validity of $d(I - P^{\lambda,n}, Q) \neq 0$ at $\lambda = 0$ in a neighborhood Q of a switched boundary equilibrium x_0. Specifically, we assume that, for $\lambda = 0$ sweeping process (1.8) takes the form

$$-\dot{x} \in N_A(x) + f_0(x), \qquad x \in E, \tag{5.37}$$

and discover conditions for asymptotic stability of $x_0 \in \partial A$. In particular, in section 5.8 we extend the two-dimensional approach of Chapter 4, Makarenkov and Niwanthi Wadippuli [46] and derive a differential equation of sliding motion along ∂A, for which x_0 is a regular equilibrium whose stability can be investigated (Theorem 5.8.2) over the eigenvalues of the respective linearization. Assuming that the real parts of these eigenvalues are negative we conclude that $d(I - P^0 \circ V_0, Q) = 1$ and establish (Theorem 5.8.3) the existence of T-periodic solutions near x_0 for all BV-continuous state-dependent sweeping processes (1.8) that approaches (5.37) when $\lambda \to 0$.

This section uses the following extension of Theorem 5.7.1 (see e.g., Krasnoselskii-Zabreiko [36, Theorem 31.1]):

Theorem 5.8.1. Let $\bar{P} : E \to E$ be a continuous map and let $Q \subset E$ be an open bounded set. If $(\bar{P})^m$ maps Q strictly into itself for all $m \in \mathbb{N}$ sufficiently large, then

$$d(I - \bar{P}, Q) = 1.$$

The main assumption of this section is that sweeping processes (5.2) reduces to

$$-\dot{x} \in N_A(x) + f_0(x), \qquad x \in E, \tag{5.38}$$

when $\lambda = 0$ and that (5.38) posses a switched equilibrium on the boundary ∂A (as was earlier introduced in Kamenskii-Makarenkov [31] in 2d). To introduce the definition of a switched boundary equilibrium $x_0 \in \partial A$, we assume that in some neighborhood $Q \subset \mathbb{R}^n$ of x_0 the boundary ∂A is smooth and can be described as

$$\partial A \cap Q = \{x \in Q : H(x) = 0\}, \quad \text{where } H \in C^1(\mathbb{R}^n, \mathbb{R}).$$

Definition 5.8.1. A point $x_0 \in \partial A$ is a switched boundary equilibrium of sweeping process (5.38), if

$$H(x) > 0, \quad \text{for all } x \in Q \backslash A,$$

and

$$H'(x_0) = \alpha f(x_0) \quad \text{for some } \alpha < 0.$$

As the definition says, x_0 is not an equilibrium of f, however the next two lemmas imply that the solution of (5.38) with the initial condition at x_0 don't leave x_0.
If x_0 is a switched equilibrium, then Q can be considered so small that

$$\langle f(x), H'(x)\rangle < 0, \quad \text{for all } x \in \partial A \cap Q. \tag{5.39}$$

The next lemma claims that $\partial A \cap Q$ is a sliding region for the sweeping process (5.38).

Lemma 5.8.1. Let $x_0 \in \partial A$ be a switched equilibrium of (5.38) and let $Q \subset E$ be such a neighborhood of x_0 that (5.39) holds. Consider a solution x of (5.38) with an initial condition $x_0 \in \partial A \cap Q$. Let $t_1 > 0$ be such that $x(t) \in Q$ for all $t \in [0, t_1]$. Then $x(t) \in \partial A$ for all $t \in [0, t_1]$.

Proof. Let us assume, by contradiction, that there exists $t_{escape} \in [0, t_1]$ where $x(t)$ escapes from ∂A, i.e.,

$$t_{escape} = \max\{t_0 \geq 0 : x(t) \in Q, H(x(t)) = 0,\ t \in [0, t_0]\} < t_1.$$

By the definition of t_{escape}, for any $\delta > 0$ there exist $t_\delta \in [t_{escape}, t_{escape} + \delta]$ such that $H(x(t)) < 0$ for each $t \in (t_{escape}, t_\delta]$. Since, the solution $x(t)$ satisfies $\dot{x}(t) = -f_0(x(t))$ on $(t_{escape}, t_\delta]$, by the Mean-Value Theorem

$$H(x(t_\delta)) - H(x(t_{escape}) = -H'(x(t_\delta^*))f_0(x(t_\delta^*))(t_\delta - t_{escape}),$$

for some $t_\delta^* \in (t_{escape}, t_\delta)$. This yields

$$H'(x(t_{escape}))f_0(x(t_{escape})) \geq 0,$$

as $\delta \to 0$, contradicting (5.39).

The proof of the lemma is complete. $\square$

As it happens in the theory of Filippov systems (see Filippov [28]), the dynamics of (5.38) in the sliding region is described by a smooth differential equation. Indeed, let us introduce the differential equation

$$\begin{aligned} & -\dot{x} = \bar{f}(x), \\ & \text{where } \bar{f}(x) = f_0(x) - \pi_{H'(x)}(f_0(x)) \text{ and } \pi_L(\xi) = \frac{1}{\|L\|^2}\langle \xi, L\rangle L. \end{aligned} \tag{5.40}$$

Next lemma says that (5.40) is the equation of sliding motion for sweeping process (5.38) in the neighborhood of switched equilibrium $x_0 \in \partial A_0$.

Lemma 5.8.2. Let the conditions of Lemma 5.8.1 hold and let $x(t)$ be the sliding solution $x(t)$, $t \in [0, t_1]$, of sweeping process (5.38) as introduced in Lemma 5.8.1. Then $x(t)$ is a solution of (5.40) on $[0, t_1]$.

Proof. Fix $t \in [0, t_1]$ such that $\dot{x}(t)$ exists. Then, from (5.38),

$$-\dot{x}(t) = \alpha H'(x(t)) + f_0(x(t)), \quad \text{with some } \alpha > 0,$$

or

$$\alpha H'(x(t)) = -\pi_{H'(x(t))}(f_0(x(t))) + \left[-f_0(x(t)) + \pi_{H'(x(t))}(f_0(x(t)))\right] - \dot{x}(t). \tag{5.41}$$

From the definition of $\pi_L(\xi)$ we have

$$\left\langle -f_0(x(t)) + \pi_{H'(x(t))}(f_0(x(t))), H'(x(t))\right\rangle = 0.$$

On the other hand, from Lemma 5.8.1,

$$\langle \dot{x}(t), H'(x(t))\rangle = 0.$$

Therefore, taking the scalar product of (5.41) with $H'(x(t))$, we get

$$\alpha = -\frac{1}{\|H'(x(t))\|^2} \langle f_0(x(t)), H'(x(t)) \rangle ,$$

which completes the proof. $\square$

Lemma 5.8.2 implies that the boundary ∂A is an invariant manifold for the differential equation (5.40). The definition (5.40) reduces the dimension of the image of f_0 by 1. Therefore, the image of the map $\bar{f}$ acts to a space of dimension $dimE - 1$, which implies that one eigenvalue of the Jacobian $\bar{f}'(x_0)$ is always zero.

We now offer an asymptotic stability result which can be of independent interest in applications of perturbed sweeping processes.

Theorem 5.8.2. Let $x_0 \in \partial A$ be a switched equilibrium of (5.38). If real parts of $dimE - 1$ eigenvalues of the Jacobian $\bar{f}'(x_0)$ are negative, then x_0 is a uniformly asymptotically stable point of sweeping process (5.38).

Proof. Step 1. *Convergence to* ∂A. Let $B_r(x_0)$ be a ball of radius r centered at x_0. Let us show that there exists $r > 0$ such that for any $\xi \in B_r(x_0) \cap A$, the solution $t \mapsto X(t, \xi)$ of

$$\dot{x} = -f_0(x) \tag{5.42}$$

with the initial condition $X(0, \xi) = \xi$ reaches ∂A at time some time $\tau(\xi) > 0$. The proof will be through the Implicit Function Theorem applied to

$$F(t, x) = H(X(t, x)).$$

We have $F(0, x_0) = 0$ and $F_t(0, x_0) = -H'(x_0)f_0(x_0) \neq 0$ by the definition of switched equilibrium. Therefore, Implicit Function Theorem (see e.g., Rudin [57, Theorem 9.28]) ensures the existence of $\xi \to \tau(\xi)$ defined and continuous on a sufficiently small ball $B_r(x_0)$ and such that $\tau(x_0) = 0$.

It remains to show that $\tau(\xi) > 0$ for all $\xi \in B_r(x_0) \cap A$. Since, according to the definition of switched equilibrium, $H'(x_0)^T$ is a normal to A pointing outwards to A, it is sufficient to prove that $\tau(\xi) > 0$ for $\xi = x_0 - \lambda H'(x_0)^T$ with all $\lambda > 0$ sufficiently small. So we introduce a scalar function

$$G(\lambda) = \tau(x_0 - \lambda H'(x_0)^T)$$

and want to prove that $G'(0) > 0$. Using the formula for the derivative of the implicit function (see Rudin [57, Theorem 9.28])

$$\tau'(x_0) = -(H'(x_0)f_0(x_0))^{-1}H'(x_0)$$

and so

$$G'(0) = -(H'(x_0)f_0(x_0))^{-1}H'(x_0)(-H'(x_0)^T) = H'(x_0)f_0(x_0)\|H'(x_0)\|^2,$$

which is indeed positive according to Definition 5.8.1.

Finally, let us fix $\xi \in B_r(x_0) \cap A$ and let $x(t)$ be the solution of (5.38) with the initial condition $x(0) = \xi$. Since the conclusion of the Implicit Function Theorem comes with uniqueness, we have that $X(t,\xi) \notin \partial A$, $t \in [0, \tau(\xi))$. Therefore, $X(t,\xi) = x(t)$, for any $t \in [0, \tau(\xi))$, which implies that $\lim_{t \to \tau(\xi)} X(t,\xi) = \lim_{t \to \tau(\xi)} x(t)$ and so $x(\tau(\xi)) \in \partial A$.

Step 2. *Convergence along* ∂A. Lemmas 5.8.1 and 5.8.2 combined with the negativeness of real parts of $dimE - 1$ eigenvalues of $\bar{f}'(x_0)$ imply that there exists an neighborhood $x_0 \in Q \subset E$ such that any solution of (5.38) with the initial condition $x(0) \in Q \cap \partial A$ converges to x_0 along ∂A as $t \to \infty$ and the convergence is uniform with respect to the initial condition.

Making now $r > 0$ in Step 1 so small that $\cup_{\xi \in B_r(x_0)} X(\tau(\xi), \xi) \in Q$ (which is possible by continuity of $\xi \to \tau(\xi)$), we combine Step 1 and Step 2 to conclude that any solution of (5.38) with $x(0) \in B_r(x_0)$ approaches x_0 as $t \to \infty$.

The proof of the theorem is complete. □

We are now in the position to combine theorems 5.5.2, 5.7.1, and 5.8.2 when the following condition holds for (5.2) at $\lambda = 0$:

$$a(t, 0) \equiv 0, \quad c(x, 0) \equiv 0, \quad f(t, x, 0) \equiv f_0(x) \text{ with } f_0 \in C^1(E, E). \tag{5.43}$$

Theorem 5.8.3. Assume that condition I) of Theorem 5.5.1 holds. Assume, that for $\lambda = 0$ sweeping process (5.2) is smooth autonomous, i.e., satisfies (5.43). If real parts of $n - 1$ eigenvalues of $\bar{f}'(x_0)$ are negative for some switched equilibrium $x_0 \in \partial A$, then there exists $\lambda_1 > 0$ such that for all $\lambda \in (0, \lambda_1]$ sweeping process (5.2) admits a periodic solution $x_\lambda(t) \to x_0$ as $\lambda \to 0$.

Proof. Let $\bar{P}(x) = P^0(V^0(x))$. By Theorem 5.8.2, there exists an open bounded set $x_0 \in Q \subset E$ such that $(\bar{P})^m$ maps Q strictly into itself for all $m \in \mathbb{N}$ sufficiently large. Therefore, Theorem 5.8.1 ensures that condition II) of Theorem 5.5.1 holds, so Theorem 5.5.2 applies. $\square$

Similar to Theorem 5.8.3 results have been obtained for ordinary differential equations by Berstein-Halanai [11] and Cronin [22].

CHAPTER 6

CONCLUSION

In this dissertation, we established new results on the existence, periodicity, almost periodicity, stability, and bifurcations of solutions of the perturbed sweeping process (1.1). Our stability results assume that the right-hand-side of (1.1) is Lipschitz continuous and the constraint is state-independent, while the existence results assume minimum regularity such as just BV-continuity in time and dependence on the state in the constraint. In what follows we discuss conclusions for each chapter of the dissertation.

Chapter 2: Here we established the existence and global exponential stability of bounded and almost periodic solutions to perturbed sweeping process (1.1) with globally Lipschitz monotone right-hand-sides.

When the right-hand-sides of (1.1) are non-monotone, but close to monotone, we discovered that all the solutions to (1.1) are close to the unique bounded (or almost periodic) solution of the respective monotone sweeping process for large values of time. In particular, we initiated the development of the averaging theory for Moreau sweeping process (1.1) with high-frequency almost periodic excitation $g\left(\frac{t}{\varepsilon}, x\right)$, where only monotonicity of the average $g_0(x) = \lim\limits_{T\to\infty} \frac{1}{T}\int\limits_0^T g(s,x)ds$ is required. This result can be used for the design of vibrational control strategies for Moreau sweeping processes (see e.g., Bullo [17] for the respective theory in the case of the differential equation).

Building upon the modeling approach of Bastein et al ([8], [7]) the results of the chapter can help in predicting the long-term response of elastoplastic materials to combined excitation of forces of different periods. Further potential applications of the results of this chapter are in studying the dynamics of a circuit involving devices like diodes, thyristors and diacs (see Addi et al [2]) when ampere-volt characteristics (for the set function) and voltage supply (for the perturbation) receive time-periodic excitations of different periods. Such a study will require extending our theory to sweeping processes with state-dependent convex constraints.

Chapter 3: In this chapter, we introduced a framework that can be used for extending the results of Chapter 2 to sweeping processes with non-convex constraints. We further proved the existence of at least one bounded global solution to perturbed sweeping processes with so-called prox-regular moving set $C(t)$.

We then proved that the unique global solution is periodic when $t \mapsto C(t)$ and $f \mapsto F(t,x)$ are periodic with the same period. Following the lines of Chapter 2, the ideas of the present work can be extended to almost periodic solutions and to sweeping processes with small non-monotone ingredients.

An illustrative example has been also provided. At the same time, we indicated why the current approach is not capable to deal with the crowd motion model, which can be used for the development of an alternative approach.

Chapter 4: In Chapter 4 we initiated the development of bifurcation theory for sweeping process (1.1). We established a result on bifurcation of limit cycles in a suitable version of (1.1) in $\mathbb{R}^2$ from a boundary equilibrium of focus type. In particular, we derived an equation of sliding along the boundary of a unilateral constraint and observed that the action of the unilateral constraint is equivalent to an action of an orthogonal vector field pointing towards the unilateral constraint from the outside.

In this way, we linked the development of bifurcation theory for sweeping processes to the analysis of differential equations with discontinuous right-hand-sides (Filippov system, see Filippov [28]).

Chapter 5: In this chapter, we considered sweeping processes (1.1), which are just BV-continuous in time and contain a state-dependent ingredient in the moving constraint. By extending the implicit catching-up scheme of Kunze and Monteiro Marques [40] to perturbed sweeping processes, we proved solvability of BV-continuous state-dependent sweeping processes with a Lipschitz dependence on the state.

We further used topological degree arguments to establish the existence of periodic solutions to sweeping processes of this type. The analysis is carried out for the simplest possible moving set $C(t) = A + a(t) + c(x)$ throughout the entire chapter, that allowed us to focus on the development of core mathematical ideas rather than on its possible generalizations. We explained in Remarks 5.6.1 and 5.6.3 how the existence result (Theorem 5.3.1) immediately extends to the moving set of the form $C(t) = A(t) + c(x)$. We don't know whether or not an alternative approach (for example, formula (5.27) quoted from [47, Proposition 4.7, p. 26]) can deal with any more general state-dependent moving constraints.

The existence of a T-periodic solutions to a sweeping process with T-periodic right-hand-sides and convex moving set would be an immediate result when uniqueness and continuous dependence of solutions on initial conditions hold. The difficulty we overcame when proving the existence of periodic solutions comes from the fact that uniqueness and continuous dependence on initial conditions of solutions of BV-continuous state-dependent sweeping processes is still an open question even when the dependence on the state is Lipschitz continuous (for state-independent sweeping processes, the uniqueness and continuous dependence is established e.g., in Castaing and Monteiro Marques [19] and Adly et al [3]).

The second part of the chapter concerned sweeping processes with a parameter λ, for which we developed a topological degree based continuation principle. As an application of the continuation principle, we proved the occurrence of periodic solutions at a specific location is a neighborhood of a switched boundary equilibrium. Specifically, we assumed that for $\lambda = 0$, the sweeping process is autonomous and admits an asymptotically stable switched boundary equilibrium x_0. We then proved the occurrence of T-periodic solutions from x_0 when the parameter λ increases and the sweeping process becomes non-autonomous (and T-periodic).

The condition for asymptotic stability of x_0 can be replaced by assuming that the topological index of x_0 is different from 0. Such a condition can be also expressed in terms of the eigenvalues of the linearization $\bar{f}(x_0)$ of sliding differential equation (5.40), see e.g., Krasnoselskii-Zabreiko [36, Theorem 6.1] and [36, Theorem 7.4].

We anticipate that the results of the dissertation will play a stimulating role in the development of a qualitative theory of perturbed sweeping processes.

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BIOGRAPHICAL SKETCH

W.A. Lakmi Niwanthi was born in Colombo, Sri Lanka. She was admitted to the University of Colombo in Sri Lanka and obtained her Bachelor of Science in Mathematics in 2011. Then in 2014 she obtained a post graduate diploma from ICTP in Italy. Later in 2014, she started working towards her PhD in Mathematics at The University of Texas at Dallas under the guidance of Dr. Oleg Makarenkov. Her research focuses on the stability of solutions of sweeping processes.

CURRICULUM VITAE

Wadippuli Achchige Lakmi Niwanthi

November 10, 2018

Contact Information:

Department of Mathematical Sciences
The University of Texas at Dallas
800 W. Campbell Rd.
Richardson, TX 75080-3021, U.S.A.
Email: lnw140030@utdallas.edu

Educational History:

BSc. Special Degree in Mathematics, University of Colombo, Sri Lanka, 2011
Postgraduate diploma in Mathematics, Abdus Salam International Center for Theoretical Physics (ICTP), Italy, 2014
Ph.D. in Mathematics, The University of Texas at Dallas, 2014-present

Teaching Experience:

Teaching Assistant in Department of Mathematical Sciences, The University of Texas at Dallas, 2014 – present
Assistant lecturer in Department of Mathematics, University of Colombo, 2011 – 2013

Awards:

Dharmadasa Punchihewa Memorial Price for Mathematics, 2011
Douglas Amarasekara Price for the Best Student in Mathematics, 2011
The Gulamhusein A.J. Noorbhai Gold Medal for the Best Student, Specializing in Mathematics, 2011

Professional Memberships:

Institute of Electrical and Electronics Engineers (IEEE), 2010–present
Association of Computing Machinery (ACM), 2008–present

Publications:

Kamenskii,M.; Makarenkov,O.; **Niwanthi,L.**; Raynaud de Fitte,P. *Global stability of almost periodic solutions of monotone sweeping processes and their response to non-monotone perturbations.*Nonlinear Analysis: Hybrid Systems 30, (2018), 213-224.
Makarenkov, O.; **Niwanthi,L.** *Bifurcations of finite-time stable limit cycles from focus boundary equilibria in impacting systems, Filippov systems, and sweeping processes* International Journal of Bifurcation and Chaos 28, no 10(2018), 1850126.

Professional Memberships:

Local organizer of School/Workshop on Applicable Theory of Switched Systems, 2016
http://www.utdallas.edu/ oxm130230/sw16/
Student Member of American Mathematical Society (AMS), 2014–present