QUADRICS IN PSEUDO-EUCLIDEAN SPACES, INTEGRABLE BILLIARDS AND EXTREMAL POLYNOMIALS

by

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This work is dedicated to my parents, Somaly Essi and Kwadzo Adabrah, my son, Kokou Samuel Adabrah, my family members and my friends.

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Anani Komla Adabrah, PhD
The University of Texas at Dallas, 2019

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We study the geometry of confocal quadrics in pseudo-Euclidean spaces of dimensions 2, 3, and 4, respectively. Along with the notion of geometric quadrics, we also investigate the relativistic quadrics which provide tools for further investigations of billiard dynamics. The geometric quadrics of a confocal pencil and their types in pseudo-Euclidean spaces do not share all of the usual properties with confocal quadrics in Euclidean spaces, including those necessary for applications in billiard dynamics and separable mechanical systems in general. For instance, in n-dimensional Euclidean space, there are n geometric types of quadrics, whereas in n-dimensional pseudo-Euclidean space, there are $n+1$ geometric types of quadrics. Relativistic quadrics enable us to define and use Jacobi coordinates in pseudo-Euclidean settings. In the study of periodic billiard trajectories, we distinguish two cases: trajectories which are periodic with respect to Cartesian coordinates, which are the usual periodic trajectories, and the so-called elliptic periodic trajectories, which are periodic with respect to Jacobi coordinates.

In the Minkowski plane, we derive necessary and sufficient conditions for periodic and elliptic periodic trajectories of billiards within an ellipse in terms of an underlying elliptic curve. We derive equivalent conditions in terms of polynomial equations as well. The corresponding polynomials are related to the classical extremal polynomials. We have indicated the

similarities and differences with respect to previously studied periodic billiard trajectories in Euclidean cases.

The classification of hypersurfaces of degree 2 in four-dimensional pseudo-Euclidean space has been done in signatures $(3,1)$ and $(2,2)$.

TABLE OF CONTENTS

LIST OF FIGURES

CHAPTER 1

INTRODUCTION

1.1 Confocal conics in the Euclidean plane

Consider

$$\varepsilon : \quad \frac{x^2}{a} + \frac{y^2}{b} = 1, \ \ a > b > 0 \tag{1.1}$$

an ellipse in the plane with a and b fixed.

$$C_\lambda : \quad \frac{x^2}{a-\lambda} + \frac{y^2}{b-\lambda} = 1, \quad \lambda \in \mathbb{R}. \tag{1.2}$$

The family C_λ has two non-degenerate subfamilies:

- For $\lambda < b$, C_λ is an ellipse.
- For $\lambda \in (b, a)$, C_λ is a hyperbola with the x-axis as the major axis.

The family C_λ is shown in Figure 1.1.

In the Euclidean plane, there are two types of conics in C_λ : ellipses and hyperbola with the

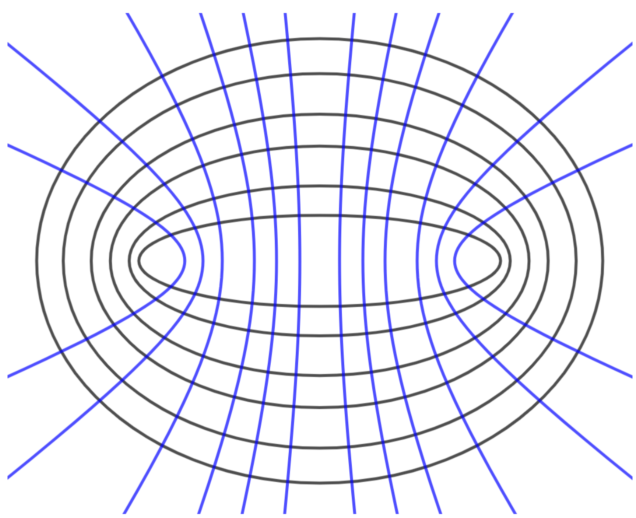

Figure 1.1: Family of confocal conics in the Euclidean plane.

x-axis as the major axis. We observed from Figure 1.1 that conics of the same type do not intersect one another, whereas conics of different types do intersect one another. Each point in the Euclidean plane is an intersection of two distinct conics.

Let

$$\frac{x^2}{a-\lambda} + \frac{y^2}{b-\lambda} = 1, \text{ where } \lambda \in \mathbb{R},\, a > b.$$

It follows that

$$\begin{aligned} &x^2(b-\lambda) + y^2(a-\lambda) = (b-\lambda)(a-\lambda), \\ &ab + \lambda^2 - (a+b)\lambda = x^2 b + y^2 a - (x^2+y^2)\lambda, \end{aligned}$$

which yields

$$\begin{aligned} &\lambda^2 - (a+b)\lambda + (x^2+y^2)\lambda + ab - (x^2 b + y^2 a) = 0, \\ &\lambda^2 + (x^2+y^2-a-b)\lambda + ab - bx^2 - ay^2 = 0. \end{aligned}$$

We computed the discriminant Δ as follows:

$$\begin{aligned} \Delta &= (x^2+y^2-a-b)^2 - 4(ab - bx^2 - ay^2), \\ &= x^4 + y^4 + a^2 + b^2 - 2ab + 2x^2y^2 + 2(b-a)x^2 + 2(a-b)y^2, \end{aligned}$$

where

$$\begin{aligned} x^4 + 2(b-a)x^2 + a^2 + b^2 - 2ab &= x^4 + 2(b-a)x^2 + (b-a)^2, \\ &= (x^2+b-a)^2, \end{aligned}$$

hence

$$\Delta = (x^2+b-a)^2 + y^4 + 2x^2y^2 + 2(a-b)y^2, \text{ where } a-b > 0, \text{ since } a > b,$$

therefore

$$\Delta > 0,$$

which means a point (x, y) in the plane is an intersection of two distinct conics.

The following section generalizes the two dimensional Euclidean plane.

1.2 Confocal quadrics and their types in the Euclidean space

Consider the family of confocal quadrics

$$\frac{x_1^2}{a_1 - \lambda} + \cdots + \frac{x_g^2}{a_g - \lambda} = 1, \quad \lambda \in \mathbb{R} \tag{1.3}$$

in a g-dimensional Euclidean space, where $a_1 > a_2 > \cdots > a_g > 0$.

The family (1.3) has the following properties:

- The intersection of exactly g quadrics from (1.3) of different geometric types defines a point of the space $\mathbb{E}^g$.

- The family (1.3) contains exactly g-geometric types of non-degenerate quadrics. Each type corresponds to one of the disjoint intervals $(-\infty, a_g), (a_g, a_{g-1}), \cdots, (a_2, a_1)$ of the parameter λ.

Definition 1.2.1. (**Jacobi coordinates**)
The Jacobi coordinates are the parameters $(\lambda_1, \cdots, \lambda_g)$ that correspond to the quadrics of (1.3) that contain a given point in $\mathbb{E}^g$.

In a g-dimensional Euclidean space, a general family of confocal quadrics contains g-geometric types of quadrics.

1.3 Billiards in the Euclidean plane

A mathematical billiard, is a dynamical system, where a particle (a dimensionless billiard ball) moves inside the domain (a billiard table) without a constraint. The particle moves in a straight line with a constant speed until it hits the boundary. The reflection off the boundary is elastic and subject to a billiard reflection law: *the impact and reflection angles are congruent to each other* [26, 40], see Figure 1.2

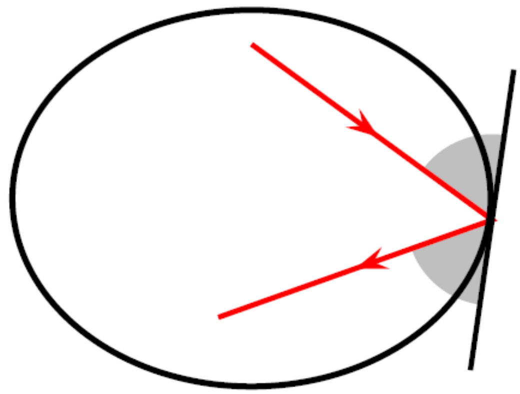

Figure 1.2: Billiard reflection law

At the impact point, the velocity of the particle decomposed into the normal and tangential components. Upon reflection, the normal component instantaneously changes sign, while the tangential one remains the same. The speed of the particle therefore does not change. Such billiard system in the Euclidean space is a good model for the motion of light rays, with mirror boundary [26, 40].

The above description of the billiard reflection does not only apply to the Euclidean geometry but it also applies to other geometries in particular the pseudo-Euclidean geometry.

1.4 Pseudo-Euclidean spaces

In this section, we defined a pseudo-Euclidean space and a pseudo-Euclidean distance between two points in the space.

Definition 1.4.1. A pseudo-Euclidean space $\mathbb{E}^{p,q}$ is a g-dimensional space $\mathbb{R}^g$ with a pseudo-Euclidean scalar product: $\langle x, y\rangle_{p,q} = x_1y_1 + \cdots + x_py_p - x_{p+1}y_{p+1} - \cdots - x_gy_g$, where $p, q \in \{1, \cdots, g-1\}$, $p + q = g$. The pair (p, q) is called the signature of the space $\mathbb{E}^{p,q}$.

Definition 1.4.2. A pseudo-Euclidean distance between two points x and y is defined by: $dist_{p,q}(x,y) = \sqrt{\langle x-y, x-y\rangle_{p,q}}$.

Note that the distance can take imaginary value since the scalar product can be negative.

Let l be a line in the pseudo-Euclidean space and let u be its vector, l is called:

- *space-like* if $\langle u,u\rangle_{p,q} > 0$,
- *time-like* if $\langle u,u\rangle_{p,q} < 0$,
- *light-like* if $\langle u,u\rangle_{p,q} = 0$.

Two vectors u and v are orthogonal in the pseudo-Euclidean space if $\langle u,v\rangle_{p,q} = 0$. A light-like line is therefore orthogonal to itself.

1.5 Confocal conics in the Minkowski plane

We study the properties of family of confocal conics in the Minkowski plane and derived focal properties of such families.

Let

$$\varepsilon_l : \quad \frac{x^2}{a} + \frac{y^2}{b} = 1, \;\; a > b > 0 \tag{1.4}$$

be an ellipse in the Minkowski plane with a and b fixed.

The associated family of confocal conics is

$$Co_\lambda : \quad \frac{x^2}{a-\lambda} + \frac{y^2}{b+\lambda} = 1, \quad \lambda \in \mathbb{R}. \tag{1.5}$$

The family Co_λ is shown in Figure 1.3 below.

The family of conics Co_λ has three non-degenerate subfamilies:

- For $\lambda < -b$, Co_λ is a hyperbola with the x-axis as the major axis.
- For $\lambda \in (-b, a)$, Co_λ is an ellipse.

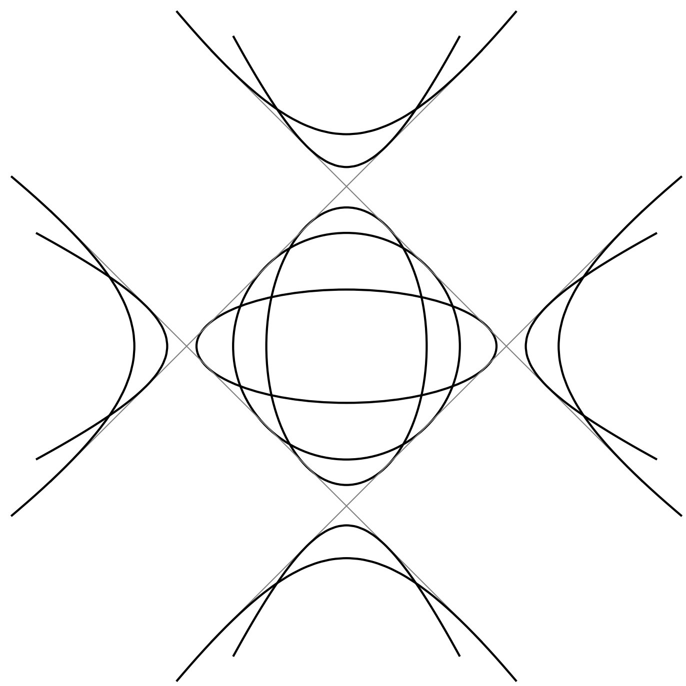

Figure 1.3: Family of confocal conics in the Minkowski plane.

- For $\lambda > a$, Co_λ is a hyperbola with the y-axis as major axis.

The quadrics Co_a, Co_b and Co_∞ corresponding to the y-axis, the x-axis and the line at infinity respectively are the degenerate quadrics. The three pairs of foci $F_1(\sqrt{a+b}, 0)$, $F_2(-\sqrt{a+b}, 0)$; $G_1(0, \sqrt{a+b})$, $G_2(0, -\sqrt{a+b}, 0)$; and $H_1(1 : -1 : 0)$, $H_2(1 : 1 : 0)$ are on the line at infinity.

Each non-degenerate member of the family Co_λ is tangent to the following four lines called the null lines.

$$x + y = \sqrt{a+b}, \;\; x + y = -\sqrt{a+b}$$

$$x - y = \sqrt{a+b}, \;\; x - y = -\sqrt{a+b}$$

These elementary results follow:

Proposition 1.5.1. 1) For each point on ellipse Co_λ, $\lambda \in (-b, a)$, either the sum or the difference of its Minkowski distances from the foci F_1 and F_2 is equal to $2\sqrt{a-\lambda}$; either the sum or the difference of the distances from the other pair of foci G_1 and G_2 is equal to $2i\sqrt{b+\lambda}$.

2) For each point on the hyperbola Co_λ, $\lambda \in (-\infty, -b)$, either the sum or difference of its Minkowski distances from the foci F_1 and F_2 is equal to $2\sqrt{a-\lambda}$; for the other pair of foci G_1 and G_2, it is equal to $2\sqrt{-b-\lambda}$.

3) For each point on the hyperbola Co_λ, $\lambda \in (a, \infty)$, either the sum or the difference of its Minkowski distances from the foci F_1 and F_2 is equal to $2i\sqrt{\lambda - a}$; for the other pair of foci G_1 and G_2, it is equal to $2i\sqrt{b+\lambda}$.

The proof is straightforward.

Proof. 1) Let X be a point on Co_λ for $\lambda \in (-b, a)$, therefore $a - \lambda > 0$.

Let us evaluate $dist_{1,1}(X, F_1) + dist_{1,1}(X, F_2)$. Set $dist_{1,1}(X, F_1) + dist_{1,1}(X, F_2) = d$.

$$\begin{aligned}
dist_{1,1}(X, F_1) + dist_{1,1}(X, F_2) &= d \\
\sqrt{(x-\sqrt{a+b})^2 - y^2} + \sqrt{(x+\sqrt{a+b})^2 - y^2} &= d \\
\sqrt{(x-\sqrt{a+b})^2 - y^2} &= d - \sqrt{(x+\sqrt{a+b})^2 - y^2} \\
d^2 + 4x\sqrt{a+b} &= 2d\sqrt{(x+\sqrt{a+b})^2 - y^2} \\
x^2[16(a+b) - 4d^2] + 4d^2y^2 &= 4d^2(a+b) - d^4 \\
x^2[16(a+b) - 4d^2] + 4d^2y^2 &= d^2[-d^2 + 4(a+b)] \\
\frac{x^2}{\frac{d^2}{4}} + \frac{y^2}{\frac{-d^2+4(a+b)}{4}} &= 1.
\end{aligned}$$

But since $X(x, y)$ is on Co_λ, it follows that

$$\frac{d^2}{4} = a - \lambda, \text{ with } a - \lambda > 0$$

$$\text{therefore } d = 2\sqrt{a-\lambda} \quad \text{as expected.}$$

Similarly

$$\begin{aligned}
dist_{1,1}(X,F_1) - dist_{1,1}(X,F_2) &= d\\
\sqrt{(x-\sqrt{a+b})^2 - y^2} - \sqrt{(x+\sqrt{a+b})^2 - y^2} &= d\\
\sqrt{(x-\sqrt{a+b})^2 - y^2} &= d + \sqrt{(x+\sqrt{a+b})^2 - y^2}\\
d^2 + 4x\sqrt{a+b} &= -2d\sqrt{(x+\sqrt{a+b})^2 - y^2}\\
x^2[16(a+b) - 4d^2] + 4d^2y^2 &= 4d^2(a+b) - d^4\\
x^2[16(a+b) - 4d^2] + 4d^2y^2 &= d^2[-d^2 + 4(a+b)]\\
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But since $X(x,y)$ is on Co_λ, it follows that:

$$\frac{d^2}{4} = a - \lambda, \quad \text{with } a - \lambda > 0$$

$$\text{therefore } d = 2\sqrt{a-\lambda} \text{ as we expected.}$$

Let us evaluate $dist_{1,1}(X,G_1) + dist_{1,1}(X,G_2)$. Set $dist_{1,1}(X,G_1) + dist_{1,1}(X,G_2) = d$.

$$\begin{aligned}
dist_{1,1}(X,G_1) + dist_{1,1}(X,G_2) &= d\\
\sqrt{x^2 - (y-\sqrt{a+b})^2} + \sqrt{x^2 - (y+\sqrt{a+b})^2} &= d\\
\sqrt{x^2 - (y-\sqrt{a+b})^2} &= d - \sqrt{x^2 - (y+\sqrt{a+b})^2}\\
d^2 - 4y\sqrt{a+b} &= 2d\sqrt{x^2 - (y+\sqrt{a+b})^2}\\
y^2[16(a+b) + 4d^2] - 4d^2x^2 &= -4d^2(a+b) - d^4\\
4d^2x^2 - y^2[16(a+b) + 4d^2] &= d^2[d^2 + 4(a+b)]\\
\frac{x^2}{\frac{d^2+4(a+b)}{4}} - \frac{y^2}{\frac{d^2}{4}} &= 1.
\end{aligned}$$

But since $X(x,y)$ is on the ellipse Co_λ, it follows that

$$-\frac{d^2}{4} = b + \lambda, \quad \text{with } b + \lambda > 0 \text{ and } d^2 < 0$$

$$\text{therefore } d = 2i\sqrt{b+\lambda}. \quad \text{as expected.}$$

Similarly

$$
\begin{aligned}
dist_{1,1}(X,G_1) - dist_{1,1}(X,G_2) &= d \\
\sqrt{x^2-(y-\sqrt{a+b})^2} - \sqrt{x^2-(y+\sqrt{a+b})^2} &= d \\
\sqrt{x^2-(y-\sqrt{a+b})^2} &= d + \sqrt{x^2-(y+\sqrt{a+b})^2} \\
d^2 - 4y\sqrt{a+b} &= -2d\sqrt{x^2-(y+\sqrt{a+b})^2} \\
y^2[16(a+b)+4d^2] - 4d^2x^2 &= -4d^2(a+b) - d^4 \\
4d^2x^2 - y^2[16(a+b)+4d^2] &= d^2[d^2+4(a+b)] \\
\frac{x^2}{\frac{d^2+4(a+b)}{4}} - \frac{y^2}{\frac{d^2}{4}} &= 1.
\end{aligned}
$$

But since $X(x,y)$ is on the ellipse Co_λ, it follows that

$$
-\frac{d^2}{4} = b+\lambda, \quad \text{with } b+\lambda>0 \text{ and } d^2<0
$$

therefore $d = 2i\sqrt{b+\lambda}$, as expected.

2) Let X be a point on Co_λ for $\lambda \in (-\infty,-b)$, therefore $b+\lambda<0$.

Let us evaluate $dist_{1,1}(X,F_1) + dist_{1,1}(X,F_2)$. Set $dist_{1,1}(X,F_1) + dist_{1,1}(X,F_2) = d$.

$$
\begin{aligned}
dist_{1,1}(X,F_1) + dist_{1,1}(X,F_2) &= d \\
\sqrt{(x-\sqrt{a+b})^2-y^2} + \sqrt{(x+\sqrt{a+b})^2-y^2} &= d \\
\sqrt{(x-\sqrt{a+b})^2-y^2} &= d - \sqrt{(x+\sqrt{a+b})^2-y^2} \\
d^2 + 4x\sqrt{a+b} &= 2d\sqrt{(x+\sqrt{a+b})^2-y^2} \\
x^2[16(a+b)-4d^2] + 4d^2y^2 &= 4d^2(a+b) - d^4 \\
x^2[16(a+b)-4d^2] + 4d^2y^2 &= d^2[-d^2+4(a+b)] \\
\frac{x^2}{\frac{d^2}{4}} - \frac{y^2}{\frac{d^2-4(a+b)}{4}} &= 1.
\end{aligned}
$$

But since $X(x,y)$ is on Co_λ, it follows that:

$$\frac{d^2-4(a+b)}{4}=-b-\lambda, \quad \text{with } b+\lambda<0$$

$$\text{therefore } d=2\sqrt{a-\lambda} \text{ as expected.}$$

Similarly

$$\begin{aligned}
dist_{1,1}(X,F_1)-dist_{1,1}(X,F_2)&=d\\
\sqrt{(x-\sqrt{a+b})^2-y^2}-\sqrt{(x+\sqrt{a+b})^2-y^2}&=d\\
\sqrt{(x-\sqrt{a+b})^2-y^2}&=d+\sqrt{(x+\sqrt{a+b})^2-y^2}\\
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Let us evaluate $dist_{1,1}(X,G_1)+dist_{1,1}(X,G_2)$. Set $dist_{1,1}(X,G_1)+dist_{1,1}(X,G_2)=d$.

$$\begin{aligned}
dist_{1,1}(X,G_1)+dist_{1,1}(X,G_2)&=d\\
\sqrt{x^2-(y-\sqrt{a+b})^2}+\sqrt{x^2-(y+\sqrt{a+b})^2}&=d\\
\sqrt{x^2-(y-\sqrt{a+b})^2}&=d-\sqrt{x^2-(y+\sqrt{a+b})^2}\\
d^2-4y\sqrt{a+b}&=2d\sqrt{x^2-(y+\sqrt{a+b})^2}\\
y^2[16(a+b)+4d^2]-4d^2x^2&=-4d^2(a+b)-d^4\\
4d^2x^2-y^2[16(a+b)+4d^2]&=d^2[d^2+4(a+b)]\\
\frac{x^2}{\frac{d^2+4(a+b)}{4}}-\frac{y^2}{\frac{d^2}{4}}&=1.
\end{aligned}$$

But since $X(x, y)$ is on the ellipse Co_λ, it follows that:

$$\frac{d^2 + 4(a+b)}{4} = a - \lambda, \quad \text{with } b + \lambda < 0$$
$$\text{therefore} \qquad d = 2\sqrt{-b-\lambda} \quad \text{as expected.}$$

Similarly

$$\begin{aligned}
dist_{1,1}(X, G_1) - dist_{1,1}(X, G_2) &= d \\
\sqrt{x^2 - (y - \sqrt{a+b})^2} - \sqrt{x^2 - (y + \sqrt{a+b})^2} &= d \\
\sqrt{x^2 - (y - \sqrt{a+b})^2} &= d + \sqrt{x^2 - (y + \sqrt{a+b})^2} \\
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-\frac{x^2}{\frac{-d^2}{4}} + \frac{y^2}{\frac{-d^2 + 4(a+b)}{4}} &= 1.
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$$\frac{-d^2 + 4(a+b)}{4} = b + \lambda, \quad \text{with } a - \lambda < 0$$

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\sqrt{x^2-(y-\sqrt{a+b})^2} + \sqrt{x^2-(y+\sqrt{a+b})^2} &= d \\
\sqrt{x^2-(y-\sqrt{a+b})^2} &= d - \sqrt{x^2-(y+\sqrt{a+b})^2} \\
d^2 - 4y\sqrt{a+b} &= 2d\sqrt{x^2-(y+\sqrt{a+b})^2} \\
y^2[16(a+b)+4d^2] - 4d^2x^2 &= -4d^2(a+b) - d^4 \\
4d^2x^2 - y^2[16(a+b)+4d^2] &= d^2[d^2+4(a+b)] \\
-\frac{x^2}{\frac{-d^2-4(a+b)}{4}} + \frac{y^2}{\frac{-d^2}{4}} &= 1
\end{aligned}$$

But since $X(x,y)$ is on the ellipse Co_λ, it follows that:

$$\frac{-d^2-4(a+b)}{4} = \lambda-a, \quad \text{with } a-\lambda<0$$

$$\text{therefore } d = 2i\sqrt{b+\lambda} \text{ as expected.}$$

Similarly

$$\begin{aligned}
dist_{1,1}(X,G_1) - dist_{1,1}(X,G_2) &= d \\
\sqrt{x^2-(y-\sqrt{a+b})^2} - \sqrt{x^2-(y+\sqrt{a+b})^2} &= d \\
\sqrt{x^2-(y-\sqrt{a+b})^2} &= d + \sqrt{x^2-(y+\sqrt{a+b})^2} \\
d^2 - 4y\sqrt{a+b} &= -2d\sqrt{x^2-(y+\sqrt{a+b})^2} \\
y^2[16(a+b)+4d^2] - 4d^2x^2 &= -4d^2(a+b) - d^4 \\
4d^2x^2 - y^2[16(a+b)+4d^2] &= d^2[d^2+4(a+b)] \\
-\frac{x^2}{\frac{-d^2-4(a+b)}{4}} + \frac{y^2}{\frac{-d^2}{4}} &= 1.
\end{aligned}$$

But since $X(x, y)$ is on the ellipse Co_λ, it follows that:

$$\frac{-d^2 - 4(a+b)}{4} = \lambda - a, \quad \text{with } a - \lambda < 0$$

$$\text{therefore } \; d = 2i\sqrt{b+\lambda} \; \text{ as expected.}$$

□

1.6 Billiards in the Minkowski plane

The description here is not different from the one in Section 1.3. We add the fact that if the normal component of the velocity of the particle is light-like i.e. belongs to the line that contains the tangential component, then the billiard reflection is not defined. Also two lines l and l' are billiard reflection of each other if their intersection point $l \cap l'$ belongs to the boundary of the conic ε and the vectors of l and l' are billiard reflections to each other.

1.7 Relativistics quadrics

The geometric quadrics of a confocal pencil and their types in the pseudo-Euclidean spaces do not satisfy all of the usual properties of confocal quadrics in the Euclidean spaces. for instance, in g-dimensional Euclidean space there are g-geometric types of quadrics while in g-dimensional pseudo-Euclidean space there are $(g+1)$-geometric types of quadrics.

1.7.1 Relativistic conics

Consider the two dimensional pseudo-Euclidean plane $\mathbb{E}^{1,1}$, called the Minkowski plane. We have already mentioned that a family of confocal conics in the Minkowski plane contains conics of three geometric types: ellipses, hyperbolas with the x-axis as the major axis and hyperbola with the y-axis as the major axis as shown in Figure 1.3. It is however more natural to consider the relativistic conics as analyzed by Birkhoff and Morris[10]. An account of that analysis is given in details.

Definition 1.7.1. Let x and y in $\mathbb{E}^{1,1}$ then $\langle x, y\rangle_{1,1} = x_1y_1 - x_2y_2$.

Definition 1.7.2. We define the Minkowki distance between two points x, y by:

$$dist_{1,1}(x,y) = \begin{cases} \sqrt{\langle x-y, x-y\rangle_{1,1}}, & \text{if } \langle x-y, x-y\rangle_{1,1} \geq 0, \\ i\sqrt{-\langle x-y, x-y\rangle_{1,1}}, & \text{if } \langle x-y, x-y\rangle_{1,1} < 0. \end{cases}$$

It follows that $dist_{1,1}(x, y) \in \mathbb{R}^+ \cup i\mathbb{R}^+$.

Consider two points $F_1(\sqrt{a+b}, 0)$ and $F_2(\sqrt{a+b}, 0)$ in the Minkowski plane and $c \in \mathbb{R}^+ \cup i\mathbb{R}^+$.

Definition 1.7.3. A relativistic ellipse is the set of points X satisfying:

$$dist_{1,1}(F_1, X) + dist_{1,1}(F_2, X) = 2c. \tag{1.6}$$

Definition 1.7.4. A relativistic hyperbola is the union of sets given by:

$$\begin{aligned} dist_{1,1}(F_1, X) - dist_{1,1}(F_2, X) = 2c, \\ dist_{1,1}(F_2, X) - dist_{1,1}(F_1, X) = 2c. \end{aligned} \tag{1.7}$$

We can clearly derive from the definition that Equation (1.6) and Equation (1.7) lead to:

$$Z: \ \frac{x^2}{c^2} + \frac{y^2}{a+b-c^2} = 1 \tag{1.8}$$

Equation (1.8) is easily obtained by setting $d = 2c$ in:

$$\frac{x^2}{\frac{d^2}{4}} + \frac{y^2}{\frac{-d^2+4(a+b)}{4}} = 1$$

which is an equation proved in Proposition 1.5.1.

Each geometrical conics Z is the union of pieces consisting of confocal relativistic conics (1.6) and (1.7). The relativistic conics can be described as follows:

Let $c \in \mathbb{R}^+$,

- if $< c < \sqrt{a+b}$, then (1.8) is a geometrical ellipse Co_{a-c^2} from the family (1.5) and the relativistic conics lie on it.

  If $c^2 > a+b-c^2$, the geometrical foci are at $(\pm\sqrt{2c^2-a-b}, 0)$ whereas the relativistic foci are at F_1 and F_2.

  If $c^2 < a+b-c^2$, the geometrical foci are at $(0, \pm\sqrt{a+b-2c^2}, 0)$ whereas the relativistic are at F_1 and F_2.

- if $c > \sqrt{a+b}$, then (1.8) is a geometrical hyperbola Co_{a-c^2} with x-axis as major axis from the family (1.5) and the relativistic conics lie on it.

  Since $c^2 > c^2 - a - b$, the geometrical foci are at $(\pm\sqrt{2c^2-a-b}, 0)$ whereas the relativistic foci are still at F_1 and F_2.

Let $c \in i\mathbb{R}^+$.

- For $c \in i\mathbb{R}^+$, $c^2 < 0$ and therefore (1.8) is a geometrical hyperbola Co_{a-c^2} with y-axis as major axis and the relativistic conics lie on it.

The following result follows:

Theorem 1.7.1. The relativistic conics that lie on each geometrical conic Z are geometrically tangent to the null lines through the foci F_1 and F_2.

Proof. Let
$$\frac{x^2}{c^2} + \frac{y^2}{a+b-c^2} = 1.$$
By implicit differentiation, one has the following:
$$\begin{aligned} 2x\frac{dx}{c^2} + 2y\frac{dy}{a+b-c^2} &= 0 \\ 2x\frac{dx}{c^2} &= -2y\frac{dy}{a+b-c^2} \\ \frac{dy}{dx} &= -\frac{x}{y}\frac{a+b-c^2}{c^2}. \end{aligned} \tag{1.9}$$

Let us now consider the geometric ellipses (1.8) where $0 < c < \sqrt{a+b}$ with slope

$$\lambda = \frac{dy}{dx} = -\frac{x}{y}\frac{a+b-c^2}{c^2}.$$

Let $(x, x-\sqrt{a+b})$ be the points on the null line $x - y = \sqrt{a+b}$ through F_1. Therefore

$$\begin{aligned}
\frac{x^2}{c^2} + \frac{(x-\sqrt{a+b})^2}{a+b-c^2} &= 1 \\
(a+b-c^2)x^2 + c^2(x^2 + (a+b) - 2x\sqrt{a+b}) &= c^2(a+b-c^2) \\
(a+b)x^2 - 2xc^2\sqrt{a+b} + c^4 &= 0 \quad \text{Therefore} \\
(c^2 - x\sqrt{a+b})^2 &= 0 \quad \text{i.e.} \\
c^2 &= x\sqrt{a+b}.
\end{aligned}$$

Hence the slope

$$\begin{aligned}
\frac{dy}{dx} &= -\frac{c^2}{\sqrt{a+b}}\frac{1}{\frac{c^2}{\sqrt{a+b}} - \sqrt{a+b}}\frac{a+b-c^2}{c^2} \\
\frac{dy}{dx} &= -\frac{c^2}{\sqrt{a+b}}\frac{\sqrt{a+b}}{c^2-(a+b)}\frac{a+b-c^2}{c^2} \\
\frac{dy}{dx} &= 1.
\end{aligned}$$

The slope of the ellipse (1.8) where $0 < c < \sqrt{a+b}$ equals that of the null line at their intersection point. We conclude that the null line is tangent to the ellipse at that point.

Let $(x, -x-\sqrt{a+b})$ be the points on the null line $x+y = -\sqrt{a+b}$ through F_2. Therefore

$$\begin{aligned}
\frac{x^2}{c^2} + \frac{(x+\sqrt{a+b})^2}{a+b-c^2} &= 1 \\
(a+b-c^2)x^2 + c^2(x^2 + (a+b) + 2x\sqrt{a+b}) &= c^2(a+b-c^2) \\
(a+b)x^2 + 2xc^2\sqrt{a+b} + c^4 &= 0 \quad \text{Therefore} \\
(c^2 + x\sqrt{a+b})^2 &= 0 \quad \text{i.e.} \\
c^2 &= -x\sqrt{a+b}.
\end{aligned}$$

Hence the slope

$$\frac{dy}{dx} = \frac{c^2}{\sqrt{a+b}} \frac{1}{\frac{c^2}{\sqrt{a+b}} - \sqrt{a+b}} \frac{a+b-c^2}{c^2}$$
$$\frac{dy}{dx} = \frac{c^2}{\sqrt{a+b}} \frac{\sqrt{a+b}}{c^2-(a+b)} \frac{a+b-c^2}{c^2}$$
$$\frac{dy}{dx} = -1.$$

The slope of the ellipse (1.8) where $0 < c < \sqrt{a+b}$ equals that of the null line at their intersection point. We conclude that the null line is tangent to the ellipse at that point. Tangency with the other null lines through the foci follows by symmetry in the axes. The case of the geometric hyperbolas (1.8) where $c > \sqrt{a+b}$ follows similarly. □

Corollary 1.7.1. [10] The relativistic foci of any conic are located at the intersections of the tangent null lines.

One can pass from relativistic ellipse (1.6) to a hyperbola by analytic continuation. more precisely, the following is true:

Theorem 1.7.2. [10] On the geometrical ellipses (1.8) where $0 < c < \sqrt{a+b}$, the segments where the slope is less than 45° are relativistic ellipses; those having a slope of more than 45° are relativistic hyperbola. On the geometrical hyperbolas (1.8) where $c > \sqrt{a+b}$, the reverse is true.

The above facts are graphically depicted in the following Figure 1.4 in which the solid lines represent the relativistic ellipses and the dotted lines represent the relativistic hyperbolas.

From Figure 1.4, and applying Theorem 1.7.1, one has the following summary:

- For $0 < c < \sqrt{a+b}$: the relativistic conics lie on the ellipse Co_{a-c^2} which consists of four arcs by touching points with the common four tangent null lines. The relativistic ellipse is the union of the two arcs intersecting the y-axis while the relativistic hyperbola is the union of the other two arcs.

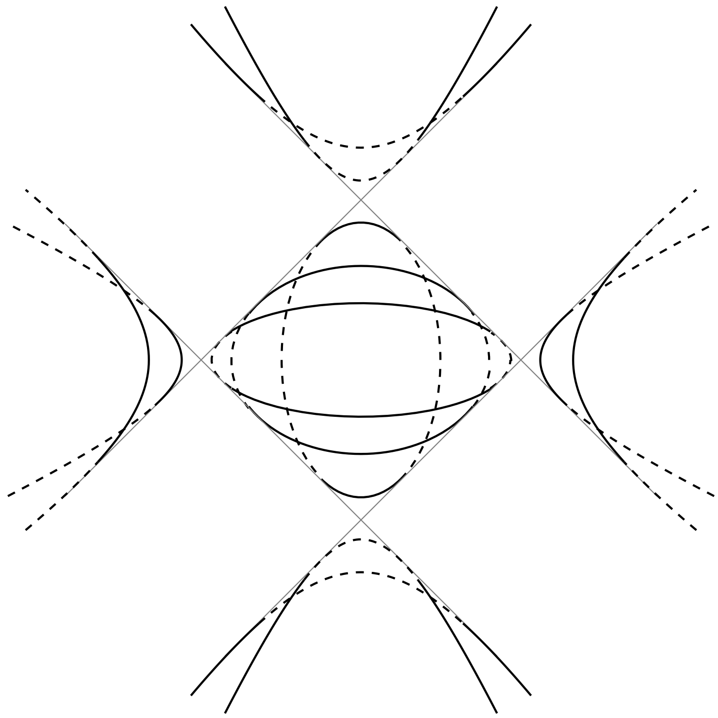

Figure 1.4: Family of confocal conics in the Minkowski plane. Solid lines represent relativistic ellipses, and dashed ones relativistic hyperbolas.

- For $c > \sqrt{a+b}$: the relativistic conics lie on the hyperbolas Co_{a-c^2} with the x-axis as the major axis. Each branch of the hyperbola is split into three arcs by touching points with common tangent null lines. Thus, the relativistic ellipse is the union of the two finite arcs, while the relativistic hyperbola is the union of the four infinite arcs.

- For $c \in i\mathbb{R}^+$: the relativistic conics lie on the hyperbola Co_{a-c^2} with the y-axis as the major axis. Each branch is split into three arcs by touching points with the common tangent null lines. The relativistic ellipse is the union of the four infinite arcs while the relativistic hyperbola is the union of the other two finite arcs.

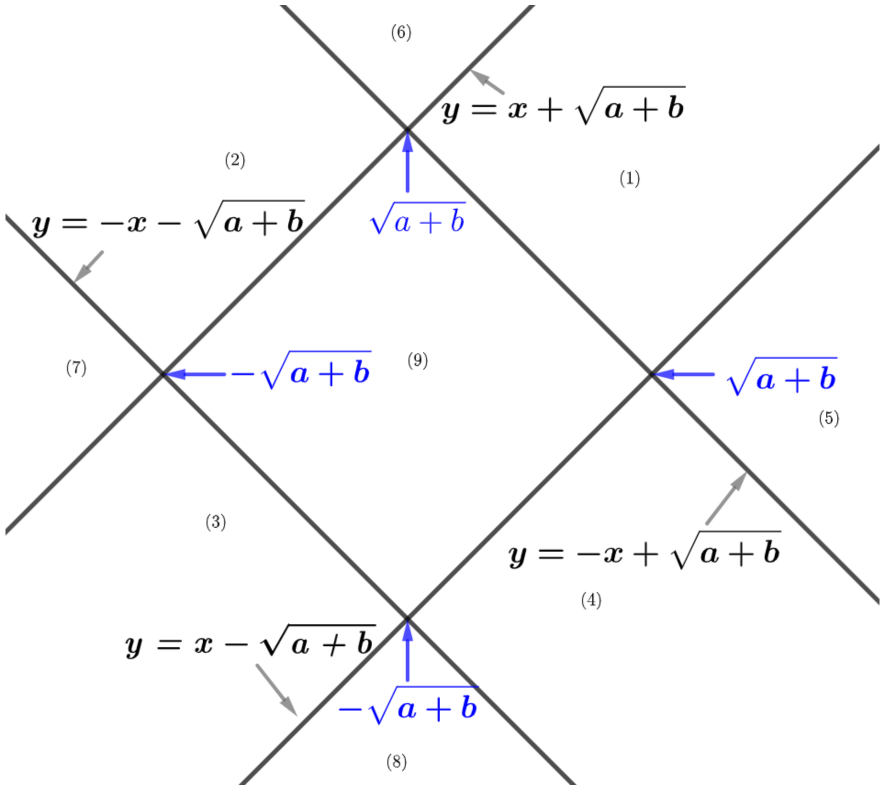

Figure 1.5: Minkowski plane divided into 9 regions by the four tangent lines to each conics.

Remark 1.7.1. In Figure 1.5, every point in the regions (5), (6), (7), (8) and (9) is an intersection of two distinct non-degenerate conics from the family (1.5) while every points in (1), (2), (3) and (4) is either an intersection of two imaginary conics or degenerates conics .

Let us check the above results.

Proof. We consider the following family

$$\frac{x^2}{a-\lambda} + \frac{y^2}{b+\lambda} = 1, \quad \lambda \in \mathbb{R},\, a > 0,\, b > 0.$$

It follows that:

$$x^2(b+\lambda) + y^2(a-\lambda) = (a-\lambda)(b+\lambda)$$

$$\lambda^2 + (x^2 - y^2 + b - a)\lambda + bx^2 + ay^2 - ab = 0.$$

The discriminant is

$$\begin{aligned}
\Delta &= (x^2 - y^2 + b - a)^2 - 4(bx^2 + ay^2 - ab) \\
&= x^4 + y^4 + (a+b)^2 - 2(a+b)x^2 - 2(a+b)y^2 - 2x^2y^2 \\
&= \begin{cases} (x^2 - a - b)^2 + y^4 - 2(a+b)y^2 - 2x^2y^2 \\ \text{or} \\ (y^2 - a - b)^2 + x^4 - 2(a+b)x^2 - 2x^2y^2 \end{cases} \\
&= \begin{cases} (x^2 - y^2 - a - b)^2 - 4(a+b)y^2, \\ \text{or} \\ (y^2 - x^2 - a - b)^2 - 4(a+b)x^2. \end{cases}
\end{aligned}$$

Let (x, y) **be in region** (1) then

$$\begin{cases} y > x - \sqrt{a+b}, \\ y > -x + \sqrt{a+b}, \\ y < x + \sqrt{a+b}. \end{cases}$$

case 1: $\quad 0 < x < \sqrt{a+b}$

$$-x + \sqrt{a+b} < y < x + \sqrt{a+b}$$

$$(-x + \sqrt{a+b})^2 < y^2 < (x + \sqrt{a+b})^2$$

$$(-x + \sqrt{a+b})^2 - (x^2 + a + b) < y^2 - (x^2 + a + b) < (x + \sqrt{a+b})^2 - (x^2 + a + b)$$

$$-2\sqrt{a+b}x < y^2 - (x^2 + a + b) < 2\sqrt{a+b}x$$

$$|\ y^2 - (x^2 + a + b)\ | < 2\sqrt{a+b}x$$

$$(y^2 - (x^2 + a + b))^2 < 4(a+b)x^2,$$

that is $\Delta < 0$.

case 2: $\quad \sqrt{a+b} < x$ **i.e.** $x - \sqrt{a+b} > 0$ **and** $x + \sqrt{a+b} > 0$

we have

$$x-\sqrt{a+b}<y<x+\sqrt{a+b}$$

$$(x-\sqrt{a+b})^2<y^2<(x+\sqrt{a+b})^2$$

$$-2\sqrt{a+b}x<y^2-(x^2+a+b)<2\sqrt{a+b}x$$

$$\mid y^2-(x^2+a+b)\mid<2\sqrt{a+b}x$$

$$(y^2-(x^2+a+b))^2<4(a+b)x^2,$$

that is $\Delta<0$,

hence $\forall(x,y)\in(1)$, $\Delta<0$.

Let (x,y) **be in region** (2) then

$$\begin{cases} y<x-\sqrt{a+b}, \\ y>-x-\sqrt{a+b}, \\ y>x+\sqrt{a+b}. \end{cases}$$

case 1: $-\sqrt{a+b}<x<0$ **i.e.** $-x+\sqrt{a+b}>0$ **and** $x+\sqrt{a+b}>0$.

$$x+\sqrt{a+b}<y<-x+\sqrt{a+b}$$

$$(x+\sqrt{a+b})^2<y^2<(-x+\sqrt{a+b})^2$$

$$2\sqrt{a+b}x<y^2-(x^2+a+b)<-2\sqrt{a+b}x$$

$$\mid y^2-(x^2+a+b)\mid<-2\sqrt{a+b}x$$

$$(y^2-(x^2+a+b))^2<4(a+b)x^2,$$

that is $\Delta < 0$.

case 2: $\quad x < -\sqrt{a+b}$ we have

$$-x - \sqrt{a+b} < y < -x + \sqrt{a+b}$$

$$(-x - \sqrt{a+b})^2 < y^2 < (-x + \sqrt{a+b})^2$$

$$2\sqrt{a+b}x < y^2 - (x^2 + a + b) < -2\sqrt{a+b}x$$

$$\mid y^2 - (x^2 + a + b) \mid < -2\sqrt{a+b}x$$

$$(y^2 - (x^2 + a + b))^2 < 4(a+b)x^2,$$

that is $\Delta < 0$.

hence $\forall (x, y) \in (2)$, $\Delta < 0$.

Let (x, y) **be in region** (3) then

$$\begin{cases} y < x + \sqrt{a+b}, \\ y < -x - \sqrt{a+b}, \\ y > x - \sqrt{a+b}. \end{cases}$$

case 1: $\quad -\sqrt{a+b} < x < 0$**,** $y < x + \sqrt{a+b}$ **always holds since** $y < 0$.

$$x - \sqrt{a+b} < y < -x - \sqrt{a+b}$$

$$(x + \sqrt{a+b})^2 < y^2 < (x - \sqrt{a+b})^2$$

$$2\sqrt{a+b}x < y^2 - (x^2 + a + b) < -2\sqrt{a+b}x$$

$$\mid y^2 - (x^2 + a + b) \mid < -2\sqrt{a+b}x$$

$$(y^2 - (x^2 + a + b))^2 < 4(a+b)x^2,$$

that is $\Delta < 0$.

case 2: $x < -\sqrt{a+b}$**,** $y < -x - \sqrt{a+b}$ **always holds** we have

$$x - \sqrt{a+b} < y < x + \sqrt{a+b}$$

$$(x + \sqrt{a+b})^2 < y^2 < (x + \sqrt{a+b})^2$$

$$2\sqrt{a+b}x < y^2 - (x^2 + a + b) < -2\sqrt{a+b}x$$

$$\mid y^2 - (x^2 + a + b) \mid < -2\sqrt{a+b}x$$

$$(y^2 - (x^2 + a + b))^2 < 4(a+b)x^2,$$

that is $\Delta < 0$.

hence $\forall (x, y) \in (3)$, $\Delta < 0$.

Let (x, y) **be in region** (4) then

$$\begin{cases} y < x - \sqrt{a+b}, \\ y < -x - \sqrt{a+b}, \\ y > -x - \sqrt{a+b}. \end{cases}$$

case 1: $0 < x < \sqrt{a+b}$**,** $y < -x + \sqrt{a+b}$ **always holds**

$$-x - \sqrt{a+b} < y < x - \sqrt{a+b}$$

$$(x - \sqrt{a+b})^2 < y^2 < (x + \sqrt{a+b})^2$$

$$-2\sqrt{a+b}x < y^2 - (x^2 + a + b) < 2\sqrt{a+b}x$$

$$\mid y^2 - (x^2 + a + b) \mid < 2\sqrt{a+b}x$$

$$(y^2 - (x^2 + a + b))^2 < 4(a+b)x^2,$$

that is $\Delta < 0$.

case 2: $> \sqrt{a+b}$ we have

$$-x - \sqrt{a+b} < y < -x + \sqrt{a+b}$$
$$(x - \sqrt{a+b})^2 < y^2 < (x + \sqrt{a+b})^2$$
$$-2\sqrt{a+b}x < y^2 - (x^2 + a + b) < 2\sqrt{a+b}x$$
$$\mid y^2 - (x^2 + a + b) \mid < 2\sqrt{a+b}x$$
$$(y^2 - (x^2 + a + b))^2 < 4(a+b)x^2,$$

that is $\Delta < 0$.

hence $\forall (x, y) \in (4)$, $\Delta < 0$.

Let (x, y) **be in region** (5) then

$$\begin{cases} y < x - \sqrt{a+b}, \\ y > -x + \sqrt{a+b}, \\ x > \sqrt{a+b}. \end{cases}$$

case 1: $-x + \sqrt{a+b} < y < 0$.

$$0 < y^2 < (\sqrt{a+b} - x)^2$$
$$-(x^2 + \sqrt{a+b}) < y^2 - (x^2 + \sqrt{a+b}) < (-x + \sqrt{a+b})^2 - (x^2 + \sqrt{a+b})$$
$$-(x^2 + \sqrt{a+b}) < y^2 - (x^2 + \sqrt{a+b}) < -2\sqrt{a+b}$$
$$4(a+b)x^2 < (y^2 - (x^2 + \sqrt{a+b}))^2 < (x^2 + a + b)^2 \text{ hence}$$
$$0 < (y^2 - (x^2 + \sqrt{a+b}))^2 - 4(a+b)x^2,$$

that is $\Delta > 0$.

case 2: $\quad 0 < y < x - \sqrt{a+b}$.

$$0 < y^2 < (\sqrt{a+b} - x)^2$$

$$-(x^2 + \sqrt{a+b}) < y^2 - (x^2 + \sqrt{a+b}) < (x - \sqrt{a+b})^2 - (x^2 + \sqrt{a+b})$$

$$-(x^2 + \sqrt{a+b}) < y^2 - (x^2 + \sqrt{a+b}) < -2\sqrt{a+b}$$

$$4(a+b)x^2 < (y^2 - (x^2 + \sqrt{a+b}))^2 < (x^2 + a + b)^2 \text{ hence}$$

$$0 < (y^2 - (x^2 + \sqrt{a+b}))^2 - 4(a+b)x^2,$$

that is $\Delta > 0$.

hence $\forall (x,y) \in (5)$, $\Delta > 0$. **Let** (x,y) **be in region** (6) then

$$\begin{cases} y > -x + \sqrt{a+b}, \\ y > x + \sqrt{a+b}, \\ y > \sqrt{a+b}. \end{cases}$$

$$|\, x \,| < y - \sqrt{a+b}$$

$$(|\, x \,| + \sqrt{a+b})^2 < y^2$$

$$(|\, x \,| + \sqrt{a+b})^2 - (x^2 + a + b) < y^2 - (x^2 + a + b)$$

$$2\sqrt{a+b}\,|\, x \,| < y^2 - (x^2 + a + b)$$

$$0 < (y^2 - (x^2 + a + b))^2 - 4(a+b)x^2,$$

that is $\Delta > 0$.

hence $\forall (x,y) \in (6)$, $\Delta > 0$.

Similarly **Let** (x,y) **be in region** (8) then

$$\begin{cases} y < -x - \sqrt{a+b}, \\ y < x - \sqrt{a+b}, \\ y < -\sqrt{a+b}. \end{cases}$$

$$| x | < -y - \sqrt{a+b}$$

$$x^2 < (y + \sqrt{a+b})^2$$

$$x^2 - (y^2 + \sqrt{a+b}) < (y + \sqrt{a+b})^2 - (y^2 + \sqrt{a+b})$$

$$x^2 - (y^2 + \sqrt{a+b}) < 2\sqrt{a+b}y$$

$$x^2 - (y^2 + \sqrt{a+b}) < 4y^2(a+b),$$

that is $\Delta > 0$.

hence $\forall (x,y) \in (8)$, $\Delta > 0$.

Let (x,y) **be in region** (7) then

$$\begin{cases} y < -x - \sqrt{a+b}, \\ y > x + \sqrt{a+b}, \\ x < -\sqrt{a+b}. \end{cases}$$

case 1: $\quad x + \sqrt{a+b} < y < 0$**.**

$$0 < y^2 < (\sqrt{a+b} + x)^2$$

$$-(x^2 + \sqrt{a+b}) < y^2 - (x^2 + \sqrt{a+b}) < (x + \sqrt{a+b})^2 - (x^2 + \sqrt{a+b})$$

$$-(x^2 + \sqrt{a+b}) < y^2 - (x^2 + \sqrt{a+b}) < 2\sqrt{a+b}$$

$$4(a+b)x^2 < (y^2 - (x^2 + \sqrt{a+b}))^2 < (x^2 + a + b)^2, \text{ hence}$$

$$0 < (y^2 - (x^2 + \sqrt{a+b}))^2 - 4(a+b)x^2,$$

that is $\Delta > 0$.

case 2: $0 < y < -x - \sqrt{a+b}$**.**

$$0 < y^2 < (\sqrt{a+b} + x)^2$$

$$-(x^2 + \sqrt{a+b}) < y^2 - (x^2 + \sqrt{a+b}) < (x + \sqrt{a+b})^2 - (x^2 + \sqrt{a+b})$$

$$-(x^2 + \sqrt{a+b}) < y^2 - (x^2 + \sqrt{a+b}) < 2\sqrt{a+b}$$

$$4(a+b)x^2 < (y^2 - (x^2 + \sqrt{a+b}))^2 < (x^2 + a + b)^2, \text{ hence}$$

$$0 < (y^2 - (x^2 + \sqrt{a+b}))^2 - 4(a+b)x^2,$$

that is $\Delta > 0$.

hence $\forall (x, y) \in (7)$, $\Delta > 0$.

Let (x, y) **be in region** (9)

case 1:

$$\begin{cases} y < x + \sqrt{a+b}, \\ y > -x - \sqrt{a+b}, \\ -\sqrt{a+b} < x < 0. \end{cases}$$

Subcase 1:

$$-x - \sqrt{a+b} < y < 0$$

$$0 < y^2 < (\sqrt{a+b} + x)^2$$

$$-(x^2 + \sqrt{a+b}) < y^2 - (x^2 + \sqrt{a+b}) < (x + \sqrt{a+b})^2 - (x^2 + \sqrt{a+b})$$

$$-(x^2 + \sqrt{a+b}) < y^2 - (x^2 + \sqrt{a+b}) < 2\sqrt{a+b}$$

$$4(a+b)x^2 < (y^2 - (x^2 + \sqrt{a+b}))^2 < (x^2 + a + b)^2, \text{ hence}$$

$$0 < (y^2 - (x^2 + \sqrt{a+b}))^2 - 4(a+b)x^2,$$

Subcase 2:

$$0 < y < x + \sqrt{a+b}$$

$$0 < y^2 < (\sqrt{a+b} + x)^2$$

$$-(x^2 + \sqrt{a+b}) < y^2 - (x^2 + \sqrt{a+b}) < (x + \sqrt{a+b})^2 - (x^2 + \sqrt{a+b})$$

$$-(x^2 + \sqrt{a+b}) < y^2 - (x^2 + \sqrt{a+b}) < 2\sqrt{a+b}$$

$$4(a+b)x^2 < (y^2 - (x^2 + \sqrt{a+b}))^2 < (x^2 + a + b)^2, \text{ hence}$$

$$0 < (y^2 - (x^2 + \sqrt{a+b}))^2 - 4(a+b)x^2,$$

case 2:

$$\begin{cases} y < -x + \sqrt{a+b}, \\ y > x - \sqrt{a+b}, \\ 0 < x < \sqrt{a+b}. \end{cases}$$

Subcase 1:

$$x - \sqrt{a+b} < y < 0$$

$$0 < y^2 < (-\sqrt{a+b} + x)^2$$

$$-(x^2 + \sqrt{a+b}) < y^2 - (x^2 + \sqrt{a+b}) < (x - \sqrt{a+b})^2 - (x^2 + \sqrt{a+b})$$

$$-(x^2 + \sqrt{a+b}) < y^2 - (x^2 + \sqrt{a+b}) < -2\sqrt{a+b}$$

$$4(a+b)x^2 < (y^2 - (x^2 + \sqrt{a+b}))^2 < (x^2 + a + b)^2, \text{ hence}$$

$$0 < (y^2 - (x^2 + \sqrt{a+b}))^2 - 4(a+b)x^2.$$

Subcase 2:

$$
\begin{aligned}
0 &< y < -x + \sqrt{a+b} \\
0 &< y^2 < (\sqrt{a+b} - x)^2 \\
-(x^2 + \sqrt{a+b}) &< y^2 - (x^2 + \sqrt{a+b}) < (x - \sqrt{a+b})^2 - (x^2 + \sqrt{a+b}) \\
-(x^2 + \sqrt{a+b}) &< y^2 - (x^2 + \sqrt{a+b}) < -2\sqrt{a+b} \\
4(a+b)x^2 &< (y^2 - (x^2 + \sqrt{a+b}))^2 < (x^2 + a + b)^2, \text{ hence} \\
0 &< (y^2 - (x^2 + \sqrt{a+b}))^2 - 4(a+b)x^2.
\end{aligned}
$$

Whence the confirmation of Remark 1.7.1.

□

Remark 1.7.2. We observed that all relativistic ellipses are disjoint from each other as well as all relativistic hyperbolas. Moreover, at the intersection point between a relativistic ellipse that is part of the geometric conic $Co_{\lambda 1}$ from the family (1.5) and a relativistic hyperbola that is part of the geometric conic $Co_{\lambda 2}$ from the family (1.5), $\lambda_1 < \lambda_2$ always holds, since $Co_{\lambda 1}$ always traces the interval $(-b, 0)$ and $Co_{\lambda 2}$ traces the interval $(0, a)$.

The above remark serves as a motivation for the introduction of relativistic types of quadrics in higher dimensional pseudo-Euclidean spaces.

The quadrics in three dimensional Minkowski space is completely studied in [23] by V. Dragović and M. Radnović. The Thesis focuses on the study of the two and four dimensional Minkowski spaces in chapter two and three respectively. The most interesting thing about the four dimensional Minkowski space is that, it has two signatures, $(3, 1)$ and $(2, 2)$. The space with signature $(3, 1)$ is fundamentally related to the study of relativity theory while the space with signature $(2, 2)$ has some good mathematical results.

CHAPTER 2

PERIODIC BILLIARDS WITHIN CONICS IN THE MINKOWSKI PLANE AND AKHIEZER AND ZOLOTAREV POLYNOMIALS.

This chapter is extracted from the following two submitted manuscripts: Anani Komla Adabrah, Vladimir Dragović and Milena Radnović for the Proceedings of the International Conference " Scientific Heritage of Sergey A. Chaplygin: nonholonomic mechanics, vortex structures and hydrodynamics", June 2-6, 2019, I. N. Ulianov Chuvash State University, Cheboksary, Russia, and Anani Komla Adabrah, Vladimir Dragović and Milena Radnović, Periodic billiards within conics in the Minkowski plane and Akhiezer categories: "math.AG, ...nlin.SI", "42 pages, 22 figures, 1 table" and "arXiv:1906.04911".

Introduction

Billiards within quadrics in pseudo-Euclidean spaces were studied in [33, 23, 24]. In [25, 26], the relationship between the billiards within quadrics in the Euclidean spaces and extremal polynomials has been studied. The aim of this chapter is to develop the connection between extremal polynomials and billiards in the Minkowski plane.

Apart from similarities with previously studied Euclidean space, see [26], there are also significant differences: for example, among the obtained extremal polynomials are such with winding numbers $(3, 1)$, which was never the case in the Euclidean setting.

Confocal conics

We recall the following:

$$\mathcal{E}\ :\ \frac{\mathsf{x}^2}{a}+\frac{\mathsf{y}^2}{b}=1, \tag{2.1}$$

is an ellipse in the plane, with a, b being fixed positive numbers and the associated family of confocal conics is

$$\mathcal{C}_\lambda\ :\ \frac{\mathsf{x}^2}{a-\lambda}+\frac{\mathsf{y}^2}{b+\lambda}=1,\quad \lambda\in\mathbf{R}. \tag{2.2}$$

The family is shown on Figure 2.1. We may distinguish the following three subfamilies in

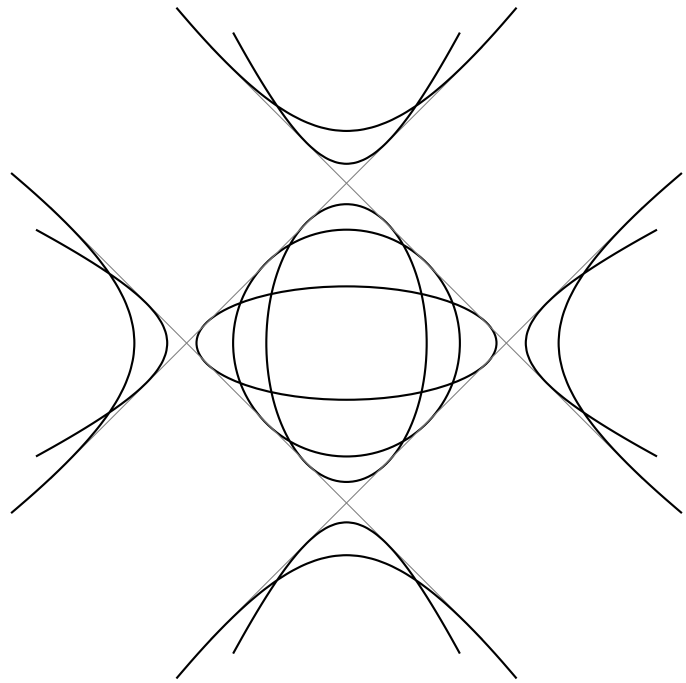

Figure 2.1: Family of confocal conics in the Minkowski plane.

the family $\mathcal{C}_\lambda$:

- for $\lambda \in (-b, a)$, conic $\mathcal{C}_\lambda$ is an ellipse;
- for $\lambda < -b$, conic $\mathcal{C}_\lambda$ is a hyperbola with x-axis as the major one;
- for $\lambda > a$, it is a hyperbola again, but now its major axis is y-axis.

In addition, there are three degenerated quadrics: $\mathcal{C}_a$, $\mathcal{C}_b$, $\mathcal{C}_\infty$ corresponding to y-axis, x-axis, and the line at the infinity respectively.

Each point inside the ellipse $\mathcal{E}$ has elliptic coordinates (λ_1, λ_2), such that $-b < \lambda_1 < 0 < \lambda_2 < a$.

The differential equation of the lines touching a given conic $\mathcal{C}_\gamma$ is:

$$\frac{d\lambda_1}{\sqrt{(a-\lambda_1)(b+\lambda_1)(\gamma-\lambda_1)}} + \frac{d\lambda_2}{\sqrt{(a-\lambda_2)(b+\lambda_2)(\gamma-\lambda_2)}} = 0. \tag{2.3}$$

2.1 Periodic trajectories

Section 2.1–2.6 deal with the trajectories with non-degenerate caustic $\mathcal{C}_\gamma$, which will mean that $\gamma \in \mathbf{R} \setminus \{-b, a\}$. Such trajectories are either space-like or time-like. The case of light-like trajectories, which correspond to the degenerate caustic $\mathcal{C}_\infty$ is considered separately, in Section 2.7.

The periodic trajectories of elliptical billiards in the Minkowski plane can be characterized in algebro-geometric terms using the underlying elliptic curve:

Theorem 2.1.1. The billiard trajectories within $\mathcal{E}$ with caustic $\mathcal{C}_\gamma$ are n-periodic if and only if $nQ_0 \sim nQ_\gamma$ on the elliptic curve:

$$\mathscr{C} \ : \ y^2 = \varepsilon(a - x)(b + x)(\gamma - x), \tag{2.4}$$

with Q_0 being a point of $\mathscr{C}$ corresponding to $x = 0$, and Q_γ the point corresponding to $x = \gamma$, and $\varepsilon = \operatorname{sign} \gamma$.

Proof. Along a billiard trajectory within $\mathcal{E}$ with caustic $\mathcal{C}_\gamma$, the elliptic coordinate λ_1 traces the segment $[b_0, 0]$, and λ_2 the segment $[0, b_1]$, where b_0 is the largest negative and b_1 the smallest positive member of the set $\{a, -b, \gamma\}$.

Case 1. If $\mathcal{C}_\gamma$ is an ellipse and $\gamma < 0$, then $b_0 = \gamma$, $b_1 = a$. The coordinate λ_1 takes value $\lambda_1 = \gamma$ at the touching points with the caustic and value $\lambda_1 = 0$ at the reflection points off the arcs of $\mathcal{E}$ where the restricted metric is time-like. On the other hand, λ_2 takes value $\lambda_2 = a$ at the intersections with y-axis, and $\lambda_2 = 0$ at the reflection points off the arcs of $\mathcal{E}$ where the restricted metric is space-like.

Case 2. If $\mathcal{C}_\gamma$ is an ellipse and $\gamma > 0$, then $b_0 = -b$, $b_1 = \gamma$. The coordinate λ_1 takes value $\lambda_1 = -b$ at the intersections with x-axis and value $\lambda_1 = 0$ at the reflection points off the arcs of $\mathcal{E}$ where the restricted metric is time-like. On the other hand, λ_2 takes value $\lambda_2 = \gamma$ at the touching points with the caustic, and $\lambda_2 = 0$ at the reflection points off the arcs of $\mathcal{E}$ where the restrictes metric is space-like.

Case 3. If $\mathcal{C}_\gamma$ is a hyperbola, then $b_0 = -b$, $b_1 = a$. The coordinate λ_1 takes value $\lambda_1 = -b$ at the intersections with x-axis and value $\lambda_1 = 0$ at the reflection points off the arcs of $\mathcal{E}$ where the restricted metric is time-like. On the other hand, λ_2 takes value $\lambda_2 = a$ at the intersections with y-axis, and $\lambda_2 = 0$ at the reflection points off the arcs of $\mathcal{E}$ where the restricted metric is space-like.

In each case, the elliptic coordinates change monotonously between their extreme values.

Consider an n-periodic billiard trajectory and denote by n_1 the number of reflections off time-like arcs and n_2 the number of reflections off space-like ones, $n_1 + n_2 = n$. Integrating (2.3) along the trajectory, we get:

$$n_1 \int_{b_0}^{0} \frac{d\lambda_1}{\sqrt{\varepsilon(a-\lambda_1)(b+\lambda_1)(\gamma-\lambda_1)}} + n_2 \int_{b_1}^{0} \frac{d\lambda_2}{\sqrt{\varepsilon(a-\lambda_2)(b+\lambda_2)(\gamma-\lambda_2)}} = 0, \qquad (2.5)$$

i.e.

$$n_1(Q_0 - Q_{c_1}) + n_2(Q_0 - Q_{b_1}) \sim 0.$$

In Case 1, this is equivalent to

$$n_1(Q_0 - Q_\gamma) + n_2(Q_0 - Q_a) \sim n(Q_0 - Q_\gamma),$$

since a closed trajectory crosses the y-axis even number of times, i.e n_2 must be even, and $2Q_a \sim 2Q_\gamma$.

Similarly, in Case 2, it follows since n_1 is even, and in Case 3 both n_1 and n_2 need to be even. $\square$

From the proof of Theorem 2.1.1, we have:

Corollary 2.1.1. The period of a closed trajectory with a hyperbola as caustic is even.

Theorem 2.1.2. The billiard trajectories within $\mathcal{E}$ with caustic $\mathcal{C}_\gamma$ are n-periodic if and only if:

$$C_2 = 0, \quad \begin{vmatrix} C_2 & C_3 \\ C_3 & C_4 \end{vmatrix} = 0, \quad \begin{vmatrix} C_2 & C_3 & C_4 \\ C_3 & C_4 & C_5 \\ C_4 & C_5 & C_6 \end{vmatrix} = 0, \dots \quad \text{for} \quad n = 3, 5, 7, \dots$$

$$B_3 = 0, \quad \begin{vmatrix} B_3 & B_4 \\ B_4 & B_5 \end{vmatrix} = 0, \quad \begin{vmatrix} B_3 & B_4 & B_5 \\ B_4 & B_5 & B_6 \\ B_5 & B_6 & B_7 \end{vmatrix} = 0, \dots \quad \text{for} \quad n = 4, 6, 8, \dots.$$

Here, we denoted:

$$\sqrt{\varepsilon(a-x)(b+x)(\gamma-x)} = B_0 + B_1 x + B_2 x^2 + \dots,$$
$$\frac{\sqrt{\varepsilon(a-x)(b+x)(\gamma-x)}}{\gamma - x} = C_0 + C_1 x + C_2 x^2 + \dots,$$

the Taylor expansions around $x = 0$.

Proof. Denote by Q_∞ the point of $\mathscr{C}$ (2.4) corresponding to $x = \infty$ and notice that

$$2Q_\gamma \sim 2Q_\infty. \tag{2.6}$$

Consider first n even. Because of (2.6), the condition $nQ_0 \sim nQ_\gamma$ is equivalent to $nQ_0 \sim nQ_\infty$, which is equivalent to the existence of a meromorphic function of $\mathscr{C}$ with the unique pole at Q_∞ and unique zero at Q_0, such that the pole and the zero are both of the multiplicity n. The basis of $\mathscr{L}(nQ_\infty)$ is:

$$1, x, x^2, \dots, x^{n/2}, y, xy, x^{n/2-2}y, \tag{2.7}$$

thus a non-trivial linear combination of those functions with a zero of order n at $x = 0$ exists if and only if:

$$\begin{vmatrix} B_{n/2+1} & B_{n/2} & \dots & B_3 \\ B_{n/2+2} & B_{n/2+1} & \dots & B_4 \\ \dots & & & \\ B_{n-1} & B_n & \dots & B_{n/2+1} \end{vmatrix} = 0.$$

Now, suppose n is odd. Because of (2.6), the condition $nQ_0 \sim nQ_\gamma$ is equivalent to $nQ_0 \sim (n-1)Q_\infty + Q_\gamma$, which is equivalent to the existence of a meromorphic function of $\mathscr{C}$ with only two poles: of order $n-1$ at Q_∞ and a simple pole at Q_γ, and unique zero at Q_0. The basis $\mathscr{L}((n-1)Q_\infty + Q_\gamma)$ is:

$$1, x, x^2, \dots, x^{(n-1)/2}, \frac{y}{\gamma - x}, \frac{xy}{\gamma - x}, \dots, \frac{x^{(n-1)/2-1}y}{\gamma - x}, \tag{2.8}$$

thus a non-trivial linear combination of those functions with a zero of order n at $x = 0$ exists if and only if:

$$\begin{vmatrix} C_{(n-1)/2+1} & C_{(n-1)/2} & \dots & C_2 \\ C_{(n-1)/2+2} & C_{(n-1)/2+1} & \dots & C_3 \\ \dots & & & \\ C_{n-1} & C_n & \dots & C_{(n-1)/2+1} \end{vmatrix} = 0.$$

$\square$

2.2 Examples of periodic trajectories with small periods: $3 \leq n \leq 10$

3-periodic trajectories

There is a 3-periodic trajectory of the billiard within (2.1), with a non-degenerate caustic $\mathcal{C}_{\lambda_0}$ in the Minkowski plane if and only if

- the caustic is an ellipse, i.e. $\lambda_0 \in (-b, a)$; and

- $C_2 = 0$.

We solve the equation

$$C_2 = \frac{3a^2b^2 + 2a^2b\lambda_0 - a^2\lambda_0^2 - 2ab^2\lambda_0 - 2ab\lambda_0^2 - b^2\lambda_0^2}{8(ab)^{\frac{3}{2}}\lambda_0^{\frac{5}{2}}} = 0, \tag{2.9}$$

which yields the following two solutions for the parameter λ_0 for the caustic:

$$\lambda_{01} = \frac{ab}{(a+b)^2}(a - b + 2\sqrt{a^2 + ab + b^2}). \tag{2.10}$$

$$\lambda_{02} = -\frac{ab}{(a+b)^2}(-a + b + 2\sqrt{a^2 + ab + b^2}). \tag{2.11}$$

Notice that both caustics $\mathcal{C}_{\lambda_{02}}$ and $\mathcal{C}_{\lambda_{01}}$ are ellipses since $-b < \lambda_{02} < 0 < \lambda_{01} < a$.

Two examples of a 3-periodic trajectories are shown in Figure 2.2.

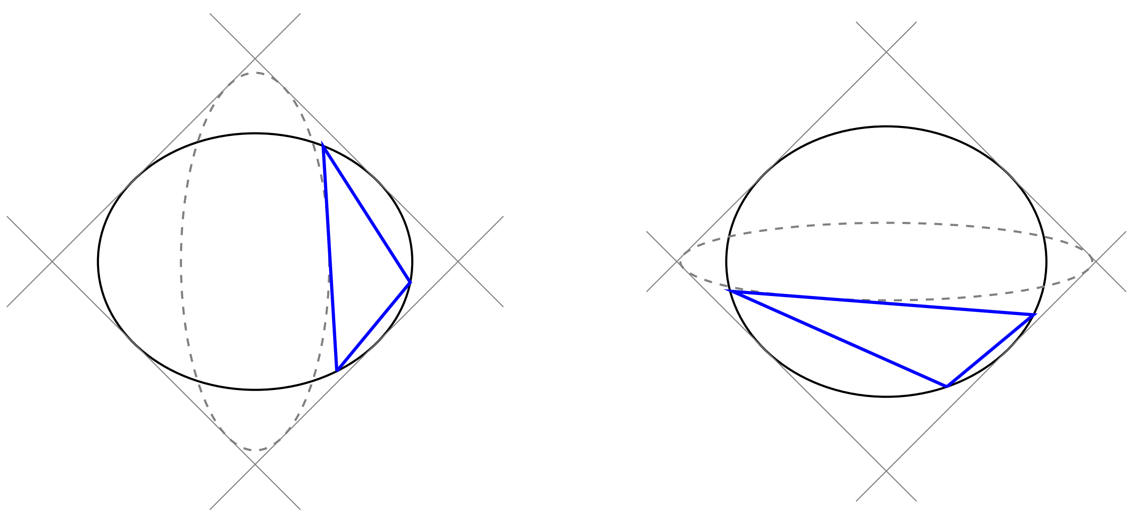

Figure 2.2: A 3-periodic trajectory with an ellipse along the y-axis as caustic ($a = 3$, $b = 2$, $\gamma \approx 2.332$) is shown on the left, while another trajectory with an ellipse along the x-axis as caustic ($a = 7$, $b = 5$, $\gamma \approx -4.589$) is on the right.

4-periodic trajectories

There is a 4-periodic trajectory of the billiard within (2.1), with a non-degenerate caustic $\mathcal{C}_{\lambda_0}$ in the Minkowski plane if and only if $B_3 = 0$. We solve the equation

$$B_3 = -\frac{(ab + a\lambda_0 + b\lambda_0)(ab + a\lambda_0 - b\lambda_0)(ab - a\lambda_0 - b\lambda_0)}{16(ab\lambda_0)^{\frac{5}{2}}} = 0, \tag{2.12}$$

which yields the following solutions for the parameter λ_0 for the caustic

$$\lambda_{01} = -\frac{ab}{a+b}, \qquad \lambda_{02} = -\frac{ab}{a-b}, \qquad \lambda_{03} = \frac{ab}{a+b}. \tag{2.13}$$

Since $\lambda_{01}, \lambda_{03} \in (-b, a)$ and $\lambda_{02} \notin (-b, a)$, therefore conic $\mathcal{C}_{\lambda_{02}}$ is a hyperbola whereas conics $\mathcal{C}_{\lambda_{01}}$ and $\mathcal{C}_{\lambda_{03}}$ are ellipses.

In Figure 2.3 and Figure 2.4, examples of a 4-periodic trajectories with each type of caustic are shown.

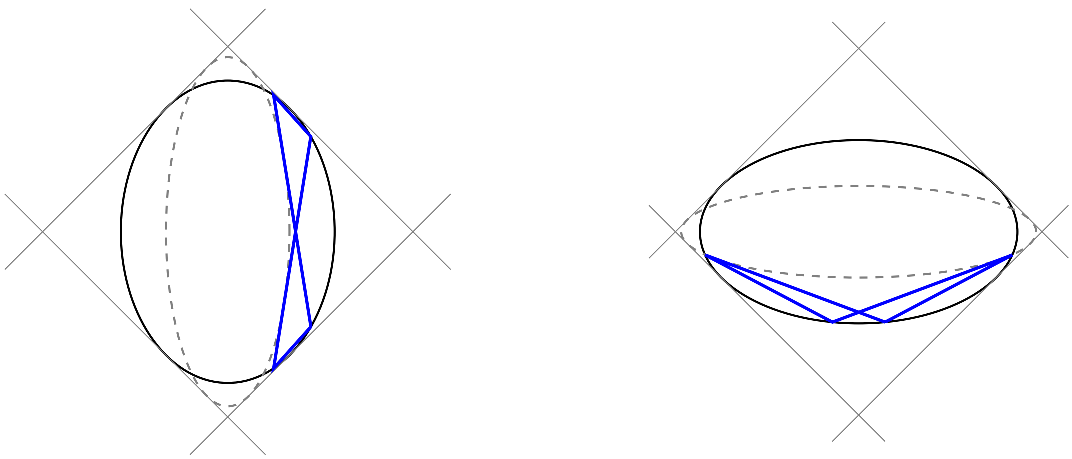

Figure 2.3: A 4-periodic trajectory with an ellipse along the y-axis as caustic ($a = 2$, $b = 4$, $\gamma = 4/3$) is shown on the left, while another trajectory with an ellipse along the x-axis as caustic ($a = 9$, $b = 3$, $\gamma = -9/4$) is on the right.

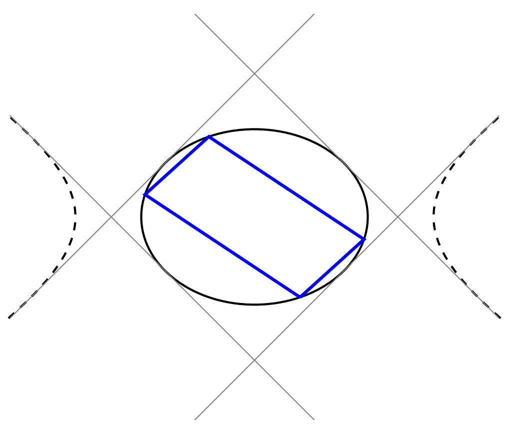

Figure 2.4: A 4-periodic trajectory with a hyperbola along the x-axis as caustic ($a = 5$, $b = 3$, $\gamma = -15/2$.

5-periodic trajectories

There is a 5-periodic trajectory of the billiard within (2.1), with a non-degenerate caustic $\mathcal{C}_{\lambda_0}$ in the Minkowski plane if and only if

- the caustic is an ellipse, i.e. $\lambda_0 \in (-b, a)$; and
- $C_2C_4 - C_3^2 = 0$.

We computed

$$\begin{aligned} C_2C_4 - C_3^2 =& \frac{2^{-10}}{a^5b^5\lambda_0^7}\Big((a+b)^6\,\lambda_0^6 - 2ab\,(-b+a)\,(-3b+a)\,(-b+3a)\,(a+b)^2\,\lambda_0^5 \\ & - a^2b^2\left(29a^2 - 54ab + 29b^2\right)(a+b)^2\,\lambda_0^4 - 36a^3b^3\,(-b+a)\,(a+b)^2\,\lambda_0^3 \\ & - a^4b^4\left(9a^2 + 34ab + 9b^2\right)\lambda_0^2 + 10a^5b^5\,(-b+a)\,\lambda_0 + 5a^6b^6\Big). \end{aligned} \tag{2.14}$$

Examples of a 5-periodic billiard trajectories are shown in Figure 2.5 and Figure 2.6.

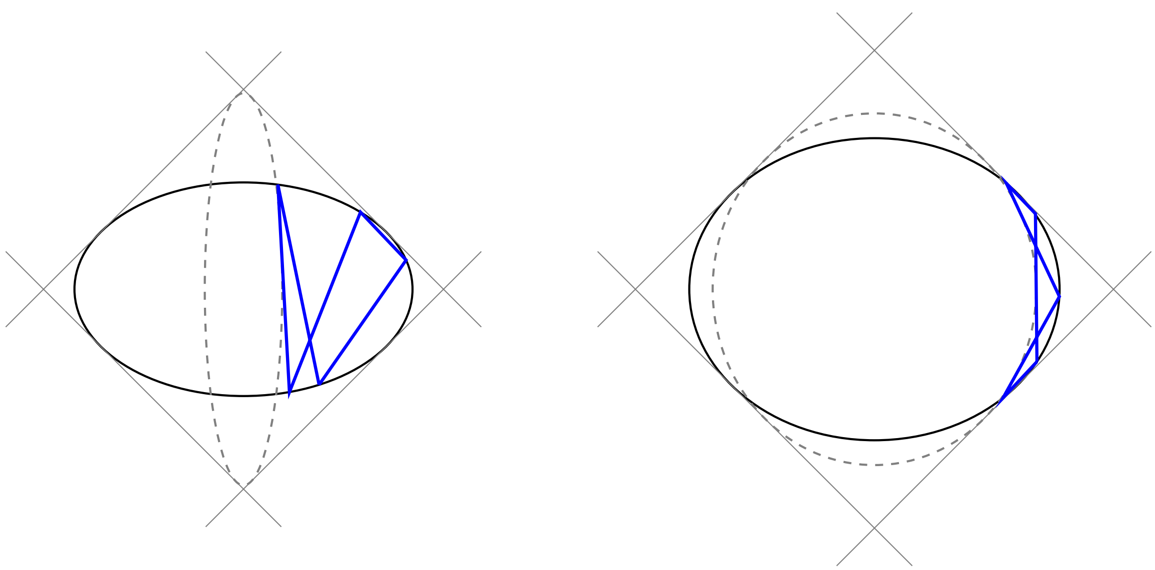

Figure 2.5: A 5-periodic trajectories with an ellipse along the y-axis as caustic. On the left, the particle is bouncing 4 times off the relativistic ellipse and once off relativistic hyperbola ($a = 5$, $b = 2$, $\gamma \approx 4.7375$), while on the right the billiard particle is reflected twice off relativistic ellipse and 3 times off relativistic hyperbola ($a = 6$, $b = 4$, $\gamma \approx 1.4205$).

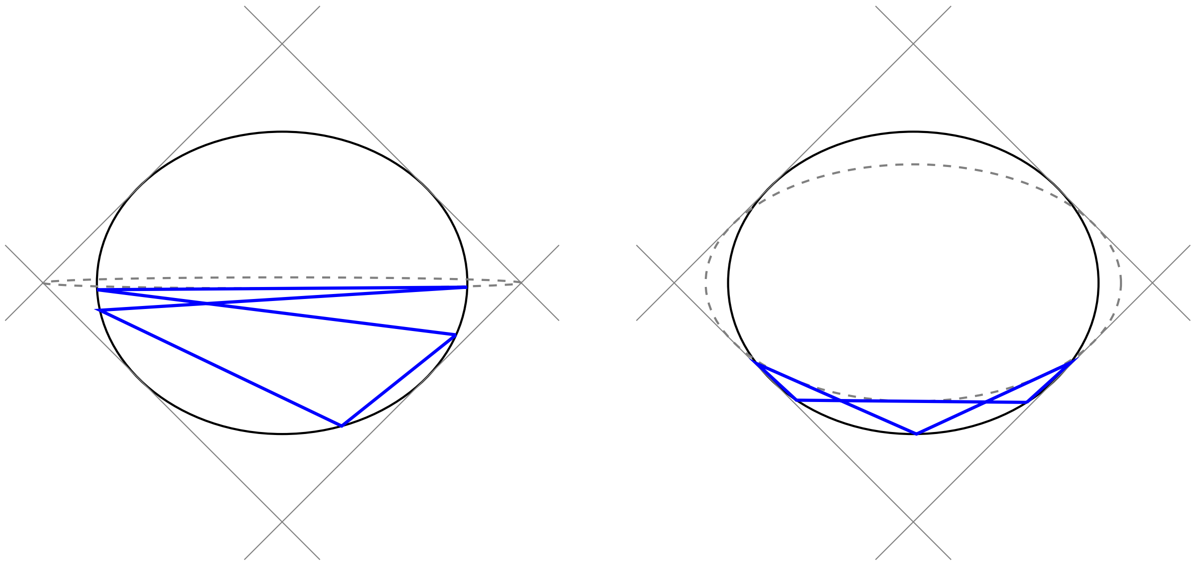

Figure 2.6: A 5-periodic trajectories with an ellipse along the x-axis as caustic. On the left, the particle is bouncing once off the relativistic ellipse and 4 times off relativistic hyperbola ($a = 6$, $b = 4$, $\gamma \approx -3.9947$), while on the right the billiard particle is reflected twice off relativistic hyperbola and 3 times off relativistic ellipse ($a = 6$, $b = 4$, $\gamma \approx -1.5413$).

6-periodic trajectories

There is a 6-periodic trajectory of the billiard within (2.1), with a non-degenerate caustic $\mathcal{C}_{\lambda_0}$ in the Minkowski plane if and only if $B_3B_5 - B_4^2 = 0$.

We computed

$$\begin{aligned} B_3B_5 - B_4^2 =& \frac{2^{-14}}{a^7b^7\lambda^7}\Big(-(a+b)^2\lambda_0^2 + 2ab(a-b)\lambda_0 + 3a^2b^2\Big) \\ & \times \big((a+b)(a-3b)\lambda_0^2 + 2ab(a+b)\lambda_0 + a^2b^2\big)\big((a+b)^2\lambda_0^2 + 2ab(a-b)\lambda_0 + a^2b^2\big) \\ & \times \big(-(a+b)(3a-b)\lambda_0^2 - 2ab(a+b)\lambda_0 + a^2b^2\big). \end{aligned} \tag{2.15}$$

Let us consider the condition

$$-(a+b)^2\lambda_0^2 + 2ab(a-b)\lambda_0. + 3a^2b^2 = 0$$

This produces 3-periodic trajectories as already studied previously.

The discriminant of the third factor $(a+b)^2\lambda_0^2 + 2ab(a-b)\lambda_0 + a^2b^2$ is $-16a^3b^3$ which is negative, the expression has therefore no real roots in λ_0.

Next, we consider

$$(a+b)(a-3b)\lambda_0^2 + 2ab(a+b)\lambda_0 + a^2b^2 = 0,$$

the above equation has two real solutions which are

$$\lambda_0 = \frac{ab}{(a+b)(a-3b)}\Big(-a - b \pm 2\sqrt{ab+b^2}\Big).$$

Finally we consider

$$-(a+b)(3a-b)\lambda_0^2 - 2ab(a+b)\lambda_0 + a^2b^2 = 0,$$

it has two real solutions

$$\lambda_0 = \frac{ab}{(a+b)(3a-b)}\Big(-a - b \pm 2\sqrt{ab+a^2}\Big).$$

An example of a 6-periodic trajectory with a hyperbola as caustic is shown in Figure 2.7.

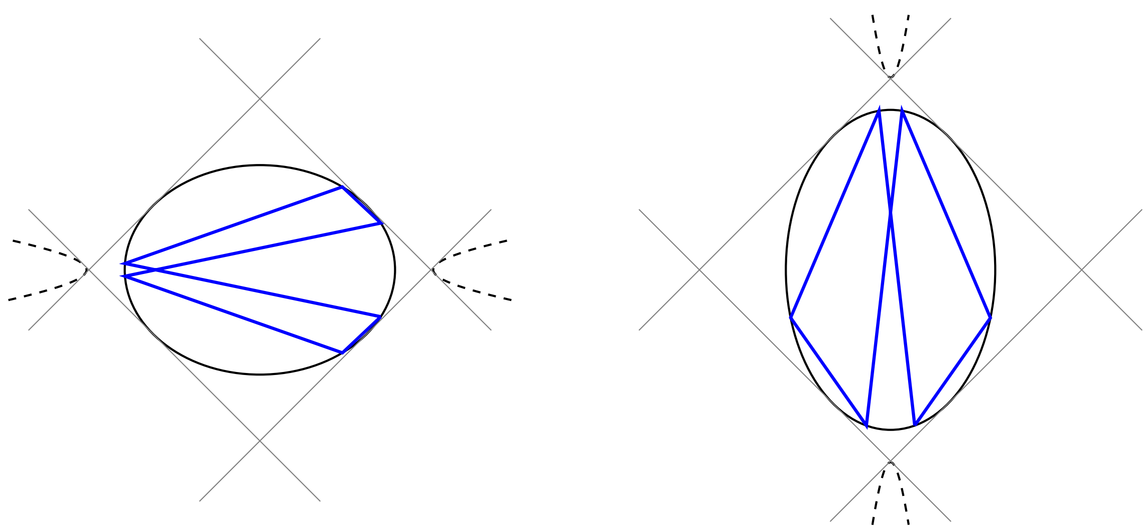

Figure 2.7: A 6-periodic trajectory with a hyperbola along the x-axis as caustic ($a = 5$, $b = 3$, $\gamma \approx -3.2264$ is shown on the left, while another trajectory with a hyperbola along the y-axis as caustic ($a = 3$, $b = 7$ and $\gamma \approx 3.1189$) is on the right. On the left, the particle bounces off the relativistic ellipse twice and 4 times the relativistic hyperbola while on the right the particle bounces off the relativistic ellipse 4 times and the relativistic hyperbola twice.

7-periodic trajectories

There is a 7-periodic trajectory of the billiard within (2.1), with a non-degenerate caustic $\mathcal{C}_{\lambda_0}$ in the Minkowski plane if and only if

- the caustic is an ellipse, i.e. $\lambda_0 \in (-b, a)$; and
- $$\begin{vmatrix} C_2 & C_3 & C_4 \\ C_3 & C_4 & C_5 \\ C_4 & C_5 & C_6 \end{vmatrix} = 0$$
  .

We computed

$$\begin{aligned}
&\begin{vmatrix} C_2 & C_3 & C_4 \\ C_3 & C_4 & C_5 \\ C_4 & C_5 & C_6 \end{vmatrix} = \\
&\frac{1}{(ab)^{2^{21}\frac{21}{2}}\lambda_0^{\frac{27}{2}}}\Big(-(a+b)^{12}\lambda_0^{12} + 4ab(a-b)(-3b+a)(-b+3a)\left(a^2-6ab+b^2\right) \\
&(a+b)^6{\lambda_0}^{11} + 2a^2b^2\left(59a^4-332a^3b+626a^2b^2-332ab^3+59b^4\right)(a+b)^6{\lambda_0}^{10} \\
&+ 28a^3b^3(a-b)\left(13a^2-38ab+13b^2\right)(a+b)^6{\lambda_0}^9 \\
&+ a^4b^4\left(7a^2+30ab+7b^2\right)\left(63a^4-84a^3b-38a^2b^2-84ab^3+63b^4\right) \\
&(a+b)^2{\lambda_0}^8 - 8a^5b^5(a-b)\left(21a^4-420a^3b-50a^2b^2-420ab^3+21b^4\right)(a+b)^2{\lambda_0}^7 \\
&- 12a^6b^6\left(105a^4-420a^3b+422a^2b^2-420ab^3+105b^4\right)(a+b)^2{\lambda_0}^6 \\
&- 24a^7b^7(a-b)\left(75a^2-106ab+75b^2\right)(a+b)^2{\lambda_0}^5 \\
&- 3a^8b^8\left(437a^2-726ab+437b^2\right)(a+b)^2{\lambda_0}^4 \\
&- 4a^9b^9(a-b)\left(121a^2+250ab+121b^2\right){\lambda_0}^3 - 14a^{10}b^{10}\left(3a^2+14ab+3b^2\right){\lambda_0}^2 \\
&+ 28a^{11}b^{11}(a-b)\lambda + 7a^{12}b^{12}\Big).
\end{aligned} \tag{2.16}$$

Examples of a 7-periodic trajectories are shown in Figure 2.8.

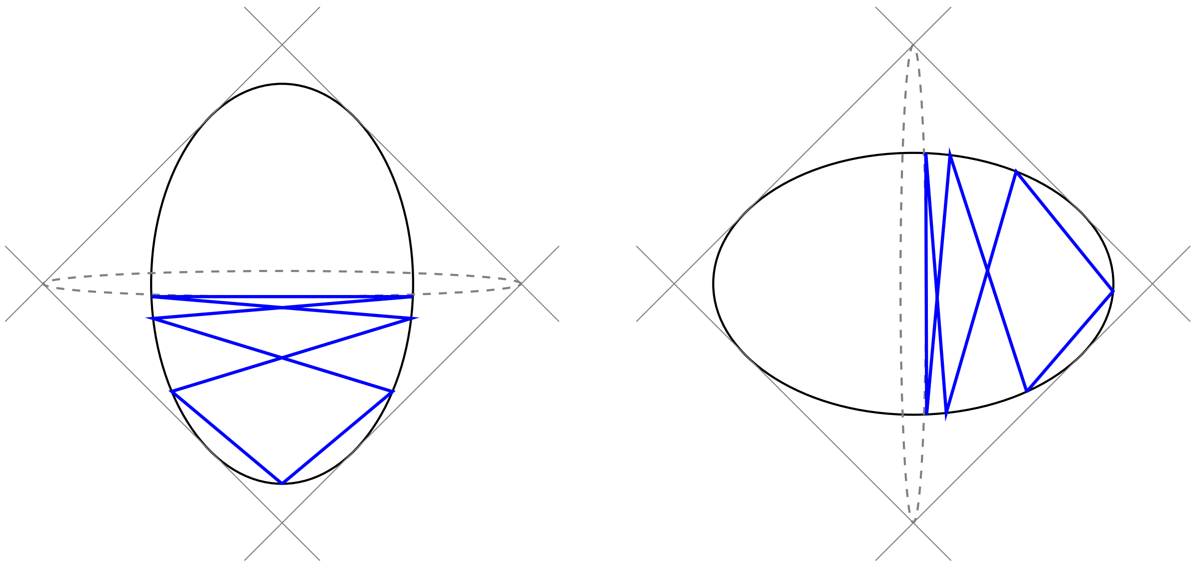

Figure 2.8: A 7-periodic trajectory with an ellipse along the x-axis as caustic ($a = 3$, $b = 7$, $\gamma \approx -6.9712$) is shown on the left, while another trajectory with an ellipse along the y-axis as caustic ($a = 7$, $b = 3$ and $\gamma \approx 6.9712$) is on the right. On the left, the particle bounces once off the relativistic ellipse and 6 times off the relativistic hyperbola while on the right the particle bounces 6 times off the relativistic ellipse and once off the relativistic hyperbola.

8-periodic trajectories

There is an 8-periodic trajectory of the billiard within (2.1), with a non-degenerate caustic $\mathcal{C}_{\lambda_0}$ in the Minkowski plane if and only if

$$\begin{vmatrix} B_3 & B_4 & B_5 \\ B_4 & B_5 & B_6 \\ B_5 & B_6 & B_7 \end{vmatrix} = 0,$$

We calculate

$$\begin{vmatrix} B_3 & B_4 & B_5 \\ B_4 & B_5 & B_6 \\ B_5 & B_6 & B_7 \end{vmatrix} =$$

$$-\frac{1}{(2^{25}ab\lambda_0)^{\frac{27}{2}}}\left(ab-a\lambda_0-b\lambda_0\right)\left(ab+a\lambda_0+b\lambda_0\right)\left(ab+a\lambda_0-b\lambda_0\right)$$
$$\Big(\left(a+b\right)^4{\lambda_0}^4-4ab\left(a+b\right)\left(-b+a\right)^2{\lambda_0}^3-2a^2b^2\left(a+b\right)\left(5a-3b\right){\lambda_0}^2-$$
$$4a^3b^3\left(a+b\right)\lambda_0+a^4b^4\Big)\Big(\left(a+b\right)^4{\lambda_0}^4+4ab\left(a+b\right)\left(-b+a\right)^2{\lambda_0}^3+$$
$$2a^2b^2\left(a+b\right)\left(3a-5b\right){\lambda_0}^2+4a^3b^3\left(a+b\right)\lambda_0+a^4b^4\Big)\Big(\left(a^2-6ab+b^2\right)\left(a+b\right)^2{\lambda_0}^4+$$
$$4ab\left(-b+a\right)\left(a+b\right)^2{\lambda_0}^3+2a^2b^2\left(3a^2+2ab+3b^2\right){\lambda_0}^2+4a^3b^3\left(-b+a\right)\lambda_0+a^4b^4\Big). \tag{2.17}$$

In Figure 2.9 and Figure 2.10, three examples of an 8-periodic trajectories are shown.

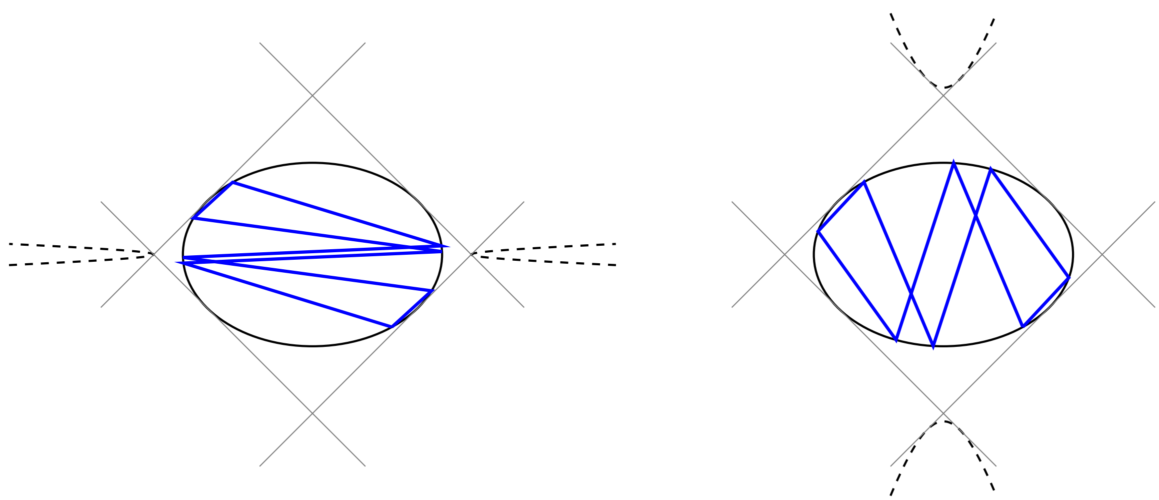

Figure 2.9: On the left, an 8-periodic trajectory with a hyperbola along x-axis as caustic ($a=6$, $b=3$, $\gamma\approx-3.0151$), with 2 vertices on relativistic ellipses and 6 on relativistic hyperbolas. On the right, an 8-periodic trajectory with a hyperbola along y-axis as caustic ($a=6$, $b=3$, $\gamma\approx 6.9168$), with 6 vertices on relativistic ellipses and 2 on relativistic hyperbolas.

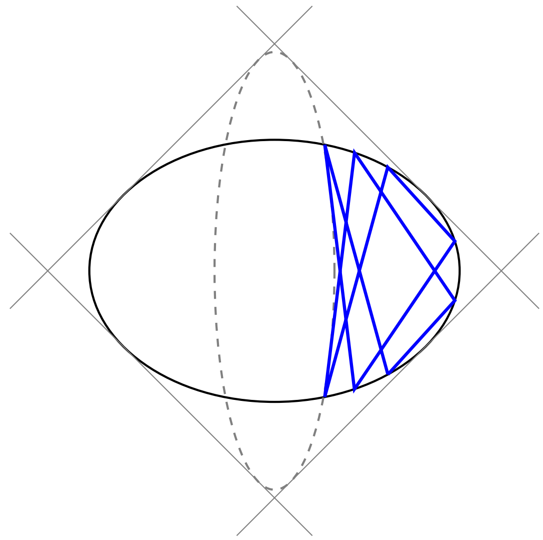

Figure 2.10: An 8-periodic trajectory with an ellipse along y-axis as caustic. There are 2 reflections off relativistic hyperbola and 6 off relativistic ellipses. ($a = 6$, $b = 3$, $\gamma \approx 5.3707$).

9-periodic trajectories

There is a 9-periodic trajectory of the billiard within (2.1), with a non-degenerate caustic $\mathcal{C}_{\lambda_0}$ in the Minkowski plane if and only if

- the caustic is an ellipse, i.e. $\lambda_0 \in (-b, a)$; and
- $$\begin{vmatrix} C_2 & C_3 & C_4 & C_5 \\ C_3 & C_4 & C_5 & C_6 \\ C_4 & C_5 & C_6 & C_7 \\ C_5 & C_6 & C_7 & C_8 \end{vmatrix} = 0$$

  .

We computed

$$
\begin{vmatrix} C_2 & C_3 & C_4 & C_5 \\ C_3 & C_4 & C_5 & C_6 \\ C_4 & C_5 & C_6 & C_7 \\ C_5 & C_6 & C_7 & C_8 \end{vmatrix} =
$$

$$
\begin{aligned}
&\frac{1}{2^{36}(ab)^{18}\lambda^{22}} \times \Big((a+b)^{20}\lambda^{20} - 4\,ab(-b+a)(a^2-6\,ab+b^2)(5\,a^2- \\
&10\,ab+b^2)(a^2-10\,ab+5\,b^2)(a+b)^{12}\lambda^{19} - 6\,a^2b^2(55\,a^6-622\,a^5b+2521\,a^4b^2 \\
&-3844\,a^3b^3+2521\,a^2b^4-622\,ab^5+55\,b^6)(a+b)^{12}\lambda^{18} - 4\,a^3b^3(-b+a)(457\,a^4 \\
&-3420\,a^3b+6838\,a^2b^2-3420\,ab^3+457\,b^4)(a+b)^{12}\lambda^{17} - a^4b^4(4555\,a^{10}+ \\
&7790\,a^9b-98897\,a^8b^2+31528\,a^7b^3-152698\,a^6b^4+475860\,a^5b^5-152698\,a^4b^6+ \\
&31528\,a^3b^7-98897\,a^2b^8+7790\,ab^9+4555\,b^{10})(a+b)^6\lambda^{16}- \\
&48\,a^5b^5(-b+a)(11\,a^8+2160\,a^7b-5980\,a^6b^2+9040\,a^5b^3-23006\,a^4b^4+ \\
&9040\,a^3b^5-5980\,a^2b^6+2160\,ab^7+11\,b^8)(a+b)^6\lambda^{15}+8\,a^6b^6(4265\,a^8- \\
&50720\,a^7b+164204\,a^6b^2-355488\,a^5b^3+497238\,a^4b^4-355488\,a^3b^5+ \\
&164204\,a^2b^6-50720\,ab^7+4265\,b^8)(a+b)^6\lambda^{14}+16\,a^7b^7(-b+a)(7855\,a^6- \\
&53094\,a^5b+131265\,a^4b^2-207444\,a^3b^3+131265\,a^2b^4-53094\,ab^5+ \\
&7855\,b^6)(a+b)^6\lambda^{13}+18\,a^8b^8(14417\,a^6-89050\,a^5b+236351\,a^4b^2-332076\,a^3b^3+ \\
&236351\,a^2b^4-89050\,ab^5+14417\,b^6)(a+b)^6\lambda^{12}+8\,a^9b^9(-b+a)(44525\,a^8- \\
&5200\,a^7b-168100\,a^6b^2+112400\,a^5b^3+445166\,a^4b^4+112400\,a^3b^5- \\
&168100\,a^2b^6-5200\,ab^7+44525\,b^8)(a+b)^2\lambda^{11}+4\,a^{10}b^{10}(82225\,a^8-
\end{aligned}
\tag{2.18}
$$

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$$407700\,a^6b^2 + 138432\,a^5b^3 + 616518\,a^4b^4 + 138432\,a^3b^5 - 407700\,a^2b^6 +$$

$$82225\,b^8)(a+b)^2\lambda^{10} + 24\,a^{11}b^{11}(-b+a)(7475\,a^6 + 24050\,a^5b - 29827\,a^4b^2 -$$

$$39556\,a^3b^3 - 29827\,a^2b^4 + 24050\,ab^5 + 7475\,b^6)(a+b)^2\lambda^9 + 2\,a^{12}b^{12}(4225\,a^6 +$$

$$282438\,a^5b - 404721\,a^4b^2 + 118932\,a^3b^3 - 404721\,a^2b^4 + 282438\,ab^5 +$$

$$4225\,b^6)(a+b)^2\lambda^8 - 16\,a^{13}b^{13}(-b+a)(5269a^4 - 24260a^3b + 8110a^2b^2 -$$

$$24260ab^3 + 5269b^4)(a+b)^2\lambda^7 - 24a^{14}b^{14}(3565a^4 - 11220a^3b + 12318a^2b^2 -$$

$$11220ab^3 + 3565b^4)(a+b)^2\lambda^6 - 16a^{15}b^{15}(-b+a)(2957a^2 - 3622ab +$$

$$2957b^2)\left(a+b\right)^2\lambda^5 - a^{16}b^{16}(16115a^4 + 6220a^3b - 20046a^2b^2 + 6220ab^3 +$$

$$16115b^4)\lambda^4 - 36a^{17}b^{17}(-b+a)(85a^2 + 178ab + 85b^2)\lambda^3 - 6a^{18}b^{18}(23a^2 +$$

$$118ab + 23b^2)\lambda^2 + 60a^{19}b^{19}(-b+a)\lambda + 9a^{20}b^{20}\Big).$$

Examples of a 9-periodic trajectories are shown in Figure 2.11.

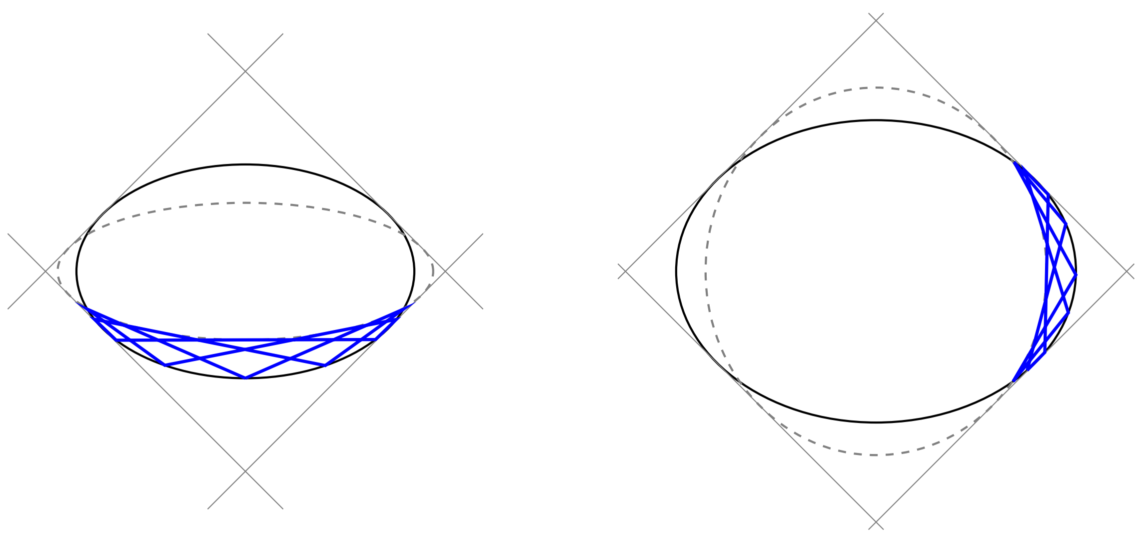

Figure 2.11: A 9-periodic trajectory with an ellipse along the x-axis as caustic ($a = 5$, $b = 2$ and $\lambda_0 \approx -1.1777$) is shown on the left, while another trajectory with an ellipse along the y-axis as caustic ($a = 7$, $b = 4$ and $\lambda_0 \approx 1.9097$) is on the right.

10-periodic trajectory

There is a 10-periodic trajectory of the billiard within (2.1), with a non-degenerate caustic $\mathcal{C}_{\lambda_0}$ in the Minkowski plane if and only if

$$\begin{vmatrix} B_3 & B_4 & B_5 & B_6 \\ B_4 & B_5 & B_6 & B_7 \\ B_5 & B_6 & B_7 & B_8 \\ B_6 & B_7 & B_8 & B_9 \end{vmatrix} = 0,$$

We computed

$$\begin{vmatrix} B_3 & B_4 & B_5 & B_6 \\ B_4 & B_5 & B_6 & B_7 \\ B_5 & B_6 & B_7 & B_8 \\ B_6 & B_7 & B_8 & B_9 \end{vmatrix} =$$

$$\frac{1}{2^{44}(ab\lambda_0)^{22}}\Big((a+b)^6\,\lambda_0^6 - 2ab\,(-b+a)\,(-3b+a)\,(-b+3a)$$
$$(a+b)^2\,\lambda^5 - a^2b^2\left(29a^2-54ab+29b^2\right)(a+b)^2\,{\lambda_0}^4 - 36a^3b^3\,(-b+a)\,(a+b)^2$$
$${\lambda_0}^3 - a^4b^4\left(9a^2+34ab+9b^2\right){\lambda_0}^2 + 10a^5b^5\,(-b+a)\,\lambda_0 + 5a^6b^6\Big) \tag{2.19}$$
$$\Big((a+b)^6\,{\lambda_0}^6 + 2ab\,(-b+a)\,(-3b+a)\,(-b+3a)\,(a+b)^2\,{\lambda_0}^5$$
$$+ 5a^2b^2\,(-b+3a)\,(-3b+a)\,(a+b)^2\,{\lambda_0}^4 + 20a^3b^3\,(-b+a)\,(a+b)^2\,{\lambda_0}^3$$
$$+ a^4b^4\left(15a^2+14ab+15b^2\right){\lambda_0}^2 + 6a^5b^5\,(-b+a)\,\lambda + a^6b^6\Big)\Big(\left(a^2-10ab+5b^2\right)$$
$$(a+b)^4\,{\lambda_0}^6 + 2ab\,(3a-5b)\,(a+b)^4\,{\lambda_0}^5$$
$$+ a^2b^2\,(a+b)\left(15a^3+5a^2b+45ab^2-9b^3\right){\lambda_0}^4$$

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$$
\begin{aligned}
&+ 4a^3b^3\,(a+b)\left(5a^2 - 10ab + 9b^2\right){\lambda_0}^3 + a^4b^4\,(a+b)\,(15a - 29b)\,{\lambda_0}^2 \\
&+ 6a^5b^5\,(a+b)\,\lambda_0 + a^6b^6\Big) \\
&\Big(\left(5a^2 - 10ab + b^2\right)(a+b)^4\,{\lambda_0}^6 + 2ab\,(5a - 3b)\,(a+b)^4\,{\lambda_0}^5 \\
&- a^2b^2\,(a+b)\left(9\,a^3 - 45\,a^2b - 5ab^2 - 15b^3\right)\lambda^4 \\
&- 4a^3b^3\,(a+b)\left(9a^2 - 10ab + 5b^2\right){\lambda_0}^3 - a^4b^4\,(a+b)\,(29a - 15b)\,{\lambda_0}^2 \\
&- 6a^5b^5\,(a+b)\,\lambda_0 + a^6b^6\Big).
\end{aligned}
$$

In Figure 2.12 and Figure 2.13, three examples of an 10-periodic trajectories are shown.

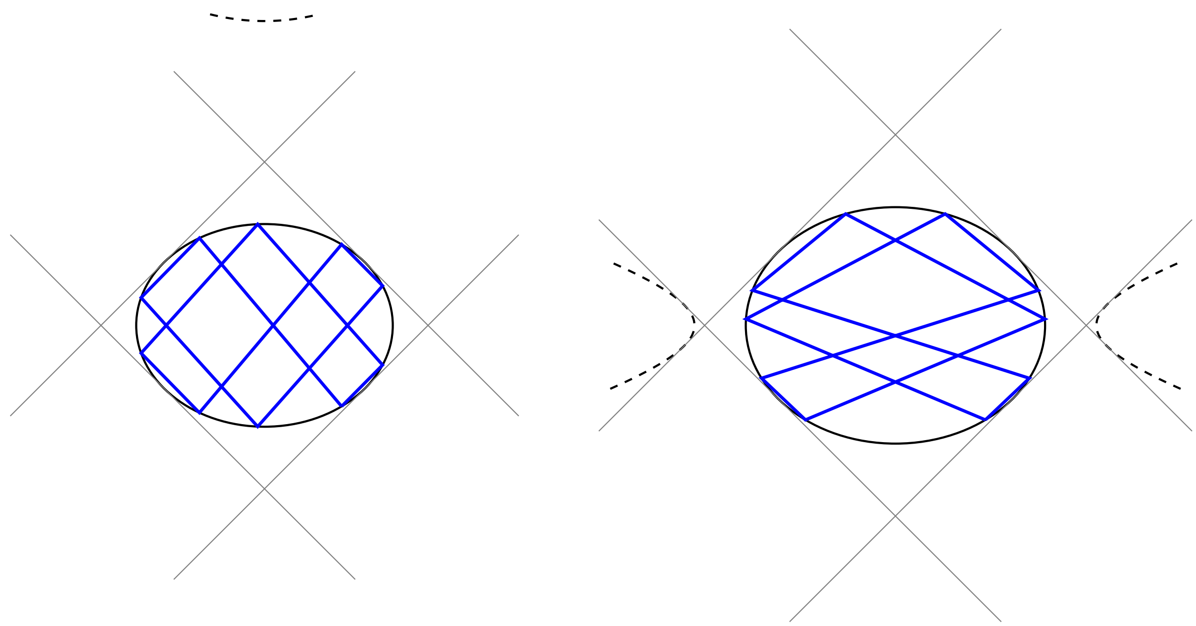

Figure 2.12: On the left, a 10-periodic trajectory with a hyperbola along the y-axis as caustic ($a = 8$, $b = 5$ and $\lambda_0 \approx 40.0965$,). On the right, a 10-periodic trajectory with a hyperbola along the x-axis as caustic ($a = 8$, $b = 5$ and $\lambda_0 = -6.4196$).

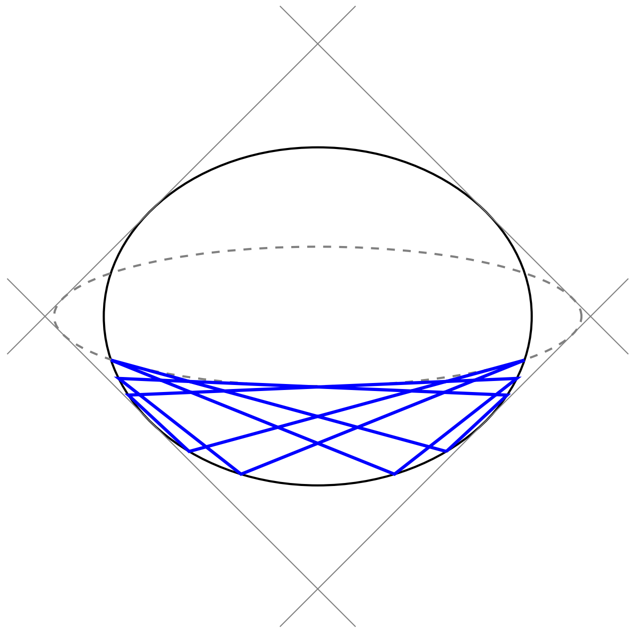

Figure 2.13: A 10-periodic trajectory with an ellipse along the x-axis as caustic ($a = 8$, $b = 5$ and $\lambda_0 = -4.1502$).

2.2.1 Table of summary on number of touching points with relativistic ellipses and hyperbolas

In the table below , N.O.R.O.R.E. and N.O.R.O.R.H. stand for the number of reflections off relativistic ellipses and the number of reflections off relativistic hyperbolas respectively.

Table 2.1: Table of summary on number of touching points with relativistic ellipses and hyperbolas

Periodicity $n = n_1 + n_2$	Caustic of the trajectory	n_1 : N.O.R.O.R.E.	n_2 : N.O.R.O.R.H.
n=3			
	Ellipse along y-axis	2	1
	Ellipse along x-axis	1	2
n=4			
	Ellipse along x-axis	2	2
	Ellipse along y-axis	2	2
	Hyperbola along x-axis	2	2
n=5			
	Ellipse along y-axis	2	3
	Ellipse along x-axis	3	2
n=5			
	Ellipse along y-axis	4	1
	Ellipse along x-axis	1	4

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Periodicity $n = n_1 + n_2$	Caustic of the trajectory	n_1 : N.O.R.O.R.E.	n_2 : N.O.R.O.R.H.
n=6	Hyperbola along x-axis	2	4
	Hyperbola along y-axis	4	2
n=7	Ellipse along x-axis	1	6
	Ellipse along y-axis	6	1
n=8	Hyperbola along x-axis	2	6
	Hyperbola along y-axis	6	2
	Ellipse along x-axis	6	2
n=9	Ellipse along x-axis	5	4
	Ellipse along y-axis	4	5
n=10	Hyperbola along y-axis	6	4
	Hyperbola along x-axis	4	6
	Ellipse along x-axis	4	6

2.2.2 Cayley-type conditions and discriminantly factorizable polynomials

Example 2.2.1. Let us denote the numerator in the expression (2.9) as $\mathsf{G}_2(\lambda_0, a, b)$:

$$\mathsf{G}_2(\lambda_0, a, b) = -\left(a+b\right)^2 {\lambda_0}^2 + 2\,ab\left(a-b\right)\lambda_0 + 3\,a^2b^2$$

Let us find the discriminant of G_2 with respect to λ_0

$$\mathscr{D}_{\lambda 0}\mathsf{G}_2 = 2^4\,\left(a^2+ab+b^2\right)a^2b^2.$$

It follows that G_2 is a discriminantly factorizable polynomial.

Example 2.2.2. Let us denote the numerator in the expression (2.12) as $\mathsf{G}_3(\lambda_0, a, b)$:

$$\begin{aligned}\mathsf{G}_3(\lambda_0, a, b) &= -(ab+a\lambda_0+b\lambda_0)(ab+a\lambda_0-b\lambda_0)(ab-a\lambda_0-b\lambda_0)\\ &= -\left(a+b\right)^2\left(a-b\right)\lambda^3 - ab\left(a+b\right)^2\lambda^2 + a^2b^2\left(a-b\right)\lambda + a^3b^3\end{aligned}$$

Let us find the discriminant of G_3 with respect to λ_0

$$\mathscr{D}_{\lambda 0}\mathsf{G}_3 = 2^6\,a^8b^8\left(a+b\right)^2.$$

It follows that G_3 is a discriminantly factorizable polynomial.

Example 2.2.3. Let us denote the numerator in the expression (2.14) as $\mathsf{G}_4(\lambda_0, a, b)$:

$$\begin{aligned}\mathsf{G}_6(\lambda_0, a, b) &= \left(a+b\right)^6\lambda_0^6 - 2ab\left(-b+a\right)\left(-3b+a\right)\left(-b+3a\right)\left(a+b\right)^2\lambda_0^5\\ &\quad - a^2b^2\left(29a^2-54ab+29b^2\right)\left(a+b\right)^2\lambda_0^4 - 36a^3b^3\left(-b+a\right)\left(a+b\right)^2\lambda_0^3\\ &\quad - a^4b^4\left(9a^2+34ab+9b^2\right)\lambda_0^2 + 10a^5b^5\left(-b+a\right)\lambda_0 + 5a^6b^6.\end{aligned}$$

Let us find the discriminant of G_6 with respect to λ_0

$$\begin{aligned}\mathscr{D}_{\lambda 0}\mathsf{G}_6 = -5(2)^{44}\,&(27\,a^6+81\,a^5b+322\,a^4b^2+509\,a^3b^3+322\,a^2b^4\\ &+81\,ab^5+27\,b^6)\left(a+b\right)^8b^{38}a^{38}.\end{aligned}$$

It follows that G_3 is a discriminantly factorizable polynomial.

Example 2.2.4. Let us denote the numerator in the expression (2.15) as $\mathsf{G}_8(\lambda_0, a, b)$:

$$\mathsf{G}_8(\lambda_0, a, b) =$$

$$(3\,a-b)(a-3\,b)(a+b)^6\lambda^8 + 8\,ab(a-b)(a+b)^6\lambda^7 - 4\,a^2b^2(3\,a^4 - 24\,a^3b + 10\,a^2b^2$$

$$-\,24\,ab^3 + 3\,b^4)\,(a+b)^2\,\lambda^6 - 8\,a^3b^3\,(a-b)\left(9\,a^2 - 14\,ab + 9\,b^2\right)(a+b)^2\,\lambda^5$$

$$-\,10\,a^4b^4\left(11\,a^2 - 18\,ab + 11\,b^2\right)(a+b)^2\,\lambda^4 - 72\,a^5b^5\,(a-b)\,(a+b)^2\,\lambda^3$$

$$-\,4\,a^6b^6\,(a+3\,b)\,(3\,a+b)\,\lambda^2 + 8\,a^7b^7(a-b)\lambda + 3\,a^8b^8$$

Let us find the discriminant of G_8 with respect to λ_0

$$\mathscr{D}_{\lambda 0}\mathsf{G}_8 = -2^{88}\,\left(a^2 + ab + b^2\right)(a+b)^{18}\,b^{74}a^{74}.$$

It follows that G_8 is a discriminantly factorizable polynomial.

Example 2.2.5. The discriminant $\mathscr{D}_{\lambda 0}\mathsf{G}_{12}$ of the polynomial numerator of the expression in (2.16) is:

$$\mathscr{D}_{\lambda 0}\mathsf{G}_{12} = -(2)^{184}(7)^2\,(84375\,a^{12} + 506250\,a^{11}b$$

$$+\,4266243\,a^{10}b^2 + 16690590\,a^9b^3 + 34989622\,a^8b^4 + 45383698\,a^7b^5 + 46564971\,a^6b^6 +$$

$$45383698\,a^5b^7 + 34989622\,a^4b^8 + 16690590\,a^3b^9 + 4266243\,a^2b^{10} + 506250\,ab^{11}$$

$$+\,84375\,b^{12})\,(a+b)^{40}\,b^{172}a^{172}$$

It follows that G_8 is a discriminantly factorizable polynomial.

Example 2.2.6. The discriminant $\mathscr{D}_{\lambda 0}\mathsf{G}_{15}$ of the polynomial numerator of the expression in (2.17) is:

$$\mathscr{D}_{\lambda 0}\mathsf{G}_{15} =$$

$$-\,2^{246}\,\left(27\,a^2 + 46\,ab + 27\,b^2\right)(a+b)^8\,(8\,a^{26} + 200\,a^{25}b + 2427\,a^{24}b^2 + 19048\,a^{23}b^3 +$$

$$108652\,a^{22}b^4 + 479688\,a^{21}b^5 + 1703702\,a^{20}b^6 + 4993208\,a^{19}b^7 + 12286692\,a^{18}b^8$$

$$+\,25688608\,a^{17}b^9 + 46007797\,a^{16}b^{10} + 70961808\,a^{15}b^{11} + 94556312\,a^{14}b^{12} + 108998288\,a^{13}b^{13}$$

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$$+ 108671412\, a^{12}b^{14} + 93545968\, a^{11}b^{15} + 69297712\, a^{10}b^{16} + 43955208\, a^{9}b^{17} +$$
$$23703317\, a^{8}b^{18} + 10761608\, a^{7}b^{19} + 4059132\, a^{6}b^{20} + 1248808\, a^{5}b^{21} + 305302\, a^{4}b^{22}$$
$$+ 57048\, a^{3}b^{23} + 7652\, a^{2}b^{24} + 656\, ab^{25} + 27\, b^{26})(27\, a^{26} + 656\, a^{25}b + 7652\, a^{24}b^{2} + 57048\, a^{23}b^{3}$$
$$+ 305302\, a^{22}b^{4} + 1248808\, a^{21}b^{5} + 4059132\, a^{20}b^{6} + 10761608\, a^{19}b^{7} + 23703317\, a^{18}b^{8}$$
$$+ 43955208\, a^{17}b^{9} + 69297712\, a^{16}b^{10} + 93545968\, a^{15}b^{11} + 108671412\, a^{14}b^{12}$$
$$+ 108998288\, a^{13}b^{13} + 94556312\, a^{12}b^{14} + 70961808\, a^{11}b^{15} + 46007797\, a^{10}b^{16}$$
$$+ 25688608\, a^{9}b^{17} + 12286692\, a^{8}b^{18} + 4993208\, a^{7}b^{19} + 1703702\, a^{6}b^{20} + 479688\, a^{5}b^{21}$$
$$+ 108652\, a^{4}b^{22} + 19048\, a^{3}b^{23} + 2427\, a^{2}b^{24} + 200\, ab^{25} + 8\, b^{26})\Big(a^{5} + 5\, a^{4}b + 10\, a^{3}b^{2}$$
$$+ 10\, a^{2}b^{3} + 5\, ab^{4} + b^{5}\Big)(a^{7} + 7\, a^{6}b + 21\, a^{5}b^{2} + 35\, a^{4}b^{3} + 35\, a^{3}b^{4} + 21\, a^{2}b^{5}$$
$$+ 7\, ab^{6} + b^{7})b^{278}a^{278}$$

It follows that G_{15} is a discriminantly factorizable polynomial.

Example 2.2.7. The discriminant $\mathscr{D}_{\lambda 0}\mathsf{G}_{20}$ of the polynomial numerator of the expression in (2.18) is:

$$\mathscr{D}_{\lambda 0}\mathsf{G}_{20} =$$
$$(2)^{520}(3)^{9}\left(a^{2} + ab + b^{2}\right)(2573571875\, a^{18} + 23162146875\, a^{17}b + 343857834375\, a^{16}b^{2} +$$
$$2225854012500\, a^{15}b^{3} + 7915637923674\, a^{14}b^{4} + 18294550565718\, a^{13}b^{5} + 35800011229590\, a^{12}b^{6}$$
$$+ 71422154979456\, a^{11}b^{7} + 123117217701777\, a^{10}b^{8} + 150424579541609\, a^{9}b^{9}$$
$$+ 123117217701777\, a^{8}b^{10} + 71422154979456\, a^{7}b^{11} + 35800011229590\, a^{6}b^{12}$$
$$+ 18294550565718\, a^{5}b^{13} + 7915637923674\, a^{4}b^{14} + 2225854012500\, a^{3}b^{15}$$
$$+ 343857834375\, a^{2}b^{16} + 23162146875\, ab^{17} + 2573571875\, b^{18})\,(a+b)^{120}\, b^{500}a^{500}$$

It follows that G_{20} is a discriminantly factorizable polynomial.

Example 2.2.8. The discriminant $\mathscr{D}_{\lambda 0}\mathsf{G}_{24}$ of the polynomial numerator of the expression in (2.19) is:

$$\mathscr{D}_{\lambda 0}\mathsf{G}_{24} = 5(2)^{776}\left(64\,a^2 - 7\,ab + 64\,b^2\right)\left(64\,a^2 + 135\,ab + 135\,b^2\right)\left(135\,a^2 + 135\,ab + 64\,b^2\right) \times \\ \left(27\,a^6 + 81\,a^5 b + 322\,a^4 b^2 + 509\,a^3 b^3 + 322\,a^2 b^4 + 81\,ab^5 + 27\,b^6\right)(a+b)^{180}\,b^{732} a^{732}$$

It follows that G_{24} is a discriminantly factorizable polynomial.

Remark 2.2.1. We observed in the above examples that all polynomials are discriminantly factorizable. However, it is important to note that their factors are homogeneous, thus, by a change of variables $(a, b) \mapsto (a, \hat{b})$, with $\hat{b} = \frac{b}{a}$, transforms the polynomials into discriminantly separable polynomials in new variables $(a, \hat{b})$.

2.2.3 Disciminantly separable polynomials

Similarly to the case of the Euclidean plane [26], the Cayley-type conditions obtained previously have a very interesting algebraic structure. Namely, the numerators of the corresponding expressions are polynomials in 3 variables. As examples below show, those polynomials have factorizable discriminants which, after a change of varibles, lead to discriminantly separable polynomials in the sense of the following definition.

Definition 2.2.1. [17] A polynomial $F(x_1, \ldots, x_n)$ is discriminantly separable if there exist polynomials $f_1(x_1), \ldots, f_n(x_n)$ such that the discriminant $\mathscr{D}_{x_i}F$ of F with respect to x_i satisfies:

$$\mathscr{D}_{x_i}F(x_1, \ldots, \hat{x}_i, \ldots, x_n) = \prod_{j \neq i} f_j(x_j),$$

for each $i = 1, \ldots, n$.

Discriminantly factorizable polynomials were introduced in [23] in connection with n-valued groups. Various applications of discriminantly separable polynomials in continuous and discrete integrable systems were presented in [27].The connection between Cayley-type conditions in the Euclidean setting and discriminantly factorizable and separable polynomials have been observed in [26]. As examples below show, the Cayley conditions in the Minkowski plane provide examples of discriminantly factorisable polynomials which, after a change of variables, have separable discriminants. It would be interesting to establish this relationship as a general statement.

After applying Remark 2.2.1 to the previous first four examples, one gets the following discriminantly separable polynomials in new variables $(a, \hat{b})$:

Example 2.2.9.

$$\mathscr{D}_{\lambda 0}\mathsf{G}_2 = 2^4 a^8 \hat{b}^2 \left(1 + \hat{b} + \hat{b}^2\right).$$

Example 2.2.10.

$$\mathscr{D}_{\lambda 0}\mathsf{G}_3 = 2^6 a^{18} \hat{b}^8 \left(1 + \hat{b}\right)^2.$$

Example 2.2.11.

$$\begin{aligned}\mathscr{D}_{\lambda 0}\mathsf{G}_6 = -52^{44} a^{90} \hat{b}^{38} (27 &+ 81\,\hat{b} + 322\,\hat{b}^2 + 509\,\hat{b}^3 + 322\,\hat{b}^4 \\ &+ 81\,\hat{b}^5 + 27\,b^6)\,(a+b)^8 .\end{aligned}$$

Example 2.2.12.

$$\mathscr{D}_{\lambda 0}\mathsf{G}_8 = -2^{88} a^{168} \hat{b}^{74} \left(1 + \hat{b} + \hat{b}^2\right) \left(1 + \hat{b}\right)^{18}.$$

2.3 Elliptic periodic trajectories

Points of the plane which are symmetric with respect to the coordinate axes share the same elliptic coordinates, thus there is no bijection between the elliptic and the Cartesian coordinates. Thus, we introduce a separate notion of periodicity in elliptic coordinates.

Definition 2.3.1. A billiard trajectory is *n-elliptic periodic* is it is n-periodic in elliptic coordinates joined to the confocal family $\mathcal{C}_\lambda$.

Now, we will derive algebro-geometric conditions for elliptic periodic trajectories.

Theorem 2.3.1. A billiard trajectory within $\mathcal{E}$ with the caustic $\mathcal{C}_\gamma$ is n-elliptic periodic without being n-periodic if and only if one of the following conditions is satisfied on $\mathscr{C}$:

(a) $\mathcal{C}_\gamma$ is an ellipse, $0 < \gamma < a$, and $nQ_0 - (n-1)Q_\gamma - Q_{-b} \sim 0$;

(b) $\mathcal{C}_\gamma$ is an ellipse, $-b < \gamma < 0$, and $nQ_0 - (n-1)Q_\gamma - Q_a \sim 0$;

(c) $\mathcal{C}_\gamma$ is a hyperbola, n is even and $nQ_0 - (n-2)Q_\gamma - Q_{-b} - Q_a \sim 0$;

(d) $\mathcal{C}_\gamma$ is a hyperbola, n is odd, and $nQ_0 - (n-1)Q_\gamma - Q_a \sim 0$;

(e) $\mathcal{C}_\gamma$ is a hyperbola, n is odd, and $nQ_0 - (n-1)Q_\gamma - Q_{-b} \sim 0$.

Moreover, such trajectories are always symmetric with respect to the origin in Case (c). They are symmetric with respect to the x-axis in Cases (b) and (d), and with respect to the y-axis in Cases (a) and (e).

Proof. Let M_0 be the initial point of a given n-elliptic periodic trajectory, and M_1 the next point on the trajectory with the same elliptic coordinates. Then, integrating (2.3) M_0 to M_1 along the trajectory, we get:

$$n_1(Q_0 - Q_{c_1}) + n_2(Q_0 - Q_{b_1}) \sim 0,$$

where $n = n_1 + n_2$, and n_1 is the number of times that the particle hit the arcs of $\mathcal{E}$ with time-like metrics, and n_2 the number of times it hit the arcs with space-like metrics. We denoted by c_1 the largest negative member of the set $\{a, -b, \gamma\}$, and by b_1 its smallest positive member.

The trajectory is not n-periodic if and only if at least one of n_1, n_2 is odd, which then leads to the stated conclusions. $\square$

The explicit Cayley-type conditions for elliptic periodic trajectories are:

Theorem 2.3.2. A billiard trajectory within $\mathcal{E}$ with the caustic $\mathcal{Q}_\gamma$ is n-elliptic periodic without being n-periodic if and only if one of the following conditions is satisfied:

(a) $\mathcal{C}_\gamma$ is an ellipse, $0 < \gamma < a$, and

$$D_1 = 0, \quad \begin{vmatrix} D_1 & D_2 \\ D_2 & D_3 \end{vmatrix} = 0, \quad \begin{vmatrix} D_1 & D_2 & D_3 \\ D_2 & D_3 & D_4 \\ D_3 & D_4 & D_5 \end{vmatrix} = 0, \dots \quad \text{for} \quad n = 2, 4, 6, \dots$$

$$E_2 = 0, \quad \begin{vmatrix} E_2 & E_3 \\ E_3 & E_4 \end{vmatrix} = 0, \quad \begin{vmatrix} E_2 & E_3 & E_4 \\ E_3 & E_4 & E_5 \\ E_4 & E_5 & E_6 \end{vmatrix} = 0, \dots \quad \text{for} \quad n = 3, 5, 7, \dots ;$$

(b) $\mathcal{C}_\gamma$ is an ellipse, $-b < \gamma < 0$, and

$$E_1 = 0, \quad \begin{vmatrix} E_1 & E_2 \\ E_2 & E_3 \end{vmatrix} = 0, \quad \begin{vmatrix} E_1 & E_2 & E_3 \\ E_2 & E_3 & E_4 \\ E_3 & E_4 & E_5 \end{vmatrix} = 0, \dots \quad \text{for} \quad n = 2, 4, 6, \dots$$

$$D_2 = 0, \quad \begin{vmatrix} D_2 & D_3 \\ D_3 & D_4 \end{vmatrix} = 0, \quad \begin{vmatrix} D_2 & D_3 & D_4 \\ D_3 & D_4 & D_5 \\ D_4 & D_5 & D_6 \end{vmatrix} = 0, \dots \quad \text{for} \quad n = 3, 5, 7, \dots ;$$

(c) $\mathcal{Q}_\gamma$ is a hyperbola, n even and

$$C_1 = 0, \quad \begin{vmatrix} C_1 & C_2 \\ C_2 & C_3 \end{vmatrix} = 0, \quad \begin{vmatrix} C_1 & C_2 & C_3 \\ C_2 & C_3 & C_4 \\ C_3 & C_4 & C_5 \end{vmatrix} = 0, \dots \quad \text{for} \quad n = 2, 4, 6, \dots$$

(d) $\mathcal{Q}_\gamma$ is a hyperbola, n is odd, and

$$D_2 = 0, \qquad \begin{vmatrix} D_2 & D_3 \\ D_3 & D_4 \end{vmatrix} = 0, \qquad \begin{vmatrix} D_2 & D_3 & D_4 \\ D_3 & D_4 & D_5 \\ D_4 & D_5 & D_6 \end{vmatrix} = 0, \dots \quad \text{for} \quad n = 3, 5, 7, \dots.$$

(e) $\mathcal{Q}_\gamma$ is a hyperbola, n is odd, and

$$E_2 = 0, \qquad \begin{vmatrix} E_2 & E_3 \\ E_3 & E_4 \end{vmatrix} = 0, \qquad \begin{vmatrix} E_2 & E_3 & E_4 \\ E_3 & E_4 & E_5 \\ E_4 & E_5 & E_6 \end{vmatrix} = 0, \dots \quad \text{for} \quad n = 3, 5, 7, \dots.$$

Here, we denoted:

$$\frac{\sqrt{\varepsilon(a-x)(b+x)(\gamma-x)}}{a-x} = D_0 + D_1 x + D_2 x^2 + \dots,$$
$$\frac{\sqrt{\varepsilon(a-x)(b+x)(\gamma-x)}}{b+x} = E_0 + E_1 x + E_2 x^2 + \dots,$$

the Taylor expansion around $x = 0$, while Bs and Cs are as in Theorem 2.1.2.

Proof. (a) Take first n even. Using Theorem 2.3.1, we have:

$$nQ_0 \sim (n-1)Q_\gamma + Q_{-b} \sim (n-2)Q_\infty + Q_{-b} + Q_\gamma \sim (n-2)Q_\infty + Q_\infty + Q_a \sim (n-1)Q_\infty + Q_a.$$

The basis of $\mathscr{L}((n-1)Q_\infty + Q_a)$ is:

$$1, x, x^2, \dots, x^{n/2-1}, \frac{y}{x-a}, \frac{xy}{x-a}, \frac{x^{n/2-1}y}{x-a},$$

thus a non-trivial linear combination of these functions with a zero of order n at $x = 0$ exists if and only if:

$$\begin{vmatrix} D_{n/2} & D_{n/2-1} & \dots & D_1 \\ D_{n/2+1} & D_{n/2} & \dots & D_2 \\ \dots & & & \\ D_{n-1} & D_{n-2} & \dots & D_{n/2} \end{vmatrix} = 0.$$

For odd n, we have:

$$nQ_0 \sim (n-1)Q_\gamma + Q_{-b} \sim (n-1)Q_\infty + Q_{-b}.$$

The basis of $\mathscr{L}((n-1)Q_\infty + Q_{-b})$ is:

$$1, x, x^2, \dots, x^{(n-1)/2}, \frac{y}{x+b}, \frac{xy}{x+b}, \frac{x^{(n-1)/2-1}y}{x+b},$$

thus a non-trivial linear combination of these functions with a zero of order n at $x = 0$ exists if and only if:

$$\begin{vmatrix} E_{(n-1)/2+1} & E_{(n-1)/2} & \dots & E_2 \\ E_{(n-1)/2+2} & E_{(n-1)/2+1} & \dots & E_3 \\ \dots & & & \\ E_{n-1} & E_{n-2} & \dots & E_{(n-1)/2+1} \end{vmatrix} = 0.$$

Case (b) is done similarly as (a).

(c) We have $nQ_0 \sim (n-2)Q_\gamma + Q_{-b} + Q_a \sim (n-1)Q_\infty + Q_\gamma$.

(d) We have $nQ_0 \sim (n-1)Q_\gamma + Q_a \sim (n-1)Q_\infty + Q_a$.

(e) We have $nQ_0 \sim (n-1)Q_\gamma + Q_{-b} \sim (n-1)Q_\infty + Q_{-b}$. □

2.4 Examples of elliptic periodic trajectories with small periods: $3 \le n \le 7$

2-elliptic periodic trajectories

There is a 2-elliptic periodic trajectory without being 2-periodic of the billiard within (2.1), with a non-degenerate caustic $\mathcal{C}_{\gamma_0}$ in the Minkowski plane if and only if one of the following is satisfied:

- the caustic is an ellipse, with $\gamma_0 \in (0, a)$; and $D_1 = 0$.
- the caustic is an ellipse, with $\gamma_0 \in (-b, 0)$; and $E_1 = 0$.
- the caustic is a hyperbola, and n even and $C_1 = 0$.

We solve the following equations

$$D_1 = \frac{(a+b)\lambda - ab}{2a^{\frac{3}{2}(b\lambda)^{\frac{1}{2}}}} = 0,$$
$$E_1 = -\frac{(a+b)\lambda + ab}{2b^{\frac{3}{2}(b\lambda)^{\frac{1}{2}}}} = 0,$$
$$C_1 = \frac{(a-b)\lambda + ab}{2\lambda^{\frac{3}{2}(ab)^{\frac{1}{2}}}} = 0,$$

which respectively yield the solutions for the parameter γ_0 for the caustic:

$$\gamma_0 = \frac{ab}{a+b},$$
$$\gamma_0 = -\frac{ab}{a+b},$$
$$\gamma_0 = -\frac{ab}{a-b}$$

Some examples of a 2-elliptic periodic trajectories without being 2-periodic are shown in Figure 2.14 and Figure 2.15.

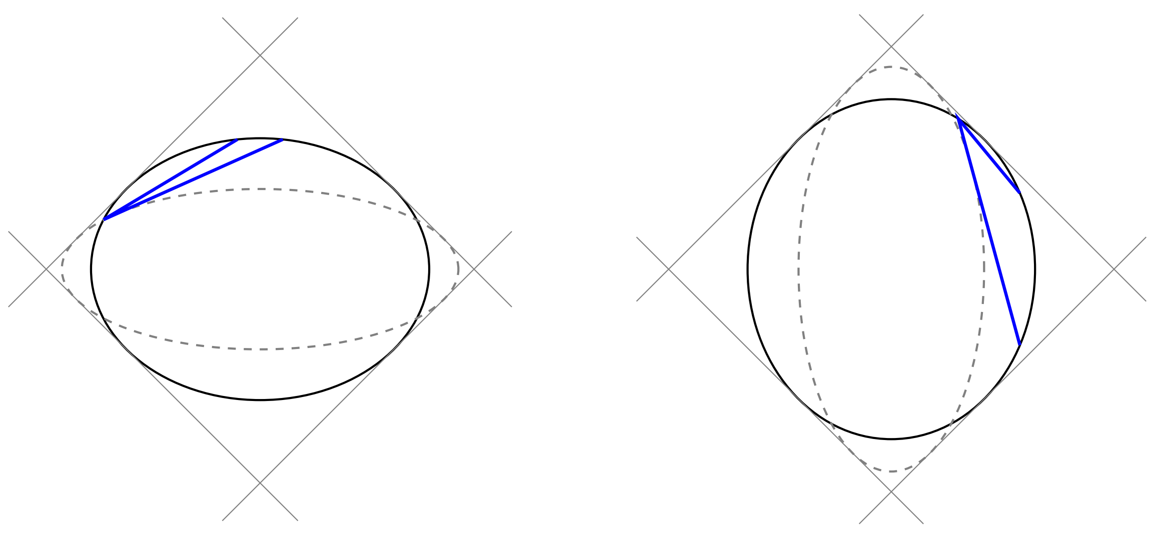

Figure 2.14: A 2-elliptic periodic trajectories with ellipses as caustics. On the left, the caustic is an ellipse along x-axis ($a = 5$, $b = 3$, $\gamma = -15/8$), and on the right an ellipse along y-axis ($a = 5$, $b = 7$ and $\gamma = 35/12$).

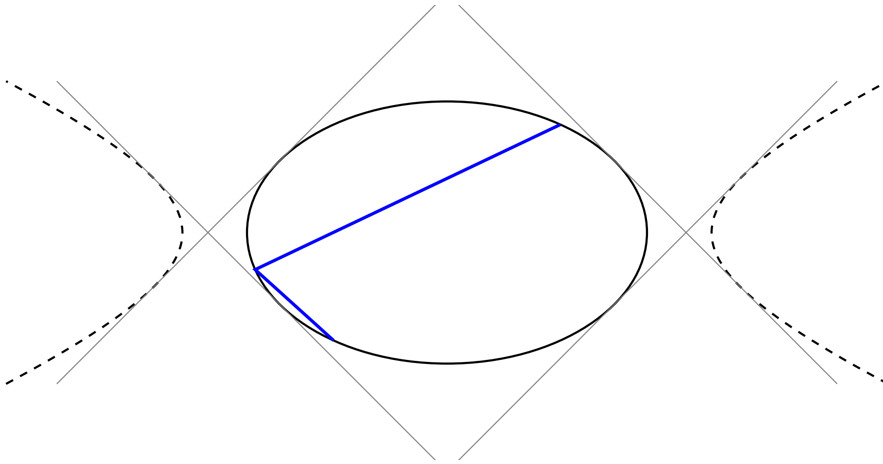

Figure 2.15: A 2-elliptic periodic trajectory with a hyperbola as caustic ($a = 7$, $b = 3$, $\gamma = -5.25$).

3-elliptic periodic trajectories

There is a 3-elliptic periodic trajectory without being 3-periodic of the billiard within (2.1), with a non-degenerate caustic $\mathcal{C}_{\gamma_0}$ in the Minkowski plane if and only if one of the following is satisfied:

- the caustic is an ellipse, with $\gamma_0 \in (0, a)$; and $E_2 = 0$, or the caustic is a hyperbola with n and $E_2 = 0$.
- the caustic is an ellipse, with $\gamma_0 \in (-b, 0)$; and $D_2 = 0$, or the caustic is a hyperbola and $D_2 = 0$.

The following equations are solved:

$$E_2 = -\frac{1}{8}\frac{1}{b^{\frac{5}{2}}(a\lambda)^{\frac{3}{2}}}(-(a+b)(3a-b)\lambda^2 - 2ab(a+b)\lambda + a^2b^2) = 0, \tag{2.20}$$

$$D_2 = -\frac{1}{8}\frac{1}{b^{\frac{5}{2}}(a\lambda)^{\frac{3}{2}}}((a+b)(a-3b)\lambda^2 + 2ab(a+b)\lambda + a^2b^2) = 0, \tag{2.21}$$

which respectively yield the pair of solutions for the parameter γ_0 for the caustic:

$$\gamma_0 = \frac{(-a-b+2\sqrt{a^2+ab})ba}{(a+b)(3a-b)}, \text{ and } \gamma_0 = -\frac{(-a-b+2\sqrt{a^2+ab})ba}{(a+b)(3a-b)}$$
$$\gamma_0 = \frac{(-a-b+2\sqrt{b^2+ab})ba}{(a+b)(a-3b)}, \text{ and } \gamma_0 = -\frac{(-a-b+2\sqrt{b^2+ab})ba}{(a+b)(a-3b)}.$$

Examples of a 3-elliptic periodic trajectories which are not 3-periodic are shown in Figure 2.16 and Figure 2.17.

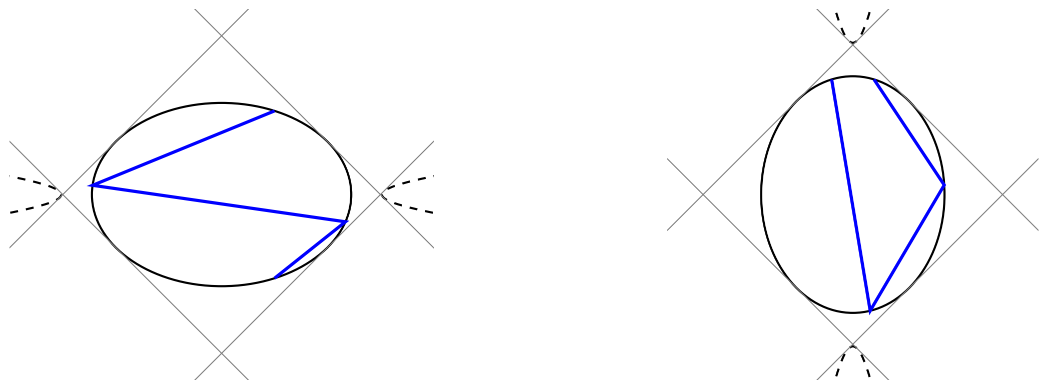

Figure 2.16: A 3-elliptic periodic trajectories with hyperbolas as caustics. On the left, the caustic is orientied along the x-axis ($a = 6$, $b = 3$, $\gamma \approx -3.1595918$), and on the right along the y-axis ($a = 3$, $b = 5$, $\gamma \approx 3.2264236$).

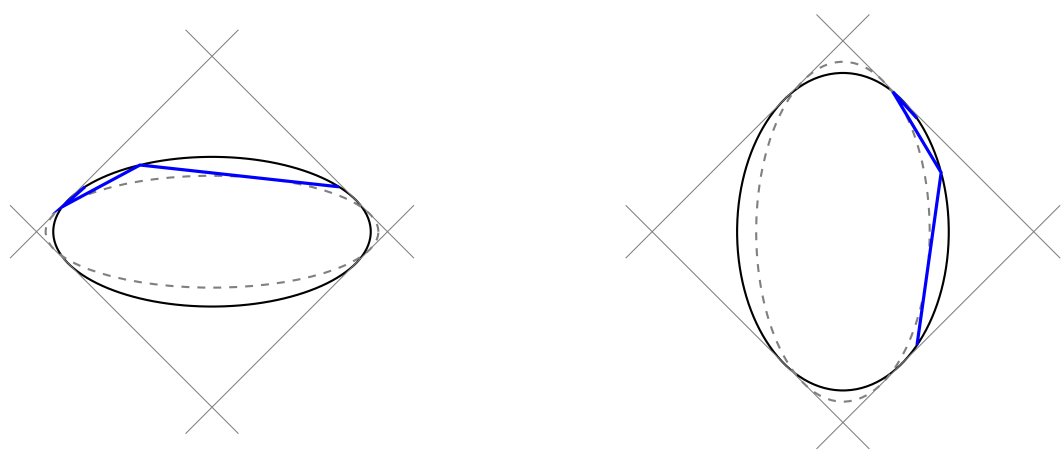

Figure 2.17: A 3-elliptic periodic trajectory without being 3-periodic with an ellipse along the x-axis as caustic ($a = 9$, $b = 2$ and $\lambda_0 = -.8831827$) on the let. On the right,a 3-elliptic periodic trajectories without being 3-periodic with an ellipse along the y-axis as caustic ($a = 4$, $b = 9$ and $\lambda_0 = 1.312805$).

4-elliptic periodic trajectories

There is a 4-elliptic periodic trajectory without being 4-periodic of the billiard within (2.1), with a non-degenerate caustic $\mathcal{C}_{\gamma_0}$ in the Minkowski plane if and only if one of the following is satisfied:

- the caustic is an ellipse, with $\gamma_0 \in (0, a)$; and $D_3D_1 - D_2^2 = 0$.
- the caustic is an ellipse, with $\gamma_0 \in (-b, 0)$; and $E_3E_1 - E_2^2 = 0$.
- the caustic is a hyperbola, and $C_3C_1 - C_2^2 = 0$.

which respectively produce the following equations

$$D_3D_1 - D_2^2 = \tag{2.22}$$

$$\frac{((a+b)^4\lambda^4 - 4ab(a+b)(a-b)^2\lambda^3 - 2a^2b^2(a+b)(5a-3b)\lambda^2 - 4a^3b^3(a+b)\lambda + a^4b^4)}{a^5(64b\lambda)^3} \tag{2.23}$$

$$E_3E_1 - E_2^2 = \tag{2.24}$$

$$\frac{((a+b)^4\lambda^4 + 4ab(a+b)(a-b)^2\lambda^3 + 2a^2b^2(a+b)(3a-5b)\lambda^2 + 4a^3b^3(a+b)\lambda + a^4b^4)}{64b^5(a\lambda)^3} \tag{2.25}$$

$$C_3C_1 - C_2^2 = \tag{2.26}$$

$$\frac{1}{64\lambda^5(ab)^3} \times ((a^2 - 6ab + b^2)(a+b)^2\lambda^4 + 4ab(a-b)(a+b)^2\lambda^3 \tag{2.27}$$

$$+ 2a^2b^2(3a^2 + 2ab + 3b^2)\lambda^2 + 4a^3b^3(a-b)\lambda + a^4b^4))$$

Each real solution γ_0 for the above equations for some fixed values of a and b will produce a 4-elliptic periodic trajectory which is not 4-periodic.

Some examples are shown in Figure 2.18 and Figure 2.19

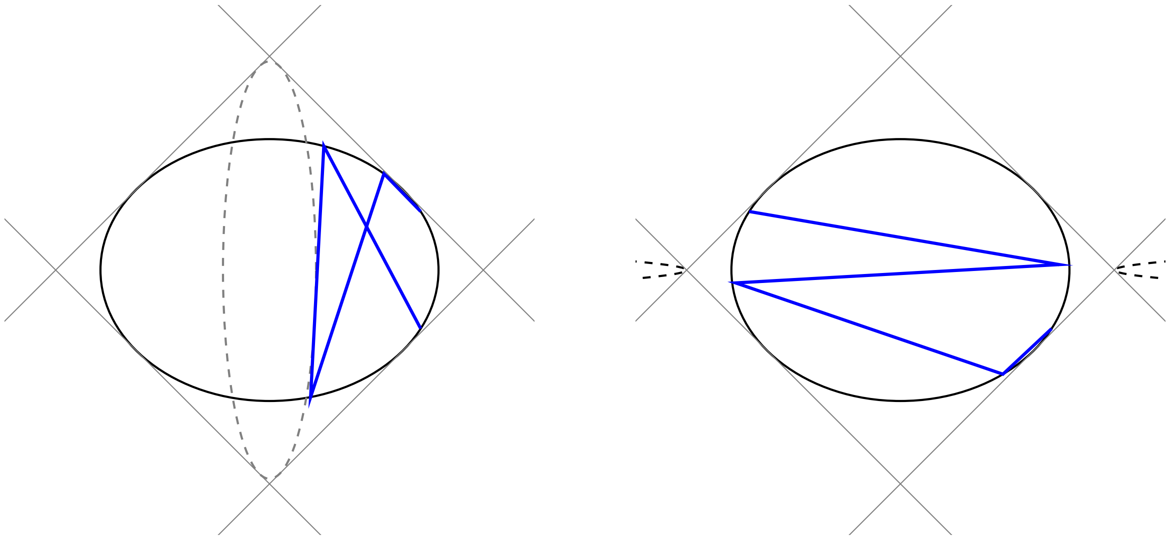

Figure 2.18: A 4-elliptic periodic trajectories. On the left, the caustic is an ellipse ($a = 5$, $b = 3$, $\gamma \approx 4.6216$), and it is a hyperbola on the right ($a = 5$, $b = 3$, $\gamma \approx -3.0243$).

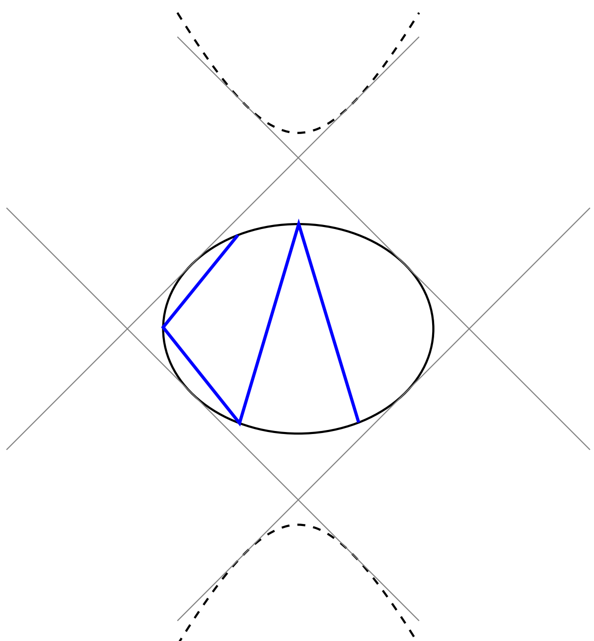

Figure 2.19: A 4-elliptic periodic trajectory without being 4-periodic with a hyperbola along the y-axis as caustic ($a = 5$, $b = 3$ and $\lambda_0 \approx 5.4942$).

5-elliptic periodic trajectories

There is a 5-elliptic periodic trajectory without being 5-periodic of the billiard within (2.1), with a non-degenerate caustic $\mathcal{C}_{\gamma_0}$ if and only if one of the following is satisfied:

- the caustic is an ellipse, with $\gamma_0 \in (0, a)$ or a hyperbola and $E_2E_4 - E_3^2 = 0$.
- the caustic is an ellipse, with $\gamma_0 \in (-b, 0)$ or a hyperbola $D_2D_4 - D_3^2 = 0$.

which yield the following equations

$$
\begin{aligned}
E_2E_4 - E_3^2 = \frac{2^{-10}}{b^7(a\lambda)^5} \times \Big(&(5a^2 - 10ab + b^2)(a+b)^4\lambda^6 + 2ab(5a - 3b)(a+b)^4\lambda^5 \\
&- a^2b^2(a+b)(9a^3 - 45a^2b - 5ab^2 - 15b^3)\lambda^4 - 4a^3b^3(a+b)(9a^2 - 10ab + 5b^2)\lambda^3 \\
&- a^4b^4(a+b)(29a - 15b)\lambda^2 - 6a^5b^5(a+b)\lambda + a^6b^6\Big) = 0 \\
D_2D_4 - D_3^2 = \frac{2^{-10}}{1024a^7(b\lambda)^5} \times \Big(&(a^2 - 10ab + 5b^2)(a+b)^4\lambda^6 + 2ab(3a - 5b)(a+b)^4\lambda^5 \\
&+ a^2b^2(a+b)(15a^3 + 5a^2b + 45ab^2 - 9b^3)\lambda^4 + 4a^3b^3(a+b)(5a^2 - 10ab + 9b^2)\lambda^3 \\
&+ a^4b^4(a+b)(15a - 29b)\lambda^2 + 6a^5b^5(a+b)\lambda + a^6b^6\Big) = 0
\end{aligned}
$$

Each real solution γ_0 for the above equations for some fixed values of a and b will produce a 4-elliptic periodic trajectory which is not 4-periodic.

Some examples are shown in Figure 2.20 and Figure 2.21

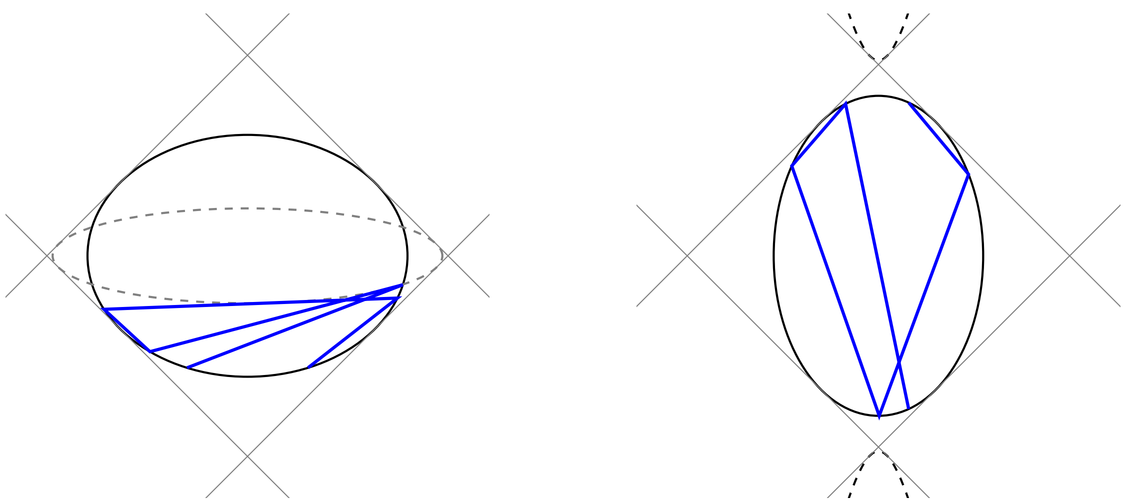

Figure 2.20: A 5-elliptic periodic trajectories. On the left, the caustic is an ellipse ($a = 7$, $b = 4$, $\gamma \approx -3.3848$) and a hyperbola on the right ($a = 3$, $b = 7$, $\gamma \approx 3.4462$).

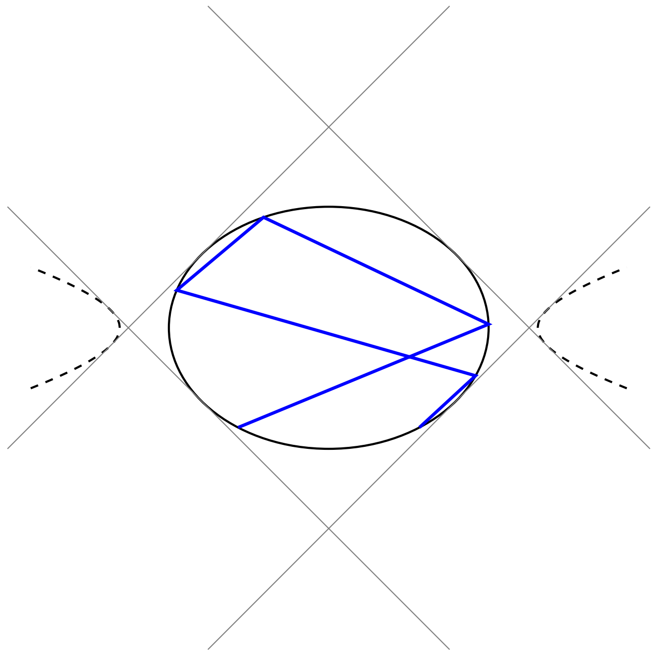

Figure 2.21: A 5-elliptic periodic trajectories without being 5-periodic with a hyperbola along the y-axis as caustic ($a = 7$, $b = 4$ and $\lambda_0 \approx -4.9683$)

A 6-elliptic periodic trajectories

There is a 6-elliptic periodic trajectory without being 6-periodic of the billiard within (2.1), with a non-degenerate caustic $\mathcal{C}_{\gamma_0}$ in the Minkowski plane if and only if one of the following is satisfied:

- the caustic is an ellipse, with $\gamma_0 \in (0, a)$; and $\begin{vmatrix} D_1 & D_2 & D_3 \\ D_2 & D_3 & D_4 \\ D_3 & D_4 & D_5 \end{vmatrix} = 0.$
- the caustic is an ellipse, with $\gamma_0 \in (-b, 0)$; and $\begin{vmatrix} E_1 & E_2 & E_3 \\ E_2 & E_3 & E_4 \\ E_3 & E_4 & E_5 \end{vmatrix} = 0.$

- the caustic is a hyperbola, and by $\begin{vmatrix} C_1 & C_2 & C_3 \\ C_2 & C_3 & C_4 \\ C_3 & C_4 & C_5 \end{vmatrix} = 0.$

which produces the following equations

- $$(13a^7 - 103a^6b + 293a^5b^2 - 823a^4b^3 + 1503a^3b^4 - 1677a^2b^5 + 3055ab^6 - 725b^7)(a+b)^2\lambda^10 - a(14a^7 - 171a^6b + 520a^5b^2 - 1481a^4b^3 + 3526a^3b^4 - 4989a^2b^5 + 6660ab^6 - 3119b^7)(a+b)^2\lambda^9 - 2a^2b(33a^6 - 172a^5b + 481a^4b^2 - 1456a^3b^3 + 2883a^2b^4 - 3268ab^5 + 2459b^6)(a+b)^2\lambda^8 - 4a^3b^2(a+b)(30a^6 - 73a^5b + 213a^4b^2 - 554a^3b^3 + 488a^2b^4 + 67ab^5 - 779b^6)\lambda^7 - 2b^3a^4(a+b)(52a^5 - 151a^4b + 370a^3b^2 - 744a^2b^3 + 474ab^4 + 127b^5)\lambda^6 - 2a^5b^4(a+b)(18a^4 - 27a^3b + 43a^2b^2 - 193ab^3 + 167b^4)\lambda^5 + 4a^6b^5(a+b)(9a^3 - 8a^2b + 47ab^2 - 16b^3)\lambda^4 + 4a^7b^6(a+b)(26a^2 + 29ab + 7b^2)\lambda^3 + a^8b^7(120a^2 + 85ab + 59b^2)\lambda^2 + a^9b^8(66a - 41b)\lambda + 14a^{10}b^9 = 0$$

- $$(a+b)^9\lambda^9 + ab(3a-b)^2(a-3b)^2(a+b)^4\lambda^8 + 4a^2b^2(9a^3 - 53a^2b + 51ab^2 - 15b^3)(a+b)^4\lambda^7 + 4b^3a^3(21a^2 - 62ab + 29b^2)(a+b)^4\lambda^6 + 2a^4b^4(a+b)(63a^4 + 60a^3b + 26a^2b^2 - 132ab^3 - 33b^4)\lambda^5 + 2a^5b^5(a+b)(63a^3 - 19a^2b + 141ab^2 - 33b^3)\lambda^4 + 4a^6b^6(a+b)(21a^2 - 38ab + 29b^2)\lambda^3 + 12a^7b^7(a+b)(3a-5b)\lambda^2 + 9a^8b^8(a+b)\lambda + a^9b^9 = 0$$

-

$$
\begin{aligned}
&(26a^6b - 178a^5b^2 + 308a^4b^3 - 308a^3b^4 + 178a^2b^5 - 26ab^6 \\
&+ 21a^6 - 154a^5b + 219a^4b^2 - 236a^3b^3 + 219a^2b^4 - 154ab^5 + 21b^6)(a+b)^4\lambda^{10} \\
&+ 2ab(37a^5b - 44a^4b^2 + 94a^3b^3 - 44a^2b^4 + 37ab^5 + 35a^5 - 47a^4b + 46a^3b^2 - 46a^2b^3 \\
&+ 47ab^4 - 35b^5)(a+b)^4\lambda^9 + a^2b^2(40a^6b - 200a^5b^2 - 240a^4b^3 + 240a^3b^4 + 200a^2b^5 \\
&- 40ab^6 + 85a^6 - 146a^5b + 187a^4b^2 - 188a^3b^3 + 187a^2b^4 - 146ab^5 + 85b^6)(a+b)^2\lambda^8 \\
&- 8a^3b^3(15a^5b - 60a^4b^2 + 106a^3b^3 - 60a^2b^4 + 15ab^5 - 5a^5 + 7a^4b - 20a^3b^2 + \\
&20a^2b^3 - 7ab^4 + 5b^5)(a+b)^2\lambda^7 - 2a^4b^4(90a^4b + 26a^3b^2 - 26a^2b^3 - 90ab^4 - 5a^4 + \\
&44a^3b + 98a^2b^2 + 44ab^3 - 5b^4)(a+b)^2\lambda^6 + 12a^5b^5(33a^3b - 190a^2b^2 + 33ab^3 - 5a^3 + \\
&79a^2b - 79ab^2 + 5b^3)(a+b)^2\lambda^5 + 2a^6b^6(772a^4b + 228a^3b^2 - 228a^2b^3 - 772ab^4 - \\
&295a^4 + 748a^3b + 2086a^2b^2 + 748ab^3 - 295b^4)\lambda^4 + 8a^7b^7(245a^3b - 30a^2b^2 + \\
&245ab^3 - 195a^3 - 69a^2b + 69ab^2 + 195b^3)\lambda^3 + a^8b^8(1130a^2b - 1130ab^2 - 1855a^2 + \\
&154ab - 1855b^2)\lambda^2 + 50a^9b^9(5ab - 21a + 21b)\lambda - 231a^{10}b^{10} = 0.
\end{aligned}
$$

Each real solution γ_0 for the above equations for some fixed values of a and b will produce a 6-elliptic periodic trajectory which is not 6-periodic.
Some examples are shown in Figure 2.22.

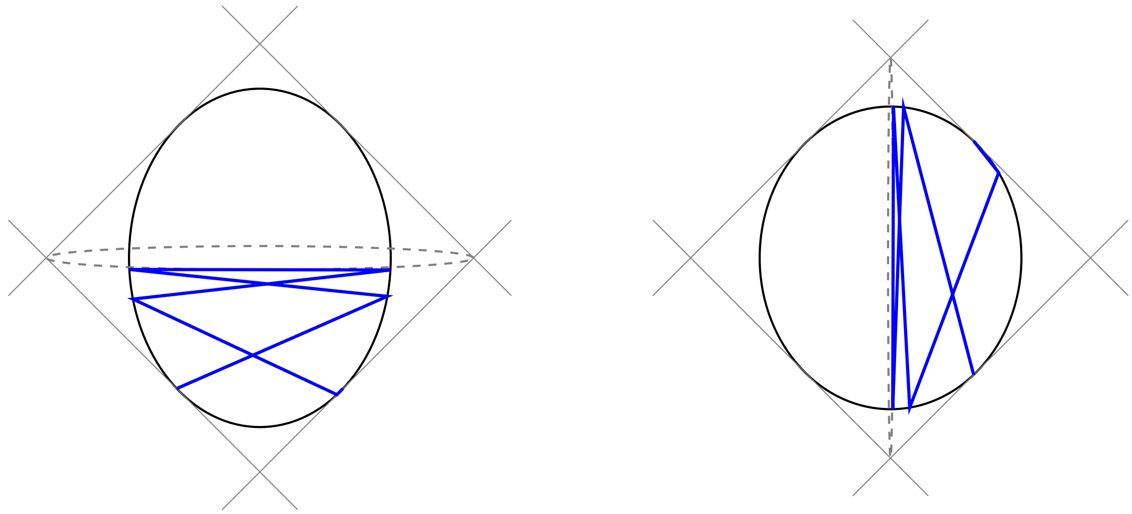

Figure 2.22: A 6-elliptic periodic trajectories. On the left, the caustic is an ellipse along the x-axis ($a = 3$, $b = 5$ and $\lambda_0 \approx -4.9755$), and it is a hyperbola along the y-axis on the right ($a = 3$, $b = 4$ and $\lambda_0 \approx 2.9989$).

7-elliptic periodic trajectories

There is a 7-elliptic periodic trajectory without being 7-periodic of the billiard within (2.1), with a non-degenerate caustic $\mathcal{C}_{\gamma_0}$ in the Minkowski plane if and only if one of the following is satisfied:

- the caustic is an ellipse, with $\gamma_0 \in (0, a)$, or a hyperbola and by
$$\begin{vmatrix} E_2 & E_3 & E_4 \\ E_3 & E_4 & E_5 \\ E_4 & E_5 & E_6 \end{vmatrix} = 0,.$$
- the caustic is an ellipse, with $\gamma_0 \in (-b, 0)$ or a hyperbola and by
$$\begin{vmatrix} D_2 & D_3 & D_4 \\ D_3 & D_4 & D_5 \\ D_4 & D_5 & D_6 \end{vmatrix} = 0.$$

which produce the following equations

- $$- (7a^3 - 35a^2b + 21ab^2 - b^3)(a+b)^9\lambda^{12} - 4ab(7a^2 - 14ab+$$
$$3b^2)(a+b)^9\lambda^{11} + 2a^2b^2(21a^6 - 342a^5b + 375a^4b^2 - 804a^3b^3 + 435a^2b^4 - 38ab^5$$
$$+ 33b^6)(a+b)^4\lambda^{10} + 4a^3b^3(121a^5 - 595a^4b + 1018a^3b^2 - 998a^2b^3 + 285ab^4$$
$$- 55b^5)(a+b)^4\lambda^9 + a^4b^4(1311a^4 - 4500a^3b + 6378a^2b^2 - 3188ab^3$$
$$+ 495b^4)(a+b)^4\lambda^8 + 8a^5b^5(225a^3 - 585a^2b + 499ab^2 - 99b^3)(a+b)^4\lambda^7$$
$$+ 4a^6b^6(a+b)(315a^5 + 483a^4b - 658a^3b^2 - 826a^2b^3 - 25ab^4 + 231b^5)\lambda^6$$
$$+ 8a^7b^7(a+b)(21a^4 + 252a^3b - 42a^2b^2 - 52ab^3 - 99b^4)\lambda^5 - a^8b^8(a+b)(441a^3$$
$$- 1197a^2b + 235ab^2 - 495b^3)\lambda^4 - 4a^9b^9(a+b)(91a^2 - 102ab + 55b^2)\lambda^3$$
$$- 2a^{10}b^{10}(a+b)(59a - 33b)\lambda^2 - 12a^{11}b^{11}(a+b)\lambda + a^{12}b^{12} = 0$$

- $$(a^3 - 21a^2b + 35ab^2 - 7b^3)(a+b)^9\lambda^12 + 4ab(3a^2 - 14ab+$$
$$7b^2)(a+b)^9\lambda^{11} + 2a^2b^2(33a^6 - 38a^5b + 435a^4b^2 - 804a^3b^3 + 375a^2b^4 - 342ab^5$$
$$+ 21b^6)(a+b)^4\lambda^{10} + 4a^3b^3(55a^5 - 285a^4b + 998a^3b^2 - 1018a^2b^3 + 595ab^4$$
$$- 121b^5)(a+b)^4\lambda^9 + a^4b^4(495a^4 - 3188a^3b + 6378a^2b^2 - 4500ab^3$$
$$+ 1311b^4)(a+b)^4\lambda^8 + 8a^5b^5(99a^3 - 499a^2b + 585ab^2 - 225b^3)(a+b)^4\lambda^7$$
$$+ 4a^6b^6(a+b)(231a^5 - 25a^4b - 826a^3b^2 - 658a^2b^3 + 483ab^4 + 315b^5)\lambda^6$$
$$+ 8a^7b^7(a+b)(99a^4 + 52a^3b + 42a^2b^2 - 252ab^3 - 21b^4)\lambda^5 + a^8b^8(a+b)(495a^3$$
$$- 235a^2b + 1197ab^2 - 441b^3)\lambda^4 + 4a^9b^9(a+b)(55a^2 - 102ab + 91b^2)\lambda^3$$
$$+ 2a^{10}b^{10}(a+b)(33a - 59b)\lambda^2 + 12a^{11}b^{11}(a+b)\lambda + a^{12}b^{12} = 0$$

Each real solution γ_0 for the above equations for some fixed values of a and b will produce a 7-elliptic periodic trajectory which is not 7-periodic.

Some examples of a 7-elliptic periodic trajectories without being 7-periodic are shown in Figure 2.23 and Figure 2.23

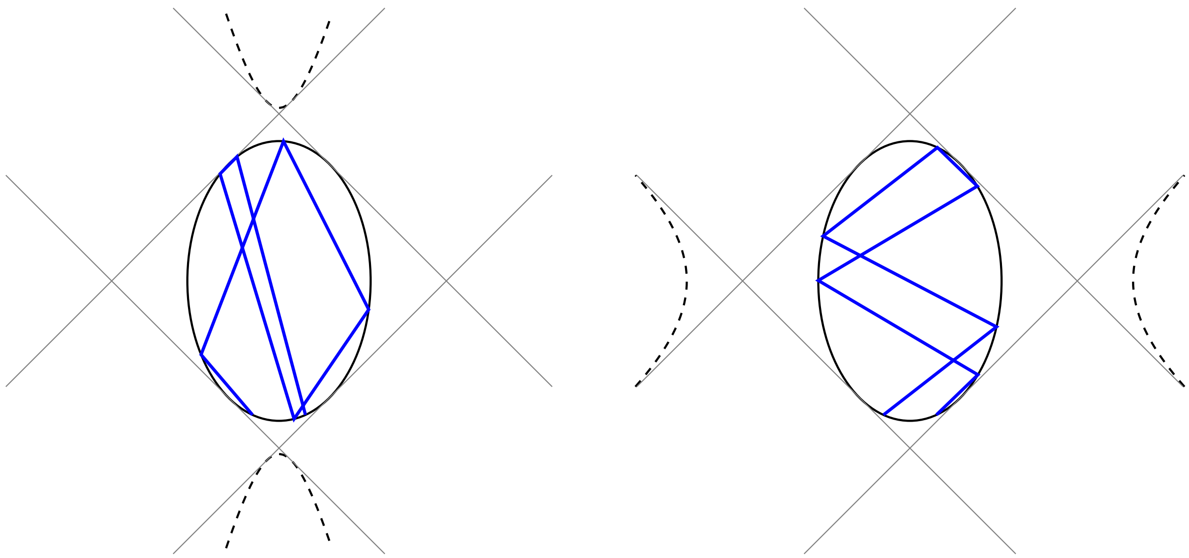

Figure 2.23: A 7-elliptic periodic trajectories. On the left, the caustic is a hyperbola along the y-axis ($a = 3$, $b = 7$ and $\lambda_0 \approx 3.7232$), and it is a hyperbola along the x-axis on the right ($a = 3$, $b = 7$ and $\lambda_0 \approx -10.7847$).

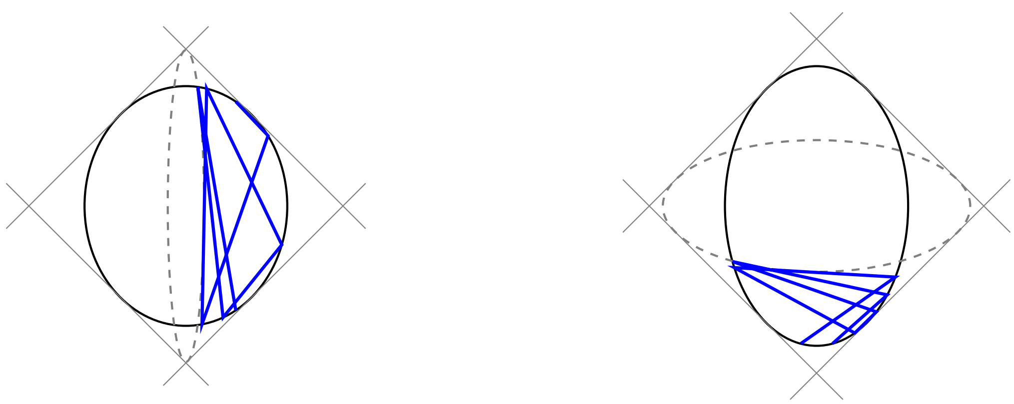

Figure 2.24: A 7-elliptic periodic trajectories. On the left, the caustic is an ellipse along the y-axis ($a = 5$, $b = 7$ and $\lambda_0 \approx 4.8394$), and it is an ellipse along the x-axis on the right ($a = 3$, $b = 7$ and $\lambda_0 \approx -5.4467$).

Discriminantly separable polynomials and elliptic periodicity

Since the case $n=2$ is trivial, we start with the case $n=3$.

From the numerator of E_2 in Equation (2.20) and D_2 in Equation (2.21), we have:

$$G_1(a,b,\gamma) = -(a+b)(3a-b)\gamma^2 - 2ab(a+b)\gamma + a^2b^2,$$
$$G_2(a,b,\gamma) = (a+b)(a-3b)\gamma^2 + 2ab(a+b)\gamma + a^2b^2,$$

and we calculate the discriminants, which factorize as follows:

$$\mathscr{D}_\gamma \mathsf{G}_1 = 16a^3b^2(a+b), \quad \mathscr{D}_\gamma \mathsf{G}_2 = 16b^3a^2(a+b).$$

Similarly, for $n=4$, from the numerator of $D_3D_1 - D_2^2$ in Equation (2.23), $E_3E_1 - E_2^2$ in Equation (2.25) and $C_3C_1 - C_2^2$ in Equation (2.27), we have:

$$G_3(a,b,\gamma) =$$
$$(a+b)^4\gamma^4 - 4ab(a+b)(a-b)^2\gamma^3 - 2a^2b^2(a+b)(5a-3b)\gamma^2 - 4a^3b^3(a+b)\gamma + a^4b^4,$$
$$G_4(a,b,\gamma) =$$
$$(a+b)^4\gamma^4 + 4ab(a+b)(a-b)^2\gamma^3 + 2a^2b^2(a+b)(3a-5b)\gamma^2 + 4a^3b^3(a+b)\gamma + a^4b^4,$$
$$G_5(a,b,\gamma) =$$
$$(a^2-6ab+b^2)(a+b)^2\gamma^4 + 4ab(a-b)(a+b)^2\gamma^3 + 2a^2b^2(3a^2+2ab+3b^2)\gamma^2$$
$$+4a^3b^3(a-b)\gamma + a^4b^4.$$

The discriminant of these polynomials factorizes as follows:

$$\mathscr{D}_\gamma \mathsf{G}_3 = -2^{16}a^{16}b^{14}(8a^2+8ab+27b^2)(a+b)^4,$$
$$\mathscr{D}_\gamma \mathsf{G}_4 = -2^{16}a^{14}b^{16}(27a^2+8ab+8b^2)(a+b)^4,$$
$$\mathscr{D}_\gamma \mathsf{G}_5 = 2^{12}(32a^6 - 491a^5b - 439a^4b^2 + 194a^3b^3 - 62a^2b^4 - 39ab^5 + 5b^6)(a+b)^3b^{15}a^{12}.$$

Using the transformation $(a, b) \mapsto (a, \hat{b})$, where $\hat{b} = \frac{b}{a}$, we get:

$$\mathscr{D}_\gamma \mathsf{G}_1 = 16a^6\hat{b}^2(1+\hat{b}),$$

$$\mathscr{D}_\gamma \mathsf{G}_2 = 16a^5\hat{b}^3(1+\hat{b}),$$

$$\mathscr{D}_\gamma \mathsf{G}_3 = -2^{16}a^{36}\hat{b}^{14}(8+8\hat{b}+27\hat{b}^2)(1+\hat{b})^4,$$

$$\mathscr{D}_\gamma \mathsf{G}_4 = -2^{16}a^{36}\hat{b}^{16}(27+8\hat{b}+8\hat{b}^2)(1+\hat{b})^4,$$

$$\mathscr{D}_\gamma \mathsf{G}_5 = 2^{12}a^{36}\hat{b}^{15}(32-491\hat{b}-439\hat{b}^2+194\hat{b}^3-62\hat{b}^4-39\hat{b}^5+5\hat{b}^6)(1+\hat{b})^3.$$

2.5 Polynomial equations

Now we want to express the periodicity conditions for billiard trajectories in the Minkowski plane in terms of polynomial functions equations.

Theorem 2.5.1. The billiard trajectories within $\mathcal{E}$ with caustic $\mathcal{C}_\gamma$ are n-periodic if and only if there exists a pair of real polynomials p_{d_1}, q_{d_2} of degrees d_1, d_2 respectively, and satisfying the following:

(a) if $n = 2m$ is even, then $d_1 = m$, $d_2 = m-2$, and

$$p_m^2(s) - s\left(s - \frac{1}{a}\right)\left(s + \frac{1}{b}\right)\left(s - \frac{1}{\gamma}\right)q_{m-2}^2(s) = 1;$$

(b) if $n = 2m+1$ is odd, then $d_1 = m$, $d_2 = m-1$, and

$$\left(s - \frac{1}{\gamma}\right)p_m^2(s) - s\left(s - \frac{1}{a}\right)\left(s + \frac{1}{b}\right)q_{m-1}^2(s) = -\operatorname{sign}\gamma.$$

Proof. We note first that the proof of Theorem 2.1.2 implies that there is a non-trivial linear combination of the bases (2.7) for n even, or (2.8) for n odd, with the zero of order n at $x = 0$.

(a) For $n = 2m$, from there we get that there are real polynomials $p_m^*(x)$ and $q_{m-2}^*(x)$ of degrees m and $m-2$ respectively, such that the expression

$$p_m^*(x) - q_{m-2}^*(x)\sqrt{\varepsilon(a-x)(b+x)(\gamma-x)}$$

has a zero of order $2m$ at $x = 0$. Multiplying that expression by

$$p^*_m(x) + q^*_{m-2}(x)\sqrt{\varepsilon(a-x)(b+x)(\gamma-x)},$$

we get that the polynomial $(p^*_m(x))^2 - \varepsilon(a-x)(b+x)(\gamma-x)(q^*_{m-2}(x))^2$ has a zero of order $2m$ at $x = 0$. Since the degree of that polynomial is $2m$, is follows that:

$$(p^*_m(x))^2 - \varepsilon(a-x)(b+x)(\gamma-x)(q^*_{m-2}(x))^2 = cx^{2m},$$

for some constant c. Notice that c is positive, since it equals the square of the leading coefficient of p^*_m. Dividing the last relation by cx^{2m} and introducing $s = 1/x$, we get the requested relation.

(b) On the other hand, for $n = 2m + 1$, we get that there are real polynomials $p^*_m(x)$ and $q^*_{m-1}(x)$ of degrees m and $m - 1$ respectively, such that the expression

$$p^*_m(x) - q^*_{m-1}(x)\frac{\sqrt{\varepsilon(a-x)(b+x)(\gamma-x)}}{\gamma-x}$$

has a zero of order $2m + 1$ at $x = 0$. Multiplying that expression by

$$(\gamma-x)\left(p^*_m(x) + q^*_{m-1}(x)\frac{\sqrt{\varepsilon(a-x)(b+x)(\gamma-x)}}{\gamma-x}\right),$$

we get that the polynomial $(\gamma-x)(p^*_m(x))^2 - \varepsilon(a-x)(b+x)(q^*_{m-1}(x))^2$ has a zero of order $2m + 1$ at $x = 0$. Since the degree of that polynomial is $2m + 1$, is follows that:

$$(\gamma-x)(p^*_m(x))^2 - \varepsilon(a-x)(b+x)(q^*_{m-1}(x))^2 = cx^{2m+1},$$

for some constant c. Notice that c is negative, since it equals the opposite of the square of the leading coefficient of p^*_m. Dividing the last relation by $-\varepsilon cx^{2m+1}$ and introducing $s = 1/x$, we get the requested relation. □

Corollary 2.5.1. If the billiard trajectories within $\mathcal{E}$ with caustic $\mathcal{C}_\gamma$ are n-periodic, then there exist real polynomials $\hat{p}_n$ and $\hat{q}_{n-2}$ of degrees n and $n - 2$ respectively, which satisfy the Pell equation:

$$\hat{p}_n^2(s) - s\left(s-\frac{1}{a}\right)\left(s+\frac{1}{b}\right)\left(s-\frac{1}{\gamma}\right)\hat{q}_{n-2}^2(s) = 1. \tag{2.28}$$

Proof. For $n = 2m$, take $\hat{p}_n = 2p_m^2 - 1$ and $\hat{q}_{n-2} = 2p_m q_{m-2}$. For $n = 2m + 1$, we set $\hat{p}_n = 2(\gamma s - 1)p_m^2 + \operatorname{sign}\gamma$ and $\hat{q}_{n-2} = 2p_m q_{m-1}$. $\square$

Theorem 2.5.2. The billiard trajectories within $\mathcal{E}$ with caustic $\mathcal{C}_\gamma$ are elliptic n-periodic without being n-periodic if and only if there exists a pair of real polynomials p_{d_1}, q_{d_2} of degrees d_1, d_2 respectively, and satisfying the following:

(a) $\mathcal{C}_\gamma$ is an ellipse, $0 < \gamma < a$, and

- $n = 2m$ is even, $d_1 = d_2 = m - 1$,

$$s\left(s - \frac{1}{a}\right)p_{m-1}^2(s) - \left(s + \frac{1}{b}\right)\left(s - \frac{1}{\gamma}\right)q_{m-1}^2(s) = 1;$$

- $n = 2m + 1$ is odd, $d_1 = m$, $d_2 = m - 1$,

$$\left(s + \frac{1}{b}\right)p_m^2(s) - s\left(s - \frac{1}{a}\right)\left(s - \frac{1}{\gamma}\right)q_{m-1}^2(s) = 1;$$

(b) $\mathcal{C}_\gamma$ is an ellipse, $-b < \gamma < 0$, and

- $n = 2m$ is even, $d_1 = d_2 = m - 1$,

$$s\left(s + \frac{1}{b}\right)p_{m-1}^2(s) - \left(s - \frac{1}{a}\right)\left(s - \frac{1}{\gamma}\right)q_{m-1}^2(s) = 1;$$

- $n = 2m + 1$ is odd, $d_1 = m$, $d_2 = m - 1$,

$$\left(s - \frac{1}{a}\right)p_m^2(s) - s\left(s + \frac{1}{b}\right)\left(s - \frac{1}{\gamma}\right)q_{m-1}^2(s) = -1;$$

(c) $\mathcal{C}_\gamma$ is a hyperbola and $n = 2m$ is even, $d_1 = d_2 = m - 1$,

$$\left(s - \frac{1}{\gamma}\right)p_{m-1}^2(s) - s\left(s - \frac{1}{a}\right)\left(s + \frac{1}{b}\right)q_{m-1}^2(s) = -\operatorname{sign}\gamma;$$

(d) $\mathcal{C}_\gamma$ is a hyperbola, $n = 2m + 1$ is odd, $d_1 = m$, $d_2 = m - 1$,

$$\left(s - \frac{1}{a}\right)p_m^2(s) - s\left(s + \frac{1}{b}\right)\left(s - \frac{1}{\gamma}\right)q_{m-1}^2(s) = -1;$$

(e) $\mathcal{C}_\gamma$ is a hyperbola, $n = 2m+1$ is odd, $d_1 = m$, $d_2 = m-1$,

$$\left(s + \frac{1}{b}\right) p_m^2(s) - s\left(s - \frac{1}{a}\right)\left(s - \frac{1}{\gamma}\right) q_{m-1}^2(s) = 1.$$

Proof. (a) For $n = 2m$, the proof of Theorem 2.3.2 implies that there are polynomials $p_{m-1}^*(x)$ and $q_{m-1}^*(x)$ of degrees $m-1$, such that the expression

$$p_{m-1}^*(x) - q_{m-1}^*(x)\frac{\sqrt{(a-x)(b+x)(\gamma-x)}}{a-x}$$

has a zero of order $2m$ at $x = 0$. Multiplying that expression by

$$(a-x)\left(p_{m-1}^*(x) + q_{m-1}^*(x)\frac{\sqrt{(a-x)(b+x)(\gamma-x)}}{a-x}\right),$$

we get that the polynomial $(a-x)(p_{m-1}^*(x))^2 - (a-x)(b+x)(q_{m-1}^*(x))^2$ has a zero of order $2m$ at $x = 0$. Since the degree of that polynomial is $2m$, is follows that:

$$(a-x)(p_{m-1}^*(x))^2 - (b+x)(\gamma-x)(q_{m-1}^*(x))^2 = cx^{2m},$$

for some constant c. Notice that c is positive, since it equals the square of the leading coefficient of q_{m-1}^*. Dividing the last relation by cx^{2m} and introducing $s = 1/x$, we get the requested relation.

For $n = 2m+1$, the proof of Theorem 2.3.2 implies that there are polynomials $p_m^*(x)$ and $q_{m-1}^*(x)$ of degrees m and $m-1$, such that the expression

$$p_m^*(x) - q_{m-1}^*(x)\frac{\sqrt{(a-x)(b+x)(\gamma-x)}}{b+x}$$

has a zero of order $2m+1$ at $x = 0$. Multiplying that expression by

$$(b+x)\left(p_m^*(x) + q_{m-1}^*(x)\frac{\sqrt{(a-x)(b+x)(\gamma-x)}}{b+x}\right),$$

we get that the polynomial $(b+x)(p_m^*(x))^2 - (a-x)(\gamma-x)(q_{m-1}^*(x))^2$ has a zero of order $2m+1$ at $x = 0$. Since the degree of that polynomial is $2m+1$, is follows that:

$$(b+x)(p_m^*(x))^2 - (a-x)(\gamma-x)(q_{m-1}^*(x))^2 = cx^{2m+1}$$

for some constant c. Notice that c is positive, since it equals the square of the leading coefficient of p_m^*. Dividing the last relation by cx^{2m+1} and introducing $s = 1/x$, we get the requested relation.

(b) For $n = 2m$, the proof of Theorem 2.3.2 implies that there are real polynomials $p_{m-1}^*(x)$ and $q_{m-1}^*(x)$ of degrees $m-1$, such that the expression

$$p_{m-1}^*(x) - q_{m-1}^*(x)\frac{\sqrt{-(a-x)(b+x)(\gamma-x)}}{b+x}$$

has a zero of order $2m$ at $x = 0$. Multiplying that expression by

$$(b+x)\left(p_{m-1}^*(x) + q_{m-1}^*(x)\frac{\sqrt{-(a-x)(b+x)(\gamma-x)}}{b+x}\right),$$

we get that the polynomial $(b+x)(p_{m-1}^*(x))^2 + (a-x)(\gamma-x)(q_{m-1}^*(x))^2$ has a zero of order $2m$ at $x = 0$. Since the degree of that polynomial is $2m$, is follows that:

$$(b+x)(p_{m-1}^*(x))^2 + (a-x)(\gamma-x)(q_{m-1}^*(x))^2 = cx^{2m},$$

for some constant c. Notice that c is positive, since it equals to the square of the leading coefficient of q_{m-1}^*. Dividing the last relation by cx^{2m} and introducing $s = 1/x$, we get the requested relation.

For $n = 2m+1$, the proof of Theorem 2.3.2 implies that there are polynomials $p_m^*(x)$ and $q_{m-1}^*(x)$ of degrees m and $m-1$, such that the expression

$$p_m^*(x) - q_{m-1}^*(x)\frac{\sqrt{-(a-x)(b+x)(\gamma-x)}}{a-x}$$

has a zero of order $2m+1$ at $x = 0$. Multiplying that expression by

$$(a-x)\left(p_m^*(x) + q_{m-1}^*(x)\frac{\sqrt{-(a-x)(b+x)(\gamma-x)}}{a-x}\right),$$

we get that the polynomial $(a-x)(p_m^*(x))^2 + (b+x)(\gamma-x)(q_{m-1}^*(x))^2$ has a zero of order $2m+1$ at $x = 0$. Since the degree of that polynomial is $2m+1$, is follows that:

$$(a-x)(p_m^*(x))^2 + (b+x)(\gamma-x)(q_{m-1}^*(x))^2 = cx^{2m+1}$$

for some constant c. Notice that c is negative, since it is opposite to the square of the leading coefficient of p_m^*. Dividing the last relation by $-cx^{2m+1}$ and introducing $s = 1/x$, we get the requested relation.

For (c), the proof of Theorem 2.3.2 implies that there are polynomials real $p_m^*(x)$ and $q_{m-1}^*(x)$ of degrees m and $m-1$, such that the expression

$$p_m^*(x) - q_{m-1}^*(x)\frac{\sqrt{\varepsilon(a-x)(b+x)(\gamma-x)}}{\gamma - x}$$

has a zero of order $2m+1$ at $x = 0$. Multiplying that expression by

$$(\gamma - x)\left(p_m^*(x) + q_{m-1}^*(x)\frac{\sqrt{\varepsilon(a-x)(b+x)(\gamma-x)}}{\gamma - x}\right),$$

we get that the polynomial $(\gamma - x)(p_m^*(x))^2 - \varepsilon(a-x)(b+x)(q_{m-1}^*(x))^2$ has a zero of order $2m+1$ at $x = 0$. Since the degree of that polynomial is $2m+1$, is follows that:

$$(\gamma - x)(p_m^*(x))^2 - \varepsilon(a-x)(b+x)(q_{m-1}^*(x))^2 = cx^{2m+1}$$

for some constant c. Notice that c is negative, since it is opposite to the square of the leading coefficient of p_m^*. Dividing the last relation by $-\varepsilon c x^{2m+1}$ and introducing $s = 1/x$, we get the requested relation.

(d) The proof of Theorem 2.3.2 implies that there are real polynomials $p_m^*(x)$ and $q_{m-1}^*(x)$ of degrees m and $m-1$, such that the expression

$$p_m^*(x) - q_{m-1}^*(x)\frac{\sqrt{\varepsilon(a-x)(b+x)(\gamma-x)}}{a - x}$$

has a zero of order $2m+1$ at $x = 0$. Multiplying that expression by

$$(a - x)\left(p_m^*(x) + q_{m-1}^*(x)\frac{\sqrt{\varepsilon(a-x)(b+x)(\gamma-x)}}{a - x}\right),$$

we get that the polynomial $(a - x)(p_m^*(x))^2 - \varepsilon(b+x)(\gamma-x)(q_{m-1}^*(x))^2$ has a zero of order $2m+1$ at $x = 0$. Since the degree of that polynomial is $2m+1$, is follows that:

$$(a - x)(p_m^*(x))^2 - \varepsilon(b+x)(\gamma-x)(q_{m-1}^*(x))^2 = cx^{2m+1}$$

for some constant c. Notice that c is negative, since it is opposite to the square of the leading coefficient of p^*_m. Dividing the last relation by $-cx^{2m+1}$ and introducing $s = 1/x$, we get the requested relation.

(e) For $n = 2m + 1$, the proof of Theorem 2.3.2 implies that there are real polynomials $p^*_m(x)$ and $q^*_{m-1}(x)$ of degrees m and $m - 1$, such that the expression

$$p^*_m(x) - q^*_{m-1}(x)\frac{\sqrt{\varepsilon(a-x)(b+x)(\gamma-x)}}{b+x}$$

has a zero of order $2m + 1$ at $x = 0$. Multiplying that expression by

$$(b+x)\left(p^*_m(x) + q^*_{m-1}(x)\frac{\sqrt{\varepsilon(a-x)(b+x)(\gamma-x)}}{b+x}\right),$$

we get that the polynomial $(b+x)(p^*_m(x))^2 - \varepsilon(a-x)(\gamma-x)(q^*_{m-1}(x))^2$ has a zero of order $2m + 1$ at $x = 0$. Since the degree of that polynomial is $2m + 1$, is follows that:

$$(b+x)(p^*_m(x))^2 - \varepsilon(a-x)(\gamma-x)(q^*_{m-1}(x))^2 = cx^{2m+1}$$

for some constant c. Notice that c is positive, since it equals the square of the leading coefficient of p^*_m. Dividing the last relation by cx^{2m+1} and introducing $s = 1/x$, we get the requested relation. $\square$

After Corollary 2.5.1 and the relation Equation (2.28), we see that the Pell equations arise as the functional polynomial conditions for periodicity. Let us recall some important properties of the solutions of pell's equations.

2.6 Classical Extremal Polynomials and Caustics

2.6.1 Fundamental Properties of Extremal Polynomials

From the previous section we know that the Pell's equation plays a key role in functional-polynomial formulation of periodicity conditions in the Minkowski plane. The solutions of the Pell's equation are so-called extremal polynomials. Denote $\{c_1, c_2, c_3, c_4\} = \{0, \frac{1}{a}, -\frac{1}{b}, \frac{1}{\lambda_0}\}$ with the ordering $c_1 < c_2 < c_3 < c_4$. The polynomials $\hat{p}_n$ are so called *generalized Chebyshev polynomials* on two intervals $[c_1, c_2] \cup [c_3, c_4]$, with an appropriate normalization. Namely, one can consider the question of finding the monic polynomial of certain degree n which minimizes the maximum norm on the union of two intervals. Denote such a polynomial as $\hat{P}_n$ and its norm L_n. The fact that polynomial $\hat{p}_n$ is a solution of the Pell's equation on the union of intervals $[c_1, c_2] \cup [c_3, c_4]$ is equivalent to the following conditions:

(i) $\hat{p}_n = \hat{P}_n / \pm L_n$

(ii) the set $[c_1, c_2] \cup [c_3, c_4]$ is the maximal subset of $\mathbf{R}$ for which $\hat{P}_n$ is the minimal polynomial in the sense above.

Chebyshev was the first who considered a similar problem on one interval, and this was how celebrated Chebyshev polynomials emerged in XIXth century. We are going to say a bit more about original Chebyshev polynomials below. Let us recall a fundamental result about generalized Chebyshev polynomials [6, 7].

Theorem 2.6.1 (A corollary of the Krein-Levin-Nudelman Theorem)**.** [35] There exists a polynomial P_n of degree n which satisfies a Pell equation on the union of intervals $[c_1, c_2] \cup [c_3, c_4]$ if and only if there exists an integer n_1 such that the equation holds:

$$n_1 \int_{c_2}^{c_3} \hat{f}(s)ds = n \int_{c_4}^{\infty} \hat{f}(s)ds, \text{ where } \hat{f}(s) = \left(\sqrt{\prod_{i=1}^{4} (s - c_i)} \right)^{-1}.. \tag{2.29}$$

The modulus of the polynomial reaches its maximal values L_n at the points $c_i : |P_n(c_i)| = L_n$. In addition, there are exactly $\tau_1 = n - n_1 - 1$ internal extremal points of the interval $[c_3, c_4]$ where $|P_n|$ reaches the value L_n, and there are $\tau_2 = n_1 - 1$ internal extremal points of $[c_1, c_2]$ with the same property.

Definition 2.6.1. [26] We call the pair (n, n_1) the partition and (τ_1, τ_2) the signature of the generalized Chebyshev polynomial P_n.

Now we are going to formulate and prove the main result of this Section, which relates n_1, n_2 the numbers of reflections off relativistic ellipses and off relativistic hyperbolas respectively with the partition and the signature of the related solution of a Pell equation.

Theorem 2.6.2. Given a periodic billiard trajectory with period $n = n_1 + n_2$, where n_1 is the number of reflections off relativistic ellipses, n_2 the number of reflections off the relativistic hyperbolas, then the partition corresponding to this trajectory is (n, n_1). The corresponding extremal polynomial $\hat{p}_n$ of degree n has $n_1 - 1$ internal extremal points in the first interval and $n - n_1 - 1 = n_2 - 1$ internal extremal points in the second interval.

Proof. Recall that $c_1 < c_2 < c_3 < c_4$. From the Equation (2.5), one has:

$$n_1 \int_{b_0}^{0} f(x)dx + n_2 \int_{b_1}^{0} f(x)dx = 0 \tag{2.30}$$

where b_0 is the largest negative value in $\{a, -b, \gamma\}$ and b_1 the smallest positive value in $\{a, -b, \gamma\}$.

Case 1: $\mathscr{C}_\gamma$ is an ellipse and $\gamma < 0$, shown on Figure 2.25

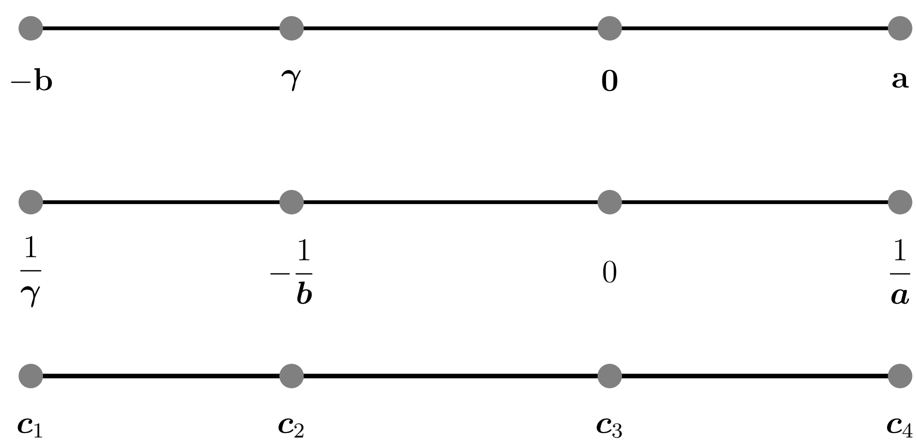

Figure 2.25: $b_0 = \boldsymbol{\gamma}$, $b_1 = \boldsymbol{a}$.

Equation (2.30) is equivalent to

$$n_1 \int_\gamma^0 f(x)dx + n_1 \int_0^a f(x)dx + n_2 \int_a^0 f(x)dx - n_1 \int_0^a f(x)dx = 0$$

$$n_1 \int_\gamma^a f(x)dx + (n_1 + n_2) \int_a^0 f(x)dx = 0$$

$$n_1 \int_\gamma^a f(x)dx = (n_1 + n_2) \int_0^a f(x)dx$$

Since the cycles around the cuts on the elliptic curve are homologous:

$$\int_\gamma^a f(x)dx = \int_{-\infty}^{-b} f(x)dx$$

Hence Equation (2.30) is equivalent to

$$n_1 \int_{-\infty}^{-b} f(x)dx = (n_1 + n_2) \int_0^a f(x)dx$$

Let $s = \frac{1}{x}$, $c_1 = \frac{1}{\gamma}$, $c_2 = -\frac{1}{b}$,$c_3 = 0$, $c_4 = \frac{1}{a}$ (see Figure 2.25) and substitute in the above to get

$$n_1 \int_{-\frac{1}{b}}^0 \tilde{f}(s)ds = (n_1 + n_2) \int_{\frac{1}{a}}^\infty \tilde{f}(s)ds \Leftrightarrow n_1 \int_{c_2}^{c_3} \tilde{f}(s)ds = (n_1 + n_2) \int_{c_4}^\infty \tilde{f}(s)ds$$

The right hand side of the above equivalent relation is tagged as follows

$$n_1 \int_{c_2}^{c_3} \tilde{f}(s)ds = (n_1 + n_2) \int_{c_4}^\infty \tilde{f}(s)ds \tag{2.31}$$

Case 2: $\mathscr{C}_\gamma$ is an ellipse and $\gamma > 0$, shown on Figure 2.26

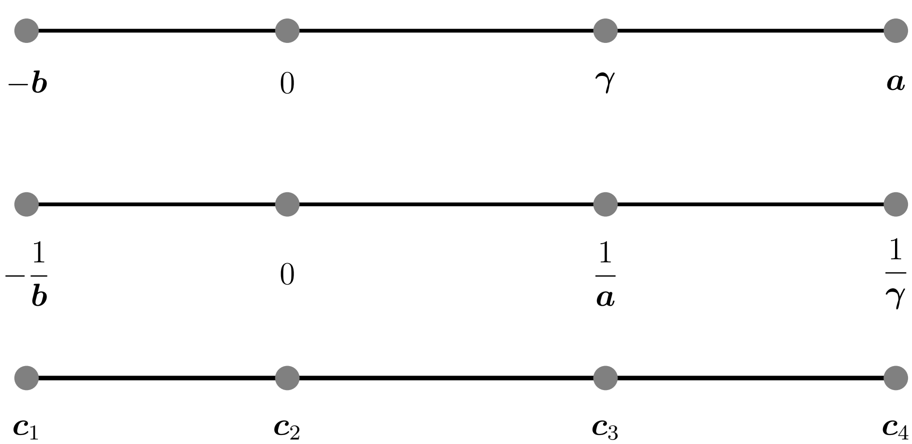

Figure 2.26: $b_0 = -\boldsymbol{b}$, $b_1 = \boldsymbol{\gamma}$.

Equation (2.30) is equivalent to

$$\begin{aligned}
n_1\int_{-b}^{0} f(x)dx + n_1\int_{0}^{\gamma} f(x)dx + n_2\int_{\gamma}^{0} f(x)dx - n_1\int_{0}^{\gamma} f(x)dx &= 0\\
n_1\int_{-b}^{\gamma} f(x)dx + (n_1+n_2)\int_{\gamma}^{0} f(x)dx &= 0\\
n_1\int_{-b}^{\gamma} f(x)dx &= (n_1+n_2)\int_{0}^{\gamma} f(x)dx
\end{aligned}$$

Since the cycles around the cuts on the elliptic curve are homologous:

$$\int_{-b}^{\gamma} f(x)dx = \int_{a}^{\infty} f(x)dx$$

Hence Equation (2.30) is equivalent to

$$n_1\int_{a}^{\infty} f(x)dx = (n_1+n_2)\int_{0}^{\gamma} f(x)dx$$

Let $s=\frac{1}{x}$, $c_1=-\frac{1}{b}$, $c_2=0$, $c_3=\frac{1}{a}$, $c_4=\frac{1}{\gamma}$, (see Figure 2.26) and substitute in the above to get

$$n_1\int_{0}^{\frac{1}{a}} \tilde{f}(s)ds = (n_1+n_2)\int_{\frac{1}{\gamma}}^{\infty} \tilde{f}(s)ds \Leftrightarrow \textit{Equation } (2.31).$$

Case 3: *i.*) $\mathscr{C}_\gamma$ **is a hyperbola and** $\gamma < -b$**, shown on Figure 2.27**

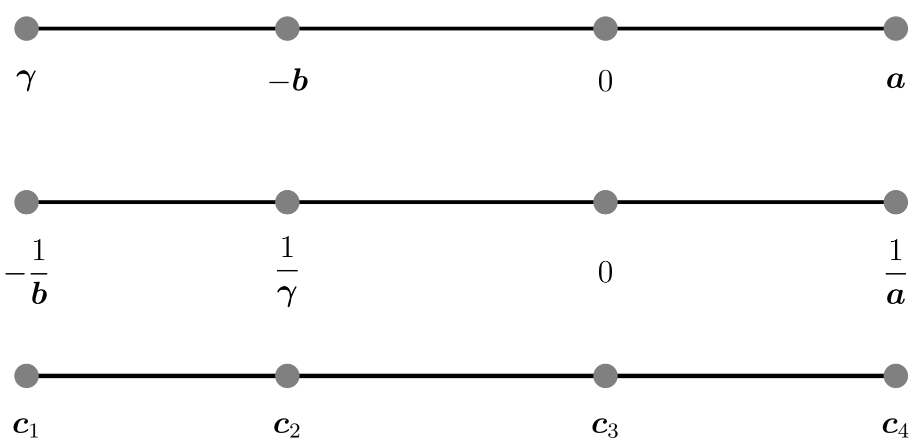

Figure 2.27: $b_0 = -\boldsymbol{b}$, $b_1 = \boldsymbol{a}$.

Equation (2.30) is equivalent to

$$
\begin{aligned}
n_1 \int_{-b}^{0} f(x)dx + n_1 \int_{0}^{a} f(x)dx + n_2 \int_{a}^{0} f(x)dx - n_1 \int_{0}^{a} f(x)dx = 0 \\
n_1 \int_{-b}^{a} f(x)dx + (n_1 + n_2) \int_{a}^{0} f(x)dx = 0 \\
n_1 \int_{-b}^{a} f(x)dx = (n_1 + n_2) \int_{0}^{a} f(x)dx
\end{aligned}
$$

Since the cycles around the cuts on the elliptic curve are homologous:

$$\int_{-b}^{a} f(x)dx = \int_{\infty}^{\gamma} f(x)dx$$

Hence Equation (2.30) is equivalent to

$$n_1 \int_{\infty}^{\gamma} f(x)dx = (n_1 + n_2) \int_{0}^{a} f(x)dx \Leftrightarrow ()$$

Let $s = \frac{1}{x}$, $c_1 = -\frac{1}{b}$, $c_2 = \frac{1}{\gamma}$ $c_3 = 0$, $c_4 = \frac{1}{a}$, , (see Figure 2.27) and substitute in the above to get

$$n_1 \int_{\frac{1}{\gamma}}^{0} \tilde{f}(s)ds = (n_1 + n_2) \int_{\frac{1}{a}}^{\infty} \tilde{f}(s)ds \Leftrightarrow \mathit{Equation}\ (2.31).$$

Case 3: *ii.*) $\mathscr{C}_\gamma$ **is a hyperbola and $\gamma > a$, shown on Figure 2.28**

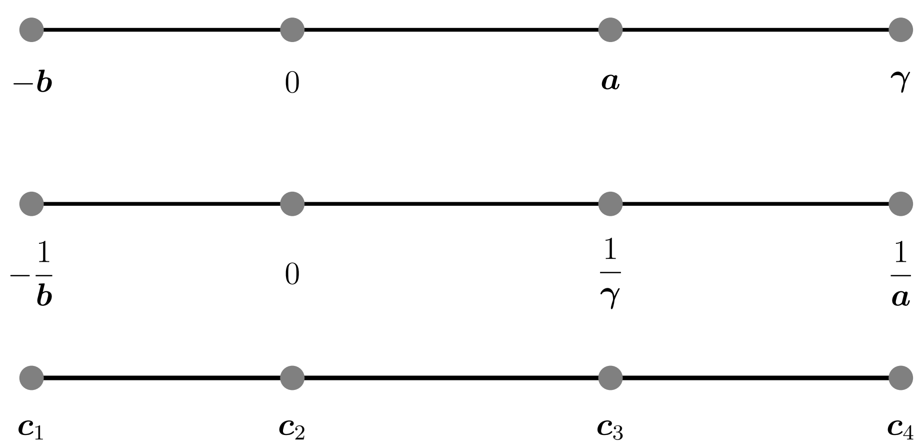

Figure 2.28: $b_0 = -\boldsymbol{b}$, $b_1 = \boldsymbol{a}$.

Equation (2.30) is equivalent to

$$\begin{aligned} n_1\int_{-b}^{0} f(x)dx + n_1\int_{0}^{a} f(x)dx + n_2\int_{a}^{0} f(x)dx - n_1\int_{0}^{a} f(x)dx = 0 \\ n_1\int_{-b}^{a} f(x)dx + (n_1+n_2)\int_{a}^{0} f(x)dx = 0 \\ n_1\int_{-b}^{a} f(x)dx = (n_1+n_2)\int_{0}^{a} f(x)dx \end{aligned}$$

Since the cycles around the cuts on the elliptic curve are homologous:

$$\int_{-b}^{a} f(x)dx = \int_{\gamma}^{\infty} f(x)dx$$

Hence Equation (2.30) is equivalent to

$$n_1\int_{\gamma}^{\infty} f(x)dx = (n_1+n_2)\int_{0}^{a} f(x)dx$$

Let $s = \frac{1}{x}$, $c_1 = -\frac{1}{b}$, $c_2 = 0$, $c_3 = \frac{1}{\gamma}$, $c_4 = \frac{1}{a}$, (see Figure 2.28) and substitute in the above to get

$$n_1\int_{0}^{\frac{1}{\gamma}} \tilde{f}(s)ds = (n_1+n_2)\int_{\frac{1}{a}}^{\infty} \tilde{f}(s)ds \Leftrightarrow \mathit{Equation}\ (2.31).$$

We see that in each case we managed to rewrite Equation (2.5) in an equivalent form of Equation (2.29). Thus the proof of the Theorem follows by applying the version of Krein-Levin-Nudelman Theorem listed above. $\square$

In particular, for $n = 3$, if the caustic $\mathscr{C}_\gamma$ is an ellipse with $\gamma < 0$, then $n_1 = 1$. The corresponding extremal polynomial $\hat{p}_3$ has the following presentation on Figure 2.29.

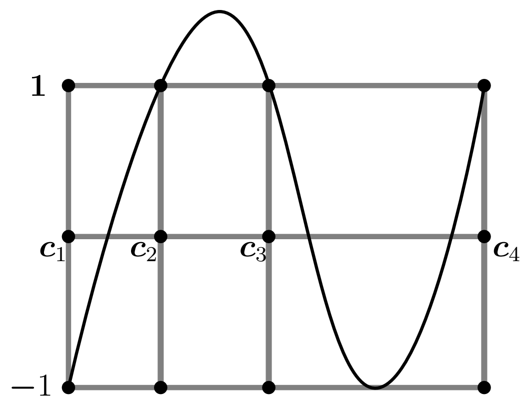

Figure 2.29: Representation of extremal polynomial $\hat{p}_3$ corresponding to $n = 3$, $n_1 = 1$ and $\gamma < 0$.

We will provide explicit formulae of such polynomials in terms of the general Akhiezer polynomial below. Such polynomials and partitions $(3, 1)$ do not arise in the study of Euclidean billiard trajectories.

In the case $n = 3$ with the caustic $\mathscr{C}_\gamma$ being an ellipse with $\gamma > 0$, we have $n_1 = 2$. The corresponding extremal polynomial $\hat{p}_3$ has the following presentation on Figure 2.30.

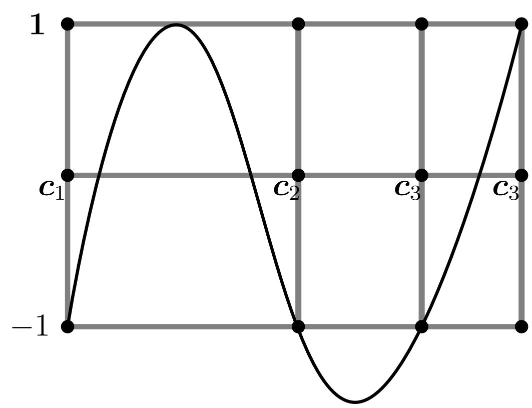

Figure 2.30: Representation of extremal polynomial $\hat{p}_3$ corresponding to $n = 3$, $n_1 = 2$ and $\gamma > 0$.

Such polynomials can be explicitly expressed in terms of the Zolatarev polynomials, see below, since their partition is $(3, 2)$, they appeared before in the Euclidean case.

Let us recall that the celebrated Chebyshev polynomials $T_n(x), n = 0, 1, 2, \ldots$ defined by the recursion:

$$T_0(x) = 1,\, T_1(x) = x,\, T_{n+1}(x) + T_{n-1}(x) = 2xT_n(x), \tag{2.32}$$

for $n = 1, 2 \ldots$ can be parameterized as

$$T_n(x) = \cos n\phi,\, x = \cos\phi, \tag{2.33}$$

or, alternatively:

$$T_n(x) = \frac{1}{2}\left(v^n + \frac{1}{v^n}\right), \quad x = \frac{1}{2}\left(v + \frac{1}{v}\right). \tag{2.34}$$

Denote $L_0 = 1$ and $L_n = 2^{1-n}, n = 1, 2, \ldots$. Then the Chebyshev Theorem states that the polynomials $L_nT_n(x)$ are characterized as the solutions of the following min-max problem:

find the polynomial of degree n with the leading coefficient equal 1 which minimizes the uniform norm on the interval $[-1, 1]$.

2.6.2 Zolotarev polynomials

Following the ideas of Chebyshev, his student Zolotarev posed and solved a handful of problems, including the following [6, 26]:

For the given real parameter σ and all polynomials of degree n of the form:

$$p(x) = x^n - n\sigma x^{n-1} + p_2x^{n-2} + \ldots p_n, \tag{2.35}$$

find the one with the minimal uniform norm on the interval $[-1, 1]$.

Denote this minimal uniform norm as $L_n = L(\sigma, n)$.

For $\sigma > \tan^2(\Pi/2n)$, the solution z_n has the following property ([6], p. 298, Fig. 9):

$\Pi 1$ – *The equation $z_n(x) = L_n$ has $n-2$ double solutions in the open interval* $(-1, 1)$ *and simple solutions at $-1, 1, \alpha, \beta$, where $1 < \alpha < \beta$, while in the union of the intervals $[-1, 1] \cup [\alpha, \beta]$ the inequality $z_n^2 \leq L_n$ is satisfied and $z_n > L_n$ in the complement.*

The polynomials z_n are given by the following explicit formulae:

$$z_n = \ell_n \left(v(u)^n + \frac{1}{v(u)^n} \right), \; x = \frac{\mathrm{sn}^2 u + \mathrm{sn}^2 \frac{K}{n}}{\mathrm{sn}^2 u - \mathrm{sn}^2 \frac{K}{n}}, \tag{2.36}$$

where

$$\ell_n = \frac{1}{2^n} \left(\frac{\sqrt{\kappa}\theta_1^2(0)}{H_1\left(\frac{K}{n}\right)\theta_1\left(\frac{K}{n}\right)} \right)^{2n}, \quad v(u) = \frac{H\left(\frac{K}{n} - u\right)}{H\left(\frac{K}{n} + u\right)}$$

and

$$\sigma = \frac{2\mathrm{sn}\frac{K}{n}}{\mathrm{cn}\frac{K}{n}\mathrm{dn}\frac{K}{n}} \left(\frac{1}{\mathrm{sn}\frac{2K}{n}} - \frac{\theta'\left(\frac{K}{n}\right)}{\theta\left(\frac{K}{n}\right)} \right) - 1.$$

Formulae for the endpoints of the second interval are

$$\alpha = \frac{1 + \kappa^2 \mathrm{sn}^2 \frac{K}{n}}{\mathrm{dn}^2 \frac{K}{n}}, \quad \beta = \frac{1 + \mathrm{sn}^2 \frac{K}{n}}{\mathrm{cn}^2 \frac{K}{n}}, \tag{2.37}$$

with

$$\kappa^2 = \frac{(\alpha - 1)(\beta + 1)}{(\alpha + 1)(\beta - 1)}.$$

According to Cayley's condition for $n = 3$ and $\lambda_0 \in (0, a)$ we have

$$\lambda_0 = \frac{ab(a - b) + 2ab\sqrt{a^2 + ab + b^2}}{(a + b)^2}.$$

In order to derive the formulas for $\hat{p}_3$ in terms of z_3, let us construct an affine transformation:

$$h : [-1, 1] \cup [\alpha, \beta] \to [-b^{-1}, 0] \cup [a^{-1}, \lambda_0^{-1}], \; h(x) = \hat{a}x + \hat{b}.$$

We immediately get

$$\hat{a} = -\hat{b}, \; \hat{a} = \frac{1}{2b}$$

and

$$\alpha = 2t + 1, \tag{2.38}$$

$$\lambda_0 = \frac{2b}{\beta - 1} \tag{2.39}$$

where $t = b/a$.

Now we get the following

Proposition 2.6.1. The polynomial $\hat{p}_3$ can be expressed through the Zolotarev polynoamil z_3 up to a nonessential constant factor:

$$\hat{p}_3(s) \sim z_3(2bs+1).$$

To verify the proposition, we should certify that the definition of α and β from Equation (2.37) for $n=3$ and the relations Equation (2.38), Equation (2.39) are compatible with the formula for λ_0 we got before as Cayley condition, see Equation (2.10)

In order to do that we will use well-known identities for the Jacobi elliptic functions:

$$\mathrm{sn}^2 u + \mathrm{cn}^2 u = 1, \tag{2.40}$$

$$\kappa^2 \mathrm{sn}^2 u + \mathrm{dn}^2 u = 1, \tag{2.41}$$

$$\mathrm{sn}(u+v) = \frac{\mathrm{sn}\, u \mathrm{cn}\, v \mathrm{dn}\, v + \mathrm{sn}\, v \mathrm{cn}\, u \mathrm{dn}\, u}{1-\kappa^2 \mathrm{sn}^2 u \mathrm{sn}^2 v}, \tag{2.42}$$

$$\mathrm{sn}(K-u) = \frac{\mathrm{cn}\, u}{\mathrm{dn}\, u}. \tag{2.43}$$

In particular, we get

$$\mathrm{sn}\left(\frac{2K}{3}\right) = \frac{2\mathrm{sn}\,\frac{K}{3}\mathrm{cn}\,\frac{K}{3}\mathrm{dn}\,\frac{K}{3}}{1-\kappa^2\mathrm{sn}^4\frac{K}{3}}, \tag{2.44}$$

$$\mathrm{sn}\left(\frac{2}{3}K\right) = \mathrm{sn}\left(K-\frac{K}{3}\right) = \frac{\mathrm{cn}\,\frac{K}{3}}{\mathrm{dn}\,\frac{K}{3}}. \tag{2.45}$$

Let us denote

$$Y = \mathrm{sn}\left(\frac{K}{3}\right),$$

then from the previous two relations we get as in [26]:

$$1 - 2Y + 2\kappa^2 Y^3 - \kappa^2 Y^4 = 0.$$

We can express κ in terms of Y and get:

$$\kappa^2 = \frac{2Y-1}{Y^3(2-Y)}. \tag{2.46}$$

By plugging the last relation into Equation (2.37) for $n = 3$ we get

$$\alpha = \frac{Y^2 - 4Y + 1}{Y^2 - 1}.$$

Since, at the same time from the Cayley condition we have $\alpha = 2t + 1$, with $t = b/a$, we can express Y in terms of t:

$$tY^2 + 2Y - (t + 1) = 0,$$

and

$$Y = \frac{-1 \pm \sqrt{1 + t + t^2}}{t}. \tag{2.47}$$

We plug the last relation into the formula for β from Equation (2.37) for $n = 3$

$$\beta = \frac{1 + Y^2}{1 - Y^2},$$

and we get another formula for β in terms of t:

$$\beta = \frac{2t^2 + t + 2 - \pm 2\sqrt{t^2 + t + 1}}{-t - 2 \pm 2\sqrt{t^2 + t + 1}}. \tag{2.48}$$

We see that the last formula with the choice of the + sign corresponds to a formula for β from Equation (2.39). This formula relates β and λ_0 from the Caley condition Equation (2.10). From Equation (2.48), taking the positive sign in β yields,

$$\beta = \frac{2t^2 + t + 2 - 2\sqrt{t^2 + t + 1}}{-t - 2 + 2\sqrt{t^2 + t + 1}}. \tag{2.49}$$

Substituting Equation (2.49) into Equation (2.39) produces

$$\lambda_0 = \frac{2b}{\beta - 1} = b\frac{-2 - t + 2\sqrt{1 + t + t^2}}{2 + t + t^2 - 2\sqrt{1 + t + t^2}} \tag{2.50}$$

But from the Cayley formula Equation (2.10)

$$\lambda_0 = \frac{ab}{(a + b)^2}(a - b + 2\sqrt{a^2 + ab + b^2})$$

Knowing that $t = \frac{b}{a}$, the equation is equivalent to

$$\lambda_0 = b\frac{1 - t + 2\sqrt{1 + t + t^2}}{(1 + t)^2} \tag{2.51}$$

In order to show that the two expressions in Equation (2.50) and Equation (2.51) are identical, we simplify their difference that yields zero.This finalizes the verification. (One can observe that the $-$ sign option from the formula Equation (2.48) would correspond to the $-$ sign in the formula for λ_0 Equation (2.11).

Among the polynomials $\hat{p}_n$ the property of type $\Pi 1$ can be attributed only to those with $n = 2k + 1$ and winding numbers $(2k + 1, 2k)$, in other words to those with the signature $(0, 2k - 1)$.

2.6.3 Akhiezer polynomials on symmetric intervals $[-1, -\alpha] \cup [\alpha, 1]$

The problem of finding polynomials of degree n with the leading coefficient 1 and minimizing the uniform norm on the union of two symmetric intervals $[-1, -\alpha] \cup [\alpha, 1]$, for given $0 < \alpha < 1$ appeared to be of a significant interest in radio-techniques applications. Following the ideas of Chebyshev and Zolotarev, Akhiezer derived in 1928 the explicit formulae for such polynomials $A_n(x; \alpha)$ with the deviation $L_n(\alpha)$ [6, 7].

These formulas are specially simple in the case of even degrees $n = 2m$, when Akhiezer polynomials A_{2m} are obtained by a quadratic substitution from the Chebyshev polynomial T_m:

$$A_{2m}(x; \alpha) = \frac{(1 - \alpha^2)^m}{2^{2m-1}} T_m\left(\frac{2x^2 - 1 - \alpha^2}{1 - \alpha^2}\right), \tag{2.52}$$

with

$$L_{2m}(\alpha) = \frac{(1 - \alpha^2)^m}{2^{2m-1}}.$$

We are going to construct $\hat{p}_4(s)$ up to a nonessential constant factor as a composition of $A_4(x; \alpha)$ for certain α and an affine transformation. We are going to study the possibility to have an affine transformation

$$g : [-1, -\alpha] \cup [\alpha, 1] \to [-b^{-1}, \lambda_0^{-1}] \cup [0, a^{-1}], \quad g(x) = \hat{a}x + \hat{b},$$

which corresponds to the case when $\lambda_0 < -b$ ie $a > b$. For $n = 4$ such caustic is Equation (2.13)

$$\lambda_0 = \frac{ab}{b-a}.$$

From $g(-1) = -b^{-1}$, $g(1) = a^{-1}$ we get

$$\hat{a} = \frac{a+b}{2ab}, \hat{b} = \frac{b-a}{2ab}.$$

Then, from $g(\alpha) = 0$ we get

$$\alpha = \frac{a-b}{a+b}.$$

Finally, we calculate:

$$g(-\alpha) = \frac{a+b}{2ab}\frac{b-a}{a+b} + \frac{b-a}{2ab} = \frac{b-a}{ab}.$$

We recognize λ_0^{-1} on the right-hand side of the last relation.

This proves the following:

Proposition 2.6.2. In this case the polynomial $\hat{p}_4(s)$ is equal up to a constant multiplier to

$$\hat{p}_4(s) \ \sim T_2(2abs^2 + 2(a-b)s + 1), \tag{2.53}$$

where $T_2(x) = 2x^2 - 1$ is the second Chebyshev polynomial and $x = \frac{1}{a+b}\big(2abs + a - b\big)$.

Let us study the possibility to have an affine transformation

$$f : [-1, -\alpha] \cup [\alpha, 1] \to [-b^{-1}, 0] \cup [\lambda_0^{-1}, a^{-1}], \quad f(x) = \hat{a}x + \hat{b},$$

which corresponds to the case when $\lambda_0 > a$ ie $a < b$. For $n = 4$ such caustic is

$$\lambda_0 = \frac{ab}{b-a}.$$

From $f(-1) = -b^{-1}$, $f(1) = a^{-1}$ we get

$$\hat{a} = \frac{a+b}{2ab}, \, \hat{b} = \frac{b-a}{2ab}.$$

Then, from $f(-\alpha) = 0$ we get

$$\alpha = \frac{b-a}{a+b}.$$

Finally, we calculate:

$$f(\alpha) = \frac{a+b}{2ab}\frac{b-a}{a+b} + \frac{b-a}{2ab} = \frac{b-a}{ab}.$$

We recognize λ_0^{-1} on the right-hand side of the last relation.

This proves the following proposition which is the same as Equation (2.53).

Proposition 2.6.3. In this case the polynomial $\hat{p}_4(s)$ is equal up to a constant multiplier to

$$\hat{p}_4(s) \ \sim T_2(2abs^2 + 2(a-b)s + 1), \tag{2.54}$$

where $T_2(x) = 2x^2 - 1$ is the second Chebyshev polynomial and $x = \frac{1}{a+b}\big(2abs + a - b\big)$.

Let us study the possibility to have an affine transformation

$$h : [-1, -\alpha] \cup [\alpha, 1] \to [\lambda_0^{-1}, -b^{-1}] \cup [0, a^{-1}], \quad h(x) = \hat{a}x + \hat{b},$$

which corresponds to the case when $\lambda_0 \in (-b, 0)$. For $n = 4$ such caustic is

$$\lambda_0 = -\frac{ab}{a+b}.$$

From $h(1) = a^{-1}$, $h(\alpha) = 0$ we get

$$\hat{a} = \frac{1}{1-\alpha}\frac{1}{a}, \, \hat{b} = -\frac{\alpha}{1-\alpha}\frac{1}{a}.$$

Then, from $h(-\alpha) = -\frac{1}{b}$ we get

$$\frac{\alpha}{1-\alpha} = \frac{a}{2b}.$$

ie

$$\alpha = \frac{a}{a+2b}.$$

Finally, we calculate:

$$h(-1) = -\big(1 + \frac{\alpha}{1-\alpha}\big)\frac{1}{a} - \frac{\alpha}{1-\alpha}\frac{1}{a},$$

$$h(-1) = -\big(1 + \frac{a}{2b}\big)\frac{1}{a} - \frac{a}{2b}\frac{1}{a} = -\frac{1}{a} - \frac{1}{b} = -\frac{a+b}{ab}.$$

We recognize λ_0^{-1} on the right-hand side of the last relation.

This proves the following:

Proposition 2.6.4. In this case the polynomial $\hat{p}_4(s)$ is equal up to a constant multiplier to

$$\hat{p}_4(s) \ \sim T_2\Big(\frac{8a^2b^2s^2 + 8a^2bs - 4b(a+b)}{4b(a+b)}\Big), \tag{2.55}$$

where $T_2(x) = 2x^2 - 1$ is the second Chebyshev polynomial and $x = \frac{1}{a+2b}\big(2abs + a\big)$.

Let us study the possibility to have an affine transformation

$$l : [-1, -\alpha] \cup [\alpha, 1] \to [-b^{-1}, 0] \cup [a^{-1}, \lambda_0^{-1}], \quad l(x) = \hat{a}x + \hat{b},$$

which corresponds to the case when $\lambda_0 \in (0, a)$. For $n = 4$ such caustic is

$$\lambda_0 = \frac{ab}{a+b}.$$

From $l(-1) = -b^{-1}$, $l(-\alpha) = 0$ we get

$$\hat{a} = \frac{1}{1-\alpha}\frac{1}{b},\ \hat{b} = \frac{\alpha}{1-\alpha}\frac{1}{b}.$$

Then, from $l(\alpha) = \frac{1}{a}$ we get

$$\frac{\alpha}{1-\alpha} = \frac{b}{2a}.$$

ie

$$\alpha = \frac{b}{b+2a}.$$

Finally, we calculate:

$$l(1) = \big(1 + \frac{\alpha}{1-\alpha}\big)\frac{1}{b} + \frac{\alpha}{1-\alpha}\frac{1}{b},$$
$$l(1) = \big(1 + \frac{b}{2a}\big)\frac{1}{b} + \frac{b}{2a}\frac{1}{b} = \frac{1}{a} + \frac{1}{b} = \frac{a+b}{ab}.$$

We recognize λ_0^{-1} on the right-hand side of the last relation.

This proves the following:

Proposition 2.6.5. In this case the polynomial $\hat{p}_4(s)$ is equal up to a constant multiplier to

$$\hat{p}_4(s) \ \sim T_2\Big(\frac{8a^2b^2s^2 - 8ab^2s - 4a(a+b)}{4a(a+b)}\Big), \tag{2.56}$$

where $T_2(x) = 2x^2 - 1$ is the second Chebyshev polynomial and $x = \frac{1}{2a+b}\big(2abs - b\big)$.

2.6.4 General Akhiezer polynomials on unions of two intervals

Following Akhiezer [3, 4, 5], let us consider the union of two intervals $[-1, \alpha] \cup [\beta, 1]$, where

$$\alpha = 1 - 2sn^2\left(\frac{m}{n}K\right), \qquad \beta = 2sn^2\left(\frac{n-m}{n}K\right) - 1. \tag{2.57}$$

Define

$$TA_n(x, m, \kappa) = L\left(v^n(u) + \frac{1}{v^n(u)}\right), \tag{2.58}$$

where

$$v(u) = \frac{H\left(u - \frac{m}{n}K\right)}{H\left(u + \frac{m}{n}K\right)},$$
$$x = \frac{sn^2(u)cn^2\left(\frac{m}{n}K\right) + cn^2(u)sn^2\left(\frac{m}{n}K\right)}{sn^2(u) - sn^2\left(\frac{m}{n}K\right)},$$

and

$$L = \frac{1}{2^{n-1}}\left(\frac{\theta(0)\theta_1(0)}{\theta\left(\frac{m}{n}K\right)\theta_1\left(\frac{m}{n}K\right)}\right), \qquad \kappa^2 = \frac{2(\beta-\alpha)}{(1-\alpha)(1+\beta)}.$$

Here, $\theta_i, \ i = 0, 1, 2, 3$, denote the standard Riemann theta functions, see for example [7] for more details. Akhiezer proved the following result:

Theorem 2.6.3 (Akhiezer). (a) The function $TA_n(x, m, \kappa)$ is a polynomial of degree n in x with the leading coefficient 1 and the second coefficient equal to $-n\tau_1$, where

$$\tau_1 = -1 + 2\frac{sn\left(\frac{m}{n}K\right) cn\left(\frac{m}{n}K\right)}{dn\left(\frac{m}{n}K\right)} \left(\frac{1}{sn(\frac{2m}{n}K)} - \frac{\theta'\left(\frac{m}{n}K\right)}{\theta\left(\frac{m}{n}K\right)} \right).$$

(b) The maximum of the modulus of T_n on the union of the two intervals $[-1, \alpha] \cup [\beta, 1]$ is L.

(c) The function T_n takes values $\pm L$ with alternating signs at $\mu = n - m + 1$ consecutive points of the interval $[-1, \alpha]$ and at $\nu = m + 1$ consecutive points of the interval $[\beta, 1]$. In addition

$$T_n(\alpha, m, \kappa) = T_n(\beta, m, \kappa) = (-1)^m L,$$

and for any $x \in (\alpha, \beta)$, it holds:

$$(-1)^m T_n(x, m, \kappa) > L.$$

(d) Let F be a polynomial of degree n in x with the leading coefficient 1, such that:

i.) $max|F(x)| = L$ for $x \in [-1, \alpha] \cup [\beta, 1]$;

ii) F(x) takes values $\pm L$ with alternating signs at n-m+1 consecutive points of the interval $[-1, \alpha]$ and at m+1 consecutive points of the interval $[\beta, 1]$.

Then $F(x) = T_n(x, m, \kappa)$.

Let us determine the affine transformations when the caustic is an ellipse.

Case $\lambda \in (-b, 0)$

For

$$h : [-1, \alpha] \cup [\beta, 1] \to [\lambda_0^{-1}, -b^{-1}] \cup [0, a^{-1}], \quad h(x) = \hat{a}x + \hat{b},$$

we get

$$\hat{a} = \frac{1}{\beta - \alpha}\frac{1}{b},\ \hat{b} = \frac{-\beta}{\beta - \alpha}\frac{1}{b},\ \frac{1-\beta}{\beta-\alpha} = \frac{b}{a}.$$

Thus:

$$\lambda = \frac{\beta - 1}{1+\beta}a = \frac{\alpha - \beta}{\beta + 1}b \tag{2.59}$$

Example 2.6.1. For $n = 3$ and $m = 2$. From Equation (2.57), one gets:

$$\alpha = 1 - 2sn^2\frac{2}{3}K,\ \beta = 2sn^2\frac{K}{3} - 1.$$

It follows that:

$$\frac{b}{a} = t = \frac{1-\beta}{\beta - \alpha} = \frac{1 - sn^2\frac{K}{3}}{sn^2\frac{2}{3}K + sn^2\frac{K}{3} - 1}, \tag{2.60}$$

Thus

$$\lambda = b\frac{\alpha - \beta}{\beta + 1} = b\frac{1 - sn^2\frac{K}{3} - sn^2\frac{2}{3}K}{sn^2\frac{K}{3}}. \tag{2.61}$$

From the addition formula:

$$sn\frac{2}{3}K = sn(K - \frac{K}{3}) = \frac{snK cn\frac{-K}{3}dn\frac{-K}{3} + sn\frac{-K}{3}cnKdnK}{1 - \kappa^2 sn^2\frac{-K}{3}sn^2K},$$

Hence

$$sn^2\frac{2}{3}K = \frac{1 - sn^2\frac{K}{3}}{1 - \kappa^2 sn^2\frac{K}{3}},$$

ie

$$sn^2\frac{K}{3} = \frac{sn^2\frac{2}{3}K - 1}{\kappa^2 sn^2\frac{2}{3}K - 1}.$$

Let $sn\frac{K}{3} = Z$, from Equation (2.46)

$$\kappa^2 = \frac{2Z - 1}{Z^3(2 - Z)}.$$

Also

$$\alpha = -2\frac{1 - Z^2}{1 - KZ^2} + 1, \tag{2.62}$$

simplifies to

$$\alpha = 2Z^2 - 4Z + 1, \tag{2.63}$$

and

$$\beta = 2Z^2 - 1.$$

Equation Equation (2.60) implies that

$$t = \frac{1 - Z^2}{2Z - 1}. \tag{2.64}$$

Denote

$$q = \frac{\alpha - \beta}{\beta + 1},$$

therefore

$$q = \frac{1 - 2Z}{Z^2}.$$

Thus, we have two expressions for λ. One is from the Cayley condition Equation (2.11) and the other is from Equation (2.59). We want to show that these two expressions are identical that is

$$b\frac{\alpha - \beta}{\beta + 1} = -\frac{ab}{(a+b)^2}(-a + b + 2\sqrt{a^2 + ab + b^2}) \tag{2.65}$$

In order to do so, we first expressed both the left hand side and the right hand side of the above in terms of $t = \frac{b}{a}$ and next transform both side in terms of Z and showed that the L.H.S and the R.H.S yields the same expression.

$$q = \frac{1 - t - 2\sqrt{1 + t + t^2}}{(1+t)^2}$$

$$q(1+t)^2 + t - 1 = -2\sqrt{1 + t + t^2}$$

$$[q(1+t)^2 + t - 1]^2 = 4(1 + t + t^2)^2$$

which is equivalent to

$$\frac{Z^2 - Z + 1}{2Z - 1} = \frac{Z^2 - Z + 1}{2Z - 1}$$

which evaluates to true, therefore Equation (2.65) holds.

Case $\lambda \in (0, a)$

For

$$l : [-1, \alpha] \cup [\beta, 1] \to [-b^{-1}, 0] \cup [a^{-1}, \lambda^{-1}], \quad l(x) = \hat{a}x + \hat{b},$$

we get

$$\hat{a} = \frac{1}{\alpha + 1}\frac{1}{b}, \; \hat{b} = \frac{-\alpha}{\alpha + 1}\frac{1}{b}, \; \frac{\alpha + 1}{\beta - \alpha} = \frac{a}{b}.$$

Thus

$$\lambda = \frac{\alpha + 1}{1 - \alpha} b \tag{2.66}$$

Example 2.6.2. For $n = 3$, and $m = 1$. From Equation (2.57), one gets:

$$\alpha = 1 - 2sn^2\frac{K}{3}, \; \beta = 2sn^2\frac{2K}{3} - 1.$$

$$\frac{b}{a} = t = \frac{\beta - \alpha}{\alpha + 1} = \frac{sn^2\frac{2}{3}K + sn^2\frac{K}{3} - 1}{1 - sn^2\frac{K}{3}} \tag{2.67}$$

Thus

$$\lambda = \frac{1 - sn^2\frac{K}{3}}{sn^2\frac{K}{3}} b.$$

From the addition formula:

$$sn\frac{2K}{3} = sn(K - \frac{K}{3}) = \frac{snK cn\frac{-K}{3} dn\frac{-K}{3} + sn\frac{-K}{3} cnK dnK}{1 - \kappa^2 sn^2\frac{-K}{3} sn^2 K}$$

Hence

$$sn^2\frac{2}{3}K = \frac{1 - sn^2\frac{K}{3}}{1 - \kappa^2 sn^2\frac{K}{3}}$$

Let $sn\frac{K}{3} = Z$, from Equation (2.46)

$$\kappa^2 = \frac{2Z-1}{Z^3(2-Z)}.$$

Also

$$\beta = 2\frac{1-Z^2}{1-KZ^2} - 1, \tag{2.68}$$

Simplifies to

$$\beta = -2Z^2 + 4Z - 1, \tag{2.69}$$

And

$$\alpha = 1 - 2Z^2.$$

Equation Equation (2.67) implies that

$$t = -\frac{2Z-1}{Z^2-1}, \tag{2.70}$$

Denote

$$p = \frac{1+\alpha}{1-\alpha}$$

Therefore

$$p = \frac{1-Z^2}{Z^2}.$$

Thus, we have two expressions for λ. One is from the Cayley condition Equation (2.10) and the other is from Equation (2.66). We want to show that these two expressions are identical that is

$$b\frac{1+\alpha}{1-\alpha} = \frac{ab}{(a+b)^2}(a - b + 2\sqrt{a^2+ab+b^2}), \tag{2.71}$$

In order to do so, we first expressed both the left hand side and the right hand side of the above in terms of $t = \frac{b}{a}$ and next transform both side in terms of Z and showed that the L.H.S and the R.H.S yields the same expression.

$$p = \frac{1-t-2\sqrt{1+t+t^2}}{(1+t)^2}$$

$$p(1+t)^2 + t - 1 = 2\sqrt{1+t+t^2}$$

$$[p(1+t)^2 + t - 1]^2 = 4(1+t+t^2)^2$$

which is equivalent to

$$\frac{Z^2 - Z + 1}{2Z - 1} = \frac{Z^2 - Z + 1}{2Z - 1}$$

which evaluates to true, therefore Equation (2.71) holds.

Proposition 2.6.6. For $n = 3$ and $\lambda \in (-b, 0)$, the polynomial $\hat{p_3}$ is up to a nonessential factor equal to:

$$\hat{p_3} \sim TA_3\Big(2a(1 - sn^2\frac{K}{3})s + 2sn^2\frac{K}{3} - 1; 2, \kappa\Big),$$

For $n = 3$ and $\lambda \in (0, a)$, the polynomial $\hat{p_3}$ is up to a nonessential factor equal to:

$$\hat{p_3} \sim TA_3\Big(2b(1 - sn^2\frac{K}{3})s + 1 - 2sn^2\frac{K}{3}; 1, \kappa\Big)$$

Now, using the Akhiezer Theorem part (c), see Theorem 2.6.3, one can compare and see that the number of internal extremal points coincides with $n_1 - 1$ and $n_2 - 1$ as proposed in Theorem 2.6.2. These numbers match with Figure 2.29 and Figure 2.30 and the Table from Section 2.2.1.

2.7 Periodic light-like trajectories and Chebyshev polynomials

Light-like billiard trajectories, by definition, have at each point the velocity v satisfying $\langle v, v\rangle = 0$. Their caustic is the conic at infinity $\mathcal{C}_\infty$. Since successive segments of such trajectories are orthogonal to each other, the light-like trajectories can close only after an even number of reflections. In ([23],Theorem 3.3), it is proved that a light-like billiard trajectory within $\mathcal{E}$ is periodic with even period n if and only if

$$arccot\sqrt{\frac{a}{b}} \in \left\{\frac{k\pi}{n} \mid 1 \le k < \frac{n}{2}, \Big(k, \frac{n}{2}\Big) = 1\right\}. \tag{2.72}$$

For k not being relatively prime with $n/2$, the corresponding light-like trajectories are also periodic, and their period is a divisor of n.

Applying the limit $\gamma \to +\infty$ in Corollary 2.5.1, we get the following proposition.

Proposition 2.7.1. A light-like trajectory within ellipse $\mathcal{E}$ is periodic with period $n = 2m$ if and only if there exist real polynomials $\hat{p}_m(s)$ and $\hat{q}_{m-1}(s)$ of degrees m and $m-1$ respectively if and only if:

- $\hat{p}_m^2(s) - \left(s - \frac{1}{a}\right)\left(s + \frac{1}{b}\right)\hat{q}_{m-1}^2(s) = 1$; and
- $\hat{q}_{m-1}(0) = 0$.

The first condition from Proposition 2.7.1 is a standard Pell's equation describing extremal polynomials on one interval $[-1/b, 1/a]$, thus polynomials $\hat{p}_m$ can be obtained as Chebyshev polynomials composed with an affine transformation $[-1/b, 1/a] \to [-1, 1]$. The additional condition $\hat{q}_{m-1}(0) = 0$, which is equivalent to $\hat{p}_m'(0) = 0$ implies an additional constraint on parameters a and b. We have the following

Proposition 2.7.2.

- $\hat{p}_m(s) = T_m\left(\frac{2ab}{a+b}s + \frac{a-b}{a+b}\right)$, where T_m is defined by (2.33);
- the condition $\hat{q}_{m-1}(0) = 0$ is equivalent to (2.72).

Proof. The increasing affine transformation $h : [-1/b, 1/a] \to [-1, 1]$ is given by the formula $h(s) = (2abs + a - b)/(a + b)$. The internal extremal points of the Chebyshev polynomial T_m of degree m on the interval $[-1, 1]$ are given by

$$x_k = \cos\left(\frac{k}{m}\pi\right), \quad k = 1, \ldots, m-1,$$

according to the formula (2.33). The second item follows from $h(0) = x_k$. This is equivalent to

$$\frac{a-b}{a+b} \in \left\{\cos\left(\frac{k}{m}\pi\right) | k = 1, \ldots, m-1\right\},$$

which is equivalent to (2.72). $\square$

CHAPTER 3

THE CLASSIFICATION OF HYPERQUADRICS AND FAMILY OF CONFOCAL QUADRICS IN FOUR DIMENSIONAL MINKOWSKI SPACE

3.1 Classification of hypersurfaces of degree two in four dimensional space

Let us consider the following two group of surfaces in which A, B, C and D are positive numbers.

$$Ax_1^2 + Bx_2^2 + Cx_3^2 + Dx_4^2 + 1 = 0, \tag{3.1}$$

$$Ax_1^2 + Bx_2^2 + Cx_3^2 - Dx_4^2 + 1 = 0, \tag{3.2}$$

$$Ax_1^2 + Bx_2^2 - Cx_3^2 - Dx_4^2 + 1 = 0, \tag{3.3}$$

$$Ax_1^2 - Bx_2^2 - Cx_3^2 - Dx_4^2 + 1 = 0, \tag{3.4}$$

$$-Ax_1^2 - Bx_2^2 - Cx_3^2 - Dx_4^2 + 1 = 0, \tag{3.5}$$

and

$$Ax_1^2 + Bx_2^2 + Cx_3^2 + Dx_4^2 - 1 = 0, \tag{3.6}$$

$$Ax_1^2 + Bx_2^2 + Cx_3^2 - Dx_4^2 - 1 = 0, \tag{3.7}$$

$$Ax_1^2 - Bx_2^2 - Cx_3^2 - Dx_4^2 - 1 = 0, \tag{3.8}$$

$$Ax_1^2 - Bx_2^2 - Cx_3^2 - Dx_4^2 - 1 = 0, \tag{3.9}$$

$$-Ax_1^2 - Bx_2^2 - Cx_3^2 - Dx_4^2 - 1 = 0. \tag{3.10}$$

Let us classify each of them to type form. We applied the classification technique developed in the section 5 of [11].

Surface Equation (3.1), $\Delta > 0$.

Let's find the reduced form of the surface Equation (3.1). The quartic equation is of the form

$$t^4 - J_0 t^3 + J_1 t^2 - J_2 t + A_{55} = 0,$$

where

$$
\begin{aligned}
J_0 &= A + B + C + D, \\
J_1 &= AB + AC + AD + BC + BD + CD, \\
J_2 &= ABC + ABD + BCD + ACD, \\
\Delta &= ABCD, \\
A^{55} &= ABCD.
\end{aligned}
$$

From what is above, it follows that

$$\frac{\Delta}{A^{55}} = \frac{ABCD}{ABCD} = 1,$$

and that A, B, C and D are the roots of the discriminating quartic equation, hence the surface Equation (3.1) is of the form

$$Ax_1^2 + Bx_2^2 + Cx_3^2 + Dx_4^2 + \frac{\Delta}{A^{55}} = 0,$$

Surface Equation (3.2), $\Delta < 0$.

Let's find the reduced form of the surface (Equation (3.2)).The quartic equation is of the form

$$t^4 - J_0t^3 + J_1t^2 - J_2t + A_{55} = 0,$$

$$
\begin{aligned}
J_0 &= A + B + C - D, \\
J_1 &= AB + AC - AD + BC - BD - CD, \\
J_2 &= ABC - ABD - BCD - ACD, \\
\Delta &= -ABCD, \\
A^{55} &= -ABCD.
\end{aligned}
$$

From what is above, it follows that

$$\frac{\Delta}{A^{55}} = \frac{ABCD}{ABCD} = 1,$$

and that A, B, C and $-D$ are the roots of the discriminating quartic equation, hence the surface (Equation (3.2)) is of the form

$$Ax_1^2 + Bx_2^2 + Cx_3^2 - Dx_4^2 + \frac{\Delta}{A^{55}} = 0,$$

Surface Equation (3.3), $\Delta < 0$.

Let's find the reduced form of the surface Equation (3.3).The quartic equation is of the form

$$t^4 - J_0 t^3 + J_1 t^2 - J_2 t + A_{55} = 0,$$

$$\begin{aligned}
J_0 &= A + B - C - D, \\
J_1 &= AB - AC - AD - BC - BD + CD, \\
J_2 &= -ABC - ABD + BCD + ACD, \\
\Delta &= ABCD, \\
A^{55} &= ABCD.
\end{aligned}$$

From what is above, it follows that

$$\frac{\Delta}{A^{55}} = \frac{ABCD}{ABCD} = 1,$$

and that A, B, $-C$ and $-D$ are the roots of the discriminating quartic equation, hence the surface Equation (3.2) is of the form

$$Ax_1^2 + Bx_2^2 + Cx_3^2 - Dx_4^2 + \frac{\Delta}{A^{55}} = 0,$$

Similar reductions were performed on the remaining surfaces and the results are put into two group of surfaces, those of negative discriminant and those of positive discriminant. In fact, the given surfaces are already in the reduced form, we just need to determine their type.

Surfaces of positive discriminant and their types

$$
\begin{array}{rcccccccl}
Ax_1^2 & + & Bx_2^2 & + & Cx_3^2 & + & Dx_4^2 & + & \dfrac{\Delta}{ABCD}=0 \quad \textbf{is of type 1}, \\
Ax_1^2 & + & Bx_2^2 & - & Cx_3^2 & - & Dx_4^2 & + & \dfrac{\Delta}{ABCD}=0 \quad \textbf{is of type 3}, \\
-Ax_1^2 & - & Bx_2^2 & - & Cx_3^2 & - & Dx_4^2 & + & \dfrac{\Delta}{ABCD}=0 \quad \textbf{is of type 5}, \\
Ax_1^2 & + & Bx_2^2 & + & Cx_3^2 & - & Dx_4^2 & - & \dfrac{\Delta}{ABCD}=0 \quad \textbf{is of type 2}, \\
Ax_1^2 & - & Bx_2^2 & - & Cx_3^2 & - & Dx_4^2 & - & \dfrac{\Delta}{ABCD}=0 \quad \textbf{is of type 4}.
\end{array}
$$

Surfaces of negative discriminant and their types

$$
\begin{array}{rcccccccl}
Ax_1^2 & + & Bx_2^2 & + & Cx_3^2 & - & Dx_4^2 & - & \dfrac{\Delta}{ABCD}=0 \quad \textbf{is of type 4}, \\
Ax_1^2 & - & Bx_2^2 & - & Cx_3^2 & - & Dx_4^2 & - & \dfrac{\Delta}{ABCD}=0 \quad \textbf{is of type 2}, \\
Ax_1^2 & + & Bx_2^2 & + & Cx_3^2 & + & Dx_4^2 & + & \dfrac{\Delta}{ABCD}=0 \quad \textbf{is of type 5}, \\
Ax_1^2 & + & Bx_2^2 & - & Cx_3^2 & - & Dx_4^2 & + & \dfrac{\Delta}{ABCD}=0 \quad \textbf{is of type 3}, \\
-Ax_1^2 & - & Bx_2^2 & - & Cx_3^2 & - & Dx_4^2 & + & \dfrac{\Delta}{ABCD}=0 \quad \textbf{is of type 1}.
\end{array}
$$

ie

$$Ax_1^2 \quad + \quad Bx_2^2 \quad + \quad Cx_3^2 \quad + \quad Dx_4^2 \quad + \quad 1=0 \quad \textbf{is of type 1}, \tag{3.11}$$

$$Ax_1^2 \quad + \quad Bx_2^2 \quad + \quad Cx_3^2 \quad - \quad Dx_4^2 \quad - \quad 1=0 \quad \textbf{is of type 2}, \tag{3.12}$$

$$Ax_1^2 \quad + \quad Bx_2^2 \quad - \quad Cx_3^2 \quad - \quad Dx_4^2 \quad + \quad 1=0 \quad \textbf{is of type 3}, \tag{3.13}$$

$$Ax_1^2 \quad - \quad Bx_2^2 \quad - \quad Cx_3^2 \quad - \quad Dx_4^2 \quad - \quad 1=0 \quad \textbf{is of type 4}, \tag{3.14}$$

$$-Ax_1^2 \quad - \quad Bx_2^2 \quad - \quad Cx_3^2 \quad - \quad Dx_4^2 \quad + \quad 1=0 \quad \textbf{is of type 5}. \tag{3.15}$$

and

$$\begin{aligned}
-Ax_1^2 &- Bx_2^2 - Cx_3^2 - Dx_4^2 + 1 = 0 \quad \textbf{is of type 1}, && (3.16)\\
Ax_1^2 &- Bx_2^2 - Cx_3^2 - Dx_4^2 - 1 = 0 \quad \textbf{is of type 2}, && (3.17)\\
Ax_1^2 &+ Bx_2^2 - Cx_3^2 - Dx_4^2 + 1 = 0 \quad \textbf{is of type 3}, && (3.18)\\
Ax_1^2 &+ Bx_2^2 + Cx_3^2 - Dx_4^2 - 1 = 0 \quad \textbf{is of type 4}, && (3.19)\\
Ax_1^2 &+ Bx_2^2 + Cx_3^2 + Dx_4^2 + 1 = 0 \quad \textbf{is of type 5}. && (3.20)
\end{aligned}$$

The remaining chapters will utilize the following result in order to classify hyperquadrics.

$$\begin{aligned}
-Ax_1^2 &- Bx_2^2 - Cx_3^2 - Dx_4^2 + 1 = 0 \quad \textbf{is of type 1}, && (3.21)\\
Ax_1^2 &+ Bx_2^2 + Cx_3^2 - Dx_4^2 - 1 = 0 \quad \textbf{is of type 2}, && (3.22)\\
Ax_1^2 &+ Bx_2^2 - Cx_3^2 - Dx_4^2 + 1 = 0 \quad \textbf{is of type 3}, && (3.23)\\
Ax_1^2 &- Bx_2^2 - Cx_3^2 - Dx_4^2 - 1 = 0 \quad \textbf{is of type 4}, && (3.24)\\
Ax_1^2 &+ Bx_2^2 + Cx_3^2 + Dx_4^2 + 1 = 0 \quad \textbf{is of type 5}. && (3.25)
\end{aligned}$$

where A, B, C and D are all positive numbers.

3.2 Family of Confocal Quadrics in Four Dimensional Minkowski space

The four dimensional Minkowski space has two signatures: $(3, 1)$ and $(2, 2)$. The family of confocal quadrics are studied in both $\mathbb{E}^{3,1}$ and $\mathbb{E}^{2,2}$.

3.2.1 The case of signature (3,1)

Let us consider the four dimensional Minkowski space $\mathbb{E}^{3,1}$. A general family of confocal hyperquadrics in $\mathbb{E}^{3,1}$ is given by

$$(\mathscr{R}_\lambda): \quad \frac{x^2}{a-\lambda}+\frac{y^2}{b-\lambda}+\frac{z^2}{c-\lambda}+\frac{w^2}{d+\lambda}=1, \;\; \lambda\in\mathbb{R} \text{ and } a>b>c>0, \;\; d>0. \tag{3.26}$$

The family $(\mathscr{R}_\lambda)$ contains the following quadrics of five geometric types:

- For $\lambda\in(-\infty,-d)$: the family $(\mathscr{R}_\lambda)$ is of type 2 oriented along the w-axis.
- For $\lambda\in(-d,c)$: the family $(\mathscr{R}_\lambda)$ is of type 1.
- For $\lambda\in(c,b)$: the family $(\mathscr{R}_\lambda)$ is of type 2 oriented along the z-axis.
- For $\lambda\in(b,a)$: the family $(\mathscr{R}_\lambda)$ is of type 3.
- For $\lambda\in(a,\infty)$: the family $(\mathscr{R}_\lambda)$ is of type 4.

In addition, there are five degenerate quadrics: $\mathscr{R}_a$, $\mathscr{R}_b$, $\mathscr{R}_c$, $\mathscr{R}_{-d}$ and $\mathscr{R}_\infty$ that are the hyperplane $x=0$, $y=0$, $z=0$, $w=0$ and the hyperplane at infinity respectively. The following quadrics are single out in the coordinate hyperplane.

- Hyperboloid of two sheets oriented along the w-axis $\mathscr{R}_a^{yzw}: -\frac{y^2}{a-b}-\frac{z^2}{a-c}+\frac{w^2}{a+d}=1$ in the hyperplane $x=0$.
- Hyperboloid of one sheet oriented along the z-axis $\mathscr{R}_b^{xzw}: \frac{x^2}{a-b}-\frac{z^2}{b-c}+\frac{w^2}{b+d}=1$ in the hyperplane $y=0$.
- Ellipsoid $\mathscr{R}_c^{xyw}: \frac{x^2}{a-c}+\frac{y^2}{b-c}+\frac{w^2}{c+d}=1$ in the hyperplane $z=0$.
- Ellipsoid $\mathscr{R}_{-d}^{xyz}: \frac{x^2}{a+d}+\frac{y^2}{b+d}+\frac{w^2}{c+d}=1$ in the hyperplane $w=0$.

Tropic curves on quadrics in four dimensional Minkowski space and discriminant set

Tropic curves are set of points at which the metrics induced on the tangent hyperplane are degenerate.

Tangent hyperplane at (x_0, y_0, z_0, w_0) of $(\mathscr{R}_\lambda)$ is given by

$$\frac{xx_0}{a-\lambda} + \frac{yy_0}{b-\lambda} + \frac{zz_0}{c-\lambda} - \frac{ww_0}{d+\lambda} = 1. \tag{3.27}$$

The induced metric is degenerate if and only if the parallel hyperplane that contains the origin is tangential to $x^2 + y^2 + z^2 - w^2 = 0$ ie:

$$\frac{x_0^2}{(a-\lambda)^2} + \frac{y_0^2}{(b-\lambda)^2} + \frac{z_0^2}{(c-\lambda)^2} - \frac{w_0^2}{(d+\lambda)^2} = 1. \tag{3.28}$$

Proposition 3.2.1. The tropical manifold on $(\mathscr{R}_\lambda)$ are the intersection of the hyperquadrics with the hypercone

$$\frac{x^2}{(a-\lambda)^2} + \frac{y^2}{(b-\lambda)^2} + \frac{z^2}{(c-\lambda)^2} - \frac{w^2}{(d+\lambda)^2} = 1. \tag{3.29}$$

Proposition 3.2.2. The union of the tropical manifold on all hyperquadrics Equation (3.26) is a union of two hypersurfaces Σ^+ and Σ^-, which can be respectively parametrically represented as

$$\begin{cases} x = \dfrac{a-\lambda}{\sqrt{a+d}}\rho sin(\psi)cos(\theta), \\ y = \dfrac{b-\lambda}{\sqrt{b+d}}\rho sin(\psi)sin(\theta), \\ z = \dfrac{c-\lambda}{\sqrt{c+d}}\rho cos(\psi), \\ w = \pm(d+\lambda)\sqrt{\dfrac{\rho^2 sin^2(\psi)cos^2(\theta),}{a+d} + \dfrac{\rho^2 sin^2(\psi)sin^2(\theta)}{b+d} + \dfrac{\rho^2 cos^2(\psi)}{c+d}}, \end{cases} \tag{3.30}$$

where $\rho > 0$, $\theta \in [0, 2\pi)$, $\lambda \in \mathbb{R}$.

The intersection of the hypersurface in the hyperplane $w = 0$ is

$$\frac{x^2}{a+d} + \frac{y^2}{b+d} + \frac{z^2}{c+d} = \rho^2.$$

Lemma 3.2.1. The tropical manifolds of the hyperquadric $(\mathscr{R}_\lambda)$ represent exactly the locus of points (x, y, z, w) for which the equation

$$(\mathscr{R}_\lambda): \quad \frac{x^2}{a-\lambda} + \frac{y^2}{b-\lambda} + \frac{z^2}{c-\lambda} + \frac{w^2}{d+\lambda} = 1, \text{ has } \lambda_0 \text{ as multiple root.} \tag{3.31}$$

Proof. The Equation (3.31) is equivalent to

$$\begin{aligned}(a-\lambda)(b-\lambda)(c-\lambda)(d+\lambda) &= (b-\lambda)(c-\lambda)(d+\lambda)x^2 + (a-\lambda)(c-\lambda)(d+\lambda)y^2 \\ &\quad + (a-\lambda)(b-\lambda)(d+\lambda)z^2 + (a-\lambda)(b-\lambda)(c-\lambda)w^2,\end{aligned}$$

which is equivalent to:

$$\lambda^4 + p_3\lambda^3 + p_2\lambda^2 + p_1\lambda + p_0 = 0, \tag{3.32}$$

where

$$p_3 = x^2 + y^2 + z^2 - w^2 - a - b - c + d,$$

$$p_2 =$$

$$x^2(-b-c+d) + y^2(-a-c+d) + z^2(-a-b+d) - w^2(a+b+c) + (bc+ab+ac-$$

$$bd - dc - ad),$$

$$p_1 =$$

$$x^2(bc-bd-cd) + y^2(ac-ad-cd) + z^2(-bd+ab-ad) - w^2(-bc-ab-ac)$$

$$+ (-abc + dbc + adb + adc),$$

$$p_0 = x^2(dbc) + y^2(acd) + z^2(abd) + w^2(abd) - (abcd).$$

Equation (3.32) has $\lambda_0 = 0$ as triple zero if and only if $p_0 = p_1 = p_2 = 0$. This is equivalent to belonging to $(\mathscr{R}_0)$.

Additionally, we have:

$$
\begin{aligned}
p_1 &= x^2(bc-bd-cd)+y^2(ac-ad-cd)+z^2(-bd+ab-ad)-w^2(-bc-ab-ac)\\
&+(-abc+dbc+adb+adc),\\
&= x^2\frac{(abc-abd-acd)}{a}+y^2\frac{(abc-abd-bcd)}{b}+z^2\frac{(-bcd+abc-acd)}{c}\\
&-w^2\frac{(bcd+abd+acd)}{d}+(-abc+dbc+abd+acd),\\
&= x^2\frac{abc-abd-acd-dbc+dbc}{a}+y^2\frac{abc-abd-bcd-adc+adc}{b}\\
&+z^2\frac{-bcd+abc-acd-adb+adb}{c}-w^2\frac{abc-abc+bcd+abd+acd}{d}\\
&-(abc-adb-adc-dbc)\\
&=(abc-abd-acd-bcd)[\frac{x^2}{a}+\frac{y^2}{b}+\frac{z^2}{c}+\frac{w^2}{d}-1]+x^2\frac{dbc}{a}+y^2\frac{adc}{b}+z^2\frac{adb}{c}-w^2\frac{abc}{d},\\
&=(abc-abd-acd-bcd)[\frac{x^2}{a}+\frac{y^2}{b}+\frac{z^2}{c}+\frac{w^2}{d}-1]+abcd[\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-\frac{w^2}{d^2}].
\end{aligned}
$$

□

3.2.2 The case of signature (2,2)

We consider the four dimensional Minkowski space $\mathbb{E}^{3,1}$. A general family of confocal hyperquadrics in $\mathbb{E}^{2,2}$ is given by:

$$
(\mathscr{S}_\lambda): \quad \frac{x^2}{a-\lambda}+\frac{y^2}{b-\lambda}+\frac{z^2}{c+\lambda}+\frac{w^2}{d+\lambda}=1,\ \ \lambda\in\mathbb{R} \text{ and } a>b>0,\ \ d>c>0. \tag{3.33}
$$

The family $(\mathscr{S}_\lambda)$ contains the following quadrics of five geometric types:

- For $\lambda \in (-\infty, -c)$: the family $(\mathscr{S}_\lambda)$ is of type 3.
- For $\lambda \in (-c, -d)$: the family $(\mathscr{S}_\lambda)$ is of type 2 oriented along w-axis.
- For $\lambda \in (-d, b)$: the family $(\mathscr{S}_\lambda)$ is of type 1.

- For $\lambda \in (b, a)$: the family $(\mathscr{S}_\lambda)$ is of type 2 oriented along y-axis.

- For $\lambda \in (a, \infty)$: the family $(\mathscr{S}_\lambda)$ is of type 4.

In addition, there are five degenerate quadrics: $\mathscr{S}_a$, $\mathscr{S}_b$, $\mathscr{S}_c$, $\mathscr{S}_{-d}$ and $\mathscr{S}_\infty$ that are the hyperplane $x = 0$, $y = 0$, $z = 0$, $w = 0$ and the hyperplane at infinity respectively. The following quadrics are single out in the coordinate hyperplane.

- Hyperboloid of one sheet oriented along the y-axis $\mathscr{S}_a^{yzw} : -\dfrac{y^2}{a-b} + \dfrac{z^2}{a+c} + \dfrac{w^2}{a+d} = 1$ in the hyperplane $x = 0$.

- Ellipsoid $\mathscr{S}_b^{xzw} : \dfrac{x^2}{a-b} + \dfrac{z^2}{b+c} + \dfrac{w^2}{b+d} = 1$ in the hyperplane $y = 0$.

- Ellipsoid $\mathscr{S}_c^{xyw} : \dfrac{x^2}{a+c} + \dfrac{y^2}{b+c} + \dfrac{w^2}{d-c} = 1$ in the hyperplane $z = 0$.

- Hyperboloid of one sheet along the y-axis $\mathscr{S}_{-d}^{xyz} : \dfrac{x^2}{a+d} + \dfrac{y^2}{b+d} - \dfrac{z^2}{d-c} = 1$ in the hyperplane $w = 0$.

Tropic curves on quadrics in four dimensional Minkowski space and discriminant set

Tropic curves are set of points at which the metrics induced on the tangent hyperplane are degenerate.

Tangent hyperplane at (x_0, y_0, z_0, w_0) of $(\mathscr{R}_\lambda)$ is given by

$$\frac{xx_0}{a-\lambda} + \frac{yy_0}{b-\lambda} - \frac{zz_0}{c+\lambda} - \frac{ww_0}{d+\lambda} = 1. \tag{3.34}$$

The induced metric is degenerate if and only if the parallel hyperplane that contains the origin is tangential to $x^2 + y^2 - z^2 - w^2 = 0$ ie

$$\frac{x_0^2}{(a-\lambda)^2} + \frac{y_0^2}{(b-\lambda)^2} - \frac{z_0^2}{(c+\lambda)^2} - \frac{w_0^2}{(d+\lambda)^2} = 1. \tag{3.35}$$

Proposition 3.2.3. The tropic curves on $(\mathscr{R}_\lambda)$ are the intersection of the hyperquadrics with the hypercone

$$\frac{x^2}{(a-\lambda)^2} + \frac{y^2}{(b-\lambda)^2} - \frac{z^2}{(c+\lambda)^2} - \frac{w^2}{(d+\lambda)^2} = 1. \tag{3.36}$$

Proposition 3.2.4. The union of the tropical manifold on all hyperquadrics Equation (3.26) is a union of two hypersurfaces Σ^+ and Σ^-, which can be respectively parametrically represented as

$$\begin{cases} x = \dfrac{a-\lambda}{\sqrt{a+d}}\rho sin(\psi)cos(\theta), \\ y = \dfrac{b-\lambda}{\sqrt{b+d}}\rho sin(\psi)sin(\theta), \\ z = \dfrac{c+\lambda}{\sqrt{c+d}}\rho cos(\psi), \\ w = \pm(d+\lambda)\sqrt{\dfrac{\rho^2 sin^2(\psi)cos^2(\theta),}{a+d} + \dfrac{\rho^2 sin^2(\psi)sin^2(\theta)}{b+d} - \dfrac{\rho^2 cos^2(\psi)}{c+d}}. \end{cases} \tag{3.37}$$

where $\rho > 0$, $\theta \in [0, 2\pi)$, $\lambda \in \mathbb{R}$.

The intersection of the hypersurface in the hyperplane $w = 0$ is

$$\frac{x^2}{a+d} + \frac{y^2}{b+d} - \frac{z^2}{d-c} = \rho^2.$$

Lemma 3.2.2. The tropical manifolds of the hyperquadric $(\mathscr{S}_\lambda)$ represent exactly the locus of points (x, y, z, w) for which the equation

$$(\mathscr{R}_\lambda): \quad \frac{x^2}{a-\lambda} + \frac{y^2}{b-\lambda} + \frac{z^2}{c+\lambda} + \frac{w^2}{d+\lambda} = 1, \text{ has } \lambda_0 \text{ as multiple root.} \tag{3.38}$$

Proof. The Equation (3.38) is equivalent to:

$$\begin{aligned}(a-\lambda)(b-\lambda)(c+\lambda)(d+\lambda) &= (b-\lambda)(c+\lambda)(d+\lambda)x^2 + (a-\lambda)(c+\lambda)(d+\lambda)y^2 \\ &\quad + (a-\lambda)(b-\lambda)(d+\lambda)z^2 + (a-\lambda)(b-\lambda)(c+\lambda)w^2,\end{aligned}$$

which is equivalent to:

$$\lambda^4 + l_3\lambda^3 + l_2\lambda^2 + l_1\lambda + l_0 = 0, \tag{3.39}$$

where

$l_3 = -x^2 - y^2 + z^2 + w^2 + a + b - c - d,$

$l_2 =$

$x^2(b - c + d) + y^2(a - c + d) + z^2(-a - b + d) + w^2(-a - b + c) - (-bc + ab - ac - bd + dc$

$- ad),$

$l_1 =$

$x^2(bc + bd - cd) + y^2(ac + ad - cd) + z^2(-bd + ab - ad) + w^2(-bc + ab - ac) + (abc - dbc$

$+ adb - adc),$

$l_0 = x^2(dbc) + y^2(acd) + z^2(abd) + w^2(abc) - (abcd).$

Equation (3.39) has $\lambda_0 = 0$ as triple zero if and only if $l_0 = l_1 = l_2 = 0$. This is equivalent to belonging to $(\mathscr{S}_0)$.

Additionally, we have:

$$\begin{aligned}
l_1 &= x^2(bc + bd - cd) + y^2(ac + ad - cd) + z^2(-bd + ab + ad) + w^2(-bc + ab - ac) \\
&\quad - (abc - dbc + adb - adc), \\
&= x^2\frac{(abc + abd - acd)}{a} + y^2\frac{(abc + abd - bcd)}{b} + z^2\frac{(-bcd + abc - acd)}{c} \\
&\quad - w^2\frac{(-bcd + abd - acd)}{d} - (abc - dbc + abd - acd), \\
&= x^2\frac{abc + abd - acd - dbc + dbc}{a} + y^2\frac{abc + abd - bcd - adc + adc}{b} \\
&\quad + z^2\frac{-bcd + abc - acd - adb + adb}{c} + w^2\frac{abc - abc - bcd + abd - acd}{d} \\
&\quad - (abc + adb - adc - dbc), \\
&= (abc + abd - acd - bcd)[\frac{x^2}{a} + \frac{y^2}{b} + \frac{z^2}{c} + \frac{w^2}{d} - 1] + x^2\frac{dbc}{a} + y^2\frac{adc}{b} - z^2\frac{adb}{c} - w^2\frac{abc}{d}, \\
&= (abc + abd - acd - bcd)[\frac{x^2}{a} + \frac{y^2}{b} + \frac{z^2}{c} + \frac{w^2}{d} - 1] + abcd[\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} - \frac{w^2}{d^2}].
\end{aligned}$$

□

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Anani Komla Aabrah was born in the city of Tsévié more specifically in Assiama (his hometown), in Togo (his country). He spent most of his life in Daviémodzi which is a five minute walk from his hometown. He was admitted to the University of Lomé in Togo and obtained his maîtrise degree in mathematics which is equivalent to the Bachelor of Science in Mathematics, in 2009. In 2012, he obtained his master's degree from the African Institute of Mathematical Science in Senegal (AIMS Senegal in Mbour). In 2013, he started his PhD program at The University of Texas at Dallas and obtained his master's degree in 2016 and his data science certificate in 2018. His dissertation was under the supervison of Dr. Vladimir Dragović. His research focuses on Quadrics in Pseudo-Euclidean spaces, Integrable Billiards and Extremal Polynomials.

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Skills

Problem Solving
Excellent analytical and logical reasoning skills. Able to multi-task. Can learn new skills quickly. Able to lead or work within a group environment.

Computer Languages
Some experience with C++, R, Sql, Spark, Scala. Can become proficient in any of these (or other) languages upon request.

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Creative, motivated, innovative, LaTeX, mathematical ability. Teaching skills.

Language Skills

- **English : Fluent.**
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