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Technical Note—On Optimal Policies for Inventory Systems with Batch Ordering

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We study a periodically reviewed multiechelon inventory system in series such that order quantities at every stage have to be multiples of a given stage-specific batch size. The batch sizes are nested in the sense that the batch size for every stage is an integer multiple of the batch size for its downstream stage. The problem is that of determining the policy that minimizes the expected discounted sum of costs over a finite horizon. The result is that an echelon (R, nQ) policy is optimal when demands are independent across periods or, more generally, Markov-modulated. We also comment on algorithmic implications of our result and on extensions.

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1. Introduction and Model

In this paper we study a periodically reviewed multiechelon inventory system in series with batch ordering constraints. The batch sizes are nested, that is, the batch size for every stage is an integer multiple of the batch size for its downstream stage. We show that an echelon (R, nQ) policy minimizes the expected discounted sum of costs incurred by this system over a finite horizon. This policy prescribes the following action for every echelon: if the echelon inventory position exceeds its target level, then this echelon should not order; otherwise, this echelon should order the smallest number of batches required to raise the inventory position to this target level or above, subject to inventory availability upstream. As far as motivation for the problem studied here is concerned, we refer the readers to Chen (2000), who provides an excellent discussion on it. In the remainder of this section, we present our model, assumptions, and notation in detail.

We index our stages by $i \in \{1, 2, \dots, N\}$, where N is the number of stages in the system. Stage N is supplied by an external supplier with unlimited inventory, stage N feeds stage $N - 1$, and so on, whereas stage 1 faces external demand. External demand that occurs at the most downstream stage is backordered in the event that it exceeds the available supply in a period. Holding costs are charged at all stages, and a backorder cost is charged at the demand-facing stage. The objective is to find a policy that minimizes the expected discounted sum of costs incurred over a finite planning horizon of T periods, indexed by

$t \in \{1, 2, \dots, T\}$, subject to two constraints. The first constraint is the usual material availability constraint that the amount of inventory that any stage can obtain in a period is limited by the amount of inventory available at its upstream stage. The second constraint is a batch ordering constraint that requires every order placed by any stage i to be a multiple of a given batch size Q^i . We assume throughout that the batch sizes are nested in the sense that batch sizes at upstream stages are multiples of batch sizes at downstream stages, i.e., $Q^{i+1} \in \{1 \cdot Q^i, 2 \cdot Q^i, \dots\}$. Demands in different periods are assumed to be probabilistically independent of each other.¹ We assume a one-period lead-time at every stage; this assumption is made to simplify the notation and exposition—our analysis and results extend readily to arbitrary integer lead-times.

Because of the batch ordering constraint, modular arithmetic becomes important in our analysis. For any number z and any number $s > 0$, we use $[z]^s$ to denote the remainder after dividing z by s , i.e., $[z]^s = \min\{\theta \geq 0: (z - \theta)/s \in \{\dots, -2, -1, 0, +1, +2, \dots\}\}$. Note that, by definition, $0 \leq [z]^s < s$ for all z and s .

The sequence of events in each period t is as follows. (1) Each stage i receives the delivery of the order placed in the previous period, denoted by q_{t-1}^i . Backordered demand at stage 1, if any, is satisfied to the extent possible. Then, the inventory levels and the outstanding backorder level, if any, are observed. Let $\mathbf{x}_t = (x_t^1, x_t^2, \dots, x_t^N)$ denote the vector of echelon inventory levels, where x_t^i represents the sum of inventory in stages 1 through i minus any backorder

at stage 1. (2) Then, the ordering quantity $q_t^i \geq 0$ at each stage i is decided; the ordering cost is $c_t^i \cdot q_t^i$ for some nonnegative constant c_t^i . We require that the ordering quantity q_t^i is a nonnegative integer multiple of Q^i and that these units are available in the immediate upstream stage, i.e., $[q_t^i]^{Q^i} = 0$ for $i \in \{1, \dots, N\}$, and $q_t^i \leq x_t^{i+1} - x_t^i$ for $i \in \{1, \dots, N-1\}$. We denote the after-ordering inventory position by $y_t^i = x_t^i + q_t^i$ and let $\mathbf{y}_t = (y_t^1, y_t^2, \dots, y_t^N)$. (3) Demand D_t is realized. Demand is satisfied to the extent possible with the inventory at stage 1, and any excess demand is backlogged. In the next period, $x_{t+1}^i = y_t^i - D_t$.

We assume that the holding and shortage cost associated with each period depends on the echelon inventory vector \mathbf{x}_t in a separable manner: $\tilde{G}_t(\mathbf{x}_t) = \sum_{i=1}^N \tilde{G}_t^i(x_t^i)$, where each \tilde{G}_t^i is a function that depends only on x_t^i . An example of such a function is the following, which is a standard cost function used in multiechelon inventory theory:

$$\tilde{G}_t^i(x_t^i) = \begin{cases} h^i \cdot x_t^i & \text{if } i \in \{2, \dots, N\} \\ E[h^1 \cdot (x_t^1 - D_t)^+ + (b + h^1 + \dots + h^N) \cdot (D_t - x_t^1)^+] & \text{if } i = 1, \end{cases} \quad (1)$$

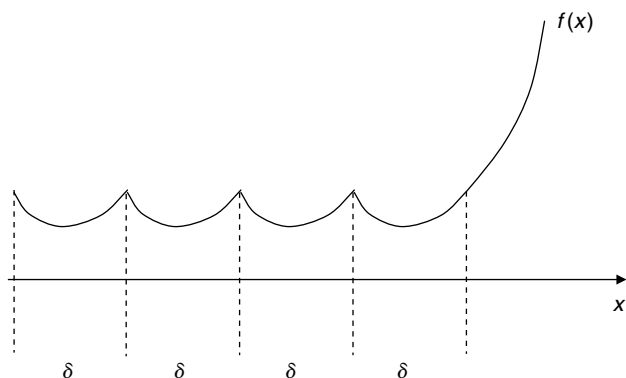
where h^i represents the per-unit echelon holding cost at echelon $i \in \{1, \dots, N\}$, and b represents the per-unit-period backorder cost.

We introduce an important concept used throughout the paper. For any number $\delta > 0$, we say a function f is δ -difference-increasing if $f(x + \delta) - f(x)$ is weakly increasing in x (please see Figure 1 for an example). This notion was used in Gallego and Toktay (2004) to analyze a single-stage model. Next, we present our assumptions on the single-period cost functions, i.e. $\tilde{G}_t^i(\cdot)$.

ASSUMPTION 1. For every i and every t , \tilde{G}_t^i is a Q^i -difference-increasing function such that, for some \bar{x} , $\tilde{G}_t^i(x + Q^i) - \tilde{G}_t^i(x) > 0$ holds whenever $x \geq \bar{x}$.

Observe that the \tilde{G}_t^i function in (1) satisfies the above assumption. To see this, note that it is convex and thus δ -difference-increasing for any δ .

Figure 1. Example of a δ -difference-increasing function.



It is more convenient to express the cost function in terms of the after-ordering echelon inventory position vector \mathbf{y}_t rather than \mathbf{x}_t . If the echelon- i inventory position in period t is y_t^i , then the echelon- i inventory level in the next period (period $t + 1$) is $y_t^i - D_t$. Because the holding and shortage cost incurred in period $t + 1$ is not affected by any decision in period $t + 1$, we define

$$G_t^i(y_t^i) = \alpha \cdot E[\tilde{G}_t^i(y_t^i - D_t)] + c_t^i \cdot y_t^i,$$

where $\alpha \in (0, 1]$ denotes the discount factor. Here, the term $c_t^i \cdot y_t^i$ is used to recognize the variable cost associated with the inventory position y_t^i . Let $G_t(\mathbf{y}_t) = \sum_{i=1}^N G_t^i(y_t^i)$. Because each \tilde{G}_t^i is Q^i -difference-increasing, each G_t^i is also Q^i -difference-increasing. (It is easy to verify that the Q^i -difference-increasing property is preserved under the expectation operator.)

Now we present the dynamic program. For any $t \in \{1, \dots, T\}$, let

$$\begin{aligned} f_t^\circ(x_t^1, \dots, x_t^N) \\ = \min_{\mathbf{y}_t} \sum_{i=1}^N G_t^i(y_t^i) + \alpha E[f_{t+1}^\circ(y_t^1 - D_t, \dots, y_t^N - D_t)] \\ - \sum_{i=1}^N c_t^i \cdot x_t^i, \\ \text{s.t. } x_t^i \leq y_t^i \leq x_t^{i+1} \quad \text{and} \quad [y_t^i]^{Q^i} = [x_t^i]^{Q^i} \\ \text{for each } i \in \{1, \dots, N\}, \end{aligned} \quad (2)$$

where we define $x_t^{N+1} = \infty$ for convenience, and $f_{T+1}^\circ(\cdot) = 0$ everywhere.

In addition, we adopt the “plausible initial state” assumption of Chen (1999), which requires that the stage on-hand inventory at stage $i + 1$ satisfies $[x_1^{i+1} - x_1^i]^{Q^i} = 0$. (Without this assumption, there will always be at least $[x_1^{i+1} - x_1^i]^{Q^i}$ units of inventory kept at stage $i + 1$ in every period.)

REMARK 1. We make an interesting observation with respect to the sequence $\{x_t^i \mid t = 1, \dots, T\}$, for any $i \in \{1, \dots, N\}$. Because the ordering quantity in each stage i is an integer multiple of Q^i , it can be shown that the sequence $\{[x_t^i]^{Q^i} \mid t = 1, \dots, T\}$ does not depend on any decision and is an exogenous sequence of random variables. In particular, $[x_t^i]^{Q^i} = [x_1^i - (D_1 + \dots + D_{t-1})]^{Q^i}$. Also, it follows that $[y_t^i]^{Q^i} = [x_t^i]^{Q^i}$. Furthermore, due to the integer ratio constraint, it follows that $[z]^{Q^i} = [z']^{Q^i}$ holds whenever $[z]^{Q^{i+1}} = [z']^{Q^{i+1}}$ for any pair of numbers z and z' .

REMARK 2. The batch ordering constraint and the plausible initial state assumption ensure that the installation- $(i + 1)$ inventories are always multiples of Q^i under any feasible policy, i.e., $[x_t^{i+1} - x_t^i]^{Q^i} = 0$ holds for any $t \in \{1, \dots, T\}$. This implies $[x_t^{i+1}]^{Q^i} = [x_t^i]^{Q^i}$.

The remainder of the paper is organized as follows. In §2, we explain how our paper is related to the literature. We analyze the model and present our results for the single echelon case in §3. This analysis and these results are then generalized to the multiechelon case in §4. Finally, in §5 we discuss extensions of our results.

2. Related Literature and Our Contribution

Veinott (1965) studies a single-stage inventory problem with batch ordering and shows that an (R, nQ) policy is optimal when the demands are *stationary*, that is, *independently and identically distributed* (i.i.d.) across periods. He also shows that the result applies to the case of nonstationary demands under a technical condition on the demand distributions—the case in which demands are stochastically increasing through time is the main example in which this condition is satisfied. To our knowledge, there has been no proof of the optimality of (R, nQ) policies for the general case of nonstationary demands for single-stage systems. Our result that such a policy is optimal even in multistage serial systems with nonstationary demands is therefore a new result, even for single-stage systems.

The model studied by Clark and Scarf (1960) is the same as ours, with the exception that it does not include the batch ordering constraint—this can be viewed as a special case of our model in which the batch size is one for every stage if the demands are integral. In this special case, our result and proof are identical to theirs. However, for the general case (i.e., with arbitrary but nested batch sizes), their proof that involves decomposing the cost-to-go function for the entire system into echelon-specific cost functions that are one-dimensional and convex does not extend immediately. It will be seen in our proof that the cost decomposition idea continues to hold but that these echelon-specific cost functions are not convex. We surmount this difficulty by showing that for every i , echelon- i 's cost function possesses a weaker property, which we call Q^i -difference-increasing, and that this property is sufficient for the optimality of (R, nQ) policies. This property was first introduced by Gallego and Toktay (2004), who study a single-location batch ordering problem with the constraint that at most one batch can be ordered in a period—their result is that a threshold policy (i.e., a policy of ordering a batch if the inventory position in a period is smaller than the threshold) is optimal.

The model studied by Chen (2000) is identical to ours, with three differences. The first difference is that the demands in different periods are assumed to be i.i.d. in his paper. We assume only independence. The second difference is that the performance measure used there is the infinite horizon, average cost per period, whereas we consider the finite horizon. The third difference is that Chen assumes demands to be integer-valued, whereas we do not require this assumption. For this problem, Chen's result is identical to ours—echelon (R, nQ) policies are optimal. The second and third differences are superficial because his proof can be modified to address those differences without changing its main structure. The first difference, on the other hand, is crucial because Chen makes use of the concept of *myopic optimality* (see Heyman and Sobel 1984), which is applicable when demands are i.i.d. but not applicable in our setting (because we assume only independence of demands).²

Our contribution to the literature is twofold. In terms of the results, ours is the first paper that establishes the optimality of (R, nQ) policies for *single-stage* and *multi-stage* systems with nonstationary demands. In terms of methodology, our contribution is the identification that the Q -difference-increasing property is an appropriate generalization of convexity that enables the analysis of inventory models without batch ordering constraints to carry over to models with these constraints. Moreover, we show in §5 that our results extend to single-stage capacitated systems, assembly systems, and systems with fixed replenishment schedules.

We conclude this section with a comment on a different stream of related literature on single-stage systems. An alternate model (which is similar in spirit to batch ordering models) is to allow arbitrary order sizes but include a fixed cost (or set-up cost) for placing orders. The standard result in this stream is that an (s, S) policy is optimal. The proofs are usually based on a property called K -convexity introduced and used by Scarf (1960) for the case of independent demands; later, this was extended to Markov-modulated demand environments to show the optimality of *state-dependent* (s, S) policies (Song and Zipkin 1993, Sethi and Cheng 1997).

3. Single-Echelon System

We consider the special case of the single-echelon problem, i.e., $N = 1$. The analysis of this case is useful for identifying structural properties for the multiechelon problem. Moreover, our results are new even for the single-stage problem as discussed in §2.

For $t \in \{1, \dots, T\}$, define

$$f_t^1(x_t^1) = -c_t^1 \cdot x_t^1 + \min_{y_t^1} \{g_t^1(y_t^1) \mid y_t^1 \geq x_t^1, [y_t^1]^{Q^1} = [x_t^1]^{Q^1}\}, \quad (3)$$

where

$$g_t^1(y_t^1) = G_t^1(y_t^1) + \alpha E[f_{t+1}^1(y_t^1 - D_t)], \quad (4)$$

and $f_{T+1}^1(\cdot) = 0$. This formulation is a special case of (2) for $N = 1$.

Proposition 3.2 below shows the optimal policy for the above formulation. We first need a preliminary definition and a lemma. It is easy to show, using Assumption 1 and induction, that for every t , $g_t^1(y + Q^1) - g_t^1(y) > 0$ for sufficiently large y . We now define

$$r_t^1 = \inf\{y \mid g_t^1(y + Q^1) - g_t^1(y) \geq 0\} \quad \text{if it exists and } r_t^1 = -\infty \text{ otherwise.} \quad (5)$$

LEMMA 3.1. Assume that a function g is Q -difference-increasing such that there exists \bar{y} satisfying $g(y + Q) - g(y) \geq 0$ for all $y \geq \bar{y}$. Let r be defined as $\inf\{y \mid g(y + Q) - g(y) \geq 0\}$ if it exists, and let $r = -\infty$ otherwise (i.e., $g(y + Q) \geq g(y)$ for all y). For any $x \in [0, Q)$, let

$s(x) = -\infty$ if $r = -\infty$; otherwise, let it be the unique number in $[r, r + Q)$ such that $[s(x)]^Q = x$. Then, the following statements hold:

- (i) For every $x \in [0, Q)$, if $r > -\infty$, then $s(x)$ is the smallest y such that $g(y + Q) - g(y) \geq 0$ and $[y]^Q = x$.
- (ii) For every $u \in \mathbb{R}$, $\max\{u, s([u]^Q)\}$ minimizes $g(y)$ over y subject to $[y]^Q = [u]^Q$ and $u \leq y$.
- (iii) For every $v \in \mathbb{R}$, $\min\{v, s([v]^Q)\}$ minimizes $g(y)$ over y subject to $[y]^Q = [v]^Q$ and $y \leq v$.

All our proofs can be found in the online appendix. An electronic companion to this paper is available as part of the online version at <http://dx.doi.org/10.1287/opre.1120.1060>. Lemma 3.1 implies that if g_t^1 is a Q^1 -difference-increasing function, then the interval $[r_t^1, r_t^1 + Q^1)$ forms an interval of minimizers for the collection of problems of the form $\min\{g_t^1(y) : [y]^{Q^1} = x\}$ for all $x \in [0, Q^1)$.

PROPOSITION 3.2. *The following statements hold for any $t \in \{1, \dots, T\}$:*

- (i) Both $f_t^1(\cdot)$ and $g_t^1(\cdot)$ are Q^1 -difference-increasing.
- (ii) An optimal solution to (3) is to order, in each period t , the minimum number of batches of size Q^1 such that the after-ordering inventory position y_t^1 is at least r_t^1 . That is, the (R, nQ) policy with $R = r_t^1$ and $Q = Q^1$ in every period t is optimal.

4. Multiechelon System

We now consider the system with N echelons. We initially assume $N = 2$. We show that an echelon (R, nQ) policy is optimal for this system. Our proof is a generalization of Clark and Scarf's (1960) proof that echelon base-stock policies are optimal (without batch ordering).

For $t \in \{1, \dots, T\}$, we define the *induced penalty function* for echelon 2 as follows:

$$\lambda_t^2(x_t^2) = \min_{y_t^1} \{g_t^1(y_t^1) \mid y_t^1 \leq x_t^2, [y_t^1]^{Q^1} = [x_t^2]^{Q^1}\} - \min_{y_t^1} \{g_t^1(y_t^1) \mid [y_t^1]^{Q^1} = [x_t^2]^{Q^1}\}, \quad (6)$$

and let $\lambda_{T+1}^2(\cdot) = 0$. Above, the constraint $[y_t^1]^{Q^1} = [x_t^2]^{Q^1}$ is equivalent to $[y_t^1]^{Q^1} = [x_t^1]^{Q^1}$ because we recall from Remark 2 that $[x_t^2]^{Q^1} = [x_t^1]^{Q^1}$. In the first minimization problem above, the after-ordering inventory position of echelon 1 is constrained by x_t^2 , the echelon-2 inventory level. This upper bound constraint does not exist in the second minimization problem. Thus, each λ_t^2 is clearly nonnegative, and λ_t^2 captures the additional cost to the single-stage system due to the upper bound constraint imposed by x_t^2 .

PROPOSITION 4.1. *For any $t \in \{1, \dots, T\}$, $\lambda_t^2(\cdot)$ is Q^1 -difference-increasing.*

We define the following dynamic program that captures the impact of the echelon-2 inventory decisions. Let $f_{T+1}^2(\cdot) = 0$. For any $t \in \{1, \dots, T\}$, define

$$f_t^2(x_t^2) = \min_{y_t^2} \{g_t^2(y_t^2) \mid y_t^2 \geq x_t^2, [y_t^2]^{Q^2} = [x_t^2]^{Q^2}\} - c_t^2 \cdot x_t^2, \quad (7)$$

where

$$g_t^2(y_t^2) = G_t^2(y_t^2) + \alpha E[\lambda_{t+1}^2(y_t^2 - D_t)] + \alpha E[f_{t+1}^2(y_t^2 - D_t)]. \quad (8)$$

It will be shown later, in Theorem 4.3, how $f_t^2(x_t^2)$ is related to the dynamic programming function $f_t^o(x_t^1, x_t^2)$ given in (2).

Note that in (8) the first term represents the impact of the current echelon-2 inventory position y_t^2 on the echelon-2 cost, the second term represents its impact on the echelon-1 cost in the next period ($t + 1$), and the third term represents the future costs. The following proposition shows that the optimal solution for the above dynamic programming formulation (7) is once again an (R, nQ) policy. It is easy to show, using Assumption 1 and induction, that for every t , $g_t^2(y + Q^2) - g_t^2(y) > 0$ for sufficiently large y . We now define

$$r_t^2 = \inf\{y \mid g_t^2(y + Q^2) - g_t^2(y) \geq 0\} \text{ if it exists and let } r_t^2 = -\infty \text{ otherwise.} \quad (9)$$

PROPOSITION 4.2. *The following statements hold for any $t \in \{1, \dots, T\}$:*

- (i) All $f_t^2(\cdot)$, $g_t^2(\cdot)$, and $\lambda_t^2(\cdot)$ are Q^2 -difference-increasing.
- (ii) An optimal policy to (7) is to order, in each period t , the minimum number of Q^2 -unit batches such that the after-ordering inventory position y_t^2 is at least r_t^2 .

We recall the dynamic program for the multiechelon system given in (2). Let

$$g_t^o(y_t^1, y_t^2) = G_t^1(y_t^1) + G_t^2(y_t^2) + \alpha E[f_{t+1}^o(y_t^1 - D_t, y_t^2 - D_t)]. \quad (10)$$

Then for $t \in \{1, \dots, T\}$,

$$f_t^o(x_t^1, x_t^2) = \min_{y_t^1, y_t^2} \{g_t^o(y_t^1, y_t^2) \mid x_t^1 \leq y_t^1 \leq x_t^2 \leq y_t^2, [y_t^i]^{Q^i} = [x_t^i]^{Q^i} \forall i \in \{1, 2\}\} - c_t^1 \cdot x_t^1 - c_t^2 \cdot x_t^2. \quad (11)$$

The following result shows that $f_t^o(x_t^1, x_t^2)$ can be written as a separable function of x_t^1 and x_t^2 , and the optimal choice of y_t^1 and y_t^2 is closely related to the optimal policy of the two single-dimensional dynamic programs (3) and (7). Recall the definitions of $f_t^1(\cdot)$, $g_t^1(\cdot)$, and r_t^1 from (3)–(5), and also the definitions of $f_t^2(\cdot)$, $g_t^2(\cdot)$, and r_t^2 from (7)–(9).

THEOREM 4.3. *The following statements hold for any $t \in \{1, \dots, T\}$:*

- (i) $g_t^o(y_t^1, y_t^2) = g_t^1(y_t^1) + g_t^2(y_t^2)$.
- (ii) An optimal policy to (11) is the following: stage 2 orders the minimum number of Q^2 -unit batches such that y_t^2 is at least r_t^2 , and stage 1 orders the minimum number of Q^1 -unit batches such that y_t^1 is at least $\min\{r_t^1, x_t^2\}$. That is, an echelon (R, nQ) policy is optimal.
- (iii) $f_t^o(x_t^1, x_t^2) = f_t^1(x_t^1) + f_t^2(x_t^2) + \lambda_t^2(x_t^2)$.

The result that an echelon (R, nQ) policy is optimal holds even when the number of stages N exceeds 2. The main addition required is to replicate (6)–(9) once for each $i \in \{3, \dots, N\}$ and to replace the superscript 2 (superscript 1) by the superscript i ($i - 1$) in these equations. Furthermore, while we have assumed one-period lead-times at all stages, our result holds when every stage i faces a procurement lead-time of $\tau^i \geq 1$ periods. Now, y_t^i denotes the echelon- i inventory position after ordering and $G_t^i(y_t^i)$ denotes $c_t^i \cdot y_t^i$ plus the discounted value of the expected holding and shortage costs experienced in echelon i in period $t + \tau^i$. In (8), the quantity $\alpha E[\lambda_{t+1}^2(y_t^2 - D_t)]$ should be replaced by $\alpha^{\tau^2} E[\lambda_{t+\tau^2}^2(y_t^2 - \sum_{t'=t}^{t+\tau^2-1} D_{t'})]$. Now the echelon (R, nQ) policy refers to the policy of raising the inventory position of every echelon i to the interval $[r_t^i, r_t^i + Q^i)$ in every period t . Finally, we remark that our results are also useful in obtaining efficient algorithms; a discussion is included in the online appendix.

5. Extensions

Capacitated Systems

For the single-echelon system described in §3, let us consider a new constraint that the ordering quantity in each period t cannot exceed $U_t \cdot Q^1$. Then Proposition 3.2 can be adapted to show that f_t^1 and g_t^1 are Q^1 -difference-increasing, and the optimal policy is a *modified* (R, nQ) policy—this policy is the same as an (R, nQ) policy if the production requirement in a period is within the capacity limit in that period; the *modification* is that if the requirement exceeds the capacity limit, then the policy prescribes that the entire capacity be used in that period. (Federgruen and Zipkin 1986 show the same result without batch ordering.) The special case in which $U_t = 1$ for all t was analyzed in Gallego and Toktay (2004). Moreover, for the multiechelon system of §4, if there is a production limit of $U_t \cdot Q^N$ for stage N while the other stages are not capacity constrained, our analysis in that section continues to hold; the optimal policy is an echelon (R, nQ) policy for echelons $1, 2, \dots, N - 1$ and it is a *modified echelon* (R, nQ) policy for echelon N .

Assembly Systems

Rosling (1989) shows how Clark and Scarf's (1960) analysis of serial systems applies to assembly systems under a reasonable assumption on the starting inventory state of the system. His main result is that echelon base-stock policies are optimal for assembly systems also. Chen (2000) applies similar ideas to show the optimality of echelon (R, nQ) policies for assembly systems with backordering, again with i.i.d. demands. Under our more general assumptions on demands (that they are independent), Rosling's ideas can be used in a straightforward manner along with our analysis in §4 to show the same result.

Fixed Replenishment Intervals

Recent papers by Van Houtum et al. (2007), Chao and Zhou (2009), and Shang and Zhou (2010) study multiechelon models in which every echelon is constrained to order according to a *fixed replenishment schedule*. By letting c_t^i take prohibitively high values for those periods in which echelon i is not allowed to order, such scheduling constraints can be accommodated by our model. Thus, our result on the optimality of echelon (R, nQ) policies continues to hold. We illustrate this idea through an example: Let echelon i be allowed to order with an ordering frequency of T^i periods and at a unit cost of c^i . Let the holding and shortage cost functions $\tilde{G}_t^i(\cdot)$ be those defined in (1). If we relax this model by allowing echelon i to order in the *other periods* also but at a unit cost $c_t^i = b \cdot T^i + c^i$, we obtain an instance of the model studied in this paper.

Moreover, it is easy to verify that the optimal policy will be such that no orders are placed in these other periods, i.e., $r_t^i = -\infty$ in these periods. This is because the maximum benefit we obtain by ordering a unit in one of these other periods, instead of waiting until the next scheduled ordering opportunity, is $b \cdot T^i$. Thus, the scheduling constraint is satisfied by the optimal policy to our model, and the optimal policy in our model coincides with the optimal policy in the schedule-constrained model. It should be noted that there is one main difference between our results and assumptions on fixed replenishment intervals and the three papers mentioned above. Those papers required that the ordering epochs for the various echelons are *nested*, in the sense that the ordering frequency at any stage is an integer multiple of the ordering frequency at its upstream stage. We do not require this assumption. The reason for this difference is the following: The papers above focus on computing the optimal policy when minimizing the infinite horizon average cost. Under the assumption of nested ordering epochs, this minimization can be achieved by solving the myopic problem of minimizing the costs over one ordering cycle for echelon N . In our case, on the other hand, we focus only on minimizing the finite horizon cost and establish only the structure of the optimal policy rather than provide a special algorithm to compute it or show myopic optimality; this is why we do not need to assume nesting.

Electronic Companion

An electronic companion to this paper is available as part of the online version at <http://dx.doi.org/10.1287/opre.1120.1060>.

Endnotes

1. It should be noted that cyclical demands are special cases of independent demands. Our model and results (§§3 and 4) easily extend to the more general case, where demands are Markov-modulated. We find it convenient for the sake of exposition to limit our discussion to independent demands.
2. A detailed explanation is provided in the online appendix.

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