SOME TOPOLOGICAL ASPECTS OF INTEGRABLE RIGID BODY DYNAMICS

by

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To my family:

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SOME TOPOLOGICAL ASPECTS OF INTEGRABLE RIGID BODY DYNAMICS

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The main aim of this dissertation is to describe topology of isoenergy manifolds of the classical Kirchhoff case of Kirchhoff equations of rigid body motion in an ideal, incompressible fluid.

The first chapter introduces the basic notions of rigid body dynamics and integrable Hamiltonian systems. To that end it also introduces the concept of symplectic and Poisson manifolds and the analytical mechanics theorems and definitions that are needed to formulate and understand the models of rigid body systems characterized by three different systems of six non-linear differential equations: the Euler-Poisson equations, the Kirchhoff equations, and the Poincaré-Zhukovsky equations. The role of underlying Lie-Poisson algebras is stressed.

In Chapter 2 we studied the Goryachev-Chaplygin top. This system is completely integrable if reduced to a level set of one first integral only. The bifurcation diagram of this completely integrable system is the region of possible motion on the plane of first integrals together with the image of the critical set.

Chapter 3 gives a complete description of the topology of the iso-energy manifolds of the Kirchhoff system of the Kirchhoff equations of rigid body motion in an ideal, incompressible fluid. This is a Hamiltonian system on Lie-Poisson algebra $e(3)$ with a Hamlitonian which is quadratic in mixed terms as well. For such general quadratic Hamiltonians on $e(3)$ we first

construct so-called reduced potential.

In the special case of the Kirchhoff system we use the reduced potential to construct its Reeb graphs. Based on a theorem of Smale, we use the combinatorics of the constructed Reeb graphs to compute the topology of the isoenergy manifolds. The challenge of the presence of a large number of parameters has been compensated by a relatively simple form of the reduced potential in this case.

In Chapter 4, we investigate the bifurcations of the momentum mapping for the Poincaré model of rigid body filled with ideal incompressible vortex fluid. The equations of motion are the Poincaré-Zhukovky equations. They can be seen as a Hamiltonian system on the Lie-Poisson algebra $so(4)$ with a quadratic Hamiltonian. For this purpose, we find the critical points of rank zero and rank one . Finally, the bifurcations are studied for the Kirchhoff case on the Lie algebra $e(3)$. We find critical points of rank zero and rank one.

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CHAPTER 1

INTRODUCTION

In order to study the bifurcations of systems of rigid body, let us first introduce the class of manifolds and the type of systems that describe this motion. For this purpose, we will first start with the notion of a manifold in symplectic geometry.

1.1 Poisson manifolds and symplectic manifolds

Symplectic and Poisson manifolds both arise from classical mechanics. As we will see symplectic manifolds come equipped with a natural vector field called the Hamiltonian vector field that corresponds to the Hamiltonian equations of a system. Symplectic manifolds are a special case of Poisson manifolds, as we will state in theorem 1. Indeed, a motivation for studying Poisson manifolds is that frequently it can be simpler to study a Poisson manifold that a symplectic manifold is embedded into, than it is to study a symplectic manifold itself. For example, it is often easier to analyze stability of the dynamics on a Poisson manifold than on its symplectic leaves.

Let us define a binary operation $\{\cdot,\cdot\}$ called the *Poisson bracket* such that this operation satisfies the following properties:

1. $\{f,g\} = -\{g,f\}$ $(skew-symmetry)$,

2. $\{af+bg,h\} = a\{f,h\}+b\{g,h\},\ \{h,af+bg\} = a\{h,f\}+b\{h,g\},\ a,b \in \mathbb{R}$ $(bilinearity)$,

3. $\{fg,h\} = \{f,h\}g + f\{g,h\}(Leibniz's\ rule)$,

4. $\{f,\{g,h\}\} + \{g,\{h,f\}\} + \{h,\{f,g\}\} = 0$ $(Jacobi\ identity)$

Now, suppose we have a manifold S with the operation defined by Poisson bracket and its properties defined as in 1-4, we call S a *Poisson manifold.* Considering the definition of a Poisson bracket, the structure can be degenerate and possess *Casimir functions* $f_K(\mathbf{x})$ that commute with all variables x_i. This means Casimir functions would commute with all functions $g(\mathbf{x})$ on S, i.e., $\{f_k, g\} = 0$. We will just call Casimir functions, *Casimirs* from now on, in this manuscript.

Definition 1. *A symplectic manifold S is a differentiable manifold with a global closed 2-form Ω of maximal rank, meaning $d\Omega = 0$, $\Omega^n \neq 0$.*

To include the linear algerba behind this, assuming a vector space V^{2n}, if $\Omega \in \bigwedge^2 V$ with $rk\ \Omega = 2n$, then there exist $\theta^1, ..., \theta^{2n} \in V^*$, linearly independent and with labeling such that

$$\Omega = \theta^1 \wedge \theta^{n+1} + ... + \theta^n \wedge \theta^{2n}.$$

The following also called the splitting theorem and demonstrates the relation between Poisson and symplectic manifolds.

Theorem 1. *Let $(P, \{\cdot, \cdot\})$ be a Poisson manifold, and let $p \in P$. Then there exists an open neighbourhood $U \subset P$ containing p and a unique diffeomorphic Poisson mapping*

$$\Phi = \Phi_S \times \Phi_N : U \to S \times N$$

where S is a symplectic manifold and N is a Poisson manifold with rank zero at $\Phi_N(p)$.

S in the above theorem is called a symplectic leaf of P. In other words, this theorem states that every Poisson manifold is naturally partitioned into regularly immersed symplectic leaves.

Furthermore, the structural group of the tangent bundle of S may be reduced to $U(n)$ with the use of a Reimannian metric and its orientibility. This means that the symplectic manifold

S carries an almost complex structure, hence leading to the German mathematician H. Weyl giving it the name *symplectic* from the Latin *com-plex* changing to Greek *sym-plectic.*[1]
In other words, a **symplectic manifold** is a pair (S, ω) where S is a manifold and ω is a non-degenerate closed 2-form on S.[44]

Definition 2. ***Non-degeneracy****: Let V be a vector space. Let*

$$\omega : V \times V \to \mathbb{R}$$

be a skew-symmetric, bilinear 2-form with $\omega \in \wedge^2 V^\star$. The form ω is ***nondegenerate*** *if for every $v \in V$,*

$$\omega(v, u) = 0 \ , \ \forall u \in V \Rightarrow v = 0.$$

Definition 3. ***A symplectic vector space*** *is a pair (V, ω) where V is a vector space and*

$$\omega \in \wedge^2(V^\star)$$

is a non-degenerate bilinear skew-symmetric form.

A two-form $\omega \in \Omega^2(S)$ is **nondegenerate** if and only if for any point $s \in S$, the bilinear form ω_s on the tangent space $T_s S$ is nondegenerate.
If we consider $V = \mathbb{R}^2$ with coordinates x and y and $\omega = dx \wedge dy$ as a (constant coefficient)differential form, then it is easy to see that

$$\omega^\sharp(\frac{\partial}{\partial x}) = \iota(\frac{\partial}{\partial x})(dx \wedge dy) = dy$$

and that similarly

$$\omega^\sharp(\frac{\partial}{\partial y}) = -dx.$$

So $\omega^\sharp$ is bijective. Therefore $(\mathbb{R}^2, dx \wedge dy)$ is a symplectic manifold.
Let (S, ω) be a symplectic manifold. Then for every point $s \in S$,

$$(\omega)^\sharp : T_s S \to T_s^\star S$$

is an isomorphism so there is a correspondence between 1-forms and vector fields. In particular, given a function $f \in C^\infty(S)$, the differential df of f is a one-form

$$s \mapsto df_s = \Sigma \frac{\partial f}{\partial x_i}(dx_i)_s.$$

This gives us a vector field

$$X_f(s) := ((\omega^\sharp)^{-1}(df_s).$$

Equivalently, X_f is defined by $\iota(X_f)\omega = df$.

The vector field X_f defined above is called the **Hamiltonian vector field** of the function f on a symplectic manifold (S, ω).

1.2 Hamiltonian Systems

In section 1.1, we defined the Poisson bracket and we will use this bracket to describe the motion of rigid bodies. First, let us discuss the basics of Hamiltonian dynamics to shed light on the description of a system defining rigid body motion.

As we know, in Newtonian mechanics motion is described in terms of acceleration, masses, time and velocities for the sake of tangibility in every day use. Years after Newton, Joseph-Louise Lagrange came up with a formulation for dynamics of a system with k degrees of freedom. Let $q_i,\ i = 1, ..., k$ define our position coordinates, $\dot{q}_i = \frac{dq_i}{dt}$ define our generalized velocity coordinates and let t be the independent time variable. The Lagrangian or Lagrange function

$$L(q_i, \dot{q}_i, t)$$

is then a function of $2k + 1$ dynamical variables. Lagrangian, L is defined as the difference between Kinetic energy and potential energy, for a general time dependent system in a electromagnetic field, is formulated as:

$$L(q, \dot{q}, t) = T(q, \dot{q}, t) - U(q, \dot{q}, t),$$

where T is the kinetic and U is the potential energy.

Hamilton[7] shows in his book, that for a conservative system, the Lagrangian equations of motion satisfy:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0. \tag{1.1}$$

In (1.1), the quantity $\frac{\partial L}{\partial \dot{q}_i}$ is the generalized momentum as we know it, in the absence of vector potential, otherwise it is called magnetic momentum. Lagrangian can now be used to describe the motion of a system. Here, Lagrangian would be a function of $2k$ dynamical variables $(q_1, ..., q_k, \dot{q}_1, ..., \dot{q}_k)$ with the dot representing time derivative. The motion of the system then can be described with k second order differential equations.

The formulation used in Hamiltonian dynamics to describe the motion of a system includes the Lagrange function. For the system coordinates $(q_1, ..., q_k)$ and generalized momenta $(p_1, ..., p_k)$, the momenta in (1.1) are defined in the following way:

$$p_i = \frac{\partial L}{\partial \dot{q}_i},$$

Change of basis from $(q, \dot{q}, t)$ to (q, p, t) is performed using a Legendre transformation gives:

$$H(q, p, t) = \left(\sum_i p_i \dot{q}_i - L(q, \dot{q}, t)\right), \;\; \dot{q} = \dot{q}(p, q, t). \tag{1.2}$$

After finding the differential of the left hand side and right hand side of (1.2) and setting them equal, we obtain:

$$\frac{\partial H}{\partial t}dt + \sum_i \frac{\partial H}{\partial q_i}dq_i + \sum_i \frac{\partial H}{\partial p_i}dp_i = \sum_i p_i d\dot{q}_i + \sum_i \dot{q}_i dp_i - \sum_i \frac{\partial L}{\partial q_i}dq_i - \sum_i \frac{\partial L}{\partial \dot{q}_i}d\dot{q}_i - \frac{\partial L}{\partial t}dt. \tag{1.3}$$

Simplifying the expression in (1.3) and equating the two sides of the equation, results in

$$\frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t}, \tag{1.4}$$

and for $i = 1, ..., k$:

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \tag{1.5}$$

$$\dot{p}_i = -\frac{\partial H}{\partial q_i}. \tag{1.6}$$

Taking the assumptions and results above into consideration, the function $H(q, p, t)$ is called the Hamiltonian and equations (1.5) and (1.6) are Hamilton equations of motion. Not considering the theory of relativity, the Hamiltonian is the sum of the potential and kinetic energies:

$$H(q, p) = T + U.$$

The Poisson bracket as defined in section 1.1, for two functions $f(q_i, p_i, t)$ and $g(q_i, p_i, t)$ is defined by

$$\{f, g\} \equiv \sum_i \left(\frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q_i} \right),$$

where obviously $\{f, f\} \equiv 0$ and if we let $f = H$, we have

$$\frac{dH}{dt} = \frac{\partial H}{\partial t}.$$

If H doesn't depend on time explicitly , it would be a constant of motion. Similarly, any invariant of the motion that doesn't depend explicitly on time, would have a vanishing Poisson bracket with H.

1.3 Integrable Hamiltonian Systems

Differential equations and as a result, Hamiltonian systems are divided into two classes, non-integrable versus integrable systems. Birkhoff discovered that a differential equations system is solved when there is some relation between the pattern of the motion and the phase space and this relation was clear when the system has sufficient number of conservation laws including first integrals, symmetry fields, or other tensor invariants[5]. In this case, we say a system is solved by "quadratures" which means the solutions are found by doing a finite number of algebraic operations and calculations of integrals of known functions, Bour and Liouville structured these conditions and relationships and later they were shaped into what we call the *Liouville- Arnold theorem.*[3].

Theorem 2. *Assume that on a symplectic manifold* $M^{2n}(\boldsymbol{p},\ \boldsymbol{q}) = M^{2n}(p_1, ..., p_n, q_1, ..., q_n)$, *$n$ functions in involution are given, i.e.,*

$$F_1, ..., F_n : \{F_i, F_j\} \equiv 0,\ i, j = 1, ..., n.$$

We also assume that on the level manifold M_f of the integrals $\{x \in M^{2n} : F_i = c_i,\ i = 1, ..., n\}$, where the n functions F_i are independent. Then:

1. *M_f is a smooth manifold invariant under the phase flow with the Hamiltonian function $H = F_1$.*

2. *If the manifold M_f is connected and compact, then it is diffeomorphic to the n-dimensional torus*

$$T_n = \{(\varphi_1, ..., \varphi_n)\ mod\ 2\pi\},$$

with φ_i being the action angles.

3. *Locally there exists a canonical coordinate transformation $(\mathbf{p}, \mathbf{q}) \mapsto (\varphi, \mathbf{E})$ (called 'action-angle' coordinates)*

$$(\varphi, E) = (\varphi_1, ..., \varphi_n, E_1, ..., E_n) \in T^n \times \mathbb{R}^n$$

such that the angles $\varphi_i,\ i = 1, ..., n$ are coordinates on M_c, the actions $E_i,\ i = 1, ..., n$ are first integrals and $H(q, p) = H(E)$. The phase flow with the Hamiltonian function $H = F_1$ defines on M_f a conditionally periodic motion, i.e., in some angular coordinates $\varphi = (\varphi_1, ..., \varphi_n)$, we have the Hamilton equations

$$\frac{d\boldsymbol{\varphi}}{dt} = \boldsymbol{\omega},\ \boldsymbol{\omega} = \omega(c_1, ..., c_n) = (\omega_1, ..., \omega_n),$$

$$\frac{d\mathbf{E}}{dt} = 0,\ \mathbf{E} = (\mathbf{E_1}, ..., \mathbf{E_n})$$

4. The canonical equations with Hamiltonian function H are integrable by quadratures.

The classical proof of the statement of Liouville - Arnold theorem can be found in Whittaker's treatise[6]. A Hamiltonian system satisfying the conditions of the Arnold- Liouville thereom is called completely integrable.

Assume a dynamical system is given with two degrees of freedom and with the Hamiltonian $H(\mathbf{q}, \mathbf{p})$ as a function of canonical variables $\mathbf{q} = (q_1, q_2)$ and $\mathbf{p} = (p_1, p_2)$, in the form:

$$\dot{\mathbf{q}} = \frac{\partial H}{\partial \mathbf{p}}, \quad \dot{\mathbf{p}} = -\frac{\partial H}{\partial \mathbf{p}}. \tag{1.7}$$

If this system has an additional integral $K(\mathbf{q}, \mathbf{p})$, by Arnold - Liouville theorem, the system is integrable and can be integrated by quadrature as long as H and K are in involution on the 4 dimensional symplectic manifold M^4 defined by the system and the Casimirs[3]. The examples we will discuss in this dissertation are integrable systems.

1.4 Rigid body dynamics

In this section we will review the physics of moments of inertia, kinetic energy and angular momentum and angular velocity to arrive at the system of 6 differential equations defining motion of a rigid body.

Moment of inertia of a rigid body about an axis is the inertia needed to carry it by rotation about that axis. We define moment of inertia for a 3 dimensional body along x-axis to be

$$\iiint_\tau (y^2 + z^2) dm = A,$$

the moment of inertia along y-axis to be

$$\iiint_\tau (x^2 + z^2) dm = B,$$

and the one along the z-axis to be

$$\iiint_\tau (x^2 + y^2) dm = C.$$

We know from physics that in the absence of applied torques, the angular kinetic energy T is conserved so $\frac{dT}{dt} = 0$.

Let us define a few terms here. The angular momentum of a rigid body rotating about an axis passing through the origin of the local reference frame is in fact the product of the inertia tensor of the body and the angular velocity. The moment of inertia tensor in 3D Cartesian coordinates, is a three-by-three matrix $\mathbf{I}$ that can be multiplied by angular velocity vector of the rigid body to produce the corresponding angular momentum vector for the rigid body at its center of mass.

Definition 4. *The diagonal elements in the inertia tensor are called the angular moments of inertia. There exists an orthogonal basis on which* $\mathbf{I}$ *tensor has a diagonal form. Diagonal elements are the* ***principal moments of inertia****. The coordinate axes are principal axes.*

Definition 5. *A* ***principal axis of rotation*** *is an eigenvector of the mass moment of inertia tensor, defined relative to the center of mass of the body.*

In the above definition, the corresponding eigenvalues are the principal moments of inertia. To formulate this, we use the fact that the angular momentum vector is given by the moment of inertia tensor times the angular velocity vector, i.e., $L = \mathbf{I}\ \omega$. If ω is an eigenvector of $\mathbf{I}$, then we have

$$L = \mathbf{I}\ \omega = \lambda\omega,$$

where the scalar eigen value λ is the principal moment of inertia. There are always three mutually orthogonal principal axes of rotation and three corresponding principal moments of inertia in $3D$ space.

The angular kinetic energy may be expressed in terms of an inertia tensor $\mathbf{I}$ and the angular velocity vector $\boldsymbol{\omega}$.

$$T = \frac{1}{2}\boldsymbol{\omega} \cdot \mathbf{I} \cdot \boldsymbol{\omega} = \frac{1}{2}A\omega_1^2 + \frac{1}{2}B\omega_2^2 + \frac{1}{2}C\omega_3^2,$$

where ω_k are the components of the angular velocity vector $\boldsymbol{\omega}$ along the principal axes, and the A, B *and* C are the principal moments of inertia. Thus, the conservation of kinetic energy imposes a constraint on the three-dimensional angular velocity vector $\boldsymbol{\omega}$ that in the principal axis frame, it must lie on an ellipsoid, called *inertia ellipsoid.*

If x, y and z axis are oriented along the principal axes of the ellipsoid of inertia, given $\vec{r}$ the radius vector and $\vec{v}$, the linear velocity, for $\vec{G}$,the angular momentum, we get :

$$\vec{G} = \iiint\limits_{\tau} \vec{r} \times \vec{v}\,\mathrm{d}m = \iiint\limits_{\tau} \vec{r} \times (\vec{\omega} \times \vec{r})\,\mathrm{d}m = \iiint\limits_{\tau} [\vec{\omega}|\vec{r}|^2 - \vec{r}(\vec{r}.\vec{\omega})]\,\mathrm{d}m,$$

using the property of cross product that says

$$\vec{a} \times (\vec{b} \times \vec{c}) = \vec{b}(\vec{a} \cdot \vec{c}) - \vec{c}(\vec{a} \cdot \vec{b}).$$

Doing the following computations,

$$\vec{\omega}|\vec{r}|^2 = (p\hat{\imath} + q\hat{\jmath} + r\hat{k})(x^2 + y^2 + z^2)$$

where $\hat{\imath}, \hat{\jmath}, \hat{k}$ are the units vectors of the stationary coordinate system where O is the the origin of both systems(stationary and the one rigidly connected to the body). and

$$\begin{aligned}\vec{\omega}|\vec{r}|^2 - \vec{r}(\vec{r}.\vec{\omega}) &= \hat{\imath}[p(x^2 + y^2 + z^2) - px^2 - x(qy + rz)] + \\ &\hat{\jmath}[q(x^2 + y^2 + z^2) - qy^2 - y(px + rz)] + \hat{k}[r(x^2 + y^2 + z^2) - rz^2 - z(px + qy).] \end{aligned} \quad (1.8)$$

Applying the triple integrals to the region τ, we get

$$\vec{G} = \hat{\imath}(I_{xx}p - I_{xy}q - I_{xz}r) + \hat{\jmath}(I_{yy}q - I_{yz}r - I_{yx}p) + \hat{k}(I_{zz}r - I_{zx}p - I_{zy}q),$$

where

$$\iiint\limits_{\tau} (y^2 + z^2)dm = A = I_{xx}, \qquad \iiint\limits_{\tau} xydm = I_{xy} = I_{yx},$$

$$\iiint\limits_{\tau} (x^2 + z^2)dm = B = I_{yy}, \qquad \iiint\limits_{\tau} xzdm = I_{xz} = I_{zx},$$

$$\iiint\limits_{\tau} (x^2+y^2)dm = C = I_{zz}, \qquad \iiint\limits_{\tau} yzdm = I_{yz} = I_{zy},$$

with the remark that if x, y, z axes are oriented along the principal axes of ellipsoid of inertia, then the coefficients I_{xy}, I_{yz} and I_{xz} become zero.

The angular momentum is defined to be the vector: $\vec{G} = Ap\hat{i} + Bq\hat{j} + Cr\hat{k}$ where angular velocity vector is $\vec{\omega} = p\hat{i} + q\hat{j} + r\hat{k}$, assuming $\hat{\imath}, \hat{\jmath}, \hat{k}$ are the unit vectors of the coordinate system connected rigidly to the rigid body and A, B and C are defined as above.

If $\vec{R}$ is some variable vector originating from the point O and let x, y, z be the coordinates of its vertex in the moving system of coordinates, rigidly connects with the body, such that:

$$\vec{R} = \hat{\imath}x + \hat{\jmath}y + \hat{k}z,$$

then:

$$\frac{d\vec{R}}{dt} = \hat{\imath}\frac{dx}{dt} + \hat{\jmath}\frac{dy}{dt} + \hat{k}\frac{dz}{dt} + x\frac{d\hat{i}}{dt} + y\frac{d\hat{j}}{dt} + z\frac{d\hat{k}}{dt}.$$

Since $\frac{d\hat{i}}{dt}$ is the velocity of the vertex of the unit vector $\hat{\imath}$ of the moving system and applying the same thing to $\hat{\jmath}$ and $\hat{k}$, we have

$$\frac{d\hat{i}}{dt} = \vec{\omega} \times \hat{\imath},$$

$$\frac{d\hat{j}}{dt} = \vec{\omega} \times \hat{\jmath}$$

and

$$\frac{d\hat{k}}{dt} = \vec{\omega} \times \hat{k}.$$

The $\vec{\omega}$ is the angular velocity vector of the body. Therefore, we have

$$\frac{d\vec{R}}{dt} = \hat{\imath}\frac{dx}{dt} + \hat{\jmath}\frac{dy}{dt} + \hat{k}\frac{dz}{dt} + \vec{\omega} \times (x\hat{i} + y\hat{j} + z\hat{k}).$$

This means that since

$$\hat{\imath}\frac{dx}{dt} + \hat{\jmath}\frac{dy}{dt} + \hat{k}\frac{dz}{dt}$$

is the relative derivative of the vector $\vec{R}$ with respect to time, in general we have:

$$\frac{d\vec{R}}{dt} = \frac{\delta \vec{R}}{\delta t} + \vec{\omega} \times \vec{R}. \tag{1.9}$$

Now fixing $\bar{z}$, which is the coordinate of the fixed point of the body in direction of z axis,

$$\bar{k} = \gamma_1 \hat{i} + \gamma_2 \hat{j} + \gamma_3 \hat{k},$$

where γ_1, γ_2 and γ_3 are the coordinates of the direction cos vectors with respect to the moving axes Ox, Oy, Oz.

Since $\bar{k}$ is fixed,

$$\frac{d\bar{k}}{dt} = 0$$

and

$$\frac{d\bar{k}}{dt} = \frac{\delta \bar{k}}{\delta t} + \vec{\omega} \times \bar{k} \Rightarrow \frac{\delta \bar{k}}{\delta t} = -\vec{\omega} \times \bar{k}.$$

So,

$$\hat{i}\frac{d\gamma_1}{dt} + \hat{j}\frac{d\gamma_2}{dt} + \hat{k}\frac{d\gamma_3}{dt} = -\vec{\omega} \times \bar{k} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \gamma_1 & \gamma_2 & \gamma_3 \\ p & q & r \end{vmatrix}.$$

This gives:

$$\begin{cases} \dfrac{d\gamma_1}{dt} = \gamma_2 r - q\gamma_3, \\ \dfrac{d\gamma_2}{dt} = p\gamma_3 - \gamma_1 r, \\ \dfrac{d\gamma_3}{dt} = q\gamma_1 - p\gamma_2. \end{cases} \tag{1.10}$$

The differential equations in (1.10), are the first three equations in the system of equations corresponding to the motion of rigid bodies. Now, let us find the next set of equations for this motion.

Let us note that we are considering the motion of a heavy rigid body here, given the mass in a gravitational field. In general, $\vec{L}$(moment of force of gravity) is the cross product of the position vector $\vec{r_0}$ of a point and the force applied to that point:

$$\vec{L} = \vec{r_0} \times \vec{F}.$$

Let us suppose $\vec{F} = mg$ is in direction of negative $\bar{z}$,

$$\vec{F} = mg(\gamma_1 \hat{i} + \gamma_2 \hat{j} + \gamma_3 \hat{k})$$

and

$$L = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x_0 & y_0 & z_0 \\ mg\gamma_1 & mg\gamma_2 & mg\gamma_3 \end{vmatrix} = \hat{i}mg(y_0\gamma_3 - z_0\gamma_2) + \hat{j}mg(z_0\gamma_1 - x_0\gamma_3) + \hat{k}mg(x_0\gamma_2 - y_0\gamma_1). \tag{1.11}$$

Also, we can show that $\frac{d\vec{G}}{dt} = \vec{L}$, since we have

$$\iiint_{\tau} \vec{r} \times \vec{v} \, \mathrm{d}m = \iiint_{\tau} \vec{r} \times \vec{v} \rho d\tau,$$

and now deriving this with respect to time, we have

$$\frac{d\vec{G}}{dt} = \iiint_{\tau} \left(\frac{d\vec{r}}{dt} \times \vec{v} + \vec{r} \times \frac{d\vec{v}}{dt} \right) \rho d\tau.$$

However, since $\frac{d\vec{r}}{dt}$ and $\vec{v}$ are in the same direction

$$\frac{d\vec{r}}{dt} \times \vec{v} = 0.$$

Then in

$$\iiint_{\tau} \vec{r} \times \frac{d\vec{v}}{dt} \rho d\tau,$$

since

$$\rho d\tau \frac{d\vec{v}}{dt} = dm\vec{a} = d\vec{F},$$

we obtain

$$\iiint_\tau \vec{r} \times d\vec{F}$$

and this is nothing but $\vec{L}$.

Using (1.9) and (1.11), We can rewrite $\frac{d\vec{G}}{dt} = \vec{L}$ as

$$\frac{d\vec{G}}{dt} = A\frac{dp}{dt}\hat{\imath} + B\frac{dq}{dt}\hat{\jmath} + C\frac{dr}{dt}\hat{k} + \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ p & q & r \\ Ap & Bq & Cr \end{vmatrix}.$$

This gives the other three differential equations corresponding to motion of the rigid body, which is:

$$\begin{cases} A\dfrac{dp}{dt} + (C - B)qr = mg(y_0\gamma_3 - z_0\gamma_2) \\ B\dfrac{dq}{dt} + (A - C)pr = mg(z_0\gamma_1 - x_0\gamma_3) \\ C\dfrac{dr}{dt} + (B - A)pq = mg(x_0\gamma_2 - y_0\gamma_1). \end{cases} \tag{1.12}$$

The Goryachev Chaplygin system would occur when $A = B = 4C$ in (1.12).

1.5 The Euler-Poisson equations of rigid body dynamics

The closed form of the system of equations represented by (1.10) and (1.12) are called the *Euler-Poisson equations* defining the motion of a rigid body about a fixed point in a uniform gravitational field. The *Euler-Poisson equations* are given below:

$$\begin{cases} \boldsymbol{I}\dot{\boldsymbol{\omega}} + \boldsymbol{\omega} \times \boldsymbol{I}\boldsymbol{\omega} = \mu\mathbf{r} \times \boldsymbol{\gamma} \\ \dot{\boldsymbol{\gamma}} = \boldsymbol{\gamma} \times \boldsymbol{\omega}. \end{cases} \tag{1.13}$$

In (1.13), $\boldsymbol{\omega} = (p, q, r)$ is the angular velocity vector, $\boldsymbol{r} = (r_1, r_2, r_3)$ is the position vector of the center of mass and $\boldsymbol{\gamma} = (\gamma_1, \gamma_2, \gamma_3)$ is the unit vertical vector, with origin at the fixed point of the rigid body, $\mu = mg$ is the body's weight and $\boldsymbol{I} = diag(a_1, a_2, a_3)$ is the inertia tensor relative to the fixed point in these axes.

Remark 1. *Let us note that* $I_{xx} = a_1$, $I_{yy} = a_2$ *and* $I_{zz} = a_3$ *in the above representation of the system, with* I *being the inertia tensor.*

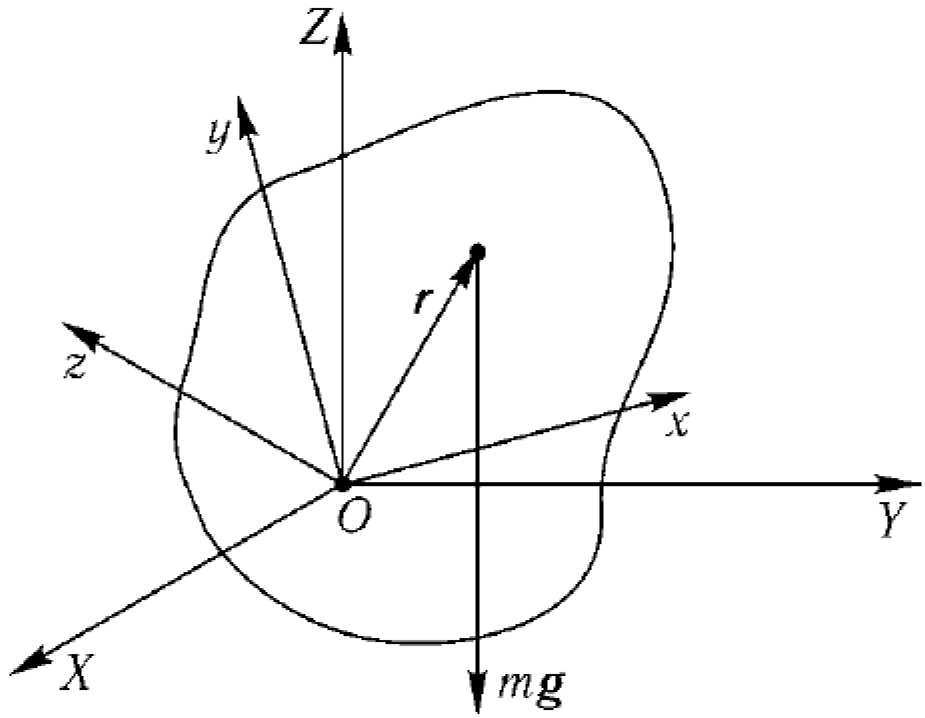

Figure 1.1: A rigid body with a fixed point in a gravitational field

Letting $\boldsymbol{M} = \boldsymbol{I\omega}$ be the angular momentum vector, (1.13) can be written in the following Hamiltonian form:

$$\dot{M}_i = \{M_i, H\},\ \dot{\gamma}_i = \{\gamma_i, H\},\ i = 1, 2, 3.$$

Let us define the Lie–Poisson bracket on the Lie algebras $so(4)$, $e(3)$ and $so(3,1)$.

$$\{M_i, M_j\} = -\epsilon_{ijk}M_k,\ \{M_i, \gamma_j\} = -\epsilon_{ijk}\gamma_k,\ \{\gamma_i, \gamma_j\} = -\varkappa\epsilon_{ijk}M_k. \tag{1.14}$$

Cases $\varkappa > 0$, $\varkappa = 0$ and $\varkappa < 0$ in (1.14) correspond to the Lie algebra $so(4), e(3)$ and $so(3,1)$, respectively.

The Hamiltonian, representing the total energy of the system is defined as:

$$H = \frac{1}{2}\langle \boldsymbol{M}, \boldsymbol{I^{-1}M}\rangle - \mu\langle \boldsymbol{r}, \boldsymbol{\gamma}\rangle. \tag{1.15}$$

The Lie–Poisson bracket in this case being

$$\{M_i, M_j\} = -\epsilon_{ijk}M_k,\ \{M_i, \gamma_j\} = -\epsilon_{ijk}\gamma_k,\ \{\gamma_i, \gamma_j\} = 0, \tag{1.16}$$

defined on Lie algebra $e(3)$, is degenerate and has two Casimirs

$$F_1 = \langle \boldsymbol{M}, \boldsymbol{\gamma}\rangle,\ F_2 = \boldsymbol{\gamma}^2 \tag{1.17}$$

which commute with any function of (M, γ). In (1.17), F_1 is called the integral of areas and it's the projection of angular momentum on a fixed vertical axis. The integral $F_2 = const$ is the square of the absolute value of the unit vertical vector and can be normalized to 1.

1.6 Integrable cases of the Euler-Poisson equations: the Euler case, the Lagrange case, the Kowalevski case and the Goryachev – Chaplygin case

The Euler case occurs when there are no fields acting on the body, i.e., $\boldsymbol{r} = 0$. The Hamiltonian and the additional integral in this case are

$$H = \frac{1}{2}\langle \boldsymbol{M}, \boldsymbol{I^{-1}M}\rangle,\ F_3 = \boldsymbol{M}^2 = const.$$

The geometric interpretation of this case was given by L. Poinsot in 1851, stating that the inertial ellipsoid with the fixed center $\frac{1}{2}(a_1p^2 + a_2q^2 + a_3r^2) = h$ rolls without slipping on a plane fixed in the absolute space and perpendicular to the angular momentum vector.

In the Lagrange case, the body possesses dynamical symmetry due to $a_1 = a_2$ and the center of mass lies on the axis of dynamical symmetry $r_1 = r_2 = 0$ and the additional integral is $F_3 = M_3 = const$.

In Kowalevski case just like Lagrange's case, the body has a dynamical symmetry with

$a_1 = a_2$ and the center of mass lies in the equatorial plane of the ellipsoid of inertia $r_3 = 0$. In addition $\frac{a_1}{a_3} = 2$ holds. The Hamiltonian and the additional integral found by Kowalevski are

$$H = \frac{1}{2}(M_1^2 + M_2^2 + 2M_3^2) - x\gamma_1,$$
$$F_3 = (\frac{M_1^2 - M_2^2}{2} + x\gamma_1)^2 + (M_1 M_2 + x\gamma_2)^2 = k^2,$$

with the coordinates chosen in a way that the position vector of the center of mass has coordinates $\boldsymbol{r} = (x, 0, 0)$ and the weight of the body is assumed to equal 1.

The Goryachev –Chaplygin case is an integrable case where the assumption for integrability is $\langle \boldsymbol{M}, \boldsymbol{\gamma} \rangle = 0$ meaning that the angular momentum vector is forced to lie in the horizontal plane. Also, the ratio of the moments of inertia here is $\frac{a_1}{a_3} = 4$ and we have dynamical symmetry due to $a_1 = a_2$. The additional integral and the Hamiltonian are given below:

$$H = \frac{1}{2}(M_1^2 + M_2^2 + 4M_3^2) - x\gamma_1,$$

$$F_3 = M_3(M_1^2 + M_2^2) + xM_1\gamma_3.$$

1.7 The Zhukovsky – Poincaré equations and the Poincaré case

Fundamental questions of astrophysics and geophysics about the phenomena of precessional motions of celestial bodies, including the Earth, led to theoretical studies and formulation of the assumptions that a planet is a body which consists of a hard shell (mantle) surrounding the liquid core. Modelling these questions further led to the study of rigid bodies with cavities filled with fluids.

One of the first comprehensive studies of this subject was done by N. Y. Zhukovsky in 1880's [43]. He studied uniform vortex motion in the case of ellipsoidal cavities and also potential motion in the case of non-simply connected cavities. Parallel in time to that was unpublished treatise of another great A. M. Lyapunov [21]. The celebrated monograph on

Hydrodynamics by H. Lamb collected a significant amount of material, including important historic and background results and references. The whole last Chapter XII of that book is devoted to rotating masses of liquid, [20]. Similar questions were studied by various authors in the last decade of XIX century like V. A. Steklov [38, 40], V. Volterra [42], and others, see also [17, 41]. The geophysical questions of the Earth precession also motivated H. Poincaré to study the subject in 1910. We are going to follow closely here Poincaré's work [22].

Let us also mention some more recent works. In [36] the model problem of Mercury's librations was considered. In [16] the problem of dynamics of a rigid body with a liquid-filled cavity was considered in the framework of nonholonomic dynamics. Following Zhukovsky [43] and Poincaré [22], we consider a rigid body with an ellipsoidal cavity filled with ideal incompressible fluid. There is a solution to the Euler equations for the ideal fluid, for which velocities satisfy the hydrodynamics equations. The boundary conditions are linear in coordinates. Due to the results of Helmholtz [19], it is also known that the vortex flow being homogeneous initially, remains homogeneous thereafter.

Let us now set the scene for Zhukovsky's work and his assumptions that helped build the model for motion of a rigid body with cavity filled with fluid.

Letting O be the origin fixed at the center of the cavity and X be the kinematic moment matrix of the body with cavity filled with fluid and

$$M_i = \frac{1}{2}\epsilon_{ijk}X_{jk},\ \ P_i = X_{oi},\ \ i,j,k = 1,2,3,$$

the equations of motion are represented in the following way:

$$\dot{\boldsymbol{M}} = \boldsymbol{M}\times\frac{\partial H_1}{\partial \boldsymbol{M}} + \boldsymbol{P}\times\frac{\partial H_1}{\partial \boldsymbol{P}},\ \ \dot{\boldsymbol{P}} = \boldsymbol{P}\times\frac{\partial H_1}{\partial \boldsymbol{M}} + \boldsymbol{M}\times\frac{\partial H_1}{\partial \boldsymbol{P}}, \tag{1.18}$$

where:

$$H_1 = \frac{1}{2}\langle \boldsymbol{M}, \boldsymbol{I}^{-1}\boldsymbol{M}\rangle + \langle \boldsymbol{M}, \boldsymbol{B}\boldsymbol{P}\rangle + \frac{1}{2}\langle \boldsymbol{P}, \boldsymbol{C}\boldsymbol{P}\rangle, \tag{1.19}$$

see [5]. The equations (1.18) are equivalent to the Hamiltonian equations on the Poisson algebra of $so(4)$, with the Hamiltonian H_1. The diagonal matrix I is the inertia tensor that is positive definite and of the form $I = \mathrm{diag}(a_1, a_2, a_3)$ where $a_i > 0$. We will suppose the matrices B and C are also diagonal and of the form $B = \mathrm{diag}(b_1, b_2, b_3)$ and $C = \mathrm{diag}(c_1, c_2, c_3)$. Here c_i correspond to the coefficients of the relative potential defined below and b_i are the moments of rotation of the relative motion. Poincaré calculated the relative force of the motion of the ideal fluid in a rigid body with an ellipsoidal cavity defined as

$$\alpha x^2 + \beta y^2 + \gamma z^2 = 1. \tag{1.20}$$

This calculation was done in terms of the density of the liquid and the principal axes of the ellipsoidal cavity. The relative potential energy is

$$\frac{1}{2}(c_1 P_1^2 + c_2 P_2^2 + c_3 P_3^2),$$

where

$$c_1 = \frac{4\pi d}{15}\frac{1}{\alpha\beta\gamma}\left(\frac{1}{\beta} + \frac{1}{\gamma}\right), \quad c_2 = \frac{4\pi d}{15}\frac{1}{\alpha\beta\gamma}\left(\frac{1}{\alpha} + \frac{1}{\gamma}\right), \quad c_3 = \frac{4\pi d}{15}\frac{1}{\alpha\beta\gamma}\left(\frac{1}{\alpha} + \frac{1}{\beta}\right). \tag{1.21}$$

Here d is the density of the fluid and α, β, γ are the coefficients of the ellipsoidal cavity (1.20). Poincaré calculated

$$b_1 = \frac{8\pi d}{15}\frac{1}{\sqrt{\alpha\beta\gamma}}\frac{1}{\sqrt{\beta\gamma}}, \quad b_2 = \frac{8\pi d}{15}\frac{1}{\sqrt{\alpha\beta\gamma}}\frac{1}{\sqrt{\alpha\gamma}}, \quad b_3 = \frac{8\pi d}{15}\frac{1}{\sqrt{\alpha\beta\gamma}}\frac{1}{\sqrt{\alpha\beta}}. \tag{1.22}$$

The Poisson commutator relations on $so(4)$ have the form

$$\{M_i, M_j\} = -\epsilon_{ijk} M_k, \;\; \{M_i, P_j\} = -\epsilon_{ijk} P_k, \;\; \{P_i, P_j\} = -\epsilon_{ijk} M_k. \tag{1.23}$$

The Casimirs of the Poisson algebra $so(4)$ are defined in the standard way [4, 5]:

$$f_1 = P_1^2 + P_2^2 + P_3^2 + M_1^2 + M_2^2 + M_3^2, \qquad f_2 = M_1 P_1 + M_2 P_2 + M_3 P_3. \tag{1.24}$$

1.7.1 The Poincaré case

Poincaré is well-known for his conjecture and his 3-body problem, but in this text we want to focus more on his accomplishments in celestial and fluid mechanics leading to a simplified model of the earth motion, considering it as a ellipsoidal rigid body with a symmetric ellipsoidal cavity filled with liquid. Having seen the equations of motion that was established for motion of a rigid body with cavity filled with fluid, here we want to demonstrate Poincaré main work in defining this system and establishing its integrability after finding the additional first integral.

The Poincaré case [22] assumes an additional, axial symmetry and can be seen as the simplest integrable example, where, for each of the diagonal matrices A, B, C the corresponding pairs of eigen-values coincide:

$$a_1 = a_2, \quad b_1 = b_2, \quad c_1 = c_2.$$

Along with these conditions, it is also assumed that the cavity is symmetric, $\alpha = \beta$. This symmetry together with (1.21) and (1.22) imply $b_3 = c_3$. We will refer to the model which satisfies

$$a_1 = a_2, \quad b_1 = b_2, \quad c_1 = c_2, \quad b_3 = c_3$$

as the Poincaré model of rigid body on $so(4)$. Along with the Hamiltonian (1.19), there is an additional first integral of motion in the Poincaré model, M_3. We will denote it as K

$$K = M_3. \tag{1.25}$$

Thus in the Poincaré case, there are four conserved quantities, H_1 and K are two first integrals and f_1 and f_2 are two Casimirs:

$$f_1(\mathbf{M}, \mathbf{P}) = 1, \quad f_2(\mathbf{M}, \mathbf{P}) = g, \quad H_1(\mathbf{M}, \mathbf{P}) = h_1, \quad K(\mathbf{M}, \mathbf{P}) = k. \tag{1.26}$$

Thus, the Poincaré case is completely integrable, see [22], [5]. Using the Casimirs and the conditions of axial symmetry, we transform the Hamiltonian (1.19) and get the Hamiltonian for the Poincaré case:

$$H = \frac{1}{2}(M_1^2 + M_2^2 + aM_3^2) + b(M_1P_1 + M_2P_2) + \frac{1}{2}c(P_1^2 + P_2^2), \tag{1.27}$$

provided that

$$a = \frac{(a_3^{-1} - c_3)}{(a_1 - c_3)}, \quad b = \frac{(b_1 - b_3)}{(a_1^{-1} - c_3)}, \quad c = \frac{(c_1 - c_3)}{(a_1^{-1} - c_3)}. \tag{1.28}$$

Remark 2. *From $a = (a_3^{-1} - c_3)/(a_1 - c_3)$, we conclude that $a < 0$ if $a_3^{-1} > c_3 > a_1$ or $a_3^{-1} < c_3 < a_1$; otherwise $a \geq 0$.*

Lemma 1. *For b and c in the Hamiltonian H (1.27), the relations hold:*

$$a_3^{-1} > c_3 \Rightarrow c \geq b;$$

$$a_3^{-1} < c_3 \Rightarrow c \leq b.$$

Proof. With the use of the inequality between the arithmetic and geometric mean and given that β and γ are positive coefficients in the equation of ellipsoid (1.20), we get:

$$\frac{1}{\beta} + \frac{1}{\gamma} = \frac{\beta + \gamma}{\beta\gamma} \geq \frac{2\sqrt{\beta\gamma}}{\beta\gamma} = \frac{2}{\sqrt{\beta\gamma}}.$$

This means $c_1 \geq b_1$ based on the relations (1.21) and (1.22). Since $b_3 = c_3$ and $a_1 > 0$, we get $c \geq b$. $\square$

1.8 Kirchhoff equations on dynamics of a rigid body in an ideal incompressible fluid

This section is dedicated to the original prototype of a system of rigid body that was invented by the German mathematician G. Kirchhoff who modeled the dynamics of motion of a rigid body in an ideal incompressible fluid.

In effect, the Hamiltonian systems defined on the Lie-Poisson space $e(3)^*$ are labeled *Kirchhoff equations* with the Hamiltonian being a quadratic function with respect to the variables impulsive momentum and implulsive force on $e(3)^*$.

Kirchhoff's model assumes an ideal fluid which can be described by the type of fluid that is incompressible, irrotational and at rest at infinity. The velocity potential of this kind of fluid is single-valued. The equations of motion of the rigid body in an ideal fluid interestingly decouple from the partial differential equations describing the fluid motion and they result in a system of six ordinary differential equations.

Following Kirchhoff [14], we consider motion of a rigid body τ in $\mathbb{R}^3$ in an ideal incompressible fluid with density ρ. It is supposed that motion of fluid is potential with φ as a potential function and with the velocity $v = \frac{\partial \varphi}{\partial x}$, where $x \in \mathbb{R}^3$. Since the fluid is incompressible, we have that $\triangle \varphi = 0$, since the divergence of v is zero. It is also assumed that $v \to 0$ when $|x| \to \infty$.

The equations of the rigid body defined by Kirchhoff can be written in Hamiltonian form on the Lie algebra $e(3) = so(3) \bigoplus_s \mathbb{R}^3$.

$$\begin{cases} \dot{\mathbf{M}} = \mathbf{M} \times \frac{\partial \mathbf{H}}{\partial \mathbf{M}} + \mathbf{P} \times \frac{\partial \mathbf{H}}{\partial \mathbf{P}}. \\ \dot{\mathbf{P}} = \mathbf{P} \times \frac{\partial \mathbf{H}}{\partial \mathbf{M}}. \end{cases} \tag{1.29}$$

The equations (1.29) are the Kirchhoff equations of motions of rigid body in an ideal fluid with $\mathbf{M}$ and $\mathbf{P}$ representing *impulsive momentum* and *impulsive force*, respectively. The system is defined in the reference frame fixed with respect to the body.

The Hamiltonian H_1 is a quadratic form in the variables $\mathbf{M}$ and $\mathbf{P}$ and is defining the Kinetic energy T:

$$H_1 = \frac{1}{2}\langle \mathbf{I^{-1}M}, \mathbf{M} \rangle + \langle \mathbf{BM}, \mathbf{P} \rangle + \frac{\mathbf{1}}{\mathbf{2}}\langle \mathbf{CP}, \mathbf{P} \rangle, \tag{1.30}$$

with I and C being symmetric matrices and B an arbitrary one. As H is defining the kinetic energy of the body and the fluid, I, B and C are such that the Hamiltonian would be positive-definite.

Equation (1.29) have the following Casimirs:

$$F_1 = \langle M.P \rangle = c_1,\ F_2 = \mathbf{P^2} = \mathbf{c_2}, \tag{1.31}$$

and as usual one of the integrals of the system is $H = h$. Unlike Euler-Poisson equations where c_2 in the Casimir is equal to 1, for Kirchhoff's equations, this is not necessarily the case.

1.8.1 The Kirchhoff case of Kirchhoff equations on $e(3)$

The Kirchhoff case of Kirchhoff equations on $e(3)$ is an integrable case of Kirchhoff equations on $e(3)$ and was discovered by G. Kirchhoff for a dynamically symmetric body moving in an ideal fluid. Kirchhoff managed to integrate the equations of motion in terms of elliptic functions. This case is analogous to the Lagrange case of the Euler-Poisson equations and has the Casimirs mentioned in (1.31), with the extra integral of the system being

$$K = M_3.$$

The Kirchhoff case is defined by conditions:

$$I = \mathrm{diag}(a_1, a_1, a_3), \quad B = \mathrm{diag}(b_1, b_1, b_3), \quad C = \mathrm{diag}(c_1, c_1, c_3). \tag{1.32}$$

The extra integral M_3 is related to the existence of a cyclic coordinate which is the angle of proper rotation.

1.9 Overview of the questions, methods and results that are treated in this dissertation

In the next chapter, we will delve deeper into the Goryachev- Chaplygin case. The process to find the first integrals of the system is reviewed and then the solution by quadrature is obtained. After that, the critical points of rank zero and rank one and the equilibrium analysis is performed. We will finally obtain the bifurcations of the Liouville Tori of this case. The study of bifurcations of Liouville Tori of integrable system is of great importance in understanding the stability of the system. For instance, these bifurcations have been studied in Elliptical billiards in [33].

In Chapter 3, we will study topology of the Kirchhoff case of rigid body motion in an ideal incompressible fluid. We introduce the reduced potential for general Hamiltonian systems on $e(3)$ with mixed quadratic terms. In application to the Kirchhoff case, we describe the Reeb graphs of the reduced potential. We provide a complete topological description of the three-dimensional isoenergy manifolds for that system, based on a combinatorial study of the Reeb graphs. Studying its momentum map, we describe the points of ranks zero and one[12].

In Chapter 4, we are going to study topology of the Poincaré model of a rigid body with an ellipsoidal cavity filled with an ideal incompressible liquid. The Poincaré model is integrable due to certain additional symmetry conditions. We introduce the reduced potential for general Hamiltonian systems on $so(4)$ with mixed quadratic terms. In application to the Poincaré case, we describe the Reeb graphs of the reduced potential. We provide a complete topological description of the three-dimensional isoenergy manifolds for that system, based on a combinatorial study of the Reeb graphs[12]. Studying its momentum map, we describe points of ranks zero and one and present corresponding bifurcation diagrams.

In order to facilitate the study of bifurcation diagrams and Reeb graphs, let us define these concepts here.

Definition 6. *Let (M^{2n}, ω) be a symplectic manifold corresponding to an integrable system with the Hamiltonian H. Let $f_1, f_2, ..., f_n$ be its independent integrals in involution. Let us define the smooth mapping*

$$F : M^{2n} \to \mathbb{R}^n, \ where \ F(x) = (f_1(x), f_2(x), ..., f_n(x)) \, .$$

The mapping F is called the momentum mapping.

Definition 7. *A point $x \in M$ is called a critical(or singular) point of the momentum mapping F if* $\operatorname{rank} dF(x) < n$. *Its image $F(x)$ is called a critical value.*

Now, let $K \subset M$ be the set of all critical points of the momentum mapping F.

Definition 8. *The set $F(K) \subset \mathbb{R}^n$ which is the image of K under the momentum mapping, is called the bifurcation diagram.*

After drawing the bifurcation diagrams of the integrable system for different values of parameters of the system, we can then acquire the structure of their Reeb graphs based on their critical points, Afterwards, these Reeb graphs and bifurcations diagrams are put together and analyzed based on the regions of the bifurcation diagrams to conclude the topology of Liouville tori as defined in theorem 2. Reeb graphs are defined in Chapter 3 as this method of finding topology of Liouville tori of the Poincaré system is implemented there.

CHAPTER 2

THE GORYACHEV-CHAPLYGIN SYSTEM

2.1 Model of rigid body motion

The Goryachev-Chaplygin case is obtained by letting $A = B = 4C$ in (1.12) and it requires that the center of gravity of the rigid body lies in the equatorial plane of the ellipsoid of inertia, i.e., $y_0 = z_0 = 0$. This means that (1.12) becomes:

$$\begin{cases} 4\dfrac{dp}{dt} = 3qr \\ 4\dfrac{dq}{dt} = -3rp - a\gamma_3 \\ \dfrac{dr}{dt} = a\gamma_2, \end{cases} \tag{2.1}$$

with $\frac{mgx_0}{C} = a$.

2.2 First Integrals

We know that in general to integrate this system with six differential equations completely, we need five first integrals. We know that three of these first integrals are easily obtained from mechanical and geometrical considerations. In our case, since time doesn't enter in our equations explicitly, we can replace this system by a system of 5 equations in the symmetrical form:

$$\frac{dp}{P} = \frac{dq}{Q} = \frac{dr}{R} = \frac{d\gamma_1}{\Gamma_1} = \frac{d\gamma_2}{\Gamma_2} = \frac{d\gamma_3}{\Gamma_3} = dt,$$

where

$$\begin{cases} AP = -(C-B)qr + mg(y_0\gamma_3 - z_0\gamma_2) \\ BQ = -(A-C)pr + mg(z_0\gamma_1 - x_0\gamma_3) \\ CR = -(B-A)pq + mg(x_0\gamma_2 - y_0\gamma_1) \\ \Gamma_1 = r\gamma_2 - q\gamma_3 \\ \Gamma_2 = p\gamma_3 - r\gamma \\ \Gamma_3 = q\gamma_1 - p\gamma_2. \end{cases}$$

And, since t doesn't enter the equations for $P, Q, R, \Gamma_1, \Gamma_2, \Gamma_3$, we basically only have:

$$\frac{dp}{P} = \frac{dq}{Q} = \frac{dr}{R} = \frac{d\gamma_1}{\Gamma_1} = \frac{d\gamma_2}{\Gamma_2} = \frac{d\gamma_3}{\Gamma_3}.$$

Another remarkable property here is that since P doesn't contain p, Q doesn't contain q and etc., so that

$$\frac{\partial P}{\partial p} = \frac{\partial Q}{\partial q} = \frac{\partial R}{\partial r} = \frac{\partial \Gamma_1}{\partial \gamma_1} = \frac{\partial \Gamma_2}{\partial \gamma_2} = \frac{\partial \Gamma_3}{\partial \gamma_3} = 0 \tag{2.2}$$

From (2.2), we have:

$$\frac{\partial P}{\partial p} + \frac{\partial Q}{\partial q} + \frac{\partial R}{\partial r} + \frac{\partial \Gamma_1}{\partial \gamma_1} + \frac{\partial \Gamma_2}{\partial \gamma_2} + \frac{\partial \Gamma_3}{\partial \gamma_3} = 0.$$

By theory of postmultipliers, if we have four first integrals not containing t, one more first integral can be found by means of integration of certain ordinary equations in total differentials using integrating factors which is carried out by quadratures. This means the system can be completely integrated as its equivalent to a system of five differential equations. The theory of postmultiplier states that in the system of n equations:

$$\frac{dx_1}{X_1} = \frac{dx_2}{X_2} = \dots = \frac{dx_n}{X_n} = dt,$$

if we know the postmultiplier of the system and $n-2$ first integrals, then the system reduces to integrating one equation with a known integrating factor which is always carried out by quadrature. This is guaranteed by the Arnold-Liouville theorem. Here, we have a 6 dimensional space, but fixing two of the geometrical Casimirs(two of the first integrals, one of which is $\gamma^2 + \gamma_2^2 + \gamma_3^2 = 1$), we get a 4 dimensional common symplectic manifold on the common level set. By Arnold-Liouville theorem, we only need two more first integrals to be able to completely integrate the system.

Definition 9. *Let P be a Poisson manifold. A function $j \in C^\infty(P)$ such that $\{j, f\} = 0$ for all $f \in C^\infty(M)$ is called a Casimir function on P.*

Now let's find the four first integrals in Goryachev-Chaplygin case. We know that $\gamma^2 + \gamma_2^2 + \gamma_3^2 = 1$ is **always a Casimir** for rigid body motion system. To find the second integral, just multiply the equations in (2.1) respectively by p, q and r. This will give

$$4p\frac{dp}{dt} + 4q\frac{dq}{dt} + r\frac{dr}{dt} = 3pqr - 3pqr - aq\gamma_3 + ar\gamma_2.$$

Integrating the above result, we get

$$2p^2 + 2q^2 + \frac{r^2}{2} = \int a(-q\gamma_3 + r\gamma_2)dt$$

and since from (1.10), we know that

$$-q\gamma_3 + r\gamma_2 = \frac{d\gamma_1}{dt},$$

we end up getting the Hamiltonian of the system to be

$$4(p^2 + q^2) + r^2 = 2a\gamma_1 + k.$$

Next, we multiply the equations in (2.1) respectively by γ_1, γ_2 and γ_3 and add them to the equations in (1.10) respectively multiplied by $4p, 4q$ and r. This gives

$$4(\gamma_1\frac{dp}{dt} + p\frac{d\gamma_1}{dt}) + 4(\gamma_2\frac{dq}{dt} + q\frac{d\gamma_2}{dt}) + (\gamma_3\frac{dr}{dt} + r\frac{d\gamma_3}{dt}) =$$

$$3qr\gamma_1 - 3pr\gamma_2 - a\gamma_2\gamma_3 + a\gamma_2\gamma_3 + 4pr\gamma_2 - 4pq\gamma_3 + 4pq\gamma_3 - 4qr\gamma_1 + \gamma_1 qr - \gamma_2 pr = 0.$$

Integrating the above equation we get one of the Casimirs of the system to be

$$4(p\gamma_1 + q\gamma_2) + r\gamma_3 = h$$

where h is the constant of integration.

To get the fourth integral, as denoted by Goryachev, in the case that $h = 0$ which is when the principal angular momentum lies in a horizontal plane, we can find the fourth invariant

$$4(p\gamma_1 + q\gamma_2) + r\gamma_3 = 0. \tag{2.3}$$

Multiplying the first two equations of (2.1) by p and q respectively, we get :

$$4p\frac{dp}{dt} + 4q\frac{dq}{dt} = -aq\gamma_3$$

and simplifying it, we have

$$2\frac{d}{dt}(p^2 + q^2) = -aq\gamma_3. \tag{2.4}$$

Multiply (2.4) by $2r$ and we get

$$4r\frac{d}{dt}(p^2 + q^2) = -2arq\gamma_3. \tag{2.5}$$

Then let's multiply the third equation in (2.1) by $4(p^2 + q^2)$, and we have

$$4(p^2 + q^2)\frac{dr}{dt} = 4a\gamma_2(p^2 + q^2). \tag{2.6}$$

Adding (2.5) and (2.6), the result is

$$4\frac{d}{dt}(r(p^2 + q^2)) = -2aqr\gamma_3 + 4a\gamma_2(p^2 + q^2). \tag{2.7}$$

Now, multiplying the third equation of (1.10) by $4ap$ and the first equation in (2.1) by $a\gamma_3$ and adding them, we obtain:

$$4a\frac{d}{dt}(p\gamma_3) = 4apq\gamma_1 - 4ap^2\gamma_2 + 3aqr\gamma_3. \tag{2.8}$$

Sum of (2.7) and (2.8), gives:

$$4\frac{d}{dt}[r(p^2+q^2)+ap\gamma_3] = aqr\gamma_3 + 4apq\gamma_1 + 4a\gamma_2 q^2 = aq[r\gamma_3 + 4p\gamma_1 + 4q\gamma_2] \tag{2.9}$$

and with the assumption that $h = 0$, we know that

$$r\gamma_3 + 4p\gamma_1 + 4q\gamma_2 = 0. \tag{2.10}$$

After plugging in (2.10) into (2.9) and integrating (2.9) with respect to t, we obtain **the fourth integral**:

$$r(p^2+q^2)+ap\gamma_3 = g,$$

where g is the constant of integration. Therefore, we have found two Casimirs

$$\langle \vec{\gamma}, \vec{\gamma} \rangle = 1,$$

$$\langle \vec{M}, \vec{\gamma} \rangle = 0$$

and a Hamiltonian

$$H = 4p^2 + 4q^2 + r^2 - 2a\gamma$$

and the integral

$$f = r(p^2+q^2) + ap\gamma_3.$$

2.3 Solution

In *Lectures on Integration of the Equations of Motion of Rigid Body about a Fixed Point* by Golubev, the details of the solution bu quadratures is mentioned [11]. Goryachev uses a substitution using two new variables u and v, given the role of Kovalevskaya's variables

s_1 and s_2. Let $u - v = r$ and $uv = 4(p^2 + q^2)$ and introducing the following functions of u and v:

$$\begin{cases} U = u^3 - ku - 4g, \quad V = v^3 - kv + 4g, \\ U_1^2 = U - 2au, \quad V_1^2 = V - 2av, \\ -U_2^2 = U + 2au, \quad -V_2^2 = V + 2av. \end{cases} \tag{2.11}$$

Now, using the four first integrals we found, and substituting for u and v, we obtain three new equations for γ_1, γ_2 and γ_3:

$$\begin{cases} 2a\gamma_1 = uv + (u - v)^2 - k, \\ 2a\gamma_3 = \dfrac{4g - (u - v)uv}{2p}, \\ 2a\gamma_2 = -\dfrac{4g - (u - v)uv}{8pq}(u - v) - \dfrac{uv + (u - v)^2 - k}{q}p. \end{cases} \tag{2.12}$$

This substitution helps us completely integrate the system. First eliminating p and then q, using the u and v substitutions, we get

$$\begin{cases} 8ap = U_1V_2 - V_1U_2 \\ 8aq = U_1V_1 + U_2V_2 \\ r = u - v \\ 2a\gamma = \dfrac{U + V}{u + v} \\ 2a\gamma_2 = \dfrac{U_1U_2 - V_1V_2}{u + v} \\ 2a\gamma_3 = \dfrac{U_1V_2 + V_1U_2}{u + v}. \end{cases} \tag{2.13}$$

From the first two equations in (2.1), we get $8(p\frac{dp}{dt} + q\frac{dq}{dt}) = -2aq\gamma_3$ and taking derivative of the second u,v substitution, $\frac{du}{dt}v + u\frac{dv}{dt} = -2aq\gamma_3$. From the third equation of (2.1), with the assumption that $r = u - v$, we have:

$$\frac{du}{dt} - \frac{dv}{dt} = a\gamma_2.$$

Hence,

$$\begin{cases} \dfrac{du}{dt}(u+v) = au\gamma_2 - 2a\gamma_3 \\ \dfrac{dv}{dt}(u+v) = -av\gamma_2 - 2a\gamma_3. \end{cases}$$

From equations (2.13), we have:

$$2a\gamma_2 = \frac{(U_1U_2 - V_1V_2)}{u+v}$$

and

$$4aq\gamma_3 = \frac{(U_1^2 + U_2^2)V_1V_2 + (V_1^2 + V_2^2)U_1U_2}{4a(u+v)}.$$

Manipulating these, we get Chaplygin's remarkable equations that can be solved using hyperelliptic integrals in the case of genus 2 which is a problem quite analogous to the case of Kovalevskaya:

$$\begin{cases} \dfrac{du}{U_1U_2} - \dfrac{dv}{V_1V_2} = 0 \\ \dfrac{2udu}{U_1U_2} + \dfrac{2vdv}{V_1V_2} = dt. \end{cases} \tag{2.14}$$

2.4 Equilibrium Analysis

To find the equilibrium solutions, we set the right hand side of our system of differential equations equal to zero.

$$\begin{cases} 3qr = 0, (1') \\ -3rp - a\gamma_3 = 0, (2') \\ a\gamma_2 = 0, (3') \\ r\gamma_2 - q\gamma_3 = 0, (4') \\ p\gamma_3 - r\gamma_1 = 0, (5') \\ q\gamma_1 - p\gamma_2 = 0.(6') \end{cases} \tag{2.15}$$

From $(3')$, we have $\gamma_2 = 0$.

From $(1')$, we have either $q = 0$ or $r = 0$.

2.4.1 $q \neq 0$ but $r = 0$

If $q \neq 0$ but $r = 0$, then we have equilibrium

$$(p_*, q_*, r_*, \gamma_1^*, \gamma_2^*, \gamma_3^*) = (p, q, 0, \gamma_1, 0, 0).$$

In this case, using the fourth integral we get $g = 0$ which can simplify the substitution and consequently the integration.Plugging in to the last equation from (2.1), we get $q\gamma_1 = 0$ which gives either $\gamma_1 = 0$(the first integral not satisfied) or $q = 0$ but this is contradicting the assumption which takes us to the last possible scenario.

2.4.2 $q = r = 0$

In case $q = r = 0$, then the equilibrium occurs at

$$(p_*, q_*, r_*, \gamma_1^*, \gamma_2^*, \gamma_3^*) = (p, 0, 0, \gamma_1, 0, 0).$$

Here the fourth integral again gives g=0. From the second integral we have $4p^2 = 2a\gamma_1 + k$, and from the third integral with $h = 0$, p has to be zero,which leads to the equilibrium becoming $(0, 0, 0, \frac{-k}{2a}, 0, 0)$. This together with the second integral give $k = \pm 2a$ which gives us the final result $(0, 0, 0, \pm 1, 0, 0)$ for the equilibrium. Since at this point the determinant of stability matrix is zero, this is a stable equilibrium.

2.4.3 Conclusion

Putting all stability diagrams together, the conclusion is that in space of real numbers, the only stable equilibriums of this system are $(0, 0, 0, 1, 0, 0)$ and $(0, 0, 0, -1, 0, 0)$.

2.4.4 Bifurcation diagram of Goryachev-Chaplygin top

In order to investigate the stability of the equilibrium solutions of the Goryachev- Chaplygin system, we are following the footsteps of Mamaev, Bolsinov and Borisov in [8]. To construct the bifurcation diagram of Goryachev-Chaplygin top, a representation of the system in separating variables is used. These canonical variables are defined on the symplectic leaf

$$\mathcal{M}_0 = \{\mathbf{M}, \gamma | \gamma^{\mathbf{2}} = \mathbf{1}, (\mathbf{M}, \gamma) = \mathbf{0}\}$$

with constant zero area, $\mathbf{M} = (\mathbf{p}, \mathbf{q}, \mathbf{r})$ and $\gamma = (\gamma_1, \gamma_2, \gamma_3)$. The definitions are as follows:

$$p = 2\sqrt{p_1 p_2}\sin\frac{q_1 - q_2}{2}\ ,\ q = 2\sqrt{p_1 p_2}\cos\frac{q_1 - q_2}{2}\ ,\ r = p_1 - p_2,$$
$$\gamma_1 = \frac{p_1\sin q_1 + p_2 \sin q_2}{p_1 + p_2}\ ,\ \gamma_2 = \frac{p_1 \cos q_1 + p_2 \cos q_2}{p_1 + p_2},$$

and

$$\gamma_3 = \frac{-2\sqrt{p_1 p_2}}{p_1 + p_2}\cos\frac{q_1 + q_2}{2}.$$

Here, $q_1, q_2 \in [0, 2\pi)$ are the angular variables and $p_1, p_2 > 0$ are their momenta, respectively. Consequently, the variables p_i and q_i are defined on a half-cylinder. The Hamiltonian and the first integral corresponding to this change of variables are:

$$H = \frac{2}{p_1 + p_2}(p_1^3 + p_2^3 - \frac{\mu}{2}(p_1 \sin q_1 + p_2 \sin q_2)),$$
$$F = \frac{4p_1 p_2}{p_1 + p_2}(p_1^2 - p_2^2 - \frac{\mu}{2}(\sin q_1 - \sin q_2)).$$

It's easy to see that we have a common level set

$$\mathcal{M}_{h,f} = \{\mathbf{M}, \gamma | H = h, F = f\}$$

of the functions is described by

$$\varphi_1(q_1, p_1) = p_1(p_1^2 - \frac{\mu}{2}\sin q_1 - \frac{h}{2}) - \frac{f}{4} = 0,$$

$$\varphi_2(q_1,p_1) = p_2(p_2^2 - \frac{\mu}{2}\text{sin}q_2 - \frac{h}{2}) + \frac{f}{4} = 0.$$

Since these equations are invariant under the change of variables f to $-f$ and φ_1 to φ_2 and φ_2 to φ_1, we only need to look at the case $f > 0$, so we have the following level curves:

$$\bar{\varphi}(p,q) = p(p^2 - \frac{\mu}{2}\text{sin}q - \frac{h}{2})$$

which are closed on the cylinder and depending on h being between $-\mu$ and μ or being greater than μ, they have different forms.

Now, we know that degeneration of curves above happens when the rank of the map drops. Investigating the degeneration of $\varphi_1 = 0$, and later $\varphi_2 = 0$ we get two cases, with the first one being

$$\begin{cases} \dfrac{\partial\varphi_1}{\partial p_1} = \dfrac{\partial\varphi_1}{\partial q_1} = 0 \\ \varphi_1 = 0\ ,\ \varphi_2 = 0 \end{cases}$$

and the second one being

$$\begin{cases} \dfrac{\partial\varphi_2}{\partial p_2} = \dfrac{\partial\varphi_2}{\partial q_2} = 0, \\ \varphi_1 = 0\ ,\ \varphi_2 = 0, \end{cases}$$

in both of which, if we let $2p_1 = \lambda$ and respectively $2p_2 = \lambda$, we get:

$$f = \lambda^3\ ,\ h = \pm\mu + \frac{3}{2}\lambda^2.$$

Here μ corresponds to the constant 4 in the bifurcation diagram. This result leads to the following figure:

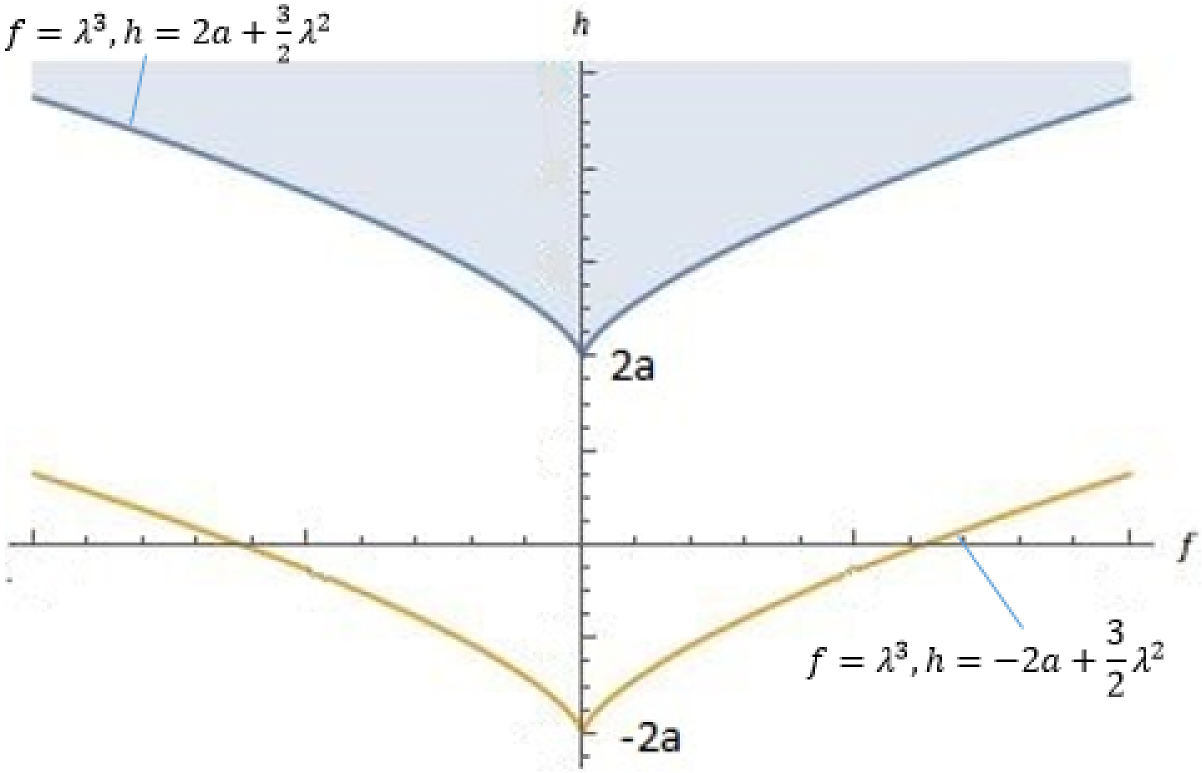

Figure 2.1: Bifurcation diagram of the Goryachev-Chaplygin case

2.5 Dependence of first integrals

Let us list the first integrals we got as follows:

$$\begin{cases} 4(p^2+q^2)+r^2 = 2a\gamma_1 + k, (1) \\ 4(p\gamma_1+q\gamma_2)+r\gamma_3 = 0, (2) \\ \gamma_1^2+\gamma_2^2+\gamma_3^2 = 1, (3) \\ r(p^2+q^2)+ap\gamma_3 = g. (4) \end{cases}$$

To investigate what happens if a pair of first integrals are dependent, we introduce:

$$\begin{cases} m_1 = 4(p^2+q^2) + r^2 - 2a\gamma_1 - k = 0, \\ m_2 = 4(p\gamma_1 + q\gamma_2) + r\gamma_3 = 0, \\ m_3 = \gamma_1^2 + \gamma_2^2 + \gamma_3^2 - 1 = 0, \\ m_4 = r(p^2+q^2) + ap\gamma_3 - g = 0. \end{cases}$$

we know that $m_1,\ m_2,\ m_3,\ m_4$ are functionally independent, if we show that $\Delta m_1,\ \Delta m_2,\ \Delta m_3,$ Δm_4 are linearly independent. This is equivalent to the following determinant becoming zero, i.e.,

$$\begin{vmatrix} 8p & 8q & 2r & -2a & 0 & 0 \\ 4\gamma_1 & 4\gamma_2 & \gamma_3 & 4p & 4q & r \\ 0 & 0 & 0 & 2\gamma_1 & 2\gamma_2 & 2\gamma_3 \\ 2rp + a\gamma_3 & 2rq & p^2+q^2 & 0 & 0 & ap \end{vmatrix} = 0.$$

Our MATLAB calculations show that this matrix has rank 4 and doesn't drop rank, affirming that the integrals are functionally independent.

2.6 When does the hyper-elliptic, genus 2 case convert to elliptic case and drop genus?

Definition 10. *The resultant of two univariate polynomials over a field or over a commutative ring is commonly defined as the determinant of their Sylvester matrix. More precisely, let*

$$A = a_0x^d + a_1x^{d-1} + \cdots + a_d$$

and

$$B = b_0x^e + b_1x^{e-1} + \cdots + b_e$$

be nonzero polynomials of degrees d and e respectively. Let us denote by $\mathcal{P}_i$ the vector space (or free module if the coefficients belong to a commutative ring) of dimension i whose elements are the polynomials of degree strictly less than i. The map

$$\varphi : \mathcal{P}_e \times \mathcal{P}_d \to \mathcal{P}_{d+e}$$

such that

$$\varphi(P, Q) = AP + BQ$$

is a linear map between two spaces of the same dimension. Over the basis of the powers of x (listed in descending order), this map is represented by a square matrix of dimension $d + e$, which is called the Sylvester matrix of A and B (for many authors and in the article Sylvester matrix, the Sylvester matrix is defined as the transpose of this matrix; this convention is not used here, as it breaks the usual convention for writing the matrix of a linear map). [39]

The resultant of A and B is thus the determinant

$$\begin{vmatrix} a_0 & 0 & \cdots & 0 & b_0 & 0 & \cdots & 0 \\ a_1 & a_0 & \cdots & 0 & b_1 & b_0 & \cdots & 0 \\ a_2 & a_1 & \ddots & 0 & b_2 & b_1 & \ddots & 0 \\ \vdots & \vdots & \ddots & a_0 & \vdots & \vdots & \ddots & b_0 \\ a_d & a_{d-1} & \cdots & \vdots & b_e & b_{e-1} & \cdots & \vdots \\ 0 & a_d & \ddots & \vdots & 0 & b_e & \ddots & \vdots \\ \vdots & \vdots & \ddots & a_{d-1} & \vdots & \vdots & \ddots & b_{e-1} \\ 0 & 0 & \cdots & a_d & 0 & 0 & \cdots & b_e \end{vmatrix},$$

which has e columns of a_i and d columns of b_j (the fact that the first column of a's and the first column of b's have the same length, that is $d = e$, is here only for simplifying the display of the determinant).

In the solution provided by Goryachev and Chaplygin, we have:

$$\begin{cases} U_1^2 = 4u^3 - (k+2a)u - g \\ U_2^2 = 4u^3 - (k-2a)u - g. \end{cases}$$

As a reminder, $a = \frac{mgx_0}{C}$, k is the constant of integration in the first integral and g is the constant of integration in the 4th integral. Now, the genus will drop when U_1^2 and U_2^2 have a common root or when either of U_1^2 or U_2^2 have a double root. U_1^2 and U_2^2 have a common root, when their *resultant* is zero. That is:

$$\operatorname{res}(U_1^2, U_2^2) = \begin{vmatrix} 4 & 0 & -k-2a & -g & 0 & 0 \\ 0 & 4 & 0 & -k-2a & -g & 0 \\ 0 & 0 & 4 & 0 & -k-2a & -g \\ 4 & 0 & -k+2a & -g & 0 & 0 \\ 0 & 4 & 0 & -k+2a & -g & 0 \\ 0 & 0 & 4 & 0 & -k+2a & -g \end{vmatrix} = 0.$$

But, $\operatorname{res}(U_1^2, U_2^2) = 1024a^3g = 0$, which means we need either $g = 0$ which is when the constant of integration in the 4th integral is zero or when $a = \frac{mgx_0}{C} = 0$. Since mg represents the weight of the rigid body and it can never be zero, we conclude $x_0 = 0$.

U_1^2 has a double root when the resultant (determinant of the 5×5 Sylvester matrix) we obtain from U_1^2 and $\frac{dU_1^2}{du}$ is zero, i.e.,

$$\operatorname{res}(U_1^2, \frac{dU_1^2}{du}) = \begin{vmatrix} 4 & 0 & 12 & 0 & 0 \\ 0 & 4 & 0 & 12 & 0 \\ -k-2a & 0 & -k-2a & 0 & 12 \\ -g & -k-2a & 0 & -k-2a & 0 \\ 0 & -g & 0 & 0 & -k-2a \end{vmatrix} = 0.$$

This happens when $-512a^3 - 768a^2k - 384ak^2 + 1728g^2 - 64k^3 = 0$.

Similarly U_2^2 has a double root when

$$\operatorname{res}(U_2^2, \frac{dU_2^2}{du}) = \begin{vmatrix} 4 & 0 & 12 & 0 & 0 \\ 0 & 4 & 0 & 12 & 0 \\ -k+2a & 0 & -k+2a & 0 & 12 \\ -g & -k+2a & 0 & -k+2a & 0 \\ 0 & -g & 0 & 0 & -k+2a \end{vmatrix} = 0.$$

This means U_2^2 has a double root when $512a^3 - 768a^2k + 384ak^2 + 1728g^2 - 64k^3 = 0$

CHAPTER 3

TOPOLOGICAL ANALYSIS OF ISOENERGY SURFACES OF THE KIRCHHOFF CASE ON $e(3)$

The aim of this chapter is to study the topology of the Kirchhoff case of a rigid body in an ideal incompressible fluid, subject to the additional symmetry conditions as explained in Chapter 1. A modern account of the rigid body dynamics and various generalizations can be found in [5, 18, 19] and references therein. The topological methods were applied to rigid body dynamics for example in [23], [34], [4], [30], [15], [13], [24], [26], [27], [9] and the references therein. In particular, the topological methods were applied to rigid body dynamics in fluid in [28], [29], [25].

Our approach belongs to the setting proposed and developed by the Moscow State University topology school, by A. T. Fomenko, A. V. Bolsinov and their students and collaborators. It was broadly presented in [4]. From the methodological point of view, the topological analysis of the Kirchhoff system is interesting because its Hamiltonian has a nontrivial mixed quadratic term, which set it a bit outside the boundaries of the material presented in [4]. Nevertheless, by applying a theorem of Smale (see [37] and below), and slightly generalizing the technique from [4], we reduce the topological study of isoenergy 3-manifolds to a combinatorial analysis of the Reeb graphs of the reduced potential. Another challenge is the number of natural parameters present in this system – which is equal to three. The crux of the matter which allowed us to perform a quite explicit topological analysis relied on the surprising fact that the study of the so-called reduced potential got reduced to a study of a fifth degree polynomial which naturally factorizes into a linear term and a biquadratic function, see (3.35).

We introduced the Casimirs, the Hamiltonian and extra integral of the Kirchhoff system in Chapter 1 and defined the parameters of the Hamiltonian (1.27), precisely in (1.28). Considering those definition, we can make a couple of conclusions in the following that will be useful in investigating the topology of Kirchhoff system, given different relationships between the parameters a, b and c.

In the following, we are going to study now the topology of the isoenergy manifolds of the Kirchhoff case, as we studied in [12].

3.1 Topology of isoenergy 3-manifolds

Let us consider a systems with a Hamiltonian of the form (1.30) with I and C being symmetric matrices and B an arbitrary one. An isoenergy manifold Q^3 is defined to be a common level surface of the functions f_1, f_2 and the Hamiltonian H in the Euclidean space $\mathbb{R}^6(\mathbf{M}, \mathbf{P})$. Since we assume that $f_1 = 1$, different manifolds Q^3 are determined by two parameters g and h, the values of the functions f_2 and H:

$$Q^3_{g,h} = \{(\mathbf{M}, \mathbf{P}) | f_1 = 1, f_2 = g, H = h\}.$$

The description of the topological types of Q^3 is related to the bifurcation diagram for the Casimir f_2 and the Hamiltonian H. Consider the mapping

$$F = f_2 \times H : S^2 \times \mathbb{R}^3 \to \mathbb{R}^2(g, h).$$

The critical values of F form the bifurcation diagram $\Sigma \subset \mathbb{R}^2(g, h)$. The pre-image of an arbitrary point $(g, h) \notin \Sigma$ is a non-singular isoenergy manifold $Q^3_{g,h}$. As a result, the complement of Σ in the plane $\mathbb{R}^2(g, h)$ is divided into connected components. For all points (g, h) from the same region, the topological type of the corresponding isoenergy manifolds $Q^3_{g,h}$ is the same.

The two-dimensional Poisson sphere is given in $\mathbb{R}^3(\mathbf{P})$ by the equation $P_1^2 + P_2^2 + P_3^2 = 1$. Let $\pi(\mathbf{M}, \mathbf{P}) = \mathbf{P}$ be the projection from $\mathbb{R}^6(\mathbf{M}, \mathbf{P})$ onto $S^2 \subset \mathbb{R}^3(\mathbf{P})$. Consider the projection $\pi(Q^3)$ of an isoenergy manifold Q^3. Given that $\pi(Q^3)$ is homeomorphic to the sphere with m holes, the Smale theorem ([37] and [4]) gives:

(1) If $m = 0$, then Q^3 is diffeomorphic to $\mathbb{RP}^3$.

(2) If $m = 1$, then Q^3 is diffeomorphic to S^3.

(3) If $m > 1$, then Q^3 is diffeomorphic to $m-1$ copies of $(S^1 \times S^2)$.

3.1.1 Reduced potential for Hamiltonians with quadratic mixed term

In (1.19), the matrices I,B and C are diagonal, thus the Hamiltonian is of the form:

$$H_1 = \frac{1}{2}(a_1^{-1}M_1^2 + a_2^{-1}M_2^2 + a_3^{-1}M_3^2) + b_1M_1P_1 + b_2M_2P_2 + b_3M_3P_3 + U(P), \tag{3.1}$$

where $U(P)$ represents potential energy of the system. For a fixed value of the energy, $H_1 = h$, we have

$$a_1^{-1}M_1^2 + a_2^{-1}M_2^2 + a_3^{-1}M_3^2 + 2(b_1M_1P_1 + b_2M_2P_2 + b_3M_3P_3) = 2(h - U(P)). \tag{3.2}$$

We transform (3.2) by completing the squares using the change of variables

$$\sqrt{a_1^{-1}}M_1 + b_1\sqrt{a_1}P_1 = \hat{M}_1,\ \ \sqrt{a_2^{-1}}M_2 + b_2\sqrt{a_2}P_2 = \hat{M}_2,\ \ \sqrt{a_3^{-1}}M_3 + b_3\sqrt{a_3}P_3 = \hat{M}_3.$$

We see that a given point in the Poisson sphere, $P = (P_1, P_2, P_3) \in S^2$ belongs to $\pi(Q^2_{h,g})$ if and only if there exists a common solution to the equations:

$$\hat{M}_1^{\,2} + \hat{M}_2^{\,2} + \hat{M}_3^{\,2} = 2(h - U(P)) + b_1^2a_1P_1^2 + b_2^2a_2P_2^2 + b_3^2a_3P_3^2$$

and

$$\sum_{i=1}^{3} \hat{M}_iP_i\sqrt{a_i} = g + \sum_{i=1}^{3} b_iP_i^2a_i.$$

Proposition 1. *Given a point of the Poisson sphere, $P = (P_1, P_2, P_3) \in S^2$. The plane $f_2(M, P) = g$ intersects the ellipsoid (3.2) in $\mathbb{R}^3(M)$ if and only if*

$$\left(g + \sum_{i=1}^{3} b_iP_i^2a_i\right)^2 \leq \sum_{i=1}^{3} \left(2(h - U(P)) + b_1^2P_1^2a_1 + b_2^2P_2^2a_2 + b_3^2P_3^2a_3\right) P_i^2a_i.$$

We therefore introduce:

Definition 1. *The function*

$$\varphi_g(P) = \frac{1}{2}\frac{\left(g + \sum\limits_{i=1}^{3} b_i P_i^2 a_i\right)^2}{\sum\limits_{i=1}^{3} P_i^2 a_i} + U(P) - \frac{1}{2}(b_1^2 P_1^2 a_1 + b_2^2 P_2^2 a_2 + b_3^2 P_3^2 a_3) \tag{3.3}$$

is called the reduced potential of the Hamiltonian (3.1)*.*

The above definition extends the notion of reduced potential from [4] to the case of Hamiltonians with quadratic mixed terms. Knowing the reduced potential of the system and the above mentioned Smale's theorem, reduce the study of the topology of isoenergy manifolds to the construction of the Reeb graph of the reduced potential $\varphi_g(P)$, [4]. The Reeb graph is a simple combinatorial object which describes the structure of the extremal points of a given function. The description of the topology of isoenergy manifolds thus reduces to a simple analysis of extremal points of the reduced potential, construction of its Reeb graph, and combinatorial analysis of the graph. The set of isoenergy manifolds would be empty when the Hamiltonian is less than the minimum value of the reduced potential. Otherwise, the isoenergy manifold will be either $\mathbb{RP}^3$ or several copies of $S^1 \times S^2$ or S^3. As in [4], we get:

Theorem 3. *Let the reduced potential $\varphi_g(P)$ be defined in* (3.3)*. Then:*

(1) If $h > \max\ \varphi_g(P)$, then $Q^3_{g,h} \simeq \mathbb{RP}^3$.

(2) If $\min\ \varphi_g(P) < h < \max\ \varphi_g(P)$ and h is a regular value of $\varphi_g(P)$, then the set $\{\varphi_g(P) \leq h\}$ is a disjoint union of two-dimensional manifolds with boundary $B_{i_1}, \ldots, B_{i_m}$ embedded into the Poisson sphere, where B_k is a 2-disk with k holes. In this case, the isoenergy manifold $Q^3_{h,g}$ is a smooth three-dimensional manifold which is homeomorphic to a disjoint union of three dimensional manifolds $N_{i_1}, \ldots, N_{i_m}$, where N_0 is the

three-dimensional sphere, and $N_k (k \geq 1)$ is the connected sum of k copies of $S^1 \times S^2$. If one cuts the Reeb graph at a level h, the number of connected components of the lower part of the graph is equal to the number of connected components of $Q^3_{g,h}$. If the connected component of the lower part of the graph has k boundary points (not counting the original vertices of the graph), then the corresponding connected component is homeomorphic to the connected sum of $k-1$ copies of $S^1 \times S^2$ for $k > 1$ and to S^3 for $k = 1$.

(3) If $h < \min\ \varphi_g(P)$, then $Q^3_{g,h}$ is empty.

3.1.2 Reduced potential for the Kirchhoff case. Bifurcation diagrams

We are going to apply the above considerations to the Kirchhoff case with the Hamiltonian (1.27):

$$H = \frac{1}{2}(M_1^2 + M_2^2 + aM_3^2) + b(M_1P_1 + M_2P_2) + \frac{1}{2}c(P_1^2 + P_2^2).$$

Here $\hat{I} = \mathrm{diag}(1, 1, \frac{1}{a})$, $\hat{B} = \mathrm{diag}(b, b, 0)$, $\hat{C} = \mathrm{diag}(c, c, 0)$ and $U(P) = \frac{1}{2}c(P_1^2 + P_2^2)$.
We get:

$$\varphi_g(\mathbf{P}) = \frac{(g + b(P_1^2 + P_2^2))^2}{2\langle \hat{A}\mathbf{P}, \mathbf{P}\rangle} + \frac{c - b^2}{2}(P_1^2 + P_2^2), \tag{3.4}$$

on the Poisson sphere, which is called *the reduced potential for the Kirchhoff case.* The reduced potential in this case can be seen as a function of P_3 only, see formula (3.32) below. It determines the topology of isoenergy manifolds $Q^3_{h,g}$ according to theorem 3.

In order to find the bifurcations of the map $f_2 \times H$, we need to describe the critical points of the mapping. The critical points of the mapping $f_2 \times H : S^2 \times \mathbb{R}^3 \to \mathbb{R}^2$ satisfy the following ([4]):

$$\mathrm{grad}\, H = \mu_1 \mathrm{grad}\, f_1 + \mu_2 \mathrm{grad}\, f_2, \quad f_1 = 1, \tag{3.5}$$

where $\mu_1$and μ_2 are some real numbers. This leads to the following two equations, first from the derivatives with respect to M and then to P:

$$\mu_2 P_3 = aM_3, \tag{3.6}$$

$$2\mu_1 P_3 = \mu_2 M_3. \tag{3.7}$$

(1) Assume $P_3 \neq 0$. We get from the previous two equations:

$$2a\mu_1 = -\mu_2^2. \tag{3.8}$$

From (3.5), we get also

$$M_1 = (\mu_2 - b)P_1, \quad (b - \mu_2)M_1 = P_1(2\mu_1 - c), \tag{3.9}$$

$$M_2 = (\mu_2 - b)P_2, \quad (b - \mu_2)M_2 = P_2(2\mu_1 - c). \tag{3.10}$$

Thus

$$(b - \mu_2)^2 P_1 = -P_1(2\mu_1 - c), \tag{3.11}$$

$$(b - \mu_2)^2 P_2 = -P_2(2\mu_1 - c). \tag{3.12}$$

(1.1) Assume $P_1 = P_2 = M_1 = M_2 = 0$. Then $P_3 = \pm 1$, $M_3 = \pm\mu_2/a$, and

$$h = \frac{\mu_2^2}{2a}, \quad g = \pm\frac{\mu_2}{a}.$$

We get $\mu_2^2 = a^2 g^2$ and substituting in the above formula for h, we get the parabola Π_1:

$$\Pi_1 : h(g) = \frac{a}{2}g^2. \tag{3.13}$$

(1.2) Assume $P_1^2 + P_2^2 \neq 0$. From (3.11) one gets

$$(b - \mu_2)^2 = c - 2\mu_1. \tag{3.14}$$

Using (3.8), we get

$$\mu_2^2 \geq -ac, \tag{3.15}$$

and

$$\mu_2^2(a-1) - 2ab\mu_2 + a(b^2 - c) = 0. \tag{3.16}$$

(1.2.1) Assume $a \neq 1$. Then (3.16) is a quadratic equation with the discriminant $D_a = 4a(b^2 - c(1-a))$. For (3.16) to have real solutions, the condition is

$$D_a \geq 0 \Leftrightarrow b^2 \geq c(1-a). \tag{3.17}$$

We get

$$h = \frac{\mu_2^2 - b^2 + c}{2} + P_3^2 \frac{\mu_2^2(1-a^2) + b^2a^2 - ca^2}{2a^2}, \tag{3.18}$$

$$g = (\mu_2 - b) + P_3^2 \left(\frac{\mu_2}{a} - \mu_2 + b\right). \tag{3.19}$$

We get

$$P_3^2 = a\frac{g - \mu_2 + b}{\mu_2(1-a) + ba}, \tag{3.20}$$

with the condition

$$0 \leq a\frac{g - \mu_2 + b}{\mu_2(1-a) + ba} \leq 1. \tag{3.21}$$

Substituting (3.20) into (3.18) we get a line $h = h(g)$, which, together with the conditions (3.21) defines a line segment Λ:

$$\Lambda : h = h(g) = \frac{\mu_2^2 - b^2 + c}{2} + \frac{g - \mu_2 + b}{\mu_2(1-a) + ba}\frac{\mu_2^2(1-a^2) + b^2a^2 - ca^2}{2a}. \tag{3.22}$$

The line segment Λ is defined, according to (3.21), for g, such that

$$\hat{g}_0 \leq g \leq \hat{g}_1, \quad \hat{g}_0 = \mu_2 - b, \quad \hat{g}_1 = \frac{\mu}{a}. \tag{3.23}$$

If $D_a = 0$, which is equivalent to $b^2 = c(1-a)$, then $\mu_2 = ab/(a-1)$ and $\hat{g}_0 = \hat{g}_1 = b/(a-1)$. Thus, in this case the segment Λ degenerate to a point of tangency of parabolas Π_0 and Π_1.

If $D_a > 0$, which is equivalent to $b^2 > c(1-a)$, then the equation (3.16) has two distinct solutions for μ_2, denoted $\mu_2^{(i)}$, $i = 1, 2$. They will lead to two line segments $\Lambda^{(i)}$, $i = 1, 2$, defined with equations (3.22) and (3.23) with $\mu = \mu_2^{(i)}$, $i = 1, 2$.

(1.2.2) Assume $a = 1$ and $b \neq 0$. Then

$$\mu_2 = \frac{b^2 - c}{2b}.$$

From

$$M_1 = -\frac{b^2 + c}{2b} P_1,\ M_2 = -\frac{b^2 + c}{2b} P_2,\ M_3 = \frac{b^2 - c}{2b} P_3,$$

one gets

$$\Lambda_1 : h = h(g) = \frac{\mu_2^2 - b^2 + c}{2} + \frac{(g + b - \mu_2)(b\mu_2 - c)}{2b}, \tag{3.24}$$

with the conditions:

$$\begin{aligned} b > 0 &: -\frac{b}{2} - \frac{c}{2b} < g < \frac{b}{2} - \frac{c}{2b}, \\ b < 0 &: \frac{b}{2} - \frac{c}{2b} < g < -\frac{b}{2} - \frac{c}{2b}. \end{aligned} \tag{3.25}$$

The subcase $a = 1, b = 0$ due to (3.16) leads to $c = 0$ and gives the parabola

$$\Pi_0 = \Pi_1 : h = h(g) = \frac{g^2}{2},$$

forms the bifurcation diagram.

(2) Assume $P_3 = M_3 = 0$. Then $P_1^2 + P_2^2 = 1$ and:

$$M_1 = (\mu_2 - b)P_1,\ M_2 = (\mu_2 - b)P_2,\ M_3 = P_3 = 0.$$

Thus:

$$h = \frac{(\mu_2 - b)^2}{2} + b(\mu_2 - b) + \frac{c}{2}, \tag{3.26}$$

$$g = \mu_2 - b. \tag{3.27}$$

Thus, the bifurcation diagram contains the parabola Π_0:

$$\Pi_0 : h = h(g) = \frac{g^2}{2} + bg + \frac{c}{2}. \tag{3.28}$$

Summarizing the above results, we get:

Proposition 2. *The bifurcation diagram of the mapping $f_2 \times H$ consists of the union of two parabolas $\Pi_0 : h = \frac{1}{2}g^2 + bg + \frac{c}{2}$, and $\Pi_1 : h = \frac{a}{2}g^2$ and in the case $b^2 > c(1-a)$ two line segments $\Lambda^{(i)}$, $i = 1, 2$, defined for $a \neq 1$ with* (3.22) *and* (3.23) *and for $a = 1$ one line segment Λ_1 given with* (3.24) *and* (3.25).

Example 1. *Let us consider the case $a > 1, b < 0, c = 0$ and study the line segments $\Lambda^{(i)}$, $i = 1, 2$ with* (3.22) *and* (3.23). *We get*

$$\mu_2^{(1)} = \frac{b\sqrt{a}}{\sqrt{a}-1}, \qquad \mu_2^{(2)} = \frac{b\sqrt{a}}{\sqrt{a}+1}; \tag{3.29}$$

$$\hat{g}_0^{(1)} = \frac{b}{\sqrt{a}-1}, \qquad \hat{g}_0^{(2)} = -\frac{b}{\sqrt{a}+1}; \tag{3.30}$$

$$\hat{g}_1^{(1)} = \frac{b}{\sqrt{a}(\sqrt{a}-1)}, \qquad \hat{g}_0^{(2)} = \frac{b}{\sqrt{a}(\sqrt{a}+1)}. \tag{3.31}$$

One can easily get:

$$\hat{g}_0^{(1)} < \hat{g}_1^{(1)} < \hat{g}_1^{(2)} < 0 < \hat{g}_0^{(2)},$$

see Fig. 3.7.

Proposition 3. *For various values of the parameters a, b, c, such that $a > 0, c \geq b$, the bifurcation diagrams of the mapping $f_2 \times H$ are presented by the following figures:*

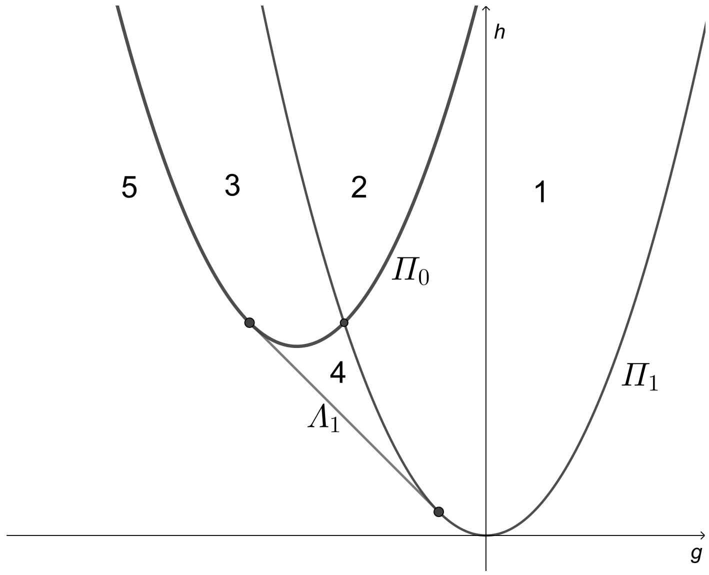

Figure 3.1: The case when $a = 1, b \neq 0, c > 0$

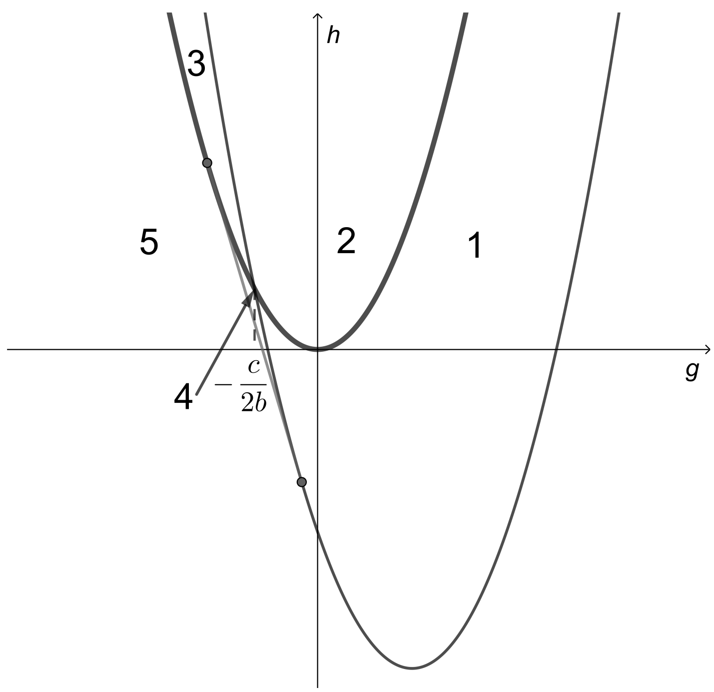

Figure 3.2: The case when $a = 1, b \neq 0, c < 0$

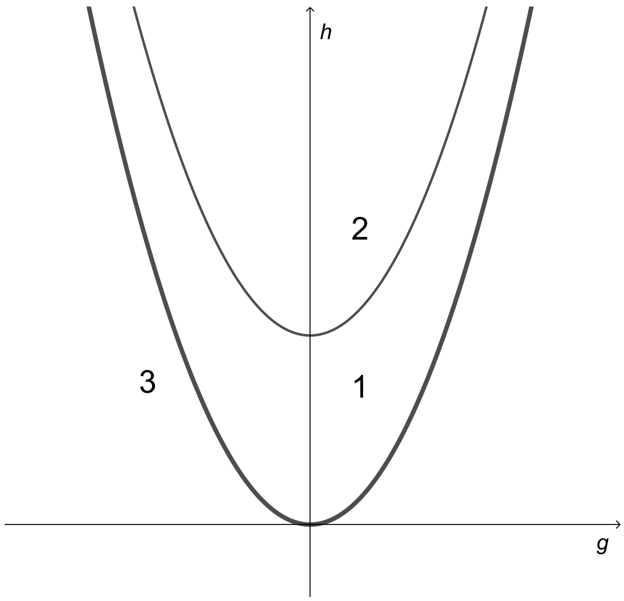

Figure 3.3: The case when $a = 1, b = 0, c > 0$

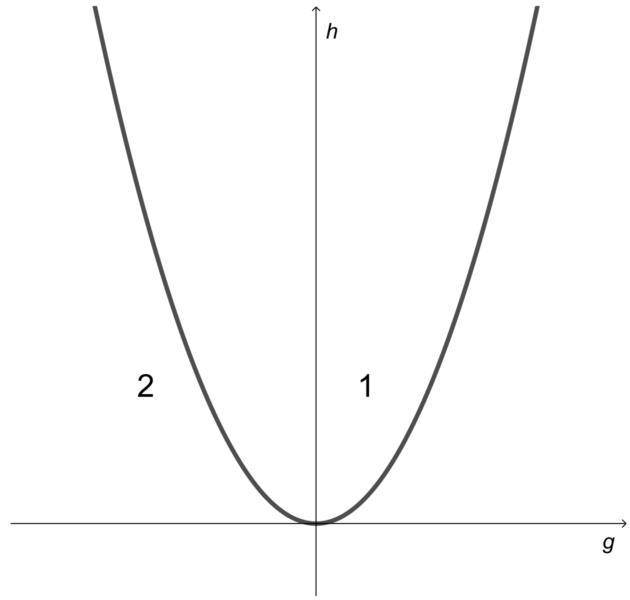

Figure 3.4: The case when $a = 1, b = c = 0$

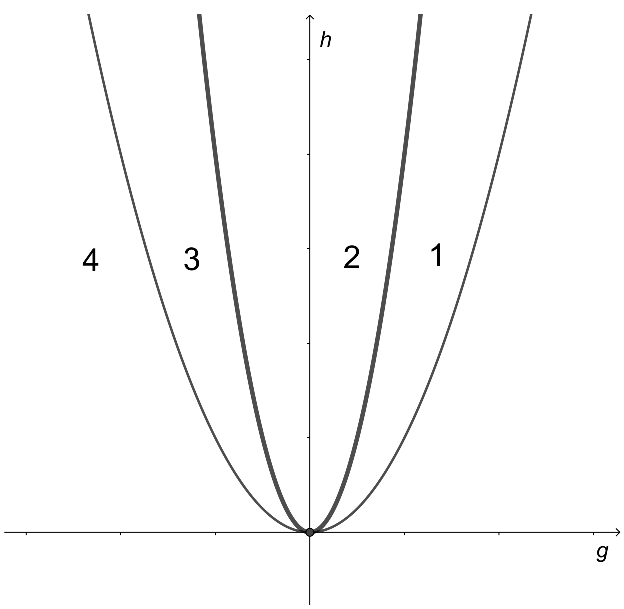

Figure 3.5: The case when $a > 1, b = c = 0$

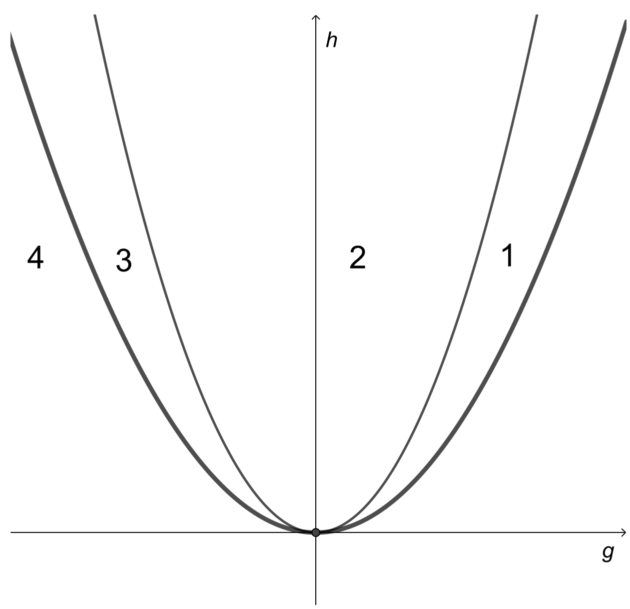

Figure 3.6: The case when $a < 1, b = c = 0$

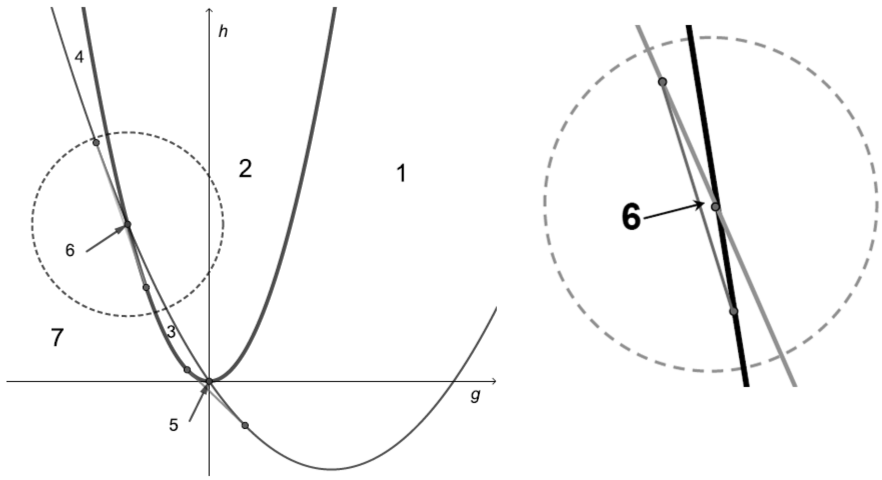

Figure 3.7: The case when $a > 1, b < 0, c = 0$

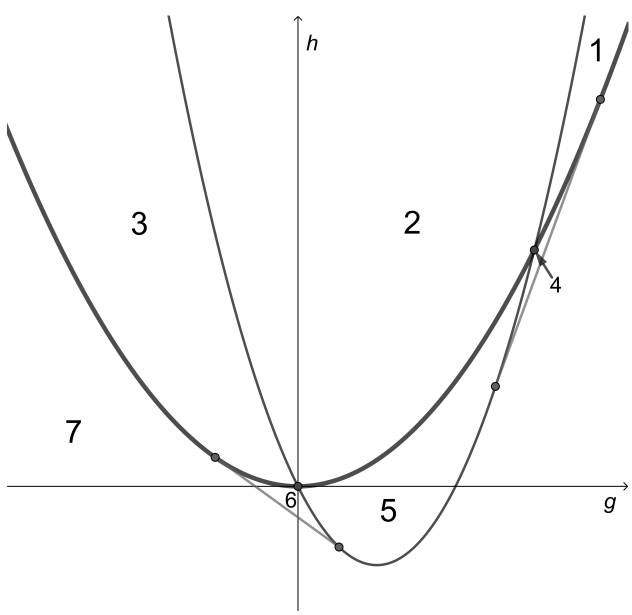

Figure 3.8: The case when $a < 1, b < 0, c = 0$

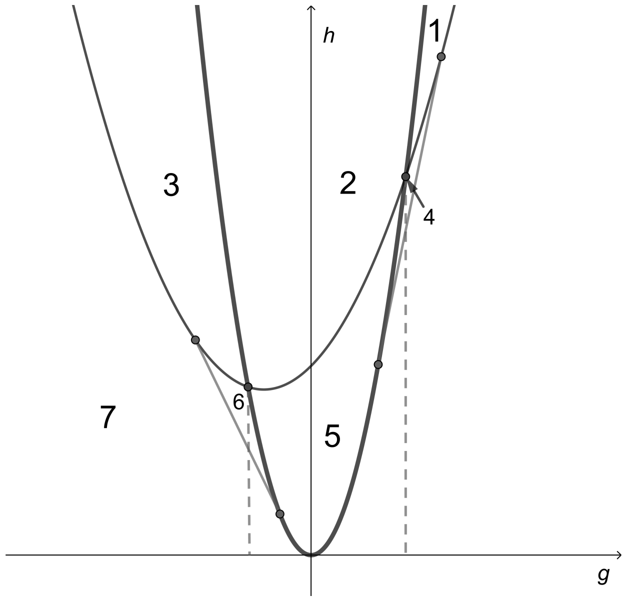

Figure 3.9: The case when $c > 0, a \neq 1, b^2 > c(1-a)$ with $g_{1,2} = \frac{b \mp \sqrt{b^2 - c(1-a)}}{a-1}$

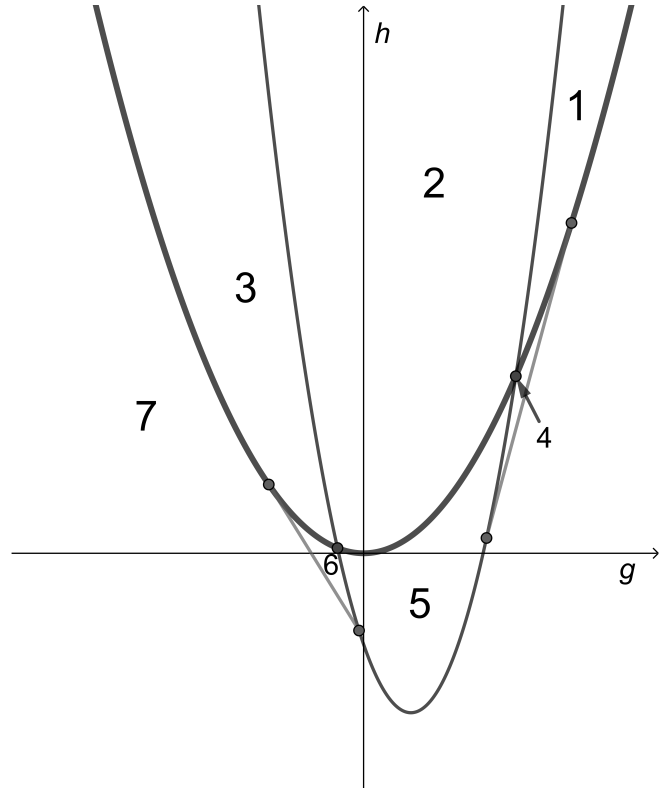

Figure 3.10: The case when $b \leq c < 0, a \neq 1, b^2 > c(1-a)$ with $g_{1,2} = \frac{b \mp \sqrt{b^2 - c(1-a)}}{a-1}$

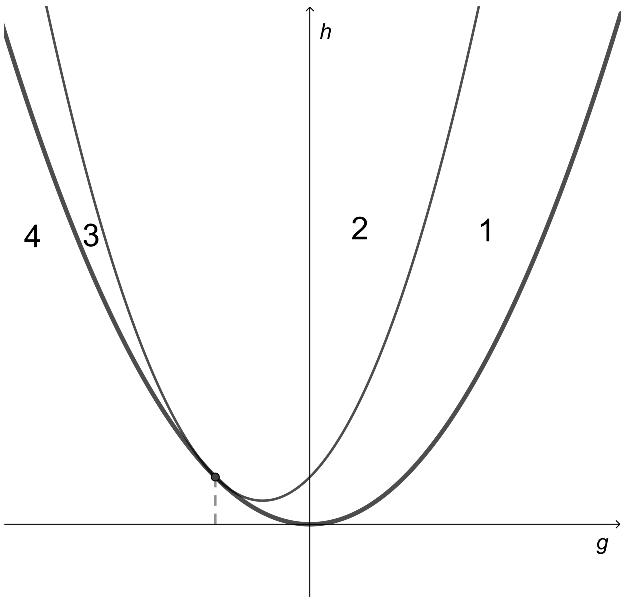

Figure 3.11: The case when $c > 0, b > 0, a < 1, b^2 = c(1-a)$

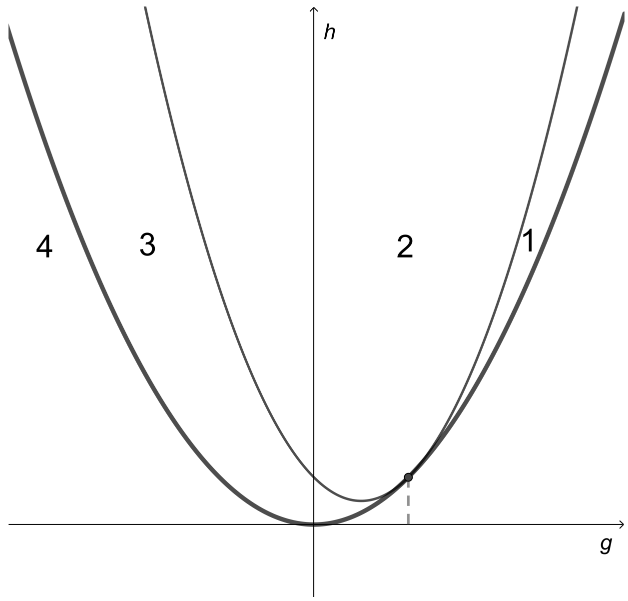

Figure 3.12: The case when $c > 0, b < 0, a < 1, b^2 = c(1-a)$

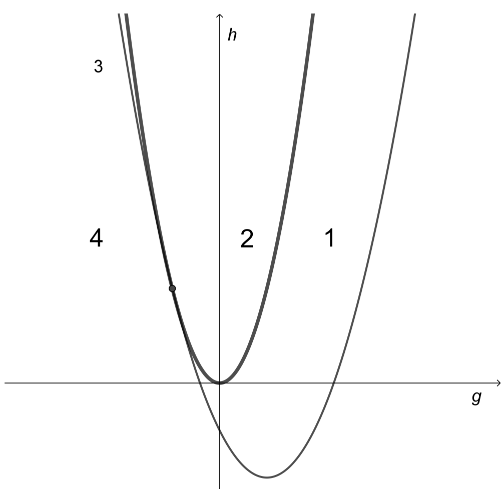

Figure 3.13: The case when $b \leq c < 0, a > 1, b^2 = c(1-a)$

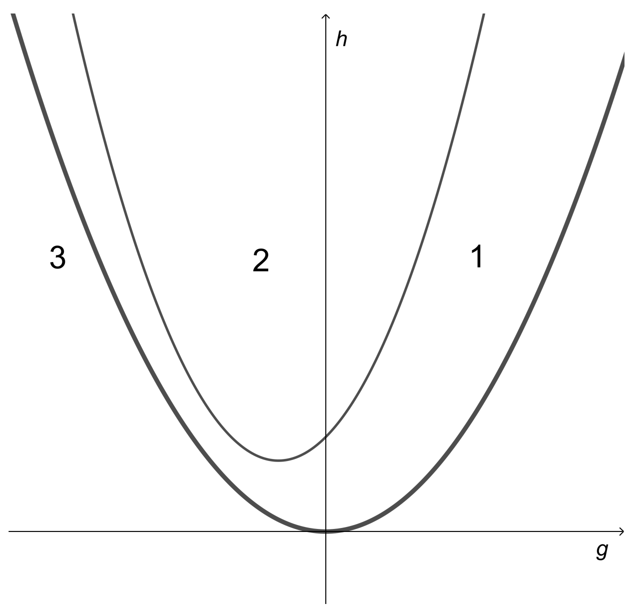

Figure 3.14: The case when $c > 0, a < 1, b^2 < c(1-a)$

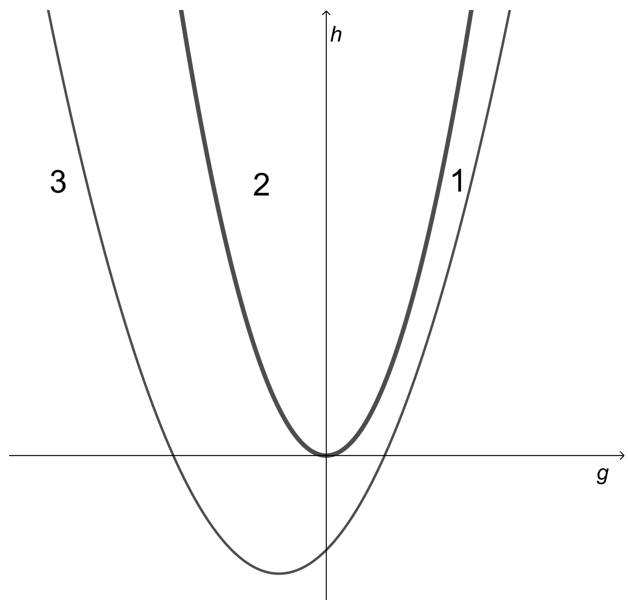

Figure 3.15: The case when $c < 0, a > 1, b^2 < c(1-a)$

3.1.3 Analysis of the reduced potential and the Reeb graphs

Let us define Reeb graphs officially here. For this purpose, we first need to introduce Morse functions.

Consider a smooth function $f(x)$ on a smooth manifold X^n, and let $x_1, x_2, ..., x_n$ be smooth regular coordinates in a neighborhood of a point $p \in X^n$. The Point p is called critical if the differential

$$df = \sum \frac{\partial f}{\partial x_i} dx_i$$

vanishes at the point p. The critical point is called *non-degenerate* if the second differential

$$d^2 f = \sum \frac{\partial^2 f}{\partial x_i \partial x_j} dx_i dx_j$$

is non-degenerate at this point. This is equivalent to the fact that the determinant of the second derivative matrix is non-zero.

Definition 11. *A smooth function is called a Morse function if all its critical points are non-degenerate.[4]*

Now, given that f is a Morse function of a compact smooth manifold x^n, for any $a \in \mathbb{R}$, consider the level surface $f^{-1}(a)$ and its connected components, which will be called *fibers*. As a result, on the manifold there appears the structure of a foliation with singularities. By declaring each fiber to be a point and introducing the natural quotient topoloy in the space Γ of fibers, we obtain some quotient space that can be considered as the base of the foliation. For a Morse function Γ is a finite graph.

Definition 12. *The graph Γ is called the Reeb graph of the Morse function f on the manifold X^n. A vertex of the Reeb graph is the point corresponding to the singular fiber of the function f. [4]*

To demonstrate this in an example, let us consider the two-dimensional torus, embedded in $\mathbb{R}^3$ as shown in figure 3.16 and take the natural height function to be a Morse function on this torus. Then, its Reeb graph has the form shown to the right of the torus. Similarly, the Reeb graph of the sphere with two handles is shown to the right of it in figure 3.16.

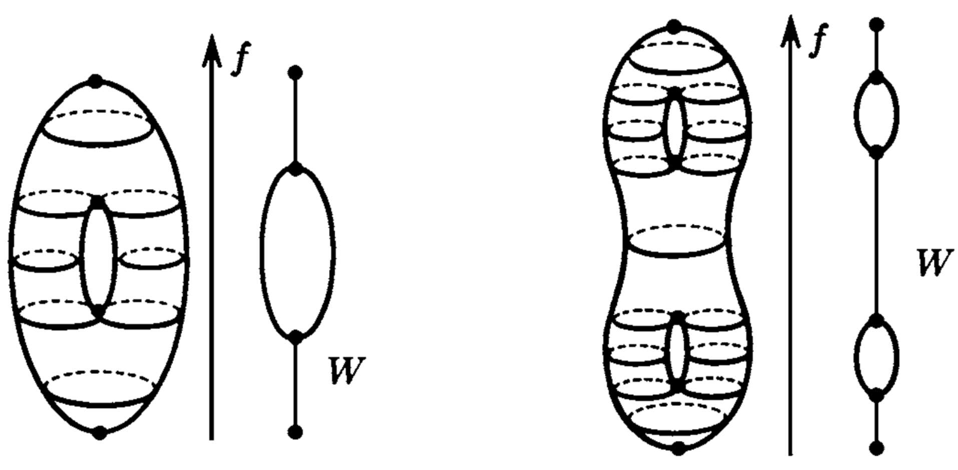

Figure 3.16: Reeb graph of a torus and sphere with two handles

Let us now go back to the concept of reduced potential and plot the different possible Reeb graphs in case of different values of the parameters a, b and c.

The reduced potential for the Kirchhoff case (3.4) can be written as a function of P_3 only:

$$\varphi_g(P_3) = \frac{1}{2}\left(\frac{(\frac{b^2}{a} - \frac{c}{a} + c)P_3^4 + (\frac{c}{a} - \frac{b^2}{a} - 2c - 2bg)P_3^2 + g^2 + bg + c}{1 + (\frac{1}{a} - 1)P_3^2}\right). \tag{3.32}$$

Let us note that φ_g is an even function and after taking the derivative with respect to P_3 and simplification, we get:

$$\varphi_g'(P_3) = \frac{P_3 f_g(P_3^2)}{4a(1 + (\frac{1}{a} - 1)P_3^2)^2}. \tag{3.33}$$

where

$$f_g(x) = (b^2 + (a-1)c)(\frac{1}{a} - 1)x^2 + 2(b^2 + (a-1)c)x + (a-1)g^2 - 2bg - b^2 - ac, \tag{3.34}$$

is a quadratic function. Thus the analysis of extremal points of the reduced potential and the structure of its Reeb graph is based on the analysis of a polynomial of order five

$$P_3 f_g(P_3^2), \tag{3.35}$$

which is a product of a linear factor and a biquadratic. This fact is essential for the effectiveness of the results which we are going to obtain.

In order to investigate the structure of the Reeb graphs corresponding to the reduced potential, we treat the quartic $f_g(P_3^2)$ as a bi-quadratic. We consider the different possible scenarios depending on parameters $a > 0$, b and c, $c \geq b$, according to Remark 2 and Lemma 1. We also assume $P_3 \in [-1, 1]$ since P_3 is the third coordinate of a point on the unit Poisson sphere.

Lemma 2. *The polynomial f_g given in* (3.34) *with $a > 0$ has at most one root in the interval* $(0, 1)$.

Proof. Suppose that $x_1, x_2 \in (0,1)$ are solutions of the equation $f_g = 0$. Then $a \neq 1$ and $T = (x_1 + x_2)/2$ should satisfy $0 < T < 1$. From (3.34) we have $T = a/(a-1)$. From $0 < T$ we get $a > 1$. Then from $T < 1$ we get contradiction. $\square$

From the previous Lemma the options for f_g and the Reeb graphs are described by the following Lemma.

Lemma 3. *There are four potential types of the behaviour of the polynomial f_g given by (3.34) on the interval $(0,1)$ with $a > 0$:*

I *Type I: $f_g < 0$ on $(0,1)$;*

II *Type II: $f_g > 0$ on $(0,1)$;*

III *Type III: $f_g(0) < 0$ and $f_g(1) > 0$;*

IV *Type IV: $f_g(0) > 0$ and $f_g(1) < 0$.*

The following Reeb graphs and topologies correspond to each of the types I, II, III, IV:

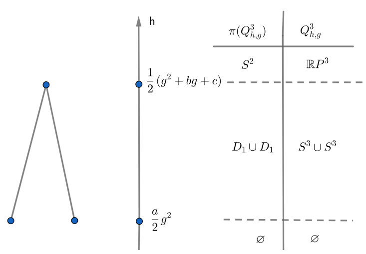

Figure 3.17: The Reeb graph and topology of type I

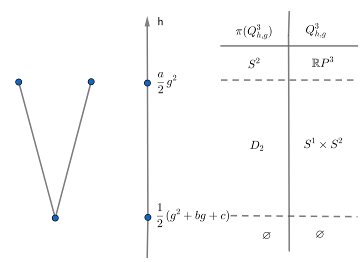

Figure 3.18: The Reeb graph and topology of type II

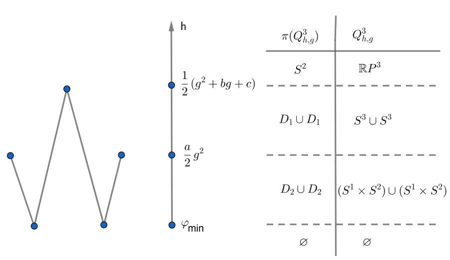

Figure 3.19: The Reeb graph and topology of type IIIa

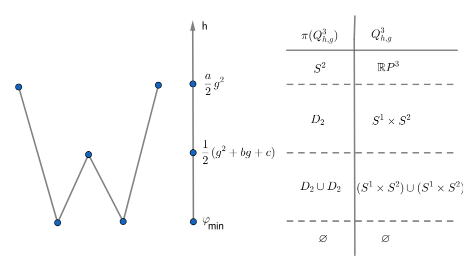

Figure 3.20: The Reeb graph and topology of type IIIb

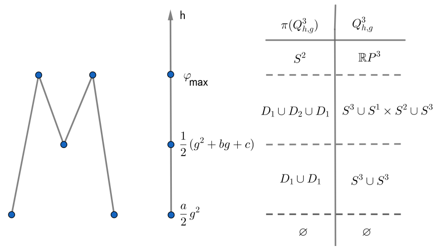

Figure 3.21: The Reeb graph and topology of type IVa

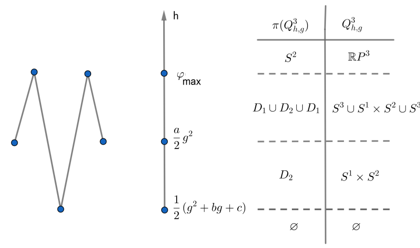

Figure 3.22: The Reeb graph and topology of type IVb

The types I and II correspond to the situations with no solutions of the equation $f_g(x) = 0$ in the interval $(0, 1)$, while the types III and IV correspond to the situations with one solution of the equation $f_g(x) = 0$ in the interval $(0, 1)$. The next Lemma will show that the type IV is impossible in the Kirchhoff case.

Lemma 4. *For $a > 0$, the condition $f_g(0) > 0$ implies $f_g(1) > 0$.*

Proof. The condition $f_g(0) > 0$ is equivalent to

$$(a-1)ag^2 - 2abg - b^2a - a^2c > 0.$$

The condition $f_g(1) < 0$ is equivalent to

$$(a-1)ag^2 - 2abg + b^2 - c > 0.$$

The conditions $f_g(0) > 0$ and $f_g(1) < 0$ imply

$$b^2 < c(1-a). \tag{3.36}$$

The parabolas describing $f_g(1)$ and $f_g(0)$ as quadratic functions in g are simultaneously convex and their discriminants are both negative under the condition (3.36). Thus, under this condition, they cannot have opposite signs as functions of g. Thus, the statement follows. $\square$

Corollary 1. *In the Kirchhoff case the Reeb graphs of the type IV (presented at 3.21, and 3.22) are impossible.*

Lemma 5. *(I) The conditions for type I are:*

$$(a-1)ag^2 - 2abg + b^2 - c < 0, \tag{3.37}$$

$$(a-1)ag^2 - 2abg - b^2a - a^2c < 0. \tag{3.38}$$

(II) The conditions for type II are:

$$(a-1)ag^2 - 2abg + b^2 - c > 0, \tag{3.39}$$

$$(a-1)ag^2 - 2abg - b^2a - a^2c > 0. \tag{3.40}$$

(III) The conditions for type III are:

$$(a-1)ag^2 - 2abg + b^2 - c > 0, \tag{3.41}$$

$$(a-1)ag^2 - 2abg - b^2a - a^2c < 0, \tag{3.42}$$

which imply

$$b^2 > c(1-a). \tag{3.43}$$

By applying theorem 3 and Lemmata 2, 3, 4, and 5, we get the following

Proposition 4. *For the Kirchhoff case, possible Reeb graphs with the corresponding topologies are of types I, II, IIIa, and IIIb. They are presented in the figures: 3.17, 3.18, 3.19, and 3.20.*

Example 2. *Assume $a = 1$ and $b > 0$. Then as before $\mu_2 = (b^2 - c)/(2b)$, and*

$$\Lambda_1 : h = h(g) = \frac{\mu_2^2 - b^2 + c}{2} + \frac{(g + b - \mu_2)(b\mu_2 - c)}{2b},$$

with:

$$g_0 = -\frac{b}{2} - \frac{c}{2b}, \quad g_1 = \frac{b}{2} - \frac{c}{2b}.$$

The parabolas Π_0 and Π_1 intersect at the pint with its g-coordinate equal to $\hat{g} = -c/(2b)$. Thus, we consder the followong zones for g:

$$g < g_0, \quad g_0 < g < \hat{g}, \quad \hat{g} < g < g_1, \quad g_1 < g.$$

By applying the above results, we get the following Reeb graphs in each of the zones:

$$g < g_0 : II, \quad g_0 < g < \hat{g} : IIIb, \quad \hat{g} < g < g_1 : IIIa, \quad g_1 < g : I.$$

Then the connected component of the complement of the bifurcation diagram as numerated in Fig. 3.1 have the following corresponding topologies:

$$1 - S^3 \cup S^3; \quad 2 - \mathbb{RP}^3; \quad 3 - S^1 \times S^2; \quad 4 - S^1 \times S^2 \cup S^1 \times S^2; \quad 5 - \emptyset.$$

As one can see at Fig. 3.23, some connected components of the complement of the bifurcation diagram belong to more that one zone (compare with Fig. 3.1). Their topologies can be determined using corresponding Reeb graphs in more than one way. But, the result does not depend on that choice.

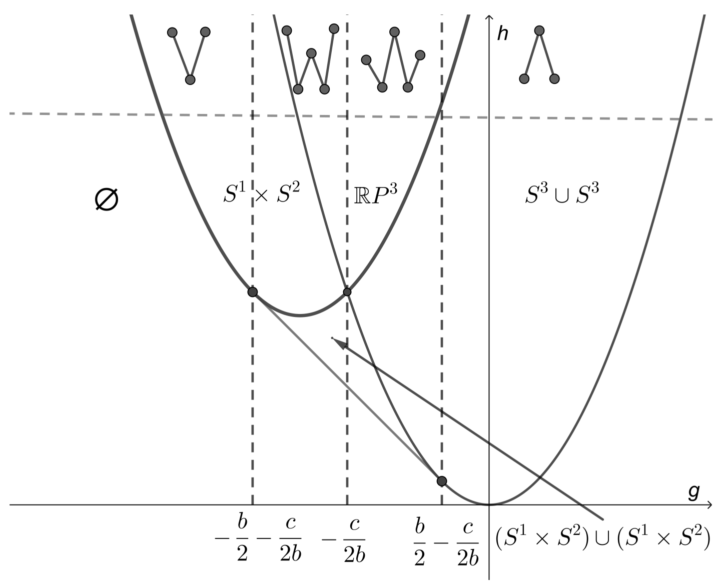

Figure 3.23: Zones, Reeb graphs, regions and topologies for $a = 1, c \geq b > 0$.

Synthesizing the results from Propositions 3 and 4, as we did in Example 2, we finally get this theorem.

Theorem 4. *The topologies of the isoenergy manifolds $Q^3_{g,h}$ are described depending on the values of the parameters and regions of the corresponding bifurcation diagrams as follows, with the types of the Reeb graphs listed from the left to the right:*

1 *Case A, Fig. 3.1, the Reeb graph types: II, IIIb, IIIa, I; topologies per region: 1-$S^3 \cup S^3$; 2-$\mathbb{RP}^3$; 3-$S^1 \times S^2$; 4-$S^1 \times S^2 \cup S^1 \times S^2$; 5-$\emptyset$.*

2 *Case A_1, Fig. 3.2, the Reeb graph types: I, IIIa, IIIb, II; topologies per region: 1-$S^1 \times S^2$; 2-$\mathbb{RP}^3$; 3-$S^3 \cup S^3$; 4-$S^1 \times S^2 \cup S^1 \times S^2$; 5-$\emptyset$.*

3 *Case B, Fig. 3.3, the Reeb graph type: I; topologies per region: 1-$S^3 \cup S^3$; 2-$\mathbb{RP}^3$; 3-$\emptyset$.*

4 *Case C, Fig. 3.4, the Reeb graph: a single point; topologies per region: 1-$\mathbb{RP}^3$; 2-$\emptyset$.*

5 *Case C_2, Fig. 3.5, the Reeb graph type: II; topologies per region: 1- $S^1 \times S^2$; 2-$\mathbb{RP}^3$; 3-$S^1 \times S^2$; 4-$\emptyset$.*

6 *Case C_3, Fig. 3.6, the Reeb graph type: I; topologies per region: 1-$S^3 \cup S^3$; 2-$\mathbb{RP}^3$; 3-$S^3 \cup S^3$; 4-$\emptyset$.*

7 *Case C_4, Fig. 3.7, the Reeb graph types: II, IIIb, IIIa, I, IIIa, IIIb, II; topologies per region: 1- $S^1 \times S^2$; 2-$\mathbb{RP}^3$; 3-$S^3 \cup S^3$; 4-$S^1 \times S^2$; 5-$S^1 \times S^2 \cup S^1 \times S^2$; 6-$S^1 \times S^2 \cup S^1 \times S^2$; 7-$\emptyset$.*

8 *Case C_5, Fig. 3.8, the Reeb graph types: II, IIIb, IIIa, I; topologies per region: 1-$S^3 \cup S^3$; 2-$\mathbb{RP}^3$; 3-$S^1 \times S^2$; 4-$S^1 \times S^2 \cup S^1 \times S^2$; 5-$\emptyset$.*

9 *Case D, Fig. 3.9, the Reeb graph types: II, IIIa, IIIb, I, IIIa, IIIb, II; topologies per region: 1-$S^1 \times S^2$; 2-$\mathbb{RP}^3$; 3-$S^1 \times S^2$; 4-$S^1 \times S^2 \cup S^1 \times S^2$; 5-$S^3 \cup S^3$; 6-$S^1 \times S^2 \cup S^1 \times S^2$; 7-$\emptyset$.*

10 *Case* D_1, *Fig. 3.10, the Reeb graph types: II, IIIb, IIIa, I; topologies per region: 1-*$S^3 \cup S^3$; *2-*$\mathbb{RP}^3$; *3-*$S^1 \times S^2$; *4-*$S^1 \times S^2 \cup S^1 \times S^2$; *5-*$\emptyset$.

11 *Case* E, *Fig. 3.11, the Reeb graph types: I, I; topologies per region: 1-*$S^3 \cup S^3$; *2-*$\mathbb{RP}^3$; *3-*$S^3 \cup S^3$; *4-*$\emptyset$.

12 *Case* E_2, *Fig. 3.12, the Reeb graph types: I, I; topologies per region: 1-*$S^3 \cup S^3$; *2-*$\mathbb{RP}^3$; *3-*$S^3 \cup S^3$; *4-*$\emptyset$.

13 *Case* E_1, *Fig. 3.13, the Reeb graph types: II, II; topologies per region: 1-*$S^1 \times S^2$; *2-*$\mathbb{RP}^3$; *3-*$S^1 \times S^2$; *4-*$\emptyset$.

14 *Case* F, *Fig. 3.14, the Reeb graph type: I; topologies per region: 1-*$S^3 \cup S^3$; *2-*$\mathbb{RP}^3$; *3-*$\emptyset$.

15 *Case* F_1, *Fig. 3.15, the Reeb graph type: II; topologies per region: 1-*$S^1 \times S^2$; *2-*$\mathbb{RP}^3$; *3-*$\emptyset$.

CHAPTER 4

BIFURCATION ANALYSIS OF THE POINCARÉ CASE ON $so(4)$ AND THE KIRCHHOFF CASE ON $e(3)$

The Poincaré model of a rigid body with an ellipsoidal cavity filled with an ideal incompressible liquid has a Hamiltonian of the same form as the Kirchhoff Hamiltonian (1.29) with the additional conditions similar to (1.32), and with the underlying Poisson algebra being $so(4)$ [5]. We have studied the bifurcation analysis of the Poincaré model on the Lie algebra $so(4)$ in detail with an account of critical points leading to conclusions about the momentum mapping and the bifurcation diagrams given different conditions on the parameters of the Poincaré model[12].

4.1 Momentum map for the Poincaré model of rigid body on $so(4)$

The Poincaré case is integrable. Thus, we are going to study also the bifurcation diagram of the map $\phi_g = H \times K : M^4_{1,g} \to \mathbb{R}^2(h,k)$, where the maps $H : M^4_{1,g} \to R(h)$ and $K : M^4_{1,g} \to R(k)$ are restrictions of the Hamiltonian H (1.27) and the first integral K from (1.25) respectively. Here $M^4_{1,g}$ is a symplectic manifold, the common level surface of two Casimirs $M^4_{1,g} = f_1^{-1}(1) \cap f_2^{-1}(g)$. The corresponding family of momentum mappings has the form

$$\phi_g : M^4_{1,g} \to \mathbb{R}^2(h,k), \quad x \mapsto (H(x), K(x)).$$

Thus, the mapping ϕ_g depends on the parameter g. The rank of a critical point of the momentum mapping is the rank of the differential of the momentum mapping at this point. It is clear that for the system under consideration, the ranks of critical points of the momentum mapping ϕ_g are zero or one. We consider these two cases separately.

4.1.1 Critical points of rank 0

In order to find the critical points of rank zero, we need to find out when the skew gradients sgrad H and sgrad K both vanish simultaneously. As a result, let us consider the following calculations of Poisson bracket:

$$\{M_1, H\} = (a-1)\, M_2 M_3 - b\, (P_2 M_3 + P_3 M2) - c P_2 P_3$$

$$\{M_2, H\} = (1-a)\, M_1 M_3 + b\, (P_1 M_3 + P_3 M1) + c P_1 P_3$$

$$\{M_3, H\} = 0$$

$$\{P_1, H\} = -P_3 M_2 - b P_2 P_3 + a P_2 M_3 - \varkappa b M_2 M_3 - \varkappa c P_2 M_3$$

$$\{P_2, H\} = P_3 M_1 + b P_1 P_3 - a P_1 M_3 + \varkappa b M_1 M_3 + \varkappa c P_1 M_3$$

$$\{P_3, H\} = P_1 M_2 - P_2 M_1 + \varkappa c\, (M_1 P_2 - M_2 P_1)$$

Therefore, the field sgrad H is written on $so(4)$ explicitly:

$$\begin{aligned}
&\{M_1, H\} = (a-1) M_2 M_3 - b(P_2 M_3 + P_3 M_2) - c P_2 P_3.\\
&\{M_2, H\} = (1-a) M_1 M_3 + b(P_1 M_3 + P_3 M_1) + c P_1 P_3.\\
&\{M_3, H\} = 0.\\
&\{P_1, H\} = -P_3 M_2 - b P_2 P_3 + a P_2 M_3 - b M_2 M_3 - c P_2 M_3.\\
&\{P_2, H\} = P_3 M_1 + b P_1 P_3 - a P_1 M_3 + b M_1 M_3 + c P_1 M_3.\\
&\{P_3, H\} = P_1 M_2 - P_2 M_1 + c(M_1 P_2 - M_2 P_1).
\end{aligned}$$

Similarly, we need to write the field sgrad$K$. Since $F = M_3$, we have:

$$\{M_1, M_3\} = \frac{\partial M_1}{\partial M_1}\frac{\partial M_3}{\partial M_3}\, \{M_1, M_3\} = M_2$$

$$\{M_2, M_3\} = \frac{\partial M_2}{\partial M_2}\frac{\partial M_3}{\partial M_3}\, \{M_2, M_3\} = -M_1$$

$$\{M_3, M_3\} = 0$$

$$\{P_1, M_3\} = \frac{\partial P_1}{\partial P_1}\frac{\partial M_3}{\partial M_3}\{P_1, M_3\} = -\{M_3, P_1\} = P_2$$

$$\{P_2, M_3\} = \frac{\partial P_2}{\partial P_2}\frac{\partial M_3}{\partial M_3}\{P_2, M_3\} = -\{M_3, P_2\} = -P_1$$

$$\{P_3, M_3\} = \frac{\partial P_3}{\partial P_3}\frac{\partial M_3}{\partial M_3}\{P_3, M_3\} = 0$$

The skew gradient of $K = M_3$ is:

$$\{M_1, K\} = M_2.$$

$$\{M_2, K\} = -M_1.$$

$$\{M_3, K\} = 0.$$

$$\{P_1, K\} = P_2.$$

$$\{P_2, K\} = -P_1.$$

$$\{P_3, K\} = 0.$$

Proposition 5. *For the Lie algebra $so(4)$, the set of points where both skew gradient vector fields,* $\operatorname{sgrad} H$ *and* $\operatorname{sgrad} K$ *with Hamiltonians H and K vanish simultaneously, is the family of points $(0, 0, M_3, 0, 0, P_3)$ in the space $\mathbb{R}^6(H, K)$.*

Theorem 5. *For the Lie algebra $so(4)$, the image of critical points of rank 0 is:*

$$h = \frac{a}{4}(1 \pm \sqrt{1-4g^2}) + bg \quad and \quad k = \pm\frac{1 \pm \sqrt{1-4g^2}}{2},$$

for $-\frac{1}{2} < g < \frac{1}{2}$.

4.1.2 Critical points of rank one

The bifurcation diagram of an integrable system is defined to be the region of possible motion depicted on the plane of the first integrals (h, k) where the curves are images of the critical points of rank one. For the purpose of finding the critical points of rank one, we need to find out where $\operatorname{sgrad} H$ and $\operatorname{sgrad} K$ are linearly dependent while at least one of them is not zero.

The critical points of rank 1 are found using the minors of the following matrix:

$$\begin{bmatrix} \{M_1, H\} & \{M_1, F\} \\ \{M_2, H\} & \{M_2, F\} \\ \{M_3, H\} & \{M_3, F\} \\ \{P_1, H\} & \{P_1, F\} \\ \{P_2, H\} & \{P_2, F\} \\ \{P_3, H\} & \{P_3, F\} \end{bmatrix}$$

The calculations using

$$H = \frac{1}{2}(M_1^2 + M_2^2 + M_3^2) + b(M_1P_1 + M_2P_2) + \frac{1}{2}c(P_1^2 + P_2^2)$$

and

$$F = M_3$$

gives:

$$\begin{bmatrix} (a-1)M_2M_3 - b(P_2M_3 + P_3M_2) - cP_2P_3 & M_2 \\ (1-a)M_1M_3 + b(P_1M_3 + M_1P_3) + cP_1P_3 & -M_1 \\ 0 & 0 \\ -P_3(M_2 + bP_2) + M_3(aP_2 - \varkappa bM_2 - \varkappa cP_2) & P_2 \\ P_3(M_1 + bP_1) + M_3(\varkappa bM_1 + \varkappa cP_1 - aP_1) & -P_1 \\ (\varkappa c - 1)(M_1P_2 - M_2P_1) & 0 \end{bmatrix}$$

In order to find the critical points of rank 1, we need all principal minors of this matrix to become 0 and in this case the least complicated minor is $\Delta_{1,6} = (\varkappa c - 1)(M_1P_2 - M_2P_1)$. As long as $\varkappa c - 1 \neq 0$, for this minor to be 0, we have two cases:

$$\mathbf{1)} M_2 = 0 \; and M_1P_2 = 0$$
$$\mathbf{2)} M_2 \neq 0 \; and \; P_1 = \frac{M_1P_2}{M_2}.$$

These are the only cases provided we choose to do the calculations on the standard Lie group $so(4)$ when $\varkappa = 1$ and $c \neq 1$.

Let us investigate case 1) in detail with comparison to all other minors. Letting $M_2 = 0 = M_1$ and assume $\varkappa = 1$, we have:

$$\begin{bmatrix} -bP_2M_3 - cP_2P_3 & 0 \\ bP_1M_3 + cP_1P_3 & 0 \\ 0 & 0 \\ -bP_2P_3 + aM_3P_2 - cM_3P_2 & P_2 \\ bP_3bP_1 + M_3(cP_1 - aP_1) & -P_1 \\ 0 & 0 \end{bmatrix}$$

And from here, we get the following principal minors needing to be zero:

$$\Delta_{1,4} = P_2^2(-bM_3 - cP_3)$$

$$\Delta_{1,5} = P_1(bP_2M_3 + cP_2P_3)$$

$$\Delta_{2,5} = bP_1^2M_3 + cP_1^2P_3 + M_3(cP_1 - aP_1)$$

$$\Delta_{2,4} = bP_2P_1M_3 + cP_1P_2P_3$$

$$\Delta_{4,5} = P_1(-bP_2P_3 + aM_3P_2 - cM_3P_2) + bP_2P_3P_1 + M_3P_2(cP_1 - aP_1))$$

Letting $\Delta_{1,5} = 0$ looking at all other principal minors , we have two new cases:

$$\textbf{1i)} P_1 = P_2 = 0,$$

This means the first family of solutions would be:

$$(M_1, M_2, M_3, P_1, P_2, P_3) = (0, 0, M_3, 0, 0, P_3).$$

Or

$$\textbf{1ii)} bM_3 = -cP_3 \text{ and } c = a,$$

and this results in the second family of points which would be

$$(M_1, M_2, M_3, P_1, P_2, P_3) = (0, 0, M_3, P_1, P_2, -\frac{b}{c}M_3).$$

Now, we need to investigate the case $M_1 = P_2 = 0$ which leads to the following matrix:

$$\begin{bmatrix} (a-1)M_2M_3 - bP_3M_2 & M_2 \\ bP_1M_3 + cP_1P_3 & 0 \\ 0 & 0 \\ -P_3M_2 + M_3bM_2 & 0 \\ bP_3P_1 + M_3P_1(c-a) & -P_1 \\ 0 & 0 \end{bmatrix}$$

Investigating the minors, we have

$$\Delta_{1,2} = M_2P_1(bM_3 + cP_3)$$

$$\Delta_{1,4} = M_2^2(-P_3 + bM_3)$$

$$\Delta_{1,5} = -P_1((a-1)M_2M_3 - bP_3M_2) - M_2(bP_3P_1 + M_3P_1(c-a))$$

$$\Delta_{2,5} = -P_1(bP_1M_3 + cP_1P_3)$$

Similarly, case 2) was investigated in more detail. This is when $M_2 \neq 0$ and $P_1 = \frac{M_1P_2}{M_2}$.

Proposition 6. *On the Lie algebra so(4), the set of points at which the skew-gradients of the Hamiltonian (1.27) and the first integral $K = M_3$ are dependent is the union of the following family of points in the space $\mathbb{R}^6(\mathbf{M}, \mathbf{P})$:*

(I) $M_1 = P_1 = 0$ *and* $M_3(c-1)P_2M_2 - bP_2^2 + bM_2^2 = (cP_2^2 - M_2^2)P_3$.

(II) $M_1 = P_1 = P_2 = 0$ *and* $P_3 = -bM_3$.

(III) $M_1 = M_3 = P_1 = P_3 = 0$.

(IV) $M_1 = P_1 = 0$ *and* $M_2 = -bP_2$ *and* $P_3 = -bM_3$.

(V) $M_1 = P_1 = 0$ *and* $P_3 = -bM_3$ *where* $c = 1$.

(VI) $P_1 = P_2 = 0$ *and* $P_3 = -bM_3$.

(VII) $M_1P_2 = M_2P_1$ *and* $M_3(c-1)P_2M_2 - bP_2^2 + bM_2^2 = (cP_2^2 - M_2^2)P_3$.

The intersection of each family of critical points from Proposition 6 with $M^4_{1,g}$ gives a set of critical points of the momentum mapping Φ_g. The images of these sets in the plane $\mathbb{R}^2(h,k)$ form the bifurcation diagram $\Sigma_{h,k}$. In order to find the bifurcation curves corresponding to Φ_g, to each of the families of equations in Proposition 6, the four equations in (1.26) are added and then the variables $M_1, M_2, M_3, P_1, P_2, P_3$ are eliminated from the system of equations. The lengthy calculations lead to the following:

Proposition 7. *The critical values the momentum mapping Φ_g obtained from Proposition 6 are defined by the equations on the plane $\mathbb{R}^2(h,k)$ as follows:*

(I) $h = \frac{1}{2}(1-(a-1)k^2)$.

(II) $h = \frac{1}{2}(g^2+2bg+1) = \frac{1}{2}[(g+b)^2+1-b^2]$.

(III) $h = \frac{1}{2}(\frac{c}{b^2}-1+k^2(1+a-\frac{c}{b^2}))$.

(IV) $h = \frac{1}{2}+(a+b^2-1)k^2+bg+\frac{g^2+b^2k^4+2gbk^2}{2(1-k^2)}$.

(V) $h = \frac{1}{2}(1+(a-1)k^2)$.

(VI) $h = \frac{1}{2}$.

(VII) $h = \frac{1}{2}(1+cg^2)$.

Remark 3. *Without loss of generality, we can assume $g \geq 0$ since the coordinate transformation*

$$(M_1, M_2, M_3, P_1, P_2, P_3) \mapsto (-M_1, M_2, M_3, P_1, -P_2, -P_3)$$

preserves the Casimir $f_1 = \boldsymbol{M}^2 + \boldsymbol{P}^2$ and the integral K, the Hamiltonian $H = H(a,b,c)$ transforms to $\hat{H} = H(a,-b,c)$ and changes the sign of the Casimir $f_2 = \langle \boldsymbol{M}, \boldsymbol{P} \rangle$.

Theorem 6. *The bifurcation diagrams of the map $\phi_g = H \times K : M^4_{1,g} \to \mathbb{R}^2(h,k)$ are given through the following cases:*

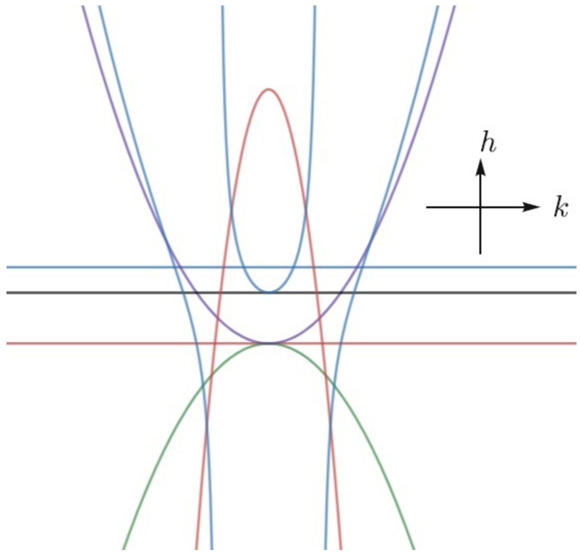

Figure 4.1: $c > 0$ and $b^2 > 1 - a$ and $b \neq 0$

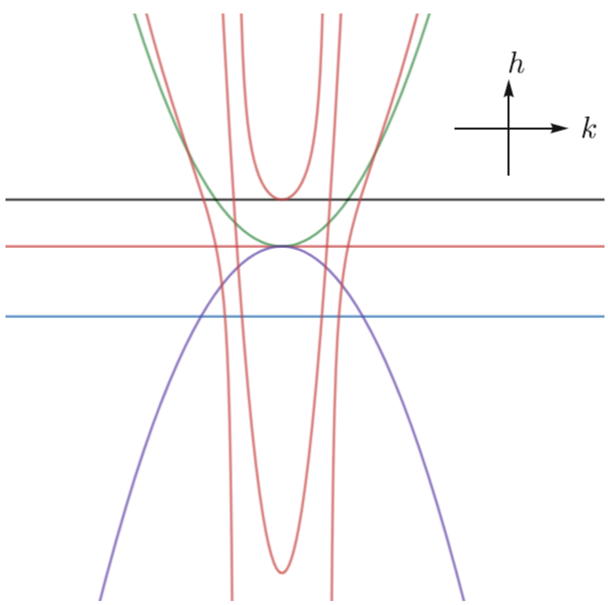

Figure 4.2: $c < 0$ and $b^2 > 1 - a$ and $b \neq 0$

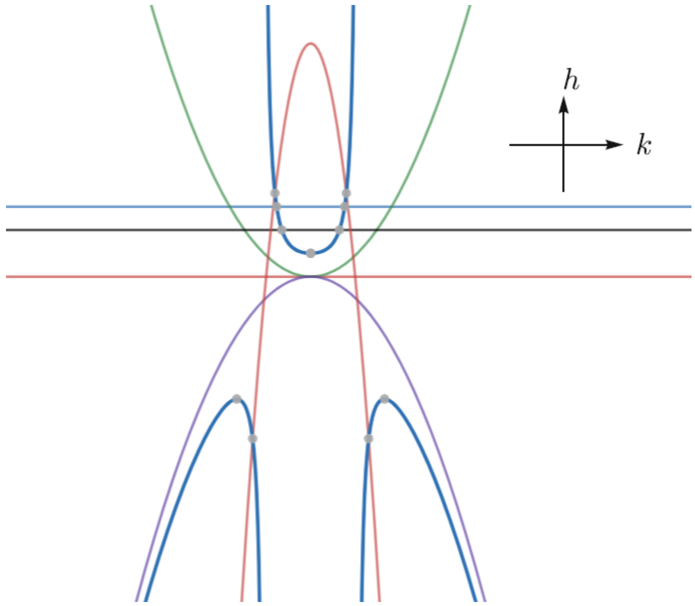

Figure 4.3: $c > 0$ and $b^2 < 1 - a$ and $b \neq 0$

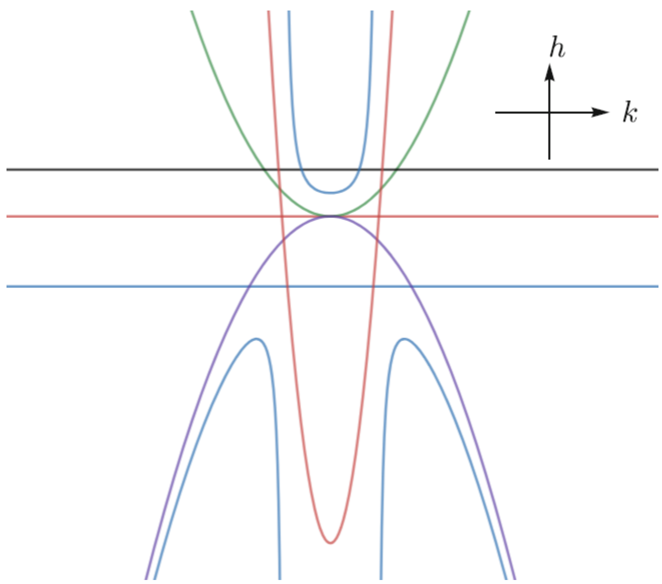

Figure 4.4: $c < 0$ and $b^2 < 1 - a$ and $b \neq 0$

4.2 Momentum map for the Kirchhoff case on $e(3)$ of rigid body motion in fluid

The Kirchhoff case of Kirchhoff equations was investigated together with Chaplygin case in [32]. Here we are going to find the critical points of the momentum mapping of Kirchhoff case of Kirchhoff equations and explore the the momentum mapping of this case. The Kirchhoff case is completely integrable with two first integrals, the Hamiltonian H and $K = M_3$, see for example [5, 18, 19]. We are going to study the bifurcation diagram of the map $\phi_g = H \times K : M^4_{1,g} \to \mathbb{R}^2(h,k)$, where the maps $H : M^4_{1,g} \to \mathbb{R}(h)$ and $K : M^4_{1,g} \to \mathbb{R}(k)$ are restrictions of the Hamiltonian H (1.27) and the first integral K from (1.25) respectively. Here $M^4_{1,g}$ is a symplectic manifold, the common level surface of two Casimirs, i.e., $M^4_{1,g} = f_1^{-1}(1) \cap f_2^{-1}(g)$. The corresponding family of momentum mappings has the form

$$\phi_g : M^4_{1,g} \to \mathbb{R}^2(h,k), \quad x \mapsto (H(x), K(x)).$$

Thus, the mapping ϕ_g depends on the parameter g. The rank of a critical point of the momentum mapping is the rank of the differential of the momentum mapping at this point. It is clear that for the system under consideration the ranks of critical points of the momentum mapping ϕ_g are zero or one. In this section, we have studied the bifurcation analysis of the Kirchhoff case on the Lie algebra $e(3)$ with the need for a resultant determinant to be computed using MATLAB and other piece of momentum mapping given as relations on h and k [12].

4.2.1 Critical points of rank zero on the Lie algebra $e(3)$

Let us keep in mind the Hamiltonian and extra integral in Kirchhoff's case:

$$H = \frac{1}{2}(M_1^2 + M_2^2 + aM_3^2) + b(M_1P_1 + M_2P_2) + \frac{1}{2}c(P_1^2 + P_2^2)$$

and

$$F = M_3.$$

Since we are working on the Lie algebra $e(3)$ here, we will also be making use of the Casimirs

$$\mathbf{P^2 = 1} \tag{4.1}$$

and

$$M_1P_1 + M_2P_2 + M_3P_3 = g. \tag{4.2}$$

In order to find the critical points of rank zero, we need to find out when the skew gradients sgrad H and sgrad K both vanish simultaneously. In general, the field sgrad H is written explicitly in the following:

$\{M_1, H\} = (a-1)M_2M_3 - b(P_2M_3 + P_3M_2) - cP_2P_3$

$\{M_2, H\} = (1-a)M_1M_3 + b(P_1M_3 + P_3M_1) + cP_1P_3$

$\{M_3, H\} = 0$

$\{P_1, H\} = -P_3M_2 - bP_2P_3 + aP_2M_3 - \varkappa bM_2M_3 - \varkappa cP_2M_3$

$\{P_2, H\} = P_3M_1 + bP_1P_3 - aP_1M_3 + \varkappa bM_1M_3 + \varkappa cP_1M_3$

$\{P_3, H\} = P_1M_2 - P_2M_1 + \varkappa c(M_1P_2 - M_2P_1).$

Letting $\varkappa = 0$ for the Lie algebra $e(3)$, we have:

$\{M_1, H\} = (a-1)M_2M_3 - b(P_2M_3 + P_3M_2) - cP_2P_3$

$\{M_2, H\} = (1-a)M_1M_3 + b(P_1M_3 + P_3M_1) + cP_1P_3$

$\{M_3, H\} = 0$

$\{P_1, H\} = -P_3M_2 - bP_2P_3 + aP_2M_3$

$\{P_2, H\} = P_3M_1 + bP_1P_3 - aP_1M_3$

$\{P_3, H\} = P_1M_2 - P_2M_1.$

Also, since $K = M_3$, the skew gradient of K is written explicitly in the follwoing manner:

$\{M_1, K\} = M_2$

$\{M_2, K\} = -M_1$

$\{M_3, K\} = 0$

$\{P_1, K\} = -\{M_3, P_1\} = P_2$

$\{P_2, K\} = -\{M_3, P_2\} = -P_1$

$\{P_3, K\} = 0.$

Proposition 8. *Letting $\varkappa = 0$ considering the Lie algebra e(3), the set of points where both skew gradient vector fields,* sgrad H *and* sgrad K *with Hamiltonians H and K vanish simultaneously, is the following family of points in the space $\mathbb{R}^6(H, K)$:*
$(0, 0, M_3, 0, 0, P_3)$.

Theorem 7. *For $\varkappa = 0$, the image of critical points of rank 0 is*

$$h = \frac{a}{2}g^2 \text{ and } k = \pm g.$$

4.3 Critical points of rank one on the Lie algebra $e(3)$

In order to find the critical points of rank one, we need to find out for what values of the variables, the skew gradients of the Hamiltonian and the skew gradients of the extra integral $K = M_3$ are independent. Therefore, the critical points of rank one are found using the minors of the following matrix:

$$\begin{bmatrix} \{M_1, H\} & \{M_1, K\} \\ \{M_2, H\} & \{M_2, K\} \\ \{M_3, H\} & \{M_3, K\} \\ \{P_1, H\} & \{P_1, K\} \\ \{P_2, H\} & \{P_2, K\} \\ \{P_3, H\} & \{P_3, K\}. \end{bmatrix}$$

This leads to the following matrix:

[4].

$$\begin{bmatrix} (a-1)M_2M_3 - b(P_2M_3 + P_3M_2) - cP_2P_3 & M_2 \\ (1-a)M_1M_3 + b(P_1M_3 + M_1P_3) + cP_1P_3 & -M_1 \\ 0 & 0 \\ -P_3(M_2 + bP_2) + aM_3P_2 & P_2 \\ P_3(M_1 + bP_1) - aM_3P_1 & -P_1 \\ -(M_1P_2 - M_2P_1) & 0 \end{bmatrix}. \tag{4.3}$$

This means the following minors are acquired:

$$\Delta_{1,2} = bM_3(M_1P_2 - M_2P_1) + cP_3(M_1P_2 - M_2P_1)$$

$$\Delta_{1,4} = -P_2^2(bM_3 + cP_3) + M_2(M_2P_3 - P_2M_3)$$

$$\Delta_{1,5} = M_2(P_1M_3 - M_1P_3) + P_1P_2(bM_3 + cP_3)$$

$$\Delta_{2,4} = M_1(P_2M_3 - M_2P_3) + P_1P_2(bM_3 + cP_3)$$

$$\Delta_{2,5} = M_1(M_1P_3 - M_3P_1) - P_1^2(bM_3 + cP_3)$$

$$\Delta_{1,6} = M_2(M_1P_2 - M_2P_1)$$

$$\Delta_{2,6} = -M_1(M_1P_2 - M_2P_1)$$

$$\Delta_{4,5} = P_3(P_1M_2 - M_1P_2)$$

$$\Delta_{4,6} = P_2(M_1P_2 - M_2P_1)$$

$$\Delta_{5,6} = -P_1(M_1P_2 - M_2P_1).$$

From the minors above and further calculations we end up with the following proposition. The set of points of rank 1 is the set of points which are not of rank 0 and at which the skew-gradients of the Hamiltonian (1.27) and the first integral $K = M_3$ are dependent, see [4].

Proposition 9. *The set of points of rank 1 is the union of the following families of points in the space* $\mathbb{R}^6(\mathbf{M}, \mathbf{P})$:

I The generic case

$$M_1P_2 = M_2P_1 \text{ and } \Delta_{2,4} = 0 \tag{4.4}$$

with $M_1 \neq 0$ *or* $P_2 \neq 0$ *and* $P_3 \neq \pm 1$*. The points satisfy the relation:*

$$(M_1^*, M_2^*, M_3^*, P_1^*, P_2^*, P_3^*) = (M_1, M_2, k, \frac{(P_3^2-1)M_1}{kP_3-g}, \frac{(P_3^2-1)M_2}{kP_3-g}, P_3) \tag{4.5}$$

II Singular case $P_1 = P_2 = 0$ *and* $P_3 = \pm 1$ *with* $M_1 \neq 0$*:*

$$(M_1^*, M_2^*, M_3^*, P_1^*, P_2^*, P_3^*) = (M_1, 0, k, 0, 0, \pm 1).$$

III Singular case $P_1 = M_1 = 0$*:*

$$(M_1^*, M_2^*, M_3^*, P_1^*, P_2^*, P_3^*) = (0, \pm\frac{g-kP_3}{\sqrt{1-P_3^2}}, k, 0, \pm\sqrt{1-P_3^2}, P_3).$$

IV Singular case $P_1 = M_1 = P_3 = 0$*:*

$$(M_1^*, M_2^*, M_3^*, P_1^*, P_2^*, P_3^*) = (0, M_2, k, 0, \pm 1, 0).$$

V Singular case $M_1 = M_2 = 0, P_1 \neq 0$*:*

Va If $P_2 = 0$ *then*

$$(M_1^*, M_2^*, M_3^*, P_1^*, P_2^*, P_3^*) = (0, 0, k, \pm\sqrt{1-\frac{g^2}{k^2}}, 0, \frac{g}{k}).$$

Vb If $P_2 \neq 0$ *then*

$$(M_1^*, M_2^*, M_3^*, P_1^*, P_2^*, P_3^*) = (0, 0, k, P_1, P_2, -b\frac{k}{c}).$$

VI Singular case $P_2 = M_2 = 0$*:*

$$(M_1^*, M_2^*, M_3^*, P_1^*, P_2^*, P_3^*) = (\pm\frac{g-kP_3}{\sqrt{1-P_3^2}}, 0, k, \pm\sqrt{1-P_3^2}, 0, P_3).$$

Proof. We are going to focus on the generic case. Assuming

$$\Delta_{2,4} = 0,\ \ P_1 M_2 = M_1 P_2, \tag{4.6}$$

we have:

$$M_1(P_2 M_3 - M_2 P3) + P_1 P_2(bM_3 + cP_3) = 0.$$

One checks

$$\Delta_{1,4} = M_2(M_2P_3 - P_2M_3) - P_2^2(bM_3 + cP_3) = M_2(M_2P_3 - P_2M_3) + \frac{P_2M_1}{P_1}(P_2M_3 - M_2P_3) = 0.$$

This means $\Delta_{14} = 0$ if $P_1 \neq 0$. The other minors also vanish under the assumptions (4.7):

$$\Delta_{1,2} = bM_3(M_1P_2 - M_2P_1) + cP_3(M_1P_2 - M_2P_1) = (bM_3 + cP_3)(M_1P_2 - M_2P_1) = 0.$$

Similarly

$$\Delta_{1,5} = M_2(P_1M_3 - M_1P_3) + P_1P_2(bM_3 + cP_3) = M_3(M_2P_1 - M_1P_2) = 0.$$

Also,

$$\begin{aligned}\Delta_{2,5} &= M_1(M_1P_3 - M_3P_1) - P_1^2(bM_3 + cP_3) = M_1(M_1P_3 - M_3P_1) - \frac{P_1M_1P_2}{M_2}(bM_3 + cP_3)\\ &= M_2M_1(M_1P_3 - M_3P_1) + M_1^2(P_2M_3 - M_2P_3) = M_1M_3(M_1P_2 - M_2P_1) = 0,\end{aligned}$$

provided $M_2 \neq 0$. It is straightforward that under the assumptions (4.7),

$$\Delta_{1,6} = \Delta_{2,6} = \Delta_{4,5} = \Delta_{4,6} = \Delta_{5,6} = 0.$$

We conclude that, when

$$P_1M_2 = P_2M_1, \Delta_{2,4} = 0 \ \textit{and} \ P_1, M_2 \neq 0,$$

the matrix (4.3) is of rank 1. Then, the singular cases follow directly. $\square$

Assuming

$$\Delta_{2,4} = 0, \ P_1 M_2 = M_1 P_2, \tag{4.7}$$

we have:

$$M_1(P_2M_3 - M_2P3) + P_1P_2(bM_3 + cP_3) = 0,$$

and now:

$$\Delta_{1,4} = M_2(M_2P_3 - P_2M_3) - P_2^2(bM_3 + cP_3)$$

$$= M_2(M_2P_3 - P_2M_3) + \frac{P_2M_1}{P_1}(P_2M_3 - M_2P_3)$$

$$= M_2\left[(M_2P_3 - P_2M_3) + (P_2M_3 - M_2P_3)\right] = 0.$$

This means $\Delta_{14} = 0$ if $P_1 \neq 0$.

Let us now investigate the other minors under the assumptions (4.7).

$$\Delta_{1,2} = bM_3(M_1P_2 - M_2P_1) + cP_3(M_1P_2 - M_2P_1)$$

$$= (bM_3 + cP_3)(M_1P_2 - M_2P_1) = 0.$$

And

$$\Delta_{1,5} = M_2(P_1M_3 - M_1P_3) + P_1P_2(bM_3 + cP_3)$$

$$= M_2(P_1M_3 - M_1P_3) - M_1(P_2M_3 - M_2P_3) =$$

$$= M_3(M_2P_1 - M_1P_2) = 0.$$

Also,

$$\Delta_{2,5} = M_1(M_1P_3 - M_3P_1) - P_1^2(bM_3 + cP_3)$$

$$= M_1(M_1P_3 - M_3P_1) - \frac{P_1M_1P_2}{M_2}(bM_3 + cP_3).$$

This gives

$$M_2M_1(M_1P_3 - M_3P_1) + M_1^2(P_2M_3 - M_2P_3)$$

$$= M_1(M_1P_2M_3 - M_2P_1M_3)$$

$$M_1M_3(M_1P_2 - M_2P_1) = 0, \ provided\ M_2 \neq 0.$$

It is easy to see that under the assumptions (4.7),

$$\Delta_{1,6} = \Delta_{2,6} = \Delta_{4,5} = \Delta_{4,6} = \Delta_{5,6} = 0.$$

We conclude that, when

$$P_1M_2 = P_2M_1, \Delta_{2,4} = 0 \ and\ P_1, M_2 \neq 0,$$

the matrix (4.3) is of rank 1.

In order to find the curves corresponding to Φ_g, to each of the families of equations in proposition 9, the Hamiltonian, extra integral and the Casimirs are added and then the variables $M_1, M_2, M_3, P_1, P_2, P_3$ are eliminated from the system of equations.

Lemma 6. *If polynomials F_1 and F_2 both have a common zero at x,then their resultant is zero:*

$$\mathrm{res}(F_1, F_2) = 0.$$

Proposition 10. *The images under the momentum mapping Φ_g of families of critical points from proposition 9 are respectively contained in the curves defined by the equations on the plane $\mathbb{R}^2(h,k)$as follows:*

1. *Under the conditions (4.4), the relation between h, k is given as $R(h,k) = \mathrm{res}(F_1, F_2) = 0$ which depends on the parameters a, b, c, g;*

2. $h = \frac{ag^2}{2}$.

3. *The values for P_3 are given by $F_3(P_3) = 0$ with the condition that $-1 < P_3 < 1$, when*

$$F_3(P_3) := cP_3^5 + bkP_3^4 + P_3^3(k^2 - 2c - k) + P_3^2(-kg - 2bk + 2gk) + P_3(c - k^2 - g^2) + k(b - g) = 0.$$

4. *For $k = 0$, we have $h = \frac{g^2}{2} + bg + \frac{c}{2}$, and for $k \neq 0$, we have $b = -g$ and $h = \frac{-g^2}{2} + \frac{a}{2}k^2 + \frac{c}{2}$.*

5. *If* $M_1 = M_2 = P_2 = 0,\ P_1 \neq 0$, *then we have* $h = \frac{1}{2}ak^2 - \frac{cg^2}{2k^2} + \frac{c}{2}$ *and if* $M_1 = M_2 = 0, P_1 \neq 0, P_2 \neq 0$, *then* $h = \frac{k^2}{2}(a - \frac{b^2}{c}) + \frac{c}{2}$ *and* $g = \frac{-bk^2}{c}$.

6. *If* $M_2 = P_2 = 0,\ P_1 \neq 0, P_2 \neq 0$, *we have* $F_3(P_3) := cP_3^5 + bkP_3^4 + P_3^3(k^2 - 2c - k) + P_3^2(-kg - 2bk + 2gk) + P_3(c - k^2 - g^2) + k(b - g) = 0$, *with* $-1 < P_3 < 1$.

Proof. In order to prove the case 1, we proceed in the following way:

From (4.4), we have

$$M_1(M_2P_3 - P_2M_3) = P_1P_2(bM_3 + cP_3)$$

and with the extra integral $k = M_3$, we obtain

$$P_1^2(bk + cP_3) = M_1^2P_3 \tag{4.8}$$

and then with reference to (4.5) and under the condition that $P_1, M_1, M_2 \neq 0.$ we have:

$$P_1 = \frac{(P_3^2 - 1)M_1}{kP_3 - g}. \tag{4.9}$$

Now, combining equations (4.8) and (4.9), we obtain:

$$(P_3^2 - 1)^2M_1^2(bk + cP_3) - M_1^2P_3(kP_3 - g)^2 + (P_3^2 - 1)M_1^2k(kP_3 - g) = 0,$$

which results in

$$F_1(P_3) := cP_3^5 + bkP_3^4 + (k - k^2 - 2c)P_3^3 + (2kg - 2bk - g)P_3^2 + (c - g^2 - k^2)P_3 + (bk + gk) = 0. \tag{4.10}$$

Let us keep(4.10) in mind for the time being and work with the Hamiltonian and rewrite it using our Casimirs, so that we have

$$h = \frac{1}{2}(M_1^2 + \frac{M_1^2P_2^2}{P_1^2} + ak^2) + b(g - kP_3) + \frac{1}{2}c(1 - P_3^2). \tag{4.11}$$

Multiplying (4.11) by $2P_1^2$, we obtain:

$$2hP_1^2 = M_1^2P_1^2 + M_1^2(1 - P_3^2 - P_1^2) + ak^2P_1^2 + 2bP_1^2(g - kP_3) + cP_1^2(1 - P_3^2)$$

$$2hP_1^2 = M_1^2(1-P_3^2) + ak^2P_1^2 + 2bgP_1^2 - 2bkP_1^2P_3 + cP_1^2 - cP_1^2P_3^2$$

$$\left[2h - ak^2 - 2bg + 2bkP_3 - c + cP_3^2\right] P_1^2 = M_1^2(1-P_3^2).$$

Finally we arrive at:

$$P_1^2 = \frac{M_1^2(1-P_3^2)}{2h - ak^2 - 2bg + 2bkP_3 - c + cP_3^2}. \tag{4.12}$$

Let us combine equations (4.9) and (4.12) now and therefore, we have:

$$\frac{(P_3^2-1)^2M_1^2}{(kP_3-g)^2} = \frac{M_1^2(1-P_3^2)}{2h - ak^2 - 2bg + 2bkP_3 - c + cP_3^2}.$$

Under the condition $M_1 \neq 0$ and $P_3^2 \neq 1$, we obtain:

$$(P_3^2-1)(2h - ak^2 - 2bg + 2bkP_3 - c + cP_3^2) = k^2P_3^2 - 2kgP_3 + g^2,$$

which results in:

$$F_2(P_3) := cP_3^4 + 2bkP_3^3 + (2h-(a+1)k^2 - 2bg - 2c)P_3^2 + (2kg - 2bk)P_3 - 2h + ak^2 + 2bg + c = 0. \tag{4.13}$$

The polynomials F_1 and F_2 from (4.10) and (4.13) both have a common zero at P_3. This means we can apply lemma 6. This means that the determinant of the 9×9 Sylvester matrix found from the coefficients of polynomials(4.10) and (4.13)is zero.

In our case, having polynomials of degrees 5 and 4 with

$$F_1(P_3) = a_0P_3^5 + a_1P_3^4 + a_2P_3^3 + a_3P_3^2 + a_4P_3 + a_5$$

and

$$F_2(P_3) = b_0P_3^4 + b_1P_3^3 + b_2P_3^2 + b_3P_3 + b_4,$$

for the resultant of F_1 and F_2, we get a Sylvester determinant of the form:

$$R(h,k) := \operatorname{res}(F_1,F_2) = \begin{vmatrix} a_0 & 0 & 0 & 0 & b_0 & 0 & 0 & 0 & 0 \\ a_1 & a_0 & 0 & 0 & b_1 & b_0 & 0 & 0 & 0 \\ a_2 & a_1 & a_0 & 0 & b_2 & b_1 & b_0 & 0 & 0 \\ a_3 & a_2 & a_1 & a_0 & b_3 & b_2 & b_1 & b_0 & 0 \\ a_4 & a_3 & a_2 & a_1 & b_4 & b_3 & b_2 & b_1 & b_0 \\ a_5 & a_4 & a_3 & a_2 & 0 & b_4 & b_3 & b_2 & b_1 \\ 0 & a_5 & a_4 & a_3 & 0 & 0 & b_4 & b_3 & b_2 \\ 0 & 0 & a_5 & a_4 & 0 & 0 & 0 & b_4 & b_3 \\ 0 & 0 & 0 & a_5 & 0 & 0 & 0 & 0 & b_4 \end{vmatrix} = 0. \tag{4.14}$$

This equation is of the form

$$R(h,k) = \sum a_{nm} h^n k^m,$$

with $n \leq 5$, $m \leq 14$ and $n+m \leq 14$. The full equation that was obtained using the MATLAB software is included in the Appendix.

In order to prove case 2, let $P_1 = P_2 = 0$ and $P_3 = \pm 1$, while $M_1 \neq 0$. In order for $\Delta_{1,4} = 0$, we need

$$M_2^2 P_3 = 0,$$

which gives

$$M_2 = 0 \; and \; \pm k = g.$$

The Hamiltonian relation in this case becomes $h = \frac{1}{2}(M_1^2 + ag^2)$ which results in

$$M_1 = \pm\sqrt{2h - ag^2}.$$

In order for $\Delta_{2,5}$ to vanish, we need $M_1 P_3 = 0$ which leads to $M_1 = 0$. This means we have

$$(M_1^*, M_2^*, M_3^*, P_1^*, P_2^*, P_3^*) = (0, 0, k, 0, 0, \pm 1),$$

and as a result

$$h = \frac{ag^2}{2}.$$

Now, let us prove the case 3. Assuming $P_1 = M_1 = 0$, and requiring $\Delta_{1,4} = 0$, we get

$$P_2^2(bk + cP_3) = M_2(M_2P_3 - kP_2)$$

$$(1 - P_3^2)(bk + cP_3) = M_2^2P_3 - k(g - kP_3),$$

resulting in

$$M_2^2 = \frac{(1 - P_3^2)(bk + cP_3) + k(g - kP_3)}{P_3}, \ \textit{where } P_3 \neq 0. \tag{4.15}$$

Taking the (4.2) into account, we have:

$$M_2P_2 = g - kP_3,$$

leading to

$$M_2^2P_2^2 = (g - kP_3)^2,$$

which leads to

$$M_2^2 = \frac{(g - kP_3)^2}{1 - P_3^2}, \ \textit{where } P_2 \neq 0. \tag{4.16}$$

From (4.15) and (4.16), we get

$$(1 - P_3^2)^2(bk + cP_3) + k(1 - P_3^2)(g - kP_3) = P_3(g - kP_3)^2,$$

resulting in

$$F_3(P_3) := cP_3^5 + bkP_3^4 + P_3^3(k^2 - 2c - k) + P_3^2(-kg - 2bk + 2gk) + P_3(c - k^2 - g^2) + k(b - g) = 0$$

with $-1 < P_3 < 1$. Let us now look at case 4 and apply the conditions of the case as in $P_1 = M_1 = P_3 = 0$, this means that from Casimirs (4.1)and (4.2), we conclude:

$$M_2P_2 = g, \ P_2 = \pm 1,$$

which results in $M_2 = \pm g$. Now, in order for $\Delta_{1,4}$ to vanish, we have

$$bk = -kg,$$

leading to

$$k = 0 \ or \ b = -g.$$

This results in

$$h = \frac{1}{2}(g^2 + ak^2) + bg + \frac{c}{2},$$

leading to

$$k = 0, h = \frac{g^2}{2} + bg + \frac{c}{2},$$

and

$$k \neq 0, \ b = -g \ and \ h = \frac{-g^2}{2} + \frac{a}{2}k^2 + \frac{c}{2}.$$

For case 5, there are two scenarios to look at. For both cases, the assumption is that $M_1 = M_2 = 0, \ P_1 \neq 0$. Applying this assumption to $\Delta_{2,4} = 0$, we get

$$P_2 = 0 \ or \ P_3 = -b\frac{k}{c}.$$

In subcategory of case 5 when $P_2 = 0$, from Casimirs (4.1) and (4.2) we have :

$$kP_3 = g,$$

which gives

$$P_3 = \frac{g}{k},$$

and therefore

$$P_1 = \pm\sqrt{1 - \frac{g^2}{k^2}}.$$

Finally,

$$a\frac{k^2}{2} - c\frac{g^2}{2k^2} + \frac{c}{2}.$$

For subcategory of case 5 when $P_2 \neq 0,\ P_3 = -b\frac{k}{c}$, we have

$$P_1^2 + P_2^2 = 1 - \frac{b^2k^2}{c^2}; P_3 = \frac{g}{k} = -b\frac{k}{c}.$$

As a result we have

$$cg = -bk^2,$$

leading to

$$k^2 = -c\frac{g}{b},$$

and finally

$$h = \frac{1}{2}(ak^2) + \frac{c}{2}(1 - \frac{b^2k^2}{c^2}),$$

and

$$g = -\frac{bk^2}{c}.$$

And finally to prove 6, the conditions $M_2 = P_2 = 0$ and $M_1P_2 = M_2P_1$ already guarantee $\Delta_{1,4} = 0$, too. We therefore have

$$h = \frac{1}{2}(M_1^2 + ak^2) + b(g - kP_3) + \frac{c}{2}(1 - P_3^2).$$

Now to satisfy $\Delta_{2,5} = 0$, we need

$$M_1^2P_3 - kM_1P_1 = P_1^2(bk + cP_3)$$

giving

$$M_1^2P_3 - k(g - kP_3) = (1 - P_3^2)(bk + cP_3),$$

which leads to

$$M_1^2 = \frac{(1 - P_3^2)(bk + cP_3) + k(g - kP_3)}{P_3}.$$

We then have

$$M_1^2P_1^2 = (g - kP_3)^2$$

which gives

$$M_1^2 = \frac{(g - kP_3)^2}{1 - P_3^2},$$

and finally

$$F_3(P_3) \coloneqq cP_3^5 + bkP_3^4 + P_3^3(k^2 - 2c - k) + P_3^2(-kg - 2bk + 2gk) + P_3(c - k^2 - g^2) + k(b - g) = 0$$

with $-1 < P_3 < 1$. $\square$

APPENDIX

THE FULL EQUATION OF RESULTANT MATRIX $R(h,k)$

The equation we obtain from the resultant as stated in Chapter 4 in (4.14) is given here:

$$R(h,k) = -c(-c^4g^4+2c^4g^6-c^4g^8-16b^6k^8+6b^4k^{11}+64b^6k^9-3b^4k^{12}-64b^6k^{10}-c^4k^4+4c^4k^5+$$

$$2c^3k^7-6c^4k^6-4c^3k^8+4c^4k^7-c^2k^{10}+2c^3k^9-c^4k^8-4bc^3g^5+6bc^3g^7-2bc^3g^9+$$

$$32b^5cg^7-32b^5cg^9-4ab^4k^{10}-2ab^2k^{13}+ab^2k^{14}+16ab^4k^{12}-3ac^3k^6+$$

$$12ac^3k^7+4ac^2k^9-14ac^3k^8-8ac^2k^{10}+4ac^3k^9+ac^3k^{10}+2a^2ck^{11}-$$

$$a^3ck^{10}-4a^2ck^{12}+4a^3ck^{11}-2a^2ck^{13}-2a^4ck^{11}-8a^3ck^{13}+8a^4ck^{12}-$$

$$a^3ck^{14}-6a^4ck^{13}-a^5ck^{12}-4a^4ck^{14}+4a^5ck^{13}-4a^5ck^{14}-8b^4ck^7+$$

$$3b^2ck^{10}+24b^4ck^8-2b^2ck^{11}-8b^4ck^9+b^2ck^{12}-16b^4ck^{10}-32cg^2h^5+$$

$$4c^3g^4h+32cg^4h^5-6c^3g^6h+2c^3g^8h+16b^5gk^8-32b^5gk^9+24b^5gk^{10}+2c^3g^6k+$$

$$4cg^4k^7-cg^2k^{10}-2cg^4k^8+8b^4hk^8+4b^2hk^{11}-2b^2hk^{12}-32b^4hk^{10}+8ch^3k^4-$$

$$32ch^4k^3+32ch^5k^2+6c^3hk^4-32ch^3k^5+128ch^4k^4-128ch^5k^3-24c^3hk^5+8ch^2k^7-96ch^4k^5+$$

$$128ch^5k^4-8c^2hk^7+28c^3hk^6-16ch^2k^8+64ch^3k^7-64ch^4k^6+16c^2hk^8-8c^3hk^7-8ch^2k^9+$$

$$8ch^3k^8-2c^3hk^8-4b^2c^2g^6-8b^2c^3g^6+4b^2c^2g^8-16b^3c^2g^7+16b^4c^2g^6+8b^2c^3g^8+$$

$$16b^3c^2g^9-16b^4c^2g^8+4a^2b^2k^{12}+8a^2b^4k^{10}-2a^3b^2k^{11}-10a^2b^2k^{13}-32a^2b^4k^{11}+$$

$$9a^3b^2k^{12}+3a^2b^2k^{14}+32a^2b^4k^{12}-10a^3b^2k^{13}-a^4b^2k^{12}+4a^4b^2k^{13}-4a^4b^2k^{14}-$$

$$3a^2c^2k^8-2a^2c^3k^7+12a^2c^2k^9+8a^2c^3k^8-8a^2c^2k^{10}-10a^2c^3k^9-4a^3c^2k^9-8a^2c^2k^{11}+$$

$$4a^2c^3k^{10}+16a^3c^2k^{10}+a^2c^2k^{12}-16a^3c^2k^{11}-a^4c^2k^{10}+4a^4c^2k^{11}-4a^4c^2k^{12}-$$

$$8b^2c^3k^5-2b^2c^2k^7+32b^2c^3k^6+8b^2c^2k^8-40b^2c^3k^7-16b^4c^2k^6-10b^2c^2k^9+16b^2c^3k^8+64b^4c^2k^7+$$

$$12b^2c^2k^{10} - 64b^4c^2k^8 + 16c^2g^2h^4 - 4c^2g^4h^2 + 16c^2g^4h^3 - 8c^3g^4h^2 - 16c^2g^4h^4 + 4c^2g^6h^2 -$$

$$16c^2g^6h^3 + 8c^3g^6h^2 - 8b^4g^4k^5 + 8b^2g^4k^8 + 8b^3g^5k^6 - 8b^4g^2k^8 + 28b^4g^4k^6 + 16b^5g^3k^6 - 32b^5g^5k^4 +$$

$$48b^6g^2k^6 - 48b^6g^4k^4 + 16b^6g^6k^2 - 2b^2g^2k^{11} - 8b^2g^4k^9 - 8b^3g^3k^9 + 32b^3g^5k^7 - 32b^4g^2k^9 + 20b^4g^4k^7 +$$

$$24b^4g^6k^5 - 136b^5g^3k^7 + 152b^5g^5k^5 - 16b^5g^7k^3 - 192b^6g^2k^7 + 192b^6g^4k^5 - 64b^6g^6k^3 +$$

$$b^2g^2k^{12} + 2b^2g^4k^{10} - 18b^3g^5k^8 + 30b^4g^2k^{10} - 36b^4g^4k^8 + 36b^4g^6k^6 + 96b^5g^3k^8 - 64b^5g^5k^6 + 40b^5g^7k^4 +$$

$$208b^6g^2k^8 - 240b^6g^4k^6 + 112b^6g^6k^4 - 16b^6g^8k^2 + 2c^4g^2k^2 - c^3g^2k^4 - 4c^4g^2k^3 +$$

$$16b^2h^3k^5 - 16b^2h^4k^4 + 2c^3g^2k^5 - 2c^3g^4k^3 - 72b^2h^3k^6 + 64b^2h^4k^5 + 6c^2g^2k^7 - 6c^2g^4k^5 + 4c^2g^6k^3 -$$

$$6c^3g^2k^6 - 8c^3g^4k^4 + 11c^3g^6k^2 + 4c^4g^2k^5 + 4c^4g^4k^3 + 16b^2h^2k^8 + 80b^2h^3k^7 - 64b^2h^4k^6 +$$

$$32b^4h^2k^6 - 4c^2g^2k^8 - 10c^2g^4k^6 + 14c^2g^6k^4 + 6c^3g^2k^7 + 16c^3g^4k^5 + 4c^3g^6k^3 - 2c^4g^2k^6 - 3c^4g^4k^4 -$$

$$2c^4g^6k^2 - 40b^2h^2k^9 - 128b^4h^2k^7 + 2c^2g^2k^9 + 6c^2g^4k^7 + 4c^2g^6k^5 - c^3g^2k^8 - 4c^3g^4k^6 -$$

$$5c^3g^6k^4 - 2c^3g^8k^2 + 12b^2h^2k^{10} + 128b^4h^2k^8 - 12c^2h^2k^4 + 32c^2h^3k^3 - 16c^2h^4k^2 - 8c^3h^2k^3 +$$

$$48c^2h^2k^5 - 128c^2h^3k^4 + 64c^2h^4k^3 + 32c^3h^2k^4 - 32c^2h^2k^6 + 128c^2h^3k^5 - 64c^2h^4k^4 - 40c^3h^2k^5 -$$

$$32c^2h^2k^7 + 16c^3h^2k^6 + 4c^2h^2k^8 - ack^{12} + 2chk^{10} - 8ab^2c^2k^7 + 32ab^2c^2k^8 - 56ab^2c^2k^9 - 2a^2b^2ck^9 +$$

$$48ab^2c^2k^{10} + 10a^2b^2ck^{10} - 34a^2b^2ck^{11} + 8a^3b^2ck^{10} + 44a^2b^2ck^{12} - 32a^3b^2ck^{11} +$$

$$32a^3b^2ck^{12} - 64bc^2g^3h^3 - 48bc^2g^5h^2 - 320b^2cg^4h^3 + 64bc^2g^5h^3 + 48b^2c^2g^6h + 320b^3cg^5h^2 -$$

$$64b^3c^2g^5h + 48bc^2g^7h^2 + 320b^2cg^6h^3 - 48b^2c^2g^8h - 320b^3cg^7h^2 + 64b^3c^2g^7h - 8ab^3g^3k^7 -$$

$$8ab^4g^2k^7 + 8ab^4g^4k^5 + 12ab^2g^2k^{10} - 4ab^2g^4k^8 + 40ab^3g^3k^8 - 8ab^3g^5k^6 + 52ab^4g^2k^8 -$$

$$64ab^4g^4k^6 - 64ab^5g^3k^6 + 32ab^5g^5k^4 - 8a^2b^3gk^9 - 16ab^2g^2k^{11} + 12ab^2g^4k^9 - 14ab^3g^3k^9 + 12ab^3g^5k^7 -$$

$$104ab^4g^2k^9 + 184ab^4g^4k^7 - 32ab^4g^6k^5 + 256ab^5g^3k^7 - 128ab^5g^5k^5 + 36a^2b^3gk^{10} + 4ab^2g^2k^{12} -$$

$$5ab^2g^4k^{10} - 14ab^3g^3k^{10} + 16ab^3g^5k^8 + 16ab^4g^2k^{10} - 68ab^4g^4k^8 + 36ab^4g^6k^6 - 288ab^5g^3k^8 +$$

$$192ab^5g^5k^6 - 32ab^5g^7k^4 - 40a^2b^3gk^{11} - 8a^3b^3gk^{10} - 2a^2b^3gk^{12} + 32a^3b^3gk^{11} -$$

$$32a^3b^3gk^{12}+5ac^3g^2k^4-2ac^3g^4k^2-2ac^2g^2k^6-14ac^3g^2k^5-24ab^2h^2k^7+32ab^2h^3k^6+$$

$$6ac^2g^2k^7-4ac^2g^4k^5+2ac^2g^6k^3+4ac^3g^2k^6+8ac^3g^4k^4+3ac^3g^6k^2+$$

$$108ab^2h^2k^8-128ab^2h^3k^7+2ac^2g^2k^8-2ac^2g^4k^6+9ac^2g^6k^4+6ac^3g^2k^7-8ac^3g^4k^5+$$

$$8ac^3g^6k^3-a^2cg^2k^8-120ab^2h^2k^9+128ab^2h^3k^8+4ac^2g^2k^9-14ac^2g^4k^7+18ac^2g^6k^5-ac^3g^2k^8+$$

$$6ac^3g^4k^6+ac^3g^6k^4-ac^3g^8k^2+4a^2cg^2k^9-2a^2cg^4k^7+12a^2b^2hk^9+a^3cg^2k^8+2ac^2g^2k^{10}+$$

$$6ac^2g^4k^8+4ac^2g^6k^6+13a^2cg^2k^{10}-4a^2cg^4k^8-54a^2b^2hk^{10}-6a^3cg^2k^9-10a^2cg^2k^{11}+10a^2cg^4k^9+$$

$$60a^2b^2hk^{11}+13a^3cg^2k^{10}-5a^3cg^4k^8+8a^3b^2hk^{10}+2a^4cg^2k^9-a^2cg^2k^{12}-$$

$$2a^2cg^4k^{10}+8a^3cg^2k^{11}-2a^3cg^4k^9-32a^3b^2hk^{11}-9a^4cg^2k^{10}-5a^3cg^2k^{12}+3a^3cg^4k^{10}+32a^3b^2hk^{12}+$$

$$14a^4cg^2k^{11}-4a^4cg^4k^9+a^5cg^2k^{10}+a^4cg^2k^{12}-4a^5cg^2k^{11}+5a^5cg^2k^{12}-a^5cg^4k^{10}+8bc^3g^3k^2-$$

$$48ac^2h^2k^5+32ac^2h^3k^4-4bc^2g^3k^4-18bc^3g^3k^3-8b^2c^3g^4k+192ac^2h^2k^6-128ac^2h^3k^5+18bc^2g^3k^5-$$

$$8bc^2g^5k^3-34bc^3g^3k^4+28bc^3g^5k^2-4b^2cg^4k^4-4b^2c^2g^6k-16b^3c^2gk^5-16b^3c^2g^5k-$$

$$192ac^2h^2k^7+128ac^2h^3k^6-42bc^2g^3k^6+18bc^2g^5k^4+18bc^2g^7k^2+42bc^3g^3k^5+2bc^3g^5k^3-48a^2ch^2k^7+$$

$$80a^2ch^3k^6+24a^2c^2hk^7-4b^2cg^2k^7+12b^2cg^4k^5+40b^2c^3g^6k-24b^3cg^5k^3+72b^3c^2gk^6+$$

$$16b^4cg^2k^5-8b^4cg^4k^3+12bc^2g^3k^7-6bc^2g^5k^5+40bc^2g^7k^3+2bc^3g^3k^6+8bc^3g^5k^4+8bc^3g^7k^2+$$

$$192a^2ch^2k^8-320a^2ch^3k^7-96a^2c^2hk^8-6b^2cg^2k^8+4b^2cg^4k^6+28b^2cg^6k^4+36b^2c^2g^8k-$$

$$32b^3cg^3k^6+80b^3cg^5k^4-80b^3c^2gk^7+64b^3c^2g^7k+8b^4cg^2k^6+40b^4cg^4k^4-72b^4cg^6k^2-64b^4c^2g^6k+$$

$$96b^5cg^3k^4-96b^5cg^5k^2+8bc^2g^3k^8+26bc^2g^5k^6+20bc^2g^7k^4-144a^2ch^2k^9+320a^2ch^3k^8+96a^2c^2hk^9-$$

$$40a^3ch^2k^8+8a^3c^2hk^8-8b^2cg^2k^9-36b^2cg^4k^7+88b^2cg^6k^5-142b^3cg^3k^7+88b^3cg^5k^5+64b^3cg^7k^3+$$

$$24b^3c^2gk^8-112b^4cg^2k^7-288b^4cg^4k^5+344b^4cg^6k^3-384b^5cg^3k^5+384b^5cg^5k^3-96a^2ch^2k^{10}+$$

$$160a^3ch^2k^9-32a^3c^2hk^9+12b^2cg^2k^{10}+15b^2cg^4k^8-28b^2cg^6k^6+150b^3cg^3k^8-90b^3cg^5k^6+92b^3cg^7k^4+$$

$$128b^4cg^2k^8+192b^4cg^4k^6-104b^4cg^6k^4+88b^4cg^8k^2+416b^5cg^3k^6-480b^5cg^5k^4+224b^5cg^7k^2-$$

$$160a^3ch^2k^{10} + 32a^3c^2hk^{10} - 8b^2ch^2k^5 - 64b^2ch^3k^4 + 16b^2c^2hk^5 + 40b^2ch^2k^6+$$

$$256b^2ch^3k^5 - 64b^2c^2hk^6 - 136b^2ch^2k^7 - 256b^2ch^3k^6 + 112b^2c^2hk^7 + 176b^2ch^2k^8 - 96b^2c^2hk^8-$$

$$32b^3gh^2k^5 + 64b^3gh^3k^4 + 144b^3gh^2k^6 - 256b^3gh^3k^5 + 16b^3g^3hk^5 + 16b^4g^2hk^5 - 16b^4g^4hk^3-$$

$$24b^2g^2hk^8 + 8b^2g^4hk^6 - 160b^3gh^2k^7 + 256b^3gh^3k^6 - 80b^3g^3hk^6 + 16b^3g^5hk^4 - 104b^4g^2hk^6+$$

$$128b^4g^4hk^4 + 128b^5g^3hk^4 - 64b^5g^5hk^2 + 32b^2g^2hk^9 - 24b^2g^4hk^7 - 8b^3gh^2k^8 + 28b^3g^3hk^7-$$

$$24b^3g^5hk^5 + 208b^4g^2hk^7 - 368b^4g^4hk^5 + 64b^4g^6hk^3 - 512b^5g^3hk^5 + 256b^5g^5hk^3 - 8b^2g^2hk^{10}+$$

$$10b^2g^4hk^8 + 28b^3g^3hk^8 - 32b^3g^5hk^6 - 32b^4g^2hk^8 + 136b^4g^4hk^6 - 72b^4g^6hk^4+$$

$$576b^5g^3hk^6 - 384b^5g^5hk^4 + 64b^5g^7hk^2 - 8cg^2h^3k^2 - 32c^2g^2h^3k - 10c^3g^2hk^2+$$

$$8c^3g^2h^2k - 4cg^2h^2k^4 + 48cg^2h^3k^3 - 144cg^2h^4k^2 + 4c^2g^2hk^4 - 64c^2g^2h^4k + 28c^3g^2hk^3 + 16cg^2h^2k^5-$$

$$104cg^2h^3k^4 + 224cg^2h^4k^3 - 160cg^2h^5k^2 - 8cg^4h^2k^3 + 40cg^4h^3k^2 - 12c^2g^2hk^5+$$

$$8c^2g^4hk^3 - 16c^2g^4h^3k - 8c^3g^2hk^4 - 16c^3g^4hk^2 + 24c^3g^4h^2k + 52cg^2h^2k^6 - 64cg^2h^3k^5 + 16cg^2h^4k^4-$$

$$16cg^4h^2k^4 + 16cg^4h^3k^3 - 4c^2g^2hk^6 + 4c^2g^4hk^4 - 18c^2g^6hk^2 + 32c^2g^6h^2k - 12c^3g^2hk^5+$$

$$16c^3g^4hk^3 - 40cg^2h^2k^7 + 40cg^2h^3k^6 + 40cg^4h^2k^5 - 24cg^4h^3k^4 - 8c^2g^2hk^7 + 28c^2g^4hk^5 - 36c^2g^6hk^3+$$

$$2c^3g^2hk^6 - 12c^3g^4hk^4 - 2c^3g^6hk^2 - 4cg^2h^2k^8 - 8cg^4h^2k^6 - 4c^2g^2hk^8 - 12c^2g^4hk^6-$$

$$8c^2g^6hk^4 - 8achk^9 + 16achk^{10} + 8achk^{11} + 8a^2b^2c^2k^8 - 32a^2b^2c^2k^9 + 32a^2b^2c^2k^{10}+$$

$$96b^2c^2g^4h^2 - 96b^2c^2g^6h^2 - 2a^2b^2g^2k^9 + 8a^2b^3g^3k^7 + 9a^2b^2g^2k^{10} - 4a^2b^2g^4k^8-$$

$$44a^2b^3g^3k^8 - 32a^2b^4g^2k^8+$$

$$24a^2b^4g^4k^6 + 2a^3b^2g^2k^9 + 4a^2b^2g^2k^{11} + 90a^2b^3g^3k^9 - 20a^2b^3g^5k^7 + 128a^2b^4g^2k^9 - 96a^2b^4g^4k^7-$$

$$10a^3b^2g^2k^{10} + 8a^3b^3g^3k^8 - 6a^2b^2g^2k^{12} + 3a^2b^2g^4k^{10} - 26a^2b^3g^3k^{10} + 10a^2b^3g^5k^8-$$

$$136a^2b^4g^2k^{10} + 128a^2b^4g^4k^8 - 24a^2b^4g^6k^6 + 18a^3b^2g^2k^{11} - 4a^3b^2g^4k^9 - 32a^3b^3g^3k^9+$$

$$a^4b^2g^2k^{10} - 4a^3b^2g^2k^{12} + a^3b^2g^4k^{10} + 40a^3b^3g^3k^{10} - 8a^3b^3g^5k^8 - 4a^4b^2g^2k^{11} + 5a^4b^2g^2k^{12}-$$

$$a^4b^2g^4k^{10} + 4a^2c^2g^2k^6 - a^2c^2g^4k^4 + 2a^2c^3g^2k^5 - 16a^2c^2g^2k^7 - 6a^2c^3g^2k^6 - 2a^2c^3g^4k^4 -$$

$$24a^2b^2h^2k^8 + 20a^2c^2g^2k^8 + a^2c^2g^6k^4 + 2a^2c^3g^2k^7 + 6a^2c^3g^4k^5 + 4a^3c^2g^2k^7 + 96a^2b^2h^2k^9 -$$

$$2a^2c^2g^2k^9 - 2a^2c^2g^4k^7 + 8a^2c^2g^6k^5 + 2a^2c^3g^2k^8 - 2a^2c^3g^4k^6 + 2a^2c^3g^6k^4 - 15a^3c^2g^2k^8 -$$

$$2a^3c^2g^4k^6 - 96a^2b^2h^2k^{10} + 8a^2c^2g^2k^{10} - 3a^2c^2g^4k^8 + 6a^2c^2g^6k^6 + 16a^3c^2g^2k^9 + 2a^3c^2g^4k^7 +$$

$$a^4c^2g^2k^8 + 3a^3c^2g^2k^{10} - 2a^3c^2g^4k^8 + 2a^3c^2g^6k^6 - 4a^4c^2g^2k^9 + 5a^4c^2g^2k^{10} -$$

$$a^4c^2g^4k^8 - 4b^2c^2g^2k^4 + 8b^2c^2g^4k^2 + 16b^2c^3g^2k^3 + 34b^2c^2g^2k^5 - 36b^2c^2g^4k^3 - 40b^2c^3g^2k^4 +$$

$$16b^2c^3g^4k^2 + 32b^3c^2g^3k^3 - 24a^2c^2h^2k^6 - 50b^2c^2g^2k^6 - 6b^2c^2g^4k^4 + 44b^2c^2g^6k^2 +$$

$$16b^2c^3g^2k^5 - 48b^2c^3g^4k^3 - 48b^3c^2g^3k^4 - 8b^3c^2g^5k^2 + 48b^4c^2g^2k^4 - 48b^4c^2g^4k^2 + 96a^2c^2h^2k^7 -$$

$$14b^2c^2g^2k^7 - 46b^2c^2g^4k^5 + 76b^2c^2g^6k^3 + 8b^2c^3g^2k^6 + 48b^2c^3g^4k^4 + 16b^2c^3g^6k^2 -$$

$$168b^3c^2g^3k^5 + 88b^3c^2g^5k^3 - 192b^4c^2g^2k^5 + 192b^4c^2g^4k^3 - 96a^2c^2h^2k^8 + 2b^2c^2g^2k^8 +$$

$$122b^2c^2g^4k^6 + 26b^2c^2g^6k^4 + 36b^2c^2g^8k^2 + 184b^3c^2g^3k^6 - 16b^3c^2g^5k^4 + 80b^3c^2g^7k^2 +$$

$$208b^4c^2g^2k^6 - 240b^4c^2g^4k^4 + 112b^4c^2g^6k^2 + 32b^2c^2h^2k^4 - 128b^2c^2h^2k^5 + 128b^2c^2h^2k^6 -$$

$$16b^2g^2h^3k^3 + 16b^2g^2h^4k^2 - 8b^2g^2h^2k^5 + 80b^2g^2h^3k^4 - 64b^2g^2h^4k^3 + 32b^3g^3h^2k^3 -$$

$$64b^3g^3h^3k^2 + 36b^2g^2h^2k^6 - 144b^2g^2h^3k^5 + 80b^2g^2h^4k^4 - 16b^2g^4h^2k^4 + 32b^2g^4h^3k^3 -$$

$$16b^2g^4h^4k^2 - 176b^3g^3h^2k^4 + 256b^3g^3h^3k^3 - 128b^4g^2h^2k^4 + 96b^4g^4h^2k^2 + 16b^2g^2h^2k^7 +$$

$$32b^2g^2h^3k^6 - 8b^2g^4h^3k^4 + 360b^3g^3h^2k^5 - 320b^3g^3h^3k^4 - 80b^3g^5h^2k^3 + 64b^3g^5h^3k^2 +$$

$$512b^4g^2h^2k^5 - 384b^4g^4h^2k^3 - 24b^2g^2h^2k^8 + 12b^2g^4h^2k^6 - 104b^3g^3h^2k^6 + 40b^3g^5h^2k^4 -$$

$$544b^4g^2h^2k^6 + 512b^4g^4h^2k^4 - 96b^4g^6h^2k^2 + 16c^2g^2h^2k^2 - 64c^2g^2h^2k^3 + 120c^2g^2h^3k^2 -$$

$$24c^3g^2h^2k^2 + 80c^2g^2h^2k^4 - 128c^2g^2h^3k^3 + 80c^2g^2h^4k^2 + 8c^3g^2h^2k^3 - 8c^2g^2h^2k^5 - 24c^2g^2h^3k^4 -$$

$$8c^2g^4h^2k^3 + 16c^2g^4h^3k^2 + 8c^3g^2h^2k^4 - 8c^3g^4h^2k^2 + 32c^2g^2h^2k^6 - 12c^2g^4h^2k^4 +$$

$$24c^2g^6h^2k^2 - 2ab^2ck^9 + 7ab^2ck^{10} - 16ab^4ck^8 - 8ab^2ck^{11} + 64ab^4ck^9 + 15ab^2ck^{12} -$$

$$64ab^4ck^{10}+160bcg^3h^4+8bc^2g^5h+16bc^3g^5h-160bcg^5h^4-8bc^2g^7h-16bc^3g^7h-160b^4cg^6h+$$

$$160b^4cg^8h+32ab^5gk^8-4ab^3gk^{11}-128ab^5gk^9-2ab^3gk^{12}+128ab^5gk^{10}+6acg^2k^9-$$

$$2acg^4k^7-5acg^2k^{10}+11acg^4k^8-16ab^2hk^{10}-32ab^4hk^8-2acg^2k^{11}-4acg^4k^9+$$

$$40ab^2hk^{11}+128ab^4hk^9-12ab^2hk^{12}-128ab^4hk^{10}-4bc^3gk^4-12ach^2k^6+$$

$$64ach^3k^5-80ach^4k^4+12ac^2hk^6+8ac^3hk^5+20bc^3gk^5-2bc^3g^5k+48ach^2k^7-256ach^3k^6+$$

$$320ach^4k^5-48ac^2hk^7-32ac^3hk^6+4bc^2g^7k-20bc^3gk^6+192ach^3k^7-$$

$$320ach^4k^6+32ac^2hk^8+40ac^3hk^7+4bcg^5k^5+4bc^2gk^8-4bc^3gk^7+18bc^3g^7k+$$

$$6a^2chk^8+8b^3cgk^7-96ach^2k^9+128ach^3k^8+32ac^2hk^9-16ac^3hk^8-4bcg^3k^8+26bcg^5k^6-$$

$$4bc^2gk^9+8bc^3gk^8-24a^2chk^9-12b^3cgk^8-32b^5cgk^6-12ach^2k^{10}-4ac^2hk^{10}-4bcg^3k^9-$$

$$8bcg^5k^7+16a^3chk^9+20b^3cgk^9-32b^4cg^8k+128b^5cgk^7-128b^5cg^7k+48a^2chk^{11}-$$

$$64a^3chk^{10}-8b^3cgk^{10}-128b^5cgk^8+6a^2chk^{12}+48a^3chk^{11}+10a^4chk^{10}+$$

$$32a^3chk^{12}-40a^4chk^{11}+40a^4chk^{12}+4b^2chk^7-14b^2chk^8+32b^4chk^6+$$

$$16b^2chk^9-128b^4chk^7-30b^2chk^{10}+128b^4chk^8-64b^5ghk^6+8b^3ghk^9+256b^5ghk^7+$$

$$4b^3ghk^{10}-256b^5ghk^8+32cg^2h^4k+128cg^2h^5k-64cg^4h^4k-4c^2g^6hk-12cg^2hk^7+$$

$$4cg^4hk^5-16c^3g^6hk+10cg^2hk^8-22cg^4hk^6+4cg^2hk^9+8cg^4hk^7-8abc^2gk^6-$$

$$4abcg^3k^6+36abc^2gk^7+8abc^3gk^6+18abcg^3k^7-4abcg^5k^5-38abc^2gk^8-16abc^3gk^7-$$

$$4a^2bcgk^8+14abcg^3k^8+6abcg^5k^6+8abc^3gk^8+32ab^3cgk^8+16a^2bcgk^9-28abcg^3k^9+56abcg^5k^7+$$

$$2abc^2gk^{10}-112ab^3cgk^9-8a^2bcgk^{10}-12a^3bcgk^9-6abcg^3k^{10}-12abcg^5k^8+$$

$$128ab^3cgk^{10}-20a^2bcgk^{11}+48a^3bcgk^{10}-6a^2bcgk^{12}-36a^3bcgk^{11}-10a^4bcgk^{10}-$$

$$26a^3bcgk^{12}+40a^4bcgk^{11}-40a^4bcgk^{12}+8ab^2chk^7-40ab^2chk^8+136ab^2chk^9-$$

$$176ab^2chk^{10}+32ab^3ghk^7-144ab^3ghk^8+160ab^3ghk^9+8ab^3ghk^{10}+4acg^2hk^6-16acg^2hk^7+$$

$$8acg^4hk^5 - 52acg^2hk^8 + 16acg^4hk^6 + 40acg^2hk^9 - 40acg^4hk^7 + 4acg^2hk^{10} + 8acg^4hk^8 -$$

$$16bcgh^2k^4 + 96bcgh^3k^3 - 160bcgh^4k^2 - 96bcg^3h^3k + 16bc^2ghk^4 + 64bcgh^2k^5 - 384bcgh^3k^4 +$$

$$640bcgh^4k^3 + 8bcg^3hk^4 - 640bcg^3h^4k - 72bc^2ghk^5 + 4bc^2g^5hk - 16bc^3ghk^4 - 32bcgh^2k^6 +$$

$$288bcgh^3k^5 - 640bcgh^4k^4 - 36bcg^3hk^5 + 8bcg^5hk^3 + 224bcg^5h^3k + 76bc^2ghk^6 + 32bc^3ghk^5 -$$

$$64bc^3g^5hk - 32b^3cg^5hk - 80bcgh^2k^7 + 208bcgh^3k^6 - 28bcg^3hk^6 - 12bcg^5hk^4 - 68bc^2g^7hk -$$

$$16bc^3ghk^6 - 64b^3cghk^6 - 24bcgh^2k^8 + 56bcg^3hk^7 - 112bcg^5hk^5 - 4bc^2ghk^8 + 224b^3cghk^7 +$$

$$160b^3cg^7hk + 640b^4cg^6hk + 12bcg^3hk^8 + 24bcg^5hk^6 - 256b^3cghk^8 - 32a^2b^2c^2g^2k^6 +$$

$$24a^2b^2c^2g^4k^4 + 128a^2b^2c^2g^2k^7 - 96a^2b^2c^2g^4k^5 - 136a^2b^2c^2g^2k^8 + 128a^2b^2c^2g^4k^6 - 24a^2b^2c^2g^6k^4 +$$

$$24a^2b^2g^2h^2k^6 - 96a^2b^2g^2h^2k^7 + 120a^2b^2g^2h^2k^8 - 24a^2b^2g^4h^2k^6 + 24a^2c^2g^2h^2k^4 - 96a^2c^2g^2h^2k^5 +$$

$$120a^2c^2g^2h^2k^6 - 24a^2c^2g^4h^2k^4 - 128b^2c^2g^2h^2k^2 + 512b^2c^2g^2h^2k^3 - 544b^2c^2g^2h^2k^4 +$$

$$512b^2c^2g^4h^2k^2 + 12abc^2g^3k^4 - 4abc^2g^5k^2 - 46abc^2g^3k^5 - 2abc^2g^5k^3 - 8abc^3g^5k^2 - 4ab^2cg^2k^6 +$$

$$4ab^2cg^4k^4 + 40abc^2g^3k^6 + 22abc^2g^5k^4 + 4abc^2g^7k^2 - 32abc^3g^3k^5 + 32abc^3g^5k^3 +$$

$$20ab^2cg^2k^7 - 44ab^2cg^4k^5 - 16ab^3cg^3k^5 + 16ab^3cg^5k^3 + 32ab^3c^2gk^6 + 4a^2bcg^3k^6 -$$

$$12a^2bc^2gk^7 - 38abc^2g^3k^7 + 30abc^2g^5k^5 + 34abc^2g^7k^3 + 32abc^3g^3k^6 + 8abc^3g^7k^2 -$$

$$32ab^2cg^2k^8 + 128ab^2cg^4k^6 - 20ab^2cg^6k^4 + 96ab^3cg^3k^6 - 160ab^3cg^5k^4 - 128ab^3c^2gk^7 +$$

$$112ab^4cg^2k^6 - 176ab^4cg^4k^4 + 80ab^4cg^6k^2 - 28a^2bcg^3k^7 + 58a^2bc^2gk^8 + 48abc^2g^3k^8 + 4abc^2g^5k^6 +$$

$$30abc^2g^7k^4 - 68ab^2cg^2k^9 + 46ab^2cg^4k^7 + 48ab^2cg^6k^5 - 280ab^3cg^3k^7 + 520ab^3cg^5k^5 -$$

$$80ab^3cg^7k^3 + 128ab^3c^2gk^8 - 448ab^4cg^2k^7 + 704ab^4cg^4k^5 - 320ab^4cg^6k^3 + 72a^2bcg^3k^8 -$$

$$20a^2bcg^5k^6 - 68a^2bc^2gk^9 + 48a^2b^3cgk^8 + 12a^3bcg^3k^7 - 8a^3bc^2gk^8 + 36ab^2cg^2k^{10} -$$

$$87ab^2cg^4k^8 + 90ab^2cg^6k^6 + 48ab^3cg^3k^8 - 144ab^3cg^5k^6 + 112ab^3cg^7k^4 + 464ab^4cg^2k^8 - 816ab^4cg^4k^6 +$$

$$496ab^4cg^6k^4 - 80ab^4cg^8k^2 + 20a^2bcg^3k^9 + 4a^2bcg^5k^7 + 6a^2bc^2gk^{10} - 192a^2b^3cgk^9 -$$

$$58a^3bcg^3k^8 + 32a^3bc^2gk^9 - 34a^2bcg^3k^{10} + 28a^2bcg^5k^8 + 192a^2b^3cgk^{10} + 102a^3bcg^3k^9-$$

$$28a^3bcg^5k^7 - 32a^3bc^2gk^{10} + 10a^4bcg^3k^8 - 4a^3bcg^3k^{10} + 6a^3bcg^5k^8 - 40a^4bcg^3k^9+$$

$$50a^4bcg^3k^{10} - 10a^4bcg^5k^8 + 96ab^2ch^2k^6 - 32ab^2c^2hk^6 - 384ab^2ch^2k^7 + 128ab^2c^2hk^7+$$

$$384ab^2ch^2k^8 - 128ab^2c^2hk^8 - 48a^2b^2chk^8 + 192a^2b^2chk^9 - 192a^2b^2chk^{10} + 8ab^2g^2hk^7-$$

$$96ab^3gh^2k^6 - 32ab^3g^3hk^5 - 36ab^2g^2hk^8 + 16ab^2g^4hk^6 + 384ab^3gh^2k^7 + 176ab^3g^3hk^6 + 128ab^4g^2hk^6-$$

$$96ab^4g^4hk^4 - 16ab^2g^2hk^9 - 384ab^3gh^2k^8 - 360ab^3g^3hk^7 + 80ab^3g^5hk^5 - 512ab^4g^2hk^7 + 384ab^4g^4hk^5+$$

$$48a^2b^3ghk^8 + 24ab^2g^2hk^{10} - 12ab^2g^4hk^8 + 104ab^3g^3hk^8 - 40ab^3g^5hk^6 + 544ab^4g^2hk^8-$$

$$512ab^4g^4hk^6 + 96ab^4g^6hk^4 - 192a^2b^3ghk^9 + 192a^2b^3ghk^{10} + 12acg^2h^2k^4 - 64acg^2h^3k^3+$$

$$80acg^2h^4k^2 - 16ac^2g^2hk^4 + 4ac^2g^4hk^2 - 8ac^3g^2hk^3 - 72acg^2h^2k^5 + 288acg^2h^3k^4 - 320acg^2h^4k^3+$$

$$64ac^2g^2hk^5 + 24ac^3g^2hk^4 + 8ac^3g^4hk^2 + 156acg^2h^2k^6 - 448acg^2h^3k^5 + 400acg^2h^4k^4-$$

$$60acg^4h^2k^4 + 128acg^4h^3k^3 - 80acg^4h^4k^2 - 80ac^2g^2hk^6 - 4ac^2g^6hk^2 - 8ac^3g^2hk^5 - 24ac^3g^4hk^3-$$

$$6a^2cg^2hk^6 + 96acg^2h^2k^7 - 32acg^2h^3k^6 - 24acg^4h^2k^5 + 8ac^2g^2hk^7 + 8ac^2g^4hk^5-$$

$$32ac^2g^6hk^3 - 8ac^3g^2hk^6 + 8ac^3g^4hk^4 - 8ac^3g^6hk^2 + 36a^2cg^2hk^7 - 60acg^2h^2k^8 + 36acg^4h^2k^6-$$

$$32ac^2g^2hk^8 + 12ac^2g^4hk^6 - 24ac^2g^6hk^4 - 78a^2cg^2hk^8 + 30a^2cg^4hk^6 - 16a^3cg^2hk^7 - 48a^2cg^2hk^9+$$

$$12a^2cg^4hk^7 + 72a^3cg^2hk^8 + 30a^2cg^2hk^{10} - 18a^2cg^4hk^8 - 112a^3cg^2hk^9 + 32a^3cg^4hk^7-$$

$$10a^4cg^2hk^8 - 8a^3cg^2hk^{10} + 40a^4cg^2hk^9 - 50a^4cg^2hk^{10} + 10a^4cg^4hk^8 + 16bcg^3h^2k^2-$$

$$48bc^2gh^2k^3 + 64bc^2gh^3k^2 - 24bc^2g^3hk^2 + 48bc^2g^3h^2k - 112bcg^3h^2k^3 + 464bcg^3h^3k^2 + 232bc^2gh^2k^4-$$

$$256bc^2gh^3k^3 + 92bc^2g^3hk^3 + 256bc^2g^3h^3k + 8b^2cg^2hk^4 - 8b^2cg^4hk^2 + 96b^2cg^4h^2k+$$

$$288bcg^3h^2k^4 - 816bcg^3h^3k^3 + 800bcg^3h^4k^2 - 80bcg^5h^2k^2 - 272bc^2gh^2k^5 + 256bc^2gh^3k^4-$$

$$80bc^2g^3hk^4 - 44bc^2g^5hk^2 + 96bc^2g^5h^2k + 64bc^3g^3hk^3 - 40b^2cg^2hk^5+$$

$$88b^2cg^4hk^3 + 1280b^2cg^4h^3k + 192b^3cgh^2k^4 + 32b^3cg^3hk^3 - 64b^3c^2ghk^4 + 80bcg^3h^2k^5+$$

$32bcg^3h^3k^4+16bcg^5h^2k^3-48bcg^5h^3k^2+24bc^2gh^2k^6+76bc^2g^3hk^5-60bc^2g^5hk^3-64bc^3g^3hk^4+$

$64b^2cg^2hk^6-256b^2cg^4hk^4+40b^2cg^6hk^2-288b^2cg^6h^2k-144b^2c^2g^6hk-768b^3cgh^2k^5-$

$192b^3cg^3hk^4+320b^3cg^5hk^2-1280b^3cg^5h^2k+256b^3c^2ghk^5+256b^3c^2g^5hk-224b^4cg^2hk^4+$

$352b^4cg^4hk^2-136bcg^3h^2k^6+112bcg^5h^2k^4-96bc^2g^3hk^6-8bc^2g^5hk^4-60bc^2g^7hk^2+136b^2cg^2hk^7-$

$92b^2cg^4hk^5-96b^2cg^6hk^3+768b^3cgh^2k^6+560b^3cg^3hk^5-1040b^3cg^5hk^3-256b^3c^2ghk^6+$

$896b^4cg^2hk^5-1408b^4cg^4hk^3-72b^2cg^2hk^8+174b^2cg^4hk^6-180b^2cg^6hk^4-96b^3cg^3hk^6+$

$288b^3cg^5hk^4-224b^3cg^7hk^2-928b^4cg^2hk^6+1632b^4cg^4hk^4-992b^4cg^6hk^2-2abcgk^{10}-4abcgk^{11}+$

$64ab^3c^2g^3k^4+32ab^3c^2g^5k^2+12a^2bc^2g^3k^5-120ab^2c^2g^2k^7+88ab^2c^2g^4k^5+72ab^2c^2g^6k^3+$

$256ab^3c^2g^3k^5-128ab^3c^2g^5k^3-54a^2bc^2g^3k^6-12a^2bc^2g^5k^4-22a^2b^2cg^2k^7+24a^2b^2cg^4k^5+$

$36ab^2c^2g^2k^8+28ab^2c^2g^4k^6+56ab^2c^2g^6k^4+24ab^2c^2g^8k^2-288ab^3c^2g^3k^6+192ab^3c^2g^5k^4-$

$32ab^3c^2g^7k^2+54a^2bc^2g^3k^7+24a^2bc^2g^5k^5+100a^2b^2cg^2k^8-142a^2b^2cg^4k^6-128a^2b^3cg^3k^6+$

$80a^2b^3cg^5k^4+8a^3bc^2g^3k^6+20a^2bc^2g^3k^8+4a^2bc^2g^5k^6+12a^2bc^2g^7k^4-110a^2b^2cg^2k^9+$

$322a^2b^2cg^4k^7-72a^2b^2cg^6k^5+512a^2b^3cg^3k^7-320a^2b^3cg^5k^5-32a^3bc^2g^3k^7-48a^3b^2cg^2k^8+$

$40a^3b^2cg^4k^6-64a^2b^2cg^2k^{10}-56a^2b^2cg^4k^8+46a^2b^2cg^6k^6-560a^2b^3cg^3k^8+448a^2b^3cg^5k^6-$

$80a^2b^3cg^7k^4+40a^3bc^2g^3k^8-8a^3bc^2g^5k^6+192a^3b^2cg^2k^9-160a^3b^2cg^4k^7-$

$200a^3b^2cg^2k^{10}+208a^3b^2cg^4k^8-40a^3b^2cg^6k^6+24ab^2g^2h^2k^5-32ab^2g^2h^3k^4-120ab^2g^2h^2k^6+$

$128ab^2g^2h^3k^5+96ab^3g^3h^2k^4+216ab^2g^2h^2k^7-160ab^2g^2h^3k^6-48ab^2g^4h^2k^5+32ab^2g^4h^3k^4-$

$384ab^3g^3h^2k^5-12a^2b^2g^2hk^7-48ab^2g^2h^2k^8+12ab^2g^4h^2k^6+480ab^3g^3h^2k^6-96ab^3g^5h^2k^4+$

$60a^2b^2g^2hk^8-48a^2b^3g^3hk^6-108a^2b^2g^2hk^9+24a^2b^2g^4hk^7+192a^2b^3g^3hk^7-8a^3b^2g^2hk^8+$

$24a^2b^2g^2hk^{10}-6a^2b^2g^4hk^8-240a^2b^3g^3hk^8+48a^2b^3g^5hk^6+32a^3b^2g^2hk^9-40a^3b^2g^2hk^{10}+$

$$8a^3b^2g^4hk^8 + 48ac^2g^2h^2k^3 - 32ac^2g^2h^3k^2 - 180ac^2g^2h^2k^4 + 128ac^2g^2h^3k^3 - 24ac^2g^4h^2k^2 +$$

$$192ac^2g^2h^2k^5 - 160ac^2g^2h^3k^4 + 24ac^2g^4h^2k^3 + 32ac^2g^4h^3k^2 + 48a^2cg^2h^2k^5 -$$

$$80a^2cg^2h^3k^4 - 24a^2c^2g^2hk^5 + 36ac^2g^2h^2k^6 - 24ac^2g^4h^2k^4 + 24ac^2g^6h^2k^2 - 216a^2cg^2h^2k^6 +$$

$$320a^2cg^2h^3k^5 + 90a^2c^2g^2hk^6 + 12a^2c^2g^4hk^4 + 336a^2cg^2h^2k^7 - 400a^2cg^2h^3k^6 - 96a^2cg^4h^2k^5 +$$

$$80a^2cg^4h^3k^4 - 96a^2c^2g^2hk^7 - 12a^2c^2g^4hk^5 + 40a^3cg^2h^2k^6 - 8a^3c^2g^2hk^6 + 24a^2cg^2h^2k^8 -$$

$$18a^2c^2g^2hk^8 + 12a^2c^2g^4hk^6 - 12a^2c^2g^6hk^4 - 160a^3cg^2h^2k^7 + 32a^3c^2g^2hk^7 + 200a^3cg^2h^2k^8 -$$

$$40a^3cg^4h^2k^6 - 40a^3c^2g^2hk^8 + 8a^3c^2g^4hk^6 - 216bc^2g^3h^2k^2 - 88b^2cg^2h^2k^3 + 384b^2cg^2h^3k^2 -$$

$$16b^2c^2g^2hk^3 + 216bc^2g^3h^2k^3 - 320bc^2g^3h^3k^2 + 400b^2cg^2h^2k^4 - 1536b^2cg^2h^3k^3 - 568b^2cg^4h^2k^2 -$$

$$56b^2c^2g^2hk^4 + 104b^2c^2g^4hk^2 - 384b^2c^2g^4h^2k - 512b^3cg^3h^2k^2 + 128b^3c^2g^3hk^2 +$$

$$80bc^2g^3h^2k^4 + 16bc^2g^5h^2k^2 - 440b^2cg^2h^2k^5 + 1600b^2cg^2h^3k^4 + 1288b^2cg^4h^2k^3 - 1664b^2cg^4h^3k^2 +$$

$$240b^2c^2g^2hk^5 - 176b^2c^2g^4hk^3 + 2048b^3cg^3h^2k^3 - 512b^3c^2g^3hk^3 - 256b^2cg^2h^2k^6 - 224b^2cg^4h^2k^4 +$$

$$184b^2cg^6h^2k^2 - 72b^2c^2g^2hk^6 - 56b^2c^2g^4hk^4 - 112b^2c^2g^6hk^2 - 2240b^3cg^3h^2k^4 +$$

$$1792b^3cg^5h^2k^2 + 576b^3c^2g^3hk^4 - 384b^3c^2g^5hk^2 + 16abcghk^6 - 64abcghk^7 + 32abcghk^8 +$$

$$80abcghk^9 + 24abcghk^{10} + 96abc^2g^3h^2k^2 - 384abc^2g^3h^2k^3 - 576ab^2cg^2h^2k^4 + 480ab^2cg^4h^2k^2 +$$

$$128ab^2c^2g^2hk^4 - 96ab^2c^2g^4hk^2 + 480abc^2g^3h^2k^4 - 96abc^2g^5h^2k^2 + 2304ab^2cg^2h^2k^5 -$$

$$1920ab^2cg^4h^2k^3 - 512ab^2c^2g^2hk^5 + 384ab^2c^2g^4hk^3 + 240a^2bcg^3h^2k^4 - 48a^2bc^2g^3hk^4 -$$

$$2400ab^2cg^2h^2k^6 + 2496ab^2cg^4h^2k^4 - 480ab^2cg^6h^2k^2 + 544ab^2c^2g^2hk^6 - 512ab^2c^2g^4hk^4 +$$

$$96ab^2c^2g^6hk^2 - 960a^2bcg^3h^2k^5 + 192a^2bc^2g^3hk^5 + 288a^2b^2cg^2hk^6 - 240a^2b^2cg^4hk^4 +$$

$$1200a^2bcg^3h^2k^6 - 240a^2bcg^5h^2k^4 - 240a^2bc^2g^3hk^6 + 48a^2bc^2g^5hk^4 - 1152a^2b^2cg^2hk^7 +$$

$$960a^2b^2cg^4hk^5 + 1200a^2b^2cg^2hk^8 - 1248a^2b^2cg^4hk^6 + 240a^2b^2cg^6hk^4 - 144abcgh^2k^5 +$$

$$320abcgh^3k^4 - 16abcg^3hk^4 + 48abc^2ghk^5 + 576abcgh^2k^6 - 1280abcgh^3k^5 + 112abcg^3hk^5 -$$

$$232abc^2ghk^6 - 432abcgh^2k^7 + 1280abcgh^3k^6 - 288abcg^3hk^6 + 80abcg^5hk^4 + 272abc^2ghk^7 -$$

$$192ab^3cghk^6 + 72a^2bcghk^7 - 312abcgh^2k^8 - 80abcg^3hk^7 - 16abcg^5hk^5 - 24abc^2ghk^8 + 768ab^3cghk^7 -$$

$$288a^2bcghk^8 + 136abcg^3hk^8 - 112abcg^5hk^6 - 768ab^3cghk^8 + 216a^2bcghk^9 + 80a^3bcghk^8 +$$

$$156a^2bcghk^{10} - 320a^3bcghk^9 + 320a^3bcghk^{10} + 144abcg^3h^2k^3 - 320abcg^3h^3k^2 - 96abc^2gh^2k^4 -$$

$$48abc^2g^3hk^3 - 696abcg^3h^2k^4 + 1280abcg^3h^3k^3 + 384abc^2gh^2k^5 + 216abc^2g^3hk^4 + 48abc^2g^5hk^2 +$$

$$88ab^2cg^2hk^5 - 96ab^2cg^4hk^3 + 1224abcg^3h^2k^5 - 1600abcg^3h^3k^4 - 336abcg^5h^2k^3 +$$

$$320abcg^5h^3k^2 - 384abc^2gh^2k^6 - 216abc^2g^3hk^5 - 96abc^2g^5hk^3 - 400ab^2cg^2hk^6 + 568ab^2cg^4hk^4 +$$

$$512ab^3cg^3hk^4 - 320ab^3cg^5hk^2 - 240a^2bcgh^2k^6 - 72a^2bcg^3hk^5 + 48a^2bc^2ghk^6 - 48abcg^3h^2k^6 +$$

$$72abcg^5h^2k^4 - 80abc^2g^3hk^6 - 16abc^2g^5hk^4 - 48abc^2g^7hk^2 + 440ab^2cg^2hk^7 - 1288ab^2cg^4hk^5 +$$

$$288ab^2cg^6hk^3 - 2048ab^3cg^3hk^5 + 1280ab^3cg^5hk^3 + 960a^2bcgh^2k^7 + 348a^2bcg^3hk^6 -$$

$$192a^2bc^2ghk^7 + 256ab^2cg^2hk^8 + 224ab^2cg^4hk^6 - 184ab^2cg^6hk^4 + 2240ab^3cg^3hk^6 - 1792ab^3cg^5hk^4 +$$

$$320ab^3cg^7hk^2 - 960a^2bcgh^2k^8 - 612a^2bcg^3hk^7 + 168a^2bcg^5hk^5 + 192a^2bc^2ghk^8 - 80a^3bcg^3hk^6 +$$

$$24a^2bcg^3hk^8 - 36a^2bcg^5hk^6 + 320a^3bcg^3hk^7 - 400a^3bcg^3hk^8 + 80a^3bcg^5hk^6).$$

Tables A.1 and A.2 show the coefficients of the terms solely the in the variables k and h

respectively.

Table A.1: The coefficients of terms in the resultant polynomial that are solely terms in k

Terms	Coefficients
k	$-c(2c^3g^6 - 8b^2c^3g^4 - 4b^2c^2g^6 - 16b^3c^2g^5 + 40b^2c^3g^6 + 64b^3c^2g^7 + 36b^2c^2g^8 + ...)$
k^2	$-c(16b^6g^6 - 16b^6g^8 + 2c^4g^2 + 11c^3g^6 - 2c^4g^6 - 2c^3g^8 - 2ac^3g^4 + 3ac^3g^6 + ...)$
k^3	$-c(-16b^5g^7 - 64b^6g^6 - 16b^6g^8 - 4c^4g^2 - 2c^3g^4 + 4c^2g^6 + 4c^4g^4 + 4c^3g^6 + ...)$
k^4	$-c(c^5 + 32b^5g^5 - 48b^6g^4 + 40b^5g^7 - c^3g^2 + 14c^2g^6 - 5c^3g^6 + 32ab^5g^5 + ...)$
k^5	$-c(4c^4 - 8b^2c^3 - 8b^4g^4 + 24b^4g^6 + 152b^5g^5 + 192b^6g^4 + 2c^3g^2 - 32ab^4g^6 + ...)$
k^6	$-c(-6c^4 3ac^3 + 32b^2c^3 - 16b^4c^2 + 8b^3g^5 + +28b^4g^4 + 16b^5g^3 + 48b^6g^2 + ...)$
k^7	$-c(c^3 + 4c^4 + 12ac^3 - 8b^4c + 4cg^4 - 2a^2c^3 - 2b^2c^2 - 40b^2c^3 + 64b^4c^2 + 32b^3g^5 + ...)$
k^8	$-c(-16b^6 - 4c^3 - c^4 - 14ac^3 + 24b^4c + 16b^5g - 2cg^4 - 3a^2c^2 + 8a^2c^3 + 8b^2c^2 + ...)$
k^9	$-64b^6c - 2c^4 - 4ac^3 - 4ac^4 + 8b^4c^2 - 12a^2c^3 + 10a^2c^4 + 4a^3c^3 + 10b^2c^3 + ...$
k^{10}	$64b^6c + c^3 + 4ab^4c$
k^{11}	$-2a^2c^2$
k^{12}	$3b^4c - 16ab^4c$
k^{13}	$2ab^2c$
k^{14}	$-ab^2c$

Table A.2: The coefficients of terms in the resultant polynomial that are solely terms in h

Terms	Coefficients
h	$-c(4c^3g^4 - 6c^3g^6 + 2c^3g^8 + 48b^2c^2g^6 - 64b^3c^2g^5 + 8bc^2g^5 - 160b^4cg^6 + 160b^4cg^8)$
h^2	$-c(-4c^2g^4 - 8c^3g^4 + 4c^2g^6 + 8c^3g^6 - 48bc^2g^5 + 320b^3cg^5 + 48bc^2g^7 - 320b^3cg^7 + ...)$
h^3	$-c(16c^2g^4 - 16c^2g^6 - 64bc^2g^3 - 320b^2cg^4 + 64bc^2g^5 + 320b^2cg^6)$
h^4	$-c(-16c^2g^4 + 160bcg^3 - 160bcg^5 + 16c^2g^2)$
h^5	$-32c^2g^4 + 32c^2g^2$

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BIOGRAPHICAL SKETCH

Fariba Khoshnasib-Zeinabad was born in Tehran, Iran. Her last name has two parts. The first part (Khoshnasib) has Indian origins as her father's side of the family has Indian origins and Zeinabad is the name of the town her father was born in. It's customary for Iranians to have last names that indicate their roots.

Fariba immigrated to the US when she was 21, having to quit her studies in Iran in the middle of her education as her visa application had taken years to go through and she couldn't miss her chance. She started out as a biotechnology major, while working in a hospital as a GI technician aiming to apply to medical school. Soon, she realized she was too emotional to be in healthcare, as seeing patients struggling with their health made Fariba feel powerless and disappointed. She decided to choose a profession in which she could train the physicians of future and at the same time do research that could potentially help patients.

She did some casual research in data analytics for clinical studies and supervised undergraduate research on dynamics of a long-lived pore in a liposome which can have potential applications in cancer therapy. She is going to expand on this research in the near future with her students.

She got her Master of Science in Engineering Mathematics from UT Dallas in 2010. She chose to have a gap of about 3 years to teach and tutor mathematics and see if it was the profession that she wanted. She then decided to apply to UT Dallas for a PhD in mathematics in 2013. During that time she also audited and took some statistics and data science courses at UTD and on Coursera. In 2016 she moved to Wisconsin after having been offered an opportunity to teach mathematics and statistics at the University of Wisconsin- Eau Claire. She decided to continue her PhD part time at that time. She is currently teaching at Earlham College. Dr. Dragovic, her PhD advisor supported her during this time through the rough times and challenges and she is very appreciative of him for not letting her give up so easily.

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