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MEAN FIELD STACKELBERG GAMES: AGGREGATION OF DELAYED INSTRUCTIONS*

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Abstract. In this paper, we consider an N-player interacting strategic game in the presence of a (endogenous) dominating player, who gives direct influence on individual agents, through its impact on their control in the sense of Stackelberg game, and then on the whole community. Each individual agent is subject to a delay effect on collecting information, specifically at a delay time, from the dominating player. The size of his delay is completely known by the agent, while to others, including the dominating player, his delay plays as a hidden random variable coming from a common fixed distribution. By invoking a noncanonical fixed point property, we show that for a general class of finite N-player games, each of them converges to the mean field counterpart which may possess an optimal solution that can serve as an ϵ-Nash equilibrium for the corresponding finite N-player game. Second, we provide, with explicit solutions, a comprehensive study on the corresponding linear quadratic mean field games of small agents with delay from a dominating player. Given the information flow obtained from both the dominating player and the whole community via the mean field term, the filtration to which the control of the representative agent adapted is non-Brownian. Therefore, we propose to utilize backward stochastic dynamics (instead of the common approach through backward stochastic differential equations) for the construction of adjoint process for the resolution of his optimal control. A simple sufficient condition for the unique existence of mean field equilibrium is provided by tackling a class of nonsymmetric Riccati equations. Finally, via a study of a class of forward-backward stochastic functional differential equations, the optimal control of the dominating player is granted given the unique existence of the mentioned mean field equilibrium for small players.

Key words. mean field games, Stackelberg games, dominating player, delay information from dominating player, backward stochastic dynamics, nonsymmetric Riccati equations, forward-backward stochastic functional differential equations

AMS subject classifications. 34K50, 60H10, 60F99, 93E20

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1. Introduction.

Heinrich von Stackelberg [23] introduced a hierarchical solution for markets with leader and followers in 1934, which is now known as the *Stackelberg equilibrium.* In the context of a two-person nonzero-sum game, a follower would pick an optimal strategy based on the policies approved by the leader, who may anticipate the follower's rational reaction and announce the policies that optimize his performance index. The notion of the Stackelberg solution was later extended to a multiperiod setting; see Simaan and Cruz [22]. For the continuous time version, one may find Başar, Bensoussan, and Sethi [1] and Bensoussan, Chen, and Sethi [4] and

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the references therein. In practice, due to the heterogeneous technological advance of different agents and the presence of transaction costs, it is natural to assume that individuals have no choice but to respond slowly (with various magnitudes) to policy changes. Individuals can also gather information through the interactions with the community. This raises the importance in studying Stackelberg games which consist of many followers that receive information from the leader with various magnitudes of delay. To make it consistent with the general context of mean field games, "agents" (resp., "dominating player") would be regarded as synonymous with "followers" (resp., "leader") in this article.

On the other hand, providing a tractable analysis of interactive strategic behavior of a group of agents is normally challenging. One of the most popular modeling frameworks is through the use of stochastic differential games (SDGs) to mimic the evolutionary dynamics of interacting agents, each of whom aims to optimize his own objective functional. For example, Elliott [12] examined the relationship between the existence of the values of the zero-sum SDGs with two players. Bensoussan and Frehse [5, 6] use the dynamic programming approach to solve the nonzero-sum SDGs with N players over an infinite time horizon. In general, the nonzero-sum game problem is getting harder to tackle with the number of agents; in contrast, the problem becomes much more computable for infinitely many players due to the corresponding Hamiltonian–Jacobi–Bellman and Fokker–Planck (HJB-FP) systems. This approach was founded by Huang, Caines, and Malhamé [13, 14], who investigated SDG problems in an infinitely many players setting; independently, Lasry and Lions [16, 17, 18] studied similar problems from the viewpoint of *mean field theory* originating from physics, and they coined the novel study as *mean field games* (MFGs). Under the mean field framework, instead of highlighting the interaction between any two agents explicitly, each individual now interacts with a common medium created in the community, precisely, the *mean field term*; mathematically, this mean field term converges to a functional of the probability distribution of the whole population as the number of agents N goes to infinity.

In the contemporary MFG problems, agents are assumed to have homogeneous objectives and state dynamics, yet with independently and identically distributed noises. Each agent makes a decision purely based on his own state and the mean field term from the community; in particular, in explaining its own interaction with the community, each individual considers the mean field term as exogenous. That is to say, each individual's decision has negligible effect on the whole community—the mean field term. Thus, without loss of generality, we can just focus on one agent in an MFG and call him the *representative agent*. Mathematically, given the mean field term (as a functional of probability measure), one first solves the optimal control problem for the representative agent. By equating the given mean field term with the functional of the probability measure of all the agents caused by their optimal trajectories, i.e., the corresponding fixed point, we obtain the desired mean field term; this constitutes an equilibrium for the mean field problem and serves as an ϵ-Nash equilibrium for the finite-player counterparts. In general, the common MFG possesses the following forward-backward structure: (1) a forward dynamic describes the individual strategic behavior; (2) a backward equation describes the evolution of individual optimal strategy. For the details of the derivation of this system of equations with a forward-backward feature, consult the works of Huang, Caines, and Malhamé [13, 14], Lasry and Lions [16, 17, 18], Bensoussan, Chau, and Yam [3], Bensoussan, Frehse, and Yam [8], and Bensoussan et al. [9].

With no doubt, one can easily imagine several dimensions on generalizing the first batch of MFG results such as those mentioned above. For example, it is interesting to study the problems with agents coming from heterogeneous sources; see [8]. Besides, Huang [15] also investigated linear quadratic MFGs that consist of a significantly "big" (major) player and a huge number of "small" (minor) players, in which they considered the mean field term as *exogenous* to the major player. Nourian and Caines [20] consider a similar problem under a generalized setting again with an exogenous mean field term. *To be precise, given the mean field term, they first solve the optimal control problem for the major player. Then they proceed to solve for the optimal control for the minor player.* Under their proposed framework, the decision of the major player cannot impose his immediate influence right away on the mean field term, and this limitation restricts its scope of applications in economics and finance, as it is easily perceived that most governors have certain power (even though are not almighty) on overriding and guiding the future route of the whole community. Motivated by the latter consideration, we propose a substantially different general framework in [3], the MFGs in the presence of a "dominating player" (we rephrase it as "dominating" in order to emphasize our distinction from all previous works in the literature, such as that in [15] and [20]). Compared with the community of small agents in the MFG, the nature of the dominating role of the "big" player is clear in the sense that changes in the behavior of this dominating player would immediately and directly affect both the perception of all individual agents and the aggregated (coalesced) public information through the evolution of the mean field term. More precisely, both the optimal controls of the small agents and the mean field term are functionals of the optimal dynamics of the dominating player; see Lemma 3.4 and (3.9), respectively. In this regard, in [3], the mean field term is *endogenous* in the control problem for the dominating player. *That is, given the mean field term and the policies set by the dominating player, we first solve the optimal control for the representative agent. Regarding the aggregated (optimized) mean field term as a functional of the dominating player, we next proceed to solve for the dominator's optimal control.* The functional form of the optimal control of the representative agent is adapted to the filtration influenced by the dominating player. Our setting agrees with the philosophy of Stackelberg games, in which the dominating player has to anticipate representative agents' rational reaction before making his own optimal policy.

In this paper, we make a noticeable step forward by fundamentally generalizing the dominating small players setting in [3]. In the presence of a dominating player, we assume that there is a technological limitation to each agent so that he can only grasp the information coming from the dominating player at a delayed time. In the literature for common delay problems in stochastic controls or MFGs, they usually refer to the delay in the states or controls for agents, which are clearly adapted to Brownian-type filtrations, at most, up to a delay. In our problem formulation, the delay information is generated by a third party, that is, the dominating player. Each agent then solves his own optimal control problem based on three sources of information flows, namely,

(1) agent's own noise;
(2) delayed information coming from the dominating player;
(3) the aggregated (coalesced) public information via the mean field term.

The filtration generated by the third source through the mean field term carries extra information over the first and the second source as other agents with smaller size of delay would also contribute to the aggregated (coalesced) information, that is, the mean field term. And it is clear that this new information flow to each agent would be neither Brownian nor a postponed filtration (up to a time lag); this kind of context normally happens in the rate-dependent hysteresis phenomena in economics and finance. To one specific agent, he understands his own constraints and therefore has full knowledge of his own delay time; on the other hand, the same agent will try his best to hide this information from others, just like as happened in adverse selection markets (or principal-agent models). In principle, it is natural to assume that this private information of the exact delay time of any single agent would be a hidden random variable Δ to all other parties including the dominating player, and this Δ can also be interpreted as the distance of a random agent from the dominating player; we further assume that these hidden random variables originating from all other players follow a common distribution π_Δ independently. And the knowledge of the exact form of π_Δ is supposed to be known by the dominating player and all small agents from day one on.

In the present article, in section 2, we consider an N-player game, in the presence of a dominating player. Each individual agent collects information from the dominating player. To the same agent, the magnitude of such delay is completely known, while to all other agents and the dominating player, this delay time is a hidden random variable following a fixed distribution π_Δ. It is noted that in the present setting, each agent solves his own optimal control problem based on the mentioned three sources of information flows, which altogether should result in a non-Brownian filtration that makes the common application of the standard maximum principle via the celebrated martingale representation theorem not quite immediate. We establish the convergence of the empirical system (one over the finite number of players) to the mean field analogical one, under a noncanonical fixed point property which is rare even in the traditional stochastic control theory, indeed. Furthermore, unlike the common considerations in the existing literature, we can only conclude that this convergence is only a topological one instead of one with convergence rate of order $1/\sqrt{N}$. We also show that the mentioned novel fixed point property ensures a ϵ-Nash equilibrium for the similar game with finitely many agents. In section 3, we explicitly solve this new MFG under the linear quadratic setting. By first regarding both the mean field term and the dominating player's influence as exogenous, we solve the control problem for the representative agent which yields the stochastic version of a coupled HJB-FP equation. Since the control for each agent is now adapted to a non-Brownian filtration, we utilize the lately developed technology, called backward stochastic dynamics (see Liang, Lions, and Qian [19]), to tackle the construction issue on the underlying adjoint process, which is now satisfying a backward (stochastic) equation on a preassigned filtration. The ultimate importance and necessity of the use of backward stochastic dynamics is that it overcomes the shortcoming of the reliance of the traditional martingale representation theorem on the Brownian filtration for the construction of backward stochastic differential equations (BSDEs) satisfied by the usual adjoint processes. After the establishment of the optimal control of each agent, we apply the mentioned fixed point property in section 2 to obtain the equilibrium trajectory of the desired mean field term. We then proceed to resolve the corresponding optimal control problem for the dominating player, now by regarding the mean field term as endogenous, again via the stochastic maximum principle. By constructing an appropriate linear functional, we show that the optimal solution for the dominating player has to correspond to the solution of a particular system of forward-backward stochastic functional differential equations (FBSFDEs). Finally, a simple sufficient condition on guaranteeing the unique existence of the solution for this FBSFDE, and hence the original MFG problem, will be further discussed in section 4.

2. ϵ-Nash equilibrium. Consider an N-player game and let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space. For a fixed $T > 0$, let W_0 and W_1^i be independent Wiener processes in $\mathbb{R}^{d_0}$ and $\mathbb{R}^{d_1}$, respectively, for $i \in \{1, \dots, N\}$. Assume that the initial path, $\{\xi_0(t) : t \in [-b, 0]\}$, satisfies the average Hölder continuity, such that there exists $L > 0$,

$$\mathbb{E}|\xi_0(t) - \xi_0(s)|^2 \leq L|t - s|, \quad t, s \in [-b, 0]. \tag{2.1}$$

The initial random variables $\{\xi_1^i\}_{i \in \{1,\dots,N\}}$ are square integrable, identically and independently distributed, and independent of ξ_0. The empirical state evolutions of the dominating player and the ith player (with delay δ_i) are, respectively, described by y_0 and $y_1^{i,\delta_i,N}$, or simply y_1^{i,δ_i} if there is no ambiguity, which satisfy the SDEs

$$(2.2)\quad \begin{cases} dy_0 = g_0\Big(y_0(t), \dfrac{\sum_{j=1}^N y_1^{j,\Delta_j}(t)}{N}, v_0(t)\Big)dt + \sigma_0 dW_0(t), \\ y_0(t) = \xi_0(t), \quad t \in [-b, 0]. \end{cases}$$

$$\begin{cases} dy_1^{i,\delta_i} = g_1\Big(y_1^{i,\delta_i}(t), \dfrac{\sum_{j=1, j\neq i}^N y_1^{j,\Delta_j}(t)}{N-1}, v_1^{i,\delta_i}(t), y_0(t-\delta_i)\Big)dt + \sigma_1 dW_1^i(t), \\ y_1^{i,\delta_i}(0) = \xi_1^i. \end{cases}$$

From the ith player's perspective, $\{\Delta_j\}_{j \in \{1,\dots,i-1,i+1,\dots,N\}}$ is a sequence of identically and independently distributed random variables on $\mathbb{R}$, where Δ_j represents the delay parameter for the jth player. One may naturally assume that each participant together with the dominating player has the knowledge of the prior probability measure of Δ, which is denoted by π_Δ. Each ith player, however, only knows the magnitude of his own delay, not that of the others. Equivalently, each player's delay is private information (hidden variable) to others, which resembles an adverse selection market. In particular, we set $\Delta \in [a, b]$, where $0 \leq a \leq b$ are some fixed finite positive numbers.

Define the following filtrations:

$$\begin{aligned} \mathcal{F}_t^0 &= \begin{cases} \sigma(\xi_0(s), s \leq t), & t \in [-b, 0]; \\ \sigma(\xi_0(s), W_0(s) : s \leq t), & t > 0; \end{cases} \\ \mathcal{F}_t^{1,i} &= \sigma(\xi_1, W_1^i(s) : s \leq t), \quad t > 0. \end{aligned} \tag{2.3}$$

To the ith player, all others' delay times are hidden random variables. Without loss of generality, we just focus on the ith player, with his delay $\Delta_i = \delta_i$. Let $\mathbf{v} = (v_1^{1,\delta_1}, v_1^{2,\delta_2}, \dots, v_1^{N,\delta_N})$. The objective for the ith player is to minimize the cost functional:

$$\mathcal{J}^{i,\delta_i,N}(\mathbf{v}) = \mathbb{E}\int_0^T f_1\Big(y_1^{i,\delta_i}(t), \frac{\sum_{j=1,j\neq i}^N y_1^{j,\Delta_j}(t)}{N-1}, v_1^{i,\delta_i}(t), y_0(t-\delta_i)\Big)dt. \tag{2.4}$$

Since the ith player interacts with the population through the term $\frac{1}{N-1}\sum_{j=1,j\neq i}^N y_1^{j,\Delta_j}(t)$, the Nash equilibrium could hardly be established when N is large. Nonetheless, the mean field framework allows the complex system to provide with the approximate equilibrium, i.e., the ϵ-Nash equilibrium. For details of the ϵ-Nash equilibrium, one can refer to [8, 9, 13, 14, 16, 17, 18].

Let x_0 and x_1^{i,δ_i} be the mean field counterpart of y_0 and y_1^{i,δ_i}, respectively, satisfying the SDEs

$$
\begin{cases} \quad dx_0 = g_0(x_0(t), z(t), v_0(t))dt + \sigma_0 dW_0(t), \\ x_0(t) = \xi_0(t), \quad t \in [-b, 0]. \end{cases}
$$

(2.5)

$$
\begin{cases} \quad dx_1^{i,\delta_i} = g_1(x_1^{i,\delta_i}(t), z(t), v_1^{i,\delta_i}(t), x_0(t-\delta_i))dt + \sigma_1 dW_1^i(t), \\ x_1^{i,\delta_i}(0) = \xi_1^i. \end{cases}
$$

Here z is an appropriate process to be introduced later. In analogy, the cost functional for the ith player is given by

$$
J^{i,\delta_i}(v_1^{i,\delta_i}) = \mathbb{E}\int_0^T f_1\Big(x_1^{i,\delta_i}(t), z(t), v_1^{i,\delta_i}(t), x_0(t-\delta_i)\Big)dt. \tag{2.6}
$$

As shown later in this section, we can choose z to be a process adapted to $\mathcal{F}^0_{t-a}$ such that the empirical state y_1^{i,δ_i} converges to its mean field analogy x_1^{i,δ_i}. We call this choice of z the *mean field term*, which is not quite canonical in the contemporary literature. We consider a new filtration generated by this mean field term, which is supposed to be observable to all ith players. $\mathcal{F}^z_t := \sigma(z(s) : s \le t)$. At time t, the ith player can make his own decision based on the following information:

$\mathcal{F}^{1,i}_t$: the ith player's own noise;
$\mathcal{F}^0_{t-\delta_i}$: the delayed information from the dominating player;
$\mathcal{F}^z_t$: the public summarized information from the whole community through the mean field term.

As a consequence, it is natural to assume that $v_1^{i,\delta_i}(t)$ is $\mathcal{G}^i_t := \mathcal{F}^{1,i}_t \vee \mathcal{F}^0_{t-\delta_i} \vee \mathcal{F}^z_t$ adapted. In principle, the functional form of the optimal control of the representative agent is adapted to $\mathcal{F}^0_{t-\delta} \vee \mathcal{F}^z_t$, which is under the direct influence of the dominating player, and hence within the spirit of Stackelberg games setting.

The mean field analogy system (2.5) and (2.6) is less complex than the empirical system (2.2) and (2.4) in the sense that

1. in the mean field analogy system, for any fixed $\delta \in [a, b]$, given $\mathcal{F}^0_{t-\delta} \vee \mathcal{F}^z_t$, $\{x_1^{k,\delta}\}_k$ are independent;
2. the mean field analogy evolution of the dominating player, x_0, is free from each individuals' x_1^{i,Δ_i}, while the empirical evolution, y_0, requires individual states through the term $\frac{\sum_{j=1}^N y_1^{j,\Delta_j}(t)}{N}$.

Since the interacting term is now replaced by z, the equilibrium could be attained as shown in the following. We work on the spaces for the states and controls

$$
\begin{aligned}
&x_0 \in L^2_{\mathcal{F}^0}([-b, T]; \mathbb{R}^{n_0}); \quad v_0 \in L^2_{\mathcal{F}^0}([0, T]; \mathbb{R}^{p_0}); \\
&x_1^{i,\delta_i} \in L^2_{\mathcal{G}^i}([0, T]; \mathbb{R}^{n_1}); \quad v_1 \in L^2_{\mathcal{G}^i}([0, T]; \mathbb{R}^{p_1}); \quad z \in L^2_{\mathcal{F}^0_{\cdot-a}}([0, T]; \mathbb{R}^{n_1})
\end{aligned}
$$

with similar norms

$$
\|V\|^2 := \mathbb{E}\int |V(t)|^2 dt,
$$

where each integration takes over the respective appropriate domain of definition. We impose some standard assumptions on the coefficient functions in the SDEs. For all $x_0, x_0' \in \mathbb{R}^{n_0}$; $x_1, x_1', z, z' \in \mathbb{R}^{n_1}$; $v_0, v_0' \in \mathbb{R}^{p_0}$; and $v_1, v_1' \in \mathbb{R}^{p_1}$, we assume the following:

A.1. *Lipschitz continuity.* g_0 and g_1 are globally Lipschitz continuous in all arguments, i.e., there exists $L > 0$, such that

$$
\begin{aligned}
|g_0(x_0, z, v_0) - g_0(x_0', z', v_0')| \\
&\le L(|x_0 - x_0'| + |z - z'| + |v_0 - v_0'|);\\
|g_1(x_1, z, v_1, x_0) - g_1(x_1', z', v_1', x_0')| \\
&\le L(|x_1 - x_1'| + |z - z'| + |v_1 - v_1'| + |x_0 - x_0'|).
\end{aligned}
$$

A.2. *Linear growth.* g_0 and g_1 are of linear growth in all arguments, i.e., there exists $L > 0$, such that

$$
\begin{aligned}
|g_0(x_0, z, v_0)| &\le L(1 + |x_0| + |z| + |v_0|);\\
|g_1(x_1, z, v_1, x_0)| &\le L(1 + |x_1| + |z| + |v_1| + |x_0|).
\end{aligned}
$$

A.3. *Quadratic condition* (see (A.5) in Carmona and Delarue [10]). There exists $L > 0$ such that

(2.7)
$$
\begin{aligned}
|f_1(x_1, z, v_1, x_0) - f_1(x_1', z', v_1', x_0')| \le L\Big[1 + |x_1| + |x_1'| + |z| + |z'| + |v_1| + |v_1'| + |x_0| \\
+ |x_0'|\Big] \\
\cdot \Big[|x_1 - x_1'| + |z - z'| + |v_1 - v_1'| + |x_0 - x_0'|\Big].
\end{aligned}
$$

2.1. Convergence of an N-player system. Assume that the ith player adopts the optimal controls u_1^{i,δ_i} defined by (2.5) and (2.6), with the corresponding trajectories y_1^{i,δ_i} and x_1^{i,δ_i}. By choosing a suitable process $z(t)$ adapted to $\mathcal{F}^0_{t-a}$, we can establish the convergence of $y_0 \to x_0$ and $y_1^{i,\delta_i} \to x_1^{i,\delta_i}$ by showing a stronger result that

$$
\mathbb{E}\sup_{u\le T}|y_0(u) - x_0(u)|^2 + \mathbb{E}\sup_{\delta\in[a,b]}\sup_{u\le T}|y_1^{i,\delta}(u) - x_1^{i,\delta}(u)|^2 \to 0 \quad \text{as } N \to +\infty.
$$

In the following, $K(c)$ denotes a constant only depending on c (but not N), which may vary from line to line. First, we observe that for any $t \le T$,

(2.8)
$$
\begin{aligned}
&\mathbb{E}\sup_{u\le t}|y_0(u) - x_0(u)|^2 + \mathbb{E}\sup_{\delta\in[a,b]}\sup_{u\le t}|y_1^{i,\delta}(u) - x_1^{i,\delta}(u)|^2\\
&\le K(T,L)\mathbb{E}\int_0^t \Bigg[\sup_{u\le s}|y_0(u) - x_0(u)|^2 + \sup_{\delta\in[a,b]}\sup_{u\le s}|y_1^{i,\delta}(u) - x_1^{i,\delta}(u)|^2\\
&\qquad + \left|\frac{\sum_{j=1}^N y_1^{j,\Delta_j}(s)}{N} - z(s)\right|^2 + \left|\frac{\sum_{j=1, j\ne i}^N y_1^{j,\Delta_j}(s)}{N-1} - z(s)\right|^2\Bigg] ds.
\end{aligned}
$$

For the third term, observe that

$$
\begin{aligned}
&\mathbb{E}\left|\frac{\sum_{j=1}^N y_1^{j,\Delta_j}(s)}{N} - z(s)\right|^2\\
&\le 2\mathbb{E}\frac{\sum_{j=1}^N |y_1^{j,\Delta_j}(s) - x_1^{j,\Delta_j}(s)|^2}{N} + 2\mathbb{E}\left|\frac{\sum_{j=1}^N x_1^{j,\Delta_j}(s)}{N} - z(s)\right|^2\\
&= 2\mathbb{E}|y_1^{i,\Delta_i}(s) - x_1^{i,\Delta_i}(s)|^2 + 2\mathbb{E}\left|\frac{\sum_{j=1}^N x_1^{j,\Delta_j}(s)}{N} - z(s)\right|^2\\
&\le 2\mathbb{E}\sup_{\delta\in[a,b]}\sup_{u\le s}|y_1^{i,\delta}(u) - x_1^{i,\delta}(u)|^2 + 2\mathbb{E}\left|\frac{\sum_{j=1}^N x_1^{j,\Delta_j}(s)}{N} - z(s)\right|^2,
\end{aligned}
$$

where the equality results from symmetry as each player adopts the corresponding optimal controls. Similar inequality could be obtained for the forth term. Hence, using the Gronwall's inequality, (2.8) becomes

(2.9)
$$\begin{aligned}
&\mathbb{E}\sup_{u\le t}|y_0(u)-x_0(u)|^2+\mathbb{E}\sup_{\delta\in[a,b]}\sup_{u\le t}|y_1^{i,\delta}(u)-x_1^{i,\delta}(u)|^2\\
&\le K(T,L)\mathbb{E}\int_0^t\Bigg[\sup_{u\le s}|y_0(u)-x_0(u)|^2+\sup_{\delta\in[a,b]}\sup_{u\le s}|y_1^{i,\delta}(u)-x_1^{i,\delta}(u)|^2\\
&\qquad+\left|\frac{\sum_{j=1}^N x_1^{j,\Delta_j}(s)}{N}-z(s)\right|^2+\left|\frac{\sum_{j=1,j\neq i}^N x_1^{j,\Delta_j}(s)}{N-1}-z(s)\right|^2\Bigg]ds\\
&\le K(T,L)\mathbb{E}\int_0^t\Bigg[\sup_{u\le s}|y_0(u)-x_0(u)|^2+\sup_{\delta\in[a,b]}\sup_{u\le s}|y_1^{i,\delta}(u)-x_1^{i,\delta}(u)|^2\\
&\qquad+\left|\frac{x_1^{i,\Delta_i}(s)-z(s)}{N}\right|^2+\left(\left(\frac{N-1}{N}\right)^2+1\right)\left|\frac{\sum_{j=1,j\neq i}^N x_1^{j,\Delta_j}(s)}{N-1}-z(s)\right|^2\Bigg]ds\\
&\le e^{K(T,L)t}\int_0^t\mathbb{E}\left|\frac{x_1^{i,\Delta_i}(s)-z(s)}{N}\right|^2+\mathbb{E}\left|\frac{\sum_{j=1,j\neq i}^N x_1^{j,\Delta_j}(s)}{N-1}-z(s)\right|^2ds.
\end{aligned}$$

The first term clearly vanishes as $N\to+\infty$. By putting $t=T$, whether (2.9) converges to zero would only depend on

$$\int_0^T\mathbb{E}\left|\frac{\sum_{j=1,j\neq i}^N x_1^{j,\Delta_j}(s)}{N-1}-z(s)\right|^2ds\to 0\quad\text{or not.}$$

We now proceed to establish this fact step by step as follows.

2.1.1. Constant Δ. We first consider the simplest case that $\Delta=\delta=a=b$, i.e., the amplitude of delay is homogeneous among all players. Under our problem formulation, since $\mathcal{F}_t^z\subset\mathcal{F}_{t-\delta_i}^0$, the control u_1^{i,δ_i} is adapted to $\mathcal{F}_{t-\delta_i}^0\vee\mathcal{F}_t^{1,i}$. Observe that given that $\mathcal{F}_{t-\delta}^0$, $\{x_1^{j,\delta}(t)\}_j$ are independent, we have

$$\begin{aligned}
&\mathbb{E}\left|\frac{\sum_{j=1,j\neq i}^N x_1^{j,\delta}(s)}{N-1}-z(s)\right|^2\\
&=\frac{1}{(N-1)^2}\mathbb{E}\Bigg[\sum_{j=1,j\neq i}^N\left|x_1^{j,\delta}(s)-z(s)\right|^2+2\sum_{j<k,j,k\neq i}^N\mathbb{E}^{\mathcal{F}_{t-\delta}^0}[x_1^{j,\delta}(s)-z(s)]\\
&\qquad\cdot\mathbb{E}^{\mathcal{F}_{t-\delta}^0}[x_1^{k,\delta}(s)-z(s)]\Bigg].
\end{aligned}$$

If the mean field term is chosen to satisfy the fixed point

$$z(t)=\mathbb{E}^{\mathcal{F}_{t-\delta}^0}x_1^{1,\delta},$$

the cross terms vanish. Hence by symmetric consideration on $\{x_1^{j,\delta}(s)-z(s)\}_j$, we obtain

$$\mathbb{E}\left|\frac{\sum_{j=1,j\neq i}^N x_1^{j,\delta}(s)}{N-1}-z(s)\right|^2=\frac{1}{N-1}\mathbb{E}\left|x_1^{1,\delta}(s)-z(s)\right|^2\to 0\quad\text{as }N\to+\infty.\tag{2.10}$$

2.1.2. Discrete Δ. We next consider Δ to be discretely distributed with

$$\mathbb{P}(\Delta=a_k)=p_k,\quad\text{where }a=a_0<a_1<\cdots<a_n=b,$$

with a fixed finite n. One can regard p_k to be the population proportion with delay a_k. Recall that π_Δ, the distribution of Δ, is known by all players, while the ith player only knows his delay magnitude δ_i. Recall that the mean field term $z(t)$ is assumed to be $\mathcal{F}^0_{t-a}$ adapted, while $u^{i,\delta_i}(t)$ is adapted to $\mathcal{G}^i_t := \mathcal{F}^{1,i}_t \vee \mathcal{F}^0_{t-\delta_i} \vee \mathcal{F}^z_t$.

LEMMA 2.1. *If the mean field term satisfies the fixed point property,*

$$z(t) = p_0 z^{a_0}(t) + p_1 z^{a_1}(t) + \cdots + p_n z^{a_n}(t),$$

where $z^{a_i}(s) = \mathbb{E}^{\mathcal{F}^0_{s-a_i} \vee \mathcal{F}^z_s} x_1^{1,a_i}(s)$, *then*

$$\mathbb{E}\left| \frac{\sum_{j=1, j\neq i}^N x_1^{j,\Delta_j}(s)}{N-1} - z(s) \right|^2 \to 0 \quad \text{as } N \to +\infty.$$

Proof. We denote $M = (M_0, M_1, \ldots, M_n)$ to be the multinomial random variable so that M_k counts the number of players in the kth hysteresis group. Given M, by permutation symmetry, we can re-index the players without altering the conditional expectation. Hence, without loss of generality, we can assume the first M_0 players have $\Delta = a_0$. Then, the next M_1 players have $\Delta = a_1$ and so on. Thus,

$$\begin{aligned} &\mathbb{E}\left| \frac{\sum_{j=1, j\neq i}^N x_1^{j,\Delta_j}(s)}{N-1} - z(s) \right|^2 \\ &= \mathbb{E}\left\{ \mathbb{E}\left[\left| \frac{\sum_{j=1, j\neq i}^N x_1^{j,\Delta_j}(s)}{N-1} - z(s) \right|^2 \middle| M \right] \right\} \\ &= \mathbb{E}\left\{ \mathbb{E}\left[\left| \frac{\sum_{M_0} x_1^{j,a_0}(s) + \sum_{M_1} x_1^{j,a_1}(s) + \cdots + \sum_{M_n} x_1^{j,a_n}(s)}{N-1} - z(s) \right|^2 \middle| M \right] \right\}. \end{aligned}$$

To show the convergence as $N \to \infty$, since $z = p_0 z^{a_0} + p_1 z^{a_1} + \cdots + p_n z^{a_n}$, we have

$$\begin{aligned} &\mathbb{E}\left\{ \mathbb{E}\left[\left| \frac{\sum_{M_0} x_1^{j,a_0}(s) + \cdots + \sum_{M_n} x_1^{j,a_n}(s)}{N-1} - z(s) \right|^2 \middle| M \right] \right\} \\ &\le n \sum_{k=0}^n \mathbb{E}\left\{ \mathbb{E}\left[\left| \frac{M_k}{N-1} \frac{\sum_{M_k} x_1^{j,a_k}(s)}{M_k} - p_k z^{a_k}(s) \right|^2 \middle| M \right] \right\} \\ &\le 2n \sum_{k=0}^n \mathbb{E}\left\{ \mathbb{E}\left[\left| \frac{M_k}{N-1} \left(\frac{\sum_{M_k} x_1^{j,a_k}(s)}{M_k} - z^{a_k}(s) \right) \right|^2 \right.\right. \\ &\qquad \left.\left. + \left| \left(\frac{M_k}{N-1} - p_{a_k} \right) z^{a_k}(s) \right|^2 \middle| M \right] \right\} \\ &= 2n \sum_{k=0}^n \mathbb{E}\left\{ \left(\frac{M_k}{N-1} \right)^2 \mathbb{E}\left[\left| \frac{\sum_{M_k} x_1^{j,a_k}(s)}{M_k} - z^{a_k}(s) \right|^2 \middle| M \right] \right\} \\ &\qquad + 2n \sum_{k=0}^n \mathbb{E}\left(\frac{M_k}{N-1} - p_{a_k} \right)^2 \mathbb{E}\left| z^{a_k}(s) \right|^2. \end{aligned} \tag{2.11}$$

Observe that given $\mathcal{F}^0_{s-a_k} \vee \mathcal{F}^z_s$, $\{x_1^{j,a_k}(s)\}_j$ are independent. By assumption $z^{a_k}(s) = \mathbb{E}^{\mathcal{F}^0_{s-a_k} \vee \mathcal{F}^z_s} x_1^{1,a_k}(s)$, together with the derivation as in (2.10) in section 2.1.1, we have

$$\begin{aligned} &\sum_{k=0}^n \mathbb{E}\left\{ \left(\frac{M_k}{N-1} \right)^2 \mathbb{E}\left[\left| \frac{\sum_{M_k} x_1^{j,a_k}(s)}{M_k} - z^{a_k}(s) \right|^2 \middle| M \right] \right\} \\ &= \sum_{k=0}^n \mathbb{E}\left\{ \left(\frac{M_k}{N-1} \right)^2 \left(\frac{1}{M_k} \right) \mathbb{E}\left| x_1^{1,a_k}(s) - z^{a_k}(s) \right|^2 \right\} \\ &= \frac{1}{N-1} \sum_{k=0}^n p_{a_k} \mathbb{E}\left| x_1^{1,a_k}(s) - z^{a_k}(s) \right|^2, \end{aligned} \tag{2.12}$$

which goes to 0 as $N \to \infty$. On the other hand, the second term

$$\sum_{k=0}^{n} \mathbb{E}\left(\frac{M_k}{N-1} - p_{a_k}\right)^2 \mathbb{E}\left|z^{a_k}(s)\right|^2 = \frac{1}{N-1}\sum_{k=0}^{n} p_{a_k}(1-p_{a_k})\mathbb{E}\left|z^{a_k}(s)\right|^2 \tag{2.13}$$

clearly vanishes as $N \to \infty$. ◻

2.1.3. Continuum Δ. In this subsection, we consider Δ being distributed on $[a,b]$ with an absolutely continuous measure π_Δ. We make some additional assumptions on the optimal control

A.4. *Continuity on optimal controls.* There exists $C > 0$ such that $[a,b] \ni \delta \mapsto u_1^{i,\delta}$ is Lipschitz continuous, i.e.,

$$\mathbb{E}\int_0^T |u_1^{i,\delta}(s) - u_1^{i,\gamma}(s)|^2 ds \le C|\delta - \gamma| \quad \forall \delta, \gamma \in [a,b].$$

A.5. *Boundedness on optimal controls.* There exists $C > 0$ such that

$$\mathbb{E}\int_0^T \sup_{\delta\in[a,b]} |u_1^{i,\delta}(s)|^2 ds < \infty.$$

This condition could be verified for the optimal solution to be obtained in the rest of this paper. Using the ideas in Lemma 2.1, we first assume the mean field term satisfies the fixed point property

$$z(s) = \int_{[a,b]} z^\delta(s) d\pi_\Delta(\delta) = \int_{[a,b]} \mathbb{E}^{\mathcal{F}^0_{s-\delta}\vee\mathcal{F}^z_s} x_1^{\cdot,\delta}(s) d\pi_\Delta(\delta). \tag{2.14}$$

To validate the limiting argument in this section, we make two claims in which we omit the proofs here, as they are pretty standard.

C.1. *Boundedness.*

$$\begin{aligned} &\mathbb{E}\sup_{u\le T} |x_0(u)|^2 + \mathbb{E}\sup_{\delta\in[a,b]}\sup_{u\le T} |x_1^{i,\delta}(u)|^2 < \infty; \\ &\limsup_{N\to\infty}\max_{1\le i\le N}\left[\mathbb{E}\sup_{u\le T}|y_0(u)|^2 + \mathbb{E}\sup_{\delta\in[a,b]}\sup_{u\le T}|y_1^{i,\delta}(u)|^2\right] < \infty. \end{aligned} \tag{2.15}$$

C.2. *Hölder continuity.*

$$\mathbb{E}\int_0^T |x_1^{i,\delta}(s) - x_1^{i,\gamma}(s)|^2 ds \le K|\delta-\gamma| \quad \forall \delta,\gamma \in [a,b], \tag{2.16}$$

where K is a constant depending on $T, L, C, \sigma_0, \sigma_1, u_0$, and $\sup_\delta u_1^{i,\delta}$.

THEOREM 2.2. *Suppose that z satisfies* (2.14). *Then we have*

$$\int_0^T \mathbb{E}\left|\frac{\sum_{j=1,j\ne i}^N x_1^{j,\Delta_j}(s)}{N-1} - z(s)\right|^2 ds \to 0, \quad \text{as } N \to +\infty.$$

Proof. Let $\{a_k^{(n)}\}_{k=0}^n$ be the level n uniform partition on $[a,b]$, i.e., $a_k^{(n)} = a + \frac{k}{n}(b-a); k = 0,1,\dots,n$. On the other hand, let $M^{(n)} = (M_1^{(n)}, M_2^{(n)}, \dots, M_n^{(n)})$ be the multinomial random variable on $\{a_1^{(n)}, a_2^{(n)}, \dots, a_n^{(n)}\}$ with event probabilities $p_k^{(n)} := \pi_\Delta(a_{k-1}^{(n)}, a_k^{(n)}]$ (hence $\sum_{k=1}^n p_k^{(n)} = 1$). Define the conditional random variable

$$(\Delta_j)_k^{(n)} = \Delta_j|_{\Delta_j\in(a_{k-1}^{(n)}, a_k^{(n)}]}, \quad j = 1,\dots,N \text{ and } k = 1,\dots,n.$$

Hence we have

$$
\begin{aligned}
&\int_0^T \mathbb{E}\left|\frac{\sum_{j=1,j\neq i}^N x_1^{j,\Delta_j}(s)}{N-1} - z(s)\right|^2 ds \\
&= \int_0^T \mathbb{E}\left\{\mathbb{E}\left[\left|\frac{\sum_{M_1^{(n)}} x_1^{j,(\Delta_j)_1^{(n)}}(s) + \cdots + \sum_{M_n^{(n)}} x_1^{j,(\Delta_j)_n^{(n)}}(s)}{N-1} - z(s)\right|^2 \middle| M^{(n)}\right]\right\} ds \\
&\le 3\int_0^T \mathbb{E}\left\{\mathbb{E}\left[\left|\frac{\sum_{M_1^{(n)}}\left(x_1^{j,(\Delta_j)_1^{(n)}}(s) - x_1^{j,a_1^{(n)}}(s)\right)}{N-1} + \cdots \right.\right.\right. \\
&\qquad\qquad \left.\left.\left. + \int_0^T \frac{\sum_{M_n^{(n)}}\left(x_1^{j,(\Delta_j)_n^{(n)}}(s) - x_1^{j,a_n^{(n)}}(s)\right)}{N-1}\right|^2 \middle| M^{(n)}\right]\right\} ds \\
&\quad + 3\int_0^T \mathbb{E}\left\{\mathbb{E}\left[\left|\frac{\sum_{M_1^{(n)}} x_1^{j,a_1^{(n)}}(s)}{N-1} + \cdots + \frac{\sum_{M_n^{(n)}} x_1^{j,a_n^{(n)}}(s)}{N-1} - z^{(n)}(s)\right|^2 \middle| M^{(n)}\right]\right\} ds \\
&\quad + 3\int_0^T \mathbb{E}|z^{(n)}(s) - z(s)|^2 ds.
\end{aligned} \tag{2.17}
$$

Here we set $z^{(n)}(t) = \sum_{k=1}^n p_k^{(n)} z^{a_k^{(n)}}(t) = \sum_{k=1}^n p_k^{(n)} \mathbb{E}^{\mathcal{F}^0_{t-a_k^{(n)}} \vee \mathcal{F}^z_t} x_1^{1,a_k^{(n)}}(t)$. The first term in (2.17) is controlled by

$$
\begin{aligned}
&\int_0^T n\sum_{k=1}^n \mathbb{E}\left\{\mathbb{E}\left[\left|\frac{\sum_{M_k^{(n)}}\left(x_1^{j,(\Delta_j)_k^{(n)}}(s) - x_1^{j,a_k^{(n)}}(s)\right)}{N-1}\right|^2 \middle| M^{(n)}\right]\right\} ds \\
&= \int_0^T n\sum_{k=1}^n \mathbb{E}\left\{\left(\frac{M_k^{(n)}}{N-1}\right)^2 \mathbb{E}\left[\left|\frac{\sum_{M_k^{(n)}}\left(x_1^{j,(\Delta_j)_k^{(n)}}(s) - x_1^{j,a_k^{(n)}}(s)\right)}{M_k^{(n)}}\right|^2 \middle| M^{(n)}\right]\right\} ds \\
&= \int_0^T n\sum_{k=1}^n \left(\frac{p_k^{(n)}(1-p_k^{(n)})}{N-1} + (p_k^{(n)})^2\right)\mathbb{E}\left[\left|x_1^{j,(\Delta_j)_k^{(n)}}(s) - x_1^{j,a_k^{(n)}}(s)\right|^2\right] ds,
\end{aligned}
$$

where the last equality holds due to the symmetry of $\{x_1^{j,(\Delta_j)_k^{(n)}}(s) - x_1^{j,a_k^{(n)}}(s)\}_j$ within each hysteresis group. Using the Lipschitz property (2.16), we have

$$
\begin{aligned}
&\int_0^T \mathbb{E}\left\{\mathbb{E}\left[\left|\frac{\sum_{M_1^{(n)}}\left(x_1^{j,(\Delta_j)_1^{(n)}}(s) - x_1^{j,a_1^{(n)}}(s)\right)}{N-1} + \cdots \right.\right.\right. \\
&\qquad \left.\left.\left. + \frac{\sum_{M_n^{(n)}}\left(x_1^{j,(\Delta_j)_n^{(n)}}(s) - x_1^{j,a_n^{(n)}}(s)\right)}{N-1}\right|^2 \middle| M^{(n)}\right]\right\} ds \\
&\le K\int_0^T n\sum_{k=1}^n \left(\frac{p_k^{(n)}(1-p_k^{(n)})}{N-1} + (p_k^{(n)})^2\right)\mathbb{E}\left|(\Delta_j)_k^{(n)} - a_k^{(n)}\right| ds \qquad (2.18) \\
&\le Kn\sum_{k=1}^n \left(\frac{p_k^{(n)}(1-p_k^{(n)})}{N-1} + (p_k^{(n)})^2\right)\frac{1}{n} \\
&\le K\left(\frac{1}{N-1} + \sup_{k\le n} p_k^{(n)}\right),
\end{aligned}
$$

For the second term, by plugging (2.12) and (2.13) into (2.11) in Lemma 2.1, we have

$$
\begin{aligned}
&\int_0^T \mathbb{E}\bigg\{\mathbb{E}\bigg[\Big|\frac{\sum_{M_1^{(n)}} x_1^{j,a_1^{(n)}}(s)}{N-1}+\cdots+\frac{\sum_{M_n^{(n)}} x_1^{j,a_n^{(n)}}(s)}{N-1}-z^{(n)}(s)\Big|^2\Big|M^{(n)}\bigg]\bigg\}ds\\
&\le \int_0^T \frac{2n}{N-1}\sum_{k=1}^n p_k^{(n)}\mathbb{E}\Big|x_1^{j,a_k^{(n)}}(s)-z^{a_k^{(n)}}(s)\Big|^2+\frac{2n}{N-1}\sum_{k=1}^n p_k^{(n)}(1-p_k^{(n)})\mathbb{E}\Big|z^{a_k^{(n)}}(s)\Big|^2 ds\\
&\le \int_0^T \frac{2n}{N-1}\sum_{k=1}^n p_k^{(n)}\mathbb{E}\Big|x_1^{j,a_k^{(n)}}(s)-z^{a_k^{(n)}}(s)\Big|^2+\frac{2n}{N-1}\sum_{k=1}^n p_k^{(n)}\mathbb{E}\Big|x_1^{j,a_k^{(n)}}(s)\Big|^2 ds\\
&= \int_0^T \frac{2n}{N-1}\sum_{k=1}^n p_k^{(n)}\Big(\mathbb{E}\Big|x_1^{j,a_k^{(n)}}(s)-z^{a_k^{(n)}}(s)\Big|^2+\mathbb{E}\Big|x_1^{j,a_k^{(n)}}(s)\Big|^2\Big)ds.
\end{aligned}
$$

Using the bound given by (2.15), we have

$$
\begin{aligned}
&\int_0^T \mathbb{E}\bigg\{\mathbb{E}\bigg[\Big|\frac{\sum_{M_1^{(n)}} x_1^{j,a_1^{(n)}}(s)}{N-1}+\cdots+\frac{\sum_{M_n^{(n)}} x_1^{j,a_n^{(n)}}(s)}{N-1}-z^{(n)}(s)\Big|^2\Big|M^{(n)}\bigg]\bigg\}ds\\
&\le \Big(10T\mathbb{E}\sup_{\delta\in[a,b]}\sup_{u\le T}|x_1^{i,\delta}(u)|^2\Big)\frac{n}{N-1}.
\end{aligned}
\tag{2.19}
$$

For the third term in (2.17), observe that

$$
\begin{aligned}
&\int_0^T \mathbb{E}\Big|z^{(n)}(s)-z(s)\Big|^2 ds\\
&= \int_0^T \mathbb{E}\Big|\sum_{k=1}^n p_k^{(n)}\mathbb{E}^{\mathcal{F}^0_{s-a_k^{(n)}}\vee\mathcal{F}^z_s} x_1^{1,a_k^{(n)}}(s)-\int_{[a,b]}\mathbb{E}^{\mathcal{F}^0_{s-\delta}\vee\mathcal{F}^z_s}x_1^{1,\delta}(s)d\pi_\Delta(\delta)\Big|^2 ds\\
&= \int_0^T \mathbb{E}\Big|\int_{[a,b]}\Big[\sum_{k=1}^n \mathbb{I}_{(a_{k-1}^{(n)},a_k^{(n)}]}(\delta)\mathbb{E}^{\mathcal{F}^0_{s-a_k^{(n)}}\vee\mathcal{F}^z_s} x_1^{1,a_k^{(n)}}(s)-\mathbb{E}^{\mathcal{F}^0_{s-\delta}\vee\mathcal{F}^z_s}x_1^{1,\delta}(s)\Big]d\pi_\Delta(\delta)\Big|^2 ds\\
&\le 2\int_0^T \mathbb{E}\int_{[a,b]}\Big|\sum_{k=1}^n \mathbb{I}_{(a_{k-1}^{(n)},a_k^{(n)}]}(\delta)\mathbb{E}^{\mathcal{F}^0_{s-a_k^{(n)}}\vee\mathcal{F}^z_s}\Big(x_1^{1,a_k^{(n)}}(s)-x_1^{1,\delta}(s)\Big)\Big|^2 d\pi_\Delta(\delta)ds\\
&\quad+2\int_0^T \mathbb{E}\int_{[a,b]}\Big|\sum_{k=1}^n \mathbb{I}_{(a_{k-1}^{(n)},a_k^{(n)}]}(\delta)\mathbb{E}^{\mathcal{F}^0_{s-a_k^{(n)}}\vee\mathcal{F}^z_s} x_1^{1,\delta}(s)-\mathbb{E}^{\mathcal{F}^0_{s-\delta}\vee\mathcal{F}^z_s}x_1^{1,\delta}(s)\Big|^2 d\pi_\Delta(\delta)ds.
\end{aligned}
\tag{2.20}
$$

For the first term on the right-hand side of (2.20), by the Lipschitz property (2.16),

$$
\begin{aligned}
&\int_0^T \mathbb{E}\int_{[a,b]}\Big|\sum_{k=1}^n \mathbb{I}_{(a_{k-1}^{(n)},a_k^{(n)}]}(\delta)\mathbb{E}^{\mathcal{F}^0_{s-a_k^{(n)}}\vee\mathcal{F}^z_s}\Big(x_1^{1,a_k^{(n)}}(s)-x_1^{1,\delta}(s)\Big)\Big|^2 d\pi_\Delta(\delta)ds\\
&= \int_0^T \mathbb{E}\sum_{k=1}^n\int_{(a_{k-1}^{(n)},a_k^{(n)}]}\Big|\mathbb{E}^{\mathcal{F}^0_{s-a_k^{(n)}}\vee\mathcal{F}^z_s}\Big(x_1^{1,a_k^{(n)}}(s)-x_1^{1,\delta}(s)\Big)\Big|^2 d\pi_\Delta(\delta)ds\\
&\le K\sum_{k=1}^n\int_{(a_{k-1}^{(n)},a_k^{(n)}]}\frac{1}{n}d\pi_\Delta(\delta)\\
&= \frac{K}{n}.
\end{aligned}
$$

For the second term on the right-hand side of (2.20), for any fixed s and δ, there exists a sequence of $(n,k;k\le n)$ such that $a_k^{(n)}\to\delta$; by applying the martingale convergence theorem,

$$\sum_{k=1}^{n} \mathbb{I}_{(a_{k-1}^{(n)}, a_k^{(n)}]}(\delta) \mathbb{E}^{\mathcal{F}^0_{s-a_k^{(n)}} \vee \mathcal{F}^z_s} x_1^{1,\delta}(s) \to \mathbb{E}^{\mathcal{F}^0_{s-\delta} \vee \mathcal{F}^z_s} x_1^{1,\delta}(s)$$

$\mathbb{P}$-almost surely and in $L^2(\Omega)$ sense, that is,

$$\lim_{n\to+\infty} \mathbb{E}\left| \sum_{k=1}^{n} \mathbb{I}_{(a_{k-1}^{(n)}, a_k^{(n)}]}(\delta) \mathbb{E}^{\mathcal{F}^0_{s-a_k^{(n)}} \vee \mathcal{F}^z_s} x_1^{1,\delta}(s) - \mathbb{E}^{\mathcal{F}^0_{s-\delta} \vee \mathcal{F}^z_s} x_1^{1,\delta}(s) \right|^2 = 0 \quad \forall(\delta, s) \quad \text{a.e.}$$

Moreover, since conditional expectation is a contraction, together with the boundedness of $x_1^{1,\delta}$ obtained in C.2, we have

$$\mathbb{E}\left| \sum_{k=1}^{n} \mathbb{I}_{(a_{k-1}^{(n)}, a_k^{(n)}]}(\delta) \mathbb{E}^{\mathcal{F}^0_{s-a_k^{(n)}} \vee \mathcal{F}^z_s} x_1^{1,\delta}(s) - \mathbb{E}^{\mathcal{F}^0_{s-\delta} \vee \mathcal{F}^z_s} x_1^{1,\delta}(s) \right|^2 \le 2\mathbb{E} \sup_{\delta\in[a,b]} \sup_{u\le T} |x_1^{1,\delta}(u)|^2,$$

where the last term is finite and independent of n. We can apply the dominated convergence theorem (on the measure $d\pi_\Delta(\delta) \otimes ds$),

$$\lim_{n\to+\infty} \int_0^T \int_{[a,b]} \mathbb{E}\left| \sum_{k=1}^{n} \mathbb{I}_{(a_{k-1}^{(n)}, a_k^{(n)}]}(\delta) \mathbb{E}^{\mathcal{F}^0_{s-a_k^{(n)}} \vee \mathcal{F}^z_s} x_1^{1,\delta}(s) - \mathbb{E}^{\mathcal{F}^0_{s-\delta} \vee \mathcal{F}^z_s} x_1^{1,\delta}(s) \right|^2 d\pi_\Delta(\delta) ds = 0.$$

Hence we have

$$\lim_{n\to\infty} \int_0^T \mathbb{E}\left| z^{(n)}(s) - z(s) \right|^2 ds = 0. \tag{2.21}$$

By combining (2.18), (2.19), and (2.21), we have

$$\begin{aligned} &\int_0^T \mathbb{E}\left| \frac{\sum_{j=1,j\ne i}^N x_1^{j,\Delta_j}(s)}{N-1} - z(s) \right|^2 ds \\ &\le K\left(\frac{1+n}{N-1} + \sup_{k\le n} p_k^{(n)} + \int_0^T \mathbb{E}|z^{(n)}(s) - z(s)|^2 ds \right), \end{aligned} \tag{2.22}$$

where K is a constant independent of n and N. Since π_Δ is an absolutely continuous measure, together with the convergence in (2.21), for any $\epsilon > 0$, we can find a sufficiently large n such that $\sup_{k\le n} p_k^{(n)} < \epsilon$ and $\int_0^T \mathbb{E}|z^{(n)}(s) - z(s)|^2 ds < \epsilon$. We then choose a sufficiently large N such that $\frac{1+n}{N-1} < \epsilon$, which completes the proof. ∎

2.1.4. General $\boldsymbol{\Delta}$.

COROLLARY 2.3. *Let π_Δ be any probability measure on $[a, b]$. If the mean field term satisfies the fixed point property* (2.14)*, then*

$$\int_0^T \mathbb{E}\left| \frac{\sum_{j=1,j\ne i}^N x_1^{j,\Delta_j}(s)}{N-1} - z(s) \right|^2 ds \to 0 \quad \text{as } N \to +\infty.$$

Proof. By the Lebesgue decomposition theorem, we can uniquely decompose the measure into $\pi_\Delta = \pi_\Delta^0 + \pi_\Delta^1$, where $\pi_\Delta^0 \ll \lambda_{[a,b]}$ and $\pi_\Delta^1 \perp \lambda_{[a,b]}$. Here $\lambda_{[a,b]}$ is the Lebesgue measure on $[a, b]$. We assume $\pi_\Delta^1 \ne 0$; otherwise it is as has been discussed in section 2.1.3. For fixed $\epsilon' > 0$, let $E_{\epsilon'} = \{\delta \in [a, b] : \pi_\Delta^1(\delta) > \epsilon'\}$ and $E_{\epsilon'}^c = \{\delta \in [a, b] : 0 < \pi_\Delta^1(\delta) \le \epsilon'\}$. Since π_Δ is a probability measure and hence finite, the cardinality $E_{\epsilon'}$ is finite. We decompose the measure π_Δ into π_Δ^{0+} and π_Δ^{1-} in which

$$\pi_\Delta^{0+} = \pi_\Delta^0 + \pi_\Delta^1 | E_{\epsilon'}^c; \qquad \pi_\Delta^{1-} = \pi_\Delta^1 | E_{\epsilon'}, \tag{2.23}$$

where $\pi_\Delta^1 | E_{\epsilon'}^c$ is just the measure π_Δ^1 restricted on $E_{\epsilon'}^c$, so does $\pi_\Delta^1 | E_{\epsilon'}$. Clearly, $\pi_\Delta = \pi_\Delta^{0+} + \pi_\Delta^{1-}$. Let the total measure for π_Δ^{0+} and π_Δ^{1-} be p^{0+} and p^{1-}, respectively (hence $p^{0+} + p^{1-} = 1$). We first consider that both $p^{0+}, p^{1-} > 0$. Denote Δ^{0+} and Δ^{1-} the random variables with respective measures $\frac{\pi_\Delta^{0+}}{p^{0+}}$ and $\frac{\pi_\Delta^{1-}}{p^{1-}}$. Consider $M = (M_{0+}, M_{1-})$ to be a binomial random variable with parameters (p^{0+}, p^{1-}), which represents the numbers of players falling into group-0+ and group-1−, respectively. Hence, we have

$$\begin{aligned} &\int_0^T \mathbb{E}\left| \frac{\sum_{j=1, j\neq i}^N x_1^{j,\Delta_j}(s)}{N-1} - z(s) \right|^2 ds \\ &\le \int_0^T 2\mathbb{E}\left\{ \mathbb{E}\left[\left| \frac{\sum_{M_{0+}} x_1^{j,\Delta_j^{0+}}(s)}{N-1} - p^{0+} z^{0+}(s) \right|^2 \middle| M \right] \right\} \\ &\qquad + 2\mathbb{E}\left\{ \mathbb{E}\left[\left| \frac{\sum_{M_{1-}} x_1^{j,\Delta_j^{1-}}(s)}{N-1} - p^{1-} z^{1-}(s) \right|^2 \middle| M \right] \right\} ds, \end{aligned} \tag{2.24}$$

where

$$z^{0+}(s) := \frac{1}{p^{0+}} \int_{[a,b]} z^\delta(s) d(\pi_\Delta^{0+})(\delta) = \frac{1}{p^{0+}} \int_{[a,b]} \mathbb{E}^{\mathcal{F}_{s-\delta}^0 \vee \mathcal{F}_s^z} x_1^{1,\delta}(s) d(\pi_\Delta^{0+})(\delta);$$

$$z^{1-}(s) := \frac{1}{p^{1-}} \int_{[a,b]} z^\delta(s) d(\pi_\Delta^{1-})(\delta) = \frac{1}{p^{1-}} \int_{[a,b]} \mathbb{E}^{\mathcal{F}_{s-\delta}^0 \vee \mathcal{F}_s^z} x_1^{1,\delta}(s) d(\pi_\Delta^{1-})(\delta).$$

It should be noted that the absolute continuity assumption on the measure π_Δ in section 2.1.3 is only required for the convergence

$$\lim_{n\to+\infty} \sup_{k\le n} p_k^{(n)} = 0$$

in (2.22). We can obtain similar estimates as in (2.22) and the first term in (2.24) becomes

$$\begin{aligned} &\mathbb{E}\left\{ \mathbb{E}\left[\left| \frac{\sum_{M_{0+}} x_1^{j,\Delta_j^{0+}}(s)}{N-1} - p^{0+} z^{0+}(s) \right|^2 \middle| M \right] \right\} \\ &\le 2\mathbb{E}\left\{ \left(\frac{M_{0+}}{N-1} \right)^2 \mathbb{E}\left[\left| \frac{\sum_{M_{0+}} x_1^{j,\Delta_j^{0+}}(s)}{M_{0+}} - z^{0+}(s) \right|^2 \middle| M \right] \right\} + 2\mathbb{E}\left[\frac{M_{0+}}{N-1} - p^{0+} \right]^2 \mathbb{E}\left| z^0(s) \right|^2 \\ &\le K\mathbb{E}\left\{ \left(\frac{M_{0+}}{N-1} \right)^2 \left(\frac{1+n}{M_{0+}} + \sup_{k\le n} p_k^{(n)} + \int_0^T \mathbb{E}|z^{(n)}(s) - z^{0+}(s)|^2 ds \right) \right\} \\ &\qquad + 2\mathbb{E}\left[\frac{M_{0+}}{N-1} - p^{0+} \right]^2 \mathbb{E}\left| z^{0+}(s) \right|^2 \\ &= K\left\{ \frac{p^{0+}(1+n)}{N-1} + \left(\sup_{k\le n} p_k^{(n)} + \int_0^T \mathbb{E}|z^{(n)}(s) - z^{0+}(s)|^2 ds \right) \mathbb{E}\left(\frac{M_{0+}}{N-1} \right)^2 \right\} \\ &\qquad + 2\mathbb{E}\left[\frac{M_{0+}}{N-1} - p^{0+} \right]^2 \mathbb{E}\left| z^{0+}(s) \right|^2 \\ &\le K\left\{ \frac{1+n}{N-1} + \sup_{k\le n} p_k^{(n)} + \int_0^T \mathbb{E}|z^{(n)}(s) - z^{0+}(s)|^2 ds \right\} + \frac{2p^{0+}(1-p^{0+})}{N-1} \mathbb{E}\left| z^{0+}(s) \right|^2, \end{aligned} \tag{2.25}$$

where $p_k^{(n)} := \frac{\pi_\Delta^{0+}(a_{k-1}^{(n)}, a_k^{(n)}]}{p^{0+}}$ and $z^{(n)}(t)$ are defined as in Theorem 2.2. By the definition of π_Δ in (2.23), we can find a sufficiently large n such that $\sup_{k\le n} p_k^{(n)} < 2\epsilon'$ (see Lemma A.1) and $\int_0^T \mathbb{E}|z^{(n)}(s) - z(s)|^2 ds < \epsilon'$. Similarly, we can choose a sufficiently large N such that the two remaining terms are smaller than ϵ'.

The treatment of the second term is similar, and since π_Δ^{1-} is a measure on a finite set, we can directly apply the result in Lemma 2.1.

Consider the second case that $p^{0+} = 0, p^{1-} = 1$. Since π_Δ^{1-} is now a measure on a finite set, we can directly apply the result in Lemma 2.1.

For the last case when $p^{0+} = 1, p^{1-} = 0$, using the same argument, we could obtain similar estimate as in (2.25), which is of order $\mathcal{O}(\epsilon')$. ∎

2.2. Equilibrium. Assume Δ to be a random variable on $[a,b]$ with any probability measure π_Δ. In section 2.1, we assume that all players adopt their own mean field analogical optimal controls to establish the convergence of $y_0 \to x_0$ and $y_1^{i,\delta_i} \to x_1^{i,\delta_i}$. Suppose now, without loss of generality, the first player uses an arbitrary control v_1, while other players still adopt the optimal control u_1. Using similar arguments as in section 2.1, the following convergence also holds.

COROLLARY 2.4. *Suppose* (2.14) *is satisfied. Then*

$$\begin{aligned}&\mathbb{E}\sup_{u\le T}|y_0(u) - x_0(u)|^2 + \mathbb{E}\sup_{\delta\in[a,b]}\sup_{u\le T}|y_1^{i,\delta}(u) - x_1^{i,\delta}(u)|^2\\ &\quad + \mathbb{E}\sup_{\delta\in[a,b]}\sup_{u\le T}|y_1^{1,\delta}(u) - x_1^{1,\delta}(u)|^2 \to 0\end{aligned}$$

as $N \to +\infty$.

On the other hand, we also have the convergence for the cost functionals.

LEMMA 2.5. *Suppose that* $\mathbf{v} = (v_1^{1,\delta_1}, u_1^{2,\delta_2}, \dots, u_1^{N,\delta_N})$, $\mathbf{u} = (u_1^{1,\delta_1}, u_1^{2,\delta_2}, \dots, u_1^{N,\delta_N})$, *where* v_1^{1,δ_1} *is an arbitrary control, and* $\{u_1^i : i \ge 1\}$ *are optimal with respect to the control problem defined by* (2.5) *and* (2.6)*; then*

$$\mathcal{J}^{1,\delta_1,N}(\mathbf{v}) \to J^{1,\delta_1}(v_1^{1,\delta_1}) \quad \textit{as } N \to +\infty.$$

Similarly,

$$\mathcal{J}^{1,\delta_1,N}(\mathbf{u}) \to J^{1,\delta_1}(u_1^{1,\delta_1}) \quad \textit{as } N \to +\infty.$$

Proof. With assumption A.3 in (2.7) and an application of the Hölder's inequality, we have

$$\begin{aligned}&|\mathcal{J}^{1,\delta_1,N}(\mathbf{v}) - J^{1,\delta_1}(v_1^{1,\delta_1})|\\ &\le K(T,L)\mathbb{E}\int_0^T \Bigg[1 + |y_1^{1,\delta_1}(s)| + |x_1^{1,\delta_1}(s)| + \left|\frac{\sum_{j=2}^N y_1^{j,\Delta_j}(s)}{N-1}\right| + |z(s)| + |v_1^{1,\delta_1}(s)|\\ &\qquad\qquad + |y_0(s-\delta)| + |x_0(s-\delta_1)|\Bigg]\\ &\qquad \cdot\Bigg[|y_1^{1,\delta_1}(s) - x_1^{1,\delta_1}(s)| + \left|\frac{\sum_{j=2}^N y_1^{j,\Delta_j}(s)}{N-1} - z(s)\right| + |y_0(s-\delta_1)\\ &\qquad\qquad - x_0(s-\delta_1)|\Bigg]ds\end{aligned}$$

$$\leq K(T,L)\left(\mathbb{E}\int_0^T\left[1+|y_1^{1,\delta_1}(s)|^2+|x_1^{1,\delta_1}(s)|^2+\left|\frac{\sum_{j=2}^N y_1^{j,\Delta_j}(s)}{N-1}\right|^2+|z(s)|^2\right.\right.$$
$$\left.\left.+|v_1^{1,\delta_1}(s)|^2+|y_0(s-\delta)|^2+|x_0(s-\delta_1)|^2\right]ds\right)^{\frac{1}{2}}$$
$$\cdot\left(\mathbb{E}\int_0^T\left[|y_1^{1,\delta_1}(s)-x_1^{1,\delta_1}(s)|^2+\left|\frac{\sum_{j=2}^N y_1^{j,\Delta_j}(s)}{N-1}-z(s)\right|^2+|y_0(s-\delta_1)\right.\right.$$
$$\left.\left.-x_0(s-\delta_1)|^2\right]ds\right)^{\frac{1}{2}}.$$

By the boundedness on controls in A.5, similarly we have

$$\begin{aligned}
&|\mathcal{J}^{1,\delta_1,N}(\mathbf{v})-J^{1,\delta_1}(v_1^{1,\delta_1})|\\
&\leq K(T,L)\left(\mathbb{E}\int_0^T\left[1+\sup_{u\leq s}|y_1^{1,\delta_1}(u)|^2+\sup_{u\leq s}|x_1^{1,\delta_1}(u)|^2+\sup_{\delta\in[a,b]}\sup_{u\leq s}|y_1^{i,\delta}(u)|^2\right.\right.\\
&\quad\left.\left.+\sup_{\delta\in[a,b]}\sup_{u\leq s}|x_1^{i,\delta}(u)|^2+|v_1^{1,\delta_1}(s)|^2+\sup_{u\leq s}|y_0(u)|^2+\sup_{u\leq s}|x_0(u)|^2\right]ds\right)^{\frac{1}{2}}\\
&\cdot\left(\mathbb{E}\int_0^T\left[|y_1^{1,\delta_1}(s)-x_1^{1,\delta_1}(s)|^2+\left|\frac{\sum_{j=2}^N y_1^{j,\Delta_j}(s)}{N-1}-z(s)\right|^2+|y_0(s-\delta_1)\right.\right.\\
&\quad\left.\left.-x_0(s-\delta_1)|^2\right]ds\right)^{\frac{1}{2}}\\
&=K\cdot\left(\mathbb{E}\int_0^T\left[|y_1^{1,\delta_1}(s)-x_1^{1,\delta_1}(s)|^2+\left|\frac{\sum_{j=2}^N y_1^{j,\Delta_j}(s)}{N-1}-z(s)\right|^2+|y_0(s-\delta_1)\right.\right.\\
&\quad\left.\left.-x_0(s-\delta_1)|^2\right]ds\right)^{\frac{1}{2}}.
\end{aligned}$$

Here, K in the last row is a constant that depends on $T, L, C, \sigma_0, \sigma_1$, and the L^2 bounded controls $v_1^{1,\delta_1}, \sup_\delta u_1^{i,\delta}$, and u_0 under assumptions A.4 and A.5, which is independent of N. The first and the last term go to zero as shown in previous arguments and Corollary 2.3, while the second term goes to zero as shown in Corollary 2.4. The second statement is then clear by following similar arguments. ∎

THEOREM 2.6. $\mathbf{u}=(u_1^{1,\delta_1}, u_1^{2,\delta_2}, \dots, u_1^{N,\delta_N})$, *where* $\{u_1^i : i\geq 1\}$ *are optimal with respect to the control problem defined by* (2.5) *and* (2.6) *is an* ϵ*-Nash equilibrium.*

Proof. Without lost of generality, we focus on the 1st player. Suppose that $\mathbf{v}=(v_1^{1,\delta_1}, u_1^{2,\delta_2}, \dots, u_1^{N,\delta_N})$, where v_1^{1,δ_1} is an arbitrary control. By optimality, we first have

$$J^{1,\delta_1}(u_1^1)\leq J^{1,\delta_1}(v_1^{1,\delta_1}).$$

By Lemma 2.5, we have the approximations

$$\begin{aligned}
|\mathcal{J}^{1,\delta_1,N}(\mathbf{v})-J^{1,\delta_1}(v_1^{1,\delta_1})|=o(1);\\
|\mathcal{J}^{1,\delta_1,N}(\mathbf{u})-J^{1,\delta_1}(u_1^{1,\delta_1})|=o(1).
\end{aligned}$$

Hence

$$\mathcal{J}^{1,\delta_1,N}(\mathbf{u})\leq\mathcal{J}^{1,\delta_1,N}(\mathbf{v})+o(1). \qquad \square$$

3. Linear quadratic case. The ϵ-Nash equilibrium established in section 2 allows us to consider the mean field analogy, $x_1^i(t)$, instead of the empirical interacting dynamics, $y_1^i(t)$. For any probability measure π_Δ, without loss of generality, it suffices to consider the ith player with the delay $\Delta_i = \delta$. We call him the *representative agent* for the whole population. For simplicity we neglect the index and write $x_1(t) \equiv x_1^i(t)$. We study a linear quadratic control problem as follows. The state evolutions of the dominating player and the representative agent are, respectively, described by

$$\begin{aligned} dx_0 &= \Big(A_0 x_0(t) + B_0 z(t) + C_0 v_0(t)\Big)dt + \sigma_0 dW_0(t), \\ x_0(t) &= \xi_0(t), \quad t \in [-b, 0]; \\ dx_1^\delta &= \Big(A_1 x_1^\delta(t) + B_1 z(t) + C_1 v_1(t) + D x_0(t-\delta)\Big)dt + \sigma_1 dW_1(t), \\ x_1^\delta(0) &= \xi_1, \end{aligned}$$

where the mean field term $z(t)$ is to be defined in Problem 3.2. For simplicity, we assume the coefficient matrices (e.g., A_0, B_0) to be constant. The control for the dominating player, $v_0(t)$, is $\mathcal{F}_t^0$ adapted, while the control for the representative agent, $v_1^\delta(t)$ is $\mathcal{G}_t^\delta := \mathcal{F}_t^1 \vee \mathcal{F}_{t-\delta}^0 \vee \mathcal{F}_t^z$ adapted. We denote M^T the transpose of any matrix M. Suppose that $Q_i, R_i > 0$; $i = 0, 1$; we consider the following problems.

PROBLEM 3.1. *Given the process x_0 and z, find a control $u_1^\delta = v_1^\delta$ which minimizes the cost functional:*

$$J_1(v_1^\delta, x_0, z) = \mathbb{E}\bigg[\int_0^T |x_1^\delta(t) - E_1 z(t) - F x_0(t-\delta) - G_1|_{Q_1}^2 + \Big\langle v_1^\delta(t), R_1 v_1^\delta(t)\Big\rangle dt\bigg],$$

where for any $Q > 0$, $|\cdot|_Q := \langle \cdot, Q\cdot\rangle$ and $\langle \cdot, \star\rangle$ is the usual Euclidean inner product.

PROBLEM 3.2. *Find the process z such that*

$$z(t) = \int_{[a,b]} \mathbb{E}^{\mathcal{F}_{t-\delta}^0 \vee \mathcal{F}_t^z} x_1^\delta(t) d\pi_\Delta(\delta),$$

where x_1^δ is the controlled SDE using u_1^δ solved by Problem 3.1.

PROBLEM 3.3. *Find a control $u_0 = v_0$ which minimizes the cost functional*

$$J_0(v_0) = \mathbb{E}\bigg[\int_0^T |x_0(t) - E_0 z(t) - G_0|_{Q_0}^2 + \Big\langle v_0(t), R_0 v_0(t)\Big\rangle dt\bigg],$$

where z is the solution given in Problem 3.2.

Observe that the representative agent's decision at time t is $\mathcal{G}_t^\delta$ adapted. If $\delta = a$, then $\mathcal{G}_t^a = \mathcal{F}_t^1 \vee \mathcal{F}_{t-a}^0 \vee \mathcal{F}_t^z = \mathcal{F}_t^1 \vee \mathcal{F}_{t-a}^0$, as $\mathcal{F}_t^z$ is a sub σ-algebra of $\mathcal{F}_{t-a}^0$, is a Brownian filtration. Otherwise, $\mathcal{G}_t^\delta$ is not necessarily Brownian. The classical FBSDE solved with the martingale representation theorem (MRT) could not be tackled in the absence of a Brownian filtration. Inspired by the ideas in [19], we can work on forward-backward stochastic dynamics on a non-Brownian filtration. To motivate our further development, we here provide a brief introduction to backward stochastic dynamics as given in [19].

In particular, we want to solve for $(y_t, M_y(t))$ satisfying a stochastic backward equation on an arbitrary filtration $\mathcal{H}_t$,

$$y_t = \xi + \int_t^T g(y_s)ds - \int_t^T dM_y(s), \tag{3.1}$$

or in differential form,

$$dy_t = -g(y_t)dt + dM_y(t),$$

where g is the generator with suitable regularity assumptions (for example, global Lipschitz and linear growth properties) to guarantee the unique existence of the adapted solution y; ξ is the terminal random variable and M_y is an $\mathcal{H}$-martingale. Taking conditional expectation on (3.1) yields

$$y_t = \mathbb{E}^{\mathcal{H}_t}\Big[\xi + \int_0^T g(y_s)ds\Big] - \int_0^t g(y_s)ds, \tag{3.2}$$

or in differential form,

$$dy_t = -g(y_t)dt + d\mathbb{E}^{\mathcal{H}_t}\Big[\xi + \int_0^T g(y_s)ds\Big].$$

Note that $\mathbb{E}^{\mathcal{H}_t}[\xi + \int_0^T g(y_s)ds]$ is clearly an $\mathcal{H}$-martingale and hence the targeted martingale is

$$M_y(t) = \mathbb{E}^{\mathcal{H}_t}\Big[\xi + \int_0^T g(y_s)ds\Big]. \tag{3.3}$$

Furthermore, if we define $V(y)_t := \int_0^t g(y_s)ds$, then Liang, Lyons, and Qian [19] established the method for solving (3.2) by tackling the fixed point problem of

$$y_t = \mathbb{E}^{\mathcal{H}_t}\Big[\xi + V(y)_T\Big] - V(y)_t,$$

in contrast to the resolution of classical BSDEs, which requires an application of MRT. Throughout this paper, for any backward equation y, we refer to M_y as the martingale defined by its terminal and generator as in (3.3). We first solve the control problem for the representative agent.

LEMMA 3.4 (control for the representative agent). *Problem* 3.1 *is uniquely solvable and the optimal control is $-R_1^{-1}C_1^T n^\delta(t)$, such that n^δ satisfies the backward stochastic dynamics:*

$$-dn^\delta = \Big(A_1^T n^\delta(t) + Q_1(x_1^\delta(t) - E_1 z(t) - Fx_0(t-\delta) - G_1)\Big)dt - dM_{n^\delta}(t), \quad n^\delta(T) = 0,$$

where M_n is a $\mathcal{G}^\delta$-martingale with

$$M_{n^\delta}(t) = \mathbb{E}^{\mathcal{G}_t^\delta}\Big[\int_0^T A_1^T n^\delta(s) + Q_1(x_1^\delta(s) - E_1 z(s) - Fx_0(s-\delta) - G_1)ds\Big].$$

Proof. Due to the convexity and coerciveness of the objective functional, we can apply the standard stochastic maximum principle. Consider a perturbation of the optimal control $u_1^\delta + \theta\tilde{u}_1^\delta$, where $\tilde{u}_1^\delta$ is adapted to the filtration $\mathcal{G}_t^\delta$. The original state x_1^δ becomes $x_1^\delta + \theta\tilde{x}_1^\delta$ with

$$d\tilde{x}_1^\delta = \Big(A_1\tilde{x}_1^\delta(t) + C_1\tilde{u}_1^\delta(t)\Big)dt, \quad \tilde{x}_1^\delta(0) = 0;$$

the optimality of u_1^δ would satisfy the following Euler condition:

$$
\begin{aligned}
0 =& \left.\frac{d}{d\theta}\right|_{\theta=0} J_1(u_1^\delta+\theta\tilde{u}_1^\delta, x_0, z)\\
=& 2\mathbb{E}\left[\int_0^T \Big\langle \tilde{x}_1^\delta(t), Q_1\left(x_1^\delta(t)-E_1z(t)-Fx_0(t-\delta)-G_1\right)\Big\rangle + \Big\langle \tilde{u}_1^\delta(t), R_1u_1^\delta(t)\Big\rangle dt\right].
\end{aligned} \tag{3.4}
$$

We have the inner product

$$
\begin{aligned}
d\langle n^\delta, \tilde{x}_1^\delta\rangle = \Big(& -\Big\langle \tilde{x}_1^\delta(t), Q_1\left(x_1^\delta(t)-E_1z(t)-Fx_0(t-\delta)-G_1\right)\Big\rangle + \Big\langle n^\delta(t), C_1\tilde{u}_1^\delta(t)\Big\rangle\Big)dt\\
&+ \tilde{x}_1^\delta(t)\cdot dM_n(t).
\end{aligned}
$$

Taking integration and expectation on both sides, and using (3.4), we obtain

$$
0=\mathbb{E}\int_0^T \Big\langle C_1^T n^\delta(t)+R_1u_1^\delta(t), \tilde{u}_1^\delta(t)\Big\rangle dt.
$$

As $\tilde{u}_1^\delta$ is arbitrary, the result follows. ∎

Remark 3.5. The optimal control u_1^δ has the representation $-R_1^{-1}C_1^Tn^\delta(t) = -R_1^{-1}C_1^T(P_tx_1^\delta(t)+g^\delta(t))$, where P satisfies the symmetric Riccati equation

$$
\frac{dP}{dt}+P_tA_1+A_1^TP_t-P_tC_1R_1^{-1}C_1^TP_t+Q_1=0, \quad P_T=0,
$$

and g^δ satisfies the backward stochastic dynamics

$$
\begin{aligned}
-dg^\delta =& \Big((A_1^T-P_tC_1R_1^{-1}C_1^T)g^\delta(t)+(P_tD-Q_1F)x_0(t-\delta)\\
&+(P_tB_1-Q_1E_1)z(t)-Q_1G_1\Big)dt-dM_g(t), \qquad g^\delta(T)=0.
\end{aligned}
$$

Clearly, g^δ is adapted to $\mathcal{F}_{t-\delta}^0\vee\mathcal{F}_t^z$, and hence the functional form of the optimal control for the representative agent $u_1^\delta(x_1) = -R_1^{-1}C_1^T(P_tx_1+g^\delta(t))$ is adapted to $\mathcal{F}_{t-\delta}^0\vee\mathcal{F}_t^z$. This agrees with the usual Stackelberg setting.

By the main result obtained in [19], we have the unique existence of the backward dynamic equation g^δ, which is also adapted to the non-Brownian filtration $\mathcal{G}^\delta$. Hence, there exists a unique solution to Problem 3.1 that is determined by the system

$$
\begin{cases}
dx_1^\delta = \Big((A_1-C_1R_1^{-1}C_1^TP_t)x_1^\delta(t)+B_1z(t)-C_1R_1^{-1}C_1^Tg^\delta(t)+Dx_0(t-\delta)\Big)dt\\
\qquad + \sigma_1dW_1(t), \qquad\qquad x_1^\delta(0)=\xi_1;\\
-dg^\delta = \Big((A_1^T-P_tC_1R_1^{-1}C_1^T)g^\delta(t)+(P_tD-Q_1F)x_0(t-\delta)\\
\qquad +(P_tB_1-Q_1E_1)z(t)-Q_1G_1\Big)dt-dM_g(t), \qquad g^\delta(T)=0.
\end{cases}
$$

The unique existence of the following equivalent forward-backward stochastic dynamical formulation follows immediately due to the unique existence of optimal control:

$$
\begin{cases}
dx_1^\delta = \Big(A_1x_1^\delta(t)+B_1z(t)-C_1R_1^{-1}C_1^Tn^\delta(t)+Dx_0(t-\delta)\Big)dt+\sigma_1dW_1(t),\\
x_1^\delta(0)=\xi_1;\\
-dn^\delta = \Big(A_1^Tn^\delta(t)+Q_1(x_1^\delta(t)-E_1z(t)-Fx_0(t-\delta)-G_1)\Big)dt-dM_{n^\delta}(t),\\
n^\delta(T)=0.
\end{cases} \tag{3.5}
$$

Remark 3.6. In general the forward-backward stochastic dynamics may not possess a unique global solution. We here provide a class of interesting examples with such unique existence.

Remark 3.7. It is possible to include the mean field term (z) in the diffusion coefficient of the dynamics for the representative agent. The adjoint equation under this case remains unchanged as the mean field term is exogenous to the agent.

To obtain the equilibrium in Problem 3.2, we take the expectation conditional on $\mathcal{F}^0_{t-\delta} \vee \mathcal{F}^z_t$ and integrate on δ against π over $[a,b]$ on both sides of (3.5), which yields

(3.6)
$$\begin{cases} dz = \Big((A_1+B_1)z(t) - C_1R_1^{-1}C_1^T m(t) + D\displaystyle\int_{[a,b]} x_0(t-\delta)d\pi_\Delta(\delta)\Big)dt, \\ z(0) = \mathbb{E}[\xi_1]; \\ -dm = \Big(A_1^T m(t) - Q_1F\displaystyle\int_{[a,b]} x_0(t-\delta)d\pi_\Delta(\delta) + Q_1(I-E_1)z(t) - Q_1G_1\Big)dt - dM_m(t), \\ m(T) = 0, \end{cases}$$

where we write $m(t) := \int_{[a,b]} \mathbb{E}^{\mathcal{F}^0_{t-\delta}\vee\mathcal{F}^z_t} n^\delta(t)d\pi_\Delta(\delta)$. The interchange of conditional expectation and differential is valid as (1) the noise from the dominating player W_0 is independent of the W_1 by definition and (2) the mean field term z is independent of any individual noises W_1 after the averaging effect. Here (z,m) are clearly adapted to $\mathcal{F}^0_{t-a}$, and hence (z,m) are ordinary FBSDE (adapted to the Brownian filtration $\mathcal{F}^0_{\cdot-a}$). Indeed, by MRT, $dM_m(t)$ has to be in the form of $Z(t)dW(t-a)$ for some $Z(t) \in \mathcal{F}^0_{t-a}$. Before proceeding to Problem 3.3, we first discuss the existence of (3.6).

Lemma 3.8. *Given any square integrable process x_0, suppose that the* nonsymmetric Riccati equation

(3.7)
$$\frac{d\Gamma}{dt} + \Gamma_t(A_1+B_1) + A_1^T\Gamma_t - \Gamma_tC_1R_1^{-1}C_1^T\Gamma_t + Q_1(I-E_1) = 0, \qquad \Gamma(T) = 0,$$

admits a unique solution on $[0,T]$*; then there uniquely exists a solution to* (3.6).

Proof. It is easy to see that if h satisfies the BSDE

$$\begin{aligned} -dh = \Big(&(A_1^T - \Gamma_tC_1R_1^{-1}C_1^T)h(t) \\ &+ (\Gamma_tD - Q_1F)\int_{[a,b]} x_0(t-\delta)d\pi_\Delta(\delta) - Q_1G_1\Big)dt - dM_h(t), \qquad h(T) = 0, \end{aligned}$$

then $\Gamma_t z(t) + h(t) = m(t)$ as defined in (3.6); the forward equation z will automatically exist. The uniqueness is clear. ∎

The unique existence of the solution of the nonsymmetric Riccati equation (3.7), Γ, depends on its dimension. For ease of immediate reference, we here sketch the proof of one of the main theorems, Theorem III.5 in [9].

Proposition 3.9. *We consider the following cases:*

Case 1. *If $n_1 = 1$, then Γ always admits a solution on* $[0,T]$.

Case 2. *If $n_1 > 1$, suppose that there is a representation $Q_1(I-E_1) = \mathcal{Q} + \mathcal{S}$, where $\mathcal{Q}$ is positive definite, such that*

$$\Big(1 + \sqrt{T}\|e^{A_1^T}\|\|BQ^{-\frac{1}{2}}\|\Big)\Big(1 + \|\mathcal{Q}^{-\frac{1}{2}}\mathcal{S}\mathcal{Q}^{-\frac{1}{2}}\|\Big) < 2; \tag{3.8}$$

then Γ admits a unique solution on $[0,T]$. *Here,*

$$\|e^{A_1^T}\| = \sup_{t\le T}\sqrt{\int_t^T |e^{A_1^T(s-t)}\mathcal{Q}^{\frac{1}{2}}|^2 ds}.$$

Proof. The first case is trivial. For the second case, we claim that if the condition (3.8) is satisfied, then for any $t_0 \in [0,T]$, there exists a unique solution to the following *forward-backward ordinary differential equation* on $[t_0,T]$:

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} A_1 + B_1 & -C_1R_1^{-1}C_1^T \\ -Q_1(I-E_1) & -A_1^T \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} =: \mathcal{M}\begin{pmatrix} x \\ y \end{pmatrix}, \quad x(t_0)=0, \quad y(T)=0.$$

For the sake of reference, a sketch of the proof of this claim is provided in the appendix. More details can be found in [9]. We thus have

$$0 = \begin{pmatrix} 0 & I \end{pmatrix}\begin{pmatrix} x(T) \\ y(T) \end{pmatrix} = \begin{pmatrix} 0 & I \end{pmatrix} e^{\mathcal{M}(T-t_0)}\begin{pmatrix} x(t_0) \\ y(t_0) \end{pmatrix} = \left[\begin{pmatrix} 0 & I \end{pmatrix} e^{\mathcal{M}(T-t_0)}\begin{pmatrix} 0 \\ I \end{pmatrix}\right] y(t_0).$$

Since the above equation admits a unique solution $y(t_0)$, we have that the matrix

$$\left[\begin{pmatrix} 0 & I \end{pmatrix} e^{\mathcal{M}(T-t_0)}\begin{pmatrix} 0 \\ I \end{pmatrix}\right]$$

is invertible for any $t_0 \in [0,T]$. One can set a well-defined process:

$$\hat{\Gamma}_t = -\left[\begin{pmatrix} 0 & I \end{pmatrix} e^{\mathcal{M}(T-t)}\begin{pmatrix} 0 \\ I \end{pmatrix}\right]^{-1}\left[\begin{pmatrix} 0 & I \end{pmatrix} e^{\mathcal{M}(T-t)}\begin{pmatrix} I \\ 0 \end{pmatrix}\right].$$

By simply taking differentiation with respect to t, we have

$$\begin{aligned} \frac{d\Gamma'}{dt} &= \left[\begin{pmatrix} 0 & I \end{pmatrix} e^{\mathcal{M}(T-t)}\begin{pmatrix} 0 \\ I \end{pmatrix}\right]^{-1}\begin{pmatrix} 0 & I \end{pmatrix} e^{\mathcal{M}(T-t)}\begin{pmatrix} -C_1R_1^{-1}C_1^T \\ -A_1^T \end{pmatrix}\hat{\Gamma}_t \\ &\quad + \left[\begin{pmatrix} 0 & I \end{pmatrix} e^{\mathcal{M}(T-t)}\begin{pmatrix} 0 \\ I \end{pmatrix}\right]^{-1}\begin{pmatrix} 0 & I \end{pmatrix} e^{\mathcal{M}(T-t)}\begin{pmatrix} A_1 + B_1 \\ -Q_1(I-E_1) \end{pmatrix} \\ &= -A_1^T\hat{\Gamma}_t + \hat{\Gamma}_tC_1R_1^{-1}C_1^T\hat{\Gamma}_t - \hat{\Gamma}_t(A_1+B_1) - Q_1(I-E_1). \end{aligned}$$

Hence $\hat{\Gamma}$ solves the nonsymmetric Riccati equation (3.7). ∎

In the rest of this paper, we assume that the condition given by Proposition 3.9, which is independent of the choice of x_0, holds. The unique existence of system (3.6) is then guaranteed. We next turn to the control problem for the dominating player. Note that we can decompose the system into $(z_0, m_0) \in \mathcal{F}^0_{\cdot - a}$ and a deterministic component (z_c, m_c), such that $(z,m) = (z_0 + z_c, m_0 + m_c)$:

$$\begin{cases} dz_0 = \Big((A_1+B_1)z_0(t) - C_1R_1^{-1}C_1^T m_0(t) + D\displaystyle\int_{[a,b]} x_0(t-\delta)d\pi_\Delta(\delta)\Big)dt, \\ z_0(0) = 0; \\ -dm_0 = \Big(A_1^T m_0(t) - Q_1F\displaystyle\int_{[a,b]} x_0(t-\delta)d\pi_\Delta(\delta) + Q_1(I-E_1)z_0(t)\Big)dt - dM_{m_0}(t), \\ m_0(T) = 0. \end{cases}$$

$$\begin{cases} dz_c = \Big((A_1+B_1)z_c(t) - C_1R_1^{-1}C_1^T m_c(t)\Big)dt, \\ z_c(0) = \mathbb{E}[\xi_1]; \\ -dm_c = \Big(A_1^T m_c(t) + Q_1(I-E_1)z_c(t) - Q_1G_1\Big)dt, \\ m_c(T) = 0. \end{cases}$$

We have $x_0 \mapsto (z_0(x_0), m_0(x_0)) = (z_0, m_0)$ is linear. We consider the linear functional $\mathcal{L} : L^2_{\mathcal{F}^0}([-b,T];\mathbb{R}^{n_0}) \to L^2_{\mathcal{F}^0_{\cdot - a}}([0,T];\mathbb{R}^{n_1})$ defined by

$$\mathcal{L}(x_0)(t) = z_0(t). \tag{3.9}$$

It can be shown that $\mathcal{L}$ is bounded (and hence continuous); see Lemma A.3. By the Riesz representation theorem, the Hermitian adjoint operator $\mathcal{L}^* : L^2_{\mathcal{F}^0_{\cdot-a}}([0,T];\mathbb{R}^{n_1}) \to L^2_{\mathcal{F}^0}([-b,T];\mathbb{R}^{n_0})$ uniquely exists such that

$$\mathbb{E}\int_0^T \langle f(t), \mathcal{L}(g)(t)\rangle dt = \mathbb{E}\int_{-b}^T \langle \mathcal{L}^*(f)(t), g(t)\rangle dt$$

for all $f \in L^2_{\mathcal{F}^0_{\cdot-a}}([0,T];\mathbb{R}^{n_1})$, $g \in L^2_{\mathcal{F}^0}([-b,T];\mathbb{R}^{n_0})$. In particular, we have that the operator norm is preserved, i.e., $\|\mathcal{L}\| = \|\mathcal{L}^*\|$. The dynamics and the cost functional for the dominating player can be rewritten as

$$\begin{cases} dx_0 = \Big(A_0x_0(t) + B_0(\mathcal{L}(x_0)(t) + z_c(t)) + C_0u_0(t)\Big)dt + \sigma_0 dW_0(t), \\ x_0(t) = \xi_0(t), \quad t \in [-b,0], \end{cases} \tag{3.10}$$

and

$$J_0(u_0) = \mathbb{E}\bigg[\int_0^T |x_0(t) - E_0(\mathcal{L}(x_0)(t) + z_c(t)) - G_0|^2_{Q_0} + \Big\langle u_0(t), R_0u_0(t)\Big\rangle dt\bigg].$$

THEOREM 3.10 (control for the dominating player). *The dominating player's optimal control is given by $-R_0^{-1}C_0^Tp(t)$, where p satisfies the backward stochastic functional differential equation*

$$\begin{aligned} -dp = \Big(&A_0^Tp(t) + \mathcal{L}^*(B_0^Tp)(t) + Q_0(x_0(t) - E_0(\mathcal{L}(x_0)(t) + z_c(t)) - G_0) \\ &- \mathcal{L}^*\Big(E_0^TQ_0(x_0 - E_0(\mathcal{L}(x_0) + z_c) - G_0)\Big)(t)\Big)dt - dM_p(t), \qquad p(T) = 0, \end{aligned}$$

where x_0 satisfies (3.10).

Proof. Let $u_0 + \theta\tilde{u}_0$ be the perturbation of the optimal control. The original states x_0 becomes $x_0 + \theta\tilde{x}_0$ with

$$d\tilde{x}_0 = \Big(A_0\tilde{x}_0(t) + B_0\mathcal{L}(\tilde{x}_0)(t) + C_0\tilde{u}_0(t)\Big)dt, \qquad \tilde{x}_0(t) = 0, \quad t \in [-b,0].$$

Consider the first order condition,

$$\begin{aligned} 0 &= \frac{d}{d\theta}\bigg|_{\theta=0} J(u_0 + \theta\tilde{u}_0) \\ &= 2\mathbb{E}\bigg[\int_0^T \Big\langle \tilde{x}_0(t) - E_0\mathcal{L}(\tilde{x}_0)(t), Q_0\Big(x_0(t) - E_0(\mathcal{L}(x_0)(t) + z_c(t)) - G_0\Big)\Big\rangle \\ &\qquad + \Big\langle \tilde{u}_0(t), R_0u_0(t)\Big\rangle dt\bigg]. \end{aligned} \tag{3.11}$$

On the other hand,

$$\begin{aligned} d\langle p, \tilde{x}_0\rangle = \bigg\{&\Big\langle p(t), C_0\tilde{u}_0(t)\Big\rangle + \Big\langle p(t), B_0\mathcal{L}(\tilde{x}_0)(t)\Big\rangle - \Big\langle \mathcal{L}^*(B_0^Tp)(t), \tilde{x}_0(t)\Big\rangle \\ &- \Big\langle Q_0\Big(x_0(t) - E_0(\mathcal{L}(x_0)(t) + z_c(t)) - G_0\Big), \tilde{x}_0(t)\Big\rangle \\ &+ \Big\langle \mathcal{L}^*\Big(E_0^TQ_0(x_0 - E_0(\mathcal{L}(x_0) + z_c) - G_0)\Big)(t), \tilde{x}_0(t)\Big\rangle\bigg\}dt + \Big\langle \tilde{x}_0(t), dM_p(t)\Big\rangle. \end{aligned} \tag{3.12}$$

First observe that

$$\mathbb{E}\int_0^T \Big\langle p(t), B_0\mathcal{L}(\tilde{x}_0)(t)\Big\rangle dt = \mathbb{E}\int_{-b}^T \Big\langle \mathcal{L}^*(B_0^T p)(t), \tilde{x}_0(t)\Big\rangle dt = \mathbb{E}\int_0^T \Big\langle \mathcal{L}^*(B_0^T p)(t), \tilde{x}_0(t)\Big\rangle dt.$$

On the other hand,

$$\begin{aligned}
&\mathbb{E}\int_0^T \Big\langle \mathcal{L}^*\Big(E_0^T Q_0(x_0 - E_0(\mathcal{L}(x_0)+z_c) - G_0)\Big)(t), \tilde{x}_0(t)\Big\rangle dt\\
&= \mathbb{E}\int_{-b}^T \Big\langle \mathcal{L}^*\Big(E_0^T Q_0(x_0 - E_0(\mathcal{L}(x_0)+z_c) - G_0)\Big)(t), \tilde{x}_0(t)\Big\rangle dt\\
&= \mathbb{E}\int_0^T \Big\langle Q_0(x_0(t) - E_0(\mathcal{L}(x_0)(t)+z_c(t)) - G_0), E_0\mathcal{L}(\tilde{x}_0)(t)\Big\rangle dt.
\end{aligned}$$

Hence, taking integration on $[0,T]$ and expectation on both sides of (3.12) yields

$$\begin{aligned}
0 = \mathbb{E}\int_0^T &\Big\{\Big\langle p(t), C_0\tilde{u}_0(t)\Big\rangle\\
&- \Big\langle \tilde{x}_0(t) - E_0\mathcal{L}(\tilde{x}_0)(t), Q_0\Big(x_0(t) - E_0(\mathcal{L}(x_0)(t)+z_c(t)) - G_0\Big)\Big\rangle\Big\}dt.
\end{aligned}$$

Using (3.11), we finally conclude that

$$0 = \mathbb{E}\Big[\int_0^T \Big\langle C_0^T p(t) + R_0^T u_0(t), \tilde{u}_0(t)\Big\rangle dt\Big].$$

Since $\tilde{u}_0$ is arbitrary, we have that the optimal control is $-R_0^{-1}C_0^T p(t)$. ∎

Recall that the linear operator $\mathcal{L} : L^2_{\mathcal{F}^0}([-b,T];\mathbb{R}^{n_0}) \to L^2_{\mathcal{F}^0_{\cdot-a}}([0,T];\mathbb{R}^{n_1})$, for all $g \in L^2_{\mathcal{F}^0}([-b,T];\mathbb{R}^{n_0})$, is defined by

$$\mathcal{L}(g)(t) = \alpha(t) \tag{3.13}$$

with

$$\begin{cases}
d\alpha = \Big((A_1+B_1)\alpha(t) - C_1R_1^{-1}C_1^T\beta(t) + D\int_{[a,b]} g(t-\delta)d\pi_\Delta(\delta)\Big)dt, \quad \alpha(0)=0;\\
-d\beta = \Big(A_1^T\beta(t) - Q_1F\int_{[a,b]} g(t-\delta)d\pi_\Delta(\delta) + Q_1(I-E_1)\alpha(t)\Big)dt - dM_\beta(t), \quad \beta(T)=0,
\end{cases} \tag{3.14}$$

where the unique existence of (α,β) is ensured by Lemma 3.8 and Proposition 3.9.

THEOREM 3.11 (explicit form of the Hermitian adjoint $\mathcal{L}^*$). *Let $f \in L^2_{\mathcal{F}^0_{\cdot-a}}([0,T];\mathbb{R}^{n_1})$. The Hermitian adjoint $\mathcal{L}^* : L^2_{\mathcal{F}^0_{\cdot-a}}([0,T];\mathbb{R}^{n_1}) \to L^2_{\mathcal{F}^0}([-b,T];\mathbb{R}^{n_0})$ of $\mathcal{L}$ defined by* (3.13) *and* (3.14) *is given by*

$$\mathcal{L}^*(f)(t) = D^T\int_{[a,b]} \mathbb{E}^{\mathcal{F}^0_t} q(t+\delta)d\pi_\Delta(\delta) - (Q_1F)^T\int_{[a,b]} \mathbb{E}^{\mathcal{F}^0_t} r(t+\delta)d\pi_\Delta(\delta),$$

where

$$\begin{cases}
-dq = \Big((A_1+B_1)^T q(t) + f(t) + (Q_1(I-E_1))^T r(t)\Big)dt - dM_q(t),\\
q(T) = 0,\\
q(t) = 0, \quad t \in [-b,0)\cup(T,T+b].\\
dr = \Big(A_1 r(t) - C_1R_1^{-1}C_1^T q(t)\Big)dt,\\
r(0) = 0,\\
r(t) = 0, \quad t \in [-b,0)\cup(T,T+b].
\end{cases}$$

Proof. Consider the difference of inner product

$$d(\langle q,\alpha\rangle - \langle r,\beta\rangle) = \Big\{\Big\langle q(t), D\int_{[a,b]} g(t-\delta)d\pi_\Delta(\delta)\Big\rangle - \Big\langle r(t), Q_1F\int_{[a,b]} g(t-\delta)d\pi_\Delta(\delta)\Big\rangle - \Big\langle f(t),\alpha(t)\Big\rangle\Big\}dt + \Big\langle \alpha(t), dM_q(t)\Big\rangle - \Big\langle r(t), dM_\beta(t)\Big\rangle.$$

One can take integration over $[0,T]$ and expectation on both sides; then we obtain

$$\begin{aligned}
&\mathbb{E}\int_0^T \langle f(t),\alpha(t)\rangle dt\\
&= \mathbb{E}\int_{[a,b]}\int_{-\delta}^{T-\delta} \Big\langle D^T q(t+\delta), g(t)\Big\rangle - \Big\langle (Q_1F)^T r(t+\delta), g(t)\Big\rangle dt d\pi_\Delta(\delta)\\
&= \mathbb{E}\int_{-b}^{T} \Big\langle D^T\int_{[a,b]} q(t+\delta)d\pi_\Delta(\delta), g(t)\Big\rangle - \Big\langle (Q_1F)^T\int_{[a,b]} r(t+\delta)d\pi_\Delta(\delta), g(t)\Big\rangle dt\\
&= \mathbb{E}\int_{-b}^{T} \Big\langle D^T\int_{[a,b]} \mathbb{E}^{\mathcal{F}_t^0} q(t+\delta)d\pi_\Delta(\delta) - (Q_1F)^T\int_{[a,b]} \mathbb{E}^{\mathcal{F}_t^0} r(t+\delta)d\pi_\Delta(\delta), g(t)\Big\rangle dt.
\end{aligned}$$

The second equality comes from the fact that q, r vanish on appropriate domains. ∎

Observe that both $\mathcal{L}(g)(t)$ and $\mathcal{L}^*(f)(t)$ are $\mathcal{F}_t^0$ adapted. Using the result obtained in Theorem 3.11, by putting $f(t) = B_0^T p(t) - E_0^T Q_0(x_0(t) - E_0(\mathcal{L}(x_0)(t) + z_c(t)) - G_0)$, we have the explicit formulation for the adjoint process p in Lemma 3.10. Altogether, the solution is represented by the following six equations:

(3.15)

$$\begin{cases} dx_0 = \Big(A_0x_0(t) + B_0z(t) - C_0R_0^{-1}C_0^Tp(t)\Big)dt + \sigma_0 dW_0(t),\\ x_0(0) = \xi_0;\end{cases}$$

$$\begin{cases} dz = \Big((A_1+B_1)z(t) - C_1R_1^{-1}C_1^Tm(t) + D\int_{[a,b]} x_0(t-\delta)d\pi_\Delta(\delta)\Big)dt,\\ z(0) = \mathbb{E}[\xi_1];\end{cases}$$

$$\begin{cases} -dm = \Big(A_1^Tm(t) - Q_1F\int_{[a,b]} x_0(t-\delta)d\pi_\Delta(\delta) + Q_1(I-E_1)z(t) - Q_1G_1\Big)dt - dM_m(t),\\ m(T) = 0.\end{cases}$$

$$\begin{cases} -dp = \Big(A_0^Tp(t) + D^T\int_{[a,b]} \mathbb{E}^{\mathcal{F}_t^0} q(t+\delta)d\pi_\Delta(\delta) - (Q_1F)^T\int_{[a,b]} \mathbb{E}^{\mathcal{F}_t^0} r(t+\delta)d\pi_\Delta(\delta)\\ \qquad\qquad + Q_0\Big(x_0(t) - E_0z(t) - G_0\Big)\Big)dt - dM_p(t),\\ p(T) = 0;\end{cases}$$

$$\begin{cases} -dq = \Big((A_1+B_1)^Tq(t) + B_0^Tp(t) - E_0^TQ_0(x_0(t) - E_0z(t)) - G_0)\\ \qquad\qquad + (Q_1(I-E_1))^Tr(t)\Big)dt - dM_q(t),\\ q(T) = 0,\\ q(t) = 0, \quad t\in[-b,0)\cup(T,T+b];\end{cases}$$

$$\begin{cases} dr = \Big(A_1r(t) - C_1R_1^{-1}C_1^Tq(t)\Big)dt,\\ r(0) = 0,\\ r(t) = 0, \quad t\in[-b,0)\cup(T,T+b].\end{cases}$$

Remark 3.12. If $\Delta \equiv 0$, which implies $a = b = 0$, then the above six equations reduce to the results in [3] without terminal terms.

Remark 3.13. The backward equation p is also know as the “anticipated BSDE”; see Peng and Yang [21] for details.

Remark 3.14. If $D = 0$ and $F = 0$, the system is degenerated in the sense that the state of the representative agent and his objective functional are independent of those of the dominating player, and then (3.15) reduces to a system of four equations:

(z, m)	:	linear quadratic mean field game problem (see [9]);
(x_0, p)	:	standard linear quadratic control problem, given (z, m) (see [2, 11]).

The adjoints (q, r) could be neglected.

4. Unique existence for the solution of FBSFDE. It remains to discuss the unique existence for a solution derived in Theorem 3.10, which is represented by an FBSFDE:

$$\begin{cases} dx_0 = \Big(A_0 x_0(t) + B_0(\mathcal{L}(x_0)(t) + z_c(t)) - C_0 R_0^{-1} C_0^T p(t)\Big)dt + \sigma_0 dW_0(t), \\ x_0(t) = \xi_0(t), \quad t \in [-b, 0]; \end{cases}$$
$$\begin{cases} -dp = \Big(A_0^T p(t) + \mathcal{L}^*\Big(B_0^T p - E_0^T Q_0(x_0 - E_0(\mathcal{L}(x_0) + z_c) - G_0)\Big)(t) \\ \qquad + Q_0\Big(x_0(t) - E_0(\mathcal{L}(x_0)(t) + z_c(t)) - G_0\Big)\Big)dt - dM_p(t), \\ p(T) = 0, \end{cases} \tag{4.1}$$

For a discussion on the general existence of FBSFDE, see Xu [24] and the references therein. In particular, he extends the method of continuation in the literature of FBSDE. The next theorem concludes this section.

THEOREM 4.1. *Suppose that the condition in Proposition* 3.9 *holds; then the FBSFDE* (4.1) *admits a unique solution.*

Proof. The condition in Proposition 3.9 guarantees the existence of $\mathcal{L}$. In accordance with Theorem 3.1 in Xu [24], it suffices to check the monotonicity condition. Suppose that (x_0, p), (x_0', p') are two processes on $\mathbb{R}^{n_1} \times \mathbb{R}^{n_1}$ and $(\hat{x}_0, \hat{p}) := (x_0 - x_0', p - p')$. Define the operator

$$\mathbb{A}\begin{pmatrix} \hat{x}_0 \\ \hat{p} \end{pmatrix}(t) = \begin{pmatrix} -A_0^T \hat{p}(t) - \mathcal{L}^*(B_0^T \hat{p} - E_0^T Q_0(\hat{x}_0 - E_0 \mathcal{L}(\hat{x}_0)))(t) - Q_0(\hat{x}_0(t) - E_0 \mathcal{L}(\hat{x}_0)(t)) \\ A_0 \hat{x}_0(t) + B_0 \mathcal{L}(\hat{x}_0)(t) - C_0 R_0^{-1} C_0 \hat{p}(t) \end{pmatrix}.$$

Consider the inner product

$$\begin{aligned} &\mathbb{E}\int_0^T \Big\langle \mathbb{A}(\hat{x}_0, \hat{p})(t), (\hat{x}_0(t), p(t))\Big\rangle dt \\ = &\,\mathbb{E}\int_0^T \Big\langle \hat{p}(t), B_0 \mathcal{L}(\hat{x}_0)(t)\Big\rangle dt - \mathbb{E}\int_0^T \Big\langle \hat{p}(t), C_0 R_0^{-1} C_0^T \hat{p}(t)\Big\rangle dt \\ &- \mathbb{E}\int_{-b}^T \Big\langle \mathcal{L}^*\Big(B_0^T \hat{p} - E_0^T Q_0(\hat{x}_0 - E_0 \mathcal{L}(\hat{x}_0))\Big)(t), \hat{x}_0(t)\Big\rangle dt \\ &+ \mathbb{E}\int_0^T \Big\langle Q_0\Big(\hat{x}_0(t) - E_0 \mathcal{L}(\hat{x}_0)(t)\Big), \hat{x}_0(t)\Big\rangle dt \\ = &- \mathbb{E}\int_0^T \Big\langle \hat{p}(t), C_0 R_0^{-1} C_0^T \hat{p}(t)\Big\rangle dt \\ &- \mathbb{E}\int_0^T \Big\langle \hat{x}_0(t) - E_0 \mathcal{L}(\hat{x}_0)(t), Q_0(\hat{x}_0(t) - E_0 \mathcal{L}(\hat{x}_0)(t))\Big\rangle dt. \end{aligned}$$

Since R_0 and Q_0 are positive definite, we have

$$\mathbb{E}\int_0^T \Big\langle \mathbb{A}(\hat{x}_0, \hat{p})(t), (\hat{x}_0(t), p(t))\Big\rangle dt \leq -\mathbb{E}\int_0^T \Big\langle \hat{p}(t), C_0 R_0^{-1} C_0^T \hat{p}(t)\Big\rangle dt \leq -\lambda \|\hat{p}\|^2,$$

where λ is the smallest eigenvalue for $C_0 R_0^{-1} C_0^T$. With this monotonicity condition, the usual method of continuation in FBSDE gives the unique existence result and we omit the proof here. $\square$

5. Conclusion. In summary, the N-player and the mean field system are described, respectively, by Tables 1 and 2. We show that when all the ith players adopt the optimal control defined by the above mean field dynamics and objectives, and suppose further that the mean field term z satisfies the fixed point property (2.14)

$$z(t) = \int_{[a,b]} \mathbb{E}^{\mathcal{F}^0_{t-\delta} \vee \mathcal{F}^z_t} x_1^{i,\delta}(t) d\pi_\Delta(\delta),$$

then the N-player system in Table 1 would converge to the mean field analogical system in Table 2. The mentioned optimal controls serve as an ϵ-Nash equilibrium, and their function form is adapted to $\mathcal{F}^0_{t-\delta} \vee \mathcal{F}^z_t$ and hence agrees with the spirit of Stackelberg games. The explicit solution for the linear quadratic setting has also been studied comprehensively.

Table 1
N-player system.

Dominating Player	Dynamics	$dy_0 = g_0\Big(y_0(t), \frac{\sum_{j=1}^N y_1^{j,\Delta_j}(t)}{N}, v_0(t)\Big)dt + \sigma_0 dW_0(t)$
	Objective	$\mathcal{J}^{0,N}(v_0) = \mathbb{E}\int_0^T f_0\Big(y_0(t), \frac{\sum_{j=1}^N y_1^{j,\Delta_j}(t)}{N-1}, v_0(t)\Big)dt$
ith player	Dynamics	$dy_1^{i,\delta_i} = g_1\Big(y_1^{i,\delta_i}(t), \frac{\sum_{j=1,j\neq i}^N y_1^{j,\Delta_j}(t)}{N-1}, v_1^{i,\delta_i}(t), y_0(t-\delta_i)\Big)dt + \sigma_1 dW_1^i(t)$
	Objective	$\mathcal{J}^{i,\delta_i,N}(\mathbf{v}) = \mathbb{E}\int_0^T f_1\Big(y_1^{i,\delta_i}(t), \frac{\sum_{j=1,j\neq i}^N y_1^{j,\Delta_j}(t)}{N-1}, v_1^{i,\delta_i}(t), y_0(t-\delta_i)\Big)dt$

Table 2
Mean field analogical system.

Dominating Player	Dynamics	$dx_0 = g_0(x_0(t), z(t), v_0(t))dt + \sigma_0 dW_0(t)$
	Objective	$J^0(v_0) = \mathbb{E}\int_0^T f_0\Big(x_0(t), z(t), v_0(t)\Big)dt$
ith player	Dynamics	$dx_1^{i,\delta_i} = g_1(x_1^{i,\delta_i}(t), z(t), v_1^{i,\delta_i}(t), x_0(t-\delta_i))dt + \sigma_1 dW_1^i(t)$
	Objective	$J^{i,\delta_i}(v_1^{i,\delta_i}) = \mathbb{E}\int_0^T f_1\Big(x_1^{i,\delta_i}(t), z(t), v_1^{i,\delta_i}(t), x_0(t-\delta_i)\Big)dt$

Appendix A. We provide proofs of the mentioned technical lemmas.

Lemma A.1. *Let π_Δ be a probability measure on $[a,b]$. Given $\epsilon' > 0$, let π_Δ^{0+} be a restricted measure as defined in Corollary* 2.3. *There exists $N > 0$ such that*

$$\sup_{k\le n} \pi_\Delta^{0+}(a_{k-1}^{(n)}, a_k^{(n)}] \le 2\epsilon' \quad \forall\, n > N.$$

Proof. Suppose on the contrary there is a sequence of $\{(n_j, k_j)\}_j$ such that $\pi_\Delta^{0+}(a_{k_j-1}^{(n_j)}, a_{k_j}^{(n_j)}] > 2\epsilon'$. Or, we also have $\pi_\Delta^{0+}[a_{k_j-1}^{(n_j)}, a_{k_j}^{(n_j)}] > 2\epsilon'$. Let

$$m_j := \frac{a_{k_j-1}^{(n_j)} + a_{k_j}^{(n_j)}}{2}$$

be the midpoint of the interval. By the Bolzano–Weierstrass theorem, up to a subsequence of $\{m_j\}_j$, there is a limit point m. Without loss of generality we just take $m_j \to m$. For fixed $r > 0$, consider $\mathcal{A}_r = [m - \frac{1}{r}, m + \frac{1}{r}]$. Clearly, as $\{[a_{k_j-1}^{(n_j)}, a_{k_j}^{(n_j)}]\}_j$ diminishes in width, we can find a large enough J such that for all $j > J$, $[a_{k_j-1}^{(n_j)}, a_{k_j}^{(n_j)}] \subsetneqq \mathcal{A}_r$. Hence,

$$\pi_\Delta^{0+}(\mathcal{A}_r) \geq \pi_\Delta^{0+}[a_{k_j-1}^{(n_j)}, a_{k_j}^{(n_j)}] > 2\epsilon'.$$

Take $r \uparrow +\infty$, we have

$$\pi_\Delta^{0+}(\{m\}) = \lim_{r\uparrow+\infty} \pi_\Delta^{0+}(\mathcal{A}_r) \geq 2\epsilon' > \epsilon',$$

which contradicts the definition of π_Δ^{0+}. $\square$

LEMMA A.2. *If the condition* (3.8) *is satisfied, then for any $t_0 \in [0, T]$, there exists a unique solution to the following forward-backward ordinary differential equation on $[t_0, T]$:*

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} A_1 + B_1 & -C_1 R_1^{-1} C_1^T \\ -Q_1(I - E_1) & -A_1^T \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} := \mathcal{M}\begin{pmatrix} x \\ y \end{pmatrix}, \quad x(t_0) = 0, \quad y(T) = 0.$$

Proof. Let x, y be elements in the Hilbert space $L^2([t_0, T]; \mathbb{R}^{n_1})$ endowed with the inner product $\langle x, y\rangle_{\mathcal{Q}} = \int_{t_0}^T \langle x(t), \mathcal{Q}y(t)\rangle dt$. Since $\mathcal{Q} > 0$, we easily observe that the induced norm $\|\cdot\|_{\mathcal{Q}} = |\langle\cdot,\cdot\rangle_{\mathcal{Q}}|^{\frac{1}{2}}$ is equivalent to the usual L^2 norm. Consider the ordinary differential equation

$$\left\{ \frac{d}{dt}\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} A_1 & -C_1 R_1^{-1} C_1^T \\ -\mathcal{Q} & -A_1^T \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix} + \begin{pmatrix} B_1 \\ -\mathcal{S} \end{pmatrix} x, \quad X(t_0) = 0, \quad Y(T) = 0. \right.$$

As both $\mathcal{Q}$ and $C_1 R_1^{-1} C_1^T > 0$, the map $x \mapsto X$ corresponds to a well-defined control problem which guarantees the existence of (X, Y). Thus it suffices to show that it is a contraction, indeed. We first compute the differential for the inner product

$$\frac{d}{dt}\langle X, Y\rangle = \langle -C_1 R_1^{-1} C_1^T Y(t) + Bx(t), Y(t)\rangle - \langle X(t), \mathcal{Q}X(t) + \mathcal{S}x(t)\rangle.$$

Together with the initial and terminal conditions, we have

$$\int_{t_0}^T \langle C_1 R_1^{-1} C_1^T Y(t), Y(t)\rangle + \langle X(t), \mathcal{Q}X(t)\rangle dt = \int_{t_0}^T \langle Bx(t), Y(t)\rangle - \langle X(t), \mathcal{S}x(t)\rangle dt.$$

By the Cauchy–Schwarz inequality,

$$\|X\|_{\mathcal{Q}}^2 \leq \int_{t_0}^T \langle Bx(t), Y(t)\rangle - \langle X(t), \mathcal{S}x(t)\rangle dt. \tag{A.1}$$

On the other hand,

$$Y_t = \int_t^T e^{A_1^T(s-t)}\Big(\mathcal{Q}X(s) + \mathcal{S}x(s)\Big)ds,$$

which implies

$$\sup_{t_0\le t\le T}\|Y_t\| \le \|e^{A_1^T}\|\Big(\|X\|_{\mathcal{Q}} + \|\mathcal{Q}^{-\frac{1}{2}}\mathcal{S}\mathcal{Q}^{-\frac{1}{2}}\|\|x\|_{\mathcal{Q}}\Big), \tag{A.2}$$

where

$$\|e^{A_1^T}\| = \sup_{0\le t\le T}\sqrt{\int_t^T |e^{A_1^T(s-t)}\mathcal{Q}^{\frac{1}{2}}|^2 ds}.$$

Combining (A.1) and (A.2) yields

$$\begin{aligned}\|X\|_{\mathcal{Q}}^2 &\le \sqrt{T}\sup_{t_0\le t\le T}\|Y_t\|\|BQ^{-\frac{1}{2}}\|\|x\|_{\mathcal{Q}} + \|X\|_{\mathcal{Q}}\|\mathcal{Q}^{-\frac{1}{2}}\mathcal{S}\mathcal{Q}^{-\frac{1}{2}}\|\|x\|_{\mathcal{Q}}\\
&\le \sqrt{T}\|e^{A_1^T}\|\Big(\|X\|_{\mathcal{Q}} + \|\mathcal{Q}^{-\frac{1}{2}}\mathcal{S}\mathcal{Q}^{-\frac{1}{2}}\|\|x\|_{\mathcal{Q}}\Big)\|BQ^{-\frac{1}{2}}\|\|x\|_{\mathcal{Q}}\\
&\qquad + \|X\|_{\mathcal{Q}}\|\mathcal{Q}^{-\frac{1}{2}}\mathcal{S}\mathcal{Q}^{-\frac{1}{2}}\|\|x\|_{\mathcal{Q}}\\
&= \Big(\sqrt{T}\|e^{A_1^T}\|\|BQ^{-\frac{1}{2}}\| + \|\mathcal{Q}^{-\frac{1}{2}}\mathcal{S}\mathcal{Q}^{-\frac{1}{2}}\|\Big)\|X\|_{\mathcal{Q}}\|x\|_{\mathcal{Q}}\\
&\qquad + \sqrt{T}\|e^{A_1^T}\|\|\mathcal{Q}^{-\frac{1}{2}}\mathcal{S}\mathcal{Q}^{-\frac{1}{2}}\|\|BQ^{-\frac{1}{2}}\|\|x\|_{\mathcal{Q}}^2.\end{aligned}$$

Hence, for all $t_0 \in [0, T]$, the map is a contraction if

$$\Big(1 + \sqrt{T}\|e^{A_1^T}\|\|BQ^{-\frac{1}{2}}\|\Big)\Big(1 + \|\mathcal{Q}^{-\frac{1}{2}}\mathcal{S}\mathcal{Q}^{-\frac{1}{2}}\|\Big) < 2.$$

Note that this condition is in fact independent of t_0. ∎

LEMMA A.3. *The linear operator $\mathcal{L}$ defined in* (3.9) *is bounded.*

Proof. Using Lemma 3.8 and (3.7), $m_0(t) = \Gamma_t z_0(t) + h_0(t)$, where

$$\begin{aligned}-dh_0 &= \Big((A_1^T - \Gamma_t C_1 R_1^{-1}C_1^T)h_0(t) + (\Gamma_t D - Q_1 F)\int_{[a,b]} x_0(t-\delta)d\pi_\Delta(\delta)\Big)dt - dM_{h_0}(t),\\
h_0(T) &= 0.\end{aligned}$$

For some $\eta > 0$, one can apply Itô's formula on $e^{\eta t}|h_0(t)|^2$ to obtain the estimate

$$\begin{aligned}&\mathbb{E}[e^{\eta t}|h_0(t)|^2]\\
&\le \mathbb{E}\int_t^T e^{\eta s}\Big[2\Big\langle h_0(s), \mathcal{A}_s^T h_0(s) + \mathcal{B}_s\int_{[a,b]} x_0(s-\delta)d\pi_\Delta(\delta)\Big\rangle - \eta|h_0(s)|^2\Big]ds,\end{aligned}$$

where $\mathcal{A}_t^T = A_1^T - \Gamma_t C_1 R_1^{-1}C_1^T$ and $\mathcal{B}_t = \Gamma_t D - Q_1 F$. Hence, we can choose some $\eta > 4\|\mathcal{A}\|$, where $\|\mathcal{A}\| := \sup_{t\le T}\|\mathcal{A}_t\|$, such that

$$\begin{aligned}\eta\mathbb{E}\int_0^T e^{\eta s}|h_0(s)|^2 ds &\le \mathbb{E}\int_0^T e^{\eta s}\Big[2\Big\langle h_0(s), \mathcal{A}_s h_0(s) + \mathcal{B}_s\int_{[a,b]} x_0(s-\delta)d\pi_\Delta(\delta)\Big\rangle\Big]ds\\
&\le \Big(2\|\mathcal{A}\| + \frac{\eta - 4\|\mathcal{A}\|}{2}\Big)\mathbb{E}\int_0^T e^{\eta s}|h_0(s)|^2 ds\\
&\qquad + \Big(\frac{2\|\mathcal{B}\|}{\eta - 4\|\mathcal{A}\|}\Big)\mathbb{E}\int_0^T e^{\eta s}\Big|\int_{[a,b]} x_0(s-\delta)d\pi_\Delta(\delta)\Big|^2 ds\end{aligned}$$

After rearrangement, we obtain

$$\frac{\eta}{2}\mathbb{E}\int_0^T e^{\eta s}|h_0(s)|^2 ds \le \left(\frac{2\|\mathcal{B}\|}{\eta - 4\|\mathcal{A}\|}\right)\mathbb{E}\int_{[a,b]}\int_0^T e^{\eta s}|x_0(s-\delta)|^2 ds d\pi_\Delta(\delta)$$
$$\le \left(\frac{2\|\mathcal{B}\|}{\eta - 4\|\mathcal{A}\|}\right)\mathbb{E}\int_{-b}^T e^{\eta s}|x_0(s)|^2 ds.$$

Since the η-weighted norm is equivalent to the original L^2 norm, we have

$$\|h_0\| \le K\|x_0\|. \tag{A.3}$$

Similarly, letting $\zeta > 4\|\mathcal{A} + B_1\|$, apply Itô's formula on $e^{-\zeta t}|z_0(t)|^2$ yields

$$\begin{aligned}
&\zeta\mathbb{E}\int_0^T e^{-\zeta s}|z_0(s)|^2 ds\\
&\le 2\mathbb{E}\int_0^T e^{-\zeta s}\Big\langle z_0(s), (\mathcal{A}_t + B_1)z_0(s) - C_1R_1^{-1}C_1^T h_0(s) + D\int_{[a,b]} x_0(s-\delta)d\pi_\Delta(\delta)\Big\rangle ds\\
&\le \left(2\|\mathcal{A}+B_1\| + \frac{\zeta - 4\|\mathcal{A}+B_1\|}{2}\right)\mathbb{E}\int_0^T e^{-\zeta s}|z_0(s)|^2 ds\\
&\quad + \frac{4}{\zeta - 4\|\mathcal{A}+B_1\|}\Bigg\{\|C_1R_1^{-1}C_1^T\|\mathbb{E}\int_0^T e^{-\zeta s}|h_0(s)|^2 ds\\
&\quad + \|D\|\mathbb{E}\int_0^T e^{-\zeta s}\left|\int_{[a,b]} x_0(s-\delta)d\pi_\Delta(\delta)\right|^2 ds\Bigg\}.
\end{aligned}$$

After rearrangement again, we conclude that

$$\begin{aligned}
&\frac{\zeta}{2}\mathbb{E}\int_0^T e^{-\zeta s}|z_0(s)|^2 ds\\
&\le \frac{4}{\zeta - 4\|\mathcal{A}+B_1\|}\left\{\|C_1R_1^{-1}C_1^T\|\mathbb{E}\int_0^T e^{-\zeta s}|h_0(s)|^2 ds + \|D\|\mathbb{E}\int_{-b}^T e^{-\zeta s}|x_0(s)|^2 ds\right\}.
\end{aligned}$$

By the norm equivalence again, together with (A.3), we have $\|z_0\| \le K(\|h_0\|+\|x_0\|) \le 2K\|x_0\|$. We conclude that $\mathcal{L}$ and hence $\mathcal{L}^*$ are bounded. $\square$

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