2013-11

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UTD AUTHOR(S): Vladimir Dragović

Dragovic, Vladimir, and Katarina Kukic. 2014. "Role of discriminantly separable polynomials in integrable dynamical systems." AIP Conference Proceedings, 1634: 3-8.

Role of Discriminantly Separable Polynomials in Integrable Dynamical Systems

Vladimir Dragović[1,2] and Katarina Kukić [3,a)]

[1]The Department of Mathematical Sciences, University of Texas at Dallas, 800 West Campbell Road, Richardson TX 75080, USA,
[2]Mathematical Institute SANU, KnezaMihaila 36, 11000 Belgrade, Serbia
[3]Faculty for Traffic and Transport Engineering, University of Belgrade, VojvodeStepe 305, 11000 Belgrade, Serbia

[a)]Corresponding author: k.mijailovic@sf.bg.ac.rs

Abstract. Discriminantly separable polynomials of degree two in each of the three variables are considered. Those polynomials are by definition polynomials which discriminants are factorized as the products of the polynomials in one variable. Motivating example for introducing such polynomials is the famous Kowalevski top. Motivated by the role of such polynomials in the Kowalevski top, we generalize Kowalevski's integration procedure on a whole class of systems basically obtained by replacing so called the Kowalevski's fundamental equation by some other instance of the discriminantly separable polynomial. We present also the role of the discriminantly separable polynomils in twowell-known examples: the case of Kirchhoff elasticae and the Sokolov's case of a rigid body in an ideal fluid.

Keywords: discriminantly separable polynomials, Kowalevski top, integrable dynamical systems

INTRODUCTION

In a recent paper of one of the authors [2] the notion of the discriminantly separable polynomials has been introduced. The main motivation for introducing such a class of polynomials is the famous Kowalevski top. Let us recall briefly that the Kowalevski top [12] is a heavy spinning top rotating about a fixed point. We give short note on the famous Kowalevski case of a spinning top discovered in 1889.It is a rigid body that rotates in a constant gravitational field under the conditions $I_1 = I_2 = I_3, I_3 = 1$ and $y_0 = z_0 = 0$ where with (I_1, I_2, I_3)we denoted principal moments of inertia, (x_0, y_0, z_0) is center of mass, $C = Mgx_0$and (p, q, r) is vector of angular velocity. Finally with $(\gamma, \gamma', \gamma'')$ we denote cosine of angles between *z* axes of fixed coordinate system and axes of coordinate system that is attached to the top and whose origin coincides with the fixed point. Then the equations of motion take the following form: $2\dot{p} = qr, 2\dot{q} = -pr - c\gamma'', \dot{r} = c\gamma, \dot{\gamma} = r\gamma' - q\gamma'', \dot{\gamma}' = p\gamma'' - r\gamma, \dot{\gamma}'' = q\gamma - p\gamma'$.

System has three well known independent constants of motion: $2(p^2 + q^2) + r^2 = 2c\gamma + 6l_1$ – integral of energy, $2(p\gamma + q\gamma') + r\gamma'' = 2l$ – integral of angular momentum in the direction of gravity and $\gamma^2 + \gamma'^2 + \gamma''^2 = 1$ – the length of the gravity vector. The extra constant of motion discovered by Kowalevska is $[(p + iq)^2 + c(\gamma + i\gamma')][(p - iq)^2 + c(\gamma - i\gamma')] = k^2$.

After introducing new variables $x_1 = p + iq$, $x_2 = p - iq$ and $\xi_1 = x_1^2 + c(\gamma + i\gamma'), \xi_2 = x_2^2 + c(\gamma - i\gamma')$ the fourth integral discovered by Kowalevska becomes $\xi_1\xi_2 = k^2$. Further, Kowalevska rewrites the other three first integrals in the next form:

$$r^2 = E + \xi_1 + \xi_2$$
$$rc\gamma'' = G - x_2\xi_1 - x_1\xi_2$$

AIP Conf. Proc. 1634, 3-8 (2014); doi: 10.1063/1.4903006

$$c^2\gamma''^2 = F + x_2^2\xi_1 + x_1^2\xi_2.$$

Multiplying the first and the third of the previous relations and deducting the square of the second, after a short transformation, finally Kowalevska obtained following:

$$\xi_1 P(x_2) + \xi_2 P(x_1) + R_1(x_1, x_2) + k^2(x_1 - x_2)^2 = 0$$

where she denotes $E = 6l_1 - (x_1 + x_2)^2,\ F = 2lc + x_1x_2(x_1 + x_2),\ G = c^2 - k^2 - x_1^2x_2^2,\quad P(x_1) = Ex_1^2 + 2Fx_1 + G,$ symmetrically $P(x_1)$ and finally $R(x_1, x_2) = Ex_1x_2 + F(x_1 + x_2) + G,\ R_1(x_1, x_2) = EG - F^2$. Again, by using smart identities and transformations, after some calculations, Kowalevska gets so called *the Kowalevski's fundamental equation*, denoted by (1). Further connection of the Kowalevski top with discriminantly separable polynomials is explained in the next section. The novelty of our paper is developing of this connection, by applying Kowalevski's technique of integration to obtain explicit solution of some other integrable systems, such as the case of Kirchhoff elasticae and the Sokolov's case of a rigid body in an ideal fluid.

Another novelty of our approach is that Kowalevski's integration procedure from [12] may easily be generalized and applied on a whole new class of systems we introduced in [3], [5]. We called those systems the Kowalevski type systems and basically they are obtained by replacing so called *the Kowalevski fundamental equation* by an arbitrary discriminantly separable polynomial of degree two in each of three variables.Further steps of integration procedure for those systems follow Kowalevski's procedure and finally we can get explicit solution for all systems of the Kowalevski type in theta functions of genus two.

DISCRIMINANTLY SEPARABLE POLYNOMIALS

Recall that family of discriminantly separable polynomials has been constructed in [2] as a pencil equation from the theory of conics $F(w, x_1, x_2) = 0$, where w, x_1, x_2 are the pencil parameter and the Darbouxcoordinates respectively. Here we do not go into details about that correlation between discriminantly separable polynomials and pencils of conics. We just emphasize that the keyalgebraic property of the pencil equation, as quadratic equation ineach of three variables isthat *all three of its discriminants are expressed as products of two polynomials in one variable each.* The polynomial $F(w, x_1, x_2)$ from the pencil equation is the polynomial in three variables, degree two in each of them:

$$F(w, x_1, x_2) = A(x_1, x_2)w^2 + B(x_1, x_2)w + C(x_1, x_2)$$

which discriminants can be factorized as products of polynomials in one variables:

$$D_w F(x_1, x_2) = B^2(x_1, x_2) - 4A(x_1, x_2)C(x_1, x_2) = P(x_1)P(x_2),$$

$$D_{x_i} F(w, x_j) = J(w)P(x_j), (i, j) = \{c.p.\}(\{1,2\}).$$

In previous expression A, B and C are biquadratic polynomials, symmetric in x_1 and x_2, P and J are polynomials of degree 3 and 4 respectively, andthe elliptic curves given by the equations$\gamma_1: y^2 = P(x)$ and $\gamma_2: y^2 = J(s)$are isomorphic.For more details about correlation of discriminantly separable polynomials and pencil of conics, see [2] and [4].

Now we recall the definition of discriminantly separable polynomials. With P_n^m we denote polynomials of n variables of degree m in each variable. A polynomial $F(x_1, \ldots, x_n) \in P_n^m$ is discriminantly separable if there exists polynomials $f_i(x_i)$of one variable each, such that for every i=1,…,n

$$D_{x_i} F(x_1, \ldots, \hat{x}_i, \ldots, x_n) = \prod_{\{j \neq i\}} f_{j(x_j)}.$$

It is symmetrically discriminantly separable if $f_2 = f_3 = \cdots = f_n$, while it is strongly discriminantly separable if $f_1 = f_2 = \cdots = f_n$. In [2] one can see more about classes of equivalence of such polynomials and about their properties. In [4] we classified strongly discriminantly separable polynomials modulo Möbius gauge transformations, related those polynomials with corresponding types of pencils of conics and also related them with discrete integrable systems, precisely with quad-equations.

Here we do not get into detail about the famous Kowalevski case as the direct motivation for introducing this class of polynomials. Recall here that Q from the famous *Kowalevski fundamental equation* (1) is a polynomial in three variables, degree two in each of them

$$Q(s,x_1,x_2) := (x_1-x_2)^2 s^2 - 2R(x_1,x_2)s - R_1(x_1,x_2) = 0 \tag{1}$$

where $R(x_1,x_2)$ and $R_1(x_1,x_2)$ are biquadratic polynomials from previous section. The discriminantly separability condition for polynomial Q is satisfied with polynomials:

$$D_s(Q)(x_1,x_2) = 4P(x_1)P(x_2), D_{x_1}(Q)(s,x_2) = -8J(s)P(x_2), D_{x_2}(Q)(s,x_1) = -8J(s)P(x_1)$$

$$J(s) = s^3 + 3l_1^2 s + s(c^2-k^2) + 3l_1(c^2-k^2) - 2l^2c^2, P(x_i) = -x_i^4 + 6l_1x_i^2 + 4lc\,x_i + c^2 - k^2, i = 1,2.$$

Let us mention here just one relation, see Corollary 1 from [2], known in the context of the Kowalevski top as *the Kowalevski magic change of variables*:

$$\frac{dx_1}{\sqrt{P(x_1)}} + \frac{dx_2}{\sqrt{P(x_2)}} = \frac{ds_1}{\sqrt{J(s_1)}}, \qquad \frac{dx_1}{\sqrt{P(x_1)}} - \frac{dx_2}{\sqrt{P(x_2)}} = \frac{ds_2}{\sqrt{J(s_2)}}. \tag{2}$$

It is characteristic change of variables for whole class of systems of the Kovalevski type, just with different polynomials P and J. Here s_1 and s_2 denote roots of equation (1) as a quadratic equation in s. In [5] we developed theory of the systems of the Kovalevsky type and explained in detail how Kowalevski's integration procedure can be applied on a whole class of systems. In [3] we presented few new examples of such systems. It is a characteristic property of a whole class of Kowalevski type systems that they can be explicitly integrated in theta functions of genus two. We explicitly integrated two well-known examples – the case of Kirchhoff elasticae and the Sokolov's case of a rigid body in an ideal fluid using that methodology in [6]. In next section we will briefly present how those systems fit into the Kowalevski type systems and give guidelines for their integration.First we give definition of the Kowalevski type systems in the next subsection.

THE KOWALEVSKY TYPE SYSTEMS

Given a discriminantly separable polynomial of the second degree in each of three variables

$$F(x_1,x_2,s) \coloneqq A(x_1,x_2)s^2 + 2B(x_1,x_2)s + C(x_1,x_2), \tag{3}$$

such that $D_sF(x_1,x_2) = 4(B^2 - AC) = 4P(x_1)P(x_2)$ and $D_{x_1}F(s,x_2) = J(s)P(x_2), D_{x_2}F(s,x_1) = J(s)P(x_1)$. Suppose, that a given system in variables $x_1, x_2, e_1, e_2, r, \gamma_3$ after some transformations reduces to

$$\dot{x}_1 = -if_1,\ \dot{x}_2 = if_2\ , \quad \dot{e}_1 = -me_1,\ \dot{e}_2 = me_2 \tag{4}$$

where functions f_1 and f_2 satisfy relations

$$f_1^2 = P(x_1) + e_1A(x_1,x_2), \qquad f_2^2 = P(x_2) + e_2A(x_1,x_2). \tag{5}$$

Suppose additionally, that the first integrals and invariant relations of the initial systemreduce to a relation

$$P(x_2)e_1 + P(x_1)e_2 = C(x_1,x_2) - e_1e_2A(x_1,x_2). \tag{6}$$

Instead of (6) we can assume that $\dot{x}_1\dot{x}_2 = -B(x_1,x_2)$where $B(x_1,x_2)$is a coefficient of polynomial (3).The equations for $\dot{r}$ and $\dot{\gamma}_3$ are not specified for the moment and m is a function of system's variables. If a system satisfies the above assumptions (3), (4), (5) and (6) we call it *a system of the Kowalevski type*. The Kowalevski top is an example of the systems of the Kowalevski type. In [5] we proved a theorem which is quite general and concerns all the systems of the Kowalevski type. It explains in full a subtle mechanism of an integration procedure for whole class of systems and basically formalizes the original considerations of Kowalevski, in a slightly more general context of the discriminantly separable polynomials. Here we just give formulation of Theorem. The proof may be seen in [5].

*Theorem.*Given a system which reduces to (4), (5), (6). Then the system is linearized on the Jacobian of the curve $y^2 = \mathrm{J}(\mathrm{z})(\mathrm{z}-\mathrm{k})(\mathrm{z}+\mathrm{k})$, where J is a polynomial factor of the discriminant of F as apolynomial in x_1 and k is a constant such that $e_1 e_2 = k^2$.

EXAMPLES OF THE SYSTEMS OF THE KOWALEVSKI TYPE

In this Section we give a brief overview of two well-known examples in the theory of integrable dynamical systems, but pointing out that those systems are instances of the systems of the Kowalevski type and that they may be integrated generalizing Kowalevski's integration procedure, just followings the steps given in the proof of the previous Theorem. The first example is given by Sokolov in [13], [14] and later reconsidered in [11]. It describes a motion of a rigid-body in an ideal noncompressible fluid. Sokolov considered the Hamiltonian

$$\widehat{H} = \mathrm{M}_1^2 + \mathrm{M}_2^2 + 2\mathrm{M}_3^2 + 2c_1\gamma_1 + 2c_2(\gamma_2\mathrm{M}_3 - \gamma_3\mathrm{M}_2) \tag{7}$$

on e(3) with the Lie-Poisson brackets

$$\{M_i, M_j\} = \varepsilon_{ijk} M_k, \qquad \{M_i, \gamma_j\} = \varepsilon_{ijk}\gamma_k, \qquad \{\gamma_i, \gamma_j\} = 0 \tag{8}$$

where ε_{ijk} is the totally skew-symmetric tensor. The Lie-Poisson bracket (8) has two Casimir functions

$$\gamma_1^2 + \gamma_2^2 + \gamma_3^2 = \mathrm{a}$$

and

$$\gamma_1 M_1 + \gamma_2 M_2 + \gamma_3 M_3 = b.$$

Following [11] and [12] we introduce the new variables

$$\mathrm{z}_1 = \mathrm{M}_1 + \mathrm{i}\mathrm{M}_2, \qquad \mathrm{z}_2 = \mathrm{M}_1 - \mathrm{i}\mathrm{M}_2$$

and

$$e_1 = z_1^2 - 2c_1(\gamma_1 + i\gamma_2) - c_2^2 a - c_2(2\gamma_2 M_3 - 2\gamma_3 M_2 + 2i(\gamma_3 M_1 - \gamma_1 M_3)$$

$$e_2 = z_2^2 - 2c_1(\gamma_1 - i\gamma_2) - c_2^2 a - c_2(2\gamma_2 M_3 - 2\gamma_3 M_2 + 2i(\gamma_1 M_3 - \gamma_3 M_1).$$

The second integral of motion for the system (7) maybe rewritten as$e_1 e_2 = k^2$. The equations of motion in the new variables z_i, e_ican be written in a form of the equations from the definition of the Kowalevski type systems (4-6). It is easyto show that:

$$\dot{z_1^2} = -R(z_1) - e_1(z_1 - z_2)^2, \qquad \dot{z_2^2} = -R(z_2) - e_2(z_1 - z_2)^2, \quad \dot{e_1} = -4iM_3 e_1, \qquad \dot{e_2} = 4iM_3. \tag{9}$$

With R(z) in (9) we denoted a polynomial of fourth degree given by

$$\mathrm{R}(\mathrm{z}) = -\mathrm{z}^4 + 2\mathrm{H}\,\mathrm{z}^2 - 8\,c_1 b\,\mathrm{z} - \mathrm{k}^2 + 4\mathrm{a}\,c_1^2 - 2c_2^2(2b^2 - \mathrm{Ha}) + c_2^4\mathrm{a}.$$

In order to perform the integration procedure to the Sokolov system, we need to show that an additional relation is satisfied and to relate it to a certain discriminantly separable polynomial. Starting from the equations of motion

$$\dot{z_1} = -2\mathrm{M}_3(\mathrm{M}_1 - \mathrm{i}\mathrm{M}_2) + 2c_2(\gamma_1\mathrm{M}_2 - \gamma_2\mathrm{M}_1) + 2c_1\gamma_3$$

and

$$\dot{z_2} = -2\mathrm{M}_3(\mathrm{M}_1 + \mathrm{i}\mathrm{M}_2) + 2c_2(\gamma_1\mathrm{M}_2 - \gamma_2\mathrm{M}_1) + 2c_1\gamma_3$$

one can get the following relation

$$\dot{z_1}\dot{z_2} = -(F(z_1, z_2) + (H + c_2^2 a)(z_1 - z_2)^2). \tag{10}$$

The biquadratic polynomial $F(z_1, z_2)$ is given by:

$$F(z_1, z_2) = -\frac{1}{2}(R(z_1) + R(z_2) + (z_1 - z_2)^2)\,. \tag{11}$$

After equating the square of $\dot{z_1}\dot{z_2}$ from the relation (10) with product $\dot{z_1^2}\dot{z_2^2}$ from equations (9) one can check that the variables of the Sokolov's system satisfy the following identity:

$$(z_1 - z_2)^2[2F(z_1, z_2)(z_1 - z_2)^2(H + c_2^2 a)^2 - R(z_1)e_2 - R(z_2)e_1 - e_1 e_2 (z_1 - z_2)^2] + F(z_1, z_2) - R(z_1)R(z_2) = 0 \tag{12}$$

To see that Sokolov's system is an instance of the Kowalevski type systems and to apply integration procedure developed for those systems, we just need to rewrite relation (12) in the form of (6) and to relate it with corresponding discriminantly separable polynomial (3). Denote by $\mathrm{C}(z_1, z_2)$ a biquadratic polynomial such that

$$\mathrm{F}^2(\mathrm{z}_1, \mathrm{z}_2) - \mathrm{R}(\mathrm{z}_1)R(\mathrm{z}_2) = (\mathrm{z}_1 - \mathrm{z}_2)^2 \mathrm{C}(\mathrm{z}_1, \mathrm{z}_2).$$

Finally we get relation in the form of (6) form Sokolov's systems. It can be written in the form

$$\mathrm{R}(\mathrm{z}_1)\mathrm{e}_2 + \mathrm{R}(\mathrm{z}_2)\mathrm{e}_1 = \tilde{\mathrm{C}}(\mathrm{z}_1, \mathrm{z}_2) - \mathrm{e}_1\mathrm{e}_2(\mathrm{z}_1 - \mathrm{z}_2)^2, \tag{13}$$

with $\tilde{\mathrm{C}}(\mathrm{z}_1, \mathrm{z}_2) = \mathrm{C}(\mathrm{z}_1, \mathrm{z}_2) + 2\mathrm{F}(\mathrm{z}_1, \mathrm{z}_2)(\mathrm{H} + \mathrm{c}_2^2 a) + (\mathrm{H} + \mathrm{c}_2^2 a)^2(\mathrm{z}_1 - \mathrm{z}_2)^2$. Now one can recognize the Sokolov's system as the Kowalevski type system and its further integration process is completely analogue to the Kowalevski top. The discriminantly separable polynomial that takes the role of the Kowalevski fundamental equation here is

$$\tilde{\mathrm{F}}(\mathrm{z}_1, \mathrm{z}_2, s) = (\mathrm{z}_1 - \mathrm{z}_2)^2 s^2 + 2(\mathrm{F}(\mathrm{z}_1, \mathrm{z}_2) + (\mathrm{H} + \mathrm{c}_2^2\mathrm{a})(\mathrm{z}_1 - \mathrm{z}_2)^2)s + \tilde{\mathrm{C}}(\mathrm{z}_1, \mathrm{z}_2). \tag{14}$$

The discriminants of (14) are factorized in the following manner:

$$\mathrm{D}_s(\tilde{F})(\mathrm{z}_1, \mathrm{z}_2) = 4\mathrm{R}(\mathrm{z}_1)\mathrm{R}(\mathrm{z}_2),\ \mathrm{D}_{\mathrm{z}_2}(\tilde{F})(\mathrm{z}_1, \mathrm{s}) = \mathrm{R}(\mathrm{z}_1)J(s), \mathrm{D}_{\mathrm{z}_1}(\tilde{F})(\mathrm{z}_2, \mathrm{s}) = \mathrm{R}(\mathrm{z}_2)J(s),$$

where

$$J = 8s^3 - 4(H + 3ac_2^2)s^2 - (8c_2^2 b^2 + 2k^2 - 8ac_1^2 - 8c_2^4 a^2 - 8c_2^2 Ha)s + 8c_1^2 b^2 + 4c_2^4 ab^2 - 4c_1^2 a^2 c_2^2 + k^2 c_2^2 a + Hk^2 - 2aH^2 c_2^2 + 4Hb^2 c_2^2 - 4Hc_1^2 a - 4c_2^4 Ha^2 - 2c_2^6 a^3.$$

The analogue of the Kowalevski's magic change of varibles (2) here is

$$\frac{dz_1}{\sqrt{R(z_1)}} + \frac{dz_2}{\sqrt{R(z_2)}} = \frac{ds_1}{\sqrt{J(s_1)}}, \qquad \frac{dz_1}{\sqrt{R(z_1)}} - \frac{dz_2}{\sqrt{R(z_2)}} = \frac{ds_2}{\sqrt{J(s_2)}}$$

where s_1 and s_2 are the roots of the $\tilde{\mathrm{F}}(\mathrm{z}_1, \mathrm{z}_2, s) = 0$ as a quadratic equation in s. Explicit integration in theta function of genus two is done in [6].

One more example of the systems of the Kowalevski type is the integrable Kirchhoff elasticae from [8], and the similar systems [9], [10]. That system is defined by the Hamiltonians

$$H = \frac{1}{4}(M_1^2 + M_2^2 + 2M_3^2) + \gamma_1$$

with the deformed Lie-Poisson brackets $\{M_i, M_j\}_\tau = \varepsilon_{ijk} M_k$, $\{M_i, \gamma_j\}_\tau = \varepsilon_{ijk}\gamma_k$, $\{\gamma_i, \gamma_j\}_\tau = \tau\varepsilon_{ijk} M_k$, where ε_{ijk} is the totally skew-symmetric tensor. The deformation parameter takes values τ = 0, 1, -1. These structures correspond to e(3), so(4), and so(3,1) respectively. The classical Kowalevski case corresponds to the case

τ =0. The systems with τ = -1, 1 have been considered by St. Petersburg's school, see [9], [10] and references therein. One can easily show that this system is also a system of the Kowalevski type. The detailed proof of and also interesting correlation of this system with the Kowalevski top can be seen in [5].

CONCLUSIONS

The role of the discriminantly separable polynomials in the Kowalevski type systems is clearly motivated. By virtue of this kind of polynomials we constructed many new examples of integrable systems, but also explicitly integrated some of well-known examples from the theory. That list is extended nowadays and we find more examples of well-known systems that fit in the established class of the Kowalevski type systems. Here, due to lack of space, we did not get into detailed explanation about the role of discriminantly separable systems in the discrete integrable systems, especially in the theory of quad-equations. That is another application of these polynomials about which one can see more in [4].

ACKNOWLEDGMENTS

The research was partially supported by the Serbian Ministry of Science and Technological Development, Project 174020 "Geometry and Topology of Manifolds, Classical Mechanics and Integrable Dynamical Systems". One of the authors (K.K.) uses the opportunity to thank to the ICTP - SEENET-MTP Project PRJ09 "Cosmology and Strings" in frame of the Southeastern European Network in Theoretical and Mathematical Physics for giving her the opportunity to participate at the conference TIM-2013.

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