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GLOBAL STABILITY LOBES OF TURNING PROCESSES WITH STATE-DEPENDENT DELAY*

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Abstract. We obtain global stability lobes of two models of turning processes with inherit nonsmoothness due to the presence of state-dependent delays. In the process, we transform the models with state-dependent delays into systems of differential equations with both discrete and distributed delays and develop a procedure to determine analytically the global stability regions with respect to parameters. We find that the spindle speed control strategy that we investigated in [*SIAM J. Appl. Math.*, 72 (2012), pp. 1–24] can provide essential improvement on the stability of turning processes with state-dependent delay, and furthermore we show the existence of a proper subset of the stability region which is independent of system damping. Numerical simulations are presented to illustrate the general results.

Key words. turning processes, state-dependent delay, stability chart, parameter control

AMS subject classifications. 34K20, 34K60, 70E18, 70E50

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1. Introduction.

Cutting of a metal workpiece usually generates relative vibrations between the tool and the workpiece which is called machine-tool chatter. Violent chatter severely limits the productivity of the turning process because one has to avoid damage either by removing the tool from the workpiece or by reducing the rate of machining. Moreover, machine-tool chatter significantly reduces the tool's life. Mechanical and mathematical investigations on machine-tool chatter have proved to be notoriously nontrivial and delicate due to the complexity generated by the interplay among the configurations of the machine which feeds and rotates the workpiece, the tool which cuts the workpiece with certain depth and width, and the material properties of the workpiece and the machining tool, including shape and stiffness. Study of machine-tool chatter has been an active research area since the fundamental work of Taylor [28] published in 1907. Extensive efforts till recent years have contributed to our understanding of the mechanism of chatter, the suppression of vibrations, and numerical simulations. Early work during the 1960s included Tobias and Fishwick [31], Tobias [30], Koenigsberger and Tlusty [17] when delay differential equations were introduced to model machine-tool vibrations. Important mathematical work investigating regenerative chatter was also pioneered by Stépán [25] and his research group. For more recent works, see, e.g., [1, 2, 5, 7, 9, 10, 11, 12, 15, 18, 23, 27] and the references therein.

In the research of regenerative turning processes, models of differential equations involving state-dependent delays which naturally arise from relative vibrations between the tool and the workpiece have been reported (see [2, 12]). Mathematical models of turning processes are based on the assumption that the tool and the workpiece are flexible and the chip thickness varies due to the relative vibrations of the

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tool and the workpiece. The tool cuts the surface that was formed in the previous cut, and the chip thickness is determined by the current and previous positions of the tool/workpiece. The time delay between two succeeding cuts depends on the speed of the workpiece rotation and on the workpiece surface generated by the earlier cut, and for this reason it is a state-dependent delay. However, differential equations with state-dependent delays (see [8] for a review) are mathematically perplexing due to their lack of smoothness in function space, which means that there is no linearization at their stationary points. We are thus motivated to study from the point of view of applications of models with state-dependent delays.

Stable turning processes are always top priorities for machinists since an accurate and ample stability region in the parameter space helps in the design of robust machining tools in real-world applications. In the work [12], a system of differential equations with a state-dependent delay governed by an algebraic equation (see (2.6)) was derived, and stability analysis of the linearized system was performed analytically; it was shown that the incorporation of the state-dependent delay into the model slightly affects linear stability properties of the system in the parameter domains appearing in practical applications. Following the idea of a variable spindle speed control strategy which has been investigated in many papers (see, e.g., [11, 13, 18, 19, 21]), we proposed in [9] a spindle speed control law for the state-dependent model and determined the stability region, indicating that stabilization can be achieved in high speed turning processes.

In this paper we are interested in studying how to conduct linear stability analysis of state-dependent models of turning processes, how to analytically determine the global stability lobes and stability regions for turning processes, how models with state-dependent delays can improve the linear stability of classical models of differential equations with constant delays, and furthermore, continuing the work in [9], how a spindle speed control can improve the linear stability of models with state-dependent delays. We show that a change of variables can transform the state-dependent model into a system of differential equations with both constant and distributed delays, which can be readily analyzed by many available mathematical tools.

We organize the remaining part of the paper as follows. In the next section we establish state-dependent models of turning processes with and without spindle velocity control. We show that these models can be transformed into systems of differential equations with both constant and distributed delays. In section 3, we linearize the models with both constant and distributed delays at the same equilibrium and develop an analytical description of the stability lobes in the parameter space. Based on the analytical description of the stability lobes, we show in section 4 that every two adjacent stability lobes have a unique intersection, which in turn verifies the validity of standard numerical simulations for the approximate determination of stability regions. We also show the existence of a sweet stability region which is independent of system damping. In the last two sections, we compare stability regions of different models of turning processes and provide some concluding remarks.

2. Mechanical models with state-dependent delay.

The tool is assumed to be compliant and has bending oscillations in directions x and y. See [9, 12] for an illustrative figure. The governing equations read

$$m\ddot{x}(t) + c_x\dot{x}(t) + k_x x(t) = F_x, \tag{2.1}$$

$$m\ddot{y}(t) + c_y\dot{y}(t) + k_y y(t) = -F_y. \tag{2.2}$$

The x and y components of the cutting process force can be written as

$$F_x = K_x \omega d^q, \tag{2.3}$$

$$F_y = K_y \omega d^q, \tag{2.4}$$

where m is the mass of the tool; K_x and K_y are the cutting coefficients in the x and y directions; k_x, k_y are the stiffness coefficients; c_x, c_y are the damping coefficients; ω is the depth of cut; q is a constant with empirical value 0.75; and d is the chip thickness. The chip thickness d is determined by the feed motion, the current tool position, and the earlier position of the tool, and is given as follows:

$$d(t) = \nu\tau(t) + y(t) - y(t - \tau(t)), \tag{2.5}$$

where ν is the speed of the feed. The time delay τ between the present and the previous cut is determined by the equation (see [12])

$$R\Omega\tau(t) = 2R\pi + x(t) - x(t - \tau), \tag{2.6}$$

where R is the radius of the workpiece and Ω is the spindle velocity.

Remark 2.1. From (2.6) we know that if τ is assumed to be constant for all possible solutions of the model described by (2.1)–(2.6), then we can determine the value of τ by choosing the stationary state of x and obtain $\tau = \tau_0 := 2\pi/\Omega$. If τ is not assumed to be constant, then the time delay τ is implicitly determined by the oscillations in the x direction. That is, the time delay τ is *a state-dependent delay*.

Remark 2.2. Differential equations with state-dependent delay in general have no linearized systems near the stationary states in the classical sense. To overcome this difficulty the following formal linearization process was investigated in [6]: First the delay is frozen at its stationary state, and then the resulting nonlinear system with constant delay is linearized. This method was also employed in [12] for turning processes. We note that the linear system obtained by the above formal linearization technique approximates the original system only near a stationary state and cannot provide much information of the system at a state far away from that. In this paper we are interested in finding a system with non–state-dependent delays which is equivalent to the original system not only near a stationary state but also possibly far away from it under some mild restrictions. With this equivalent system we will be able to analyze the linear stability with respect to parameters in model (2.1)–(2.6) with state-dependent delay, i.e., without using the formal linearization technique presented in [6] and [12].

2.1. Equivalent model with discrete and distributed delays. Let $C([-r_0, 0]; \mathbb{R}^n)$ be the space of bounded continuous functions from $[-r_0, 0]$ to $\mathbb{R}^n$, where $r_0 > 0$ is a constant. For every $\mathbf{x} \in C([-r_0, 0]; \mathbb{R}^n)$ and $t \in \mathbb{R}$, we define $\mathbf{x}_t \in C([-r_0, 0]; \mathbb{R}^n)$ by $\mathbf{x}_t(s) = x(t + s)$ for all $s \in [-r_0, 0]$.

Assuming that the spindle velocity Ω (expressed in rad/s) is constant, we can rewrite (2.6) as

$$\int_{t-\tau(t)}^{t} \frac{R\Omega - \dot{x}(s)}{2R\pi} \mathrm{d}s = 1. \tag{2.7}$$

Let $\dot{x}(t) = u(t)$, $\dot{y}(t) = v(t)$. System (2.1)–(2.6) can be rewritten as

$$
(2.8)\qquad \begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} x \\ y \\ u \\ v \end{bmatrix} = \begin{bmatrix} u \\ v \\ -\frac{c_x}{m}u - \frac{k_x}{m}x + \frac{K_x\omega}{m}(\nu\tau + y(t) - y(t-\tau(t)))^q \\ -\frac{c_y}{m}v - \frac{k_y}{m}y - \frac{K_y\omega}{m}(\nu\tau + y(t) - y(t-\tau(t)))^q \end{bmatrix}, \\ 1 = \displaystyle\int_{t-\tau(t)}^{t} \frac{R\Omega - u(s)}{2\pi R}\mathrm{d}s. \end{cases}
$$

The unique stationary point of system (2.8) is

$$
(2.9)\qquad (\bar{x},\, \bar{y},\, \bar{\tau},\, \bar{u},\, \bar{v}) = \left(\frac{K_x\omega\nu^q}{k_x}\tau_0^q,\, -\frac{K_y\omega\nu^q}{k_y}\tau_0^q,\, \tau_0,\, 0,\, 0\right).
$$

Assume that $R\Omega > u(t)$ for all $t \in \mathbb{R}$. Motivated by Smith [22], we put $\eta = \int_0^t \frac{R\Omega - u(s)}{2\pi R}\mathrm{d}s$ and consider the following change of variables for system (2.8):

$$
(2.10)\qquad r(\eta) = x(t), \quad \rho(\eta) = y(t), \quad j(\eta) = u(t), \quad l(\eta) = v(t), \quad k(\eta) = \tau(t).
$$

Then by (2.7) and (2.10) we have $\eta - 1 = \int_0^{t-\tau(t)} \frac{R\Omega - u(s)}{2\pi R}\mathrm{d}s$, $r(\eta - 1) = x(t - \tau(t))$, and $\rho(\eta - 1) = y(t - \tau(t))$. The equation for τ in (2.8) can be rewritten as

$$
\tau(t) = t - (t - \tau(t)) = \int_{\eta-1}^{\eta} \frac{dt}{d\bar{\eta}}\mathrm{d}\bar{\eta} = \int_{\eta-1}^{\eta} \frac{2\pi R}{R\Omega - j(\bar{\eta})}\mathrm{d}\bar{\eta} = \int_{-1}^{0} \frac{2\pi R}{R\Omega - j_\eta(s)}\mathrm{d}s.
$$

It follows that $k(\eta) = \int_{-1}^{0} \frac{2\pi R}{R\Omega - j_\eta(s)}\mathrm{d}s$. Furthermore, taking the derivative with respect to t on both sides of $r(\rho) = x(t)$, we have $\frac{\mathrm{d}r}{\mathrm{d}\eta}\frac{\mathrm{d}\eta}{\mathrm{d}t} = \dot{x}(t)$, which leads to $\frac{\mathrm{d}r}{\mathrm{d}\eta} = \dot{x}(t)\frac{\mathrm{d}t}{\mathrm{d}\eta} = j(\eta)\frac{2\pi R}{R\Omega - j(\eta)}$. Similarly we have $\frac{\mathrm{d}\rho}{\mathrm{d}\eta} = \dot{y}(t)\frac{\mathrm{d}t}{\mathrm{d}\eta} = l(\eta)\frac{2\pi R}{R\Omega - j(\eta)}$. Therefore, system (2.8) can be rewritten as

$$
(2.11)\qquad \begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}\eta}\begin{bmatrix} r \\ \rho \\ j \\ l \end{bmatrix} = \begin{bmatrix} j \\ l \\ -\frac{c_x}{m}j - \frac{k_x}{m}r + \frac{K_x\omega}{m}(\nu\, k + \rho - \rho(\eta - 1))^q \\ -\frac{c_y}{m}l - \frac{k_y}{m}\rho - \frac{K_y\omega}{m}(\nu\, k + \rho - \rho(\eta - 1))^q \end{bmatrix} \dfrac{2\pi R}{R\Omega - j(\eta)}, \\ k(\eta) = \displaystyle\int_{-1}^{0} \frac{2\pi R}{R\Omega - j_\eta(s)}\mathrm{d}s, \end{cases}
$$

where $j_\eta(s) = j(\eta + s)$. The unique stationary point of system (2.11) is

$$
(2.12)\qquad (\bar{r},\, \bar{\rho},\, \bar{k},\, \bar{j},\, \bar{l}) = \left(\frac{K_x\omega\nu^q}{k_x}\tau_0^q,\, -\frac{K_y\omega\nu^q}{k_y}\tau_0^q,\, \tau_0,\, 0,\, 0\right),
$$

where $\tau_0 = \frac{2\pi}{\Omega}$ is the period of the rotation of the workpiece.

LEMMA 2.3. *The stationary state of system* (2.8) *is stable if and only if the stationary state of system* (2.11) *is stable.*

Proof. If the stationary state of system (2.8) is stable, then there exists a neighborhood U of the stationary state of system (2.8) such that every solution $(x,\, y,\, \tau,\, u,\, v)$ with initial state in U satisfies $R\Omega > u(t)$ for all $t \in \mathbb{R}$. Then the change of variables (2.10), which transforms system (2.8) into system (2.11), is invertible for all $t \in \mathbb{R}$ and $\frac{d\eta}{dt} > 0$ for all $t \in \mathbb{R}$. Therefore (2.10) defines a homeomorphism between U and some neighborhood of the stationary state of system (2.11), which implies that the stationary state of system (2.11) is stable. The converse of the statement follows by a similar argument. ▯

2.2. Model with spindle speed control. We attempt to control the vibrations of the tool when the workpiece is turning at a high speed, by appropriately changing the turning speed Ω. Namely, instead of assuming that Ω is a constant, we suppose that Ω is a function of the time, t. Then we can rewrite (2.6) as

$$R\int_{t-\tau(t)}^{t}\Omega(s)\mathrm{d}s = 2R\pi + x(t) - x(t-\tau(t)). \tag{2.13}$$

Let $\Omega(t) = \frac{1}{R}(\dot{x}(t) + c\,x(t))$, where $c \in \mathbb{R}$ is a parameter. Then by (2.13) we have

$$\int_{t-\tau(t)}^{t}\frac{c}{2\pi R}\cdot x(s)\mathrm{d}s = 1. \tag{2.14}$$

System (2.1)–(2.5), (2.13) with the spindle speed control strategy can be rewritten as

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} x\\ y\\ u\\ v\end{bmatrix} = \begin{bmatrix} u\\ v\\ -\frac{c_x}{m}u - \frac{k_x}{m}x + \frac{K_x\omega}{m}(\nu\tau + y(t) - y(t-\tau(t)))^q\\ -\frac{c_y}{m}v - \frac{k_y}{m}y - \frac{K_y\omega}{m}(\nu\tau + y(t) - y(t-\tau(t)))^q\end{bmatrix},\\ \qquad 1 = \displaystyle\int_{t-\tau(t)}^{t}\frac{c}{2\pi R}\cdot x(s)\mathrm{d}s, \end{cases} \tag{2.15}$$

where $u(t) = \dot{x}(t)$, $v(t) = \dot{y}(t)$ for $t > 0$. Assuming that $x(t) > 0$ for all $t > 0$, we set $\eta = \int_0^t \frac{c}{2\pi R}\cdot x(s)\mathrm{d}s$ and consider the change of variables $r(\eta) = x(t)$, $\rho(\eta) = y(t)$, $j(\eta) = u(t)$, $l(\eta) = v(t)$, $k(\eta) = \tau(t)$. Then by (2.14) we have $\eta - 1 = \int_0^{t-\tau(t)}\frac{c}{2\pi R}\cdot x(s)\mathrm{d}s$, $r(\eta-1) = x(t-\tau(t))$, and $\rho(\eta-1) = y(t-\tau(t))$. The second equation of (2.8) for τ can be rewritten as

$$\tau(t) = t - (t-\tau(t)) = \int_{\eta-1}^{\eta}\frac{\mathrm{d}t}{\mathrm{d}\bar{\eta}}\mathrm{d}\bar{\eta} = \int_{\eta-1}^{\eta}\frac{2\pi R}{c}\frac{1}{r(\bar{\eta})}\mathrm{d}\bar{\eta} = \int_{-1}^{0}\frac{2\pi R}{c}\frac{1}{r_\eta(s)}\mathrm{d}s.$$

It follows that $k(\eta) = \int_{-1}^{0}\frac{2\pi R}{c}\frac{1}{r_\eta(s)}\mathrm{d}s$. Taking derivative with respect to t on both sides of $r(\rho) = x(t)$, we have $\frac{\mathrm{d}r}{\mathrm{d}\eta}\frac{\mathrm{d}\eta}{\mathrm{d}t} = \dot{x}(t)$, which leads to $\frac{\mathrm{d}r}{\mathrm{d}\eta} = \dot{x}(t)\frac{\mathrm{d}t}{\mathrm{d}\eta} = j(\eta)\frac{2\pi R}{c}\frac{1}{r(\eta)}$. Similarly we have $\frac{\mathrm{d}\rho}{\mathrm{d}\eta} = \dot{y}(t)\frac{\mathrm{d}t}{\mathrm{d}\eta} = l(\eta)\frac{2\pi R}{c}\frac{1}{r(\eta)}$. Therefore, system (2.8) can be rewritten as

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}\eta}\begin{bmatrix} r\\ \rho\\ j\\ l\end{bmatrix} = \begin{bmatrix} j\\ l\\ -\frac{c_x}{m}j - \frac{k_x}{m}r + \frac{K_x\omega}{m}(\nu\,k + \rho - \rho(\eta-1))^q\\ -\frac{c_y}{m}l - \frac{k_y}{m}\rho - \frac{K_y\omega}{m}(\nu\,k + \rho - \rho(\eta-1))^q\end{bmatrix}\dfrac{2\pi R}{c}\dfrac{1}{r(\eta)},\\ k(\eta) = \displaystyle\int_{-1}^{0}\frac{2\pi R}{c}\frac{1}{r_\eta(s)}\mathrm{d}s. \end{cases} \tag{2.16}$$

The unique stationary point of system (2.16) is

$$(\bar{r},\,\bar{\rho},\,\bar{k},\,\bar{j},\,\bar{l}) = \left(\frac{K_x\omega\nu^q}{k_x}\tau_1^q,\, -\frac{K_y\omega\nu^q}{k_y}\tau_1^q,\, \tau_1,\, 0,\, 0\right), \tag{2.17}$$

where $\tau_1 = (\frac{2\pi R}{c})^{\frac{1}{q+1}}(\frac{K_x\omega\nu^q}{k_x})^{-\frac{1}{q+1}}$. With an argument similar to that in Lemma 2.3, we have the next result.

Lemma 2.4. *The stationary state of system* (2.15) *is stable if and only if the stationary state of system* (2.16) *is stable.*

For the equality of the stationary states of systems (2.11) and (2.16), it follows from (2.9) and (2.17) that the following lemma holds.

Lemma 2.5. *Systems* (2.11) *and* (2.16) *have the same stationary state if and only if* $c = \frac{R\Omega^{q+1}}{(2\pi)^q}(\frac{K_x \omega \nu^q}{k_x})^{-1}$.

In subsequent sections, we compare the stability charts of system (2.11) and system (2.16) along the same stationary states.

3. Stability lobes. In this section, we conduct a stability analysis by linearizing systems (2.11) and (2.16) at their same stationary states. In particular, we derive an analytical description of the stability lobes by writing the characteristic equations associated with the respective linearized systems.

For system (2.11), we set $\mathbf{x} = (x_1, x_2, x_3, x_4) = (r, \rho, j, l) - (\bar{r}, \bar{\rho}, \bar{j}, \bar{l})$. Let I_{23} be the matrix when the second and the third columns of the 4×4 identity matrix I are interchanged, and I_{12} be the matrix when the first and the second columns of the 4×4 identity matrix I are interchanged. Then we have the following linearization of system (2.11) at the stationary state:

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}\eta} = \tau_0 \left(M\mathbf{x} + N\mathbf{x}(\eta - 1)\right) - \frac{{\tau_0}^2 \nu}{R\Omega} \int_{-1}^{0} N I_{23} \mathbf{x}_\eta(s) \mathrm{d}s, \tag{3.1}$$

where

$$M = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{k_x}{m} & \frac{q\, K_x \omega (\nu\tau_0)^{q-1}}{m} & -\frac{c_x}{m} & 0 \\ 0 & -\frac{k_y}{m} - \frac{q\, K_y \omega (\nu\tau_0)^{q-1}}{m} & 0 & -\frac{c_y}{m} \end{bmatrix}, \quad N = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -\frac{q\, K_x \omega (\nu\tau_0)^{q-1}}{m} & 0 & 0 \\ 0 & \frac{q\, K_y \omega (\nu\tau_0)^{q-1}}{m} & 0 & 0 \end{bmatrix},$$

and

$$N I_{23} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{q\, K_x \omega (\nu\tau_0)^{q-1}}{m} & 0 \\ 0 & 0 & \frac{q\, K_y \omega (\nu\tau_0)^{q-1}}{m} & 0 \end{bmatrix}.$$

Similarly, system (2.16) can be linearized at the stationary state into

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}\eta} = \frac{2\pi R}{c} \frac{1}{\bar{r}} \left(\overline{M}\mathbf{x} + \bar{N}\mathbf{x}(\eta - 1)\right) - \left(\frac{2\pi R}{c}\right)^2 \frac{\nu}{\bar{r}^3} \int_{-1}^{0} \overline{N} I_{12} \mathbf{x}_\eta(s) \mathrm{d}s, \tag{3.2}$$

where

$$\overline{M} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{k_x}{m} & \frac{q\, K_x \omega (\nu\tau_1)^{q-1}}{m} & -\frac{c_x}{m} & 0 \\ 0 & -\frac{k_y}{m} - \frac{q\, K_y \omega (\nu\tau_1)^{q-1}}{m} & 0 & -\frac{c_y}{m} \end{bmatrix}, \quad \overline{N} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -\frac{q\, K_x \omega (\nu\tau_1)^{q-1}}{m} & 0 & 0 \\ 0 & \frac{q\, K_y \omega (\nu\tau_1)^{q-1}}{m} & 0 & 0 \end{bmatrix},$$

and

$$\overline{N} I_{12} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -\frac{q\, K_x \omega (\nu\tau_1)^{q-1}}{m} & 0 & 0 & 0 \\ \frac{q\, K_y \omega (\nu\tau_1)^{q-1}}{m} & 0 & 0 & 0 \end{bmatrix}.$$

Assuming that systems (2.11) and (2.16) have the same stationary state, we have $\tau_1 = \tau_0 = \frac{2\pi}{\Omega}$. By Lemma 2.5, the spindle velocity control parameter c satisfies $c = \frac{R\Omega^{q+1}}{(2\pi)^q}(\frac{K_x\omega\nu^q}{k_x})^{-1}$ and hence $c\bar{r} = R\Omega$. The linear system (3.2) becomes

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}\eta} = \tau_0\left(M\mathbf{x} + N\mathbf{x}(\eta-1)\right) - c_1\int_{-1}^{0} NI_{12}\mathbf{x}_\eta(s)\mathrm{d}s, \tag{3.3}$$

where $c_1 = (\frac{2\pi}{\Omega})^{2-q}\nu(\frac{K_x\omega\nu^q}{k_x})^{-1} = \tau_0^{2-q}\nu(\frac{K_x\omega\nu^q}{k_x})^{-1}$. Now we calculate the characteristic equations of systems (3.1) and (3.3). Denote by k_r the cutting force ratio K_y/K_x, by K_1 the dimensionless depth of cut $qK_y\omega(2\pi R)^{q-1}/k_x$, and by p the dimensionless feed per revolution $\nu/(R\Omega)$. Then we have

$$\begin{cases} \dfrac{q\,K_x\omega(\nu\tau_0)^{q-1}}{m} = \dfrac{k_x}{m}\dfrac{K_1}{k_r}p^{q-1}, \\ \dfrac{q\,K_y\omega(\nu\tau_0)^{q-1}}{m} = \dfrac{k_x}{m}K_1p^{q-1}, \\ c_1 = q\tau_0\left(\dfrac{K_1}{k_r}\right)^{-1}p^{1-q}. \end{cases} \tag{3.4}$$

By transforming system (3.1) into second order scalar equations of (x_1, x_2), we obtain

$$\begin{cases} \ddot{x}_1(\eta) + \dfrac{c_x\tau_0}{m}\dot{x}_1(\eta) + \dfrac{k_x\tau_0}{m}x_1(\eta) - \dfrac{k_x}{m}\dfrac{K_1p^q\tau_0^2}{k_r}\big(x_1(\eta) - x_1(\eta-1)\big) \\ \qquad = \dfrac{k_x}{m}\dfrac{K_1p^{q-1}\tau_0}{k_r}(x_2(\eta) - x_2(\eta-1)), \\ \ddot{x}_2(\eta) + \dfrac{c_y\tau_0}{m}\dot{x}_2(\eta) + \dfrac{k_y\tau_0}{m}x_2(\eta) + \dfrac{k_x}{m}K_1p^{q-1}\tau_0\big(x_2(\eta) - x_2(\eta-1)\big) \\ \qquad = -\dfrac{k_x}{m}K_1p^q\tau_0^2(x_1(\eta) - x_1(\eta-1)). \end{cases} \tag{3.5}$$

Similarly, we can transform system (3.3) into

$$\begin{cases} \ddot{x}_1(\eta) + \dfrac{c_x\tau_0}{m}\dot{x}_1(\eta) + \dfrac{k_x\tau_0}{m}x_1(\eta) - \dfrac{k_x}{m}\dfrac{K_1p^{q-1}c_1}{k_r}\big(x_1(\eta) - x_1(\eta-1)\big) \\ \qquad = \dfrac{k_x}{m}\dfrac{K_1p^{q-1}\tau_0}{k_r}(x_2(\eta) - x_2(\eta-1)), \\ \ddot{x}_2(\eta) + \dfrac{c_y\tau_0}{m}\dot{x}_2(\eta) + \dfrac{k_y\tau_0}{m}x_2(\eta) + \dfrac{k_x}{m}K_1p^{q-1}\tau_0\big(x_2(\eta) - x_2(\eta-1)\big) \\ \qquad = -\dfrac{k_x}{m}K_1p^{q-1}c_1(x_1(\eta) - x_1(\eta-1)). \end{cases} \tag{3.6}$$

By (3.4), we can substitute c_1 by $q\tau_0(\frac{K_1}{k_r})^{-1}p^{1-q}$ in system (3.6) and obtain

$$\begin{cases} \ddot{x}_1(\eta) + \dfrac{c_x\tau_0}{m}\dot{x}_1(\eta) + \dfrac{k_x\tau_0}{m}x_1(\eta) - \dfrac{q\tau_0k_x}{m}\big(x_1(\eta) - x_1(\eta-1)\big) \\ \qquad = \dfrac{k_x}{m}\dfrac{K_1p^{q-1}\tau_0}{k_r}(x_2(\eta) - x_2(\eta-1)), \\ \ddot{x}_2(\eta) + \dfrac{c_y\tau_0}{m}\dot{x}_2(\eta) + \dfrac{k_y\tau_0}{m}x_2(\eta) + \dfrac{k_x}{m}K_1p^{q-1}\tau_0\big(x_2(\eta) - x_2(\eta-1)\big) \\ \qquad = -k_r\dfrac{q\tau_0k_x}{m}(x_1(\eta) - x_1(\eta-1)). \end{cases} \tag{3.7}$$

Assume that the tool is symmetric with $c_x = c_y$, $k_x = k_y$. From the equations in system (3.5), we know that system (3.5) has no nonconstant solutions of the form $(x_1, x_2) = e^{\lambda\eta}(c_1, c_2)$, $(c_1, c_2) \in \mathbb{R}^2$ with a zero component. Otherwise, the scalar equation $\ddot{\mathbf{y}}(\eta) + \frac{c_x\tau_0}{m}\dot{\mathbf{y}}(\eta) + \frac{k_x\tau_0}{m}\mathbf{y}(\eta) = 0$ has nonconstant 1-periodic solution, which is impossible since $c_x\tau_0/m > 0$. Therefore we can bring $(x_1, x_2) = e^{\lambda\eta}(c_1, c_2)$, with $c_1 \neq 0$, $c_2 \neq 0$, into system (3.5) and take products of the right- and left-hand sides of the resulting two equations, respectively. Then we obtain the characteristic equation of system (2.11):

(3.8)
$$\left(\lambda^2 + \frac{c_x\tau_0}{m}\lambda + \frac{k_x\tau_0}{m}\right)\left(\lambda^2 + \frac{c_x\tau_0}{m}\lambda + \frac{k_x\tau_0}{m} + \frac{k_x\tau_0 K_1 p^{q-1}}{m}\left(1 - \frac{p\tau_0}{k_r}\right)(1 - e^{-\lambda})\right) = 0.$$

Similarly, we bring $(x_1, x_2) = e^{\lambda\eta}(c_1, c_2)$, with $c_1 \neq 0$, $c_2 \neq 0$, into system (3.7) and obtain the characteristic equation of system (2.16):

(3.9)
$$\left(\lambda^2 + \frac{c_x\tau_0}{m}\lambda + \frac{k_x\tau_0}{m}\right)\left\{\lambda^2 + \frac{c_x\tau_0}{m}\lambda + \frac{k_x\tau_0}{m} + \frac{k_x\tau_0}{m}\left(K_1 p^{q-1} - q\right)(1 - e^{-\lambda})\right\} = 0.$$

In summary, we have the next two lemmas.

Lemma 3.1. *Assume that the tool is symmetric with* $c_x = c_y$, $k_x = k_y$. *Then the characteristic equation of system* (2.11) *is given by* (3.8).

Lemma 3.2. *Assume that the tool is symmetric with* $c_x = c_y$, $k_x = k_y$ *and that systems* (2.11) *and* (2.16) *have the same stationary state. Then the characteristic equation of system* (2.16) *is given by* (3.9).

Remark 3.3. We remark that sufficient conditions of the stability of systems (3.2) and (3.3) can also be obtained through other methods, for example, using the nonoscillation properties of scalar functional differential equations, the positivity of corresponding Green's functions, and the technique of differential inequalities [3, 4]. This approach does not assume that the coefficients in (2.1)–(2.6) are constants and satisfy $c_x = c_y$ and $k_x = k_y$.

For comparison of the stability regions, we also consider the characteristic equation of the model (2.1)–(2.6) with constant delay. Let $\dot{x}(t) = u(t)$, $\dot{y}(t) = v(t)$. System (2.1)–(2.6) can be rewritten as

$$\text{(3.10)} \qquad \left\{ \frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} x \\ y \\ u \\ v \end{bmatrix} = \begin{bmatrix} u \\ v \\ -\frac{c_x}{m}u - \frac{k_x}{m}x + \frac{K_x\omega}{m}(\nu\tau + y(t) - y(t-\tau_0))^q \\ -\frac{c_y}{m}v - \frac{k_y}{m}y - \frac{K_y\omega}{m}(\nu\tau + y(t) - y(t-\tau_0))^q \end{bmatrix} \right. .$$

The unique stationary point of system (3.10) is

$$\text{(3.11)} \qquad \left(\frac{K_x\omega\nu^q}{k_x}\tau_0^q,\ -\frac{K_y\omega\nu^q}{k_y}\tau_0^q,\ 0,\ 0\right).$$

Then the characteristic equation of system (3.10) at its unique stationary point is given by

$$\text{(3.12)} \qquad \lambda^2 + \frac{c_x\tau_0}{m}\lambda + \frac{k_x\tau_0}{m} + \frac{k_x\tau_0}{m}K_1 p^{q-1}(1 - e^{-\lambda}) = 0,$$

where a change of variables $\lambda\tau_0 \mapsto \lambda$ has been carried out to obtain (3.12).

Let $\xi = \frac{c_x \tau_0}{m}$, $\delta = \frac{k_x \tau_0}{m}$, and $\mathcal{P}(\lambda) = \lambda^2 + \xi\lambda + \delta$. Note that $\sqrt{k_x/m}$ is the natural frequency of the tool, and $\tau_0 = 2\pi/\Omega$ is the rotation period of the workpiece. Therefore, $\delta = \frac{k_x \tau_0}{m} = \frac{\sqrt{k_x/m}}{1/\tau_0}\sqrt{k_x/m}$ can be interpreted as the relative vibration frequency of the tool with respect to the rotation of the workpiece. Then the characteristic equations of the model (2.1)–(2.6) with constant delay, system (2.11), and system (2.16) can be, respectively, written as

$$\mathcal{P}(\lambda) + \delta h_1(1 - e^{-\lambda}) = 0, \tag{3.13}$$

$$\mathcal{P}(\lambda)(\mathcal{P}(\lambda) + \delta h_2(1 - e^{-\lambda})) = 0, \tag{3.14}$$

$$\mathcal{P}(\lambda)(\mathcal{P}(\lambda) + \delta h_3(1 - e^{-\lambda})) = 0, \tag{3.15}$$

where $h_1 = K_1 p^{q-1}$, $h_2 = K_1 p^{q-1}(1 - \frac{p\tau_0}{k_r})$, $h_3 = K_1 p^{q-1} - q = K_1 p^{q-1}(1 - \frac{p^{1-q}q}{K_1})$. To simplify notations, we shall use h instead of h_j, $j \in \{1, 2, 3\}$. Since ξ and δ are all positive, the zeros of the quadratic polynomial $\mathcal{P}(\lambda)$ always have negative real parts. We then need only to investigate the distribution of the zeros of the exponential polynomials $\mathcal{P}(\lambda) + \delta h(1 - e^{-\lambda})$. It is clear that the exponential polynomials $\mathcal{P}(\lambda) + \delta h(1 - e^{-\lambda})$ are analytic in λ and continuous in the parameters. Then by Theorem 2.1 in [20], zeros with nonnegative real parts appear as the parameters vary only if a zero appears on the imaginary axis in the complex plane.

By substituting $\lambda = i\beta$, $\beta \in \mathbb{R}$ into $\mathcal{P}(\lambda) + \delta h(1 - e^{-\lambda}) = 0$, we obtain

$$\begin{cases} \delta h \cos\beta = \delta - \beta^2 + \delta h, \\ \delta h \sin\beta = -\xi\beta. \end{cases} \tag{3.16}$$

Note that $h = h_j$, $j \in \{2, 3\}$, for the original models with state-dependent delay can be nonpositive. Note that if $h = 0$, all the roots of $\mathcal{P}(\lambda) + \delta h(1 - e^{-\lambda}) = 0$ have negative real parts. In the following we distinguish the cases that $h > 0$ and $h < 0$.

3.1. Case $h > 0$.

We have the following result.

LEMMA 3.4. *Suppose that ξ and h are positive and $\delta \geq 0$. If $\lambda = i\beta$, $\beta \in \mathbb{R}$, is a zero of $\mathcal{P}(\lambda) + \delta h(1 - e^{-\lambda}) = 0$, then $\beta = 0$ if and only if $\delta = 0$. Moreover, if $\beta \neq 0$, the following statements are true:*

(i) *$\delta < \beta^2 < \delta(1 + 2h)$ and $\beta \neq n\pi$ for every $n \in \mathbb{Z}$.*

(ii) *We have*

$$\begin{cases} h = -\dfrac{\xi}{\xi(1 - \cos\beta) + \beta\sin\beta}, \\ \delta = \beta^2 + \xi\beta\tan\dfrac{\beta}{2}. \end{cases} \tag{3.17}$$

(iii) *$\beta \in ((2n - 1)\pi, 2n\pi)$ or $\beta \in (-2n\pi, -(2n - 1)\pi)$ for some $n \in \mathbb{N}$.*

Proof. If $\beta = 0$, then by the first equation of (3.16) we have $\delta = 0$. The converse also follows from the first equation of (3.16).

Taking the sum of the squares of the left- and right-hand sides of each equation in (3.16), respectively, we obtain

$$2\left(\beta^2 - \delta\right)h = \frac{1}{\delta}\xi^2\beta^2 + \left(\beta^2 - \delta\right)^2. \tag{3.18}$$

(i) By (3.16), we have $\sin\beta \neq 0$ and $\delta > 0$; otherwise $\beta = 0$. Then we have $\cos\beta \neq \pm 1$ and $\beta \neq n\pi$ for every $n \in \mathbb{Z}$. Moreover, by the first equation of (3.16), we have $-1 < \frac{\delta - \beta^2 + \delta h}{\delta h} < 1$, which leads to $\delta < \beta^2 < \delta(1 + 2h)$.

(ii) It follows from (i) and (3.18) that $h = \frac{1}{\delta}\frac{\xi^2\beta^2+(\beta^2-\delta)^2}{2(\beta^2-\delta)}$. By (i), we know that $\beta \neq n\pi$ for every $n \in \mathbb{Z}$, and hence $\sin\beta \neq 0$. Then by (3.16), we have $\delta h = -\frac{\xi\beta}{\sin\beta} = \frac{\beta^2-\delta}{1-\cos\beta}$, which yields $\delta = \beta^2 + \xi\beta\frac{1-\cos\beta}{\sin\beta} = \beta^2 + \xi\beta\tan\frac{\beta}{2}$ and consequently $h = -\frac{1}{\delta}\frac{\xi\beta}{\sin\beta} = -\frac{\xi}{\xi(1-\cos\beta)+\beta\sin\beta}$.

(iii) It follows from (ii) that $\beta^2 + \xi\beta\frac{1-\cos\beta}{\sin\beta} > 0$ and $-\frac{\xi\beta}{\sin\beta}/(\beta^2 + \xi\beta\frac{1-\cos\beta}{\sin\beta}) > 0$. Then we have $\beta/\sin\beta < 0$. It follows that $\beta \in ((2n-1)\pi, 2n\pi)$ or $\beta \in (-2n\pi, -(2n-1)\pi)$ for some $n \in \mathbb{N}$. ∎

In the following we show that (ii) and (iii) of Lemma 3.5 are sufficient to imply that $\lambda = i\beta$, $\beta \neq 0$, is a zero of $\mathcal{P}(\lambda) + \delta h(1 - e^{-\lambda}) = 0$.

LEMMA 3.5. *Assume that ξ and h are positive and $\delta \geq 0$. If β is such that* (ii) *and* (iii) *of Lemma* 3.4 *are satisfied, then $\lambda = i\beta$ is a purely imaginary zero of $\mathcal{P}(\lambda) + \delta h(1 - e^{-\lambda}) = 0$ and δ is positive.*

Proof. We show that (3.16) is true. By (iii) we know that $\sin\beta \neq 0$; then $\beta \neq 0$, and by (ii) we have

$$h = -\frac{\frac{\xi\beta}{\sin\beta}}{\beta^2 + \xi\beta\frac{1-\cos\beta}{\sin\beta}} = -\frac{1}{\delta}\frac{\xi\beta}{\sin\beta}. \tag{3.19}$$

Then it follows that $\delta \neq 0$ and hence $\delta > 0$. Moreover, we have $\delta h \sin\beta = -\xi\beta$, which is the second equation of (3.16). By (ii) and (3.19) we have

$$\delta = \beta^2 + \xi\beta\tan\frac{\beta}{2} = \beta^2 + \xi\beta\frac{1-\cos\beta}{\sin\beta} = \beta^2 - \delta h(1-\cos\beta),$$

which leads to $\delta h\cos\beta = \delta - \beta^2 + \delta h$, which is the first equation of (3.16). ∎

Note that for every $n \in \mathbb{N}$, $\beta \in ((2n-1)\pi, 2n\pi)$ is equivalent to $-\beta \in (-2n\pi, -(2n-1)\pi)$. Then by (3.17) we know that (δ, h) can be regarded as an even function of β. Moreover, in the z-β plane, the function $z = \tan\frac{\beta}{2}$ is monotonically increasing from $-\infty$ to $+\infty$ and $z = -\frac{\beta}{\xi}$ monotonically decreasing on each interval $((2n-1)\pi, 2n\pi)$ of $\beta, n \in \mathbb{N}$. It follows that $\tan\frac{\beta}{2} = -\frac{\beta}{\xi}$ has exactly one solution for β on the interval $((2n-1)\pi, 2n\pi)$, $n \in \mathbb{N}$.

For every $n \in \mathbb{N}$, we denote by β_n^* the unique solution of $\tan\frac{\beta}{2} = -\frac{\beta}{\xi}$ on the interval $((2n-1)\pi, 2n\pi)$. By Lemmas 3.4 and 3.5, we can find a parameterization of (δ, h) on β with $\beta \in ((2n-1)\pi, 2n\pi)$, $n \in \mathbb{N}$. To be more precise, we have the next result.

THEOREM 3.6. *Let ξ be positive and β_n^* be the solution of $\tan\frac{\beta}{2} = -\frac{\beta}{\xi}$ for $\beta \in ((2n-1)\pi, 2n\pi)$, $n \in \mathbb{N}$. Then all positive values of δ and h for which $\mathcal{P}(\lambda) + \delta h(1 - e^{-\lambda}) = 0$ has purely imaginary zeros can be parameterized by*

$$\begin{cases} h = -\dfrac{\xi}{\xi(1-\cos\beta)+\beta\sin\beta}, \\ \delta = \beta^2 + \xi\beta\tan\dfrac{\beta}{2}, \end{cases} \tag{3.20}$$

where $\beta \in (\beta_n^*, 2n\pi)$, $n \in \mathbb{N}$.

Proof. We first show that if (3.20) is true, then δ and h are positive, and for every $\beta \in (\beta_n^*, 2n\pi)$, $n \in \mathbb{N}$, $i\beta$ is a purely imaginary zero of $\mathcal{P}(\lambda) + \delta h(1 - e^{-\lambda}) = 0$. From

(3.20) we know that

$$\begin{aligned}(3.21)\qquad \frac{\mathrm{d}\delta}{\mathrm{d}\beta} &= 2\beta + \xi\frac{1-\cos\beta}{\sin\beta} + \xi\beta\frac{\sin^2\beta - (1-\cos\beta)\cos\beta}{\sin^2\beta}\\ &= 2\beta + \frac{\delta-\beta^2}{\beta} + \frac{\xi\beta}{1+\cos\beta}\\ &= \frac{\delta+\beta^2}{\beta} + \frac{\xi\beta}{1+\cos\beta} > 0 \qquad \text{for every } \beta\in(\beta_n^*,\, 2n\pi),\ n\in\mathbb{N}.\end{aligned}$$

Note that $\lim_{\beta\to\beta_n^*}\delta = 0$ and $\lim_{\beta\to 2n\pi}\delta = (2n\pi)^2$. Therefore for every $n\in\mathbb{N}$ the mapping $(\beta_n^*,\, 2n\pi)\ni\beta\to\beta^2+\xi\beta\tan\frac{\beta}{2}\in(0,\,(2n\pi)^2)$ is a one-to-one correspondence. Then we have $\delta>0$ and hence $h>0$. By Lemma 3.5, $\lambda=i\beta$ is a purely imaginary zero of $\mathcal{P}(\lambda)+\delta h(1-e^{-\lambda})=0$.

By Lemma 3.4, it clear that if δ and h are positive and $i\beta$ with $\beta\in(\beta_n^*,\, 2n\pi)$, $n\in\mathbb{N}$, is a purely imaginary zero of $\mathcal{P}(\lambda)+\delta h(1-e^{-\lambda})=0$, then (3.20) is true. ∎

Denote by $\mathbb{R}_+$ the open interval $(0,\,+\infty)$, and by $\mathbb{R}_-$ the open interval $(-\infty,\,0)$. Recall that the epigraph of a function $\varphi:\mathbb{R}\supset D\ni x\to\varphi(x)\in\mathbb{R}$ is defined by $\mathrm{epi}(\varphi)=\{(x,\,t)\in D\times\mathbb{R}: t\geq\varphi(x)\}$. Now we are able to state the main result of this subsection.

THEOREM 3.7. *Let ξ be positive. For every $n\in\mathbb{N}$, the parameterization* (3.20) *of* $(\delta,\,h)$ *determines a continuously differentiable mapping*

$$(3.22)\qquad \iota_n^+ : (0,\,(2n\pi)^2)\ni\delta\to h\in(0,\,+\infty).$$

Moreover, the region $\mathcal{S}_+=\mathbb{R}_+^2\setminus\bigcup_{n=1}^{+\infty}\mathrm{epi}(\iota_n^+)$ is the stability region where all the zeros of the characteristic equation $\mathcal{P}(\lambda)(\mathcal{P}(\lambda)+\delta h(1-e^{-\lambda}))=0$ have negative real parts.

Proof. By the chain rule for $\frac{\mathrm{d}h}{\mathrm{d}\delta}$ and (3.21), it follows from Theorem 3.6 that the parameterization (3.20) of $(\delta,\,h)$ determines a continuously differentiable mapping,

$$\iota_n^+ : (0,\,(2n\pi)^2)\ni\delta\to h\in(0,\,+\infty).$$

We know from (3.13)–(3.15) that if $h=0$, all the zeros of $\mathcal{P}(\lambda)(\mathcal{P}(\lambda)+\delta h(1-e^{-\lambda}))=0$ have negative real parts for every $\delta>0$. Then the set $\mathcal{S}_+\cup\{(\delta,\,h):\delta>0,\,h=0\}$ is path-connected.

Now we show that $\mathcal{S}_+$ is the region where all the zeros of characteristic equation $\mathcal{P}(\lambda)(\mathcal{P}(\lambda)+\delta h(1-e^{-\lambda}))=0$ have negative real parts. Otherwise, there exists $(\tilde\delta,\,\tilde h_j)\in\mathcal{S}_+$ such that a zero of the characteristic equation $\mathcal{P}(\lambda)(\mathcal{P}(\lambda)+\delta h(1-e^{-\lambda}))=0$ has positive real part. Then, as $(\delta,\,h)$ varies in the path-connected set $\mathcal{S}_+\cup\{(\delta,\,h):\delta>0,\,h=0\}$ from $(\tilde\delta,\,\tilde h_j)$ to $(\tilde\delta,\,0)$, there exists $(\delta^*,\,h^*)\in\mathcal{S}_+$ such that $\mathcal{P}(\lambda)(\mathcal{P}(\lambda)+\delta h(1-e^{-\lambda}))=0$ has a purely imaginary eigenvalue. This is impossible since by Theorem 3.6 every positive value of $(\delta,\,h)$ lies in the graph of ι_n^+ for some $n\in\mathbb{N}$. ∎

Remark 3.8. For every $n\in\mathbb{N}$, the graph of the mapping ι_n^+ is called a stability lobe in the literature. (See, e.g., Stépán [25].) From the parameterization (3.20) we know that $\delta\to 0$, $h\to+\infty$ as $\beta\to(\beta_n^*)^+$, and $\delta\to(2n\pi)^2$, $h\to+\infty$ as $\beta\to(2n\pi)^-$. It follows that the graph of the mapping $\iota_n^+:(0,\,(2n\pi)^2)\ni\delta\to h\in(0,\,+\infty)$ has two vertical asymptotes at $\delta=0$ and $\delta=(2n\pi)^2$ in the δ-h plane. Figure 3.1 illustrates a family of stability lobes where $\xi=0.02$.

3.2. Case $h<0$. We first state the results parallel to Lemmas 3.4 and 3.5, respectively. The corresponding proofs are similar and hence are omitted.

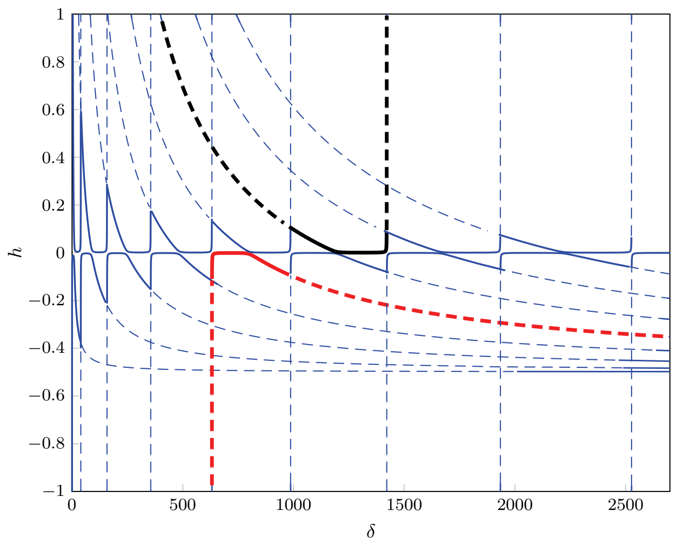

Fig. 3.1. *Stability chart with $\xi = 0.02$. The vertical asymptotes are $\delta = 0$ and $\delta = (2n\pi)^2$, $n = 1, 2, \ldots, 8$. The area between the solid saw-shape curves is the stability region of $(\delta, h) \in (0, +\infty) \times \mathbb{R}$. The bold ν-shape curves are stability lobes in the half planes where $h > 0$ and $h < 0$, respectively.*

Lemma 3.9. *Assume that $\xi > 0$, $\delta \geq 0$, and $h < 0$. If $\lambda = i\beta$, $\beta \in \mathbb{R}$, is a zero of $\mathcal{P}(\lambda) + \delta h(1 - e^{-\lambda}) = 0$, then $\beta = 0$ if and only if $\delta = 0$. Moreover, if $\beta \neq 0$, the following statements are true:*

(i) *$\delta(1 + 2h) < \beta^2 < \delta$ and $\beta \neq n\pi$ for every $n \in \mathbb{Z}$.*

(ii) *We have*

$$\begin{cases} h = -\dfrac{\xi}{\xi(1 - \cos\beta) + \beta \sin\beta}, \\ \delta = \beta^2 + \xi\beta \tan\dfrac{\beta}{2}. \end{cases} \tag{3.23}$$

(iii) *$\beta \in (2(n-1)\pi, (2n-1)\pi)$ or $\beta \in (-(2n-1)\pi, 2(n-1)\pi)$ for some $n \in \mathbb{N}$.*

Lemma 3.10. *Assume that $\xi > 0$, $\delta \geq 0$, and $h < 0$. If β is such that* (ii) *and* (iii) *of Lemma* 3.9 *are satisfied, then $\lambda = i\beta$ is a purely imaginary zero of $\mathcal{P}(\lambda) + \delta h(1 - e^{-\lambda}) = 0$ and δ is positive.*

Next we note that $\tan\frac{\beta}{2} = -\beta/\xi$ has no solution for $\beta \in (2(n-1)\pi, (2n-1)\pi)$ or $\beta \in (-(2n-1)\pi, 2(n-1)\pi)$ for any $n \in \mathbb{N}$. We have the following claim.

Theorem 3.11. *Let ξ be positive. Then all positive values of δ and negative values of h for which $\mathcal{P}(\lambda) + \delta h(1 - e^{-\lambda}) = 0$ has purely imaginary zeros can be parameterized by*

$$\begin{cases} h = -\dfrac{\xi}{\xi(1 - \cos\beta) + \beta \sin\beta}, \\ \delta = \beta^2 + \xi\beta \tan\dfrac{\beta}{2}, \end{cases} \tag{3.24}$$

where $\beta \in (2(n-1)\pi, (2n-1)\pi)$, $n \in \mathbb{N}$.

Proof. We first show that if (3.24) is true, then $\delta > 0$, $h < 0$, and for every $\beta \in (2(n-1)\pi, (2n-1)\pi)$, $n \in \mathbb{N}$, $i\beta$ is a purely imaginary zero of $\mathcal{P}(\lambda) + \delta h(1 - e^{-\lambda}) = 0$.

From (3.24) we know that for every $n \in \mathbb{N}$, $\frac{d\delta}{d\beta} = \frac{\delta+\beta^2}{\beta} + \frac{\xi\beta}{1+\cos\beta} > 0$ for every $\beta \in (2(n-1)\pi,\, (2n-1)\pi)$, and $\lim_{\beta\to 2(n-1)\pi} \delta = (2(n-1)\pi)^2$ and $\lim_{\beta\to(2n-1)\pi} \delta = +\infty$. Therefore for every $n \in \mathbb{N}$ the mapping $((2(n-1)\pi,\, (2n-1)\pi) \ni \beta \to \beta^2 + \xi\beta\tan\frac{\beta}{2} \in (2(n-1)\pi)^2,\, +\infty)$ is one-to-one. Then we have $\delta > 0$ and hence

$$h = -\frac{\xi}{\xi(1-\cos\beta)+\beta\sin\beta} = -\frac{\frac{\xi\beta}{\sin\beta}}{\beta^2+\xi\beta\tan\frac{\beta}{2}} = -\frac{1}{\delta}\frac{\xi\beta}{\sin\beta} < 0. \tag{3.25}$$

By Lemma 3.10, $\lambda = i\beta$ is a purely imaginary zero of $\mathcal{P}(\lambda) + \delta h(1-e^{-\lambda}) = 0$.

By Lemma 3.9, it clear that if $\delta > 0$, $h < 0$, and $i\beta$ with $\beta \in ((2(n-1)\pi,\, (2n-1)\pi)$, $n \in \mathbb{N}$, is a purely imaginary zero of $\mathcal{P}(\lambda) + \delta h(1-e^{-\lambda}) = 0$, then (3.24) is true. ∎

THEOREM 3.12. *Let ξ be positive. For every $n \in \mathbb{N}$, the parameterization* (3.24) *of* $(\delta,\, h)$ *determines a continuously differentiable mapping*

$$\iota_n^- : ((2(n-1)\pi)^2,\, +\infty) \ni \delta \to h \in (-\infty,\, 0). \tag{3.26}$$

Moreover, the region of $\mathcal{S}_- = (\mathbb{R}_+ \times \mathbb{R}_-) \setminus \bigcup_{n=1}^{+\infty} \mathrm{epi}(\iota_n^-)$ is the stability region where all the zeros of the characteristic equation $\mathcal{P}(\lambda)(\mathcal{P}(\lambda)+\delta h(1-e^{-\lambda})) = 0$ have negative real parts.

Proof. By the chain rule for $\frac{dh}{d\delta}$ and (3.25), it follows from Theorem 3.11 that the parameterization (3.24) of $(\delta,\, h)$ determines a continuously differentiable mapping $\iota_n^- : (((2n-1)\pi)^2,\, +\infty) \ni \delta \to h \in (-\infty,\, 0)$, $n \in \mathbb{N}$. The second part of the statement follows from a similar argument for the proof of the corresponding part of Theorem 3.12. ∎

Remark 3.13. From the parameterization (3.24) we know that $\delta \to (2(n-1)\pi)^2$, $h \to -\infty$ as $\beta \to (2(n-1)\pi)^+$, and $\delta \to +\infty$, $h \to -1/2$ as $\beta \to ((2n-1)\pi)^-$. It follows that for every $n \in \mathbb{N}$ the graph of the mapping $\iota_n^- : (((2n-1)\pi)^2,\, +\infty) \ni \delta \to h \in (-\infty,\, 0)$ has a vertical asymptote at $\delta = (2(n-1)\pi)^2$ and a horizontal asymptote $h = -1/2$ in the δ-h plane. Figure 3.1 illustrates a family of stability lobes where $\xi = 0.02$.

4. Sweet region. Prediction of stability regions for various models has been extensively investigated in the literature (see, among many others, [1, 15, 16, 26, 29, 31] and [24] for a recent review). With the state-dependent delay models involved in this paper, we investigate characteristics of the stability charts for the models in question (see section 2 for details), which have been established through parameterization in section 3. We first show that adjacent stability lobes have a unique intersection, which in turn implies that the stability region is encompassed by a chain of graph segments of the stability lobes restricted to consecutive intersections. This is a numerically and experimentally observed pattern of the stability lobes. But it seems that none of the above papers contains a mathematical verification of this phenomenon. We then investigate the position pattern of the minimum depth of the stable cut for each stability lobe. We then show that there exists a family of hyperbolas between the stability lobes. The family of separating hyperbolas provides an immediate benefit, meaning that it helps us construct a subset of the stability region which is independent of damping.

Theorems 3.7 and 3.12 have theoretically determined the stability regions, for we can enumerate all the infinitely many stability lobes. We can approximate the stability regions with the aid of numerical simulations in practice. However, there are infinitely many stability lobes which interact with each other. In principle, an intersection located below one lobe could be above another one. Nevertheless, it seems that intersections of adjacent lobes are always located on the boundary of the stability region. We verify this pattern so that we are able to determine the stability region corresponding to every interval of finite length of the parameter δ without enumerating all the infinitely many stability lobes.

THEOREM 4.1. *Let $\iota_n^+, n \in \mathbb{N}$, be the stability lobe defined at* (3.22). *Then there exists a unique zero δ_0^+ of $l_n^+ - l_{n+1}^+$ on the interval* $(0,\,(2n\pi)^2)$ *and*

$$\begin{cases} l_n^+(\delta) < l_{n+1}^+(\delta) & \text{for every } 0 < \delta < \delta_0^+, \\ l_n^+(\delta) > l_{n+1}^+(\delta) & \text{for every } \delta_0^+ < \delta < (2n\pi)^2. \end{cases}$$

Proof. We prove the existence and uniqueness of the zero of $l_n^+ - l_{n+1}^+$ on the interval $(0,\,(2n\pi)^2)$ of δ. Note that l_n^+ and l_{n+1}^+ are implicitly defined at (3.22) through the parameterization (3.20), where δ is parameterized with different domains of β. For every $n \in \mathbb{N}$, define the mappings f_n^+ by

$$f_n^+ : (\beta_n^*,\, 2n\pi) \ni \beta \to \beta^2 + \xi\beta \tan\frac{\beta}{2} \in (0,\,(2n\pi)^2) \tag{4.1}$$

and

$$g_n^+ : (\beta_n^*,\, 2n\pi) \ni \beta \to (f_{n+1}^+)^{-1}(f_n^+(\beta)) \in (\beta_{n+1}^*,\, 2(n+1)\pi), \tag{4.2}$$

where $(f_{n+1}^+)^{-1}$ denotes the inverse of f_{n+1}^+, the existence of which is guaranteed by (3.21). Then for every $\delta \in (0,\,(2n\pi)^2)$ there exists $\beta \in (\beta_n^*,\, 2n\pi)$ such that $\delta = f_n^+(\beta)$, and by (3.19) we have

$$l_n^+(\delta) - l_{n+1}^+(\delta) = \frac{\xi}{f_n^+(\beta)}\left(\frac{g_n^+(\beta)}{\sin(g_n^+(\beta))} - \frac{\beta}{\sin\beta}\right). \tag{4.3}$$

Note that $\tan\frac{\beta_n^*}{2} = -\frac{\beta_n^*}{\xi}$, $n \in \mathbb{N}$. Then we have

$$\lim_{\beta\to(\beta_n^*)^+} \frac{g_n^+(\beta)}{\sin g_n^+(\beta)} = \frac{\beta_{n+1}^*}{\sin\beta_{n+1}^*} = -\frac{1}{2}(\xi^2 + \beta_{n+1}^*),$$
$$\lim_{\beta\to(\beta_n^*)^+} \frac{\beta}{\sin\beta} = \frac{\beta_n^*}{\sin\beta_n^*} = -\frac{1}{2}(\xi^2 + \beta_n^*) > -\frac{1}{2}(\xi^2 + \beta_{n+1}^*).$$

Then by continuity, there exists $\epsilon > 0$ so that $\frac{g_n^+(\beta)}{\sin(g_n^+(\beta))} - \frac{\beta}{\sin\beta} < 0$ for every $\beta \in (\beta_n^*,\, \beta_n^* + \epsilon)$. Note that f_n^+ is increasing and continuous. It follows from (4.3) that

$$l_n^+(\delta) - l_{n+1}^+(\delta) < 0 \qquad \text{for every } \delta \in (0,\, f_n^+(\beta_n^* + \epsilon)). \tag{4.4}$$

Similarly, noting that for every $\delta \in (0,\,(2n\pi)^2)$ there exists $\beta \in (\beta_n^*,\, 2n\pi)$ such that $\delta = f_n^+(\beta) = f_{n+1}(g_n^+(\beta))$, we have $\lim_{\beta\to(2n\pi)^-} l_n^+(f_n^+(\beta)) = +\infty$ and

$$\lim_{\beta\to(2n\pi)^-} l_{n+1}^+(f_{n+1}^+(g_n^+(\beta))) = l_{n+1}^+(f_n^+(2n\pi)) = l_{n+1}^+((2n\pi)^2) < +\infty.$$

It follows that there exists $\epsilon' > 0$ such that

$$l_n^+(\delta) - l_{n+1}^+(\delta) > 0 \qquad \text{for every } \delta \in (f_n^+(2n\pi - \epsilon'),\, (2n\pi)^2). \tag{4.5}$$

By (4.4) and (4.5), there exists $\delta_0 \in (0, (2n\pi)^2)$ so that

$$l_n^+(\delta_0) - l_{n+1}^+(\delta_0) = 0. \tag{4.6}$$

In the remainder of the proof, we show the uniqueness of the zero of $l_n^+ - l_{n+1}^+$. Before we proceed, we show the following claim.

CLAIM. *For every* $\beta \in (\beta_n^*, 2n\pi)$ *we have*

$$\sin g_n^+(\beta) < \sin\beta < 0 \quad \textit{and} \quad \cos g_n^+(\beta) < \cos\beta, \qquad n \in \mathbb{N}. \tag{4.7}$$

Proof. We have $\sin g_n^+(\beta) \neq \sin\beta$ for every $\beta \in (\beta_n^*, 2n\pi) \subset ((2n-1)\pi, 2n\pi)$. Otherwise, we have $\sin g_n^+(\beta') = \sin\beta'$ for some $\beta' \in (\beta_n^*, 2n\pi)$. Since $g_n^+(\beta') \in (\beta_{n+1}^*, 2(n+1)\pi) \subset ((2n+1)\pi, 2(n+1)\pi)$, we have $g_n^+(\beta') - \beta' \in (\pi, 3\pi)$. Then we have $g_n^+(\beta') = \beta' + 2\pi$ or $g_n^+(\beta') = 3\pi - \beta'$. If $g_n^+(\beta') = 3\pi - \beta'$, then we have $\beta' \in (0, \pi)$, which is impossible. If $g_n^+(\beta') = \beta' + 2\pi$, then we have

$$\begin{aligned} f_n^+(\beta') &= \beta'^2 + \xi\beta' \tan\frac{\beta'}{2} \\ &< \beta'^2 + \xi\beta' \tan\frac{\beta'}{2} + 4\pi\beta' + (2\pi)^2 + 2\pi\xi \tan\frac{\beta'}{2} \\ &= (\beta' + 2\pi)^2 + \xi(\beta' + 2\pi)\tan\frac{\beta' + 2\pi}{2} \\ &= f_{n+1}^+(g_n^+(\beta')) = f_n^+(\beta'), \end{aligned} \tag{4.8}$$

where the inequality follows from $\tan\frac{\beta'}{2} > -\frac{\beta'}{\xi} > -\frac{2\pi}{\xi} - \frac{2\beta'}{\xi}$. But (4.8) is also impossible. Therefore, we have $\sin g_n^+(\beta) \neq \sin\beta$ for every $\beta \in (\beta_n^*, 2n\pi)$.

By continuity of $\sin g_n^+(\beta) - \sin\beta$ with respect to β on the interval $(\beta_n^*, 2n\pi)$, $\sin g_n^+(\beta) - \sin\beta$ is either positive or negative definite. If $\sin g_n^+(\beta) - \sin\beta$ is positive definite, then for every $\beta \in (\beta_n^*, 2n\pi)$ we have

$$0 > \frac{\beta}{\sin\beta} > \frac{\beta}{\sin g_n^+(\beta)} > \frac{g_n^+(\beta)}{\sin g_n^+(\beta)}.$$

Then, by (4.3), $l_n^+ - l_{n+1}^+$ has no zero on $(0, (2n\pi)^2)$. This is a contradiction with (4.6). It follows that $\sin g_n^+(\beta) < \sin\beta < 0$ for every $\beta \in (\beta_n^*, 2n\pi)$.

Similarly, we have $\cos g_n^+(\beta) \neq \cos\beta$ for every $\beta \in (\beta_n^*, 2n\pi) \subset ((2n-1)\pi, 2n\pi)$. Otherwise, we have $\cos g_n^+(\beta') = \cos\beta'$ and hence $\sin g_n^+(\beta') = \sin\beta'$ for some $\beta' \in (\beta_n^*, 2n\pi)$, which has been proved impossible.

Now we show that $\cos g_n^+(\beta) > \cos\beta$ for every $\beta \in (\beta_n^*, 2n\pi)$. Define the mapping

$$H : (\beta_n^*, 2n\pi) \ni \beta \to \frac{\sin\beta}{\beta} - \frac{\sin g_n^+(\beta)}{g_n^+(\beta)} \in \mathbb{R}. \tag{4.9}$$

By the parameterization (3.20) we know that

$$\begin{aligned} H(\beta) &= \frac{\sin\beta}{\beta} - \frac{\sin g_n^+(\beta)}{g_n^+(\beta)} \\ &= \frac{\xi(1-\cos\beta)}{\delta - \beta^2} - \frac{\xi(1 - \cos g_n^+(\beta))}{\delta - (g_n^+(\beta))^2} \\ &= \xi\frac{((g_n^+(\beta))^2 - \delta)(\cos\beta - \cos g_n^+(\beta)) - ((g_n^+(\beta))^2 - \beta^2)(1 - \cos g_n^+(\beta))}{(\delta - \beta^2)(\delta - (g_n^+(\beta))^2)}, \end{aligned} \tag{4.10}$$

where $\delta = f_n^+(\beta) = f_{n+1}^+(g_n^+(\beta))$. Note that for every $\beta \in (\beta_n^*, 2n\pi)$ we have $\beta^2 > \delta$ and $g_n^+(\beta) > \beta_{n+1} > \beta$. It follows that $(g_n^+(\beta))^2 - \delta > 0$, $(g_n^+(\beta))^2 - \beta^2 > 0$, and $(\delta - \beta^2)(\delta - (g_n^+(\beta))^2) > 0$. If $\cos g_n^+(\beta) < \cos\beta$ for every $\beta \in (\beta_n^*, 2n\pi)$, then by (4.10) we have $H(\beta) < 0$ and hence $\frac{\beta}{\sin\beta} - \frac{g_n^+(\beta)}{\sin g_n^+(\beta)} > 0$ for every $\beta \in (\beta_n^*, 2n\pi)$. Then, by (4.3), $l_n^+ - l_{n+1}^+$ has no zero on $(0, (2n\pi)^2)$, which is a contradiction with (4.6). This proves the claim.

Now we prove the uniqueness of the zero of $l_n^+ - l_{n+1}^+$. Assume that there are at least two zeros of $l_n^+ - l_{n+1}^+$ on the interval $(0, (2n\pi)^2)$. Then by the analyticity of the mappings f_n^+, g_n^+, and $l_n^+ - l_{n+1}^+$, we assume that δ_1, δ_2 with $\delta_1 \neq \delta_2$ are two minimal zeros of $l_n^+ - l_{n+1}^+$ in the sense that there are no zeros other than δ_1, which is less than δ_2. Then, on the one hand, by (4.4) we have $l_n^+(\delta) - l_{n+1}^+(\delta) > 0$ for every $\delta \in (\delta_1, \delta_2)$, and, on the other hand, by (4.5), l_n^+ is eventually larger than l_{n+1}^+ as $\delta \to (2n\pi)^2$. Then either l_n^+ and l_{n+1}^+ are tangent at δ_2 or there exists at least one more zero δ_3 of $l_n^+ - l_{n+1}^+$ such that $\delta_3 > \delta_2$ and

$$l_n^+(\delta) - l_{n+1}^+(\delta) < 0 \qquad \text{for every } \delta \in (\delta_2, \delta_3). \tag{4.11}$$

In the following we distinguish two cases.

Case 1. There exists a zero δ_3 of $l_n^+ - l_{n+1}^+$ such that $\delta_3 > \delta_2$ and (4.11) is valid. Then we have $H(\beta_2) = H(\beta_3) = 0$, where $\beta_2, \beta_3 \in (\beta_n^*, 2n\pi)$ with $\beta_2 < \beta_3$ are such that $\delta_2 = f_n^+(\beta_2)$ and $\delta_3 = f_n^+(\beta_3)$. It follows that there exists $\beta_0 \in (\beta_2, \beta_3)$ so that $f_n^+(\beta_0) \in (\delta_2, \delta_3)$ and

$$\begin{aligned}
0 &= \frac{\mathrm{d}H}{\mathrm{d}\beta}\bigg|_{\beta=\beta_0} \\
&= \left(\frac{\beta\cos\beta - \sin\beta}{\beta^2} - \frac{g_n^+(\beta)\cos g_n^+(\beta) - \sin g_n^+(\beta)}{(g_n^+(\beta))^2} \frac{\frac{\mathrm{d}}{\mathrm{d}\beta} f_n^+(\beta)}{\frac{\mathrm{d}}{\mathrm{d}\beta} f_{n+1}^+(f_n^+(\beta))} \right)\bigg|_{\beta=\beta_0} \\
&= \frac{1}{\frac{\mathrm{d}}{\mathrm{d}\beta} f_n^+(\beta)} \left(\frac{\beta\cos\beta - \sin\beta}{\beta^2 \frac{\mathrm{d}}{\mathrm{d}\beta} f_n^+(\beta)} - \frac{g_n^+(\beta)\cos g_n^+(\beta) - \sin g_n^+(\beta)}{(g_n^+(\beta))^2 \frac{\mathrm{d}}{\mathrm{d}\beta} f_{n+1}^+(f_n^+(\beta))} \right)\bigg|_{\beta=\beta_0} \\
&= \frac{1}{\frac{\mathrm{d}}{\mathrm{d}\beta} f_n^+(\beta)} \left(\frac{\beta\cos\beta - \sin\beta}{\delta\beta + \beta^3 + \frac{\xi\beta^3}{1+\cos\beta}} - \frac{g_n^+(\beta)\cos g_n^+(\beta) - \sin g_n^+(\beta)}{\delta g_n^+(\beta) + (g_n^+(\beta))^3 + \frac{\xi (g_n^+(\beta))^3}{1+\cos(g_n^+(\beta))}} \right)\Bigg|_{\beta=\beta_0} \\
&= \frac{1}{\frac{\mathrm{d}}{\mathrm{d}\beta} f_n^+(\beta)} \left(\frac{\cos\beta - \frac{\sin\beta}{\beta}}{\delta + \beta^2 + \frac{\xi\beta^2}{1+\cos\beta}} - \frac{\cos g_n^+(\beta) - \frac{\sin g_n^+(\beta)}{g_n^+(\beta)}}{\delta + (g_n^+(\beta))^2 + \frac{\xi (g_n^+(\beta))^2}{1+\cos(g_n^+(\beta))}} \right)\Bigg|_{\beta=\beta_0}, \qquad (4.12)
\end{aligned}$$

where the derivatives in the large brackets are substituted using (3.21) and $\delta = f_n^+(\beta) = f_{n+1}^+(g_n^+(\beta))$. By the claim, we have $\cos\beta > \cos g_n^+(\beta)$ and $0 < \delta + \beta^2 + \frac{\xi\beta^2}{1+\cos\beta} < \delta + (g_n^+(\beta))^2 + \frac{\xi(g_n^+(\beta))^2}{1+\cos(g_n^+(\beta))}$. Then by (4.12) we must have $H(\beta_0) = \frac{\sin\beta_0}{\beta_0} - \frac{\sin g_n^+(\beta_0)}{g_n^+(\beta_0)} > 0$. Otherwise, $\frac{\mathrm{d}H}{\mathrm{d}\beta}|_{\beta=\beta_0} > 0$. Therefore we have $\frac{\beta_0}{\sin\beta_0} - \frac{g_n^+(\beta_0)}{\sin g_n^+(\beta_0)} < 0$. Then by (4.3) we have $l_n^+(f_n^+(\beta_0)) - l_{n+1}^+(f_n^+(\beta_0)) > 0$, which contradicts (4.11) since $f_n^+(\beta_0) \in (\delta_2, \delta_3)$.

Case 2. l_n^+ and l_{n+1}^+ are tangent at δ_2. In this case, we have, by (4.3), $H(\beta_2) = \frac{\mathrm{d}H}{\mathrm{d}\beta}|_{\beta=\beta_2} = 0$. From $H(\beta_2) = 0$ we obtain that $\frac{\sin\beta_2}{\beta_2} - \frac{\sin g_n^+(\beta_2)}{g_n^+(\beta_2)} = 0$. Then by replacing β_0 with β_2 in (4.12), we have $\frac{\mathrm{d}H}{\mathrm{d}\beta}|_{\beta=\beta_2} > 0$. This is a contradiction. ∎

For the stability lobes ι_n^-, $n \in \mathbb{N}$, we have the next result.

THEOREM 4.2. *Let* ι_n^-, $n \in \mathbb{N}$, *be the stability lobe defined in* (3.26). *Then there exists a unique zero* δ_0^- *of* $l_n^- - l_{n+1}^-$ *on the interval* $(0, ((2n-1)\pi)^2)$ *and*

$$\begin{cases} l_n^-(\delta) > l_{n+1}^-(\delta) & \text{for every } 0 < \delta < \delta_0^-, \\ l_n^-(\delta) < l_{n+1}^-(\delta) & \text{for every } \delta_0^- < \delta < ((2n-1)\pi)^2. \end{cases}$$

Proof. The proof is essentially the same as that of Theorem 4.1 and hence omitted. ∎

For every $n \in \mathbb{N}$ we denote by θ_n^+ the unique solution of $\tan\beta = -\frac{\beta}{1+\xi}$ on the interval $((2n-1)\pi, 2n\pi)$, and by θ_n^- the unique solution of $\tan\beta = -\frac{\beta}{1+\xi}$ on the interval $(2(n-1)\pi, (2n-1)\pi)$.

LEMMA 4.3. *Let* ι_n^+, $n \in \mathbb{N}$, *be the stability lobe defined at* (3.22), *and* θ_n^+ *the unique solution of* $\tan\beta = -\frac{\beta}{1+\xi}$ *on the interval* $((2n-1)\pi, 2n\pi)$. *Then there exists a unique minimum* $(\delta_n^{\min}, \iota_n^{+\min})$ *of* ι_n^+, *and the inequalities* $\delta_n^{\min} < \delta_{n+1}^{\min}$ *and* $\iota_n^{+\min} > \iota_{n+1}^{+\min}$ *hold for every* $n \in \mathbb{N}$, *where*

$$\begin{cases} \delta_n^{\min} = (\theta_n^+)^2 + \xi(1+\xi) - \xi\sqrt{(1+\xi)^2 + (\theta_n^+)^2}, \\ \iota_n^{+\min} = \dfrac{\xi\sqrt{(1+\xi)^2 + (\theta_n^+)^2}}{(\theta_n^+)^2 - \xi\left(\sqrt{(1+\xi)^2 + (\theta_n^+)^2} - (1+\xi)\right)}. \end{cases} \tag{4.13}$$

Proof. We first show that ι_n^+ assumes a unique minimum for $\delta \in (0, (2n\pi)^2)$. By Theorem 3.7, we can take the parameterization (3.20) of $\iota_n^+ : (0, (2n\pi)^2) \to (0, +\infty)$. By (3.21) we know that $\mathrm{d}\delta/\mathrm{d}\beta > 0$ for all $(\beta_n^*, 2n\pi)$. Therefore $\mathrm{d}\iota_n^+/\mathrm{d}\delta = 0$ if and only if $\mathrm{d}h/\mathrm{d}\beta = 0$. Then the equations

$$0 = \frac{\mathrm{d}h}{\mathrm{d}\beta} = \frac{\xi(1+\xi)\sin\beta + \xi\beta\cos\beta}{(\xi(1-\cos\beta) + \beta\sin\beta)^2} \tag{4.14}$$

lead to

$$(1+\xi)\sin\beta + \beta\cos\beta = 0. \tag{4.15}$$

If $\cos\beta = 0$, then we have $\mathrm{d}h/\mathrm{d}\beta \neq 0$ with $\xi > 0$. So $(1+\xi)\sin\beta + \beta\cos\beta = 0$ is equivalent to $\tan\beta = -\beta/(1+\xi)$, the unique solution of which is $\theta_n^+ \in ((2n-1/2)\pi, 2n\pi)$. Moreover, we have

$$\begin{cases} \tan\beta + \beta/(1+\xi) < 0 & \text{if } \beta \in ((2n-1/2)\pi, \theta_n^+), \\ \tan\beta + \beta/(1+\xi) > 0 & \text{if } \beta \in (\theta_n^+, 2n\pi). \end{cases} \tag{4.16}$$

Next, we show that $\beta_n^* < \theta_n^+$. If $\beta_n^* \leq (2n-1/2)\pi$, then by (4.16) we are done. Now we assume that $\beta_n^* > (2n-1/2)\pi$. Let β_0 be the unique solution of $\tan\beta = -\beta/\xi$ for $\beta \in ((2n-1/2)\pi, 2n\pi)$. Then, on the one hand, we have $\theta_n^+ > \beta_0$. Otherwise we have $\tan\theta_n^+ \leq \tan\beta_0$, which leads to $-\theta_n^+/(1+\xi) = \tan\theta_n^+ \leq \tan\beta_0 = -\beta_0/\xi$ and hence $\theta_n^+ > \beta_0$, which is a contradiction. On the other hand, we have $\beta_n^* < \beta_0$. Otherwise we have $-\beta_n^*/\xi \leq -\beta_0/\xi$ and $\tan(\beta_n^*/2) \geq \tan(\beta_0/2)$, which lead to

$$\tan(\beta_0/2) \leq \tan(\beta_n^*/2) = -\beta_n^*/\xi \leq -\beta_0/\xi = \tan\beta_0$$

and hence $\tan(\beta_0/2) \le \tan\beta_0$, which is impossible for $\beta_0 \in ((2n-1/2)\pi,\, 2n\pi)$. Then we have $\beta_n^* < \beta_0 < \theta_n^+$. It follows that

$$\beta_n^* < \theta_n^+. \tag{4.17}$$

It follows from (4.14), (4.15), (4.16), and (4.17) that

$$\begin{cases} \frac{\mathrm{d}h}{\mathrm{d}\beta} < 0 & \text{if } \beta \in (\beta_n^*,\, \theta_n^+), \\ \frac{\mathrm{d}h}{\mathrm{d}\beta} > 0 & \text{if } \beta \in (\theta_n^+,\, 2n\pi). \end{cases} \tag{4.18}$$

Then ι_n^+ assumes a unique minimum on $(0,\, (2n\pi)^2)$ when its parameterization (3.20) takes value at $\beta = \theta_n^+ \in (\beta_n^*,\, 2n\pi)$. That is, the unique minimum of ι_n^+ is $(\delta_n^{\min},\, \iota_n^{+\min})$, where $\delta_n^{\min} = (\theta_n^+)^2 + \xi\theta_n^+\tan(\theta_n^+/2)$, $\iota_n^{+\min} = -\frac{\xi}{\xi(1-\cos\theta_n^+)+\theta_n^+\sin\theta_n^+}$.

Now we turn to proving the inequality in the statement. For every $n \in \mathbb{N}$ we have $\theta_n^+ \in (\beta_n^*,\, 2n\pi) \subset ((2n-1)\pi,\, 2n\pi)$. It follows that

$$\theta_n^+ < \theta_{n+1}^+ \quad \text{for every } n \in \mathbb{N}. \tag{4.19}$$

By (4.15), we have $\tan(\theta_n^+) = -\theta_n^+/(1+\xi)$, which leads to $\frac{2\tan(\theta_n^+/2)}{1-\tan^2(\theta_n^+/2)} = -\frac{\theta_n^+}{1+\xi}$. Noticing that $\tan\frac{\theta_n^+}{2} < 0$, it follows that $\theta_n^+\tan\frac{\theta_n^+}{2} = 1+\xi-\sqrt{(1+\xi)^2+(\theta_n^+)^2}$, and hence we have

$$\delta_n^{\min} = (\theta_n^+)^2 + \xi(1+\xi) - \xi\sqrt{(1+\xi)^2+(\theta_n^+)^2}. \tag{4.20}$$

Considering the function $g_1 : \mathbb{R}_+ \ni \beta \to \beta^2+\xi(1+\xi)-\xi\sqrt{(1+\xi)^2+\beta^2} \in \mathbb{R}$, we know that $\frac{\mathrm{d}g_1}{\mathrm{d}\beta} = \beta(2-\frac{\xi}{\sqrt{(1+\xi)^2+\beta^2}}) > 0$, which implies that g_1 is monotonically increasing on $\mathbb{R}_+$. Therefore, by (4.19) and (4.20), we have $\delta_n^{\min} < \delta_{n+1}^{\min}$ for every $n \in \mathbb{N}$.

From $\tan(\theta_n^+) = -\theta_n^+/(1+\xi)$ with $\theta_n^+ \in ((2n-1/2)\pi,\, 2n\pi)$ we also have

$$\iota_n^{+\min} = \frac{\xi\sqrt{(1+\xi)^2+(\theta_n^+)^2}}{(\theta_n^+)^2 - \xi\left(\sqrt{(1+\xi)^2+(\theta_n^+)^2}-(1+\xi)\right)}. \tag{4.21}$$

Considering the function $g_2 : \mathbb{R}_+ \ni \beta \to \frac{\xi\sqrt{(1+\xi)^2+\beta^2}}{\beta^2-\xi(\sqrt{(1+\xi)^2+\beta^2}-(1+\xi))}$, we know that $\frac{\mathrm{d}g_2}{\mathrm{d}\beta} = -\frac{\xi\beta\left((1+\xi)(2+\xi)+\beta^2\right)}{(\beta^2-\xi(\sqrt{(1+\xi)^2+\beta^2}-(1+\xi)))^2\sqrt{(1+\xi)^2+\beta^2}} < 0$, which implies that g_2 is monotonically decreasing on $\mathbb{R}_+$. Therefore, by (4.19) and (4.21), we have $\iota_n^{\min} > \iota_{n+1}^{\min}$ for every $n \in \mathbb{N}$. ∎

Similarly, we have the following claim.

LEMMA 4.4. *Let ι_n^-, $n \in \mathbb{N}$, be the stability lobe defined at* (3.22), *and θ_n^- the unique solution of $\tan\beta = -\frac{\beta}{1+\xi}$ on the interval $(2(n-1)\pi,\, (2n-1)\pi)$. Then there exists a unique maximum $(\delta_n^{\max},\, \iota_n^{-\max})$ of ι_n^-, and the inequalities $\delta_n^{\max} < \delta_{n+1}^{\max}$, $\iota_n^{-\max} < \iota_{n+1}^{-\max}$ hold for every $n \in \mathbb{N}$, where*

$$\begin{cases} \delta_n^{\max} = (\theta_n^-)^2 + \xi(1+\xi) - \xi\sqrt{(1+\xi)^2+(\theta_n^-)^2}, \\ \iota_n^{-\max} = -\dfrac{\xi\sqrt{(1+\xi)^2+(\theta_n^-)^2}}{(\theta_n^-)^2 + \xi\left(\sqrt{(1+\xi)^2+(\theta_n^-)^2}+(1+\xi)\right)}. \end{cases} \tag{4.22}$$

Proof. With the parameterization (3.24) on the interval $(2(n-1)\pi,\, (2n-1)\pi)$, the proof is closely similar to that of Lemma 4.3 and hence omitted. ∎

THEOREM 4.5. *For every $n \in \mathbb{N}$, let ι_n^+ and ι_n^- be the stability lobes defined in* (3.22) *and* (3.26)*, respectively. Then the graph of the hyperbola*

$$\zeta_n : (0,\, +\infty) \ni \delta \to -\frac{1}{2} + \frac{(2n-1)^2\pi^2}{2\delta} \in \left(-\frac{1}{2},\, +\infty\right) \tag{4.23}$$

has the same vertical asymptote $\delta = 0$ as the graph of ι_n^+ and the same horizontal asymptote $h = -1/2$ as the graph of ι_n^- in the δ-h plane of $\mathbb{R}_+ \times \mathbb{R}$. Moreover, we have

$$\begin{cases} \iota_n^+(\delta) - \zeta_n(\delta) > 0 & \text{for every } 0 < \delta < (2n\pi)^2, \\ \iota_n^-(\delta) - \zeta_n(\delta) < 0 & \text{for every } \delta > (2(n-1)\pi)^2. \end{cases}$$

Proof. By Remarks 3.8 and 3.13, we know that ζ_n and ι_n^+ have the same vertical asymptote $\delta = 0$, and ζ_n and ι_n^- have the same the same horizontal asymptote $h = -1/2$ in $\mathbb{R}_+ \times \mathbb{R}$.

Next we show the inequalities in the second part of the statement. If $0 < \delta < (2n\pi)^2$, then by (3.21) we know that the mapping $(\beta_n^*,\, 2n\pi) \ni \beta \to \beta^2 + \xi\beta\tan\frac{\beta}{2} \in (0,\, (2n\pi)^2)$ is a one-to-one correspondence. Then the hyperbola ζ with domain restricted on $(0,\, (2n\pi)^2)$ can be parameterized by

$$\begin{cases} \zeta_n = -\dfrac{1}{2} + \dfrac{(2n-1)^2\pi^2}{2(\beta^2 + \xi\beta\tan\frac{\beta}{2})}, \\ \delta = \beta^2 + \xi\beta\tan\dfrac{\beta}{2}, \end{cases} \tag{4.24}$$

with $\beta \in (\beta_n^*,\, 2n\pi)$. Then by Theorems 3.6 and 3.7 and by (4.24), we have for every $\delta \in (0,\, (2n\pi)^2)$ that there exists $\beta \in (\beta_n^*,\, 2n\pi) \subset ((2n-1)\pi,\, 2n\pi)$ such that $\delta = \beta^2 + \xi\beta\tan\frac{\beta}{2}$ and

$$\begin{aligned} \iota_n^+(\delta) - \zeta_n(\delta) &= -\frac{\xi}{\xi(1-\cos\beta) + \beta\sin\beta} - \left(-\frac{1}{2} + \frac{(2n-1)^2\pi^2}{2\left(\beta^2 + \xi\beta\tan\frac{\beta}{2}\right)}\right) \\ &= \frac{-\frac{1}{2}\xi(1+\cos\beta) + \frac{1}{2}\beta\sin\beta}{\xi(1-\cos\beta) + \beta\sin\beta} - \frac{(2n-1)^2\pi^2}{2\left(\beta^2 + \xi\beta\tan\frac{\beta}{2}\right)} \\ &= \frac{\beta\left(-\frac{\xi}{\tan\frac{\beta}{2}} + \beta\right) - (2n-1)^2\pi^2}{2\left(\beta^2 + \xi\beta\tan\frac{\beta}{2}\right)} \\ &> \frac{\beta^2 - (2n-1)^2\pi^2}{2\left(\beta^2 + \xi\beta\tan\frac{\beta}{2}\right)} > 0. \end{aligned}$$

If $\delta > (2(n-1)\pi)^2$, then by (3.21) we know that the mapping $(2(n-1)\pi,\, (2n-1)\pi) \ni \beta \to \beta^2 + \xi\beta\tan\frac{\beta}{2} \in ((2(n-1)\pi)^2,\, +\infty)$ is a one-to-one correspondence. Then the hyperbola ζ with domain restricted on the interval $((2(n-1)\pi)^2,\, +\infty)$ can be parameterized by (3.24). By Theorems 3.6 and 3.7 and by (3.24), we have for

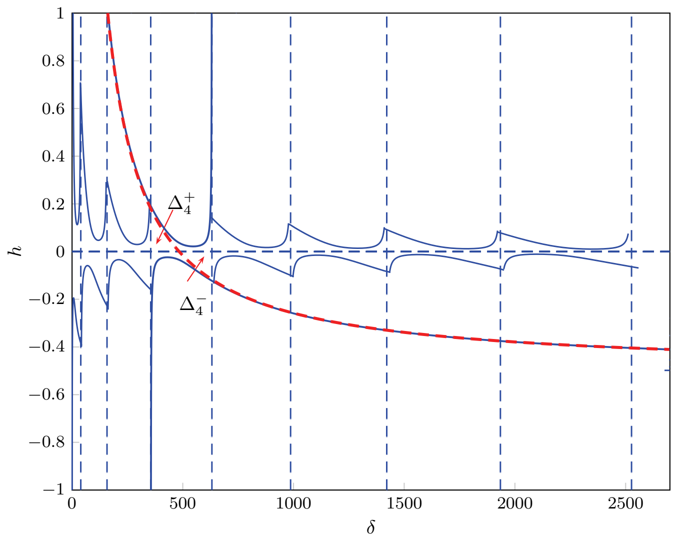

FIG. 4.1. *Stability chart with $\xi = 0.5$. The area between the solid saw-shape curves is the stability region of $(\delta, h) \in \mathbb{R}_+ \times \mathbb{R}$. The solid ν-shape curves are stability lobes ι_4^+ and ι_4^- in the half planes, where $h > 0$ and $h < 0$, respectively. The dashed bold curve is the hyperbola* $\zeta_4(\delta) = -\frac{1}{2} + \frac{49\pi^2}{2\delta}$.

every $\delta \in ((2(n-1)\pi)^2, +\infty)$ that there exists $\beta \in (2(n-1)\pi, (2n-1)\pi)$ such that $\delta = \beta^2 + \xi\beta\tan\frac{\beta}{2}$ and

$$\begin{aligned}\iota_n^-(\delta) - \zeta_n(\delta) &= -\frac{\xi}{\xi(1-\cos\beta)+\beta\sin\beta} - \left(-\frac{1}{2} + \frac{(2n-1)^2\pi^2}{2\left(\beta^2+\xi\beta\tan\frac{\beta}{2}\right)}\right)\\ &= \frac{\beta\left(-\frac{\xi}{\tan\frac{\beta}{2}}+\beta\right)-(2n-1)^2\pi^2}{2\left(\beta^2+\xi\beta\tan\frac{\beta}{2}\right)}\\ &< \frac{\beta^2-(2n-1)^2\pi^2}{2\left(\beta^2+\xi\beta\tan\frac{\beta}{2}\right)} < 0. \qquad \square\end{aligned}$$

An important application of the hyperbolas ζ_n, $n \in \mathbb{N}$, is that they help to determine an unconditionally stable region independent of the parameter ξ in the δ-h plane. We call it the *sweet region*. See Figure 4.1.

THEOREM 4.6. *For every $n \in \mathbb{N}$, let ζ_n be the the hyperbola defined at (4.23), Δ_n^+ be the closed region encompassed by the graphs of the hyperbola ζ_n and the straight lines $h = 0$, $\delta = (2(n-1)\pi)^2$ in the δ-h plane, and Δ_n^- be the closed region encompassed by the graphs of the hyperbola ζ_n and the straight lines $h = 0$, $\delta = (2n\pi)^2$ in the δ-h plane. Then $\cup_{n=1}^{+\infty}(\Delta_n^+ \cup \Delta_n^-)$ is a stability region independent of the parameter ξ.*

Proof. By definition we know that Δ_n^+ and Δ_n^-, $n \in \mathbb{N}$, are independent of ξ. By Theorem 4.5, we know that Δ_n^+ is below the stability lobe ι_n^+ defined at (3.22). Then Δ_n^+ is below every stability lobe ι_m^+, $m \geq n$; otherwise, there exists a hyperbola ζ_{m_0}, $m_0 > n$, the graph of which crosses the graph of ζ_n. This is impossible.

Note that the graphs of the stability lobes ι_m^+ with $m \leq n-1$ are to the left of the straight line $\delta = (2(n-1)\pi)^2$. It follows that $\Delta_n^+ \subset \mathbb{R}_+^2 \cup \{(\delta, h) : \delta > 0, h = 0\}$ is contained in the complement of the epigraph of the stability lobe ι_m^+ for every $m \in \mathbb{N}$. Then, by Theorem 3.7, Δ_n^+ is a stability region. A similar argument shows that Δ_n^- is also a stability region, and hence $\cup_{n=1}^{+\infty}(\Delta_n^+ \cup \Delta_n^-)$ is a stability region independent of ξ. ◻

Note that ζ_n, $n \in \mathbb{N}$, are monotonically decreasing functions on $\mathbb{R}_+$. Then evaluation of ζ_n at the vertical lines $\delta = (2(n-1)\pi)^2$ and $\delta = (2n\pi)^2$ leads to the next result.

Corollary 4.7. *Let Δ_n^+ and Δ_n^-, $n \in \mathbb{N}$, be as in Theorem* 4.6. *Then the point in Δ_n^+ with maximal h-value including $+\infty$ is $(\delta, h) = \left((2(n-1)\pi)^2, \frac{4n-3}{8(n-1)^2}\right)$; the point in Δ_n^- with minimal h-value is $(\delta, h) = \left((2n\pi)^2, \frac{1-4n}{8n^2}\right)$.*

Remark 4.8. From Lemmas 4.3 and 4.4, we conclude that as $n \to +\infty$, we have $\theta_n^{+\min} = +\infty$, $\iota_n^{+\min} \to 0$ and $\theta_n^{-\max} = +\infty$, $\iota_n^{-\max} \to 0$. By Corollary 4.7 we can also observe that the height of the stability region goes to zero as $n \to +\infty$. Recall that if $n \to +\infty$, then $\delta = k_x\tau_0/m = 2\pi k_x/(m\Omega) \to +\infty$. These observations imply that if the spindle speed is very low, the turning process without control is almost never stable.

5. Comparison of stability regions. In this section, we are interested in finding out which h_j, $j = 1, 2, 3$, defined by (3.13)–(3.15), gives a better stability region in the parameter space of the relative frequency δ and the dimensionless depth of cut K_1. In other words, we translate the results obtained in sections 3 and 4 for each of the models by substituting h_j, $j = 1, 2, 3$, for h.

For the sake of comparison, we regard the graph of the stability region with $h = K_1p^{q-1}$ as the benchmark standard which corresponds to the model with a constant delay assumption. We only need to transform the graph of ι into the graph of K_1p^{q-1} for every h.

For $j = 2$, $h_2 = K_1p^{q-1}(1 - \frac{p\tau_0}{k_r})$, which implies that if $1 > \frac{p\tau_0}{k_r} > 0$, the stability region of the model with state-dependent delay without spindle control is a vertical stretch on the the stability region of the model with a constant delay assumption. The closer $\frac{p\tau_0}{k_r}$ is to 1, the larger the stretched stability region. If $\frac{p\tau_0}{k_r} > 1$, then we have an extra stability region in the lower half plane of $\mathbb{R}_+ \times \mathbb{R}$ of (δ, h). Recall that $p = \nu/R\Omega$ is the dimensionless feed per revolution, and $\tau_0 = 2\pi/\Omega$ is the revolution period. The stability region associated with $\frac{p\tau_0}{k_r} = \frac{\nu}{R\Omega} \cdot \frac{2\pi}{\Omega} > 1$ corresponds to the situation of slow spindle speed and fast feed speed.

From the observations above we know that the model with state-dependent delay not only improves the stability region of the model with constant delay assumption by a factor $1/(1 - \frac{p\tau_0}{k_r})$ if $\frac{p\tau_0}{k_r} \neq 1$, but also provide means of investigating the low spindle speed situation.

For $j = 3$, $h_3 = K_1p^{q-1} - q$, which implies that the stability region of the model with state-dependent delay and spindle control can be obtained by up-shifting by q the boundary of the stability region of the model with a constant delay assumption. This is the most conspicuous improvement because the up-shift by a constant q produces a rectangular region of (δ, h) with height at least q in the plane of $\mathbb{R}_+ \times \mathbb{R}$. This means that, under the spindle speed control, there exists a range for the dimensionless depth of cut which is unconditionally stable for all relative vibration frequencies between the tool and the workpiece. In contrast to what we have discussed in Remark 4.8, the spindle control stability can be achieved when the very low spindle speed situation arises.

We note that K_1p^{q-1} is positively related to K_1/p, which is the ratio of the dimensionless depth of cut $K_1 = qK_y\omega(2\pi R)^{q-1}/k_x$ and the dimensionless feed per revolution $p = \nu/(R\Omega)$. The values of $K_1 p^{q-1}$ associated with stable turning processes can be interpreted as a measure of the cutting versatility of the machine-tool. If we regard h as a parameter in the model-independent stability region at δ in the δ-h plane, then for the benchmark model with a constant delay assumption we have $K_1 p^{q-1} = h$ if $h \geq 0$. For the model with state-dependent delay without spindle control, we have $K_1 p^{q-1} = h/(1 - \frac{p\tau_0}{k_r})$, which has a larger absolute value than h if $\frac{p\tau_0}{k_r} \in (0, 1)$. For the model with state-dependent delay and spindle control, we have $K_1 p^{q-1} = h + q$, which is larger than h. This comparison tells us that the models with state-dependent delay have more choices of system parameters for stable turning processes.

6. Concluding remarks. In this paper we have developed a linear stability theory of a state-dependent model of turning processes using its equivalent which is a system of differential equations with both discrete and distributed delays. We have further developed a procedure which is applicable to systems with discrete and/or distributed delays to analytically determine the stability region with respect to parameters. The linear stability analysis shows that the stability region obtained through the classical model with constant delay is smaller than that of the more accurate model with state-dependent delay and that there exists a sweet region which is independent of damping. The study on the effect of a variable spindle speed control on the stability region of the state-dependent model of turning processes shows that the spindle speed control we proposed can give an essential improvement on the stability of the turning processes. Note that when the parameters (δ, h) vary and cross the stability boundary the system undergoes Hopf bifurcation. It has been reported in [14] that apart from increased linear stability with state-dependent delay, the Hopf bifurcation being subcritical for constant delay tends to become supercritical for state-dependent delay.

We remark that the variable spindle speed control law that we proposed is a PD (proportional-differential) controller, which could be challenging to realize in practice without further feedback delays. This observation also initiates our future work on delayed feedback control for turning processes. We also remark that it is highly possible that we can extend the linear analysis to milling processes due to the similarities between these machining processes.

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