THE CHARACTER VARIETIES OF RATIONAL LINKS $C(2n, 2m+1, 2)$

by

Bradley D. Meyer

APPROVED BY SUPERVISORY COMMITTEE:

Anh T. Tran, Chair

Mieczyslaw Dabkowski

Vladimir Dragovic

Dedicated to Maria Guadalupe Cuevas-Meyer

THE CHARACTER VARIETIES OF RATIONAL LINKS $C(2n, 2m+1, 2)$

by

BRADLEY D. MEYER, BS

THESIS

Presented to the Faculty of

The University of Texas at Dallas

in Partial Fulfillment

of the Requirements

for the Degree of

MASTER OF SCIENCE IN

MATHEMATICS

THE UNIVERSITY OF TEXAS AT DALLAS

May 2019

ACKNOWLEDGMENTS

I would like to express my gratitude to my advisor Dr. Anh Tran for his guidance in the writing of this thesis. I am grateful for his patience and support throughout the entire process. I would also like to thank the friends I have made here for their many insightful discussions. Finally, I would like to thank my family for their unending support.

April 2019

THE CHARACTER VARIETIES OF RATIONAL LINKS $C(2n, 2m+1, 2)$

Bradley D. Meyer, MS
The University of Texas at Dallas, 2019

Supervising Professor: Anh T. Tran, Chair

In this thesis we study the nonabelian $SL_2(\mathbb{C})$ character varieties of an infinite family of rational links. In chapter 1 we provide background information on rational knots and links and their character varieties. We also provide some properties of Chebyshev polynomials, which will be used in calculating the character varieties. In chapter 2 we first find a presentation for the knot group of $C(2n, 2m+1, 2)$. We then calculate the nonabelian character variety and prove that the character variety of $C(2n, 2m+1, 2)$ is irreducible unless $n = 1, -1$ or $m = -1$.

TABLE OF CONTENTS

LIST OF FIGURES

CHAPTER 1

INTRODUCTION

1.1 Links

The mathematical study of knots is a precise investigation of how a 1-dimensional loop can lie in 3-dimensional space. In other words, the main question to be answered is whether any given knots are equivalent. We follow [3] in our discussion.

Formally, an m component link L is a subset of S^3 consisting of m disjoint, piecewise linear, simple closed curves. If L has one component it is called a knot. We say that two links L_1 and L_2 are equivalent if there exists a homeomorphism from S^3 to itself that maps L_1 to L_2. Such a homeomorphism is known as an ambient isotopy.

Three types of diagram moves are sufficient in order to describe links up to ambient isotopy. These diagram moves are known as Reidemeister moves (See figure 1.1). We see that any two equivalent links have diagrams related by a series of Reidemeister moves and a homeomorphism of the plane.

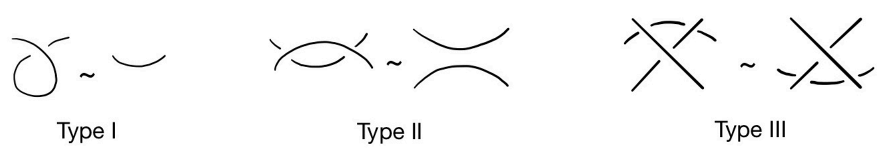

Figure 1.1. Reidemeister moves

Since equivalent links are related by an ambient isotopy, we see that if two links L_1 and L_2 are equivalent their complements in S^3 are homeomorphic as 3-manifolds.

Definition 1.1. Let $\mathcal{L}$ be a tubular neighborhood of a link L. We define the link complement X_L to be

$$X_L = S^3 - \mathcal{L}.$$

Many link invariants are, in fact, invariants of the link complement. For example, the fundamental group of a link L is defined to be the fundamental group of the link complement.

1.1.1 Rational Links

A rational link, also called a two-bridge link, is a link that admits a projection with two maxima and two minima. To every rational link we can associate a pair (p, q) of coprime integers such that $-p < q \leq p$. The link associated to the pair (p, q) is ambient isotopic to the link $L(p, q)$ defined as follows. We follow [4].

Choose $\epsilon \in \{0, 1\}$ such that $0 < q/p + \epsilon \leq 1$. Then we can write $q/p + \epsilon$ as a continued fraction

$$\frac{q}{p} + \epsilon = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots + \cfrac{1}{a_{s-1} + \cfrac{1}{a_s}}}}}$$

such that each $a_i \geq 1$. We use the sequence $[a_1, a_2, \ldots, a_{s-1}, a_s]$ to create the link shown in figure 1.2 where the number in each block gives the number of half twists. The jth block between the middle strands contains a_{2j-1} left handed half twists and the jth block between the bottom two strands contains a_{2j} right hand twists. This is the 4-plat presentation of the rational link $L(p, q)$. Note that we can always choose a sequence $[a_1, a_2, \ldots, a_{s-1}, a_s]$ such that s is odd. Indeed, if s were even with $a_s = a$ we could replace a_s by $a - 1$ and add the new entry $a_{s+1} = 1$.

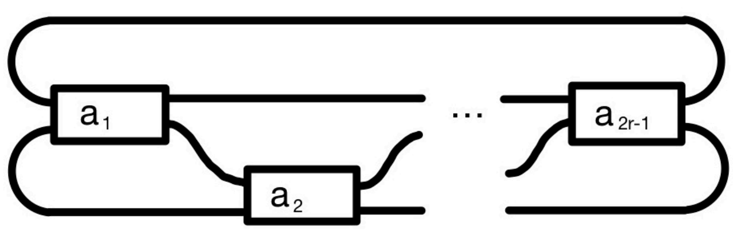

Figure 1.2. The rational link $[a_1, a_2, \ldots, a_{2r-1}]$

A rational link can have no more than two components. It is known, see [1], that the links $L(p,q)$ and $L(p',q')$ are ambient isotopic if and only if $p = p'$ and $q = q'$ or $qq' \equiv 1 \mod p$.

Note that a rational link with one component is called a rational knot. In the same way as above, we can associate a pair (p,q) of coprime integers such that $-p < q \leq p$ and the knot associated to the pair (p,q) is ambient isotopic to the knot $K(p,q)$.

1.2 Knot Group (Wirtinger Presentation)

We denote by $\pi_1(X, x_0)$ the fundamental group of X at the basepoint x_0. If X is path connected, for two points $x_0, x_1 \in X$ we have that $\pi_1(X, x_0) \simeq \pi_1(X, x_1)$. In such a case we denote the fundamental group of X simply by $\pi_1(X)$.

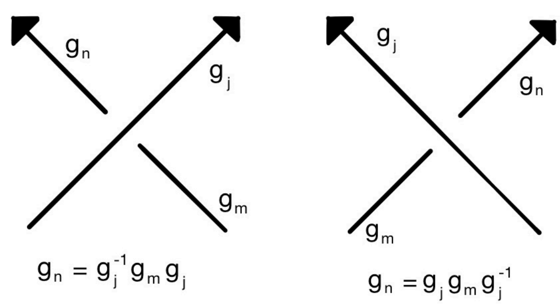

Figure 1.3. Relation for the Wirtinger Presentation

Since the knot complement X_K is path connected, we denote its fundamental group by $\pi_1(X_K)$. In order to describe $\pi_1(X_K)$, we make use of the Wirtinger presentation. The Wirtinger presentation is defined as follows [3]. Given an oriented diagram of a knot K we begin by assigning a generator g_i to each segment of the knot diagram. Each g_i represents a loop which, starting from a base point above the diagram, encircle the i-th overpassing segment of the diagram in the positive direction. At each crossing, we then obtain a relation r_k by the following rules (See figure 1.3). At the crossing c suppose the over passing segment is assigned the generator g_j and the under passing segment is assigned g_m as it approaches c

and g_n as it leaves c. Then we obtain the relation $r_k = g_j g_m g_j^{-1} g_n^{-1}$ if the sign of c is negative and $r_k = g_j^{-1} g_m g_j g_n^{-1}$ if the sign of c is positive. So we obtain the presentation of $\pi_1(X_K)$

$$\pi_1(X_K) = \langle g_1, \ldots, g_n : r_1, \ldots, r_{n-1} \rangle$$

called the Wirtinger presentation.

Note that in the same way we can define the fundamental group of a link L. In the case of a link L, we still call $\pi_1(X_L)$ the knot group of L.

1.3 Representations in $SL_2(\mathbb{C})$

An $SL_2(\mathbb{C})$ representation of a group G is a homomorphism $\rho : G \to SL_2(\mathbb{C})$. Two representations ρ and ρ' are said to be equivalent if they differ by an inner automorphism of $SL_2(\mathbb{C})$. A representation ρ is irreducible if there are no nontrivial subspaces of $\mathbb{C}^2$ invariant under the action of $\rho(G)$. Otherwise, ρ is said to be reducible. An equivalent definition of a reducible representation ρ is that it is conjugate to a representation into upper triangular matrices. We see that every abelian representation is reducible. However, the inverse is not true.

We define the character of a representation ρ as the map $\chi_\rho : G \to \mathbb{C}$, which is defined by $\chi_\rho(g) = \mathrm{tr}(\rho(g))$. Equivalent representations will have the same character since the trace operation is invariant under inner automorphism. Therefore, irreducible representations are determined up to conjugation by their character. This is not true for reducible representations. Note that any reducible representation will share its character with an abelian representation.

1.4 Character Varieties (Riley Polynomial)

We are interested in studying particular representations of the knot groups of a specific class of links. Namely, we are interested in the nonabelian representations, up to conjugation, of rational links. These representations can be described by a single polynomial called the Riley polynomial. We will follow [2], [7] in defining the Riley polynomial of rational links and knots.

1.4.1 Rational Knots

Let $K(p, q)$ be a rational knot. The knot group $\pi_1(X_{K(p,q)})$ has a representation

$$\pi_1(X_K) = \langle a, b : wa = bw \rangle$$

where a, b correspond to the meridian of K and $w = a^{\epsilon_1} b^{\epsilon_2} \cdots a^{\epsilon_{p-2}} b^{\epsilon_{p-1}}$ where $\epsilon_i = (-1)^{\lfloor iq/p \rfloor}$ for $1 \leq i \leq p-1$. The existence of such a representation follows from the algorithm in section 1.2.

Suppose $\rho : \pi_1(X_K) \to SL_2(\mathbb{C})$ is a nonabelian representation. Since a and b are conjugate in the knot group, $\rho(a)$ and $\rho(b)$ must have the same trace. We may assume, up to conjugation, that

$$\rho(a) = \begin{bmatrix} t & 1 \\ 0 & t^{-1} \end{bmatrix} \qquad \rho(b) = \begin{bmatrix} t & 0 \\ u & t^{-1} \end{bmatrix}.$$

In the case of the rational knot, the character variety is described by a polynomial in the two variables $x = \operatorname{tr}(\rho(a)) = \operatorname{tr}(\rho(b))$ and $y = \operatorname{tr}(\rho(ab))$.

Theorem 1.2. *The $SL_2(\mathbb{C})$ character variety of a rational knot is described by the set of points $(x, y) \in \mathbb{C}^2$ such that*

$$w_{11} - (t - t^{-1}) w_{21} = 0$$

where $\rho(w) = \begin{bmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \end{bmatrix}$.

Proof. We have

$$\rho(wa) - \rho(bw) = \begin{bmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \end{bmatrix} \begin{bmatrix} t & 1 \\ 0 & t^{-1} \end{bmatrix} - \begin{bmatrix} t & 0 \\ u & t^{-1} \end{bmatrix} \begin{bmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \end{bmatrix}$$
$$= \begin{bmatrix} 0 & w_{11} - (t - t^{-1})w_{12} \\ -uw_{11} + (t - t^{-1})w_{21} & w_{21} - uw_{12} \end{bmatrix}$$

By lemma 2.1 in [10], we have $w_{21} = uw_{12}$. Therefore,

$$\rho(wa) - \rho(bw) = \begin{bmatrix} 0 & w_{11} - (t - t^{-1})w_{12} \\ -u(w_{11} - (t - t^{-1})w_{12}) & 0 \end{bmatrix}$$

and the matrix equation is only satisfied when

$$w_{11} - (t - t^{-1})w_{12} = 0.$$

□

We define the polynomial

$$R_{K(p,q)}(x, y) = w_{11} + (t^{-1} - t)w_{21}$$

to be the Riley polynomial of the rational knot $K(p, q)$. In other words, the Riley polynomial describes each conjugacy class of nonabelian representations of the knot group $X_{K(p,q)}$ into $SL_2(\mathbb{C})$.

1.4.2 Rational Links

Let $L(p, q)$ be a rational link. The knot group $\pi_1(X_{L(p,q)})$ has a representation

$$\pi_1(X_{L(p,q)}) = \langle a, b : wa = aw \rangle$$

where a, b correspond to the meridians of $L(p, q)$ and $w = b^{\epsilon_1} a^{\epsilon_2} \cdots a^{\epsilon_{p-2}} b^{\epsilon_{p-1}}$ where $\epsilon_i = (-1)^{\lfloor iq/p \rfloor}$ for $1 \leq i \leq p-1$. As in the case of the rational knot, such a representation exists due to the discussion in section 1.2.

Suppose $\rho : \pi_1(X_{L(p,q)}) \to SL_2(\mathbb{C})$ is a nonabelian representation. Note that in the case of the rational link, a and b are not conjugate. We assume, up to conjugation, that

$$\rho(a) = \begin{bmatrix} s_1 & 1 \\ 0 & s_1^{-1} \end{bmatrix} \qquad \rho(b) = \begin{bmatrix} s_2 & 0 \\ u & s_2^{-1} \end{bmatrix}.$$

Then the character variety is described by the set $(x, y, z) \in \mathbb{C}^3$ such that the matrix equation $\rho(wa) = \rho(aw)$ is satisfied, where $x = \mathrm{tr}(\rho(a)), y = \mathrm{tr}(\rho(b))$, and $z = \mathrm{tr}(\rho(ab))$. Note that the matrix equation $\rho(wa) = \rho(aw)$ reduces to the two equations

$$w_{21} = 0$$

$$w_{12}(s_1 - s_1^{-1}) + w_{22} - w_{11}.$$

By lemma 1 in [7], we have

$$w_{12}(s_1 - s_1^{-1}) + w_{22} - w_{11} = (s_2 - s_2^{-1})w_{21}.$$

Therefore, $\rho(wa) = \rho(aw)$ reduces to the single equation $w_{21} = 0$. We define the polynomial $R_{L(p,q)} = w_{21}$ to be the Riley polynomial of $L(p,q)$.

1.5 Chebyshev Polynomials and Matrix Powers

We will make use of the Chebyshev polynomials of the first kind in our calculation of the character varieties. In this section we will introduce some properties of Chebyshev polynomials following [8].

Definition 1.3. We define the Chebyshev polynomials by the following recursion relation.

$$S_0(v) = 1$$

$$S_1(v) = v$$

$$S_n(v) = vS_{n-1}(v) - S_{n-2}(v)$$

for all $n \in \mathbb{Z}$.

Lemma 1.4. *Suppose $v = a + a^{-1}$ where $a \neq \pm 1$. Then*

$$S_n(v) = \frac{a^{n+1} - a^{-(n+1)}}{a - a^{-1}}.$$

Proof. We have that $S_0(v) = 1 = \frac{a-a^{-1}}{a-a^{-1}}$ and $S_1(v) = v = (a + a^{-1})(\frac{a-a^{-1}}{a-a^{-1}})$. Suppose this holds for $n = k$ and $n = k + 1$. The

$$\begin{aligned} S_{k+2}(v) &= vS_{k+1}(v) - S_k(v) \\ &= (a + a^{-1})\frac{a^{k+2} - a^{-k-2}}{a - a^{-1}} - \frac{a^{k+1} - a^{-k-1}}{a - a^{-1}} \\ &= \frac{a^{k+3} - a^{-k-3}}{a - a^{-1}}. \end{aligned}$$

Therefore, by induction, we have that $S_n(v) = \frac{a^{n+1}-a^{-(n+1)}}{a-a^{-1}}$. ☐

Lemma 1.5. *For any integer n we have*

$$S_n^2(v) - S_{n+1}(v)S_{n-1}(v) = 1 \tag{1.1}$$

$$S_n^2(v) + S_{n-1}^2(v) - vS_n(v)S_{n-1}(v) = 1. \tag{1.2}$$

Proof. By lemma 1.4 we have

$$\begin{aligned} S_n^2(v) - S_{n+1}(v)S_{n-1}(v) &= \frac{a^{2n+2} + a^{-2n-2} - 2}{(a - a^{-1})^2} - \left(\frac{a^{n+2} - a^{-n-2}}{a - a^{-1}}\right)\left(\frac{a^n - a^{-n}}{a - a^{-1}}\right) \\ &= \frac{a^{2n+2} + a^{-2n-2} - 2}{(a - a^{-1})^2} - \frac{a^{2n+2} + a^{-2n-2} - a^2 - a^{-2}}{(a - a^{-1})^2} \\ &= \frac{a^2 + a^{-2} - 2}{(a - a^{-1})^2} \\ &= 1 \end{aligned}$$

which proves (1.1).

By lemma 1.4 again

$$
\begin{aligned}
&S_n^2(v) + S_{n-1}^2(v) - vS_n(v)S_{n-1}(v) \\
&= \frac{a^{2n+2} + a^{-2n-2} - 2}{(a - a^{-1})^2} + \frac{a^{2n} + a^{-2n} - 2}{(a - a^{-1})^2} - (a + a^{-1}) \left(\frac{a^{2n+1} + a^{-2n-1} - a - a^{-1}}{(a - a^{-1})^2} \right) \\
&= \frac{a^2 + a^{-2} - 2}{(a - a^{-1})^2} \\
&= 1
\end{aligned}
$$

which proves (1.2). □

Lemma 1.6. *Suppose $A \in SL_2(\mathbb{C})$ and $v = \operatorname{tr}(A)$. For any integer n we have*

$$A^n = S_n(v)I - S_{n-1}(v)A^{-1}$$

where I is the 2×2 identity matrix.

Proof. By the Cayley-Hamilton Theorem we have

$$A^2 - vA + I = 0.$$

This implies that $A^n - vA^{n-1} + A^{n-2} = 0$. By induction we have $A^n = S_n(v)I - S_{n-1}(v)A^{-1}$. □

Lemma 1.7. *Let $A, B \in SL_2(\mathbb{C})$. Then we have the following identity:*

$$\operatorname{tr}\left(AB^{-1}\right) = \operatorname{tr}(A)\operatorname{tr}(B) - \operatorname{tr}(AB).$$

Proof. Let

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \qquad B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}.$$

By direct calculation we have

$$
\begin{aligned}
\operatorname{tr}\left(AB^{-1}\right) &= a_{11}b_{22} - a_{12}b_{21} + a_{22}b_{11} - a_{21}b_{12} \\
&= \operatorname{tr}(A)\operatorname{tr}(B) - a_{11}b_{11} - a_{22}b_{22} - a_{12}b_{21} - a_{21}b_{12} \\
&= \operatorname{tr}(A)\operatorname{tr}(B) - \operatorname{tr}(AB).
\end{aligned}
$$

□

CHAPTER 2

RATIONAL LINKS $C(2n, 2m+1, 2)$

In this chapter we calculate the character varieties of the rational links $C(2n, 2m+1, 2)$ and analyze the reducibility of the varieties. We follow methods presented in [5] and [6].

2.1 Group Presentation of the Rational Links $C(2n, 2m+1, 2)$

Let L denote the rational link $C(2n, 2m+1, 2)$ (see figure 2.1) and $\Gamma(L) = \pi_1(X_L)$. We begin by determining a presentation for $\Gamma(L)$ by following the discussion in section 1.2. We begin by giving L an orientation and considering the three sections of the link separately. When finding the presentation, we start from the right side of the link and move left.

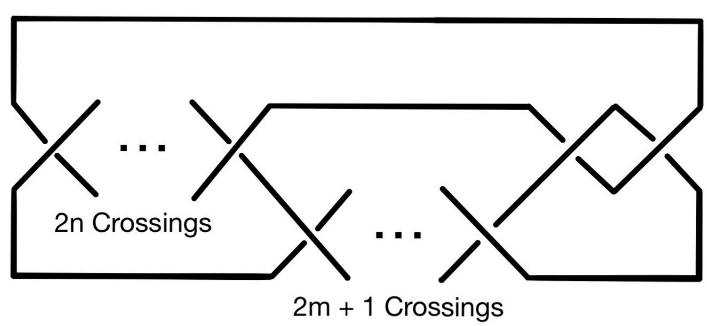

Figure 2.1. The rational link $C(2n, 2m+1, 2)$

The first section of the link consists of two crossings. We denote the loops in this section by a and b. We have the following two relations:

$$a_1 = b_2 a_2 b_2^{-1}$$

$$b_1 = a_1 b_2 a_1^{-1}.$$

The second section contains $2m+1$ crossings. We denote the loops in this section by c and d. The first two crossings of this section give the relations

$$c_1 = d_1 c_2 d_1^{-1}$$

$$d_2 = c_2 d_1 c_2^{-1}.$$

Combining these relations we obtain

$$d_2 = (c_1^{-1} d_1)^{-1} d_1 (c_1^{-1} d_1).$$

Then by induction we obtain the following

$$c_{m+1} = (c_1^{-1} d_1)^{-m} c_1 (c_1^{-1} d_1)^m$$

$$d_{m+1} = (c_1^{-1} d_1)^{-m} d_1 (c_1^{-1} d_1)^m.$$

The final section of L contains $2n$ crossings. We denote the loops in this section by e and f. Similar to the above, we consider the first two crossings of the section and by inductions we obtain the following:

$$e_{n+1} = (f_1^{-1} e_1)^{-n} e_1 (f_1^{-1} e_1)^n$$

$$f_{n+1} = (f_1^{-1} e_1)^{-n} f_1 (f_1^{-1} e_1)^n.$$

In considering figure 2.2, we have the following identifications:

$$c_1 = b_2$$

$$d_1 = b_1$$

$$e_1 = a_2$$

$$f_1 = d_{m+1}$$

$$e_{n+1} = a_1$$

$$f_{n+1} = c_{m+2}.$$

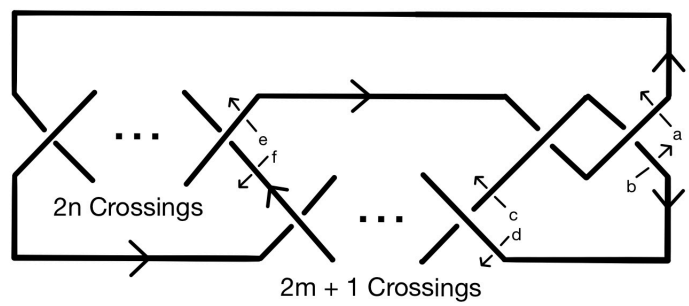

Figure 2.2. Fundamental group

Let $a_1 = a$ and $b_1 = b$. Using the identity $e_{n+1} = a$ and the relations above, we see that

$$a = (d_{m+1}^{-1}e_1)^{-n}a^{-1}b^{-1}aba(d_{m+1}^{-1}e_1)^n.$$

Which implies

$$wa = aw$$

where $w = bav^n$ and $v = d_{m+1}^{-1}e_1$. Writing $d_{m+1}^{-1}e_1$ in terms of a and b we see that

$$v = (a^{-1}b^{-1}ab)^{-m}b^{-1}(a^{-1}b^{-1}ab)^{m+1}a.$$

Hence, we have the following.

Lemma 2.1. *Let L denote the rational link $C(2n, 2m+1, 2)$ and $\Gamma(L) = \pi_1(X_L)$. We have*

$$\Gamma(L) = \langle a, b : wa = aw \rangle$$

where $w = bav^n$ and $v = (a^{-1}b^{-1}ab)^{-m}b^{-1}(a^{-1}b^{-1}ab)^{m+1}a$.

2.2 Calculating Character Variety

We will use the presentation above to calculate the character variety of $L = C(2n, 2m+1, 2)$. Since L is a rational link, the character variety will be a polynomial in the three variables $x = \mathrm{tr}(\rho(a))$, $y = \mathrm{tr}(\rho(b))$, and $z = \mathrm{tr}(\rho(ab))$.

Let $\rho : \Gamma(L) \to SL_2(\mathbb{C})$ be a nonabelian representation. Then we may assume, up to conjugation, that

$$\rho(a) = \begin{bmatrix} s_1 & 1 \\ 0 & s_1^{-1} \end{bmatrix} \qquad \rho(b) = \begin{bmatrix} s_2 & 0 \\ u & s_2^{-1} \end{bmatrix}.$$

The character variety is the polynomial determined by the matrix equation $\rho(w)\rho(a) = \rho(a)\rho(w)$. From now on, by abuse of notation, we will identify $g \in \Gamma(L)$ with its image $\rho(g) \in SL_2(\mathbb{C})$.

Proposition 2.2. *The character variety of L is given by the points $(x, y, z) \in \mathbb{C}^3$ such that $R_{n,m}(x, y, z) = 0$ where*

$$R_{n,m}(x, y, z) = S_n(\beta) - S_{n-1}(\beta)(S_m(\alpha) - S_{m-1}(\alpha))[zS_m(\alpha) - (xy - z)S_{m-1}(\alpha)].$$

Where $\beta = \operatorname{tr}(v)$ and $\alpha = \operatorname{tr}(a^{-1}b^{-1}ab)$.

Proof. We begin by finding the matrix $w = ba((a^{-1}b^{-1}ab)^{-m}b^{-1}(a^{-1}b^{-1}ab)^{m+1}a)^n$.

Let $\beta = \operatorname{tr}((a^{-1}b^{-1}ab)^{-m}b^{-1}(a^{-1}b^{-1}ab)^{m+1}a)$ and $\alpha = \operatorname{tr}(a^{-1}b^{-1}ab)$.

By lemma 1.6 we have

$$\begin{aligned} w &= ba\left[S_n(\beta)I - S_{n-1}(\beta)a^{-1}(a^{-1}b^{-1}ab)^{-(m+1)}b(a^{-1}b^{-1}ab)^m\right] \\ &= S_n(\beta)ba - S_{n-1}(\beta)\left[b(S_m(\alpha)\,b^{-1}a^{-1}ba - S_{m-1}(\alpha)I)b(S_m(\alpha)I - S_{m-1}(\alpha)\,b^{-1}a^{-1}ba)\right] \\ &= S_n(\beta)ba - S_{n-1}(\beta)\left[S_m^2(\alpha)a^{-1}bab + S_{m-1}^2(\alpha)ba^{-1}ba - S_m(\alpha)S_{m-1}(\alpha)(a^{-1}b^2a + b^2)\right]. \end{aligned}$$

The character variety is given by the matrix equation $wa - aw = 0$ where $a = \begin{bmatrix} s_1 & 1 \\ 0 & s_1^{-1} \end{bmatrix}$ and w is as above. Using Mathematica, we see that

$$wa - aw = \begin{bmatrix} s_2^{-1}u\,Q(s_1, s_2, u) & (s_1^{-2}s_2^{-2} + s_1^{-1}s_2^{-1}u - s_1^{-2})\,Q(s_1, s_2, u) \\ (s_1^{-1}s_2^{-1} - s_1s_2^{-1})u\,Q(s_1, s_2, u) & -s_2^{-1}u\,Q(s_1, s_2, u) \end{bmatrix}$$

where

$$
\begin{aligned}
Q_{n,m}(s_1,s_2,u) &= S_{n-1}(\beta)s_1^2(S_m(\alpha)S_{m-1}(\alpha)(1+s_2^2)-S_{m-1}^2(\alpha)-S_m^2(\alpha)s_2^2) \\
&\quad + S_{n-1}(\beta)(S_m(\alpha)S_{m-1}(\alpha)(1+s_2^2)-S_{m-1}^2(\alpha)s_2^2-S_m^2(\alpha)) \\
&\quad + s_1s_2(X+(S_{m-1}^2(\alpha)-S_m^2(\alpha))uS_{n-1}(\beta)).
\end{aligned}
$$

It remains to write $Q_{n,m}(s_1,s_2,u)$ in terms of x, y, and z. Note that $x=\mathrm{tr}(a)=s_1+s_1^{-1}$, $y=\mathrm{tr}(b)=s_2+s_2^{-1}$, and $z=\mathrm{tr}(ab)=s_1^{-1}s_2^{-1}+s_1s_2+u$.

Multiplying $Q_{n,m}(s_1,s_2,u)$ by $s_1^{-1}s_2^{-1}$ we get

$$
\begin{aligned}
&S_n(\beta)+S_{n-1}(\beta)\big[S_m(\alpha)S_{m-1}(\alpha)(s_1s_2+s_1s_2^{-1}+s_1^{-1}s_2+s_1^{-1}s_2^{-1}) \\
&\qquad -S_{m-1}^2(\alpha)(s_1s_2+s_1s_2^{-1}+s_1^{-1}s_2+s_1^{-1}s_2^{-1})+(S_{m-1}^2(\alpha)-S_m^2(\alpha))z\big].
\end{aligned}
$$

Since $(s_1s_2+s_1s_2^{-1}+s_1^{-1}s_2+s_1^{-1}s_2^{-1})=xy$, we have

$$
\begin{aligned}
R_{n,m}(x,y,z) &= S_n(\beta)+S_{n-1}(\beta)\left[S_m(\alpha)S_{m-1}(\alpha)xy+z(S_{m-1}^2(\alpha)-S_m^2(\alpha))-S_{m-1}^2(\alpha)xy\right] \\
&= S_n(\beta)-S_{n-1}(\beta)\left(S_m(\alpha)-S_{m-1}(\alpha)\right)\left[zS_m(\alpha)-(xy-z)S_{m-1}(\alpha)\right].
\end{aligned}
$$

□

2.3 Observations on the Reducibility of the Character Variety

Our goal is to determine the reducibility of $R_{n,m}\in\mathbb{C}[x,y,z]$. The polynomial $R_{n,m}$ contains Chebyshev polynomials in α and β, which are then polynomials in x and y. This makes it difficult to determine the reducibility of $R_{n,m}$ directly. In order to achieve our goal we will perform a birational transformation to obtain the polynomial $R_{n,m}\in\mathbb{C}[\alpha,\beta,z]$, which is easier to work with.

We first note that the cases $n=\pm1$ give the twisted Whitehead link. It is known that the character variety of the twisted Whitehead link is reducible [9]. We also note that the case $m=-1$ is ambient isotopic to $C(2(n-1))$, which is a torus link. It is also known that the character variety of torus links is reducible. Therefore, in the following discussion, we will not consider the cases $n = 1, -1$ or $m = -1$.

We begin by writing α and β in terms of x and y.

Lemma 2.3. *Let* $\alpha = \operatorname{tr}(a^{-1}b^{-1}ab)$ *and* $\beta = \operatorname{tr}(v) = \operatorname{tr}((a^{-1}b^{-1}ab)^{-m}b^{-1}(a^{-1}b^{-1}ab)^{m+1}a)$. *Then*

$$\alpha = x^2 + y^2 + z^2 - xyz - 2$$
$$\beta = (xy - z)(S_m(\alpha) - S_{m-1}(\alpha))^2 - z(\alpha - 2)S_m^2(\alpha).$$

Proof. By lemma 1.7 we have

$$\begin{aligned}
\operatorname{tr}\left(a^{-1}b^{-1}ab\right) &= \operatorname{tr}\left(a^{-1}b^{-1}\right)\operatorname{tr}\left(b^{-1}a^{-1}\right) - \operatorname{tr}\left(a^{-1}b^{-2}a^{-1}\right) \\
&= \operatorname{tr}\left(a^{-1}b^{-1}\right)^2 - \operatorname{tr}\left(a^{-1}b^{-2}\right)\operatorname{tr}(a) + \operatorname{tr}\left(b^{-2}\right) \\
&= \operatorname{tr}\left(a^{-1}b^{-1}\right)^2 - \left[\operatorname{tr}\left(a^{-1}b^{-1}\right)\operatorname{tr}(b) - \operatorname{tr}\left(a^{-1}\right)\right]\operatorname{tr}(a) + \operatorname{tr}\left(b^{-2}\right) \\
&= z^2 - [zy - x]\,x + y^2 - 2 \\
&= x^2 + y^2 + z^2 - xyz - 2.
\end{aligned}$$

In order to calculate β, we first determine the matrix $v = (a^{-1}b^{-1}ab)^{-m}b^{-1}(a^{-1}b^{-1}ab)^{m+1}a$. By lemma 1.6 we have,

$$\begin{aligned}
v &= (a^{-1}b^{-1}ab)^{-m}b^{-1}(a^{-1}b^{-1}ab)^{m+1}a \\
&= (S_m(\alpha)I - S_{m-1}(\alpha)a^{-1}b^{-1}ab)b^{-1}(S_m(\alpha)a^{-1}b^{-1}ab - S_{m-1}(\alpha)I)a \\
&= S_m^2(\alpha)b^{-1}a^{-1}b^{-1}aba + S_{m-1}^2(\alpha)a^{-1}b^{-1}a^2 - S_m(\alpha)S_{m-1}(\alpha)\left[b^{-1}a + a^{-1}b^{-2}aba\right].
\end{aligned}$$

We find the trace

$$
\begin{aligned}
\operatorname{tr}(v) &= S_m^2(\alpha)\operatorname{tr}\bigl(b^{-1}a^{-1}b^{-1}aba\bigr) + S_{m-1}^2(\alpha)\operatorname{tr}\bigl(b^{-1}a\bigr) - 2S_m(\alpha)S_{m-1}(\alpha)\operatorname{tr}\bigl(b^{-1}a\bigr)\\
&= S_m^2(\alpha)\left[\operatorname{tr}(aba)\operatorname{tr}(bab) - \operatorname{tr}\bigl((ab)^3\bigr)\right] + (xy-z)(S_{m-1}^2(\alpha) - 2S_m(\alpha)S_{m-1}(\alpha))\\
&= S_m^2(\alpha)\left[(zx-y)(zy-x) - z^3 + 3z\right] + (xy-z)(S_{m-1}^2(\alpha) - 2S_m(\alpha)S_{m-1}(\alpha))\\
&= S_m^2(\alpha)\left[(xy-z) - z(\alpha-2)\right] + (xy-z)(S_{m-1}^2(\alpha) - 2S_m(\alpha)S_{m-1}(\alpha))\\
&= (xy-z)(S_m(\alpha) - S_{m-1}(\alpha))^2 - z(\alpha-2)S_m^2(\alpha).
\end{aligned}
$$

□

Let $x = u+v$ and $y = u-v$. Then we have a birational equivalence between α, β and u^2, v^2 given by

$$
\begin{aligned}
\alpha &= 2(u^2+v^2) + z^2 - (u^2-v^2)z - 2\\
\beta &= (u^2-v^2-z)(S_m(\alpha) - S_{m-1}(\alpha))^2 - z(\alpha-2)S_m^2(\alpha).
\end{aligned}
$$

Writing $R_{n,m}$ in terms of u and v we have

$$
R_{n,m} = S_n(\beta) - S_{n-1}(\beta)(S_m(\alpha) - S_{m-1}(\alpha))[zS_m(\alpha) - (u^2-v^2-z)S_{m-1}(\alpha)].
$$

Solving β for (u^2-v^2) we get

$$
(u^2-v^2) = \frac{\beta + z\left[(S_m(\alpha) - S_{m-1}(\alpha))^2 + (\alpha-2)S_m^2(\alpha))\right]}{(S_m(\alpha) - S_{m-1}(\alpha))^2}.
$$

Putting this into $R_{n,m}$, we get the polynomial $R_{n,m} \in \mathbb{C}[\alpha, \beta, z]$ defined by

$$
\begin{aligned}
R_{n,m} &= S_n(\beta)(S_m(\alpha) - S_{m-1}(\alpha)) + \beta S_{n-1}(\beta)S_{m-1}(\alpha)\\
&\quad - z\left[(S_m(\alpha) - S_{m-1}(\alpha))^2 - (\alpha-2)S_m(\alpha)S_{m-1}(\alpha)\right] S_{n-1}(\beta)S_m(\alpha)\\
&= S_n(\beta)(S_m(\alpha) - S_{m-1}(\alpha)) + \beta S_{n-1}(\beta)S_{m-1}(\alpha) - zS_{n-1}(\beta)S_m(\alpha)
\end{aligned}
$$

Note that since we do not consider the case $n = -1$, this is a valid transformation.

Lemma 2.4. *Suppose $|n| > 1$ and $m \neq -1$. Then the polynomial $R_{n,m} \in \mathbb{C}[\alpha, \beta, z]$ is irreducible.*

Proof. We have

$$R_{n,m} = S_n(\beta)(S_m(\alpha) - S_{m-1}(\alpha)) + \beta S_{n-1}(\beta)S_{m-1}(\alpha) - zS_{n-1}(\beta)S_m(\alpha).$$

Let

$$P_{n,m} = S_n(\beta)(S_m(\alpha) - S_{m-1}(\alpha)) + \beta S_{n-1}(\beta)S_{m-1}(\alpha)$$

and

$$Q_{n,m} = S_{n-1}(\beta)S_m(\alpha).$$

Since $R_{n,m} \in \mathbb{C}[\alpha, \beta, z]$ is linear in z we only need to check that $\gcd(P_{n,m}, Q_{n,m}) = 1$.

Suppose $\gcd(P_{n,m}, Q_{n,m}) \neq 1$. Then there exists an F such that F divides P, Q and either $F \mid S_{n-1}(\beta)$ or $F \mid S_m(\alpha)$.

Suppose $F \mid S_{n-1}(\beta)$. Note that $\gcd(S_n(\beta), S_{n-1}(\beta)) = 1$. Then $F \nmid P$, which is a contradiction. So we must have $F \mid S_m(\alpha)$. But, then $F \nmid P$ for the same reason. Therefore, $\gcd(P, Q) = 1$ and $R_{n,m} \in \mathbb{C}[\alpha, \beta, z]$ is irreducible.

□

Since we have a birational equivalence between α, β and u^2, v^2, the above lemma is equivalent to $R_{n,m} \in \mathbb{C}[u^2, v^2, z]$ being irreducible unless $n = 1, -1$ or $m = -1$. Next we show that $R_{n,m} \in \mathbb{C}[u, v, z]$ is irreducible under the same conditions. In order to achieve this, we will make use of the following lemma.

Lemma 2.5. *Let $f(u, v, z) \in \mathbb{C}[u^2, v^2, z]$ be irreducible. If $f(0, 0, z) \neq g(z)^2 \in \mathbb{C}[z]$ then $f(u, v, z) \in \mathbb{C}[u, v, z]$ is irreducible.*

Proof. Suppose that $f(u, v, z) \in \mathbb{C}[u, v^2, z]$ is reducible. Then there exists $h(u, v, z) \in \mathbb{C}[u, v^2, z]$ such that $h(u, v, z)$ is prime and $h(u, v, z) \mid f(u, v, z)$. Since f is even in u, this also implies that $h(-u,v,z) \mid f(u,v,z)$. If $h(u,v,z) \neq \lambda h(-u,v,z)$, where $\lambda \in \mathbb{C}$, then

$$h(u,v,z)h(-u,v,z) \mid f(u,v,z).$$

Since $h(u,v,z)h(-u,v,z) \in \mathbb{C}[u^2,v^2,z]$ and $f(u,v,z)$ is irreducible in $\mathbb{C}[u^2,v^2,z]$ we must have $h(u,v,z)h(-u,v,z) = \lambda f(u,v,z)$. Taking $u = v = 0$ we get $h(0,0,z)^2 = \lambda f(0,0,z)$, which is a contradiction. Therefore, $f(u,v,z) \in \mathbb{C}[u,v^2,z]$ is irreducible.

Suppose that $f(u,v,z) \in \mathbb{C}[u,v,z]$ is reducible then there exists $g(u,v,z) \in \mathbb{C}[u,v,z]$ such that $g(u,v,z)$ is prime and $g(u,v,z) \mid f(u,v,z)$. If $g(u,v,z) \neq \lambda g(u,-v,z)$ then

$$g(u,v,z)g(u,-v,z) \mid f(u,v,z).$$

Since $g(u,v,z)g(u,-v,z) \in \mathbb{C}[u,v^2,z]$ and $f(u,v,z)$ is irreducible in $\mathbb{C}[u,v^2,z]$ we must have that $g(u,v,z)g(u,-v,z) = \lambda f(u,v,z)$. Taking $u = v = 0$ we get $g(0,0,z)^2 = \lambda f(0,0,z)$, which is a contradiction. $\square$

Lemma 2.6. *Suppose $|n| > 1$ and $m \neq -1$. Then $R_{n,m} \in \mathbb{C}[u,v,z]$ is irreducible.*

Proof. By the above lemma, we need to show that $R_{n,m}(u,v,z) \in \mathbb{C}[u,v,z]$ is such that $R_{n,m}(0,0,z) \neq F(z)^2$ for some $F \in \mathbb{C}[z]$. Recall that

$$\alpha = 2(u^2+v^2) + z^2 - (u^2-v^2)z - 2$$
$$\beta = (u^2-v^2-z)(S_m(\alpha) - S_{m-1}(\alpha))^2 - z(\alpha-2)S_m^2(\alpha)$$

and $R_{n,m} \in \mathbb{C}[u^2,v^2,z]$ is

$$R_{n,m} = S_n(\beta) - S_{n-1}(\beta)(S_m(\alpha) - S_{m-1}(\alpha))[zS_m(\alpha) - (u^2-v^2-z)S_{m-1}(\alpha)].$$

Let $u = 0$ and $v = 0$. Then we have

$$\alpha = z^2 - 2$$
$$\beta = -z(S_m(\alpha) - S_{m-1}(\alpha))^2 - z(z^2-4)S_m^2(\alpha)$$

and $R_{n,m}$ becomes

$$R_{n,m} = S_n(\beta) - zS_{n-1}(\beta)(S_m^2(\alpha) - S_{m-1}^2(\alpha))$$

We can write $z = a + a^{-1}$. Then by lemma 1.4 we have

$$\alpha = a^2 + a^{-2}$$

$$\beta = -a^{4m+3} - a^{-(4m+3)}.$$

Then by lemma 1.4 again $R_{n,m}$ becomes

$$R_{n,m} = \frac{(-1)^n(a^{n(4m+3)+1} - a^{-(n(4m+3)+1)})}{a - a^{-1}} = (-1)^n S_{n(4m+3)}(z).$$

This has no repeated factors. So by lemma 2.5, $R_{n,m} \in \mathbb{C}[x, y, z]$ is irreducible. $\square$

Since we have the birational equivalence between x, y and u, v given by

$$x = u + v$$

$$y = u - v$$

the above lemma shows that $R_{n,m} \in \mathbb{C}[x, y, z]$ is irreducible unless $n = 1, -1$ or $m = -1$. Therefore, we have proven the following theorem.

Theorem 2.7. *Suppose $|n| > 1$ and $m \neq -1$. Then the nonabelian character variety of the rational link $C(2n, 2m + 1, 2)$ is irreducible.*

As a corollary we have the following.

Corollary 2.8. *Suppose $|n'| > 1$ and $m' \neq 0$. Then the nonabelian character variety of the rational link $C(2n', 2m', -2)$ is irreducible.*

Proof. Consider the links $C(2n, 2m+1, 2)$. To these links we can associate the continued fraction

$$\cfrac{1}{2n + \cfrac{1}{(2m+1) + \cfrac{1}{2}}}.$$

This is equivalent to

$$\cfrac{1}{2n + \cfrac{1}{(2m+2) + \cfrac{1}{-2}}}.$$

Therefore, the links $C(2n, 2m+1, 2)$ and $C(2n, 2(m+1), -2)$ are ambient isotopic. Then let $n = n'$ and $m + 1 = m'$. $\square$

REFERENCES

[1] Burde, G. and H. Zieschang (2003). *Knots* (Second ed.). De Gruyter Studies in Mathematics 5. De Gruyter, Berlin.

[2] Culler, M. and P. B. Shalen (1983). Varieties of group representations and splittings of 3-manifolds. *Ann. of Math. 117*(1), 109–146.

[3] Lickorish, W. B. R. (1997). *An Introduction to Knot Theory*, Volume 175 of *Graduate Texts in Mathematics*. Springer-Verlag, New York.

[4] Macasieb, M. L., K. L. Petersen, and R. M. van Luijk (2011). On character varieties of two-bridge knot groups. *Proc. Lond. Math. Soc. 103*(3), 473–507.

[5] Nagasato, F., M. Suzuki, and A. T. Tran (2017). On minimality of two-bridge knots. *Internat. J. Math. 28*(03), 1750020.

[6] Nagasato, F. and A. T. Tran (2016). Some families of minimal elements for a partial ordering on prime knots. *Osaka J. Math. 53*(4), 1029–1045.

[7] Riley, R. (1992). Algebra for Heckoid groups. *Trans. Amer. Math. Soc. 334*(1), 389–409.

[8] Tran, A. T. (2015). Reidemeister torsion and Dehn surgery on twist knots. *Tokyo J. Math. 39*(2).

[9] Tran, A. T. (2016). Character varieties of $(2, 2m + 1, 2n)$ pretzel links and twisted Whitehead links. *J. Knot Theory Ramifications 25*(2), 1650007.

[10] Tran, A. T. (2017). Volumes of hyperbolic double twist knot cone-manifolds. *J. Knot Theory Ramifications 26*(11).

BIOGRAPHICAL SKETCH

Bradley Meyer received a B.S. in Biology from The University of Texas at Dallas in 2015. After receiving his B.S. degree, he taught high school science in Dallas Texas until 2017. He began his graduate work in mathematics in Fall 2017 at The University of Texas at Dallas.

Bradley Meyer

Curriculum Vitae

✉ *bradleym37@gmail.com*

Education

2019 **M.S. in Mathematics**, *The University of Texas at Dallas*, Richardson, TX.

2015 **B.S. in Biology**, *The University of Texas at Dallas*, Richardson, TX.
(magna cum laude)

Employment

2016–2017 **Secondary Science Teacher**, *Texans Can Academies*, Dallas, TX.

Teaching Experience

Texans Can Academies, *Secondary Courses*.

- Physics, 2016 - 2017
- Chemistry, 2016 - 2017
- Biology, 2017

The University of Texas at Dallas, *Teaching Assistant*.

- Biochemistry 1, Fall 2014

Languages

English Native

Spanish Basic