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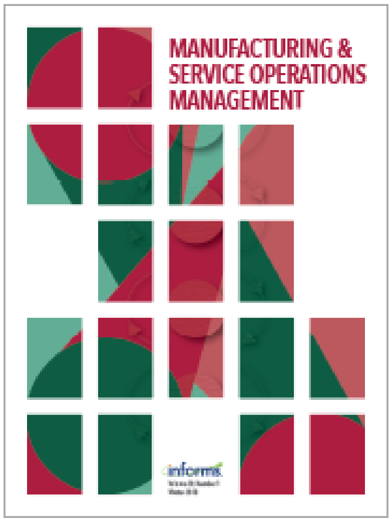
Citation:

Chen, W., M. Dawande, and G. Janakiraman. 2018. "Optimal procurement auctions under multistage supplier qualification." Manufacturing and Service Operations Management 20(3): 566-582, doi: 10.1287/msom.2017.0664



Manufacturing & Service Operations Management

Publication details, including instructions for authors and subscription information:
http://pubsonline.informs.org

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informs
http://pubsonline.informs.org/journal/msom/
MANUFACTURING & SERVICE OPERATIONS MANAGEMENT
Vol. 20, No. 3, Summer 2018, pp. 566–582
ISSN 1523-4614 (print), ISSN 1526-5498 (online)

Optimal Procurement Auctions Under Multistage Supplier Qualification

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Received: April 7, 2015
Revised: March 30, 2016; January 26, 2017; June 22, 2017
Accepted: June 23, 2017
Published Online in Articles in Advance: April 25, 2018

https://doi.org/10.1287/msom.2017.0664

Abstract. We consider a firm that solicits bids from a fixed-sized pool of yet-to-be-qualified suppliers for an indivisible contract. The contract can only be awarded to a supplier who passes a multistage qualification process. For each stage of the qualification process, the buyer incurs a fixed testing cost for each supplier she chooses to test. The buyer seeks an optimal mechanism—that is, one that minimizes her total expected cost. Motivated by the buyer's urgency (or the lack of it) of time for completing the qualification process, we obtain optimal mechanisms for two testing environments: (1) *simultaneous* testing, where in each stage, the buyer selects a subset of those suppliers who have passed all the previous stages and tests them simultaneously; and (2) *nonsimultaneous* testing, where the simultaneous-testing requirement is not imposed. Under simultaneous testing, the admission policy for selecting suppliers at each stage is based on nonuniform reserve-price thresholds. Under nonsimultaneous testing, too, the admission policy is threshold based, but the selection process is sequential in nature. The relative increase in cost due to the simultaneous-testing requirement is (under a mild condition) monotonically increasing in the number of suppliers, the expected multistage testing cost, and the overall passing probability. We also study the optimal sequencing of the qualification stages and show that the buyer should schedule the stages in increasing order of the ratio of their testing cost to their failing probability. Finally, for the simpler setting of a single-stage qualification process and a single supplier, we study a two-dimensional mechanism design problem where, in addition to cost, the passing probability is also private to the supplier. Here, too, threshold-based admission remains optimal, and the buyer offers either a pooling or a separating contract.

Supplemental Material: The online appendix is available at https://doi.org/10.1287/msom.2017.0664.

Keywords: supplier qualification • multistage testing • mechanism design • optimal auction

1. Introduction

Supplier qualification is essential to the success of any major procurement activity—no firm wants to award a procurement contract to a supplier who later fails to meet the technical requirements of the item(s) being sourced or becomes financially unstable. In addition to checks of technical capability and financial viability, the qualification process typically involves a host of other examinations, including that of past lead-time performance and the ownership structure of the supplier's business. Often, qualification also requires the contenders to demonstrate their claimed expertise via carefully chosen pilot implementations.

There is a natural progression in the tests that are involved in the qualification process. For example, technical capability, lead-time performance, and other operational metrics are perhaps the first to be investigated. If a supplier passes this stage of testing, then the financial health of the supplier's business and ownership details may be carefully reviewed in the next stage. If the supplier clears this stage too, then he may be asked to build a pilot implementation in the third stage. Thus, it is reasonable to assume that supplier qualification is a *multistage* process. Also, as is implied by the discussion above, each qualifying stage is sometimes a *time-consuming* one. Therefore, to avoid an inordinate delay in choosing the winner, the suppliers that are admitted into a stage of testing may need to be tested *simultaneously* in that stage. Finally, we recognize that the qualification process is *expensive*: the buying firm needs to invest a significant amount of time and effort on the activities in each stage and for each supplier tested in that stage.

We provide two real-world instances of the use of such qualification processes in procurement. Siemens summarizes its multistage process as follows:

> The first step . . . includes a central "manual" check of basic supplier account . . . , a background check . . . in the sanctioned parties lists under the terms . . . of local

> laws, and an intensive review of the integrity and qualifications . . . . Following successful basic qualification, additional steps may be required, depending on perceived risks. (Siemens 2018)

Kalvet and Lember (2010) describe an interesting example of the multistage supplier-selection process used for the contract of the Journey Planner software for the Helsinki metropolitan area:

> A three-stage competition was organized. Altogether 10 bids were initially received, 6 were selected for the first qualification, and then 3 were chosen for demonstration implementation. The unique aspect in this particular procurement process for innovation is that 3 bidders were asked to realize the demonstration service using the real data before the final selection. (p. 248)

The problems in this paper incorporate the key features highlighted above of supplier qualification processes: They are multistage processes, incurring significant testing costs to the buyer. In addition, depending on the time available for the procurement, the buyer may need to test the suppliers in a stage simultaneously. We consider two extremes: one in which the simultaneous testing requirement is imposed within each stage of the qualification process (i.e., all the suppliers selected for a stage of qualification are tested simultaneously) and the other in which the simultaneous testing requirement is not imposed. We refer to the former setting as *simultaneous* testing and the latter as *nonsimultaneous* testing.

Our basic setting is as follows: A buyer wishes to award an indivisible sourcing contract to a "qualified" supplier. There is a fixed-size pool of potential suppliers. To be qualified, a supplier must be selected (by the buyer) for each stage of qualification and must pass that stage; the probability of passing a stage of qualification is the same across the suppliers who enter that stage. The passing probabilities, however, may differ across stages. To be eligible for a stage of qualification, a supplier must have been selected for—and passed—all prior stages. If no supplier passes all the stages of qualification, then the buyer awards the contract to a qualified outside option. Each potential supplier is endowed with a private cost, which is a realization of a random variable that has a common distribution. For each stage of qualification, the buyer incurs a fixed testing cost for each supplier she chooses to test in that stage; these fixed costs may differ across stages. The buyer seeks an *optimal* mechanism—one that minimizes her total expected procurement cost, for the case with the simultaneous testing requirement and for the case without.

For our problem, a mechanism is defined by the following triplet: (i) an *admission* rule that specifies the selection of a supplier or multiple suppliers—from among those who have passed all previous stages—for the next stage of qualification, (ii) an *allocation* rule that specifies the choice of the winner from among the qualified suppliers and the outside option, and (iii) a *payment* rule that defines the corresponding payments. We now summarize the main results in this paper:

- *Optimal mechanisms.* We obtain optimal mechanisms (auctions) for both the simultaneous-testing and nonsimultaneous-testing settings described above. For the simultaneous-testing case, we show that the optimal admission rule is *threshold based*; that is, for each qualification stage, the buyer admits the kth-cheapest supplier only if his reported cost is below the kth threshold for that stage. These thresholds depend on the suppliers' cost distribution and not on their reported costs. Furthermore, for any stage, the kth threshold is decreasing in k. We then investigate the comparative statics of the optimal thresholds and also compare the thresholds across stages.

When the simultaneous-testing requirement is not imposed, we show that the optimal admission rule is also threshold based: the buyer only considers qualifying suppliers whose reported costs fall below a certain threshold. However, the optimal testing process is *sequential* in nature: the buyer starts with the cheapest supplier and tests him sequentially for the qualification stages. If he passes all the stages, then he gets the contract; otherwise, if he fails at any stage, the buyer tests the second-cheapest supplier sequentially through the stages. The buyer repeats this process until she finds a qualified supplier, to whom she awards the contract. If all the suppliers under consideration fail, then the buyer procures from the outside option.

- *Cost comparison.* We examine the increase in cost due to the simultaneous-testing requirement, and we show that (under a mild condition) the relative increase is monotonically increasing in the number of suppliers, the expected multistage testing cost, and the overall passing probability. In addition, we also consider a setting that explicitly considers a cost of time, and we offer some understanding on a preferable mechanism as this cost varies.
- *Sequencing of the qualification stages.* We study the optimal *sequencing* of the qualification stages: Given the passing probability and the testing cost for each stage, how should the buyer sequence the stages in an optimal mechanism? Under a mild condition, we show that under both the simultaneous-testing and the nonsimultaneous-testing settings, the stages should be sequenced in increasing order of the ratio of their testing cost to their failing probability.
- *Private information on passing probabilities.* In practice, a supplier may be better informed about his ability to pass a stage of qualification. Motivated by this, we analyze a two-dimensional mechanism design problem in which the passing probabilities are also private to the suppliers. Because of the difficulty in optimizing a general two-dimensional mechanism design problem,

we restrict attention to a single-stage qualification process with a single supplier, whose private information on each dimension (i.e., cost and passing probability) is binary. We show that the optimal admission rule is still threshold based: the buyer admits the supplier to the qualification test only if his reported cost is below a corresponding threshold. However, depending on the value of the parameters, the buyer offers either a *pooling* contract (i.e., one in which the admission criteria for some supplier types are the same) or a *separating* contract (i.e., one in which the admission criteria for different supplier types are different).

The remainder of the paper is organized as follows. We first review the related literature and differentiate our contributions in Section 2. Then, for better exposition, Section 3 presents the analysis of an optimal procurement auction for the special case of simultaneous testing in which the qualification process consists of a *single* stage. Next, Section 4 analyzes the general multistage qualification process under simultaneous testing. In Section 4, we also examine the comparative statics of the optimal thresholds and investigate how these thresholds change across stages. In Section 5, we study the multistage qualification process again by removing the simultaneous-testing requirement, and we obtain an optimal procurement mechanism. We then examine the increase in cost due to the imposition of the simultaneous-testing requirement. Finally, Section 6 examines the following additional considerations: (1) the optimal sequencing of stages; (2) the two-dimensional mechanism design problem for a single-stage, single-supplier setting, in which the passing probability is also private to the supplier; and (3) the trade-off between the simultaneous-testing and sequential-testing regimes as the cost of time varies. Section 7 concludes.

2. Related Literature and Our Contribution

We discuss work in the mechanism design literature and the procurement literature that is related to our analysis in this paper, and we distinguish our relative contribution.

In the mechanism design literature, depending on the dimensionality of the agents' private information, problems can be divided into two categories: *single dimensional* and *multidimensional*. Furthermore, depending on whether or not the private information of the agents evolves over time, we have either a *static* or a *dynamic* setting. In our analysis in this paper, we encounter both single-dimensional and multidimensional static mechanism design problems. The standard approach in static mechanism design typically reduces the original mechanism design problem to a static and deterministic optimization problem that is often easy to solve; see, for example, Myerson (1981) and Laffont and Martimort (2002). On the other hand, solutions to dynamic mechanism design problems typically require solving dynamic optimization problems, such as stochastic dynamic programs; see, for example, Gallien (2006), Chaturvedi et al. (2014), and Pavan et al. (2014). Although our problem under the simultaneous-testing setting discussed above is a static, single-dimensional mechanism design problem, the optimal mechanism requires solving a stochastic dynamic program. Moreover, this analysis becomes more challenging because the decisions in the DP are *set based*—namely, the subset of suppliers to admit into each qualification stage, from those who have passed all prior stages. Our solution to the DP involves two main steps: (1) A reduction to an equivalent DP in which the decision for each stage is a *number* rather than a set, thus significantly reducing the size of the feasible action space. (2) Establishing convexity of the cost-to-go function and other useful structural properties (such as monotonicity) of associated functions to characterize the optimal decisions.

In general, static multidimensional mechanism design problems are quite difficult to solve. This difficulty is often caused by the hardness of solving the underlying optimization problem, since several techniques that are applicable to the single-dimensional setting do not extend to multiple dimensions. Researchers have attacked such problems either by providing approximate solutions or by obtaining optimal solutions for special cases. An example of the former approach is Belloni et al. (2010), who approximate the solution of a multidimensional mechanism design problem. By adapting techniques from Border (1991), the authors reformulate the associated optimization problem and show that the reformulated problem can be approximated arbitrarily well by a sequence of finite-dimensional linear programs. As an example of the latter approach, Chaturvedi and Martínez-de-Albéniz (2011) obtain an optimal mechanism for a two-dimensional problem in the procurement setting under special technical conditions. Another example of the latter approach is Armstrong and Rochet (1999), who study a two-dimensional problem that considers a single agent with binary types on each dimension and provide an optimal mechanism for the special case when the agent's utility is additively separable in its two types. Our analysis in Section 6.2 advances the literature in that we provide a complete solution for a setting that is similar to that in Armstrong and Rochet (1999), but without the requirement that the agent's utility be additively separable in its two types.

In the procurement literature that is related to supplier qualification, a paper that is closely related to our work is Wan and Beil (2009). The authors consider a three-step procurement process: prequalification, auction, and postqualification. In the prequalification step, the buyer first announces a set of prequalification thresholds. Next, she keeps selecting suppliers from an ex ante symmetric pool of infinitely many suppliers until she finds the desired number of suppliers with quality levels higher than their corresponding thresholds. Then, these prequalified suppliers participate in an auction. On the basis of the bids from the prequalified suppliers, the buyer determines a postauction qualification sequence, in which she postqualifies suppliers until a qualified supplier is identified, or all suppliers are disqualified; in the latter case, the qualified outside option is chosen. Their goal is to analyze the trade-off between various levels of prequalification and postqualification.

Our model is different from that of Wan and Beil (2009) in the sense that the buyer only has a fixed and finite pool of yet-to-be qualified suppliers to start with. Our objective is to design an optimal procurement mechanism for the buyer. To this end, when defining the rules of the mechanism, we allow them to be dependent on the qualification outcomes. We assume that the supplier-selection processes and the results of all the qualification tests are publicly observable and verifiable. This assumption ensures that the buyer credibly commits to the preannounced rules (admission, allocation, payment) of the mechanism. By the revelation principle, it is then enough to consider only postauction qualification mechanisms. That is, instead of considering all possible mechanisms (either preauction qualification mechanisms or postauction qualification mechanisms, or a combination of both), it is enough to restrict attention to direct-revelation mechanisms, where the suppliers report their private costs directly to the buyer first, and the qualification, contract award, and payment are carried out by the buyer later. By definition, such direct-revelation mechanisms are postauction qualification mechanisms. It is important to note that when the buyer cannot credibly commit to the preannounced rules of the mechanism, the revelation principle does not necessarily hold; see, for example, Bester and Strausz (2000, 2001). We also note that our results in this paper pertain to a *multistage* qualification process, under both simultaneous testing and nonsimultaneous testing.

A part of the analysis in Wan and Beil (2009) involves optimal procurement auction design: when the decisions in the prequalification step are fixed, what is an optimal procurement auction for the buyer to use in the two remaining steps? They assume that the qualification process is *single stage* and *instantaneous* (and, thus, they use nonsimultaneous qualification testing) and show that a *threshold-based* admission rule is optimal: the buyer sorts the suppliers based on their adjusted virtual costs—the sum of their virtual costs and qualification costs—and tests all the suppliers for whom this value is below the price of the outside option sequentially, starting with the supplier with the lowest adjusted virtual cost. This result is analogous to a standard result in auction theory: in a forward auction where an auctioneer sells an item, it is optimal for her to only consider price bids that are higher than a certain reserve price. Our analysis extends this result further to the setting of multistage qualification testing, both with and without the simultaneous-testing requirement. In addition, we also show that for the two-dimensional mechanism design problem that we consider in Section 6.2, a threshold-based mechanism remains optimal.

We briefly discuss other related papers in the procurement literature. Wan et al. (2012) consider the situation where there are two suppliers: one qualified incumbent and one entrant whose qualification status is yet to be determined. The buyer wishes to use an open-descending auction to procure from these suppliers and faces the choice of whether to conduct the qualification screening of the entrant before or after the auction. The authors characterize both the buyer's optimal decision and the suppliers' optimal decisions. Moreover, they compare the theoretical predictions to results from laboratory experiments. Another related work is Wan et al. (2014); here, the authors model the situation where the buyer relies on a procurement service provider (PSP) to conduct the supplier-qualification screening and recommend a pool of suppliers. The PSP decides on his effort level to discover promising suppliers; such effort is costly for the PSP and hard to enforce in a contract. The authors use a game-theoretic analysis to characterize the optimal strategy of the buyer and also test their findings via laboratory experiments.

We refer the reader to Elmaghraby (2000) and Dimitri et al. (2006) for a general introduction to procurement research in operations and economics. Keskinocak and Tayur (2001), Elmaghraby and Keskinocak (2004), and Elmaghraby (2004, 2007) provide a good introduction to different auction formats used in industry. Beil (2010) is an excellent tutorial on procurement auctions in the operations literature.

3. Simultaneous Testing: Single-Stage Qualification Process

In this section, we consider the simpler setting where supplier qualification consists of only one stage of simultaneous testing. The model, along with the corresponding notation, is presented in Section 3.1. Our assumptions and the formulation of the optimal mechanism design problem are specified in Section 3.2. The optimal mechanism and the main steps of its proof are described in Section 3.3.

3.1. Model

The buyer seeks to procure an indivisible unit from either a supplier or an outside option. She has a pool of yet-to-be-qualified suppliers, represented by $\mathcal{N} = \{1, 2, \ldots, N\}$. Each supplier has a private cost that is drawn from a publicly known distribution with

Table 1. Our Notation

$\mathcal{N}$	The set of symmetric suppliers $\{1, 2, \ldots, N\}$
$f(\cdot), F(\cdot)$	The p.d.f and c.d.f., respectively, of the cost distribution
$\mathcal{X} = [\underline{c}, \bar{c}]$	The support of F
$\mathbf{x} = (x_1, x_2, \ldots, x_n)$	The cost vector, where $x_i, i \in N$ is private to supplier i; $x_i, i = 1, 2, \ldots, N$, are i.i.d. realizations of $F(\cdot)$
$\mathbf{x}_{-i} = (x_1, x_2, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)$	The cost vector excluding the cost of supplier i
$f(\mathbf{x}) = \prod_{i \in N} f(x_i)$	The joint density of vector $\mathbf{x}$
$f_{-i}(\mathbf{x}_{-i}) = \prod_{j=1}^{i-1} f(x_j) \prod_{k=i+1}^{n} f(x_k)$	The joint density of vector $\mathbf{x}_{-i}$
R	The cost of the qualified outside option
K	The cost of testing a supplier (in the single-stage model of Section 3)
$\beta \in (0, 1)$	The probability that a supplier passes the qualification test (in the single-stage model of Section 3)
K_j	The cost of testing a supplier in stage j (in the multistage model of Section 4)
$\beta_j \in (0, 1)$	The probability that a supplier selected for the stage-j qualification test passes that test (in the multistage model of Section 4)

support $[\underline{c}, \bar{c}]$, cumulative distribution function (c.d.f.) $F(\cdot)$, and probability density function (p.d.f.) $f(\cdot)$. The probability that a supplier passes the qualification test, if selected to participate, is β. Thus, the suppliers are symmetric with respect to the cost distribution and the probability of passing the qualification test. The buyer incurs a testing cost of K dollars for each supplier she tests. The buyer also has a qualified outside option that costs R dollars. The notation is summarized in Table 1.

The buyer is interested in (i) deciding which suppliers to test simultaneously and, based on the results of the tests, (ii) determining to which suppliers (or the outside option) to award the contract (allowing for randomized policies), and (iii) determining the payments to be made to the suppliers. The buyer's goal is to design a mechanism that minimizes her expected cost—that is, the expected sum of payments and testing costs. We will refer to such a mechanism as an *optimal* mechanism. Our analysis assumes that the buyer announces the rules of the mechanism first and *commits* to using these rules throughout the process.

3.2. Formulation of the Optimal Mechanism Design Problem

We start by stating our assumptions, which are used throughout this paper.

(i) The virtual cost function, defined by $\psi(x) = x + F(x)/f(x)$, is monotonically increasing. (By convention, we use the term *increasing* to denote *nondecreasing* and the term *decreasing* to denote *nonincreasing*.) This is a commonly used regularity condition in the auctions literature; see, for example, Krishna (2009). Many probability distributions satisfy this condition—for example, the uniform, normal, and exponential distributions.

(ii) The upper bound $\bar{c}$ of the support $\mathcal{X} = [\underline{c}, \bar{c}]$ satisfies the inequality $\bar{c} \le \psi^{-1}(R)$. We make this assumption purely for expositional convenience. As we will see later in Section 3.3, this condition ensures that a qualified supplier always has a lower virtual cost than R, which in turn guarantees that the buyer prefers a qualified supplier to the outside option in an optimal mechanism.

(iii) The processes and results of all qualification tests are publicly observable and verifiable. This assumption ensures that the buyer fully commits to the preannounced rules of the mechanism, which in turn ensures the applicability of revelation principle.

In our search for an optimal mechanism, we know from the revelation principle (see, e.g., Myerson 1981) that it is sufficient to consider incentive compatible and individually rational direct mechanisms—that is, mechanisms in which (i) each supplier is required to report a number as his cost to the buyer in a sealed-bid manner, (ii) all suppliers are willing to participate (individual rationality, or IR), and (iii) suppliers report their private cost information truthfully in a Bayesian Nash equilibrium (incentive compatibility, or IC). To present a mathematical formulation of the problem of finding an optimal mechanism, we need some additional notation, which we now assemble.

Let $\mathcal{I}(\mathbf{x}) \subseteq \mathcal{N}$ denote the subset of suppliers that the buyer chooses to test. As the notation indicates, this subset can depend on the cost vector $\mathbf{x}$. The buyer can only award the contract to a qualified supplier. If all suppliers in $\mathcal{I}(\mathbf{x})$ fail the test or if $\mathcal{I}(\mathbf{x})$ is empty, the buyer awards the contract to the outside option that is priced at R.

Let $\mathcal{S} \subseteq \mathcal{I}(\mathbf{x}) \subseteq \mathcal{N}$ denote the set of qualified suppliers. For any set $\mathcal{P}$, we use $2^{\mathcal{P}}$ to denote its power set. For each $\mathcal{S} \subseteq \mathcal{I}(\mathbf{x})$, we define an allocation rule as follows:

$$\mathbf{Q}(\mathcal{S}, \mathbf{x}):\ 2^{\mathcal{I}(x)} \times [\underline{c}, \bar{c}]^N \to [0, 1]^N.$$

Similarly, for each $\mathcal{S} \subseteq \mathcal{I}(\mathbf{x})$, we define the expected payment rule as follows:

$$\mathbf{M}(\mathcal{S}, \mathbf{x}):\ 2^{\mathcal{I}(x)} \times [\underline{c}, \bar{c}]^N \to \mathbb{R}_+^N.$$

For supplier $i \in \mathcal{N}$, $Q_i(\mathcal{S}, \mathbf{x})$ denotes the probability that supplier i is awarded the contract, and $M_i(\mathcal{S}, \mathbf{x})$ denotes the expected payment made to him.

A direct mechanism is defined by the triplet $(\mathcal{I}, \mathbf{Q}, \mathbf{M})$. We have the following feasibility constraints:

$$\sum_{i \in \mathcal{S}} Q_i(\mathcal{S}, \mathbf{x}) \le 1, \quad \forall\, \mathbf{x}, \mathcal{S} \subseteq \mathcal{I}(\mathbf{x}), \tag{1}$$

$$Q_i(\mathcal{S}, \mathbf{x}) = 0, \quad \forall\, \mathbf{x},\ \mathcal{S} \subseteq \mathcal{I}(\mathbf{x}),\ i \notin \mathcal{S}. \tag{2}$$

These constraints state that in any mechanism, the contract cannot be awarded to a supplier who is not qualified. Note that we allow the buyer to use a randomized policy in awarding the contract—for example, award the contract to supplier 1 with probability 0.5 and to supplier 2 with probability 0.5—if these suppliers are qualified.

For all $\mathcal{I} \subseteq \mathcal{N}$, and a set function $r: 2^{\mathcal{I}} \to \mathbb{R}$, let

$$\mathbb{E}_{\mathcal{S}}^{\mathcal{I}} r(\mathcal{S}) = \sum_{\mathcal{S} \subseteq \mathcal{I}} p_{\mathcal{S}}^{\mathcal{I}}\, r(\mathcal{S}),$$

where $p_{\mathcal{S}}^{\mathcal{I}} = \beta^{|\mathcal{S}|}(1-\beta)^{|\mathcal{I}|-|\mathcal{S}|}$ for all $\mathcal{S} \subseteq \mathcal{I}$. In other words, $p_{\mathcal{S}}^{\mathcal{I}(\mathbf{x})}$ denotes the probability that the set of qualified suppliers is $\mathcal{S}$, given that the buyer chooses $\mathcal{I}(\mathbf{x})$ as the subset of suppliers to be tested. By convention, we let $\mathbb{E}_{\mathcal{S}}^{\varnothing} r(\mathcal{S}) = r(\varnothing)$.

Given $\mathcal{I}(\mathbf{x})$, let

$$q_i(z_i) = \mathbb{E}_{\mathbf{x}_{-i}}[\mathbb{E}_{\mathcal{S}}^{\mathcal{I}(z_i,\mathbf{x}_{-i})} Q_i(\mathcal{S}, z_i, \mathbf{x}_{-i})], \quad \forall\, i \in \mathcal{N}, \tag{3}$$

and let

$$m_i(z_i) = \mathbb{E}_{\mathbf{x}_{-i}}[\mathbb{E}_{\mathcal{S}}^{\mathcal{I}(z_i,\mathbf{x}_{-i})} M_i(\mathcal{S}, z_i, \mathbf{x}_{-i})], \quad \forall\, i \in \mathcal{N}. \tag{4}$$

Thus, $q_i(z_i)$ (respectively, $m_i(z_i)$) denotes the probability that supplier i will be awarded the contract (respectively, the expected payment received by supplier i) if he reports a cost of z_i. Note that both $q_i(z_i)$ and $m_i(z_i)$ depend on the admission policy $\mathcal{I}(\mathbf{x})$. We suppress this dependence in our notation for simplicity.

The IC constraints can now be stated as follows:

$$m_i(x_i) - q_i(x_i)x_i \geq m_i(z_i) - q_i(z_i)x_i, \quad \forall\, x_i, z_i, i \in \mathcal{N}. \tag{5}$$

These constraints state that it is optimal for suppliers to reveal their private costs truthfully, given that all other suppliers are also doing so. That is, "truth telling" is a Bayesian Nash equilibrium.

The IR constraints can be stated as follows:

$$m_i(x_i) - q_i(x_i)x_i \geq 0, \quad \forall\, x_i, i \in \mathcal{N}. \tag{6}$$

These constraints impose that, for all suppliers, the expected payoff in a Bayesian Nash equilibrium is nonnegative.

We are now ready to formulate our mechanism design problem.

The Optimal Mechanism Design Problem. For any $\mathbf{x}$ and a triplet $(\mathcal{I}, \mathbf{Q}, \mathbf{M})$, let $\mathrm{TC}(\mathbf{x}, \mathcal{I}, \mathbf{Q}, \mathbf{M})$ denote the expected total cost incurred by the buyer. This cost is the sum of the following costs: (i) the testing cost $K|\mathcal{I}(\mathbf{x})|$, (ii) the expected payment $\sum_{i\in\mathcal{N}} \mathbb{E}_{\mathcal{S}}^{\mathcal{I}(x)} M_i(\mathcal{S}, \mathbf{x})$ made to the suppliers, and (iii) the expected cost $R\,[1 - \sum_{i\in\mathcal{N}} \mathbb{E}_{\mathcal{S}}^{\mathcal{I}(x)} Q_i(\mathcal{S}, \mathbf{x})]$ incurred if the buyer uses the outside option. Thus, using (3) and (4), we have

$$\begin{aligned}\mathbb{E}_{\mathbf{x}}&[\mathrm{TC}(\mathbf{x}, \mathcal{I}, \mathbf{Q}, \mathbf{M})]\\ &= \mathbb{E}_{\mathbf{x}}[K \cdot |\mathcal{I}(\mathbf{x})|] + \sum_{i\in\mathcal{N}} \mathbb{E}_{x_i}[m_i(x_i)]\\ &\quad + R\left[1 - \sum_{i\in\mathcal{N}} \mathbb{E}_{x_i}[q_i(x_i)]\right].\end{aligned} \tag{7}$$

The optimal mechanism design problem can be formulated as follows:

$$\begin{aligned}&\min_{\mathcal{I}, \mathbf{Q}, \mathbf{M}} \ \mathbb{E}_{\mathbf{x}}[\mathrm{TC}(\mathbf{x}, \mathcal{I}, \mathbf{Q}, \mathbf{M})]\\ &\text{s.t. } (1), (2), (3), (4), (5), \text{ and } (6).\end{aligned} \tag{P1}$$

3.3. The Optimal Mechanism

As will become clear in the proof of Theorem 1, if the buyer wants to test k suppliers, it is always beneficial for her to test the k suppliers with the lowest costs. Thus, instead of determining the optimal *subset* of suppliers to test, she only needs the optimal *number*. Next, we present a definition that will be useful later for characterizing the optimal number of suppliers to test.

Given a cost vector $\mathbf{x}$, let $x_{(i)}$, $1 \leq i \leq N$, denote the ith lowest number among $x_1, x_2, \ldots, x_N$. We will refer to the supplier with cost $x_{(i)}$ simply as supplier (i); that is, supplier (i) is the supplier with cost rank i in $\mathcal{N}$. For any cost vector $\mathbf{x}$, $\mathcal{S} \subseteq \mathcal{N}$, and $i = 1, 2, \ldots, |\mathcal{S}|$, $x_{(i)}^{\mathcal{S}}$ denotes the cost of the supplier with the ith-lowest cost in $\mathcal{S}$.

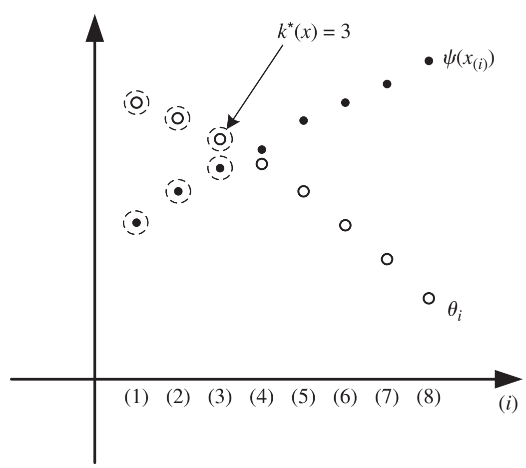
Definition 1 (Number of Suppliers to Test). For $i \in \mathcal{N}$, let $\theta_i = R - K/(\beta(1-\beta)^{i-1})$. Given a cost vector $\mathbf{x}$, let

$$k^*(\mathbf{x}) = \begin{cases} 0 & \text{if } \psi(x_{(1)}) > \theta_1, \\ i & \text{if there exists } i \in \{1, 2, \ldots, N-1\} \text{ such that} \\ & \quad \psi(x_{(i)}) \leq \theta_i \text{ and } \psi(x_{(i+1)}) > \theta_{i+1}, \\ N & \text{if } \psi(x_{(N)}) \leq \theta_N. \end{cases}$$

It can be verified that $k^*(\mathbf{x})$ is uniquely defined—this will become clear in the discussion below.

Figure 1 presents a geometric interpretation of Definition 1. In this figure, we have eight suppliers. Consider the corresponding cost vector $\mathbf{x}$. The sequence $\psi(x_{(1)}), \psi(x_{(2)}), \ldots, \psi(x_{(8)})$ is an increasing sequence (since $\psi(\cdot)$ is increasing), as shown by the solid dots in the figure. The hollow circles depict the values of the thresholds θ_i, $i = 1, 2, \ldots, 8$, as a decreasing sequence. Intuitively, if the virtual cost of supplier (i) is below the

Figure 1. A Geometric Interpretation of the Number of Suppliers to Test



Chen, Dawande, and Janakiraman: *Optimal Procurement with Supplier Qualification*
Manufacturing & Service Operations Management, 2018, vol. 20, no. 3, pp. 566–582, © 2018 INFORMS

corresponding threshold (hollow circle) in the figure, then the marginal benefit of including him in the test exceeds the marginal cost of testing him (this fact will be established in the proof of Theorem 1). Thus, it is beneficial for the buyer to include supplier (i) in the test. In the figure, we see that the virtual costs of suppliers (1), (2), and (3) are all below their corresponding thresholds, and the virtual cost of supplier (4) exceeds his threshold. Thus, $k^*(\mathbf{x}) = 3$.

We now present the complete solution of problem (P1). Consider the following mechanism for the buyer:

1. All the suppliers submit their cost bids. Let $\mathbf{x}$ denote the vector of these reported costs.
2. The buyer admits the $k^*(\mathbf{x})$ suppliers with the lowest costs into the qualification test.
3. Depending on results of the qualification test, the buyer awards the contract as follows:
   a. If there are two or more suppliers who pass the qualification test, the contract is awarded to the qualified supplier with the lowest cost; his payment is $x^{\mathcal{S}}_{(2)}$, which is the cost of the second-cheapest qualified supplier. Here, $\mathcal{S}$ denotes the set of qualified suppliers.
   b. If there is only one supplier who passes, the contract is awarded to this supplier at a price of $\min[\psi^{-1}(\theta_{k^*(\mathbf{x})}), x_{(k^*(\mathbf{x})+1)}]$, which is the highest level to which his cost can be increased, while ensuring his admission into the qualification process. (This claim is a special case of the similar claim in Section 4.2 for multistage testing. A proof of the latter claim is provided in the online appendix.)
   c. If no supplier passes, the contract is awarded to the outside option that is priced at R.
4. The buyer only pays the supplier/outside option to whom the contract is awarded; the remaining suppliers do not receive any payment.

We now state our main result of this section.

Theorem 1. *The above mechanism is optimal for problem* (P1).

For brevity, we list the main steps of the proof here, and we refer readers to the online appendix for details.

Step 1 (*Lemma* 1 *in the online appendix*). In this step, we solve the following problem:

$$\min_{\mathcal{I}(\mathbf{x})}\big[K\cdot|\mathcal{I}(\mathbf{x})| + \beta\psi(x^{\mathcal{I}(x)}_{(1)}) + (1-\beta)\beta\psi(x^{\mathcal{I}(x)}_{(2)}) + \cdots + (1-\beta)^{|\mathcal{I}(x)|-1}\beta\psi(x^{\mathcal{I}(x)}_{(|\mathcal{I}(\mathbf{x})|)}) + (1-\beta)^{|\mathcal{I}(x)|}R\big]. \quad \text{(P2)}$$

Let us examine the intuition behind this objective function, which represents the minimal expected total cost the buyer incurs if she admits the set $\mathcal{I}(\mathbf{x})$ into the qualification test. Specifically, the first term $K\cdot|\mathcal{I}(\mathbf{x})|$ is the testing cost and the remaining terms constitute the expected procurement cost. The term $\beta\psi(x^{\mathcal{I}(x)}_{(1)})$ refers to the situation when the lowest-cost supplier in $\mathcal{I}(\mathbf{x})$ passes the qualification test; here, β is the probability that this event occurs and $\psi(x^{\mathcal{I}(x)}_{(1)})$ represents the *private* cost $x^{\mathcal{I}(x)}_{(1)}$ plus the *information rent* (a standard notion in mechanism design) paid by the buyer to the supplier. The term $(1-\beta)\beta\psi(x^{\mathcal{I}(x)}_{(2)})$ corresponds to the case when the lowest-cost supplier fails but the second-lowest-cost supplier passes. The explanation for the other terms is similar. The last term $(1-\beta)^{|\mathcal{I}(x)|}R$ corresponds to the case when all suppliers in $\mathcal{I}(\mathbf{x})$ fail and the buyer awards the contract to the outside option that is priced at R.

Step 2 (*Lemmas* 2 *and* 3 in the online appendix). Using the solution $\mathcal{I}^*(\mathbf{x})$ of problem (P2) and following some standard steps in mechanism design, we construct a triplet $(\mathcal{I}^*, \mathbf{Q}^*, \mathbf{M}^*)$ that solves problem (P1). In Lemma 2, we verify a monotonicity condition (intuitively, this condition means that the lower a supplier's cost, the higher his chance of being awarded the contract) that is standard in mechanism design. Its proof is both new in our context and will be useful later in the analysis of multistage qualification (see Section 4). For this reason, we have chosen to present this proof in the main body. Lemma 3 establishes that the triplet $(\mathcal{I}^*, \mathbf{Q}^*, \mathbf{M}^*)$ solves problem (P1). The proof of this result follows a series of standard steps and is presented in the online appendix.

Step 3. Our last step is the observation that the triplet $(\mathcal{I}^*, \mathbf{Q}^*, \mathbf{M}^*)$ characterizes the mechanism proposed in Theorem 1, thus completing its proof.

4. Simultaneous Testing: Multistage Qualification Process

In this section, we generalize our analysis in the previous section to a multistage qualification process. This section is organized as follows: In Section 4.1, we present the model, notation, and formulation of the optimal mechanism design problem. The solution of this problem and the main steps of the proof are presented in Section 4.2. In Section 4.3, we study the comparative statics of the thresholds used in the optimal mechanism and compare these thresholds across the stages.

4.1. Formulation of the Optimal Mechanism Design Problem

We extend the model of Section 3.1 to a multistage qualification process. In this case, to be deemed qualified, a supplier must be selected for—and pass—the qualification test in all the stages. We consider a time horizon of $J+1$ stages, where stages 1 through J are qualification stages and stage $J+1$ is the award stage. For stage $j = 1, 2, \ldots, J$, suppose that the (conditional) probability of any supplier passing this stage of the qualification test is β_j, given that he has already passed all previous stages. The testing cost, incurred by the buyer, is K_j for each supplier tested in stage j.

Next, we introduce the notation needed to formulate the optimal mechanism design problem. Let $\mathbf{x}$ denote the vector of the reported costs. We first define the admission policy. Let $\mathcal{I}_1(\mathbf{x}) \subseteq \mathcal{N}$ denote the set of suppliers that the buyer chooses to enter into the first stage of the qualification process. As the notation indicates, we allow $\mathcal{I}_1$ to depend on $\mathbf{x}$. Let $\mathcal{S}_1 \subseteq \mathcal{I}_1(\mathbf{x})$ denote the set of first-stage-qualified suppliers. Similarly, for $j = 2, 3, \ldots, J$, let $\mathcal{I}_j(\mathcal{S}_{j-1}, \mathbf{x}) \subseteq \mathcal{S}_{j-1}$ denote the set of $(j-1)$st-stage-qualified suppliers that the buyer chooses to enter the jth-stage of qualification, and let $\mathcal{S}_j \subseteq \mathcal{I}_j(\mathcal{S}_{j-1}, \mathbf{x})$, $j = 2, 3, \ldots, J$, denote the set of suppliers who pass the stage j test.

In a direct mechanism, the buyer decides the tuple $(\mathcal{I}_1(\cdot), \mathcal{I}_2(\cdot,\cdot), \ldots, \mathcal{I}_J(\cdot,\cdot), \mathbf{Q}(\cdot,\cdot), \mathbf{M}(\cdot,\cdot))$, where the allocation rule $\mathbf{Q}$ and the payment rule $\mathbf{M}$ are defined as follows:

$$\mathbf{Q}(\mathcal{S}_J, \mathbf{x})\colon 2^{\mathcal{I}_J(\mathcal{S}_{J-1}, \mathbf{x})} \times [\underline{c}, \bar{c}]^N \to [0,1]^N,$$
$$\mathbf{M}(\mathcal{S}_J, \mathbf{x})\colon 2^{\mathcal{I}_J(\mathcal{S}_{J-1}, \mathbf{x})} \times [\underline{c}, \bar{c}]^N \to \mathbb{R}_+^N.$$

For supplier $i \in \mathcal{N}$, $Q_i(\mathcal{S}_J, \mathbf{x})$ denotes the probability that supplier i is awarded the contract and $M_i(\mathcal{S}_J, \mathbf{x})$ denotes the expected payment made to him.

The sequence of events in the mechanism design problem is as follows:

- The buyer announces the rules of the mechanism, which include
  1. the qualification admission policy for each stage: $\mathcal{I} = (\mathcal{I}_1(\mathbf{x}), \mathcal{I}_2(\mathcal{S}_1, \mathbf{x}), \ldots, \mathcal{I}_J(\mathcal{S}_{J-1}, \mathbf{x}))$;
  2. the contract allocation rule: $\mathbf{Q} = (Q_1(\mathcal{S}_J, \mathbf{x}), Q_2(\mathcal{S}_J, \mathbf{x}), \ldots, Q_n(\mathcal{S}_J, \mathbf{x}))$; and
  3. the payment rule: $\mathbf{M} = (M_1(\mathcal{S}_J, \mathbf{x}), M_2(\mathcal{S}_J, \mathbf{x}), \ldots, M_n(\mathcal{S}_J, \mathbf{x}))$.
- After learning the rules of the mechanism, each supplier $i \in \mathcal{N}$ submits a cost bid x_i that maximizes his expected utility.
- Given the reported cost vector $\mathbf{x}$, the buyer uses the announced rules $(\mathcal{I}, \mathbf{Q}, \mathbf{M})$ to carry out the qualification process, award the contract, and make payments to the suppliers.

We assume that the buyer commits to use the preannounced rules of the mechanism throughout the qualification and procurement processes. Thus, the suppliers can be viewed as playing a *static* game of incomplete information.

The feasibility constraints are as follows:

$$\sum_{i \in \mathcal{S}_J} Q_i(\mathcal{S}_J, \mathbf{x}) \le 1, \quad \forall\, \mathbf{x}, \mathcal{S}_J \subseteq \mathcal{I}_J(\mathcal{S}_{J-1}, \mathbf{x}), \tag{8}$$

$$Q_i(\mathcal{S}_J, \mathbf{x}) = 0, \quad \forall\, \mathbf{x}, \mathcal{S}_J \subseteq \mathcal{I}_J(\mathcal{S}_{J-1}, \mathbf{x}), i \notin \mathcal{S}_J. \tag{9}$$

These constraints state that the contract can only be awarded to a supplier who passes the last stage of qualification.

For $j = 1, 2, \ldots, J$, $\mathcal{I} \subseteq \mathcal{N}$, $\mathcal{S} \subseteq \mathcal{I}$, and a set function $r\colon 2^{\mathcal{I}} \to \mathbb{R}$, let

$$\mathbb{E}^{\mathcal{I}}_{\mathcal{S}, j} r(\mathcal{S}) = \sum_{\mathcal{S} \subseteq \mathcal{I}} p^{\mathcal{I}}_{\mathcal{S}, j} r(\mathcal{S}),$$

where $p^{\mathcal{I}}_{\mathcal{S}, j} = \beta_j^{|\mathcal{S}|}(1-\beta_j)^{|\mathcal{I}|-|\mathcal{S}|}$. By convention, we define $\mathbb{E}^{\varnothing}_{\mathcal{S}, j} r(\mathcal{S}) = r(\varnothing)$. Define

$$q_i(z_i) = \mathbb{E}_{\mathbf{x}_{-i}}[\mathbb{E}_{[1-J]}[Q_i(\mathcal{S}_J, z_i, \mathbf{x}_{-i})]], \quad \forall\, i \in \mathcal{N}, \tag{10}$$
$$m_i(z_i) = \mathbb{E}_{\mathbf{x}_{-i}}[\mathbb{E}_{[1-J]}[M_i(\mathcal{S}_J, z_i, \mathbf{x}_{-i})]], \quad \forall\, i \in \mathcal{N}, \tag{11}$$

where $\mathbb{E}_{[1-J]}$ is a shorthand for the iterated expectation operators:

$$\mathbb{E}_{[1-J]} := \mathbb{E}^{\mathcal{I}_1(z_i, \mathbf{x}_{-i})}_{\mathcal{S}_1, 1} \mathbb{E}^{\mathcal{I}_2(\mathcal{S}_1, z_i, \mathbf{x}_{-i})}_{\mathcal{S}_2, 2} \cdots \mathbb{E}^{\mathcal{I}_J(\mathcal{S}_{J-1}, z_i, \mathbf{x}_{-i})}_{\mathcal{S}_J, J}.$$

Here, $q_i(z_i)$ (respectively, $m_i(z_i)$) denotes the probability that supplier i will be awarded the contract (respectively, the expected payment received by supplier i) if he reports a cost of z_i. Note that both $q_i(z_i)$ and $m_i(z_i)$ depend on the admission policy $\mathcal{I}_1(\cdot), \mathcal{I}_2(\cdot,\cdot), \ldots, \mathcal{I}_J(\cdot,\cdot)$. However, we suppress this dependence in our notation for simplicity.

The IC constraints are defined by (5), and the IR constraints are defined by (6). We are now ready to formulate our mechanism design problem.

The Mechanism Design Problem. Let $\mathcal{I}$ denote the J-tuple $(\mathcal{I}_1(\cdot), \mathcal{I}_2(\cdot,\cdot), \ldots, \mathcal{I}_J(\cdot,\cdot))$. For any $\mathbf{x}$ and $(\mathcal{I}, \mathbf{Q}, \mathbf{M})$, let $\mathrm{TC}_1(\mathbf{x}, \mathcal{I}, \mathbf{Q}, \mathbf{M})$ denote the expected total cost (i.e., sum of testing cost and procurement cost) incurred by the buyer over the entire horizon. Similarly, for any $j = 2, 3, \ldots, J+1$, let $\mathrm{TC}_j(\mathcal{S}, \mathbf{x}, \mathcal{I}, \mathbf{Q}, \mathbf{M})$ denote the expected total cost incurred from stages j through $J+1$, when $\mathcal{S} \subseteq \mathcal{N}$ is the set of suppliers who have passed the $(j-1)$st-stage test. We have

$$\mathrm{TC}_1(\mathbf{x}, \mathcal{I}, \mathbf{Q}, \mathbf{M}) = K_1|\mathcal{I}_1(\mathbf{x})| + \mathbb{E}^{\mathcal{I}_1(\mathbf{x})}_{\hat{\mathcal{S}}, 1}[\mathrm{TC}_2(\hat{\mathcal{S}}, \mathbf{x}, \mathcal{I}, \mathbf{Q}, \mathbf{M})],$$
$$\mathrm{TC}_j(\mathcal{S}_{j-1}, \mathbf{x}, \mathcal{I}, \mathbf{Q}, \mathbf{M}) = K_j|\mathcal{I}_j(\mathcal{S}_{j-1}, \mathbf{x})| + \mathbb{E}^{\mathcal{I}_j(\mathcal{S}_{j-1}, \mathbf{x})}_{\hat{\mathcal{S}}, j}[\mathrm{TC}_{j+1}(\hat{\mathcal{S}}, \mathbf{x}, \mathcal{I}, \mathbf{Q}, \mathbf{M})], \quad j = 2, 3, \ldots, J,$$

where $\mathrm{TC}_{J+1}(\mathcal{S}_J, \mathbf{x}, \mathcal{I}, \mathbf{Q}, \mathbf{M}) = \sum_{i \in \mathcal{N}} M_i(\mathcal{S}_J, \mathbf{x}) + R \cdot [1 - \sum_{i \in \mathcal{N}} Q_i(\mathcal{S}_J, \mathbf{x})]$.

Our optimal mechanism design problem can now be formulated as follows:

$$\min_{\mathcal{I}, \mathbf{Q}, \mathbf{M}} \; \mathbb{E}_{\mathbf{x}}[\mathrm{TC}_1(\mathbf{x}, \mathcal{I}, \mathbf{Q}, \mathbf{M})] \quad \text{s.t. (5), (6), (8), (9), (10) and (11).} \tag{PM1}$$

4.2. The Optimal Mechanism

As will become clear in the proof of Theorem 2, if the buyer wants to test k suppliers in the jth stage, $j = 1, 2, \ldots, J$, it is optimal for her to choose the k suppliers with the lowest costs who are qualified for selection in that stage. With this knowledge, instead of deciding the optimal *subset* $\mathcal{I}_j$ of suppliers to test in stage j, the buyer only needs to determine the optimal *number* of suppliers to test. Next, we present two definitions that will be useful later to characterize this number.

For any $a \in \mathbb{R}$, let $a^+ = \max\{a, 0\}$. For any $j = 1, 2, \ldots, J-1$, $k = 0, 1, \ldots, N-1$, and $v \in (-\infty, \psi(\bar{c})]$, let

$$\delta_j(k,v) = -K_j + \beta_j \cdot \sum_{i=0}^{k} \binom{k}{i} \beta_j^i (1-\beta_j)^{k-i} (\delta_{j+1}(i,v))^+, \quad (12)$$

where

$$\delta_J(k,v) = -K_J + \beta_J (1-\beta_J)^k (R - v). \quad (13)$$

Intuitively, $\delta_j(k,v)$ represents the benefit of admitting the $(k+1)$st-lowest-cost supplier (with a virtual cost of v) in $\mathcal{S}_{j-1}$ into the stage j test, given that the k lowest-cost suppliers in $\mathcal{S}_{j-1}$ have already been admitted. (This intuition will be formally established in the proof of Theorem 2. Note that when defining $\delta_j(k,v)$, we allow $v \in (-\infty, \psi(\bar{c})]$ instead of $[\psi(\underline{c}), \psi(\bar{c})]$ for algebraic convenience.) In particular, $-K_j$ represents the testing cost of this supplier, and $\beta_j \cdot \sum_{i=0}^{k} \binom{k}{i} \beta_j^i (1-\beta_j)^{k-i} \cdot (\delta_{j+1}(i,v))^+$ represents the benefit of admitting him: with probability β_j, he passes the test and brings an expected benefit of $\sum_{i=0}^{k} \binom{k}{i} \beta_j^i (1-\beta_j)^{k-i} (\delta_{j+1}(i,v))^+$. This is because if he passes the stage j test along with an i-cardinality set $\mathcal{S}_j$, he will only bring a benefit in stage $j+1$ if $\delta_{j+1}(i,v) > 0$; otherwise, the buyer will not admit him into the stage $(j+1)$ test, and the benefit will be 0. We are now ready to define the thresholds for admitting suppliers in each stage.

Definition 2 (Thresholds for Admitting Suppliers). For any stage $j = 1, 2, \ldots, J$, let $\{\theta_{k,j}\}_{k=1,\ldots,N}$ be the zero of $\delta_j(k-1, \cdot)$ (i.e., $\delta_j(k-1, \theta_{k,j}) = 0$).

We note that these thresholds are uniquely defined and $\{\theta_{k,j}\}$ is decreasing in k; these properties have been established in the proof of Theorem 2. Also note that, by definition, a threshold $\theta_{k,j}$ can be negative; intuitively, this means that it is not beneficial to admit the (k)th-lowest-cost supplier into the stage j test.

The following definition, which has a threshold-based structure similar to that in Definition 1 of Section 3.3, will help us in deciding the optimal number of suppliers to test in each stage. Recall that for any cost vector $\mathbf{x}$, $\mathcal{S} \subseteq \mathcal{N}$, and $i = 1, 2, \ldots, |S|$, $x_{(i)}^{\mathcal{S}}$ denotes the cost of the supplier with the ith-lowest cost in $\mathcal{S}$.

Definition 3 (Number of Suppliers to Test in Each Stage). Given thresholds $\{\theta_{i,j}\}$, $i = 1, 2, \ldots, N$, which are decreasing in i for every $j = 1, 2, \ldots, J$, for any $\mathcal{S} \subseteq \mathcal{N}$, let

$$k_j^*(\mathcal{S}, \mathbf{x}) = \begin{cases} 0, & \text{if } \psi(x_{(1)}^{\mathcal{S}}) > \theta_{1,j}, \\ i, & \text{if there exists } i \in \{1, 2, \ldots, |\mathcal{S}|\} \text{ such that} \\ & \quad \psi(x_{(i)}^{\mathcal{S}}) \le \theta_{i,j} \text{ and } \psi(x_{(i+1)}^{\mathcal{S}}) > \theta_{i+1,j}, \\ |\mathcal{S}|, & \text{if } \psi(x_{(|\mathcal{S}|)}^{\mathcal{S}}) \le \theta_{|\mathcal{S}|,j}. \end{cases}$$

This definition has the same geometric interpretation as that in Figure 1: a supplier should be selected for the qualification test in a stage if and only if his virtual cost is below the threshold corresponding to his cost rank (among the suppliers who are qualified to participate in this stage).

We now present a generalization of our single-stage mechanism in Section 3.3 to multiple stages, and we prove its optimality. By convention, let $\mathcal{S}_0 = \mathcal{N}$. Consider the following mechanism for the buyer:

1. All suppliers submit their cost bids. Let $\mathbf{x}$ denote the vector of reported costs. Perform the tasks in Step 2 for $j = 1, 2, \ldots, J$, sequentially.
2. At the beginning of stage j, the buyer admits the $k_j^*(\mathcal{S}_{j-1}, \mathbf{x})$ suppliers (from the qualified pool $\mathcal{S}_{j-1}$) with the lowest costs to enter the jth-stage qualification process. Let $\mathcal{S}_j$ denote the set of suppliers who pass. If $\mathcal{S}_j = \varnothing$ (i.e., no supplier passes), proceed to Step 3(c).
3. In stage $J+1$, depending on the outcome of the qualification test, perform the following task:
   a. If there are two or more suppliers who pass the entire qualification process, then award the contract to the qualified supplier with the lowest cost. The payment to him is $x_{(2)}^{\mathcal{S}_J}$, which is the cost of the second-lowest-cost qualified supplier.
   b. If there is only one supplier who passes, then award the contract to this supplier at the price of

$$\min\big[\psi^{-1}(\theta_{k_1^*,1}), \psi^{-1}(\theta_{k_2^*,2}), \ldots, \psi^{-1}(\theta_{k_J^*,J}), x_{(k_1^*+1)}, x_{(k_2^*+1)}^{\mathcal{S}_1}, \ldots, x_{(k_J^*+1)}^{\mathcal{S}_{J-1}}\big],$$

   which is the highest value to which his cost can be increased, while ensuring his admission into every stage of the qualification process. Here, for simplicity of notation, we suppress the dependence of k_j^* on $\mathcal{S}_{j-1}$ and $\mathbf{x}$, for $j = 1, 2, \ldots, J$. (A proof of this claim is provided in the online appendix.)
   c. If no supplier passes, then award the contract to the outside option at a price of R.

Theorem 2. *With the thresholds $\{\theta_{k,j}\}$ in Definition 2, the above mechanism is optimal.*

Below, we highlight the main steps of the proof; we refer readers to the online appendix for details. We define some notation first. For any subset of suppliers $\mathcal{S} \subseteq \mathcal{N}$ and for $k = 0, 1, \ldots, |\mathcal{S}|$, let $\mathcal{S}^k$ denote the set of the k suppliers with the lowest costs in $\mathcal{S}$. Let $\mathbf{x}(\mathcal{S}^k) = (x_{(1)}^{\mathcal{S}^k}, x_{(2)}^{\mathcal{S}^k}, \ldots, x_{(k)}^{\mathcal{S}^k})$ denote the corresponding cost vector. For simplicity of exposition, we will suppress the subscripts for $\mathcal{I}_j$, $\mathcal{S}_j$, and k_j wherever no confusion arises in doing so.

The proof of Theorem 2 consists of the following main steps:

Step 1 (Lemmas 7–9). In this step, we solve the following problem: For any $\mathcal{S} \subseteq \mathcal{N}$,

$$g_j(\mathbf{x}(\mathcal{S})) = \min_{\mathcal{I} \subseteq \mathcal{S}} [K_j \cdot |\mathcal{I}| + \mathbb{E}_{\hat{\mathcal{S}},j}^{\mathcal{I}} g_{j+1}(\mathbf{x}(\hat{\mathcal{S}}))], \quad j = 1, 2, \ldots, J-1, \quad \text{(PM2)}$$

where

$$g_J(\mathbf{x}(\mathcal{S})) = \min_{\mathcal{I}\subseteq\mathcal{S}}[K_J \cdot |\mathcal{I}| + \beta_J \psi(x^{\mathcal{I}}_{(1)}) + \beta_J(1-\beta_J)\psi(x^{\mathcal{I}}_{(2)}) + \cdots + \beta_J(1-\beta_J)^{|\mathcal{I}|-1}\psi(x^{\mathcal{I}}_{(|\mathcal{I}|)}) + (1-\beta_J)^{|\mathcal{I}|}R].$$

For stage j, the first term $K_j \cdot |\mathcal{I}|$ is the testing cost, and the term $\mathbb{E}^{\mathcal{I}}_{\hat{\mathcal{S}},j} g_{j+1}(\mathbf{x}(\hat{\mathcal{S}}))$ represents the expected cost to go, which consists of both the testing costs in the subsequent stages and the expected procurement cost in the last stage. The explanation of the objective function for stage J is the same as that in Section 3.3.

Observe that the problem for the last stage in (PM2) is identical to problem (P2) in Section 3.3; thus, (PM2) can be viewed as a dynamic version of (P2). Solving (PM2) turns out to be much harder, requiring the identification of special structural properties—convexity of the cost-to-go functions, and monotonicity and special relationships in other associated functions—and establishing their preservation through the DP recursion.

Step 2 (Lemmas 10 and 11). Following a treatment similar to that in Section 3.3, we use the solution $\mathcal{I}^* = (\mathcal{I}^*_1, \mathcal{I}^*_2, \ldots, \mathcal{I}^*_J)$ of problem (PM2) to construct a tuple $(\mathcal{I}^*, \mathbf{Q}^*, \mathbf{M}^*)$ that solves problem (PM1). In Lemma 10, we verify a monotonicity condition (intuitively, this condition means that the lower a supplier's cost, the higher his chance of being awarded the contract) that is standard in mechanism design. As in Section 3.3, we complete the proof by observing that the tuple $(\mathcal{I}^*, \mathbf{Q}^*, \mathbf{M}^*)$ characterizes the mechanism proposed in Theorem 2.

4.3. Comparative Statics of the Thresholds

In this section, we study the comparative statics of the thresholds $\theta_{k,j}$, for any $k = 1, 2, \ldots, N$, and $j = 1, 2, \ldots, J$. From Definition 2 and Equations (12) and (13), for any given number k and stage j, it is easy to see that $\theta_{k,j}$ depends only on the price of the outside option R, the passing probabilities from stage j onward (including stage j) (i.e., $\beta_j, \beta_{j+1}, \ldots, \beta_J$), and the qualification costs from stage j onward (including stage j) (i.e., $K_j, K_{j+1}, \ldots, K_J$). The following result provides the comparative statics of the thresholds with respect to these parameters. The proof of the following result is in the online appendix.

Theorem 3 (Comparative Statics of the Thresholds)**.** *For any number $k = 1, 2, \ldots, N$, and stage $j = 1, 2, \ldots, J$, we have the following*:

(1) *The threshold $\theta_{k,j}$ is increasing in the price of the outside option R, keeping other parameters fixed.*

(2) *For any $l \ge j$, the threshold $\theta_{k,j}$ is decreasing in the testing cost K_l, keeping other parameters fixed.*

(3) *For any $l \ge j$, the threshold $\theta_{k,j}$ is increasing in the passing probability β_l if $\beta_l \in (0, 1/k]$, keeping other parameters fixed. For $\beta_l \in (1/k, 1)$, in general, the threshold $\theta_{k,j}$ is not monotone in β.*

The intuition behind the result above is simple: when the price of the outside option increases, the buyer is more willing to admit suppliers in the qualification tests, causing all the thresholds to increase. Similarly, for any given stage, an increase in the testing cost in that stage or any subsequent stage results in the buyer becoming more reluctant to admit suppliers, causing the thresholds in that stage to go down. For any given stage, an increase in the passing probability in that stage or any subsequent stage (when this probability is relatively small) leads the buyer to accept more suppliers into the qualification testing in that stage. However, when the passing probability is high, then an increase in this probability can cause the thresholds to change in a nonmonotone fashion. As an example, consider a simple two-stage qualification process with four symmetric suppliers. The parameters are as follows: the testing costs for the two stages are $K_1 = 5$ and $K_2 = 10$, respectively; the passing probabilities in the two stages are $\beta_1 = 0.5$ and $\beta_2 = 0.25$, respectively; and the price of the outside option is $R = 200$. As can be seen in Figure 2, the threshold $\theta_{4,1}$ (the threshold for admitting the fourth supplier in the first stage) is not monotone with respect to β_2 (the passing probability in the second stage), for $\beta_2 \in (\frac{1}{4}, 1)$.

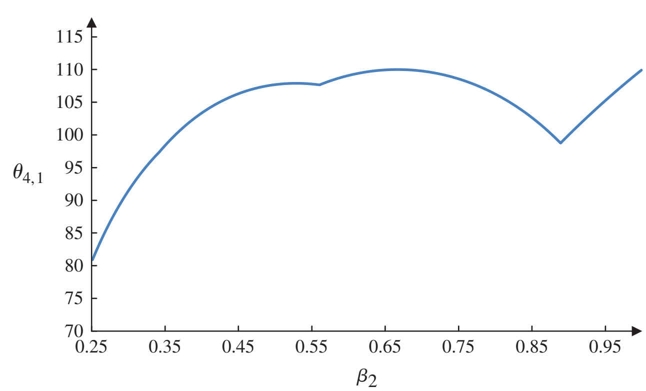
The comparison of the thresholds *across* the stages is relatively less structured. Our understanding is summarized in the following remark.

Remark. Consider two qualification stages l and j, with $l > j$. Then, we have the following observations:

(1) The threshold for admitting the *cheapest* supplier in the earlier stage (stage j) is smaller than that in the latter stage (stage l) (i.e., $\theta_{1,j} \le \theta_{1,l}$). This is intuitive: if $\theta_{1,j} > \theta_{1,l}$, then the testing cost for stage j will be wasted when the cheapest supplier's cost is above $\theta_{1,l}$ but below $\theta_{1,j}$, for this supplier has no chance of qualifying for stage l.

(2) For $k = 2, 3, \ldots, N$, one might conjecture that the thresholds $\theta_{k,l}$ and $\theta_{k,j}$ satisfy a similar monotonicity. However, this is not true in general. That is, depending on the values of the problem parameters, we could

Figure 2. (Color online) A Plot of the Threshold $\theta_{4,1}$ as a Function of the Passing Probability β_2, for $\beta_2 \in (\frac{1}{4}, 1)$



have $\theta_{k,j} \le \theta_{k,l}$ or $\theta_{k,j} \ge \theta_{k,l}$. To illustrate, we investigate the relative behavior of the thresholds using the same numerical setup as above (i.e., two qualification stages and four symmetric suppliers) and note our observations on the comparison of the two thresholds $\theta_{k,2}$ and $\theta_{k,1}$, for $k \ge 2$:

(a) For a given choice of the parameters, an increase in the price of the outside option R (starting from 0), keeping all other parameters fixed, causes both the thresholds to increase but does not reverse the ordering between the two.

(b) For a given choice of the parameters, an increase in the testing cost K_1 (starting from 0) in stage 1, keeping all other parameters fixed, has no impact on $\theta_{k,2}$ but results in a smaller $\theta_{k,1}$. For some parameter settings in which $\theta_{k,1}$ is greater than $\theta_{k,2}$ initially, the increase in K_1 eventually leads to $\theta_{k,1}$ becoming smaller than $\theta_{k,2}$, thus reversing the ordering between the two.

(c) For a given choice of the parameters, an increase in the testing cost K_2 (starting from 0) in stage 2, keeping all other parameters fixed, causes both the thresholds to decrease, with $\theta_{k,2}$ decreasing at a faster rate than that of $\theta_{k,1}$. Again, for some parameter settings in which $\theta_{k,1}$ is less than $\theta_{k,2}$ initially, the increase in K_2 reverses the ordering between the two thresholds.

(d) For a given choice of the parameters, an increase in the passing probability β_1 for stage 1, keeping all other parameters fixed, leaves $\theta_{k,2}$ unaffected but can reduce $\theta_{k,1}$. This can again reverse the ordering between the two thresholds. An increase in the passing probability β_2 for stage 2 can lead to an increase in both the thresholds, with $\theta_{k,1}$ increasing at a faster rate. This, too, can reverse the ordering between the two thresholds. □

5. Sequential Testing: Multistage Qualification Process

Thus far, motivated by the scarcity of time the buyer faces in completing the procurement process, our analysis assumed that the suppliers who are selected for a stage of qualification are tested simultaneously for that stage. In this section, we remove this requirement of simultaneous testing and study the optimal mechanism design problem under a multistage supplier qualification process. This setting is appropriate when the buyer has ample time to procure. We find that the optimal admission policy is still threshold based but *sequential* in nature: the buyer will only test the suppliers (whose costs fall below a certain threshold) sequentially until she finds a qualified one.

This section is organized as follows: The formulation of the mechanism design problem is presented in Section 5.1, followed by the optimal mechanism in Section 5.2. In Section 5.3, we examine the increase in cost resulting from imposing the requirement of simultaneous testing.

5.1. Formulation of the Mechanism Design Problem

We consider the same model and assumptions as in Section 4 of the paper, with the exception that we now remove the assumption that all chosen suppliers in any given stage must be tested simultaneously in that stage. As in the previous sections, we assume that the buyer commits to follow the preannounced rules of the mechanism throughout the qualification and procurement processes.

We first define the admission rule. Recall that we have a total number of J qualification stages. For $j = 1, 2, \ldots, J$, let

$$\mathcal{I}_{i,j}(\mathbf{x})\colon [\underline{c}, \bar{c}]^N \to [0,1]$$

denote the probability that the buyer admits supplier $i \in \mathcal{N}$ into the stage j qualification.

The allocation rule and the expected payment rule are defined as follows:

$$\mathbf{Q}(\mathbf{x})\colon [\underline{c}, \bar{c}]^N \to [0,1]^N, \qquad \mathbf{M}(\mathbf{x})\colon [\underline{c}, \bar{c}]^N \to \mathbb{R}_+^N.$$

For supplier $i \in \mathcal{N}$, $Q_i(\mathbf{x})$ denotes the probability that supplier i is awarded the contract, and $M_i(\mathbf{x})$ denotes the expected payment made to him.

We now state the feasibility constraints:

1. At any given stage of the qualification process, a supplier can only be selected if he is selected for and passes the testing in the previous stage. Similarly, a supplier can only be awarded the contract if he is selected for the last stage of qualification testing and passes it.

2. The contract can only be awarded to at most one supplier.

The following constraints formulate relaxations (via expectation) of the actual physical feasibility constraints above. We will formulate the mechanism design problem with these relaxed constraints; it is easy to verify that the optimal solution of the relaxed problem satisfies the actual physical constraints:

$$\mathcal{I}_{i,j+1}(\mathbf{x}) \le \beta_j \mathcal{I}_{i,j}(\mathbf{x}), \quad \forall\, i \in \mathcal{N}, j = 1, 2, \ldots, J-1, \tag{14}$$

$$Q_i(\mathbf{x}) \le \beta_J \mathcal{I}_{i,J}(\mathbf{x}), \quad \forall\, i \in \mathcal{N}, \tag{15}$$

$$0 \le \sum_{i \in \mathcal{S}} Q_i(\mathbf{x}) \le 1 - (1-\beta)^{|\mathcal{S}|}, \quad \forall\, \mathcal{S} \subseteq \mathcal{N}. \tag{16}$$

Here, $\beta = \beta_1 \beta_2 \ldots \beta_J$ denotes the overall probability that a yet-to-be-qualified supplier is qualified via the multistage qualification process. Constraints (14) state that for any supplier i, the probability that he is tested in stage $(j+1)$ must be less than or equal to the probability that he is tested in stage j, multiplied by his passing probability β_j in stage j. Constraints (15) state that the probability that the contract is awarded to any supplier i cannot exceed the probability that he is tested in the last stage, multiplied by his passing probability β_J in the last stage. Constraints (16) state that for any subset of suppliers $\mathcal{S} \subseteq \mathcal{N}$, the probability that the contract is awarded to a supplier in $\mathcal{S}$ cannot exceed the probability that at least one supplier in $\mathcal{S}$ is qualified. Let

$$q_i(z_i) = \mathbb{E}_{\mathbf{x}_{-i}}[Q_i(z_i, \mathbf{x}_{-i})], \quad \forall\, i \in \mathcal{N}, \tag{17}$$

$$\text{and } m_i(z_i) = \mathbb{E}_{\mathbf{x}_{-i}}[M_i(z_i, \mathbf{x}_{-i})], \quad \forall\, i \in \mathcal{N}. \tag{18}$$

Thus, $q_i(z_i)$ (respectively, $m_i(z_i)$) denotes the probability that supplier i will be awarded the contract (respectively, the expected payment received by supplier i) if he reports a cost of z_i.

The IC and IR constraints can now be stated as follows:

$$m_i(x_i) - q_i(x_i)x_i \ge m_i(z_i) - q_i(z_i)x_i, \quad \forall x_i, z_i, i \in \mathcal{N}, \tag{19}$$

$$m_i(x_i) - q_i(x_i)x_i \ge 0, \quad \forall x_i, i \in \mathcal{N}. \tag{20}$$

For any $\mathbf{x}$ and a tuple $(\mathcal{I}, \mathbf{Q}, \mathbf{M})$, let $\mathrm{TC}(\mathcal{I}, \mathbf{x}, \mathbf{Q}, \mathbf{M})$ denote the expected total cost incurred by the buyer. This cost includes the following: (i) the testing cost $K_1 \sum_{i\in\mathcal{N}} \mathcal{I}_{i,1}(\mathbf{x}) + K_2 \sum_{i\in\mathcal{N}} \mathcal{I}_{i,2}(\mathbf{x}) + \cdots + K_J \sum_{i\in\mathcal{N}} \mathcal{I}_{i,J}(\mathbf{x})$, (ii) the expected payment $\sum_{i\in\mathcal{N}} M_i(\mathbf{x})$ made to the suppliers, and (iii) the expected cost $R\,[1 - \sum_{i\in\mathcal{N}} Q_i(\mathbf{x})]$ incurred by the buyer if she uses the outside option. Using (17) and (18), we have

$$\begin{aligned} &\mathbb{E}_{\mathbf{x}}[\mathrm{TC}(\mathbf{x}, \mathcal{I}, \mathbf{Q}, \mathbf{M})] \\ &\quad = \mathbb{E}_{\mathbf{x}}\left[\sum_{j=1}^{J} \sum_{i\in\mathcal{N}} \mathcal{I}_{i,j}(\mathbf{x}) K_j\right] \\ &\qquad + \sum_{i\in\mathcal{N}} \mathbb{E}_{x_i}[(-R) q_i(x_i) + m_i(x_i)] + R. \end{aligned} \tag{21}$$

The optimal mechanism design problem can be formulated as follows:

$$\begin{aligned} &\min_{\mathcal{I}, \mathbf{Q}, \mathbf{M}} \ (21) \\ &\quad \text{s.t. } (14), (15), (16), (17), (18), (19) \text{ and } (20). \end{aligned} \tag{PMS1}$$

5.2. The Optimal Mechanism

We now present the solution of problem (PMS1). Given a cost vector $\mathbf{x}$, let $x_{(i)}, 1 \le i \le N$, denote the ith-lowest number among $x_1, x_2, \ldots, x_N$. We will refer to the supplier with cost $x_{(i)}$ simply as supplier (i); that is, supplier (i) is the supplier with cost rank i in $\mathcal{N}$. Let $\theta = R - K/\beta$, where $K = K_1 + \beta_1 K_2 + \beta_1\beta_2 K_3 + \cdots + \beta_1\beta_2 \ldots \beta_{J-1} K_J$ represents the expected multistage testing cost. Consider the following mechanism for the buyer:

1. All the suppliers submit their cost bids. Let $\mathbf{x}$ denote the vector of these reported costs.
2. The buyer does not consider any supplier who has a virtual cost greater than or equal to θ for the purpose of qualification testing and contract awarding. Let k^* be the number of remaining suppliers.
3. The buyer carries out the qualification process and awards the contract as follows: She first selects the cheapest supplier (i.e., supplier (1)) and tests him sequentially for all stages. If this supplier passes the last stage of qualification testing, then she awards him the contract. Otherwise, if he fails any stage of qualification testing, then the buyer selects the next-cheapest supplier (i.e., supplier (2)) and tests him sequentially—if he passes the last stage of qualification, then the contract is awarded to him; otherwise, the buyer selects supplier (3) and repeats this process until a qualified supplier is found. In the case that all the suppliers fail, the contract is awarded to the outside option at a price of R.
4. If supplier (i) is awarded the contract, the buyer pays her an amount

$$\begin{aligned} &\beta x_{(i+1)} + \beta(1-\beta) x_{(i+2)} + \cdots + \beta(1-\beta)^{k^*-i-1} x_{(k^*)} \\ &\quad + (1-\beta)^{k^*-i} \psi^{-1}(\theta), \end{aligned}$$

which is the expected cost for the buyer to work with the next-cheapest qualified supplier. The other suppliers do not receive any payment.

We have the following result; its proof is in the online appendix.

Theorem 4. *The above mechanism is optimal for problem* (PMS1).

5.3. Cost Comparison

Obtaining the optimal mechanism in the previous section enables us to examine the increase in cost that results from the imposition of the simultaneous-testing requirement within each stage in the model we studied in Section 4. In this section, under the assumption that the price of the outside option is "high enough," we show that the relative increase in cost is monotonically increasing in the number of suppliers, the expected multistage testing cost, and the overall passing probability. In addition, we also conduct numerical simulations to illustrate the cost and time trade-off between the simultaneous and nonsimultaneous settings.

To obtain a simple closed-form expression for the cost comparison between the two settings—with and without simultaneous testing within the qualification stages—we assume that the price of the outside option is "large enough" in each setting. We first precisely state this assumption for each of the two settings.

Assumption 1. *In the simultaneous-testing setting, the price of the outside option R is high enough so that the buyer always selects all the suppliers who pass a stage of qualification for admission to the next stage of qualification. Mathematically, R is large enough so that* $\psi(\bar{c}) \le \min\{\theta_{N,1}, \theta_{N,2}, \ldots, \theta_{N,J}\}$, *where* $\{\theta_{N,j}\}_{j=1,2,\ldots,J}$, *are defined in Definition 2.*

Assumption 2. *In the nonsimultaneous-testing setting, the price of the outside option R is high enough so that the buyer always considers all suppliers. Mathematically, R is large enough so that* $\psi(\bar{c}) \le \theta$, *where* $\theta = R - K/\beta$.

Let Cost_{sim} (respectively, Cost_{seq}) denote the buyer's expected total cost in the simultaneous (respectively, nonsimultaneous) testing setting. We have the following.

Theorem 5. *Let $K = K_1 + \beta_1 K_2 + \beta_1\beta_2 K_3 + \cdots + \beta_1\beta_2 \cdot \ldots \cdot \beta_{J-1}K_J$ be the expected multistage testing cost, and let $\beta = \beta_1\beta_2 \cdot \ldots \cdot \beta_J$ be the overall passing probability. Under Assumptions 1 and 2 and the assumption that the suppliers' costs are randomly drawn from a Uniform distribution, the relative difference in cost*

$$\frac{\text{Cost}_{\text{sim}} - \text{Cost}_{\text{seq}}}{\text{Cost}_{\text{seq}}}$$

is monotonically increasing in each of the parameters N, K, and β, keeping the other two fixed.

The proof of the above result is in the online appendix. We note that the relative difference $(\text{Cost}_{\text{sim}} - \text{Cost}_{\text{seq}})/\text{Cost}_{\text{seq}}$ could either increase or decrease with the number of testing stages J, depending on the value of parameters. Here is an example: Consider a single-stage process with testing cost C and a passing probability of α, and a two-stage process with testing costs $K_1 = C/2$ and $K_2 = C$, and passing probabilities $\beta_1 = 1/2$ and β_2. Then, for the same set of suppliers, going from the single-stage process to the two-stage process, the expected total testing cost remains the same (since $K = K_1 + \beta_1 K_2 = C$ for both the one-stage and the two-stage processes) while the overall passing probabilities are, respectively, α and $\frac{1}{2}\beta_2$. Depending on which of α and $\frac{1}{2}\beta_2$ is larger, the relative difference $(\text{Cost}_{\text{sim}} - \text{Cost}_{\text{seq}})/\text{Cost}_{\text{seq}}$ could either increase or decrease as we move from a single-stage process to a two-stage process.

Without Assumptions 1 and 2, the monotonicity properties stated in Theorem 5 may no longer hold. We provide several counterexamples below; it is easy to verify that at least one of the two assumptions is violated in each of these counterexamples.

1. An increase in the expected multistage testing cost K, while keeping N and β the same, can cause the relative cost difference to decrease. As an example, consider the following set of parameters: $N = 4$, $R = 200$, $K_1 = 5$, $K_2 = 10$, $\beta_1 = 0.5$, $\beta_2 = 0.25$, $\underline{c} = 0$, and $\bar{c} = 100$. If we increase K_1 to 6 (which will increase the value of K and will not affect the value of β), then the relative cost difference drops from 1.12% to 1.05%.

2. Similarly, an increase in the overall passing probability β, while keeping N and K the same, can cause the relative cost difference to decrease as well. For instance, consider the following set of parameters: $N = 4$, $R = 200$, $K_1 = 5$, $K_2 = 10$, $\beta_1 = 0.5$, $\beta_2 = 0.25$, $\underline{c} = 0$, and $\bar{c} = 100$. If we change K_1 to 0, K_2 to 20, and $\beta_2 = 0.251$ (which will result in a higher β and the same K), then the relative cost difference drops from 1.12% to 1.01%.

3. Finally, an increase in the number of suppliers N, while keeping K and β the same, could also cause the relative cost difference to decrease. For instance, consider the following set of parameters: $N = 20$, $R = 200$, $K_1 = 5$, $K_2 = 10$, $\beta_1 = 0.5$, $\beta_2 = 0.25$, $\underline{c} = 0$, and $\bar{c} = 100$. If we change N to 21, K_1 to 0, and K_2 to 20 (which will result in the same K and β), then the relative cost difference drops from 16.65% to 15.94%.

6. Additional Considerations

Section 6.1 studies the optimal sequencing of qualification stages under both the simultaneous-testing setting and nonsimultaneous-testing settings. Section 6.2 analyzes a single-stage, single-supplier, two-dimensional mechanism problem where, in addition to the cost, the passing probability is also private to the supplier. In Section 6.3, we explicitly introduce a cost of time and use our analysis thus far to offer some understanding on a preferable mechanism as this cost varies.

6.1. Sequencing the Qualification Stages

Given the passing probability and the testing cost for each stage, how should the buyer sequence the stages in an optimal mechanism? Under the assumption that the price of the outside option is "high enough," we show that, in the simultaneous-testing setting, the stages should be sequenced in increasing order of the ratio of their testing cost to their failing probability. The same result also holds for the nonsimultaneous-testing model, but without any assumption. This confirms the intuition that the buyer should preferentially sequence stages with cheaper testing costs and smaller passing probabilities first. Formally, we have the following result; its proof is in the online appendix.

Theorem 6. (1) *For the simultaneous-testing model of Section 4, under Assumption 1, it is optimal for the buyer to schedule the stages in the ascending order of $K_i/(1-\beta_i)$, where K_i (respectively, β_i) is the testing cost of one supplier in stage i (respectively, the passing probability of any supplier admitted to stage i).*

(2) For the nonsimultaneous-testing model of Section 5, the same result holds without any assumption.

Remark. While the result in part (1) of Theorem 6 has been established under Assumption 1, we have not been able to obtain a counterexample to this result in the *absence* of this assumption, despite considerable effort. We conjecture that the result holds even in the absence of the assumption. We also note that similar results have been established in the field of reliability testing of series systems; see, for example, Butterworth (1972). □

6.2. Private Passing Probability

In practice, a supplier may be better informed about his ability to pass a stage of qualification. Motivated by this, we analyze a two-dimensional mechanism design problem in which the passing probabilities are also private to the suppliers. Because of the difficulty in optimizing a general two-dimensional mechanism design problem, we restrict attention to a single-stage qualification process with a single supplier, whose private information on each dimension (i.e., cost and passing probability) is binary. We present our model in Section 6.2.1, the formulation of the optimal mechanism design problem in Section 6.2.2, and its solution in Section 6.2.3.

6.2.1. Model. Consider a single-stage qualification process with a single supplier. The supplier has two dimensions of private information: the first dimension reflects his cost—this can be either low (c_L) or high (c_H); the second dimension reflects his passing probability, if admitted to the qualification test—this can be either low (β_L) or high (β_H). Thus, we have a total of four types of suppliers: *LL*, *LH*, *HL*, and *HH*. Obviously, a type *LH* supplier, who has a low cost of production and a high passing probability, is the most preferred one, while a type *HL* supplier is the least preferred one. The probability that a supplier is of type *LL* (respectively, *LH*, *HL*, and *HH*) is α_{LL} (respectively, α_{LH}, α_{HL}, and α_{HH}).

Next, we discuss the buyer's decisions in designing an optimal contract. Let I_{LL} (respectively, I_{LH}, I_{HL}, and I_{HH}) denote the probability that the buyer tests a supplier if he claims to be of type *LL* (respectively, *LH*, *HL*, and *HH*). If the buyer tests a type *LL* (respectively, *LH*, *HL*, and *HH*) supplier, then let q_{LL} (respectively, q_{LH}, q_{HL}, and q_{HH}) denote the probability that the buyer awards the contract to him if he is qualified; let M_{LL} (respectively, M_{LH}, M_{HL}, and M_{HH}) denote the corresponding payment he receives from the buyer. The cost to test a chosen supplier is K. In addition to this yet-to-be-qualified supplier, we also assume that the buyer has a qualified (albeit expensive) outside option that is priced at R ($R > c_H$). As in the previous sections, we assume that the buyer commits to follow the preannounced rules of the mechanism throughout the qualification and procurement processes.

We now formulate the mechanism design problem.

6.2.2. Formulation of the Mechanism Design Problem. The buyer's total expected procurement cost TC is

$$\begin{aligned}\mathrm{TC} = {} & \alpha_{LL}[I_{LL}(K+\beta_L M_{LL}+(1-\beta_L q_{LL})R)+(1-I_{LL})R] \\ & +\alpha_{LH}[I_{LH}(K+\beta_H M_{LH}+(1-\beta_H q_{LH})R)+(1-I_{LH})R] \\ & +\alpha_{HL}[I_{HL}(K+\beta_L M_{HL}+(1-\beta_L q_{HL})R)+(1-I_{HL})R] \\ & +\alpha_{HH}[I_{HH}(K+\beta_H M_{HH}+(1-\beta_H q_{HH})R)+(1-I_{HH})R].\end{aligned} \tag{22}$$

Let U_{LL} (respectively, U_{LH}, U_{HL}, and U_{HH}) denote the utility that a type *LL* (respectively, *LH*, *HL*, and *HH*) supplier obtains by participating in the mechanism and reporting truthfully. Thus, U_{LL} (respectively, U_{LH}, U_{HL}, and U_{HH}) is also the information rent that the buyer needs to pay a supplier of type *LL* (respectively, *LH*, *HL*, and *HH*) to induce truth telling. We have

$$\begin{aligned} U_{LL} &= I_{LL}\beta_L(M_{LL}-c_L q_{LL}), \\ U_{LH} &= I_{LH}\beta_H(M_{LH}-c_L q_{LH}), \\ U_{HL} &= I_{HL}\beta_L(M_{HL}-c_H q_{HL}), \\ U_{HH} &= I_{HH}\beta_H(M_{HH}-c_H q_{HH}). \end{aligned} \tag{23}$$

To induce a supplier of any type to participate, we have the following IR constraints:

$$\begin{aligned} U_{LL} &\geq 0, \\ U_{LH} &\geq 0, \\ U_{HL} &\geq 0, \\ U_{HH} &\geq 0. \end{aligned} \tag{24}$$

To ensure that the supplier reports truthfully, we have 12 IC constraints; for brevity, we avoid specifying these constraints in the main body and instead present them in the proof of Theorem 7 (see constraints (34)–(45) in the online appendix).

Let $\mathbf{I} = (I_{LL}, I_{LH}, I_{HL}, I_{HH})$, $\mathbf{q} = (q_{LL}, q_{LH}, q_{HL}, q_{HH})$, $\mathbf{M} = (M_{LL}, M_{LH}, M_{HL}, M_{HH})$, $\mathbf{0} = (0,0,0,0)$, and $\mathbf{1} = (1,1,1,1)$. Our mechanism design problem can now be formulated as follows:

$$\begin{aligned} \min_{\mathbf{I},\mathbf{q},\mathbf{M}} \quad & (22) \\ \text{s.t.} \quad & (23), (24), (34)\text{–}(45), \\ & \mathbf{0} \leq \mathbf{I} \leq \mathbf{1}, \\ & \mathbf{0} \leq \mathbf{q} \leq \mathbf{1}. \end{aligned} \tag{P}$$

6.2.3. The Optimal Mechanism. As we show below, it is still optimal for the buyer to use a threshold-based testing policy: she should test a supplier only if his reported cost (which is part of his reported type) is below a certain threshold. For a complete solution of problem (P), we first describe the optimal mechanism and then discuss the thresholds.

Consider the following mechanism for the buyer: The supplier communicates his type to the buyer. The buyer then tests a type *LL* (respectively, *LH*, *HL*, and *HH*) supplier only if his cost c_L (respectively, c_L, c_H, and c_H) does not exceed the corresponding threshold. Depending on the values of the problem parameters, different thresholds are used; these thresholds are specified below. If the buyer admits the supplier, and he passes the qualification test, then the buyer awards the contract to him. The payment, which depends on the supplier's type, is as follows:

- If the supplier's type is *HH* or *HL*, then the payment is c_H.
- Otherwise (i.e., if the supplier's type is *LL* or *LH*),
  — if c_H is less than or equal to the threshold for type *HH*, then the payment is c_H; or
  — if c_H is greater than the threshold for type *HH*, then the payment is c_L.

If the chosen supplier fails the qualification test or the supplier's cost exceeds the corresponding threshold, then the buyer sources from the qualified outside option at a price of R.

Let $\Delta c = c_H - c_L$. We define some notation that is helpful for specifying the thresholds:

$$\theta_{LL} = R - \frac{K}{\beta_L}, \quad \theta_{LH} = R - \frac{K}{\beta_H}, \quad \theta_{HL} = R - \frac{K}{\beta_L},$$
$$\theta_{HH} = R - \frac{K}{\beta_H} - \frac{\Delta c(\beta_L \alpha_{LL} + \beta_H \alpha_{LH})}{\beta_H \alpha_{HH}},$$
$$\theta_{HH,HL} = R - \frac{(\alpha_{HH} + \alpha_{HL})K}{\beta_L \alpha_{HL} + \beta_H \alpha_{HH}} - \frac{\Delta c(\beta_L \alpha_{LL} + \beta_H \alpha_{LH})}{\beta_L \alpha_{HL} + \beta_H \alpha_{HH}}, \quad \text{and}$$
$$\theta_{HH,LL} = R - \frac{(\alpha_{HH} + \alpha_{LL})K}{\beta_L \alpha_{LL} + \beta_H \alpha_{HH}} - \frac{\Delta c \beta_H \alpha_{LH}}{\beta_L \alpha_{LL} + \beta_H \alpha_{HH}}.$$

The thresholds for all the types are specified below under three different cases, which depend on the parameters of the problem (specifically, depending on how θ_{HH} compares with θ_{HL} and $\theta_{LL} + \Delta c$):

1. (Separating contract). If $\theta_{HL} < \theta_{HH} < \theta_{LL} + \Delta c$, then the buyer should use a threshold of θ_{LL} (respectively, θ_{LH}, θ_{HL}, and θ_{HH}) for a type LL (respectively, LH, HL, and HH) supplier.
2. (Pooling contract). If $\theta_{HH} \le \theta_{HL}$, then the buyer should use a threshold of θ_{LL} (respectively, θ_{LH}, $\theta_{HH,HL}$, and $\theta_{HH,HL}$) for a type LL (respectively, LH, HL, and HH) supplier.
3. (Pooling contract). If $\theta_{HH} \ge \theta_{LL} + \Delta c$, then the buyer should use a threshold of $\theta_{HH,LL} - \Delta c$ (respectively, θ_{LH}, θ_{HL}, and $\theta_{HH,LL}$) for a type LL (respectively, LH, HL, and HH) supplier.

The proof of the following result is in the online appendix.

Theorem 7. *The above mechanism is optimal for problem* (P).

Observe that in the second case above, the admission criteria for type HH and type HL (i.e., $c_H \le \theta_{HH,HL}$ and $c_H \le \theta_{HH,HL}$, respectively) are identical, and in the third case, the admission criteria for type HH and type LL (i.e., $c_L \le \theta_{HH,LL} - \Delta c$ and $c_H \le \theta_{HH,LL}$, respectively) are identical. For this reason, we refer to these as *pooling contracts*. This is in contrast to the separating contract in the first case, where the admission criteria for the different types are different.

Intuitively, if the buying firm ranks the four supplier types, then LH (low-cost, high-quality) is the most preferred type and HL (high-cost, low-quality) is the least preferred type, with the types LL and HH ranking in between. It is easy to verify that this preference is reflected by the optimal thresholds: in all the cases, the type LH supplier has the largest threshold and the type HL supplier has the smallest threshold, and the thresholds for type LL and type HH are in between. In addition, because of the binding incentive-compatibility constraints, the optimal admission criteria of type HH may be pooled with either type HL (in the second case) or LL (in the third case), depending on the values of the problem parameters.

6.3. Incorporating the Cost of Time

In this section, we assume that for each stage of the qualification process, regardless of the number of suppliers tested in that stage, the buyer incurs a fixed cost of c_{time}. Under this consideration of the cost of time, we compare the cost associated with the *optimal* sequential-testing mechanism (Theorem 8) and an upper bound on the cost associated with the *optimal* simultaneous-testing mechanism (Theorem 9). The proofs of the following two results are in the online appendix.

Theorem 8. *In the presence of the per-stage time cost c_{time}, an optimal mechanism—over the class of mechanisms that test suppliers sequentially—is the one in Section 5.2 with the per-supplier testing cost K_j replaced by $K_j + c_{\text{time}}$, for all stages $j = 1, 2, \ldots, J$.*

If we restrict attention to the class of simultaneous-testing mechanisms, the following result provides an upper bound on the cost of an optimal mechanism in the presence of the time cost c_{time}. In the absence of this time cost, let $\mathbb{E}_{\mathbf{x}} \text{TC}(\mathbf{x})$ denote the expected total cost incurred by the buyer when she uses the optimal mechanism in Section 4.2, where $\mathbf{x}$ denotes the vector of the suppliers' private costs.

Theorem 9. *In the presence of the per-stage time cost c_{time}, the cost of an optimal mechanism—over the class of mechanisms that test suppliers simultaneously—is bounded from above by $\mathbb{E}_{\mathbf{x}}[\min(R, \text{TC}(\mathbf{x}) + J \cdot c_{\text{time}})]$.*

We now numerically compare the following two costs: the cost (denoted by TC_{seq}) of the optimal sequential-testing mechanism in Theorem 8 and the upper bound (denoted by $\overline{\text{TC}}_{\text{sim}}$) in Theorem 9 on the optimal simultaneous-testing mechanism. Our test bed consists of 5,000 instances, generated by randomly and uniformly choosing each of the problem parameters from its corresponding feasible set. These parameters and their corresponding feasible sets are as follows: number of suppliers, N: $\{2, 4, 6, 8\}$; number of stages, J: $\{2, 3, 4\}$; price of the outside option, R: $\{100, 150, 200, 250\}$; testing cost at any stage j, K_j: $\{1, 2, 4, 8\}$; and passing probability in any stage j, β_j: $\{0.6, 0.7, 0.8, 0.9\}$. The cost distribution of each supplier is Uniform$[0, 100]$.

For each generated instance and for each value of c_{time} from 0 to 40, with increments of 2, we obtain TC_{seq} and $\overline{\text{TC}}_{\text{sim}}$. Note that, depending on the realization of the parameters, there are some "trivial" instances in which both TC_{seq} and $\overline{\text{TC}}_{\text{sim}}$ are equal to the price of the outside option. For each value of c_{time}, Table 2 reports the average relative cost difference $(\overline{\text{TC}}_{\text{sim}} - \text{TC}_{\text{seq}})/\text{TC}_{\text{seq}}$

Table 2. The Behavior of the Relative Cost Difference $(\overline{TC}_{sim} - TC_{seq})/TC_{seq}$ with the Time Cost c_{time}

c_{time}	Percentage relative cost difference (average over all instances)	% of trivial instances	Percentage cost difference (average over nontrivial instances)
0	10.2298	0.02	10.2319
2	6.8204	0.00	6.8204
4	3.9740	0.04	3.9756
6	1.8352	0.34	1.8414
8	0.3910	1.48	0.3969
10	−0.6966	3.44	−0.7214
12	−1.6819	5.2	−1.7741
14	−2.1612	7.68	−2.3410
16	−2.6504	10.24	−2.9528
18	−2.9352	13.04	−3.3754
20	−3.1563	15.62	−3.7406
22	−3.3767	19.5	−4.1947
24	−3.2381	23.74	−4.2461
26	−3.2322	25.82	−4.3573
28	−3.1936	27.82	−4.4245
30	−2.9356	31.82	−4.3057
32	−2.8311	34.52	−4.3237
34	−2.7802	37.90	−4.4770
36	−2.6171	41.34	−4.4615
38	−2.3939	45.18	−4.3668
40	−2.3886	49.46	−4.7262

over the 5,000 instances. The table also reports the percentage of the trivial instances and the average relative cost difference over the nontrivial instances.

As is clear from Table 2, the relative difference between $\overline{TC}_{sim}$ and TC_{seq} decreases as the per-stage time cost c_{time} increases. When c_{time} becomes large enough (more than 8 in Table 2), this difference becomes negative, and it keeps reducing as c_{time} increases. This is intuitive, since an increase in c_{time} makes it more cost effective to use simultaneous testing instead of sequential testing. As c_{time} increases further and becomes extremely high (more than 26 in Table 2), it starts becoming increasingly cost effective in both the sequential and simultaneous regimes to work directly with the outside option; consequently, the magnitude of the relative cost difference reduces and becomes closer to 0. This also explains an increase in the number of trivial instances as c_{time} increases.

7. Concluding Remarks

The prevalence, importance, and complexity of supplier qualification in real-world procurement offer challenging directions for future work. Our analysis in this paper resulted in two "extreme" mechanisms: one in which, for any given stage, the buyer selects and tests the suppliers for that stage simultaneously, and the other in which suppliers can be tested one by one sequentially. In Section 6.3, we introduced a cost of time to compare these two environments. A more general treatment would be to obtain an optimal mechanism (over all possible mechanisms) in the presence of time cost: If this cost is negligible, then it is clear that the optimal mechanism will test the suppliers sequentially. Otherwise, if this cost is extremely high, then the optimal mechanism will directly source from the outside option. However, obtaining an optimal mechanism for intermediate values of the time cost seems to be very difficult.

Another interesting and useful generalization is to incorporate the search costs incurred by the buyer in identifying additional yet-to-be-qualified suppliers. Our analysis assumed that a fixed-size pool of such suppliers is available to the buyer; the inclusion of search costs will make the number of suppliers an endogenous decision in the buyer's mechanism design problem.

Acknowledgments

The authors thank the associate editor and two anonymous referees for their valuable suggestions, which have significantly improved the paper.

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