APPLICATIONS OF DEGREE THEORY TO

DYNAMICAL SYSTEMS WITH SYMMETRY

(WITH SPECIAL FOCUS ON COMPUTATIONAL ASPECTS

AND ALGEBRAIC CHALLENGES)

by

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by

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The study of dynamical systems with symmetry usually deals with the impact of symmetries (described by a certain group G) on the existence, multiplicity, stability and topological structure of solutions to the system, (local/global) bifurcation phenomena, etc. Among different approaches, degree theory (including Brouwer degree and equivariant degree), which involves analysis, topology and algebra, provides an effective tool for the study.

In our research, we focus on three motivating problems related to dynamical systems with symmetry: (a) bifurcation of periodic solutions in symmetric reversible FDEs; (b) existence of periodic solutions to equivariant Hamiltonian systems; (c) existence of periodic solutions to systems homogeneous at infinity. In Problem (a), equivariant degree with no free parameters provides us with the complete description of bifurcating branches of 2π-periodic solutions to reversible systems. In Problem (b), we can predict various symmetric vibrational modes of the fullerene molecule C_{60} using gradient degree. The study of Problem (c) leads to several results in algebra; two important results among them are (i) a characterization of the class of finite solvable groups in terms of lengths of non-trivial orbits in irreducible

representations, and (ii) the existence of an equivariant quadratic map between two non-equivalent $(n-1)$-dimensional S_n-representation spheres (n is odd). Finally, as the result of developing computational tools for equivariant degree, we present the related algorithms and examples.

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CHAPTER 1

INTRODUCTION

1.1 Motivation and Historical Remarks

Symmetries of dynamical systems, typically expressed in the form of *equivariance*, lead to the following paradigmatic question: *What is an impact of symmetries of a system on its dynamics, in particular, the existence and/or symmetric properties of periodic solutions?* Even in the case of a system without symmetries, this question still applies when there is hidden equivariance reflected by properties such as *periodicity* (periodic solutions), *reversibility*, *reflexibility* (odd/even functionals). Essentially, this question touches algebra, topology and analysis, and their interplay is important for answering the question. In connection with the paradigmatic question, my research focuses on the following problems:

1.1.1 Bifurcation of Periodic Solutions in Symmetric Reversible FDEs

In the problems of natural phenomena, *time-reversal symmetry* is one of the most important symmetries; it frequently arises in dynamical systems motivated by physics such as classical mechanics, thermodynamics and quantum mechanics.

To understand the concept of time-reversal symmetry, consider the dynamics of an *ideal* pendulum *without friction*: If one films the motion of the pendulum and plays the film backward in time to another person, that person will not have any clue that the film indeed goes backward in time—the motion in the reverse film satisfies the same law of physics which guides the motion in the original film. On the other hand, in the presence of friction, the amplitude of the motion of the pendulum decreases in time. In such a case, there is no way one would mistake the reverse film for the original one.

The example above illustrates a definition of time-reversal symmetry in classical mechanics: given a motion picture of a mechanical system either forward or backward in time, if one cannot decide the actual direction of time flow by watching the motion, the system is said to admit time-reversal symmetry. Such systems also show some symmetric properties in the phase space. For example, the reflection of the phase portrait with respect to a certain hyperplane coincides with the original phase portrait with reverse flow. This observation inspires a more general definition of time-reversal symmetry in dynamical systems.

In the early study of dynamical systems, Birkhoff recognized the importance of time-reversal symmetry and applied it to restricted three-body problems in classical mechanics ([21]). Forty years after Birkhoff's work, time-reversal symmetry recaught the attention in 1960s and many mathematicians contributed to this topic since then ([40, 4, 5, 76, 7]). For more historical remarks, we refer to [77].

Local bifurcation of periodic solutions to parameterized reversible ODE systems has been studied intensively by many authors (see [30, 105] and references therein). For example, the reversible codimension-one Hopf bifurcation in parameterized ODEs may occur as a result of a collision of eigenvalues of the linearization on the imaginary axis (the so-called (1:1)-resonance; see [30, 103]). Higher (1:k)-resonance were studied in [4, 5]. In contrast to the Hopf bifurcation scenarios, it should be pointed out that the eigenvalues moving along the imaginary axis may give rise to a bifurcation of periodic solutions of *constant* period (see [72, 71]).

In our research, systems of *symmetrically coupled networks* with *reversal symmetry*, where the dynamics is described by *functional differential equations* (FDEs), are considered. Note that an FDE with time-reversal symmetry involves using the information from the future, which is paradoxical in common sense; therefore, we focus on *space-reversal symmetry* rather than *time-reversal symmetry*. Such systems are associated to equations governing steady-state solutions to parabolic PDEs, where the space-reversal symmetry is related to diffusion process with two different scales: the *local diffusion* is modeled by the *continuous Laplacian* and the *non-local interaction* is modeled by discrete Laplacian. We are interested in the following problem:

Problem 1.1.1. What are the symmetric properties of 2π-periodic solutions bifurcating from the equilibrium when eigenvalues of linearization move along the imaginary axis?

1.1.2 Existence of Periodic Solutions to Equivariant Hamiltonian Systems

The problem of the existence of periodic solutions to (symmetric) Hamiltonian systems has a long history ([2, 90, 3, 17, 82]). Among them, Newtonian systems are of special interest. In the presence of group symmetries, classification of symmetric properties of periodic solutions is not an easy task even for Newtonian systems.

An important case of (symmetric) Newtonian systems is *molecular mechanics*. For example, the Nobel price awarded discovery in chemistry reveals the existence of the fullerene molecule ([73]), which can be modeled mathematically as a symmetric Newtonian system. In a fullerene molecule, the carbon atoms are arranged at 60 vertices of a truncated icosahedron, where the 90 edges represent the bonds of the molecule. The linear vibrational modes of fullerene have been measured using IR and Raman spectroscopy and group representation theory. Increasing attention has been given to the theoretical mathematical study of different aspects of this molecule, in particular, in the determination of other vibrational modes that cannot be measured experimentally (see [55, 33, 42, 107, 108, 112, 95] and the references therein). In our research, we study the following problem:

Problem 1.1.2. What are all the possible symmetric vibrational modes of fullerene?

1.1.3 Existence of Periodic Solutions to Systems Homogeneous at Infinity

The abundance of the periodic phenomena in nature (e.g., season, lunar, tide cycles) motivates the study of non-autonomous dynamical systems with periodic right-hand sides (such systems are called *periodic*). Examples of such systems naturally appear in population dynamics ([97]).

The problem of the existence of periodic solutions to periodic systems, where the right-hand side admits a principal part at infinity in the form of homogeneous polynomial map, say, P (independent of time), has been attracting considerable attention ([52, 87, 67, 89, 68, 85, 69]). In particular, the following result was obtained in [85]: there exists a periodic solution if P satisfies the conditions below:

(i) $\mathrm{ind}(0, P)$ is well-defined;

(ii) $\mathrm{ind}(0, P) \neq 0$;

(iii) P admits a homogeneous regular guiding function (the natural counterpart of Lyapunov function in the context relevant to bounded solutions).

In general, verification of Conditions (i)–(iii) is a problem of formidable complexity. If P is *quadratic*, then P gives rise (through the polarization) to the structure of a commutative (in general, non-associative) algebra, in which case, Condition (i) is equivalent to the non-existence of 2-nilpotents.

On the other hand, since Conditions (ii) and (iii) are not semi-algebraic in nature, one arrives at the following problem:

Problem 1.1.3. Given an algebra without 2-nilpotents, what are effective *sufficient* conditions (expressed in terms of the multiplication in the algebra) providing that Conditions (ii) and (iii) are satisfied?

1.2 Tools

1.2.1 Brouwer Degree in the Presence of Symmetries

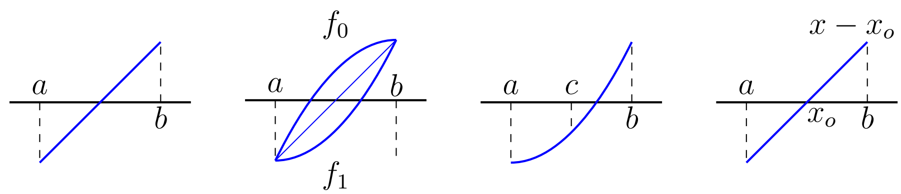
The Brouwer degree, which was first introduced by Brouwer for proving *Brouwer fixed point theorem* ([28]), can be viewed as a generalization of two important results in undergraduate math courses, i.e., *intermediate value theorem* (in short, IVT) in calculus and *argument principle* in complex analysis. The IVT is related to the following patterns of continuous functions on a closed bounded interval:

(a) **Existence.** If a function takes different signs at the endpoints, then it must admit a zero in the interval (see picture (a)).

(b) **Homotopy.** This property is preserved under continuous deformations with no zeros at the endpoints (see picture (b)).

(c) **Additivity.** To localize zeros, one can use the bisection method (see picture (c)).

(d) **Normalization.** The conclusion of the IVT is satisfied by the identity map defined on an interval containing zero (see picture (d)).



(a) Existence (b) Homotopy (c) Additivity (d) Normalization

Figure 1.1. Properties of degree

An axiomatization of properties (a)–(d) gives rise to the so-called *axiomatic approach* to the Brouwer degree ([86]).

In general, calculating the Brouwer degree of a given continuous map can be of great complexity; however, if the map respects a certain group of symmetries, there are restrictions on possible values of its degree. Borsuk was the first to discover such restrictions (the degree of an odd map is odd, see [24]). Possible extension of Borsuk's result to maps respecting more complicated symmetries was a subject of myriads of papers (see [99]). For the most general results, we refer to [60] (the so-called *Borsuk mod-p theorem* for abelian groups) and [75] (the so-called *congruence principle* for arbitrary compact Lie groups).

1.2.2 The Equivariant Degree

The equivariant degree is a counterpart of the Brouwer degree for symmetric systems which allows classifying and counting orbits of solutions according to their symmetric properties. In fact, the equivariant degree theory lies in the intersection of algebra, topology and analysis and this makes it an effective tool for studying symmetric systems.

In the application to symmetric systems, the equivariant degree has three different faces reflecting various types of problems (see [11]): equivariant degree without free parameters (application to boundary value problems), twisted equivariant degree (application to symmetric Hopf bifurcation) and Gradient degree (application to variational problems).

We refer to [72, 12] for history of development, theoretical details and examples of applications of the equivariant degree theory.

1.3 Algebraic Challenges

1.3.1 Computation of Equivariant Degrees

In the application of (equivariant) degree theory, it is crucial that the degree can be computed (or at least, estimated) effectively. The *Eisenbud-Levine-Khimshiashvili signature formula* provides an effective machinery for the computation of the *Brouwer degree* ([44]) while the *recurrence formula technique* for computing the *G-equivariant degree* is developed in [72] and [12].

Comparing to Brouwer degree, computation of equivariant degree is, however, in the early stage of its development. Although the recurrence formula provides a practical approach, its implementation may become difficult when G admits a complex structure. The computation of G-equivariant degree is quite straightforward in the following two cases:

(i) If G is a finite group, the computation can be done easily by the assistance of computer software (e.g. the GAP system [100]).

(ii) If G is an infinite Lie group with relatively simple structure (e.g., S^1, $O(2)$, $SO(3)$, $O(3)$, etc.), the computation can be done by hand.

Periodic solutions to Hamiltonian (resp. reversible) systems we are dealing with (see Sections 1.1.1 and 1.1.2) give rise to the G-equivariant degree with $G = \Gamma \times S^1$ (resp. $\Gamma \times O(2)$). In such a case, Approaches (i) and (ii) meet serious difficulties due to the complexity of Γ and the infinteness of S^1 (resp. $O(2)$).

Basic computational technique and procedures in the GAP system was developed in [80] to overcome these difficulties. Based on that, we developed new procedures in the GAP system, which are significant improvements comparing to the original ones, to assist the computation of G-equivariant degree. These procedures have been applied in [36, 70].

1.3.2 Solvable Groups and the Existence of Equivariant Quadratic Maps

It is well-known that there exists a parallelism between the structure of a group and properties of its (complex) irreducible representations. For example,

(i) any irreducible representation of an abelian group is one-dimensional ([58]);

(ii) any irreducible representation of a supersolvable group is induced from one-dimensional representation of its subgroup ([58]).

Problem 1.3.1. Does there exist an easily formulated characterization of the class of *finite solvable groups* in terms of their irreducible representations?

The classification of group representations up to linear equivalence is a classical problem (essentially, two representations are linearly equivalent if and only if their characters coincide). The well-known classification of p-group representations up to conjugate characters leads to the existence of equivariant maps whose Brouwer degree is relatively prime to p (see [6]). Since the Brouwer degree of complex homogeneous map can be easily calculated, result in [6] gives rise to the following question:

Problem 1.3.2. Given two representations, under which conditions, does there exist a quadratic equivariant map between them taking non-zero vectors to non-zero ones?

1.4 Summary of Results and Overview

1.4.1 Bifurcation of Periodic Solutions in Symmetric Reversible FDEs

As concrete examples in our study in this problem, we consider mixed delay differential equations (MDDEs) and integro-differential equations (IDEs). Under certain smoothness/genericity conditions, the standard method to study local bifurcation in symmetric reversible systems is based on combining *center manifold reduction* with *equivariant normal form classification* ([51, 56, 106, 104]). As a matter of fact, both MDDE and IDE do not admit a flow, in general; therefore, the center manifold reduction does not apply. Following the equivariant degree based method, we studied the symmetric reversible FDEs as a fixed-point problem in the Sobolev space of periodic functions and gave a complete topological classification of symmetric properties of bifurcating branches of 2π-periodic solutions.

1.4.2 Existence of Periodic Solutions to Equivariant Hamiltonian Systems

Classical topological methods used to study symmetric Hamiltonian systems are rooted in either *Ljusternik-Schnirelmann theory* (category, genus, set index theories) or *Morse theory* (Conley index, Floer homology) ([14, 16, 82, 91]). While Ljusternik-Schnirelmann theory provides only multiplicity results, Morse theory based methods, which give required classification, are not open enough for computerization since the corresponding invariants appeal to the gradient flow rather than the gradient field itself. The equivariant gradient degree introduced by Gęba keeps the balance between the strong points of both approaches ([47]).

In the study of symmetric systems, considering full symmetries, hidden and external, is the key to proper analysis of such systems. However, technical limitations lead many researchers to restrict their consideration to smaller symmetry groups, which cannot provide the full dynamical picture.

In our research, by combining the equivariant degree theory with computational techniques, which is implemented in GAP system (these procedures have also been applied in [36, 70]), we are able to analyze the non-linear dynamics of fullerene with its full symmetry and obtain its symmetric vibrational modes—some of them coincide with the experimental results; others are not shown in the existing experiments.

1.4.3 Existence of Periodic Solutions to Systems Homogeneous at Infinity

Our study of Problem 1.1.3 indicates that its answer is closely related to that of Problems 1.3.1 and 1.3.2.

For Problem 1.3.2, we applied the Norton algebra technique to establish the existence of an equivariant quadratic map between two non-equivalent $(n-1)$-dimensional S_n-representations taking non-zero vectors to non-zero ones for n being odd. Combining with the *congruence principle* for Brouwer degree (see [75]), this result suggests an effective sufficient condition to Condition (ii) (see Section 1.1.3).

On the other hand, the answer to Problem 1.3.1 is affirmative. To be more explicit, we consider *α-characteristic* of group representations (it was first introduced in [13]; see also [75]), which stands the greatest common divisor of lengths of all non-trivial orbits in the given representation. It turned out that a group is solvable if and only if all its non-trivial irreducible representations admit non-trivial α-characteristic (i.e., $\alpha > 1$). The necessity, which is the simple part, was suggested in [61]. On the other hand, by combining group algebra and the characterization of solvable groups suggested in [64], we proved the sufficiency and obtained the complete result.

1.4.4 Overview

My dissertation is organized as follows: Chapter 2 includes our work related to Problem 1.1.1, which is published in *Differential and Integral Equations* (see [15]); Chapter 3 includes our work related to Problem 1.1.2, which is close to the stage of submission; Chapter 4 includes our work related to Problem 1.1.3, which is submitted to *Journal of Algebra*; finally, Chapter 5 includes the computational aspects, while all the algorithms of the procedures developed in GAP system will be explained in detail; the material in this chapter provides the computational foundation of our results in Chapter 2, 3, 4 and [36, 70].

CHAPTER 2

BIFURCATION OF SPACE PERIODIC SOLUTIONS IN SYMMETRIC REVERSIBLE FDES[1][2]

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2.1 Introduction

Subject. Reversing symmetry is an important subject in natural science (see survey [77]). Typically, (R-symmetric) periodic solutions to one-parameter ODEs respecting the reversing symmetry appear as two-parameter families in which periodic solutions of constant period constitute a one-parameter subfamily (see Definition 2.1.1 and Theorem 4.3 in [77]). Local bifurcations of (families of) periodic solutions to parameterized reversible systems of ODEs have been studied intensively by many authors (see [30], [105] and references therein). For example, the reversible codimension-one Hopf bifurcation in parameterized ODEs may occur as a result of a collision of eigenvalues of the linearization on the imaginary axis (the so-called 1:1-resonance; see [30] and [103]). Higher 1:k resonances, related to the case $i\beta_1 = ki\beta_2$ ($k \in \mathbb{Z}$ and $i\beta_1$, $i\beta_2$ are the eigenvalues at the moment of the bifurcation) were studied in [4] and [5]. In contrast to the reversible Hopf bifurcation scenarios, in this case, the purely imaginary eigenvalues move along the imaginary axis before and after the resonance. It should be pointed out that the eigenvalues moving along the imaginary axis may give rise to a bifurcation of periodic solutions of *constant* period even without any resonance (see Section 8.6 in [72] and [71] which are a starting point for our discussion). This bifurcation, considered in systems of functional differential equations respecting a finite group of symmetries, is the main *subject* of the present paper.

General setting and motivating examples. In order to describe the general setting, denote by C_{2n} (resp. $\widetilde{C}_{2n}$) the Banach space of bounded continuous functions from $\mathbb{R}$ (resp. $[-\pi, \pi]$) to $\mathbb{R}^{2n} := \mathbb{R}^n \oplus \mathbb{R}^n$ equipped with the sup-norm. For any $u \in C_{2n}$ and $x \in \mathbb{R}$, let $u_x \in \widetilde{C}_{2n}$ be a function defined by $u_x(s) := u(x + s)$ for $s \in [-\pi, \pi]$. Assume that $f : \mathbb{R} \times \widetilde{C}_{2n} \to \mathbb{R}^{2n}$ is a continuous map and consider the following parameterized by $\alpha \in \mathbb{R}$ family of functional differential equations

$$\frac{du}{dx}(x) = f(\alpha, u_x). \tag{2.1}$$

Definition 2.1.1. System (2.1) is said to be *reversible*, if

$$f(\alpha, R\varphi(-s)) = -Rf(\alpha, \varphi(s)) \quad \text{for all } \varphi \in \widetilde{C}_{2n},\ s \in [-\pi, \pi],\ \alpha \in \mathbb{R}, \tag{2.2}$$

where $R : \mathbb{R}^n \oplus \mathbb{R}^n \to \mathbb{R}^n \oplus \mathbb{R}^n$ is given by

$$R := \begin{bmatrix} \mathrm{Id} & 0 \\ 0 & -\mathrm{Id} \end{bmatrix}. \tag{2.3}$$

We will also say that (2.1) is *R-symmetric*.

Let us present few examples of reversible systems (2.1).

Example 2.1.2 (Reversible ODEs)**.** Let $h : \mathbb{R} \oplus \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ be a continuous function satisfying $h(\alpha, Ru) = -Rh(\alpha, u)$ for all $(\alpha, u) \in \mathbb{R} \oplus \mathbb{R}^{2n}$. Then, the system of ODEs

$$\frac{du}{dx}(x) = h(\alpha, u(x))$$

is a particular case of reversible system (2.1), where $f : \mathbb{R} \times \widetilde{C}_{2n} \to \mathbb{R}^{2n}$ is given by $f(\alpha, \varphi) := h(\alpha, \varphi(0))$. Clearly, the second order system of ODEs

$$\ddot{v}(x) = g(\alpha, v(x)), \tag{2.4}$$

where $g : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ is a continuous function, can be considered as a prototypal example for (parameterized families of) reversible ODEs (here $u = (v, \dot{v}) \in C_{2n}$).

Another example of reversible system (2.1) generalizing (2.4), is the following mixed delay differential equation with both positive and negative (i.e., advanced argument) delays.

Example 2.1.3 (Reversible MDDEs)**.** Assume again that $g : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ is continuous, then the equation

$$\ddot{v}(x) = g(\alpha, v(x)) + a(v(x - \alpha) + v(x + \alpha)), \quad a \in \mathbb{R}, \tag{2.5}$$

is a particular case of system (2.1) with $u = (v, \dot{v})$ and

$$f(\alpha, v_1, v_2) = (v_2, g(\alpha, v_1)) + (0, a(v_1(\alpha) + v_1(-\alpha))), \quad (v_1, v_2) \in \widetilde{C}_{2n}.$$

Another generalization of (2.4) is the following system of integro-differential equations.

Example 2.1.4 (Reversible IDEs)**.** Let $g : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ and $k : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ be continuous. Assume that for every $\alpha \in \mathbb{R}$, k_α is even. Then, the system of integro-differential equations

$$\ddot{v}(x) = g(\alpha, v(x)) + a \int_{-\pi}^{\pi} v(x-y) k_\alpha(y)\, dy, \quad a \in \mathbb{R}, \tag{2.6}$$

is a reversible system of type (2.1).

Note that by replacing x by t in Examples 2.1.3 and 2.1.4, one obtains *time-reversible* FDEs. However, such systems involve using the information from the future by "traveling back in time", which is difficult to justify from a commonsensical viewpoint. Therefore, in the present paper, we discuss only (symmetric networks of) *space-reversible* FDEs (one can think of equations governing steady-state solutions to PDEs, see [77] and references therein). Also, the natural source of space-reversibility is related to non-local interaction. A prototypal example for equation of type (2.5) is related to diffusion process with two scales (the local diffusion is modeled by the continuous Laplacian while the non-local diffusion is related to the discrete Laplacian). For the equations close in spirit to (2.6), we refer to [48].

In this paper, we are focused on Γ-symmetric reversible systems of FDEs, where Γ is a finite group. In such systems, symmetries may come from Γ-symmetrically coupled networks. As an illustrative example of such symmetries, we consider the octahedral group $\Gamma = S_4$.

Method. The standard way to study local bifurcations in equivariant/reversible systems is rooted in the singularity theory: assuming the system satisfying certain smoothness and genericity conditions around the bifurcation point, one can combine the equivariant/reversible normal form classification with Center Manifold Theorem/Lyapunov-Schmidt Reduction. For a detailed exposition of this concept and related techniques, we refer to [51, 50, 30, 56, 78, 104, 105].

During the last twenty five years the equivariant degree theory (in short, EDT) emerged in non-linear analysis (for the detailed exposition of this theory, we refer to the monographs [12, 60, 75] and survey [11]), providing alternative methods to study bifurcation in reversible systems. In short, the equivariant degree is a topological tool allowing "counting" orbits of solutions to symmetric equations in the same way as the usual Brouwer degree does, but according to their symmetry properties. The EDT has all the needed features allowing its application in non-smooth/non-generic settings related to equivariant dynamical systems having, in general, infinite dimensional phase spaces and large spacial symmetry groups (in particular, there is no need to put extra non-resonance/simplicity requirements on eigenvalues of the linearization). Also, in many cases, the EDT based methods can be easily assisted by computer. On the other hand, the EDT cannot be applied to studying stability properties of periodic solutions (for this purpose, the Center Manifold Reduction seems to be inevitable).

In the present paper, to establish the abstract results on the existence, multiplicity and symmetric properties of bifurcating branches of periodic solutions, we use the $(\Gamma \times O(2))$-equivariant degree without free parameters, where $O(2)$ is related to the reversing symmetry while Γ reflects the symmetric character of the coupling in the corresponding network. We also present concrete examples related to S_4-symmetric coupling, for which the equivariant bifurcation invariant is fully evaluated. In order to achieve these computational goals, we developed (using some important results obtained in [36]) several new group-theoretical computational algorithms, which were implemented in the specially created GAP program.

Observe also that the EDT based method developed in this paper can be applied to establish the existence of branches of periodic solutions to reversible systems with near 2π periods (Lyapunov-type Center Theorem) usually obtained by combining the rescaling time method with the Lyapunov-Schmidt Reduction (see, for example, [105]). However, this topic goes beyond the scope of the present paper.

Overview. After the Introduction, the paper is organized as follows. In Section 2.2, we recall the standard equivariant jargon, provide isotypical decompositions of functional spaces naturally associated to periodic solutions to system (2.2) and outline the axiomatic approach to the equivariant degree without free parameters. In Section 2.3, we reformulate system (2.2) as an equivariant fixed-point problem in an appropriate Sobolev space. In Section 2.4, we apply the equivariant degree method to prove our main abstract result (see Theorem 2.4.7 and formula (2.50)) on the occurrence, multiplicity and symmetric properties of bifurcation branches of 2π-periodic solutions to equivariant system (2.2). Applications of Theorem 2.4.7 to networks of oscillators of type (2.5) and (2.6) coupled in a cube-symmetric fashion, are given in the fifth section (see Theorems 2.5.1 and 2.5.2). The explanation of computation of $(\Gamma \times O(2))$-equivariant degree can be found in Chapter 5.

2.2 Preliminaries

In this section, we review basic terminology and results from equivariant topology and representation theory as well as recall the axiomatic approach to the equivariant degree without free parameters. In addition, we provide a description of subgroups and their conjugacy classes in a direct product of groups.

2.2.1 Equivariant Jargon

Let $\mathcal{G}$ be a group. We will use the notation $H \leq \mathcal{G}$ to indicate that H is a subgroup of $\mathcal{G}$. For $H \leq \mathcal{G}$, we denote by $N_{\mathcal{G}}(H)$ the *normalizer* of H in $\mathcal{G}$, $W_{\mathcal{G}}(H) := N_{\mathcal{G}}(H)/H$ the *Weyl group* of H in $\mathcal{G}$ and by (H) the *conjugacy class* of H in $\mathcal{G}$ (we will omit the subscript "$\mathcal{G}$" if the ambient group is clear from the context). In the case $\mathcal{G}$ is a compact Lie group, we will always assume that all the considered subgroups $H \leq \mathcal{G}$ are closed.

The set $\Phi(\mathcal{G})$ of all conjugacy classes of subgroups in $\mathcal{G}$ can be naturally equipped with the partial order: $(H) \leq (K)$ if and only if $gHg^{-1} \leq K$ for some $g \in \mathcal{G}$.

In what follows, G will stand for a compact Lie group. For any integer $n \geq 0$, put $\Phi_n(G) := \{(H) \in \Phi(G) : \dim W(H) = n\}$.

Suppose X is a G-space and $x \in X$. Denote by $G_x := \{g \in G : gx = x\}$ the *isotropy* of x, by $G(x) := \{gx : g \in G\}$ the *orbit* of x, and by X/G the *orbit space*. For any isotropy G_x, call (G_x) the *orbit type* of x and put $\Phi(G;X) := \{(H) \in \Phi(G) : H = G_x \text{ for some } x \in X\}$ and $\Phi_n(G;X) := \Phi(G;X) \cap \Phi_n(G)$. Also, for any $H \leq G$, denote by $X^H := \{x \in X : G_x \geq H\}$ the set of H-fixed points and put $X^{(H)} := \{x \in X : (G_x) \geq (H)\}$, $X_H := \{x \in X : G_x = H\}$, $X_{(H)} := \{x \in X : (G_x) = (H)\}$. It is well-known that $W(H)$ acts on X^H and this action is free on X_H.

Given two subgroups $H \leq K \leq G$, define $N_G(H,K) := \{g \in G : gHg^{-1} \leq K\}$. Obviously, $N_G(H,K)$ is a left $N_G(K)$-space (also, it is a right $N_G(H)$-space). Moreover (see Proposition 2.52 in [12]), if $\dim W_G(H) = \dim W_G(K)$, then $n_G(H,K) := |N_G(H,K)/N_G(K)|$ is finite (see also [57, 75]). We will also omit the subscript "G" when the group G is clear from the context. Recall that the number $n(H,K)$ coincides with the number of conjugate copies of K in G which contains H (see [57, 75]).

Suppose Y is another G-space. A continuous map $f : X \to Y$ is called *G-equivariant* if $f(gx) = gf(x)$ for all $x \in X$ and $g \in G$. For any $H \leq G$ and equivariant map $f : X \to Y$, the map $f^H : X^H \to Y^H$, with $f^H : f|_{X^H}$, is well-defined and $W(H)$-equivariant.

Finally, given two Banach spaces E_1 and E_2 and an open bounded subset $\Omega \subset E_1$, a continuous map $f : E_1 \to E_2$ is called *Ω-admissible* if $f(x) \neq 0$ for all $x \in \partial\Omega$. In this case, (f, Ω) is called an *admissible pair*. Denote by $\mathcal{M}(E_1, E_2)$ the set of all Ω-admissible pairs.

We refer to [26, 101, 12] for additional equivariant topology background used in the present paper.

2.2.2 Subgroups in Direct Product of Groups $\mathscr{G}_1 \times \mathscr{G}_2$

Given two groups $\mathscr{G}_1$ and $\mathscr{G}_2$, consider the product group $\mathscr{G}_1 \times \mathscr{G}_2$ and define the projection homomorphisms:

$$\pi_1 : \mathscr{G}_1 \times \mathscr{G}_2 \to \mathscr{G}_1, \quad \pi_1(g_1, g_2) = g_1;$$

$$\pi_2 : \mathscr{G}_1 \times \mathscr{G}_2 \to \mathscr{G}_2, \quad \pi_2(g_1, g_2) = g_2.$$

The following well-known result (see [36] for more details), which is rooted in Goursat's Lemma (see [53]), provides the desired description of subgroups $\mathscr{H}$ of the product group $\mathscr{G}_1 \times \mathscr{G}_2$.

Theorem 2.2.1. *Let $\mathscr{H}$ be a subgroup of the product group $\mathscr{G}_1 \times \mathscr{G}_2$. Put $H := \pi_1(\mathscr{H})$ and $K := \pi_2(\mathscr{H})$. Then, there exist a group L and two epimorphisms $\varphi : H \to L$ and $\psi : K \to L$, such that*

$$\mathscr{H} = \{(h,k) \in H \times K : \varphi(h) = \psi(k)\},$$

(see Figure 2.1). In this case, we will use the notation

$$\mathscr{H} =: H\, {}^{\varphi}\!\times_L^{\psi} K.$$

$$\begin{tikzcd}
& \mathscr{G}_1 \times \mathscr{G}_2 & \\
& \mathscr{H} \arrow[u, phantom, sloped, "\leq"] \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\
\mathscr{G}_1 \geq H \arrow[r, "\varphi"'] & L & K \leq \mathscr{G}_2 \arrow[l, "\psi"]
\end{tikzcd}$$

Figure 2.1. Subgroup $\mathscr{H} \leq \mathscr{G}_1 \times \mathscr{G}_2$

In order to describe the conjugacy classes of subgroups of $\mathscr{G}_1 \times \mathscr{G}_2$, one needs the following statement (see [36]).

Proposition 2.2.2. *Let $\mathcal{G}_1$ and $\mathcal{G}_2$ be two groups. Two subgroups $H^{\varphi} \times_L^{\psi} K$, $H'^{\varphi'} \times_L^{\psi'} K'$ of $\mathcal{G}_1 \times \mathcal{G}_2$ are conjugate if and only if there exist $(a, b) \in \mathcal{G}_1 \times \mathcal{G}_2$ such that the inner automorphisms $a. : \mathcal{G}_1 \to \mathcal{G}_1$ and $b. : \mathcal{G}_2 \to \mathcal{G}_2$ given by*

$$a.g_1 = a g_1 a^{-1}, \quad b.g_2 = b g_2 b^{-1}, \quad g_1 \in \mathcal{G}_1,\ g_2 \in \mathcal{G}_2,$$

satisfy the properties: $H' = a.H$, $K' = b.K$ and $\varphi = \varphi' \circ a.$, $\psi = \psi' \circ b.$. In other words, the diagram shown in Figure 2.2 commutes.

$$\begin{tikzcd}
H \arrow[dr, "\varphi"] \arrow[dd, "a."'] & & K \arrow[dl, "\psi"'] \arrow[dd, "b."] \\
& L & \\
H' \arrow[ur, "\varphi'"'] & & K' \arrow[ul, "\psi'"]
\end{tikzcd}$$

Figure 2.2. Conjugacy classes of subgroups in direct product

2.2.3 $(\Gamma \times O(2))$-Representations

Let Γ be a *finite* group and $G := \Gamma \times O(2)$. In this subsection, we describe $(\Gamma \times O(2))$-representations on function spaces relevant for studying periodic solutions to system (2.1). We also discuss the related G-isotypical decompositions.

(a) G-Representations on Function Spaces. Denote by V^1 and V^2 two identical copies of an n-dimensional orthogonal Γ-representation. Put $V := V^1 \oplus V^2$ and consider the Sobolev spaces $\mathscr{W}^k := H^1(S^1; V^k)$ for $k = 1, 2$ (here, we use the standard identification $S^1 = \mathbb{R}/2\pi\mathbb{Z}$). Clearly, the space $\mathscr{W}^k$ is a Hilbert G-representation with the G-action defined as follows: given $w^k \in \mathscr{W}^k$,

$$((\gamma, 1)w^k)(\cdot) = \gamma w^k(\cdot) \tag{2.7}$$

$$((1_\Gamma, e^{i\theta})w^k)(\cdot) = (\phi_\theta w^k)(\cdot) \tag{2.8}$$

$$((1_\Gamma, \kappa)w^k)(\cdot) = (-1)^{k-1}(T w^k)(\cdot) \tag{2.9}$$

where $\gamma \in \Gamma$ and $e^{i\theta}$, $\kappa \in O(2)$, and

$$(\phi_\theta w^k)(t) = w^k(t+\theta) \tag{2.10}$$

$$(Tw^k)(t) = w^k(-t). \tag{2.11}$$

Put $\mathscr{W} := \mathscr{W}^1 \oplus \mathscr{W}^2$.

In a similar way, we have the Banach G-representation $C := C(S^1; V)$, where the G-action is given by the same formulas (2.7)–(2.11), and the Hilbert G-representation $E := L^2(S^1; V)$, with the G-action slightly modified, i.e., instead of (2.9) we define

$$((1_\Gamma, \kappa)w^k)(\cdot) = (-1)^k (Tw^k)(\cdot) \quad (k = 1, 2).$$

(b) Isotypical Decompositions. We assume that $\{\mathcal{V}_j\}_{j=0}^r$ is the complete list of all irreducible Γ-representations, where $\mathcal{V}_0$ stands for the trivial representation (recall, Γ is finite). Since V^1 and V^2 are equivalent Γ-representations, the Γ-isotypical decompositions of V^k are both

$$V^k = \bigoplus_{j=0}^r V_j \tag{2.12}$$

for $k = 1, 2$, where V_j is modeled on a Γ-irreducible representation $\mathcal{V}_j$. Therefore, the Γ-isotypical decomposition of $\mathscr{W}^k$ is (see (2.7)),

$$\mathscr{W}^k \simeq \bigoplus_{j=0}^r \mathscr{W}_j^k,$$

where

$$\mathscr{W}_j^k := H^1(S^1; V_j).$$

Next, consider the Fourier mode decomposition of $\mathscr{W}_j^k$

$$\mathscr{W}_j^k \simeq \mathscr{W}_{j,0}^k \oplus \overline{\bigoplus_{l=1}^\infty \mathscr{W}_{j,l}^k}, \tag{2.13}$$

where $\mathscr{W}^k_{j,0} = V_j$ and for $l > 0$,

$$\mathscr{W}^k_{j,l} := \{\cos(lt)a + \sin(lt)b : a, b \in V_j\}$$

(the closure is taken with respect to the Sobolev norm). Since the functions in $\mathscr{W}^k_{j,0}$ are constant, the component $\mathscr{W}^k_{j,0}$ can be identified with V_j. On the other hand, for $l > 0$, the component $\mathscr{W}^k_{j,l}$ can be identified with the complexification

$$(V_j)^{\mathbb{C}} := V_j \oplus iV_j = \{a + ib : a, b \in V_j\}$$

using the following relation:

$$\cos(l\cdot)a + \sin(l\cdot)b \mapsto a - ib \quad a,\, b \in V_j. \tag{2.14}$$

Define the following action of G on the space $(V_j)^{\mathbb{C}}$:

$$(\gamma, 1)z = \gamma z \tag{2.15}$$

$$(1_\Gamma, e^{i\theta})z = e^{il\theta} \cdot z \tag{2.16}$$

$$(1_\Gamma, \kappa)z = (-1)^{k-1}\overline{z}. \tag{2.17}$$

where $z := a + ib$, a, $b \in V_j$ and '$\cdot$' denotes the usual complex multiplication. Then, one can easily verify that the identification (2.14) is G-equivariant. Indeed, for example, we have for $w(t) = \cos(lt)a + \sin(lt)b$

$$\begin{aligned}(1_\Gamma, e^{i\theta})w &= \cos(lt)(\cos(l\theta)a + \sin(l\theta)b) + \sin(lt)(\cos(l\theta)b - \sin(l\theta)b)\\ &\mapsto (\cos(l\theta)a + \sin(l\theta)b) - i(\cos(l\theta)b - \sin(l\theta)b)\\ &= e^{il\theta}(a - ib).\end{aligned}$$

Remark 2.2.3. Notice that $\mathscr{W}^1_{j,0}$ and $\mathscr{W}^2_{j,0}$ are not equivalent G-representations. However, $\mathscr{W}^1_{j,l}$ and $\mathscr{W}^2_{j,l}$ (for $l > 0$) are equivalent G-representations. Indeed, define the mapping $\eta : \mathscr{W}^1_{j,l} \to \mathscr{W}^2_{j,l}$ by

$$\eta(x + iy) = i(x + iy) = -y + ix$$

Clearly, η is $(\Gamma \times SO(2))$-equivariant. However, notice that the κ-actions also commute with η, which is shown in Figure 2.3.

$$\begin{tikzcd}
\mathscr{W}^1_{j,l} & \mathscr{W}^2_{j,l} \\
x+iy \arrow[r, "\eta"] \arrow[d, "\kappa"'] & -y+ix \arrow[d, "\kappa"] \\
x-iy \arrow[r, "\eta"'] & y+ix
\end{tikzcd}$$

Figure 2.3. Isomorphism η.

Combining (2.12)–(2.17) with Remark 2.2.3 yields the following G-isotypical decomposition of $\mathscr{W}$:

$$\mathscr{W} = \bigoplus_{k=1}^{2} \bigoplus_{j=1}^{r} \mathscr{W}^k_{j,0} \oplus \overline{\bigoplus_{l=1}^{\infty} \bigoplus_{j=1}^{r} \mathscr{W}_{j,l}}, \tag{2.18}$$

where

$$\mathscr{W}_{j,l} := \mathscr{W}^1_{j,l} \oplus \mathscr{W}^2_{j,l}. \tag{2.19}$$

In the sequel, we will also use the following notation:

$$\mathscr{W}_l := \bigoplus_{j=1}^{r} \mathscr{W}_{j,l}. \tag{2.20}$$

For additional information about the representation theory, we refer to [27, 101, 12].

2.2.4 Admissible G-Pairs and Burnside Ring $A(G)$

Put $\mathcal{M} := \bigcup_V \mathcal{M}(V, V)$, where V runs over the set of Euclidean spaces (see Section 2.2.1). The Brouwer degree is the function $\deg : \mathcal{M} \to \mathbb{Z}$, which, for a given admissible pair (f, Ω), provides an algebraic count of zeroes of f in the domain Ω. Its standard properties (existence, additivity, homotopy and normalization) can be used as axioms (see, for example, [72]). In symmetric settings, when dealing with the equivariant maps f, zeroes of f usually come in orbits and in order to provide a similar algebraic count of these orbits one also needs to take into account their symmetry properties (i.e., their orbit types). An appropriate tool in these settings such a degree is the so-called equivariant degree without parameter (see, for example, [72, 11, 12, 75]).

Let us briefly recall the properties of the G-equivariant degree without parameter (which can actually be used as axioms). An admissible pair $(f, \Omega) \in \mathcal{M}(V, V)$ is called an *admissible G-pair* if Ω is G-invariant and f is G-equivariant. Denote by $\mathcal{M}^G(V)$ the set of all admissible G-pairs in V and put

$$\mathcal{M}^G := \bigcup_V \mathcal{M}^G(V),$$

where the union is taken over all orthogonal G-representations V. The collection $\mathcal{M}^G$ replaces $\mathcal{M}$ in the equivariant setting. A continuous map $h : [0,1] \times V \to V$ is called an *Ω-admissible G-homotopy* if $(h(t, \cdot), \Omega) \in \mathcal{M}^G(V)$ for any $t \in [0,1]$.

In the G-equivariant degree, the ring $\mathbb{Z}$ is replaced by the so-called *Burnside ring* $A(G)$. To be more specific, $A(G) := \mathbb{Z}[\Phi_0(G)]$ is a $\mathbb{Z}$-module, i.e., it is the free $\mathbb{Z}$-module generated by $(H) \in \Phi_0(G)$ (see Section 2.2.1). Notice that elements of $A(G)$ can be written as finite sums

$$n_1(H_1) + n_2(H_2) + \cdots + n_m(H_m) = \sum_{k=1}^{m} n_k(H_k), \quad n_k \in \mathbb{Z},\ (H_k) \in \Phi_0(G).$$

Occasionally, it will be convenient to write these elements as

$$\sum_{(H) \in \Phi_0(G)} n_H(H), \quad n_H \in \mathbb{Z},$$

with finitely many $n_H \neq 0$. To define the ring multiplication "$\cdot$" in $A(G)$, take (H), $(K) \in \Phi_0(G)$ and observe that G acts diagonally on $G/H \times G/K$ with finitely many orbit types.

Consider such an orbit type $(L) \in \Phi_0(G)$. As is well-known (see, for example, [26, 12]), $(G/H \times G/K)_{(L)}/G$ is finite. Put

$$(H) \cdot (K) := \sum_{(L) \in \Phi_0(G)} m_L(H, K)(L),$$

where $m_L(H, K) := \left|(G/H \times G/K)_{(L)}/G\right|$. Then, the $\mathbb{Z}$-module $A(G)$ equipped with the above multiplication (extended from generators by distributivity) becomes a ring. Notice that $A(G)$ is a ring with the unity (G), i.e., $(H) \cdot (G) = (H)$ for every $(H) \in \Phi_0(G)$.

2.2.5 Equivariant Degree without Parameter

Axioms. We follow the axiomatic approach to the definition of the equivariant degree without parameter given in [11] (for more information on the equivariant degree theory, see [12, 60]).

Theorem 2.2.4. *There exists a unique map (called the G-equivariant degree without parameter) $G\text{-}\deg : \mathcal{M}^G \to A(G)$, which for each admissible G-pair (f, Ω), associates an element*

$$G\text{-}\deg(f, \Omega) = \sum_{k=1}^{m} n_k(H_k) \in A(G) \tag{2.21}$$

satisfying the following properties:

(G1) (Existence) *If $G\text{-}\deg(f, \Omega) \neq 0$, i.e., $n_k \neq 0$ for some k in (2.21), then there exists $x \in \Omega$ such that $f(x) = 0$ and $(G_x) \geq (H_k)$.*

(G2) (Additivity) *If Ω_1 and Ω_2 are two disjoint open G-invariant subsets of Ω such that*

$$f^{-1}(0) \cap \Omega \subset \Omega_1 \cup \Omega_2,$$

then

$$G\text{-}\deg(f, \Omega) = G\text{-}\deg(f, \Omega_1) + G\text{-}\deg(f, \Omega_2).$$

(G3) (Homotopy) *If* $h : [0,1] \times V \to V$ *is an* Ω*-admissible* G*-homotopy, then*

$$G\text{-}\deg(h_t, \Omega) = \textit{constant}.$$

(G4) (Normalization) *If* $\Omega \subset V$ *is a* G*-invariant open bounded neighborhood of* 0*, then*

$$G\text{-}\deg(\mathrm{Id}, \Omega) = 1(G).$$

(G5) (Multiplicativity) *For any* (f_1, Ω_1), $(f_2, \Omega_2) \in \mathcal{M}^G$,

$$G\text{-}\deg(f_1 \times f_2, \Omega_1 \times \Omega_2) = G\text{-}\deg(f_1, \Omega_1) \cdot G\text{-}\deg(f_2, \Omega_2)$$

where "$\cdot$" stands for the multiplication in the Burnside ring $A_0(G)$*.*

(G6) (Suspension) *If* W *is an orthogonal* G*-representation and* $\mathcal{B}$ *is an open bounded invariant neighborhood of* $0 \in W$*, then*

$$G\text{-}\deg(f \times \mathrm{Id}_W, \Omega \times \mathcal{B}) = G\text{-}\deg(f, \Omega).$$

(G7) (Recurrence Formula) *For an admissible* G*-pair* (f, Ω)*, the* G*-degree in* (2.21) *can be computed using the following recurrence formula:*

$$n_H = \frac{\deg(f^H, \Omega^H) - \sum_{(K)>(H)} n_K n(H,K)\, |W(K)|}{|W(H)|},$$

where $|X|$ *stands for the number of elements in* X *and* $\deg(f^H, \Omega^H)$ *is the Brouwer degree of the map* $f^H := f|_{V^H}$ *on the set* $\Omega^H \subset V^H$*.*

Remark 2.2.5. Combining Property (G6) with the standard (equivariant) Leray-Schauder projection techniques (see, for example, [72, 12]), one can easily extend the equivariant degree without parameter to equivariant compact vector fields on Banach G-representations.

Basic degrees. Let V be an orthogonal G-representation and let $L : V \to V$ be a G-equivariant linear isomorphism. Keeping in mind the formula for the Brouwer degree of a linear map, denote by $\sigma_-(L)$ the set of all negative real eigenvalues of the operator L and let $E(\mu)$ be the generalized eigenspace of L corresponding to μ (which is clearly G-invariant). To obtain an effective formula for the computation of $G\text{-}\deg(L, B(V))$, where $B(V)$ stands for the unit ball in V, take the isotypical decomposition

$$V = V_0 \oplus \cdots \oplus V_r, \tag{2.22}$$

where V_i is modeled on $\mathcal{V}_i$, put

$$m_i(\mu) := \dim(E(\mu) \cap V_i)/\dim \mathcal{V}_i, \quad 0 = 1, \dots, r, \tag{2.23}$$

and call $m_i(\mu)$ the *isotypical* $\mathcal{V}_i$*-multiplicity* of μ. Then, for any irreducible representation $\mathcal{V}_i$, put

$$\deg_{\mathcal{V}_i} := G\text{-}\deg(-\operatorname{Id}, B(\mathcal{V}_i)), \tag{2.24}$$

and call $\deg_{\mathcal{V}_i}$ the basic G-degree corresponding to the irreducible representation $\mathcal{V}_i$. One can easily prove (see [12]) that for every basic degree,

$$(\deg_{\mathcal{V}_i})^2 = \deg_{\mathcal{V}_i} \cdot \deg_{\mathcal{V}_i} = (G).$$

In addition, we will also use the convention $a^0 = (G)$ for any element $a \in A(G)$.

Combining the multiplicativity and homotopy properties of the equivariant degree yields the following statement (see [12, 11]).

Proposition 2.2.6. *Let V be an orthogonal G-representation with the isotypical decomposition* (2.22) *and let $T : V \to V$ be an invertible G-equivariant linear operator. Then (see* (2.23) *and* (2.24)*),*

$$G\text{-}\deg(T, B) = \prod_{\mu \in \sigma_-(T)} \prod_{i=0}^{s} (\deg_{\mathcal{V}_i})^{m_i(\mu)} = \prod_{\mu \in \sigma_-(T)} \prod_{i=0}^{s} (\deg_{\mathcal{V}_i})^{\varepsilon_i(\mu)}$$

where the product is taken in the Burnside ring $A(G)$ and

$$\varepsilon_i(\mu) := \begin{cases} 1 & \textit{if } m_i(\mu) \textit{ is odd;} \\ 0 & \textit{if } m_i(\mu) \textit{ is even.} \end{cases}$$

2.3 Assumptions and Fixed-Point Problem Reformulation

2.3.1 Assumptions

In this subsection, we describe the setting in which the bifurcating branches of 2π-periodic solutions to (2.1) will be studied. Let Γ and V be as in Section 2.2.3(a). Consider a continuous map $f : \mathbb{R} \times \widetilde{C}_{2n} \to \mathbb{R}^{2n}$ and assume that the following conditions are satisfied.

(P1) (Branch of Equilibria) $f(\alpha, 0) = 0$ for any $\alpha \in \mathbb{R}$.

(P2) (Regularity) f is continuous, $D_u f(\alpha, 0)$ exists for any $\alpha \in \mathbb{R}$ and depends continuously on α and

$$\lim_{\|u\| \to 0} \sup_{\alpha \in [a,b]} \frac{\|f(\alpha, u) - D_u f(\alpha, 0)u\|}{\|u\|} = 0$$

for any $a, b \in \mathbb{R}$.

(P3) (No Steady-state Bifurcation) $D_u f(\alpha, 0)|_V : V \to V$ is invertible for any α.

To formulate the next condition, consider the linearized system

$$\frac{du}{dx}(x) = D_u f(\alpha, 0) u_x. \tag{2.25}$$

By substituting $u(x) = e^{\lambda x} v$ with $\lambda \in \mathbb{C}$ and $v \in V^{\mathbb{C}}$ into (2.25), we obtain the *characteristic operator* for (2.25):

$$\begin{cases} \triangle_\alpha(\lambda) : V^{\mathbb{C}} \to V^{\mathbb{C}} \\ \triangle_\alpha(\lambda) v := \lambda v - D_u f(\alpha, 0)(e^{\lambda v}) \end{cases}$$

Put

$$\Psi := \left\{ (\alpha, 0) : \det_{\mathbb{C}}[\Delta_\alpha(ik)] = 0 \text{ for some } k \in \mathbb{N} \right\}. \tag{2.26}$$

Now, we can formulate the following condition.

(P4) (Isolated Center) There exists $\alpha_o \in \mathbb{R}$ such that $(\alpha_o, 0)$ is isolated in Ψ, i.e., there exists an open neighborhood $N \subset \mathbb{R} \times C_{2n}$ of $(\alpha_o, 0)$ such that $N \cap \Psi = (\alpha_o, 0)$.

Finally, we assume the following conditions to be satisfied.

(P5) (Symmetry) f is Γ-equivariant, i.e.,

$$f(\alpha, \gamma\varphi) = \gamma f(\alpha, \varphi)$$

for any $\gamma \in \Gamma$, $\alpha \in \mathbb{R}$ and $\varphi \in \widetilde{C}_{2n}$ (Γ acts trivially on $\mathbb{R}$).

(P6) (Reversibility)

$$f(\alpha, RT\varphi) = -Rf(\alpha, \varphi)$$

(see (2.3)).

Remark 2.3.1. Let us observe that Assumption (P6) imposes some restrictions on the matrix $D_u f(\alpha, 0)$. To be more specific, by the chain rule

$$(D_u f)(\alpha, \mathbf{0})R = -R(D_u f)(\alpha, \mathbf{0}). \tag{2.27}$$

Assume that

$$(D_u f)(\alpha, \mathbf{0}) = \begin{bmatrix} A & B \\ C & D \end{bmatrix},$$

where each block in the matrix is a linear transformation from $\widetilde{C}_n$ to $\mathbb{R}^n$ (see Section 2.1). Then, the condition (2.27) implies

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} \mathrm{Id} & 0 \\ 0 & -\mathrm{Id} \end{bmatrix} = - \begin{bmatrix} \mathrm{Id} & 0 \\ 0 & -\mathrm{Id} \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix} A & -B \\ C & -D \end{bmatrix} = \begin{bmatrix} -A & -B \\ C & D \end{bmatrix},$$

which means

$$(D_u f)(\alpha, 0) = \begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}.$$

In the next subsection, assuming (P1)–(P6) to be satisfied, we will reformulate (2.1) as a G-equivariant fixed-point problem in the Sobolev space $\mathscr{W}$ (see Section 2.2.3(a)).

2.3.2 G-Equivariant Operator Reformulation of (2.1) in Functional Spaces

Let $G := \Gamma \times O(2)$ and let $\mathscr{W}$, C and E be as in Section 2.2.3(a). Consider the linear operator $L : \mathscr{W} \to E$ given by

$$Lu = \frac{du}{dx}$$

and the operator $j : \mathscr{W} \to C$ being the natural (compact) embedding of $\mathscr{W}$ into C. In addition, let $N_f : \mathbb{R} \times C \to E$ be the Nemytskii operator induced by f:

$$N_f(\alpha, u)(x) := f(\alpha, u_x). \tag{2.28}$$

Then, reformulate (2.1) as

$$Lu = N_f(\alpha, j(u)), \quad u \in \mathscr{W}. \tag{2.29}$$

Remark 2.3.2. Since the system (2.1) is reversible and Γ-symmetric (see Assumptions (P5) and (P6)), the operators L, j and N_f are G-equivariant. Indeed, the G-equivariance of j is obvious. For the operator $L : \mathscr{W} \to E$, one has:

$$\begin{aligned} L(1_\Gamma, \kappa)u(y) &= LRTu(y) = -R\frac{du}{dx}(-y) \\ &= -RTLu(y) = (1_\Gamma, \kappa)Lu(y), \end{aligned}$$

and for N_f,

$$\begin{aligned} N_f(\alpha, (1_\Gamma, \kappa)u)(y) &= f(\alpha, j(RTu)_y = f(\alpha, RTj(u)_{-y}) \\ \text{(by (P6))} \qquad &= -Rf(\alpha, j(u)_{-y}) = -RTN_f(\alpha, u)(y) \\ &= (1_\Gamma, \kappa)N_f(\alpha, u)(y). \end{aligned}$$

To convert (2.29) to a fixed-point problem in $\mathscr{W}$, consider the operator $K : \mathscr{W} \to E$ defined by

$$Ku := \frac{1}{2\pi}\int_0^{2\pi} u(x)dx.$$

Clearly, $L+K$ is invertible, but since K is not G-equivariant (and, therefore, nor is $L+K$), one needs to "adjust" the standard resolvent argument. To this end, consider an additional linear operator $S : E \to E$ given by

$$Su(x) := \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} u(x) \tag{2.30}$$

Remark 2.3.3. It is easy to see that

$$SR = -RS. \tag{2.31}$$

Hence,

$$\begin{aligned} SK(1_\Gamma,\kappa)u(x) &= SKRTu(x)\\ &= SRKu(x)\\ \text{(by (2.31))} \qquad &= -RSKu(x)\\ &= -RTSKu(x)\\ &= (1_\Gamma,\kappa)SKu(x), \end{aligned}$$

form which it follows that $L+SK:\mathscr{W}\to E$ is G-equivariant. One can easily verify that $L+SK$ is also an isomorphism.

Combining Remark 2.3.3 with (2.29), one can reformulate (2.1) as the following G-equivariant fixed-point problem in $\mathscr{W}$:

$$u=\mathcal{F}(\alpha,u), \tag{2.32}$$

where $\mathcal{F}:\mathbb{R}\times\mathscr{W}\to\mathscr{W}$ is defined by

$$\mathcal{F}(\alpha,u):=(L+SK)^{-1}\left[N_f(\alpha,u)+SKu\right]. \tag{2.33}$$

Define $\mathfrak{F}:\mathbb{R}\times\mathscr{W}\to\mathscr{W}$ by

$$\mathfrak{F}(\alpha,u):=u-\mathcal{F}(\alpha,u). \tag{2.34}$$

Then, one can rewrite (2.32) in the equivalent form as

$$\mathfrak{F}(\alpha,u)=0. \tag{2.35}$$

Remark 2.3.4. Since j is compact, $\mathfrak{F}$ is a G-equivariant compact vector field.

2.4 $(\Gamma\times O(2))$-Degree Method

In this section, we will apply the equivariant degree method to detect and classify the branches of bifurcating 2π-periodic solutions to system (2.1).

2.4.1 Linearization and Necessary Condition for the Bifurcation

Let us recall the following standard

Definition 2.4.1. Assume that the set

$$\mathfrak{C} := \overline{\{(\alpha, u) \in \Omega : \mathfrak{F}(\alpha, u) = 0,\ u \neq 0\}}$$

contains a compact connected component $\mathscr{C}$ containing nontrivial 2π-periodic functions and such that $(\alpha_o, 0) \in \mathscr{C}$. Then, $(\alpha_o, 0)$ is called a *bifurcation point* for (2.1) and $\mathscr{C}$ is said to be *branch* of nontrivial 2π-periodic solutions to (2.1) bifurcating from $(\alpha_o, 0)$.

The lemma following below provides a *necessary* condition for $(\alpha_o, 0)$ to be a bifurcation point.

Lemma 2.4.2. *Under the Assumptions* (P1)–(P3) *and* (P5)*, suppose that* $(\alpha_o, 0)$ *is a bifurcation point for* (2.1)*. Then,* $(\alpha_o, 0) \in \Psi$ *(see* (2.26)*).*

Proof. Let

$$a(\alpha) := D_u\mathfrak{F}(\alpha, 0) = \operatorname{Id} - (L + SK)^{-1}\left[D_u N_f(\alpha, 0) + SK\right]$$

be the linearization of $\mathfrak{F}$ at $(\alpha, 0)$. Then,

$$a(\alpha) = \bigoplus_{i=0}^{\infty} a_l(\alpha),$$

where $a_l(\alpha) := a(\alpha)|_{\mathscr{W}_l} : \mathscr{W}_l \to \mathscr{W}_l$ is given by

$$a_l(\alpha) := a(\alpha)|_{\mathscr{W}_l} = \begin{cases} -S\ D_u f(\alpha, 0)|_V\,, & l = 0 \\ \operatorname{Id} - \frac{1}{il}\ D_u f(\alpha, 0)|_{\mathscr{W}_l}\,, & l > 0 \end{cases} \tag{2.36}$$

(see (2.20)). By Definition 2.4.1, there exists a sequence $\{(\alpha_n, u_n)\}$ convergent to $(\alpha_o, 0)$ such that $u_n \neq 0$ and $\mathfrak{F}(\alpha_n, u_n) = 0$ for all n. Hence, by Assumptions (P1) and (P2)),

$$u_n - D_u\mathcal{F}(\alpha_n, 0)u_n + r(\alpha_n, u_n) = 0,$$

where $r(\alpha_n, u_n)$ satisfies

$$\lim_{j\to\infty} \frac{r(\alpha_n, u_n)}{\|u_n\|} = 0. \tag{2.37}$$

Then,

$$\frac{u_n}{\|u_n\|} - D_u\mathcal{F}(\alpha_n, 0)\frac{u_n}{\|u_n\|} + \frac{r(\alpha_n, u_n)}{\|u_n\|} = 0. \tag{2.38}$$

Since $D_u\mathfrak{F}(\alpha_o, 0)$ is compact and $D_u\mathfrak{F}(\alpha_n, 0)$ depends continuously on the first component (see Assumption (P2)), one has (up to choosing a subsequence)

$$\begin{aligned}
&\lim_{n\to\infty} D_u\mathcal{F}(\alpha_n, 0)\frac{u_n}{\|u_n\|}\\
&= \lim_{n\to\infty}\left(D_u\mathcal{F}(\alpha_n, 0)\frac{u_n}{\|u_n\|} - D_u\mathcal{F}(\alpha_o, 0)\frac{u_n}{\|u_n\|} + D_u\mathcal{F}(\alpha_o, 0)\frac{u_n}{\|u_n\|}\right)\\
&= \lim_{n\to\infty}\left(D_u\mathcal{F}(\alpha_n, 0)\frac{u_n}{\|u_n\|} - D_u\mathcal{F}(\alpha_o, 0)\frac{u_n}{\|u_n\|}\right) + \lim_{n\to\infty}\left(D_u\mathcal{F}(\alpha_o, 0)\frac{u_n}{\|u_n\|}\right)\\
&= \lim_{n\to\infty}\left(D_u\mathcal{F}(\alpha_o, 0)\frac{u_n}{\|u_n\|}\right) = v_*,
\end{aligned}$$

which by combining it with (2.37) and (2.38), implies

$$\lim_{n\to\infty} \frac{u_n}{\|u_n\|} = v_* \neq 0.$$

Thus

$$v_* - D_u\mathcal{F}(\alpha_o, 0)v_* = 0.$$

But this implies that $\mathrm{Id} - D_u\mathcal{F}(\alpha_o, 0)$ is not invertible, which contradicts the Assumption (P3). □

Remark 2.4.3. Notice that, we did not assume that the map f is continuously differentiable in a neighborhood of $0 \in C_{2n}$ (see Assumption (P2)), therefore one cannot apply the standard Implicit Function Theorem argument to prove Lemma 2.4.2.

2.4.2 Sufficient Condition

To apply the equivariant degree method, we need:

(a) to localize a potential bifurcation point $(\alpha_o, 0)$ in a G-invariant neighborhood $\Omega \subset \mathbb{R} \oplus \mathscr{W}$,

(b) to define a G-invariant auxiliary function ζ allowing us to detect nontrivial 2π-periodic solutions to (2.1) by applying "augmented" map $\mathfrak{F}_\zeta$.

(c) to adjust the neighborhood Ω in order to make $\mathfrak{F}_\zeta$ an Ω-admissible G-equivariant map.

To begin, define

$$\Omega(\rho, r) := \{(\alpha, u) \in \mathbb{R} \times \mathscr{W} : |\alpha - \alpha_o| < \rho,\ \|u\| < r\}\,. \tag{2.39}$$

Clearly, $\Omega(\rho, r)$ is a G-invariant open and bounded neighborhood of $(\alpha_o, 0)$.

Lemma 2.4.4. *Under Assumptions* (P1)–(P6)*, there exist* ρ*,* $r > 0$ *such that the neighborhood* $\Omega(\rho, r)$ *given by* (2.39) *satisfies the following conditions:*

(i) $\overline{\Omega(\rho, r)} \cap \Psi = \{(\alpha_o, 0)\}$*;*

(ii) $\mathfrak{F}(\alpha, u) \neq 0$ *for* $(\alpha, u) \in \overline{\Omega(\rho, r)}$ *with* $|\alpha - \alpha_0| = \rho$ *and* $u \neq 0$*.*

Proof. By Assumption (P4), there exist $\rho > 0$ and $\tilde{r} > 0$ such that

$$\forall_{r>0}\ r \leq \tilde{r} \quad \Rightarrow \quad \overline{\Omega(\rho, r)} \cap \Psi = \{(\alpha_o, 0)\}\,, \tag{2.40}$$

which implies Condition (i). Assume, for contradiction, that for all $r < \tilde{r}$, Condition (ii) is not satisfied. Then, there exists a sequence $\{(\alpha_o + \rho, u_n)\} \subset \Omega(\rho, r)$ such that $u_n \neq 0$, $\lim_{n\to\infty} |u_n| = 0$ and $\mathfrak{F}(\alpha_o + \rho, u_n) = 0$. Next, applying the the same argument as in the proof of Lemma 2.4.2, one can show that $u_n - D\mathcal{F}_u(\alpha_o + \rho, 0)$ is not invertible. Combining this with Assumption (P3) yields $(\alpha_o + \rho, 0) \in \overline{\Omega(\rho, \tilde{r})} \cap \Psi$, which contradicts (2.40). $\square$

Definition 2.4.5. Denote by $\Omega := \Omega(\rho, r)$ the neighborhood provided by Lemma 2.4.4 and call it a *special neighborhood* of $(\alpha_o, 0)$.

Next, we need to introduce the auxiliary function ζ. To this end, consider the following two subsets in $\overline{\Omega}$:

$$\partial_0 := \left\{(\alpha, u) \in \overline{\Omega} : u = 0\right\}; \tag{2.41}$$

$$\partial_r := \left\{(\alpha, u) \in \overline{\Omega} : \|u\| = r\right\}. \tag{2.42}$$

Clearly, ∂_0 and ∂_r are G-invariant disjoint closed sets. Therefore, there exists a G-invariant Urysohn function $\varsigma : \overline{\Omega} \to \mathbb{R}$ satisfying

$$\begin{cases} \varsigma(\lambda, u) > 0 & \text{for } (\lambda, u) \in \partial_r, \\ \varsigma(\lambda, u) < 0 & \text{for } (\lambda, u) \in \partial_0. \end{cases} \tag{2.43}$$

We will call such a function ς an *auxiliary function* for $\mathfrak{F}$ on $\overline{\Omega}$. Define the *augmented* G-equivariant map $\mathfrak{F}_\varsigma : \overline{\Omega} \to \mathbb{R} \times \mathscr{W}$ by

$$\mathfrak{F}_\varsigma(\alpha, u) = \big(\varsigma(\alpha, u), \mathfrak{F}(\alpha, u)\big). \tag{2.44}$$

By Lemma 2.4.4 and (2.43), the map $\mathfrak{F}_\varsigma$ is Ω-admissible. Hence, $G\text{-}\deg(\mathfrak{F}_\varsigma, \Omega)$ is well-defined (see Remark 2.2.5).

Remark 2.4.6. $G\text{-}\deg(\mathfrak{F}_\varsigma, \Omega)$ is independent of the choice of ς. Indeed, if there are two auxiliary functions ς_1 and ς_2, then $h_t := (1-t)\mathfrak{F}_{\varsigma_1} + t\mathfrak{F}_{\varsigma_2}$ is an Ω-admissible homotopy between $\mathfrak{F}_{\varsigma_1}$ and $\mathfrak{F}_{\varsigma_2}$, meaning that $G\text{-}\deg(\mathfrak{F}_{\varsigma_1}, \Omega) = G\text{-}\deg(\mathfrak{F}_{\varsigma_2}, \Omega)$ (see Property (G3)).

We are now in a position to reduce the bifurcation problem for (2.1) to the computation of the degree $G\text{-}\deg(\mathfrak{F}_\varsigma, \Omega)$. More precisely, one has the following result.

Theorem 2.4.7. *Given system* (2.1)*, suppose that f satisfies Assumptions* (P1)–(P6)*. Let $\mathfrak{F}$ be defined by* (2.33) *and* (2.34)*, and assume that the point $(\alpha_o, 0) \in \Psi$ is provided by*

Assumption (P4)*. In addition, let* Ω *be a special neighborhood of* $(\alpha_o, 0)$ *(see Definition* 2.4.5*) and consider* ς *defined by* (2.41)–(2.43)*. Then, the field* $\mathfrak{F}_\varsigma$ *defined by* (2.44) *is* G*-equivariant and* Ω*-admissible, so the equivariant degree*

$$G\text{-}\deg(\mathfrak{F}_\varsigma, \Omega) = \sum_{(H)\in\Phi_0(G)} n_H(H)$$

is well-defined. Moreover, if for some $(H_o) \in \Phi_0(G)$*, the coefficient* n_{H_o} *is non-zero, then there exists a branch* $\mathscr{C}$ *of nontrivial* 2π*-periodic solutions to* (2.1) *bifurcating from* $(\alpha_o, 0)$ *(see Definition* 2.4.1*) satisfying the conditions:*

(i) $\mathscr{C} \cap \partial_r \neq \varnothing$*;*

(ii) $\mathscr{C} \subset \mathbb{R} \times \mathscr{W}^{H_o}$ *(i.e., periodic solutions belonging to* $\mathscr{C}$ *have symmetries at least* (H_o)*);*

(iii) if, in addition, (H_o) *is a maximal orbit type in some* $\mathscr{W}_l$*,* $l = 1, 2, \dots,$ *then there are at least* $|G/H_o|_{S^1}$ *different branches of non-trivial periodic solutions with symmetries at least* (H_o) *bifurcating from* $(\alpha_o, 0)$ *(here* $|G/H_o|_{S^1}$ *stands for the number of* S^1*-orbits in* G/H_o*).*

To prove Theorem 2.4.7, we need the following statement.

Proposition 2.4.8 (Kuratowski (see [74]))**.** *Let* X *be a metric space. Suppose* $A, B \subset X$ *are two disjoint closed sets and* $K \subset X$ *is compact such that* K *intersects both* A *and* B*. If* K *does not contain a connected component* K_o *which intersects both* A *and* B*, then there exist two disjoint open sets* V_1 *and* V_2 *satisfying*

(1) $A \subset V_1$ *and* $B \subset V_2$*;*

(2) $(A \cup B \cup K) \subset (V_1 \cup V_2)$*.*

Proof of Theorem 2.4.7. Put

$$K := \overline{\{(\alpha, u) \in \Omega : u \neq 0, \mathfrak{F}(\alpha, u) = 0\}},$$

$$\varsigma_q(\alpha, u) := \|u\| - q, \quad 0 \leq q \leq r.$$

To take advantage of Proposition 2.4.8, we need to show that K intersects ∂_0 and ∂_r. Indeed, since ς_q is an auxiliary function for any $0 < q < r$, it follows that

$$G\text{-}\deg(\mathfrak{F}_{\varsigma_q}, \Omega) = G\text{-}\deg(\mathfrak{F}_{\varsigma}, \Omega) \neq 0$$

(see Remark 2.4.6). By Property (G1), there exists $(\alpha_q, u_q) \in \Omega$ such that $\mathfrak{F}_{\varsigma_q}(\alpha_q, u_q) = 0$; in particular, $\|u_q\| = q$. Observe that $\mathfrak{F}$ is a compact field, therefore, $\mathfrak{F}^{-1}(0)$ is compact; so is K. Now, by the standard compactness argument, there exist $(\alpha_0, 0) \in (K \cap \partial_0)$ and $(\alpha_r, u_r) \in (K \cap \partial_r)$.

Assume now, by contradiction, that there is no compact connected set $K_o \subset K$ which intersects ∂_0 and ∂_r simultaneously. Then, according to Proposition 2.4.8, there exist two disjoint open sets N', and N'' such that $\partial_0 \subset N'$, $\partial_r \subset C$ and $K \subset (N' \cup N'')$. Put

$$K' := \{u \in K : G(u) \cap N'' = \varnothing\}; \tag{2.45}$$

$$K'' := \{u \in K : G(u) \cap N' = \varnothing\}. \tag{2.46}$$

By construction, K' and K'' are invariant disjoint sets. Combining the openness of N' and N'' with the continuity of the G-action, one can easily show that K' and K'' are closed. Hence, the sets

$$Z' := \partial_0 \cup K';$$

$$Z'' := \partial_r \cup K''$$

are also closed, invariant and disjoint. Therefore, there exists a G-invariant Urysohn function $\mu : \overline{\Omega} \to \mathbb{R}$ with

$$\mu(\alpha, u) = \begin{cases} 1, & \text{if } (\alpha, u) \in Z', \\ 0, & \text{if } (\alpha, u) \in Z''. \end{cases}$$

Take the auxiliary function

$$\begin{cases} \varsigma : \overline{\Omega} \to \mathbb{R} \\ \varsigma(\alpha, u) = \|u\| - \mu(\alpha, u) r \end{cases} . \tag{2.47}$$

By the existence property, the set $\mathfrak{F}_\varsigma^{-1}(0) \subset K = K' \cup K''$ is non-empty. On the other hand, if $\mathfrak{F}_\varsigma(\alpha_*, u_*) = 0$, then, either $(\alpha_*, u_*) \in K'$ or $(\alpha_*, u_*) \in K''$ (recall, $K' \cap K'' = \emptyset$). If $(\alpha_*, u_*) \in K'$, then (by (2.45)–(2.47))

$$\varsigma(\alpha_*, u_*) = 0 \Rightarrow \|u_*\| = r,$$

which contradicts the choice of K', so $(\alpha, u) \notin K'$. Similarly, $(\alpha, u) \notin K''$, and we arrive at the contradiction with $\mathfrak{F}_\varsigma^{-1}(0) = \varnothing$. $\square$

Under the assumptions of Theorem 2.4.7, introduce the following concept.

Definition 2.4.9. Given a point $(\alpha_o, 0) \in \Psi$, put

$$\omega(\alpha_o) := G\text{-}\deg(\mathfrak{F}_\varsigma, \Omega) \tag{2.48}$$

and call it the *local equivariant topological bifurcation invariant* at $(\alpha_o, 0)$.

In the next subsection, we will give an effective formula for the computation of $\omega(\alpha_o)$.

2.4.3 Computation of $\omega(\alpha_o)$

(a) Reduction to Product Formula. Using the standard argument based on Property (G5) of the equivariant degree (see Section 8.5 in [72]), one can easily establish the following formula:

$$G\text{-}\deg(\mathfrak{F}_\varsigma, \Omega) = G\text{-}\deg(\mathfrak{F}_-, B(0, r)) - G\text{-}\deg(\mathfrak{F}_+, B(0, r)),$$

where $B(0,r) \subset W$ is the ball centered at 0 with radius r and

$$\begin{cases} \mathfrak{F}_\pm : \overline{B(0,r)} \to \mathscr{W} \\ \mathfrak{F}_\pm(u) := \mathfrak{F}(\alpha_o \pm \rho, u) \end{cases}$$

(provided that r is sufficiently small). Furthermore, by (P1) and (G3),

$$G\text{-}\deg(\mathfrak{F}_\pm, B(0,r)) = G\text{-}\deg(a_\pm, B(0,r)),$$

where $a_\pm := a(\alpha_o \pm \rho)$ (provided that r is sufficiently small). Next (see [12], Section 9.2), we apply the finite-dimensional approximation to $a_\pm$: there exists $m \in \mathbb{N}$ such that $a_\pm$ is $B(0,r)$-admissibly homotopic to

$$\widetilde{a}_\pm := a_\pm|_{\mathscr{W}^m} + \mathrm{Id}\,|_{(\mathscr{W}^m)^\perp},$$

where $W^m := V \oplus \bigoplus_{l=1}^m \mathscr{W}_l \subset \mathscr{W}$ (see (2.20)).

Therefore (by (G4) and (G5)),

$$G\text{-}\deg(a_\pm, B(0,r)) = G\text{-}\deg(a_\pm|_{\mathscr{W}^m}, D^m),$$

where $D^m := \{u \in \mathscr{W}^m \mid \|u\| < r\}$. Put $W_*^m := \bigoplus_{l=1}^m \mathscr{W}_l$ and $D_*^m := D^m \cap \mathscr{W}_*^m$, and let $D \subset V$ denote the unit ball. Clearly, $a_\pm|_{\mathscr{W}^m} = a_0^\pm \oplus a_*^\pm$, where $a_0^\pm := a_0(\alpha_o \pm \rho)$ and $a_*^\pm := \bigoplus_{l=1}^m a_l(\alpha_o \pm \rho)$. By (G5),

$$G\text{-}\deg(a_\pm|_{\mathscr{W}^m}, D^m) = G\text{-}\deg(a_0^\pm, D) \cdot G\text{-}\deg(a_*^\pm, D_*^m),$$

where the multiplication is taken in the Burnside ring $A(G)$.

(b) Computation of $G\text{-}\deg(a_0^\pm, D)$. By Assumption (P3),

$$G\text{-}\deg(a_0^\pm, D) = G\text{-}\deg(a_0(\alpha_o), D).$$

Take the isotypical decompositions (2.12) and (2.13) and denote by $\mathscr{W}_{j,0}^k$ the irreducible G-representation on which $\mathscr{W}_{j,0}^k$ is modeled. Then, using Proposition 2.2.6, one obtains:

$$G\text{-}\deg(a_0(\alpha_o), D) = \prod_{\mu\in\sigma_-(a_0(\alpha_o))} \prod_{k=1}^{2}\prod_{j=1}^{r} (\deg_{\mathscr{W}_{j,0}^k})^{m_{j,0}^k(\mu)},$$

where σ_- is the negative spectrum and $m_{j,0}^k(\mu)$ stands for the $\mathscr{W}_{j,0}^k$-multiplicity of μ.

Remark 2.4.10. Observe (see (2.36)) that $\mu \in \sigma_-(a_0(\alpha_o)) \iff \mu \in \sigma_+(SD_uf(\alpha_o, 0))$, where S is given by (2.30) and σ_+ stands for the positive spectrum.

(c) Computation of G- $\deg(a_*^\pm, D_*^m)$**.** Using the isotypical decompositions (2.12), (2.18) and (2.19) (see also (2.14)) and applying once again Proposition 2.2.6, one obtains:

$$G\text{-}\deg(a_*^\pm, D_*^m) = \prod_{\mu\in\sigma_-(a_*^\pm)} \prod_{l=1}^{m}\prod_{j=1}^{r} (\deg_{\mathscr{W}_{j,l}})^{m_{j,l}^\pm(\mu)}, \tag{2.49}$$

where $m_{j,l}^\pm(\mu)$ stands for the $\mathscr{W}_{j,l}$-multiplicity of the eigenvalue μ of $a_*^\pm$.

Remark 2.4.11. Put $a_{jl}^\pm := a_l(\alpha_o \pm \rho)|_{\mathscr{W}_{j,l}}$, $j = 1, \ldots, r$ and $l = 1, \ldots, m$. Then,

$$\begin{aligned}
\mu \in \sigma_-(a_*^\pm) &\iff \mu < 0 \quad \text{and} \quad \det(\mu \operatorname{Id} - a_{jl}^\pm) = 0 \\
&\iff \mu < 0 \quad \text{and} \quad \det(\mu \operatorname{Id} - a_l(\alpha_o \pm \rho)|_{\mathscr{W}_{j,l}}) = 0 \\
&\iff \mu < 0 \quad \text{and} \quad \det\left((\mu - 1)\operatorname{Id} + \frac{1}{il} D_uf(\alpha_o \pm \rho, 0)|_{\mathscr{W}_{j,l}}\right) = 0 \\
&\iff \mu < 0 \quad \text{and} \quad \det\left(il(1-\mu)\operatorname{Id} - D_uf(\alpha_o \pm \rho, 0)|_{\mathscr{W}_{j,l}}\right) = 0
\end{aligned}$$

for some $j = 1, \ldots, r$ and $l = 1, \ldots, m$. Hence, $\mu \in \sigma_-(a_*^\pm)$ if and only if $il\xi$ is an eigenvalue of $D_uf(\alpha_o \pm \rho, 0)|_{\mathscr{W}_{j,l}}$ for $\xi := 1 - \mu > 1$ and some $j = 1, \ldots, r$ and $l = 1, \ldots, m$.

Combining (2.48)–(2.49), one arrives at the following formula:

$$\begin{aligned}
\omega(\alpha_o) = &\prod_{\mu\in\sigma_-(a_0(\alpha_o))} \prod_{k=1}^{2}\prod_{j=1}^{r} (\deg_{\mathscr{W}_{j,0}^k})^{m_{j,0}^k(\mu)} \\
&\left(\prod_{\mu\in\sigma_-(a_*^-)} \prod_{l=1}^{m}\prod_{j=1}^{r} (\deg_{\mathscr{W}_{j,l}})^{m_{j,l}^-(\mu)} - \prod_{\mu\in\sigma_-(a_*^+)} \prod_{l=1}^{m}\prod_{j=1}^{r} (\deg_{\mathscr{W}_{j,l}})^{m_{j,l}^+(\mu)}\right).
\end{aligned} \tag{2.50}$$

2.5 Computation of $\omega(\alpha_0)$: Examples

In this section, we will focus on the application of Theorem 2.4.7 to networks of oscillators of types (2.5) and (2.6) coupled symmetrically. To this end, it is enough to compute the equivariant topological bifurcation invariants (see Definition 2.4.9 and (2.50)) for both cases.

2.5.1 Space-Reversal Symmetry for Second Order DDEs and IDEs

To begin with, let us show that (2.5) and (2.6) (provided that k_α is even) are reversible equations. Indeed, if v is a solution to (2.5), put $\mathfrak{v}(x) = v(-x)$. Then,

$$\begin{aligned}\ddot{\mathfrak{v}}(x) &= \ddot{v}(-x)\\ &= g(v(-x)) + a(v(-x-\alpha) + v(-x+\alpha))\\ &= g(\mathfrak{v}(x)) + a(\mathfrak{v}(x+\alpha) + \mathfrak{v}(x-\alpha)),\end{aligned}$$

therefore, $\mathfrak{v}$ is also a solution to (2.5), thus, (2.5) is reversible. Similarly, assume v is a solution to (2.6) and k_α is even. Then,

$$\begin{aligned}\ddot{\mathfrak{v}}(x) &= v''(-x)\\ &= g(v(-x)) + a\int_{-\pi}^{\pi} v(-x-y)k_\alpha(y)dy\\ &= g(\tilde{v}(x)) + a\int_{-\pi}^{\pi} v(-x+y)k_\alpha(-y)dy\\ &= g(\mathfrak{v}(x)) + a\int_{-\pi}^{\pi} \mathfrak{v}(x-y)k_\alpha(y)dy,\end{aligned}$$

therefore, $\mathfrak{v}$ is also a solution.

In what follows, we will assume for simplicity that g is a scalar function and

$$g(u) = c_1 u + q(u),$$

where $c_1 < 0$ is a parameter characterizing, for example, heat sink (see, for example, [4]), while q is a continuous function differentiable at 0 with $q'(0) = 0$. Also, put $a = 1$.

2.5.2 Coupling Systems with Octahedral Symmetry and First Order Reformulation

As a case study, in what follows, we consider networks of eight identical oscillators of type (2.5) (resp. (2.6)) coupled in the cube-like configuration (see [9]). In this way, any permutation of vertices, which preserves the coupling between them, is a symmetry of this configuration. To simplify our exposition, we consider the symmetry group of cube consisting of all transformations preserving orientation in $\mathbb{R}^3$. Obviously, this group is isomorphic to S_4. Therefore, in the context relevant to Sections 2.2.3 and 2.3.1, we have to deal with the 8-dimensional permutational Γ-representation V^1 with $\Gamma = S_4$.

As it is well-known (see, for example, [12, 45]), V_1 admits the S_4-isotypical decomposition $V^1 = V_1 \oplus V_2 \oplus V_3 \oplus V_4$, where V_1 is the trivial 1-dimensional representation, V_2 is the 1-dimensional S_4-representation where S_4 acts as S_4/A_4, V_3 is the natural 3-dimensional irreducible representation of S_4 as a subgroup of $SO(3)$, and $V_4 = V_2 \otimes V_3$.

Combining the conservation laws with the coupling symmetry allows us to choose the coupling matrix in the form $c_2 B$, where B is given by

$$B = \begin{bmatrix} -3 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & -3 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & -3 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & -3 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & -3 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & -3 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & -3 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 & -3 \end{bmatrix} \tag{2.51}$$

(the parameter $c_2 > 0$ characterizes the strength of coupling; see [9] for similar considerations, where the network of 8 coupled van der Pol oscillators was studied).

In order to be compatible with Sections 2.3 and 2.4, we need to reformulate (2.5) and (2.6) as first order differential equations. To this end, put

$$u(x) := \begin{bmatrix} u_1(x) \\ u_2(x) \end{bmatrix} = \begin{bmatrix} v(x) \\ v'(x) \end{bmatrix}.$$

Then, coupling oscillators of type (2.5) using (2.51) yields the following system:

$$\frac{d}{dx}u(x) = \begin{bmatrix} u_2(x) \\ c_1u_1(x) + q(u_1(x)) + c_2Bu_1(x) + u_1(x+\alpha) + u_1(x-\alpha) \end{bmatrix}. \tag{2.52}$$

Similarly, combining (2.6) with (2.51) yields:

$$\frac{d}{dx}u(x) = \begin{bmatrix} u_2(x) \\ c_1u_1(x) + q(u_1(x)) + c_2Bu_1(x) + \int_{-\pi}^{\pi} u_1(x-y)k_\alpha(y)\,dy \end{bmatrix}. \tag{2.53}$$

2.5.3 Equivariant Spectral Information

First, consider the equation (2.52). Following the scheme described in Section 2.3.2, reformulate equation (2.52) as the operator equation (2.35) (see also (2.32), (2.33) and (2.34)). In order to compute the topological equivariant bifurcation invariant by formula (2.50), we need the following information:

(i) eigenvalues of a (see Lemma 2.4.2) together with their $\mathscr{W}_{j,l}$-multiplicities,

(ii) basic degrees $\deg_{\mathscr{W}_{j,l}}$, and

(iii) multiplication formulae for the Burnside ring $A(S_4 \times O(2))$.

Below, we focus on (i), while (ii) and (iii) are explained in Chapter 5.

By direct computation (see (2.36)), one has

$$a_l(\alpha) = \begin{cases} -\begin{bmatrix} D_x g(0) + 2\,\mathrm{Id} & 0 \\ 0 & \mathrm{Id} \end{bmatrix}, & l = 0 \\ \begin{bmatrix} \mathrm{Id} & -\dfrac{1}{il}\,\mathrm{Id} \\ -\dfrac{1}{il}(D_x g(0) + 2\cos(\alpha l)\,\mathrm{Id}) & \mathrm{Id} \end{bmatrix}, & l > 0 \end{cases}.$$

Thus, for the characteristic equation, we obtain

$$\det(a_0(\alpha) - \lambda\,\mathrm{Id}) = \det \begin{bmatrix} -D_x g(0) - (2+\lambda)\,\mathrm{Id} & 0 \\ 0 & -(1+\lambda)\,\mathrm{Id} \end{bmatrix}$$

and

$$\begin{aligned} \det(a_l(\alpha) - \lambda\,\mathrm{Id}) &= \det \begin{bmatrix} (1-\lambda)\,\mathrm{Id} & -\dfrac{1}{il}\,\mathrm{Id} \\ -\dfrac{1}{il}(D_x g(0) + 2\cos(\alpha l)\,\mathrm{Id}) & (1-\lambda)\,\mathrm{Id} \end{bmatrix} \\ &= \det \left[(1-\lambda)^2\,\mathrm{Id} + \frac{1}{l^2}(D_x g(0) + 2\cos(\alpha l)\,\mathrm{Id}) \right] \\ &= \det \left[((1-\lambda)^2 + \frac{1}{l^2} 2\cos(\alpha l))\,\mathrm{Id} + \frac{1}{l^2} D_x g(0) \right]. \end{aligned}$$

As the result, a has the eigenvalues presented in Table 2.1, where

$$\mu_{j,0} = -c_1 + 2(j-1)c_2 - 2$$

and

$$\mu_{j,l}^{\pm} = 1 \pm \frac{1}{l}\sqrt{-c_1 + 2(j-1)c_2 - 2\cos(l\alpha)}$$

for $l > 0$ and $j = 1, 2, 3, 4$. Since the bifurcation may take place only if $\mu_{j,l}^{\pm} = 0$, it follows that only $\mu_{j,l}^{-}$ may give rise to the bifurcation points. In addition, suppose $\mu_{j,l}^{-}(\alpha_o) = 0$. If $\frac{d}{d\alpha}\cos(\alpha_o) = 0$, then α_o is not a bifurcation point. On the other hand, if $\frac{\mathrm{d}}{\mathrm{d}\alpha}\cos(\alpha_o) > 0$, then $\mu_{j,l}^{-}(\alpha_o - \delta) > 0$ and $\mu_{j,l}^{-}(\alpha_o + \delta) < 0$ for any sufficiently small $\delta > 0$; if $\cos'(\alpha_o) < 0$, then

Table 2.1. Eigenvalues of $a(\alpha)$ (MDDE)

	isotypical component X (modeled on $\mathcal{X}$)	Eigenvalue of $a\|_X$	$\mathcal{X}$-multiplicity
$l=0$	$\mathscr{W}_{1,0}^1$	$\mu_{1,0}$	1
	$\mathscr{W}_{2,0}^1$	$\mu_{2,0}$	1
	$\mathscr{W}_{3,0}^1$	$\mu_{3,0}$	1
	$\mathscr{W}_{4,0}^1$	$\mu_{4,0}$	1
	$\mathscr{W}_{1,0}^2$	-1	1
	$\mathscr{W}_{2,0}^2$	-1	1
	$\mathscr{W}_{3,0}^2$	-1	1
	$\mathscr{W}_{4,0}^2$	-1	1
$l>0$	$\mathscr{W}_{1,l}$	$\mu_{1,l}^+$	1
		$\mu_{1,l}^-$	1
	$\mathscr{W}_{2,l}$	$\mu_{2,l}^+$	1
		$\mu_{2,l}^-$	1
	$\mathscr{W}_{3,l}$	$\mu_{3,l}^+$	1
		$\mu_{3,l}^-$	1
	$\mathscr{W}_{4,l}$	$\mu_{4,l}^+$	1
		$\mu_{4,l}^-$	1

$\mu_{j,l}^-(\alpha_o-\delta)<0$ and $\mu_{j,l}^-(\alpha_o+\delta)>0$ for any sufficiently small $\delta>0$. Hence, if $\frac{\mathrm{d}}{\mathrm{d}\alpha}\cos(\alpha_o)\neq 0$, then α_o is a bifurcation point and one can effectively evaluate the negative spectra required by formula (2.50).

Consider now (2.53) and assume, for the sake of definiteness, that

$$k_\alpha(y)=\frac{1}{\sqrt{2\pi\alpha^2}}\mathrm{e}^{-\frac{y^2}{2\alpha^2}}\,\Theta(\frac{iy}{\alpha^2}\mid\frac{2\pi i}{\alpha^2}),$$

where $\Theta(z\mid\tau)=\sum_{n=-\infty}^{\infty}\mathrm{e}^{\pi i n^2\tau}\mathrm{e}^{2\pi i n z}$ is the Jacobi theta function. Note that

$$\int_{-\pi}^{\pi}\varphi(x-y)k_\alpha(y)\,\mathrm{d}y=\int_{-\infty}^{\infty}\varphi(x-y)h_\alpha(y)\,\mathrm{d}y$$

if $\varphi\in C_{2n}$ is 2π-periodic, where

$$h_\alpha(y):=\frac{1}{\sqrt{2\pi\alpha^2}}\mathrm{e}^{-\frac{y^2}{2\alpha^2}}$$

Table 2.2. Eigenvalues of $a(\alpha)$ (IDE)

	isotypical component X (modeled on $\mathcal{X}$)	Eigenvalue of $a\|_X$	$\mathcal{X}$-multiplicity
$l=0$	$\mathscr{W}^1_{1,0}$	$\mu_{1,0}$	1
	$\mathscr{W}^1_{2,0}$	$\mu_{2,0}$	1
	$\mathscr{W}^1_{3,0}$	$\mu_{3,0}$	1
	$\mathscr{W}^1_{4,0}$	$\mu_{4,0}$	1
	$\mathscr{W}^2_{1,0}$	-1	1
	$\mathscr{W}^2_{2,0}$	-1	1
	$\mathscr{W}^2_{3,0}$	-1	1
	$\mathscr{W}^2_{4,0}$	-1	1
$l>0$	$\mathscr{W}_{1,l}$	$\mu^+_{1,l}$	1
		$\mu^-_{1,l}$	1
	$\mathscr{W}_{2,l}$	$\mu^+_{2,l}$	1
		$\mu^-_{2,l}$	1
	$\mathscr{W}_{3,l}$	$\mu^+_{3,l}$	1
		$\mu^-_{3,l}$	1
	$\mathscr{W}_{4,l}$	$\mu^+_{4,l}$	1
		$\mu^-_{4,l}$	1

is the heat kernel. Then, similarly to the MDDE case, one can compute the eigenvalues of $a(\alpha)$, which are presented in Table 2.2, where

$$\mu_{j,0} = -c_1 + 2(j-1)c_2 - 1$$

and

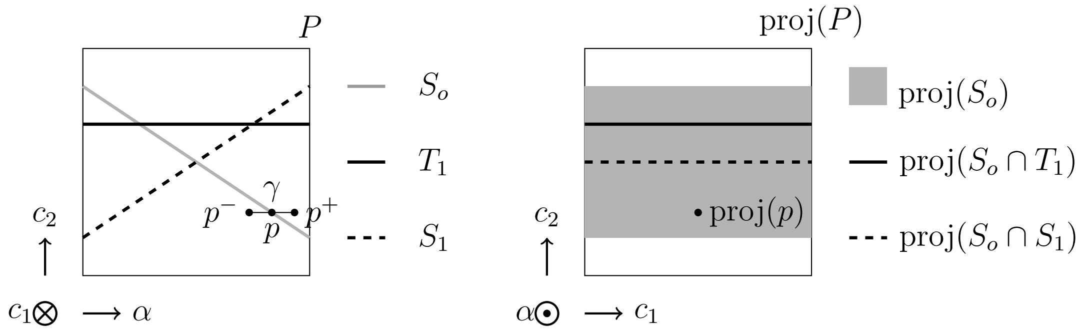
$$\mu^{\pm}_{j,l} = 1 \pm \frac{1}{l}\sqrt{-c_1 + 2(j-1)c_2 - e^{-(l\alpha)^2/2}}$$

for $l>0$ and $j=1,2,3,4$. In this case (in contrast to the DDE case), if $\mu^-_{j,l}(\alpha_o)=0$, then $\mu^-_{j,l}(\alpha_o-\delta)<0$ and $\mu^-_{j,l}(\alpha_o+\delta)>0$ (recall that $\alpha_o>0$). Hence, α_o is always a bifurcation point.

2.5.4 Parameter Space and Bifurcation Mechanism

In this subsection, we present graphical illustrations of the considered in the above examples bifurcations. Since, in both examples, $a = a(c_1, c_2, \alpha)$ depends on three parameters, take $\mathbb{R}^3$ with coordinates (c_1, c_2, α) and consider a bounded connected set $P \subset \mathbb{R}^3$.

By Assumption (P3), a_0 is non-singular, i.e., $\mu_{j,0} \neq 0$ for any j. The family of "surfaces" $\{T_j\}$, defined by the equation $\mu_{j,0} = 0$, has to be excluded from P.



(a) Splitting of P (viewing angle parallel to c_1)

(b) Splitting of S_o (viewing angle parallel to α)

Figure 2.4. Illustration of P and the Bifurcation Surfaces

On the other hand, the necessary condition for the occurrence of the bifurcation is that $\mu^-_{j,l} = 0$ for some j and $l > 0$. Thus, the equations $\mu^-_{j,l} = 0$ determines a family of "surfaces" $\{S_{j,l}\}$ in P, where potential bifurcation points are located. In other words, $\bigcup S_{j,l}$ represents the set of such potential bifurcation points in P, which we will call *critical set*. In addition, the intersection set of two or more different surfaces $\{S_{j,l}\}$ will be called a *collision set*.

To illustrate the bifurcation mechanism, assume that $P \subset \mathbb{R}^3$ is a cube with the removed $\{T_j\} := \{T_1\}$, where the critical set $\{S_{j,l}\}$ is given as $\{S_o, S_1\}$ (see Figure 2.4(a)). Note that the critical set splits P into open connected regions in which $a(\alpha, c_1, c_2)$ is non-singular. By the homotopy property of the equivariant degree, $G\text{-}\deg(a(\alpha, c_1, c_2), D)$ admits a single value in each of those regions.

Take a path $\gamma(t) = (c_1(t), c_2(t), \alpha(t))$ crossing S_o at p. Then, in a sufficiently small neighborhood of p, γ joins two points p^- and p^+ belonging to different regions. Hence, the bifurcation invariant is $\omega(p) = G\text{-}\deg(a(p^-)) - G\text{-}\deg(a(p^+)) \neq 0$ and the bifurcation takes place. Observe that in the case study, $c_1(t) \equiv c_1$, $c_2(t) \equiv c_2$ and $\alpha(t) = t$.

Unfortunately, in the above three-dimensional set P, it may be difficult to visualize how the critical set splits P. Keeping in mind that the bifurcation may occur only on one of $S_{j,l}$, we can project one of those surfaces S_o in $\{S_{j,l}\}$ onto proj(S_o) in c_1-c_2 plane and study the sets proj$(S_o \cap T_1)$ (the projection of the steady-state) and proj$(S_o \cap S_1)$ (the projection of the collision set). We refer to Figure 2.4(b), where three grey area shows the projection of S_o and the dashed line represents the projection of the collision set. Observe that for any point (c_1, c_2) in the regions (I)–(VII), by changing the value of the parameter α, one crosses the critical set, therefore for those critical points we are getting different values of the bifurcation invariant. Also, the remaining part of the square (i.e., outside the grey region) corresponds to (c_1, c_2) for which there is no bifurcation. In the next subsection, we will apply this procedure to obtain the exact values of the local equivariant bifurcation invariants for the examples related to MDDE and IDE systems.

2.5.5 Results

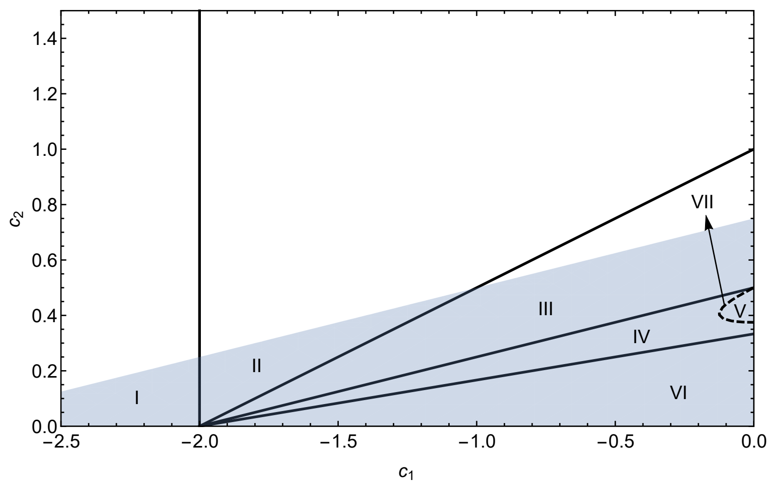
In what follows, we use the notations introduced in Section 2.5.4.

(a) Mixed Delay Differential Equation Take

$$P = \{(c_1, c_2, \alpha) : -2.5 < c_1 < 0,\ 0 < c_2 < 1.5,\ 0 < \alpha < \pi\}$$

and $S_o = S_{2,1}$. Then, the projection of S_o to the c_1-c_2 plane is as in Figure 2.5.

Given c_1 and c_2 in the shaded area, the (unique) critical value of the bifurcation parameter associated to S_o is equal to $\alpha_o(c_1, c_2) = \arccos((-c_1 + 4c_2 - 1)/2)$.



Region	Condition
I	$(-c_1 + 4c_2 < 3) \wedge (-c_1 > 2)$
II	$(-c_1 + 4c_2 < 3) \wedge (-c_1 < 2) \wedge (-c_1 + 2c_2 > 2)$
III	$(-c_1 + 4c_2 < 3) \wedge (-c_1 + 2c_2 < 2) \wedge (-c_1 + 4c_2 > 2)$
IV	$(-c_1 + 4c_2 < 2) \wedge (-c_1 + 6c_2 > 2) \wedge (-c_1 + 6c_2 - (-c_1 + 4c_2 - 1)^2 < 2)$
V	$(-c_1 + 6c_2 - (-c_1 + 4c_2 - 1)^2 > 2)$
VI	$(-c_1 + 6c_2 < 2)$
VII	$(-c_1 + 6c_2 - (-c_1 + 4c_2 - 1)^2 = 2)$

Figure 2.5. Splitting of $\text{proj}(S_o)$ associated to the MDDE system

The values of the equivariant bifurcation invariant $\omega(\alpha_o)$ are summarized in the Table 2.3.

Let us explain how the information provided by Table 2.3 can be used to classify symmetric properties of bifurcating branches of 2π-periodic solutions to system (2.52) as well as to estimate the minimal number of these branches. To simplify our exposition, we restrict ourselves with the case of region II only.

Theorem 2.5.1. *Let (c_1, c_2) be a point in region II. Then:*

(i) $(\alpha_o, 0)$*, where* $\alpha_o = \arccos((-c_1+4c_2)/2)$*, is a bifurcation point of* 2π*-periodic solutions to system* (2.52)*;*

(ii) there exist at least:

- *3 bifurcating branches with symmetry at least* $(D_4{}^{\mathbb{Z}_1}\times_{D_4} D_4)$*,*

Table 2.3. Values of $\omega(\alpha_o)$ in different regions (MDDE case)

Region	$\omega(\alpha_o)$
I	$(D_4{}^{\mathbb{Z}_4}\times_{\mathbb{Z}_2}D_2)-(D_3{}^{\mathbb{Z}_3}\times_{\mathbb{Z}_2}D_2)+(D_2{}^{D_1}\times_{\mathbb{Z}_2}D_2)-(D_2{}^{\mathbb{Z}_2}\times_{\mathbb{Z}_2}D_2)+(D_1{}^{\mathbb{Z}_1}\times_{\mathbb{Z}_2}D_2)-(\mathbb{Z}_2{}^{\mathbb{Z}_1}\times_{\mathbb{Z}_2}D_2)-(D_4{}^{\mathbb{Z}_1}\times_{D_4}D_4)+(D_3{}^{\mathbb{Z}_1}\times_{D_3}D_3)-(D_4{}^{\mathbb{Z}_2}_{V_4}\times_{D_2}D_2)-(D_4{}^{\mathbb{Z}_4}\times_{D_1}D_1)+(D_2{}^{\mathbb{Z}_2}\times_{D_1}D_1)-(D_1{}^{\mathbb{Z}_1}\times_{D_1}D_1)+(\mathbb{Z}_2{}^{\mathbb{Z}_1}\times_{D_1}D_1)$
II	$-(D_4{}^{\mathbb{Z}_4}\times_{\mathbb{Z}_2}D_2)+(D_3{}^{\mathbb{Z}_3}\times_{\mathbb{Z}_2}D_2)-(D_2{}^{D_1}\times_{\mathbb{Z}_2}D_2)+(D_2{}^{\mathbb{Z}_2}\times_{\mathbb{Z}_2}D_2)-(D_1{}^{\mathbb{Z}_1}\times_{\mathbb{Z}_2}D_2)+(\mathbb{Z}_2{}^{\mathbb{Z}_1}\times_{\mathbb{Z}_2}D_2)+(D_4{}^{\mathbb{Z}_1}\times_{D_4}D_4)-(D_3{}^{\mathbb{Z}_1}\times_{D_3}D_3)+(D_4{}^{\mathbb{Z}_2}_{V_4}\times_{D_2}D_2)+(D_4{}^{\mathbb{Z}_4}\times_{D_1}D_1)-(D_2{}^{\mathbb{Z}_2}\times_{D_1}D_1)+(D_1{}^{\mathbb{Z}_1}\times_{D_1}D_1)-(\mathbb{Z}_2{}^{\mathbb{Z}_1}\times_{D_1}D_1)$
III	$-(D_4{}^{\mathbb{Z}_4}\times_{\mathbb{Z}_2}D_2)-(D_3{}^{\mathbb{Z}_3}\times_{\mathbb{Z}_2}D_2)+(D_2{}^{D_1}\times_{\mathbb{Z}_2}D_2)+(D_1{}^{\mathbb{Z}_1}\times_{\mathbb{Z}_2}D_2)-(D_1{}^{D_1}\times_{\mathbb{Z}_1}D_1)+(D_4{}^{\mathbb{Z}_1}\times_{D_4}D_4)+(D_3{}^{\mathbb{Z}_1}\times_{D_3}D_3)+(D_4{}^{\mathbb{Z}_2}_{V_4}\times_{D_2}D_2)-(D_2{}^{\mathbb{Z}_1}_{\mathbb{Z}_2}\times_{D_2}D_2)+(D_4{}^{\mathbb{Z}_4}\times_{D_1}D_1)-(D_2{}^{\mathbb{Z}_2}\times_{D_1}D_1)-(\mathbb{Z}_2{}^{\mathbb{Z}_1}\times_{D_1}D_1)$
IV	$-(D_4{}^{\mathbb{Z}_4}\times_{\mathbb{Z}_2}D_2)-(D_3{}^{\mathbb{Z}_3}\times_{\mathbb{Z}_2}D_2)+(D_2{}^{D_1}\times_{\mathbb{Z}_2}D_2)+(D_1{}^{\mathbb{Z}_1}\times_{\mathbb{Z}_2}D_2)+(\mathbb{Z}_4{}^{\mathbb{Z}_4}\times_{\mathbb{Z}_1}D_1)+(\mathbb{Z}_3{}^{\mathbb{Z}_3}\times_{\mathbb{Z}_1}D_1)-(\mathbb{Z}_1{}^{\mathbb{Z}_1}\times_{\mathbb{Z}_1}D_1)+(D_4{}^{\mathbb{Z}_1}\times_{D_4}D_4)+(D_3{}^{\mathbb{Z}_1}\times_{D_3}D_3)+(D_4{}^{\mathbb{Z}_2}_{V_4}\times_{D_2}D_2)-(D_2{}^{\mathbb{Z}_1}_{\mathbb{Z}_2}\times_{D_2}D_2)+(D_4{}^{\mathbb{Z}_4}\times_{D_1}D_1)-(\mathbb{Z}_4{}^{\mathbb{Z}_2}\times_{D_1}D_1)-(D_2{}^{\mathbb{Z}_2}\times_{D_1}D_1)-(D_1{}^{\mathbb{Z}_1}\times_{D_1}D_1)-(\mathbb{Z}_2{}^{\mathbb{Z}_1}\times_{D_1}D_1)$
V	$-(D_4{}^{\mathbb{Z}_4}\times_{\mathbb{Z}_2}D_2)-(D_3{}^{\mathbb{Z}_3}\times_{\mathbb{Z}_2}D_2)+(D_2{}^{D_1}\times_{\mathbb{Z}_2}D_2)+(V_4{}^{\mathbb{Z}_2}\times_{\mathbb{Z}_2}D_2)+(D_1{}^{\mathbb{Z}_1}\times_{\mathbb{Z}_2}D_2)-(\mathbb{Z}_2{}^{\mathbb{Z}_1}\times_{\mathbb{Z}_2}D_2)+(\mathbb{Z}_4{}^{\mathbb{Z}_4}\times_{\mathbb{Z}_1}D_1)-(\mathbb{Z}_2{}^{\mathbb{Z}_2}\times_{\mathbb{Z}_1}D_1)-(D_4{}^{\mathbb{Z}_1}\times_{D_4}D_4)+(D_3{}^{\mathbb{Z}_1}\times_{D_3}D_3)+(D_4{}^{\mathbb{Z}_4}\times_{D_1}D_1)+(D_3{}^{\mathbb{Z}_3}\times_{D_1}D_1)-(V_4{}^{\mathbb{Z}_2}\times_{D_1}D_1)-2(D_1{}^{\mathbb{Z}_1}\times_{D_1}D_1)+(\mathbb{Z}_2{}^{\mathbb{Z}_1}\times_{D_1}D_1)$
VI	$-(D_4{}^{\mathbb{Z}_4}\times_{\mathbb{Z}_2}D_2)-(D_3{}^{\mathbb{Z}_3}\times_{\mathbb{Z}_2}D_2)+(D_2{}^{D_1}\times_{\mathbb{Z}_2}D_2)+(V_4{}^{\mathbb{Z}_2}\times_{\mathbb{Z}_2}D_2)+(D_1{}^{\mathbb{Z}_1}\times_{\mathbb{Z}_2}D_2)-(\mathbb{Z}_2{}^{\mathbb{Z}_1}\times_{\mathbb{Z}_2}D_2)+(\mathbb{Z}_4{}^{\mathbb{Z}_4}\times_{\mathbb{Z}_1}D_1)-(\mathbb{Z}_2{}^{\mathbb{Z}_2}\times_{\mathbb{Z}_1}D_1)+(D_4{}^{\mathbb{Z}_1}\times_{D_4}D_4)+(D_3{}^{\mathbb{Z}_1}\times_{D_3}D_3)+(D_4{}^{\mathbb{Z}_2}_{V_4}\times_{D_2}D_2)-(D_2{}^{\mathbb{Z}_1}_{\mathbb{Z}_2}\times_{D_2}D_2)-(V_4{}^{\mathbb{Z}_1}\times_{D_2}D_2)+(D_4{}^{\mathbb{Z}_4}\times_{D_1}D_1)-(\mathbb{Z}_4{}^{\mathbb{Z}_2}\times_{D_1}D_1)-(D_2{}^{\mathbb{Z}_2}\times_{D_1}D_1)-(V_4{}^{\mathbb{Z}_2}\times_{D_1}D_1)-(D_1{}^{\mathbb{Z}_1}\times_{D_1}D_1)+(\mathbb{Z}_2{}^{\mathbb{Z}_1}\times_{D_1}D_1)$
VII	$-(V_4{}^{\mathbb{Z}_2}\times_{\mathbb{Z}_2}D_2)+(\mathbb{Z}_2{}^{\mathbb{Z}_1}\times_{\mathbb{Z}_2}D_2)+(\mathbb{Z}_3{}^{\mathbb{Z}_3}\times_{\mathbb{Z}_1}D_1)+(\mathbb{Z}_2{}^{\mathbb{Z}_2}\times_{\mathbb{Z}_1}D_1)-(\mathbb{Z}_1{}^{\mathbb{Z}_1}\times_{\mathbb{Z}_1}D_1)+2(D_4{}^{\mathbb{Z}_1}\times_{D_4}D_4)+(D_4{}^{\mathbb{Z}_2}_{V_4}\times_{D_2}D_2)-(D_2{}^{\mathbb{Z}_1}_{\mathbb{Z}_2}\times_{D_2}D_2)-(D_3{}^{\mathbb{Z}_3}\times_{D_1}D_1)-(\mathbb{Z}_4{}^{\mathbb{Z}_2}\times_{D_1}D_1)-(D_2{}^{\mathbb{Z}_2}\times_{D_1}D_1)+(V_4{}^{\mathbb{Z}_2}\times_{D_1}D_1)+(D_1{}^{\mathbb{Z}_1}\times_{D_1}D_1)-2(\mathbb{Z}_2{}^{\mathbb{Z}_1}\times_{D_1}D_1)$

- *6 bifurcating branches with symmetry at least* $(D_2{}^{D_1}\times_{\mathbb{Z}_2}D_2)$ *and*
- *3 bifurcating branches with symmetry at least* $(D_4{}^{\mathbb{Z}_4}\times_{\mathbb{Z}_2}D_2)$

(see Sections 2.2.2 *and* 5.6*).*

Proof. Observe that all the orbit types appearing in $\omega(\alpha_o)$ in the considered case are related to the first Fourier mode. Among them, the following orbit types are maximal in $\mathscr{W}_1$: $(D_4{}^{\mathbb{Z}_1}\times_{D_4}D_4)$, $(D_2{}^{D_1}\times_{\mathbb{Z}_2}D_2)$ and $(D_4{}^{\mathbb{Z}_4}\times_{\mathbb{Z}_2}D_2)$ (see Table 5.3). To complete the proof, it remains to observe that if $(H{}^{\phi}_{\psi}\times_L K)$ is a (maximal) orbit type in $\mathscr{W}_1$, then there are $|S_4/H|$ different S^1-orbits of 2π-periodic solutions to system (2.52) (provided $H{}^{\phi}_{\psi}\times_L K$ appears in $\omega(\alpha_o)$). $\square$

(b) Integro-Differential Equation Take

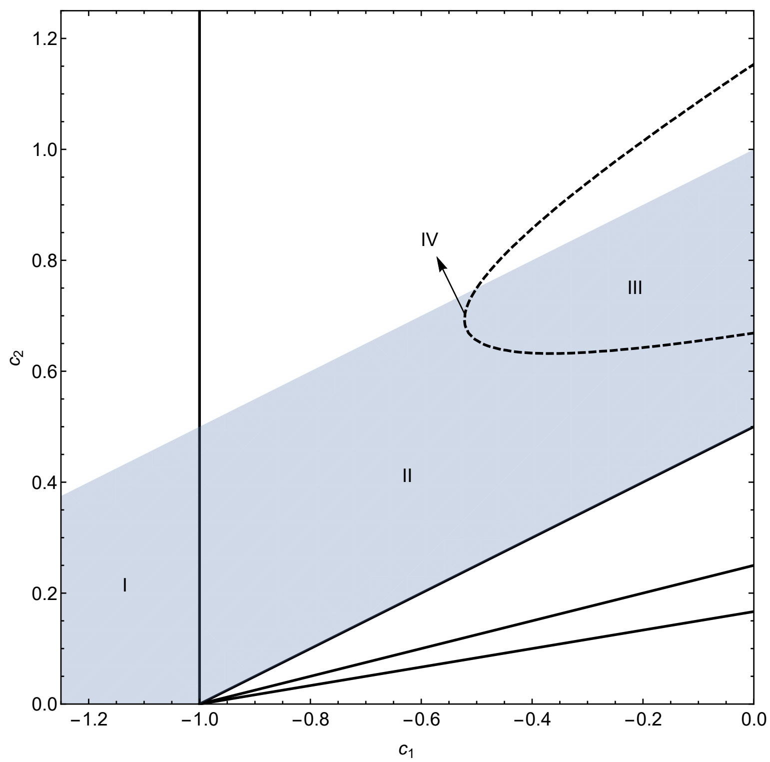
$$P = \{(c_1, c_2, \alpha) : -1.25 < c_1 < 0,\ 0 < c_2 < 1.25,\ 0 < \alpha < \infty\}$$

and $S_o = S_{3,1}$. Then, the projection of S_o to the c_1-c_2 plane is as follows (see Figure 2.4(b)):

Formally, this projection is given by

$$\text{proj}(S_o) = \{(c_1, c_2) : -1.25 < c_1 < 0,\, 0 < c_2 < 1.25 \,\text{and}\, -c_1 + 2c_2 < 2\},$$

which splits into the regions described by Figure 2.6.



Region	Condition
I	$-c_1 > 1$
II	$-c_1 + 6c_2 - 4 > (-c_1 + 2c_2 - 1)^4$
III	$-c_1 + 6c_2 - 4 < (-c_1 + 2c_2 - 1)^4$
IV	$-c_1 + 6c_2 - 4 = (-c_1 + 2c_2 - 1)^4$

Figure 2.6. Splitting of $\text{proj}(S_o)$ associated to the IDE system

Given $(c_1, c_2) \in \text{proj}(S_o)$ (shaded area), the (unique) critical value of the bifurcation parameter associated to S_o is equal to $\alpha_o(c_1, c_2) = \sqrt{-2\ln(-c_1 + 2c_2 - 1)}$. The values of the equivariant bifurcation invariant $\omega(\alpha_o)$ are summarized in the Table 2.4.

Similarly to the MDDE case, we can conclude this subsection by the theorem.

Table 2.4. Values of $\omega(\alpha_o)$ in different regions (IDE case)

Region	$\omega(\alpha_o)$
I	$(D_4{}^{D_2}\times_{\mathbb{Z}_2}D_2)-(\mathbb{Z}_4{}^{\mathbb{Z}_2}\times_{\mathbb{Z}_2}D_2)-(D_2{}^{D_1}\times_{\mathbb{Z}_2}D_2)-(V_4{}^{\mathbb{Z}_2}\times_{\mathbb{Z}_2}D_2)+(\mathbb{Z}_2{}^{\mathbb{Z}_1}\times_{\mathbb{Z}_2}D_2)+(D_3{}^{D_3}\times_{\mathbb{Z}_1}D_1)-(D_1{}^{D_1}\times_{\mathbb{Z}_1}D_1)+(\mathbb{Z}_2{}^{\mathbb{Z}_2}\times_{\mathbb{Z}_1}D_1)+(D_4{}^{\mathbb{Z}_1}\times_{D_4}D_4)-(D_3{}^{\mathbb{Z}_1}\times_{D_3}D_3)+(D_2{}^{\mathbb{Z}_1}_{D_1}\times_{D_2}D_2)+(D_2{}^{\mathbb{Z}_1}_{\mathbb{Z}_2}\times_{D_2}D_2)-(D_4{}^{D_2}\times_{D_1}D_1)+(\mathbb{Z}_4{}^{\mathbb{Z}_2}\times_{D_1}D_1)+(V_4{}^{\mathbb{Z}_2}\times_{D_1}D_1)-2(\mathbb{Z}_2{}^{\mathbb{Z}_1}\times_{D_1}D_1)$
II	$-(D_4{}^{D_2}\times_{\mathbb{Z}_2}D_2)+(\mathbb{Z}_4{}^{\mathbb{Z}_2}\times_{\mathbb{Z}_2}D_2)+(D_2{}^{D_1}\times_{\mathbb{Z}_2}D_2)+(V_4{}^{\mathbb{Z}_2}\times_{\mathbb{Z}_2}D_2)-(\mathbb{Z}_2{}^{\mathbb{Z}_1}\times_{\mathbb{Z}_2}D_2)-(D_3{}^{D_3}\times_{\mathbb{Z}_1}D_1)+(D_1{}^{D_1}\times_{\mathbb{Z}_1}D_1)-(\mathbb{Z}_2{}^{\mathbb{Z}_2}\times_{\mathbb{Z}_1}D_1)-(D_4{}^{\mathbb{Z}_1}\times_{D_4}D_4)+(D_3{}^{\mathbb{Z}_1}\times_{D_3}D_3)-(D_2{}^{\mathbb{Z}_1}_{D_1}\times_{D_2}D_2)-(D_2{}^{\mathbb{Z}_1}_{\mathbb{Z}_2}\times_{D_2}D_2)+(D_4{}^{D_2}\times_{D_1}D_1)-(\mathbb{Z}_4{}^{\mathbb{Z}_2}\times_{D_1}D_1)-(V_4{}^{\mathbb{Z}_2}\times_{D_1}D_1)+2(\mathbb{Z}_2{}^{\mathbb{Z}_1}\times_{D_1}D_1)$
III	$-(D_4{}^{D_2}\times_{\mathbb{Z}_2}D_2)+(\mathbb{Z}_4{}^{\mathbb{Z}_2}\times_{\mathbb{Z}_2}D_2)+(D_2{}^{D_1}\times_{\mathbb{Z}_2}D_2)-(D_3{}^{D_3}\times_{\mathbb{Z}_1}D_1)+(\mathbb{Z}_3{}^{\mathbb{Z}_3}\times_{\mathbb{Z}_1}D_1)+(D_1{}^{D_1}\times_{\mathbb{Z}_1}D_1)-(\mathbb{Z}_1{}^{\mathbb{Z}_1}\times_{\mathbb{Z}_1}D_1)+(D_4{}^{\mathbb{Z}_1}\times_{D_4}D_4)+(D_3{}^{\mathbb{Z}_1}\times_{D_3}D_3)+(D_4{}^{\mathbb{Z}_2}_{V_4}\times_{D_2}D_2)-(D_2{}^{\mathbb{Z}_1}_{D_1}\times_{D_2}D_2)+(D_4{}^{D_2}\times_{D_1}D_1)+(D_3{}^{\mathbb{Z}_3}\times_{D_1}D_1)-(\mathbb{Z}_4{}^{\mathbb{Z}_2}\times_{D_1}D_1)-2(D_1{}^{\mathbb{Z}_1}\times_{D_1}D_1)-(\mathbb{Z}_2{}^{\mathbb{Z}_1}\times_{D_1}D_1)$
IV	$(V_4{}^{\mathbb{Z}_2}\times_{\mathbb{Z}_2}D_2)-(\mathbb{Z}_2{}^{\mathbb{Z}_1}\times_{\mathbb{Z}_2}D_2)-(\mathbb{Z}_3{}^{\mathbb{Z}_3}\times_{\mathbb{Z}_1}D_1)-(\mathbb{Z}_2{}^{\mathbb{Z}_2}\times_{\mathbb{Z}_1}D_1)+(\mathbb{Z}_1{}^{\mathbb{Z}_1}\times_{\mathbb{Z}_1}D_1)-2(D_4{}^{\mathbb{Z}_1}\times_{D_4}D_4)-(D_4{}^{\mathbb{Z}_2}_{V_4}\times_{D_2}D_2)-(D_2{}^{\mathbb{Z}_1}_{\mathbb{Z}_2}\times_{D_2}D_2)-(D_3{}^{\mathbb{Z}_3}\times_{D_1}D_1)-(V_4{}^{\mathbb{Z}_2}\times_{D_1}D_1)+2(D_1{}^{\mathbb{Z}_1}\times_{D_1}D_1)+3(\mathbb{Z}_2{}^{\mathbb{Z}_1}\times_{D_1}D_1)$

Theorem 2.5.2. *Let (c_1, c_2) be a point in region II. Then:*

(i) $(\alpha_o, 0)$, *where* $\alpha_o = \sqrt{-2\ln(-c_1 + 2c_2 - 1)}$, *is a bifurcation point of* 2π*-periodic solutions to system* (2.53)*;*

(ii) there exist at least:

- *3 bifurcating branches with symmetry at least* $(D_4{}^{\mathbb{Z}_1}\times_{D_4}D_4)$,
- *6 bifurcating branches with symmetry at least* $(D_2{}^{D_1}\times_{\mathbb{Z}_2}D_2)$ *and*
- *3 bifurcating branches with symmetry at least* $(D_4{}^{\mathbb{Z}_4}\times_{\mathbb{Z}_2}D_2)$

(see Sections 2.2.2 *and* 5.6*).*

CHAPTER 3

NONLINEAR VIBRATIONS IN THE FULLERENE MOLECULE C_{60}

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3.1 Introduction

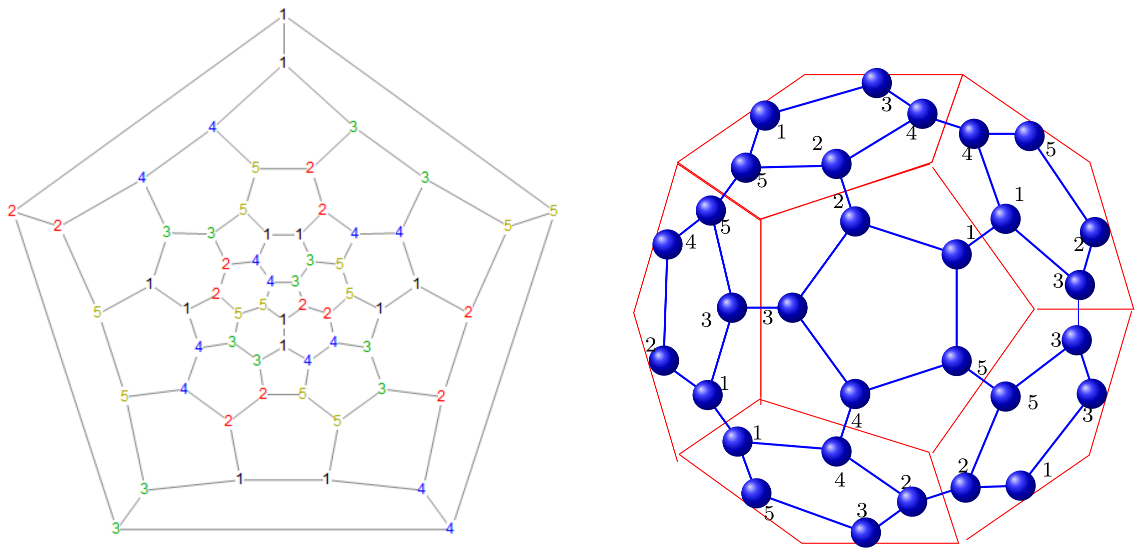

Figure 3.1. The fullerene molecule

The dynamics of the fullerene molecule is represented by Newton's equations, where the potential is given by a force field which describes how carbon atoms interact in the molecule. Many force fields have been proposed for the fullerene in terms of bond stretching, bond bending, torsion and van der Waals forces. In [112], the authors, for the first time, designed the force field specifically for the fullerene, which implement harmonic forces. The model was improved in [108] to reflect the vibrational modes obtained experimentally.

Some force fields were optimized to duplicate the normal modes obtained using IR or Raman spectroscopy, but only few of these models reflect the nonlinear characteristics of the fullerene. In order to analyze the nonlinear effects of the atomic interactions in the fullerene, we consider theoretical force field implemented in [107] for carbon nanotubes and in [20] for the fullerene. These models consider bond deformations that exceed very small fluctuations about equilibrium states, given by Morse potentials, while other force fields, such as those proposed in [108] and [112], are designed only to consider small fluctuations.

The original work [73] has currently 15 thousand of citations. To the best of our knowledge, none of these works have analyzed the real nonlinear vibrations in the the fullerene molecule. Nonlinear vibrational modes of oscillation are families of periodic solutions that appear from the equilibrium configuration with frequencies starting from the normal frequencies. These families of periodic solutions and their frequencies deviate drastically from the normal modes and frequencies for big oscillations.

In order to analyze the nonlinear vibrations in the fullerene molecule, we develop a new representation of the carbon atoms in the truncated icosahedron, which greatly simplifies the description of symmetries in the molecule. This representation uses the elements of icosahedral group as indices of the atoms, as it is illustrated in Figure 3.1.

To analyze the fullerene molecule we chose the force field without van der Waals forces given in [107]. Using this model, we compute the equilibrium configuration u_o and we find that the lengths of the single and double bonds are given by $d_S = 1.438084$ and $d_D = 1.420845$, respectively. These lengths approximate well the experimental measurements presented in [55]. Moreover, we find that the normal frequencies are within the range of 100 to 1800cm^{-1}. This range of normal frequencies fits the range obtained from the experimental and numerical data gathered in [33] .

In our approach, we formulate the problem of finding periodic solutions for the fullerene as the variational problem on the Sobolev space $H^1_{2\pi}$ (of 2π-periodic vector-valued functions) with the functional

$$J(u, \lambda) := \int_0^{2\pi} \left[\frac{1}{2}|\dot{u}(t)|^2 - \lambda^2 V(u(t))\right] dt \ ,$$

where V is the force field, λ^{-1} the frequency and $u(t) \in H^1_{2\pi}$ the renormalized 2π-periodic solution. The existence of periodic solutions is equivalent to the critical points of J. The functional J is clearly invariant under the action of the group $G := I \times O(3) \times O(2)$, which acts as permutation of the atoms, rotations in space, and translations and reflection of time, respectively.

First, we notice that the equilibrium state of the fullerene is a minimizer with icosahedral symmetries $\tilde{I} < I \times O(3)$. We compute the normal frequencies and the irreducible representation under the action of the group $\tilde{I}$. The number of each irreducible representation with non-zero normal frequencies is:

Table 3.1. Irreducible subrepresentations in the configuration space

Irreducible Rep.	$\mathcal{V}_1$	$\mathcal{V}_2$	$\mathcal{V}_3$	$\mathcal{V}_4$	$\mathcal{V}_5$	$\mathcal{V}_{-1}$	$\mathcal{V}_{-2}$	$\mathcal{V}_{-3}$	$\mathcal{V}_{-4}$	$\mathcal{V}_{-5}$
Dimension of Rep.	1	4	5	3	3	1	4	5	3	3
Number of Reps.	2	6	8	4	4	1	6	7	5	5

This result coincides with the results in [55, 33, 42, 108] which for different models, contain the classification of the eigenfrequencies associated to the character theory under the icosahedral symmetries.

To prove the global existence of the families of periodic solutions we use the equivariant gradient degree. The values of this equivariant degree can be expressed elegantly in the form

$$\nabla_G\text{-}\deg(\nabla\varphi, \Omega) = n_1(H_1) + n_2(H_2) + \cdots + n_m(H_m), \quad n_k \in \mathbb{Z},$$

where (H_j) are the orbit types in Ω, which allows to predict the existence of various critical orbits for φ and their symmetries. The gradient degree is just one of many related equivariant degrees that were introduced in the last three decades, see [11, 12, 60, 72] and references therein. In [46, 19, 18], equivariant degree has been apply to other problems in molecular dynamics.

The change of the equivariant degree at the equilibrium configuration, $\nabla_G\text{-}\deg(\nabla J, \Omega)$, for Ω being a neighborhood of u_o, is the invariant that characterize topologically the existence of families of periodic solutions and their symmetries. Moreover, this procedure has the advantage that allows us to prove the global existence of families of periodic solutions. The global property means that families of periodic solutions, represented by a continuum in $H^1_{2\pi} \times \mathbb{R}^+$, have norm or period going to the infinity, end in a collision orbit, or come back to another equilibrium. The computations of the topological G-equivariant invariants were done with GAP programming [111].

Our main result can be presented as follows:

Theorem 3.1.1. *For each normal frequency of the equilibrium configuration, corresponding to the irreducible representations of Table* 3.1*, we prove the global existence of multiple families of periodic solutions with frequency staring from the corresponding normal frequency.*

- $\mathcal{V}_{\pm 1}$*: Each representation has one family of standing wave with icosahedral symmetries (symmetries given by* (3.54)*).*

- $\mathcal{V}_{\pm 2}$*: Each representation has a total of seven families. Four families are of standing waves (symmetries given by* (3.55)*): one with tetrahedral symmetries, two with triangular symmetries and one with Klein symmetries. Three families are of traveling waves (symmetries given by* (3.56)*): two with different pentagonal symmetries and one with triangular symmetries.*

- $\mathcal{V}_{\pm 3}$*: Each representation has a total of five families. Three families are of standing waves (symmetries given by* (3.57)*): one with pentagonal, one with triangular and one with Klein symmetries. Two families are of traveling waves (symmetries given by* (3.58)*): one with pentagonal symmetries and other with different pentagonal symmetries.*

- $\mathcal{V}_{\pm 4}, \mathcal{V}_{\pm 5}$*: Each representation has a total of five families. Three families are of standing waves (symmetries given by* (3.59) *and* (3.61)*): one with pentagonal, one with triangular and one with Klein symmetries. Two families are of traveling waves (symmetries given by* (3.60) *and* (3.62)*): one with pentagonal and one with tetrahedral symmetries.*

In the standing waves, each symmetric face have the exact dynamic repeated for all times, while in the (discrete) traveling waves, there is a time shift among consecutive symmetric faces.

The difference between the solutions of the representations $\mathcal{V}_1, \ldots, \mathcal{V}_5$ and $\mathcal{V}_{-1}, \ldots, \mathcal{V}_{-5}$ is that in the representations $\mathcal{V}_1, \ldots, \mathcal{V}_5$ the solutions are symmetric by the inversion $-I \in O(3)$, while in $\mathcal{V}_{-1}, \ldots, \mathcal{V}_{-5}$ by the inversion $-I$ coupled with a π-phase shift in time.

The families of pentagonal symmetric traveling waves for the representations $\mathcal{V}_{\pm 4}$ and $\mathcal{V}_{\pm 5}$ are different from each other because there are two conjugacy classes $\mathcal{C}_4$ and $\mathcal{C}_5$ in A_5. Both symmetries are present in two different families in the representation $\mathcal{V}_{\pm 3}$.

We implement a Newton's method and pseudo length procedure to continue numerically these families of nonlinear vibrational modes. The nonlinear vibrational modes and normal modes have drastic differences for big oscillations. Icosahedral symmetries appear also in adenovirus viruses with icosahedral capsid, or other icosahedral molecules considered in [108]. The methods presented here are applicable to these cases as well.

Our main theorem predicts the global existence of families of periodic solutions from the fullerene minimal equilibrium configuration. Previous studies have considered only the existence of the linear modes which have constant frequency, but the non-linear system have periodic solutions where frequencies depend on the amplitudes of the oscillation. The study of these periodic solutions in the non-linear system for the fullerene molecule is relevant to understand the phenomena of non-linear resonances. For example, it may be possible to construct mechanisms which consider these amplitude-frequency relations to excite or blow up the vibrational states of the fullerene molecule with the nonlinear resonances. Also the vibrational excitation can identified considering the nonlinear effect of the amplitude-frequency relations of the molecule.

In Section 3.2, we include a short review of the gradient degree theory. In Section 3.3, we present the model equations necessary to study the dynamics of the fullerene molecule. In Section 3.4, we prove the equivariant bifurcation of periodic solutions from the equilibrium configuration of the fullerene molecule. In Section 3.5, we describe the symmetries of the solutions.

3.2 Equivariant Degree Theory

In this section, we introduce and discuss three types of the equivariant degree:

- Brouwer equivariant degree,
- Gradient equivariant degree,
- Twisted equivariant degree.

Although these three degrees, which were constructed with a purpose of applying them to different kinds of symmetric differential equations, it is well-known that they are interconnected. We intend to show these relations between various equivariant degrees and explore them in order to establish effective computational tools.

We begin with some theoretical concepts used for the introduction of the equivariant degrees.

3.2.1 Euler Ring

In this dissertation, we will not rely on the topological definition of the Euler ring (see [101]), for the sake of completeness, we present the original definition which was used, in various forms, in [10, 47, 92, 38, 11, 93, 101].

Definition 3.2.1. (see [101]) Let $U(G) = \mathbb{Z}[\Phi(G)]$ be a free $\mathbb{Z}$-module with basis $\Phi(G)$ and χ_c denotes the Euler characteristic defined in terms of the Alexander-Spanier cohomology with compact support (cf. [98]). Define the *multiplication* on $U(G)$ as follow: for generators (H), $(K) \in \Phi(G)$ put:

$$(H) * (K) = \sum_{(L)\in\Phi(G)} n_L(L), \tag{3.1}$$

where

$$n_L := \chi_c((G/H \times G/K)_L/N(L)), \tag{3.2}$$

and we extend it linearly to the multiplication on entire $U(G)$. We call the free $\mathbb{Z}$-module $U(G)$ equipped with multiplication (3.1) the *Euler ring* of the group G (cf. [27]).

It should be pointed out that, in general, the direct usage of this definition for computations of the multiplicative structure of the Euler ring $U(G)$ may be quite complicated. However, in some special cases, as it is indicated in this dissertation, the multiplication table for Euler ring can be established by elementary techniques.

3.2.2 Burnside Ring

Let $\Phi_0(G) = \{(H) \in \Phi(G) : \dim W(H) = 0\}$ and denote by $A(G) = \mathbb{Z}[\Phi_0(G)]$ a free $\mathbb{Z}$-module with basis $\Phi_0(G)$. Define multiplication on $A(G)$ by restricting multiplication from $U(G)$ to $A(G)$, i.e., for (H), $(K) \in \Phi_0(G)$ let

$$(H) \cdot (K) = \sum_{(L)} n_L(L), \qquad (H), (K), (L) \in \Phi_0(G), \text{ where} \tag{3.3}$$

$$n_L = \chi((G/H \times G/K)_L/N(L)) = |(G/H \times G/K)_L/N(L)| \tag{3.4}$$

(here $\chi(X)$ denotes the usual Euler characteristic of X, which in the case $X$is finite, coincides with the number $|X|$ of elements in X). Then $A(G)$ with multiplication (3.3) becomes a ring which is called the *Burnside ring* of G. As it can be shown, the coefficients (3.4) can be found (cf. [12]) using the following recursive formula:

$$n_L = \frac{n(L,K)|W(K)|n(L,H)|W(H)| - \sum_{(\widetilde{L})>(L)} n(L,\widetilde{L})n_{\widetilde{L}}|W(\widetilde{L})|}{|W(L)|}, \tag{3.5}$$

where (H), (K), (L) and $(\widetilde{L})$ are taken from $\Phi_0(G)$.

Observe that although $A(G)$ is clearly a $\mathbb{Z}$-submodule of $U(G)$, in general, it may **not** be a subring of $U(G)$.

Define $\pi_0 : U(G) \to A(G)$ as follows: for $(H) \in \Phi(G)$ let

$$\pi_0((H)) = \begin{cases} (H) & \text{if } \ (H) \in \Phi_0(G), \\ 0 & \text{otherwise.} \end{cases} \tag{3.6}$$

The map π_0 defined by (3.6) is a ring homomorphism (cf. [10]), i.e.,

$$\pi_0((H) * (K)) = \pi_0((H)) \cdot \pi_0((K)), \qquad (H), (K) \in \Phi(G).$$

The following well-known result (cf. [101], Proposition 1.14, page 231) shows a difference between the generators (H) of $U(G)$ and $A(G)$.

Proposition 3.2.2. *Let $(H) \in \Phi_n(G)$.*

(i) If $n > 0$, then $(H)^k = 0$ in $U(G)$ for some $k \in \mathbb{N}$, i.e., (H) is a nilpotent element in $U(G)$;

(ii) If $n = 0$, then $(H)^k \neq 0$ for all $k \in \mathbb{N}$.

The Burnside ring structure $A(\Gamma)$ can be effectively computed for a large class of groups of the type Γ, $\Gamma \times O(2)$, $SO(3)$ etc, where Γ is a finite group. These algorithms can be extended to more general continuous compact Lie groups G.

The multiplicative structure of $A(G)$ can be only partially used to determine the multiplication structure of $U(G)$. In Section 3.2.5, we will discuss this idea in more details in the case of the group $G := \Gamma \times O(2)$ (with Γ being a finite group).

3.2.3 Euler Ring Homomorphism

Let $\psi : G' \to G$ be a homomorphism of compact Lie groups. Then, G' acts on the left on G by $g'x := \psi(g')x$ (a similarly $xg' := x\psi(g')$ defines the right action). In particular, for any subgroup $H \leq G$, ψ induces a G'-action on G/H and the stabilizer of $gH \in G/H$ is given by

$$G'_{gH} = \psi^{-1}(gHg^{-1}). \tag{3.7}$$

Therefore, ψ induces a map $\Psi : U(G) \to U(G')$ defined by

$$\Psi((H)) = \sum_{(\tilde{H}) \in \Phi(G')} \chi_c((G/H)_{(H')}/G')(H'). \tag{3.8}$$

Proposition 3.2.3. *(see [10, 101]) The map* Ψ *defined by* (3.8) *is a homomorphism of Euler rings.*

Let us recall the following fact about Euler Ring Homomorphism $\Psi : U(G) \to U(\mathbb{T}^n)$ where $\mathbb{T}^n$ is a maximal torus in G and $\psi : \mathbb{T}^n \to G$ is the natural embedding. Then we have (cf. [10]):

Proposition 3.2.4. *Let* $\mathbb{T}^n$ *be a maximal torus in* G*. Then*

$$\Psi(\mathbb{T}^n) = |W(\mathbb{T}^n)|(\mathbb{T}^n) + \sum_{(\mathbb{T}')} n_{\mathbb{T}'}(\mathbb{T}'),$$

where $\mathbb{T}' = g\mathbb{T}^n g^{-1} \cap \mathbb{T}^n$ *for some* $g \in G$ *and* $(\mathbb{T}') \neq (\mathbb{T}^n)$*.*

3.2.4 Twisted Subgroups and Related Modules

Let Γ is a finite group and $G = \Gamma \times S^1$. In this case, there are exactly two types subgroups $H \subset G$, namely,

(a) $H = K \times S^1$ with K being a subgroup of Γ;

(b) the so-called *φ-twisted l-folded* subgroups $K^{\varphi,l}$ (in short, twisted subgroups) defined as follows: if K is a subgroup of Γ, $\varphi : K \to S^1$ a homomorphism and $l = 1, ...,$ then

$$K^{\varphi,l} := \{(\gamma, z) \in K \times S^1 : \ \varphi(\gamma) = z^l\}.$$

Then, clearly we have

$$K^{\varphi,l} = K \times^{\varphi}_{\mathbb{Z}_k} \mathbb{Z}_n,$$

where $\varphi(K) = \mathbb{Z}_k$ and $n = k \cdot l$. Put

$$\Phi^t_1(G) := \{(H) \in \Phi(G) : \ H = K^{\varphi,l} \text{ for some } K \subset \Gamma \text{ with } \dim W_\Gamma(K) = 0\},$$

One can easily observe that $\Phi_1(G) = \Phi_1^t(G)$ and $\Phi_0(G)$ can be identified with $\Phi(\Gamma)$ (with $(K \times S^1)$ identified with (K)). Therefore, the Euler ring $U(G)$, as a $\mathbb{Z}$-module is $A(\Gamma) \oplus A_1(G)$, where following the notation from [12], $A_1(G)$ stands for $\mathbb{Z}[\Phi_1(G)]$. It is well-known (see [12, 11]) that $A_1(G)$ admits a structure of $A(\Gamma)$-module. To be more precise, we have the following $A(\Gamma)$-module multiplication $\cdot : A(\Gamma) \times A_1(\Gamma) \to A_1(\Gamma)$ given by

$$(K) \cdot (H^{\varphi,l}) = \sum_{(L)\in\Phi_1(G)} n_L (L^{\varphi,l}), \quad (K) \in \Phi_0(\Gamma),\ (H^{\varphi,l}) \in \Phi_1(G),$$

where

$$n_L = \frac{n(L,K)|W(K)|n(L^{\varphi,l}, H^{\varphi,l})|W(H^{vp,l}/S^1)| - \sum_{(\widetilde{L})>(L)} n(L^{\varphi,l}, \widetilde{L}^{\varphi,l}) n_{\widetilde{L}} |W(\widetilde{L}^{\varphi,l})|}{|W(L^{\varphi,l})/S^1|}$$

The Burnside ring structure $A(\Gamma)$ as well as the $A(\Gamma)$-module structure for $A_1(G)$ were explicitly described for many specific groups Γ (see [12]) and several computational routines were created for these algebraic structures.

3.2.5 Computations of Euler Ring $U(G)$ in Special Cases

Euler Ring $U(\Gamma \times S^1)$: The following result provides a more practical description of the multiplication in the Euler ring $U(G)$ (see [92]):

Proposition 3.2.5. *Let $G = \Gamma \times S^1$ with Γ being a finite group. Then the multiplication '$*$' in the Euler ring $U(G) = A(\Gamma) \oplus A_1(G)$ can be described by the following table*

Table 3.2. Multiplication of $U(G)$

$*$	$A(\Gamma)$	$A_1(G)$
$A(\Gamma)$	Burnside ring $A(\Gamma)$ multiplication	$A(\Gamma)$-module multiplication
$A_1(G)$	$A(\Gamma)$-module multiplication	0

Euler Ring $U(\Gamma \times O(2))$**:** The classical result, Goursat's Lemma allows one to characterize subgroups of the direct product of two groups in terms of isomorphisms between their quotient groups. For two groups G_1 and G_2 and a subgroup $\mathscr{H} \subset G_1 \times G_2$, there exist Put $K_1 \leq G_1$, $K_2 \leq G_2$), a group L, and two epimorphisms $\varphi : K_1 \to L$ and $\psi : K_2 \to L$, such that

$$\mathscr{H} = \{(g_1, g_2) \in K_1 \times K_2 : \varphi(g_1) = \psi(g_2)\}. \tag{3.9}$$

Notice that:

(a) The Burnside Ring $A(\Gamma \times O(2))$ is a part of the Euler ring $U(\Gamma \times O(2))$. The available algorithms allow effective computations of $A(\Gamma \times O(2))$ for various groups Γ (see [111])..

(b) The multiplicative structure of the Euler ring $U(\Gamma \times S^1)$ is described in 3.2.5.

Denote by $\Psi : U(\Gamma \times O(2)) \to U(\Gamma \times S^1)$ the Euler ring homomorphism induced by the natural embedding $\Gamma \times SO(2) \hookrightarrow \Gamma \times O(2)$ (recall $SO(2) \simeq S^1$).

Therefore, for the group $G := \Gamma \times O(2)$, one has $\Phi_0(G) = \Phi_0^{\mathrm{I}}(G) \cup \Phi_0^{\mathrm{II}}(G) \cup \Phi_0^{\mathrm{III}}(G)$, where

$$\begin{aligned}
&\Phi_0^{\mathrm{I}}(G) := \Phi_0(\Gamma \times SO(2)); \;\; \Phi_0^{\mathrm{II}}(G) := \{(K^{\varphi} \times_L D_n) : n \in \mathbb{N}\}\,; \\
&\Phi_0^{\mathrm{III}}(G) := \{(K \times_L O(2)) : L = \mathbb{Z}_1, \, \mathbb{Z}_2\}\,.
\end{aligned}$$

On the other hand, we have $\Phi_1(G) = \Phi_1(\Gamma \times SO(2))$.

The Euler ring multiplicative structure in $U(\Gamma \times O(2))$ can be described by the Euler ring homomorphism $\Psi : U(\Gamma \times O(2)) \to U(\Gamma \times S^1)$ induced by the natural inclusion $\psi : \Gamma \times S^1 \to \Gamma \times O(2)$ and the known . structure of the Euler ring $U(\Gamma \times S^1)$ (see 3.2.5). By direct application of the definition of the homomorphism Ψ, we obtain that

$$\Psi(\mathcal{H}) = \begin{cases} 2(\mathcal{H}_0), & \text{if } \; \mathcal{H}_0 = \mathcal{H} \\ (\mathcal{H}_0), & \text{if } \; \mathcal{H}_0 \neq \mathcal{H} \text{ and } \; (\mathcal{H}_0) = (\mathcal{H}_1) \text{ in } \Gamma \times S^1 \, , \\ (\mathcal{H}_0) + (\mathcal{H}_1), & \text{if } \; \mathcal{H}_0 \neq \mathcal{H} \text{ and } \; (\mathcal{H}_0) \neq (\mathcal{H}_1) \text{ in } \Gamma \times S^1 \end{cases} \tag{3.10}$$

where $\mathcal{H}_0 := \mathcal{H} \cap (\Gamma \times SO(2))$ and $\mathcal{H}_1 := \kappa\mathcal{H}\kappa \cap (\Gamma \times SO(2))$ ($\kappa \in O(2)$).

The following result was proved in [36]:

Theorem 3.2.6. *Assume that Γ is a finite group and $G := \Gamma \times O(2)$. Then, the multiplication table of the Euler ring $U(\Gamma \times O(2))$ is completely determined by the multiplication table of the Burnside ring $A(G)$ and the structure of the $A(\Gamma)$-module $A_1^t(\Gamma \times S^1)$ via the Euler ring homomorphism $\Psi : U(\Gamma \times O(2)) \to U(\Gamma \times S^1)$. More precisely, for $(\mathcal{H}), (\mathcal{K}) \in \Phi(G)$, we can write the product of $(\mathcal{H})$ and $(\mathcal{K})$ as the sum*

$$(\mathcal{H}) * (\mathcal{K}) = \sum_{(\mathcal{L}) \in \Phi_0(G)} n_{\mathcal{L}} \, (\mathcal{L}) + \sum_{(\mathcal{L}') \in \Phi_1(G)} x_{\mathcal{L}'} \, (\mathcal{L}'),$$

where $n_{\mathcal{L}}$ are known coefficients while $x_{\mathcal{L}'}$ are unknown coefficients that can be determined by the relation

$$\sum_{(\mathcal{L}') \in \Phi_1(G)} x_{\mathcal{L}'} \, (\mathcal{L}') = \frac{1}{2} \left(\Psi(\mathcal{H}) * \Psi(\mathcal{K}) - \sum_{(\mathcal{L}) \in \Phi_0(G)} n_{\mathcal{L}} \, \Psi(\mathcal{L}) \right). \tag{3.11}$$

3.2.6 Brouwer G-Equivariant Degree

Assume that G is a compact Lie group and V is an orthogonal G-representation and $\Omega \subset V$ an open bounded G-invariant set. A G-equivariant (continuous) map $f : V \to V$ is called Ω-*admissible* if $f(x) \neq 0$ for any $x \in \partial\Omega$; in such a case, the pair (f, Ω) is called G-*admissible*. Denote by $\mathcal{M}^G(V, V)$ the set of all such admissible G-pairs, and put $\mathcal{M}^G := \bigcup_V \mathcal{M}^G(V, V)$, where the union is taken for all orthogonal G-representations V. We have the following result (see [12]):

Theorem 3.2.7. *There exists a unique map $G\text{-deg} : \mathcal{M}^G \to A(G)$, which assigns to every admissible G-pair (f, Ω) an element $G\text{-}\deg(f, \Omega) \in A(G)$, called the G-equivariant degree (or simply G-degree) of f on Ω:*

$$G\text{-}\deg(f, \Omega) = \sum_{(H_i) \in \Phi_0(G)} n_{H_i}(H_i) = n_{H_1}(H_1) + \cdots + n_{H_m}(H_m). \tag{3.12}$$

It satisfies the following properties:

(G1) **(Existence)** *If* $G\text{-}\deg(f,\Omega) \neq 0$*, i.e.,* (3.12) *contains a non-zero coefficient* n_{H_i}*, then* $\exists_{x\in\Omega}$ *such that* $f(x) = 0$ *and* $(G_x) \geq (H_i)$*.*

(G2) **(Additivity)** *Let* Ω_1 *and* Ω_2 *be two disjoint open* G*-invariant subsets of* Ω *such that* $f^{-1}(0) \cap \Omega \subset \Omega_1 \cup \Omega_2$*. Then*

$$G\text{-}\deg(f,\Omega) = G\text{-}\deg(f,\Omega_1) + G\text{-}\deg(f,\Omega_2).$$

(G3) **(Homotopy)** *If* $h : [0,1] \times V \to V$ *is an* Ω*-admissible* G*-homotopy, then*

$$G\text{-}\deg(h_t,\Omega) = \text{constant}.$$

(G4) **(Normalization)** *Let* Ω *be a* G*-invariant open bounded neighborhood of* 0 *in* V*. Then*

$$G\text{-}\deg(\mathrm{Id},\Omega) = (G).$$

(G5) **(Multiplicativity)** *For any* $(f_1,\Omega_1), (f_2,\Omega_2) \in \mathcal{M}^G$*,*

$$G\text{-}\deg(f_1 \times f_2, \Omega_1 \times \Omega_2) = G\text{-}\deg(f_1,\Omega_1) \cdot G\text{-}\deg(f_2,\Omega_2),$$

where the multiplication '$\cdot$' is taken in the Burnside ring $A(G)$*.*

(G6) **(Suspension)** *If* W *is an orthogonal* G*-representation and* $\mathcal{B}$ *is an open bounded invariant neighborhood of* $0 \in W$*, then*

$$G\text{-}\deg(f \times \mathrm{Id}_W, \Omega \times \mathcal{B}) = G\text{-}\deg(f,\Omega).$$

(G7) **(Recurrence Formula)** *For an admissible* G*-pair* (f,Ω)*, the* G*-degree* (3.12) *can be computed using the following recurrence formula*

$$n_H = \frac{\deg(f^H,\Omega^H) - \sum_{(K)>(H)} n_K \, n(H,K) \, |W(K)|}{|W(H)|}, \tag{3.13}$$

where $|X|$ *stands for the number of elements in the set* X *and* $\deg(f^H,\Omega^H)$ *is the Brouwer degree of the map* $f^H := f|_{V^H}$ *on the set* $\Omega^H \subset V^H$*.*

(G8) **(Hopf Property)** *Assume $B(V)$ is the unit ball of an orthogonal G-representation V and for $(f_1, B(V)), (f_2, B(V)) \in \mathcal{M}^G$, one has $G\text{-}\deg(f_1, B(V)) = G\text{-}\deg(f_2, B(V))$. Then, f_1 and f_2 are $B(V)$-admissible G-homotopic.*

(G9) **(Functoriality Property)** *Let $(f, \Omega) \in \mathcal{M}^G$ and $\psi : G' \to G$ a homomorphism of finite groups. Then, $(f, \Omega) \in \mathcal{M}^{G'}$ and*

$$G'\text{-}\deg(f, \Omega) = \Psi\left[G\text{-}\deg(f, \Omega)\right], \tag{3.14}$$

where $\Psi : A(G) \to A(G')$ is the ring homomorphism induced by ψ.

Remark 3.2.8. Let G and G' be two finite groups. If $\psi : G' \to G$ is a homomorphism, then ψ induces a homomorphism $\Psi : A(G) \to A(G')$ of Burnside rings (Ψ maps a finite G-set to a finite G'-set). To be more explicit, any finite G-set X can be viewed as a G'-set with the G'-action:

$$g'x := \psi(g')x, \quad g' \in G',\ x \in X. \tag{3.15}$$

In particular, for any subgroup $H \subset G$, G/H becomes a G'-set with the G'-action given by (3.15), which is described by the following formula in Burnside rings:

$$\Psi(H) = \sum_{(K) \in \Phi(G')} n_K(K), \tag{3.16}$$

where

$$n_K := \left|(G/H)_{(K)}/G'\right| = \left|(G/H)_K/N_{G'}(K)\right|. \tag{3.17}$$

3.2.7 G-Equivariant Gradient Degree

The G-equivariant degree was developed in [47] (see also [36] and [94]) The fundamental properties of the gradient equivariant degree are listed below. This degree satisfies all the standard properties expected from a degree theory.

Let V be an orthogonal G-representation. Denote by $C^2_G(V,\mathbb{R})$ the space of G-invariant real C^2-functions on V. Let $\varphi \in C^2_G(V,\mathbb{R})$ and $\Omega \subset V$ be an open bounded invariant set such that $\nabla\varphi(x) \neq 0$ for $x \in \partial\Omega$. In such a case, the pair $(\nabla\varphi, \Omega)$ is called *G-gradient Ω-admissible*. Denote by $\mathcal{M}^G_\nabla(V,V)$ the set of all G-gradient Ω-admissible pairs in $\mathcal{M}^G(V,V)$ and put $\mathcal{M}^G_\nabla := \bigcup_V \mathcal{M}^G_\nabla(V,V)$. In an obvious way, one can define a G-gradient Ω-admissible homotopy between two G-gradient Ω-admissible maps. Finally, given a G-gradient Ω-admissible homotopy class, one should have a concept of its "nice representatives." The corresponding concept of *special Ω-Morse functions* was elaborated by K.H. Mayer in [83].

Definition 3.2.9. A G-gradient Ω-admissible map $f := \nabla\varphi$ is called a *special* Ω-Morse function if

(i) $f|_\Omega$ is of class C^1;

(ii) $f^{-1}(0) \cap \Omega$ is composed of regular zero orbits;

(iii) for each (H) with $f^{-1}(0) \cap \Omega_{(H)} \neq \emptyset$, there exists a tubular neighborhood $\mathcal{N}(U,\varepsilon)$ such that f is (H)-normal on $\mathcal{N}(U,\varepsilon)$.

Theorem 3.2.10. *There exists a unique map $\nabla_G\text{-}\deg : \mathcal{M}^G_\nabla \to U(G)$, which assigns to every $(\nabla\varphi, \Omega) \in \mathcal{M}^G_\nabla$ an element $\nabla_G\text{-}\deg(\nabla\varphi,\Omega) \in U(G)$, called the G-gradient degree of $\nabla\varphi$ on Ω,*

$$\nabla_G\text{-}\deg(\nabla\varphi,\Omega) = \sum_{(H_i)\in\Phi(\Gamma)} n_{H_i}(H_i) = n_{H_1}(H_1) + \cdots + n_{H_m}(H_m), \tag{3.18}$$

satisfying the following properties:

(∇1) **(Existence)** *If $\nabla_G\text{-}\deg(\nabla\varphi,\Omega) \neq 0$, i.e.,* (3.18) *contains a non-zero coefficient n_{H_i}, then $\exists_{x\in\Omega}$ such that $\nabla\varphi(x) = 0$ and $(G_x) \geq (H_i)$.*

(∇2) (Additivity) *Let Ω_1 and Ω_2 be two disjoint open G-invariant subsets of Ω such that $(\nabla\varphi)^{-1}(0) \cap \Omega \subset \Omega_1 \cup \Omega_2$. Then,*

$$\nabla_G\text{-}\deg(\nabla\varphi, \Omega) = \nabla_G\text{-}\deg(\nabla\varphi, \Omega_1) + \nabla_G\text{-}\deg(\nabla\varphi, \Omega_2).$$

(∇3) (Homotopy) *If $\nabla_v\Psi : [0,1] \times V \to V$ is a G-gradient Ω-admissible homotopy, then*

$$\nabla_G\text{-}\deg(\nabla_v\Psi(t,\cdot), \Omega) = \text{constant}.$$

(∇4) (Normalization) *Let $\varphi \in C^2_G(V, \mathbb{R})$ be a special Ω-Morse function such that $(\nabla\varphi)^{-1}(0) \cap \Omega = G(v_0)$ and $G_{v_0} = H$. Then,*

$$\nabla_G\text{-}\deg(\nabla\varphi, \Omega) = (-1)^{m^-(\nabla^2\varphi(v_0))} \cdot (H),$$

where "$m^-(\cdot)$" stands for the total dimension of eigenspaces for negative eigenvalues of a (symmetric) matrix.

(∇5) (Multiplicativity) *For all $(\nabla\varphi_1, \Omega_1), (\nabla\varphi_2, \Omega_2) \in \mathcal{M}^G_\nabla$,*

$$\nabla_G\text{-}\deg(\nabla\varphi_1 \times \nabla\varphi_2, \Omega_1 \times \Omega_2) = \nabla_G\text{-}\deg(\nabla\varphi_1, \Omega_1) * \nabla_G\text{-}\deg(\nabla\varphi_2, \Omega_2),$$

where the multiplication '$*$' is taken in the Euler ring $U(G)$.

(∇6) (Suspension) *If W is an orthogonal G-representation and $\mathcal{B}$ an open bounded invariant neighborhood of $0 \in W$, then*

$$\nabla_G\text{-}\deg(\nabla\varphi \times \mathrm{Id}_W, \Omega \times \mathcal{B}) = \nabla_G\text{-}\deg(\nabla\varphi, \Omega).$$

(∇7) (Hopf Property) *Assume $B(V)$ is the unit ball of an orthogonal Γ-representation V and for $(\nabla\varphi_1, B(V)), (\nabla\varphi_2, B(V)) \in \mathcal{M}^G_\nabla$, one has*

$$\nabla_G\text{-}\deg(\nabla\varphi_1, B(V)) = \nabla_G\text{-}\deg(\nabla\varphi_2, B(V)).$$

Then, $\nabla\varphi_1$ and $\nabla\varphi_2$ are G-gradient $B(V)$-admissible homotopic.

(∇8) (Functoriality Property) *Let V be an orthogonal G-representation, $f : V \to V$ a G-gradient Ω-admissible map, and $\psi : G_0 \hookrightarrow G$ an embedding of Lie groups (here we assume that G and G_0 have the same dimension). Then, ψ induces a G_0-action on V such that f is an Ω-admissible G_0-gradient map, and the following equality holds*

$$\Psi[\nabla_G\text{-}\deg(f,\Omega)] = \nabla_{G_0}\text{-}\deg(f,\Omega), \tag{3.19}$$

where $\Psi : U(G) \to U(G_0)$ is the homomorphism of Euler rings induced by ψ.

(∇9) (Reduction Property) *Let V be an orthogonal G-representation, $f : V \to V$ a G-gradient Ω-admissible map, then*

$$\pi_0\left[\nabla_G\text{-}\deg(f,\Omega)\right] = G\text{-}\deg(f,\Omega). \tag{3.20}$$

where the ring homomorphism $\pi_0 : U(G) \to A(G)$ is given by (3.6).

Proof. **(Functoriality Property)** (∇8): It is sufficient to assume that f is a special Ω-Morse function such that $(\nabla\varphi)^{-1}(0) \cap \Omega = G(v_0)$ and $G_{v_0} = H$. Then clearly, the orbit $G_0(v_0)$ has the same dimension as $G(v_0)$, which implies that they share the same slice S at the point v_0. Since $(G_0)_{v_0} = G_0 \cap H =: H_0$, it follows that $S^{H_0} \supset S^H$, which implies that f is also special Ω-Morse function with respect to the group action G_0. Therefore,

$$\nabla_{G_0}\text{-}\deg(\nabla\varphi,\Omega) = (-1)^{m^-(\nabla^2\varphi(v_0))} \cdot \Psi(H).$$

$\square$

Remark 3.2.11. Let us point out that the **Functoriality Property** for the gradient degree was *overstated* in [11] (see property (∇8), page 30). This statement is not true for general Lie groups G and G_0. For example, consider $G = S^1$ and $G_0 = \mathbb{Z}_n$, $n \geq 3$, and let $V := \mathbb{C}$ with the action of S^1 and $\mathbb{Z}_3$ by complex multiplication. Then, for $f := -\operatorname{Id} : V \to V$, we have

$$\nabla_{G_0}\text{-}\deg(\nabla\varphi,\Omega) = (\mathbb{Z}_n) - (\mathbb{Z}_1), \quad \nabla_G\text{-}\deg(\nabla\varphi,\Omega) = (S^1) - (\mathbb{Z}_1).$$

On the other hand, $\Psi(S^1) = (\mathbb{Z}_n)$, $\Psi(\mathbb{Z}_1) = 0$, so

$$(\mathbb{Z}_n) = \Psi\left[\nabla_G\text{-}\deg(\nabla\varphi, \Omega)\right] \neq \nabla_{G_0}\text{-}\deg(\nabla\varphi, \Omega).$$

Remark 3.2.12. We would like to point out that the Euler homomorphism described in Functoriality Property ($\nabla 8$) may take a generator (H) to a linear combination (with positive coefficients) of more than one generator in $\Phi(G_0 \times \Gamma)$. Indeed, consider $G_0 := A_4 \times S^1 \leq S_4 \times S^1 =: G$. Then, $\Psi(\mathbb{Z}_3^t) = (\mathbb{Z}_3^{t_1}) + (\mathbb{Z}_3^{t_2})$ (see [12] for the explanation of notation).

Let $\mathscr{H}$ be a Hilbert space and consider a C^1-differentiable G-invariant functional $f : \mathscr{H} \to \mathbb{R}$ given by $f(x) = \frac{1}{2}\|x\|^2 - \varphi(x)$, $x \in \mathscr{H}$. Then,

$$\nabla f(x) = x - \nabla\varphi(x), \quad x \in \mathscr{H}.$$

We say that the functional φ is admissible if $\nabla\varphi : \mathscr{H} \to \mathscr{H}$ is a completely continuous map. Suppose $\Omega \subset \mathscr{H}$ is a G-invariant bounded open set such that the map $\nabla f : \mathscr{H} \to \mathscr{H}$ is Ω-admissible, i.e.,

$$\forall_{x\in\partial\Omega} \quad \nabla f(x) = x - \nabla\varphi(x) \neq 0.$$

By a *G-equivariant approximation scheme* $\{P_n\}_{n=1}^{\infty}$ in $\mathscr{H}$, we mean a sequence of G-equivariant orthogonal projections $P_n : \mathscr{H} \to \mathscr{H}$, $n = 1, 2, \dots$, such that:

(a) the subspaces $\mathscr{H}^n := P_n(\mathscr{H})$, $n = 1, 2, \dots$, are finite dimensional;

(b) $\mathscr{H}^n \subset \mathscr{H}^{n+1}$, $n = 0, 1, 2, \dots$;

(c) $\lim\limits_{n\to\infty} P_n x = x$ for all $x \in \mathscr{H}$.

For a G-equivariant C^1-differentiable admissible functional $\varphi : \mathscr{H} \to \mathbb{R}$ and $\Omega \subset \mathscr{H}$ – a G-invariant bounded open set such that $\nabla\varphi$ is Ω-admissible completely continuous field, the G-equivariant gradient degree $\nabla_G\text{-}\deg(\nabla\varphi, \Omega)$ is defined by

$$\nabla_G\text{-}\deg(\nabla\varphi, \Omega) := \nabla_G\text{-}\deg(\nabla\varphi_n, \Omega_n), \tag{3.21}$$

where $\varphi_n = \varphi|_{\mathscr{H}_n}$ and $\Omega_n := \Omega \cap \mathscr{H}_n$, for n being sufficiently large integer. One can easily verify that this definition does not depend on the choice of a G-approximation scheme in the space $\mathscr{H}$. We should mention that the ideas behind the usage of the approximation methods to define topological degree are rooted to [29].

The G-equivariant gradient degree defined by (3.21) has all the standard properties of a topological degree, i.e., existence, additivity, homotopy and multiplicativity.

3.2.8 Degree on the Slice

Suppose that the orbit $G(u_o)$ of $u_o \in \mathscr{H}$ is contained in a finite-dimensional G-invariant subspace, so the G-action on that subspace is smooth and $G(u_o)$ is a smooth submanifold of $\mathscr{H}$. Denote by $S_o \subset \mathscr{H}$ the slice to the orbit $G(u_o)$ at u_o. Denote by $V_o := T_{u_o}G(u_o)$ the tangent space to $G(u_o)$ at u_o. Then clearly, $S_o = V_o^{\perp}$ and S_o is a smooth Hilbert G_{u_o}-representation. Then we have (see [10]):

Theorem 3.2.13 (Slice Principle)**.** *Let $\mathscr{E}$ be an orthogonal G-representation, $\varphi : \mathscr{H} \to \mathbb{R}$ be a continuously differentiable G-invariant functional, $u_o \in \mathscr{H}$ and $G(u_o)$ be an isolated critical orbit of φ such that*

$$\Gamma := G_{u_o}.$$

Let S_o be the slice to the orbit $G(u_o)$ and $\mathcal{U}$ an isolated tubular neighborhood of $G(u_o)$. Put $\varphi_o : S_o \to \mathbb{R}$ by $\varphi_o(v) := \varphi(u_o + v)$, $v \in S_o$. Then,

$$\nabla_G\text{-}\deg(\nabla\varphi, \mathcal{U}) = \Theta(\nabla_G\text{-}\deg(\nabla\varphi_o, \mathcal{U} \cap S_o)), \tag{3.22}$$

where $\Theta : U(\Gamma) \to U(G)$ is defined on generators $\Theta(H) = (H)$, $(H) \in \Phi(\Gamma)$.

We show how to compute $\nabla_\Gamma\text{-}\deg(\mathscr{A}, B(V))$, where $\mathscr{A} : V \to V$ is a symmetric Γ-equivariant linear isomorphism and V is an orthogonal G-representation, i.e., $\mathscr{A} = \nabla\varphi$ for

$\varphi(v) = \frac{1}{2}(\mathscr{A} v \bullet v)$, $v \in V$, where "$\bullet$" stands for the inner product. Consider the G-isotypical decomposition of V and put

$$\mathscr{A}_i := \mathscr{A}|_{V_i} : V_i \to V_i, \quad i = 0, 1, \dots, r.$$

Then, by the multiplicativity property,

$$\nabla_\Gamma\text{-deg}(\mathscr{A}, B(V)) = \prod_i^r \nabla_\Gamma\text{-deg}(\mathscr{A}_i, B(V_i)) \tag{3.23}$$

Take $\xi \in \sigma_-(\mathscr{A})$, where $\sigma_-(\mathscr{A})$ stands for the negative spectrum of $\mathscr{A}$, and consider the corresponding eigenspace $E(\xi) := \ker(\mathscr{A} - \xi \mathrm{Id})$. Define the numbers $m_i(\xi)$ by

$$m_i(\xi) := \dim\left(E(\xi) \cap V_i\right) / \dim \mathcal{V}_i, \tag{3.24}$$

and the so-called basic gradient degrees by

$$\deg_{\mathcal{V}_i} := \nabla_\Gamma\text{-deg}(-\mathrm{Id}\,, B(\mathcal{V}_i)). \tag{3.25}$$

Then have that

$$\nabla_\Gamma\text{-deg}(\mathscr{A}, B(V)) = \prod_{\xi \in \sigma_-(\mathscr{A})} \prod_{i=0}^r \left(\deg_{\mathcal{V}_i}\right)^{m_i(\xi)}. \tag{3.26}$$

3.2.9 Computations of the Gradient G-Equivariant Degree

The gradient equivariant degree can be computed using standard linearization techniques.

Let V be an orthogonal G-representation and $A : V \to V$ a G-equivariant symmetric isomorphism of V. Notice that $A := \nabla\varphi$, where $\varphi(x) = \frac{1}{2}Ax \bullet x$. Consider the G-isotypical decomposition of V

$$V = \bigoplus_i V_i, \quad V_i \text{ modeled on } \mathcal{V}_i.$$

where we assume that $\{\mathcal{V}_i\}_i$ is a complete list of irreducible G-representations.

Let $\sigma(A)$ denote the spectrum of A and $\sigma_-(A) := \{\lambda \in \sigma(A) : \lambda < 0\}$, and let $E_\mu(A)$ stands for the eigenspace of A corresponding to $\mu \in \sigma(A)$. Let Ω be a G-invariant bounded neighborhood of the origin 0. Clearly, A is Ω-admissibly homotopic (in the class of gradient maps) to the linear operator $A_o : V \to V$ given by

$$A_o(v) := \begin{cases} -v, & \text{if } v \in E_\mu(A),\ \mu \in \sigma_-(A), \\ v, & \text{if } v \in E_\mu(A),\ \mu \in \sigma(A) \setminus \sigma_-(A), \end{cases}$$

i.e., $A_o|_{E_\mu(A)} = -\operatorname{Id}$ for $\mu \in \sigma_-(A)$ and $A_o|_{E_\mu(A)} = \operatorname{Id}$ for $\mu \in \sigma(A) \setminus \sigma_-(A)$. For $\mu \in \sigma_-(A)$ we define

$$m_i(\mu) := \dim(E_\mu(A) \cap V_i) / \dim \mathcal{V}_i.$$

The integer $m_i(\mu)$ is called the *$\mathcal{V}_i$-multiplicity* of μ. Since $\nabla_G\text{-}\deg(\operatorname{Id}, \mathcal{V}_i) = (G)$ is the unit element in $U(G)$, we obtain, by Multiplicativity property ($\nabla 5$), the following formula

$$\nabla_G\text{-}\deg(A, \Omega) = \prod_{\mu \in \sigma_-(A)} \prod_i \left[\nabla_G\text{-}\deg(-\operatorname{Id}, B(\mathcal{V}_i))\right]^{m_i(\mu)}, \tag{3.27}$$

where $B(W)$ is the unit ball in W.

3.2.10 Basic Degrees

Definition 3.2.14. Assume that $\mathcal{V}_i$ is an irreducible G-representation. Then, the G-equivariant gradient degree:

$$\operatorname{Deg}_{\mathcal{V}_i} := \nabla_G\text{-}\deg(-\operatorname{Id}, B(\mathcal{V}_i)) \in U(G)$$

is called the *gradient G-equivariant basic degree* for $\mathcal{V}_i$.

Similarly, we have

Definition 3.2.15. Assume that $\mathcal{V}_i$ is an irreducible G-representation. Then, the G-equivariant Brouwer degree:

$$\deg_{\mathcal{V}_i} := G\text{-}\deg(-\operatorname{Id}, B(\mathcal{V}_i)) \in U(G)$$

is called the *Brouwer G-equivariant basic degree* for $\mathcal{V}_i$.

By a simple application of the Reduction Property (see Theorem 3.2.10), we have

Proposition 3.2.16. *For any irreducible G-representation $\mathcal{V}_i$ we have*

$$\pi_0\left(\mathrm{Deg}_{\mathcal{V}_i}\right) = \deg_{\mathcal{V}_i},$$

where $\pi_0 : U(G) \to A(G)$ is the projection on $A(G)$ defined by (3.6)*.*

We have also the following simple observation (see, for example, [12]).

Proposition 3.2.17. *For any irreducible G-representation $\mathcal{V}_i$, the Brouwer basic degrees $\deg_{\mathcal{V}_i}$ satisfies $\left(\deg_{\mathcal{V}_i}\right)^2 = (G)$.*

Proposition 3.2.18. *For any irreducible G-representation $\mathcal{V}_i$, the gradient basic degree $\mathrm{Deg}_{\mathcal{V}_i}$ is an invertible elements in $U(G)$.*

Proof. Notice that

$$\mathrm{Deg}_{\mathcal{V}_i} = \deg_{\mathcal{V}_i} + a,$$

where $\deg_{\mathcal{V}_i}$ is the Brouwer G-equivariant basic degree and

$$a := \sum_k n_k(L_k), \quad \dim W(L_k) \geq 1.$$

Since $(\deg_{\mathcal{V}_i})^2 = (G)$, we have

$$(\mathrm{Deg}_{\mathcal{V}_i})^2 = (G) + 2\deg_{\mathcal{V}_i} * a + a^2.$$

Clearly,

$$b := 2\deg_{\mathcal{V}_i} * a + a^2 = \sum_l m_l(H_l), \quad \dim W(H_l) \geq 1,$$

and

$$((G)+b)((G)-b) = (G) - b^2, \quad ((G)+b^2)((G)-b^2) = (G) - b^4, \quad (G) - b^{2^k})((G)+b^{2^k}) = (G) - b^{2^{k+}}$$

One can easily notice that, by Proposition 3.2.2, for every l, there exists $N_l > 1$ such that $(H_l)^{N_l} = 0$, so we can put $n := \max\{N_l\}$. Notice that $(m_l(H_l) + m_{l'}(H_{l'}))^{2n} = 0$. Thus, by induction, for a sufficiently large m, $b^{mn} = 0$, so the invertibility of $\mathrm{Deg}_{\mathcal{V}_i}$ follows. $\square$

Remark 3.2.19. For $\widetilde{G} = \Gamma \times S^1$, where Γ is a finite group, the computation of basic gradient degrees $\widetilde{\mathrm{Deg}}_{\mathcal{V}_o}$ (here $\mathcal{V}_o$ stands for an irreducible $\Gamma \times S^1$-representation) can be completely reduced to the computation of basic degrees $\deg_{\mathcal{V}_i}$ without parameter and twisted basic degrees $\deg_{\mathcal{V}_{j,l}}$ with one free parameter. Namely, we have the following types of irreducible $\widetilde{G}$-representations: (i) irreducible Γ-representation $\mathcal{V}_i$, where S^1 acts trivially; (ii) the representations $\mathcal{V}_{j,l}$, $l \in \mathbb{N}$, where $\mathcal{V}_{j,l}$ as a Γ-representation is equivalent to a complex irreducible Γ-representation $\mathcal{U}_j$, and the S^1 action on $\mathcal{V}_{j,l}$ is given by *l-folding*, i.e., $e^{i\alpha}v := e^{il\alpha} \cdot v$, $v \in \mathcal{V}_{j,l}$, $e^{i\alpha} \in S^1$ and '$\cdot$' stands for complex multiplication. Then:

(i) $\widetilde{\mathrm{Deg}}_{\mathcal{V}_i} = \widetilde{\deg}_{\mathcal{V}_i}$;

(ii) $\widetilde{\mathrm{Deg}}_{\mathcal{V}_{j,l}} = (\widetilde{G}) - \widetilde{\deg}_{\mathcal{V}_{j,l}}$,

where $\widetilde{\deg}_{\mathcal{V}_i} = \widetilde{G}\text{-}\deg(-\mathrm{Id}, B(\mathcal{V}_i)) \in A(\widetilde{G})$ and $\deg_{\mathcal{V}_{j,l}}$ is the so-called *twisted basic degree* (see [12] for more details and definitions). More precisely, the basic degree

$$\widetilde{\deg}_{\mathcal{V}_i} = (\widetilde{G}) + n_{L_1}(L_1) + \cdots + n_{L_n}(L_n),$$

can be computed from the recurrence formula

$$n_{L_k} = \frac{(-1)^{n_k} - \sum_{L_k < L_l} n(L_k, L_k) \cdot n_{L_l} \cdot |W(L_l)|}{|W(L_k)|}, \quad n_k = \dim \mathcal{V}_i^{L_k} \tag{3.28}$$

and the twisted degree

$$\widetilde{\deg}_{\mathcal{V}_{j,l}} = n_{H_1}(H_1) + n_{H_2}(H_2) + \cdots + n_{H_m}(H_m),$$

can be computed from the recurrence formula

$$n_{H_k} = \frac{\frac{1}{2}\dim \mathcal{V}_{j,l}^{H_k} - \sum_{H_k < H_s} n_{H_s}\, n(H_k, H_s)\ |W(H_s)/S^1|}{\left|\frac{W(H_k)}{S^1}\right|}. \tag{3.29}$$

One can also find in [12] complete lists of these basic degrees for several groups.

The case $G := \Gamma \times O(2)$ with Γ being a finite group Consider a symmetric G-equivariant linear isomorphism $T : V \to V$, where V is an orthogonal G-representation, i.e., $T = \nabla\varphi$ for $\varphi(v) = \frac{1}{2}(Tv \bullet T)$, $v \in V$, where "$\bullet$" stands for the inner product. We will show how to compute $\nabla_G\text{-}\deg(T, B(V))$.

Let $\{\mathcal{V}_i\}_{i=0}^r$ denote the collection of all real Γ-irreducible representations and

$$\left\{\mathcal{U}_0, \mathcal{U}_{\frac{1}{2}}, \mathcal{U}_1, \dots, \mathcal{U}_l, \dots\right\}$$

denote the collection of all real irreducible $O(2)$-representations, where $\mathcal{U}_0 \simeq \mathbb{R}$ is the trivial representation of $O(2)$, $\mathcal{U}_{\frac{1}{2}} \simeq \mathbb{R}$ is the one-dimensional irreducible real representation on which $O(2)$ acts through the homomorphism $O(2) \to O(2)/SO(2) \simeq \mathbb{Z}_2$, and $\mathcal{U}_l \simeq \mathbb{C}$, $l \in \mathbb{N}$, is the two-dimensional irreducible real representation of $O(2)$ with the action of $O(2)$ given by

$$e^{i\theta}z := e^{il\theta} \cdot z, \quad e^{i\theta} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \in SO(2),$$
$$\kappa z := \overline{z}, \quad \kappa = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},$$

where '$\cdot$' denotes complex multiplication (see also [12]).

There are three types of irreducible G-representations: (i) irreducible G-representations on which $O(2)$ acts trivially, i.e., they are in fact irreducible Γ-representations (there is no $O(2)$-action), which we will denote by $\mathcal{V}_i$, $i = 0, 1, 2, \dots, r$, (ii) irreducible G-representations on which $SO(2)$ acts trivially and $SO(2)\kappa$ acts by multiplying -1, which we will denote by $\mathcal{V}_i^-$, $i = 0, 1, 2, \dots, r$; finally, (iii) irreducible G-representations on which $S^1 = SO(2)$ acts non-trivially. If $\mathcal{V}$ is one of such representations, then it admits a complex structure induced by the action of S^1 and there exists $l \in \mathbb{N}$ such that $zv = z^l \cdot v$ for all $z \in S^1$, $v \in \mathcal{V}$ ('$\cdot$' denotes the complex multiplication). One can show that there exists an irreducible Γ-representation

$\mathcal{V}_i$ such that $\mathcal{V} = \mathcal{V}_i \otimes_{\mathbb{R}} \mathcal{U}_l$. In such a case, we will denote $\mathcal{V}$ by $\mathcal{V}_{i,l}$, $i = 0, 1, 2, \dots, r$ and $l = 1, 2, \dots$. Note that $\mathcal{V}_{i,l}$ can be also described as a Γ-representation: $\mathcal{V}_{i,l} = \mathcal{V}_i \oplus \mathcal{V}_i$ and with $z \in \mathcal{V}_{i,l}$ written as $z = (x, y)^T$, $x, y \in \mathcal{V}_i$, the S^1-action is given by

$$e^{i\alpha} z := \begin{bmatrix} \cos(l\alpha)\ \mathrm{Id}_{\mathcal{V}_i} & -\sin(l\alpha)\ \mathrm{Id}_{\mathcal{V}_i} \\ \sin(l\alpha)\ \mathrm{Id}_{\mathcal{V}_i} & \cos(l\alpha)\ \mathrm{Id}_{\mathcal{V}_i} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

and $\kappa z = \kappa(x, y)^T = (y, x)^T$, for $(x, y)^T \in \mathcal{V}_{i,l}$.

The algorithm for computing the basic gradient degrees $\mathrm{Deg}_{\mathcal{V}_i}$, $\mathrm{Deg}_{\mathcal{V}_i^-}$ and $\mathrm{Deg}_{\mathcal{V}_{i,l}}$ was introduced in [36]. Notice that $\mathrm{Deg}_{\mathcal{V}_i}$ and $\mathrm{Deg}_{\mathcal{V}_i^-}$ belong to $A(G)$, therefore they can be computed using the recurrence formula (3.13). In the case of the gradient basic degree $\mathrm{Deg}_{\mathcal{V}_{i,l}}$, consider two elements $\mathrm{Deg}^0_{\mathcal{V}_{i,l}} \in A(G)$ and $\mathrm{Deg}^1_{\mathcal{V}_{i,l}} \in \mathbb{Z}[\Phi_1(G)]$ such that

$$\mathrm{Deg}_{\mathcal{V}_{i,l}} = \mathrm{Deg}^0_{\mathcal{V}_{i,l}} + \mathrm{Deg}^1_{\mathcal{V}_{i,l}}. \tag{3.30}$$

Then

$$\mathrm{Deg}^0_{\mathcal{V}_{i,l}} = \pi_0[\mathrm{Deg}_{\mathcal{V}_{i,l}}] = n_1(L_1) + \cdots + n_m(L_m), \tag{3.31}$$

where the coefficients n_j, $j = 1, 2, \dots, m$ can be effectively computed from the recurrence formula (3.13). Since the coefficients x_j in

$$\mathrm{Deg}^1_{\mathcal{V}_{i,l}} = x_1(H_1) + \cdots + x_s(H_s) \tag{3.32}$$

are unknown, by Functoriality Property, we can apply the Euler ring homomorphisms $\Psi : U(\Gamma \times O(2)) \to U(\Gamma \times S^1)$. More precisely, since $\mathcal{V}_{i,l}$ is also an irreducible $(\Gamma \times S^1)$-representation, we have

$$\Psi[\mathrm{Deg}_{\mathcal{V}_{i,l}}] = \widetilde{\mathrm{Deg}}_{\mathcal{V}_{i,l}},$$

where $\widetilde{\mathrm{Deg}}_{\mathcal{V}_{i,l}}$ denotes the corresponding gradient basic degree for $\widetilde{G}$. Then, by applying Ψ to (3.30), (3.31) and (3.32), we obtain

$$2x_1(H_1) + \cdots + 2x_s(H_s) = \widetilde{\mathrm{Deg}}_{\mathcal{V}_{i,l}} - n_1\Psi(L_1) + \cdots + n_m\Psi(L_m), \tag{3.33}$$

Table 3.3. Conjugacy classes of elements in A_5

$\mathcal{C}_1$	$\mathcal{C}_2$	$\mathcal{C}_3$	$\mathcal{C}_4$	$\mathcal{C}_5$
(1)	(12)(34), (13)(24), (14)(23)	(123), (132)	(12345)	(12354)
	(12)(35), (13)(25), (15)(23)	(124), (142)	(12453)	(12435)
	(12)(45), (14)(25), (15)(24)	(125), (152)	(12534)	(12543)
	(13)(45), (14)(35), (15)(34)	(134), (143)	(13254)	(13245)
	(23)(45), (24)(35), (25)(34)	(135), (153)	(13542)	(13524)
		(145), (154)	(13425)	(13452)
		(234), (243)	(14235)	(14253)
		(235), (253)	(14352)	(14325)
		(245), (254)	(14523)	(14532)
		(345), (354)	(15243)	(15234)
			(15432)	(15423)
			(15324)	(15342)

which is a system of simple linear equations and can be easily solved. In summary, we have the following result:

Theorem 3.2.20 ([36]). *For a given group $G := \Gamma \times O(2)$, all the basic degrees can be effectively computed by applying Euler ring homomorphism $\Psi : U(\Gamma \times O(2)) \to U(\Gamma \times S^1)$ and the gradient basic degrees $\widetilde{\mathrm{Deg}}_{\mathcal{V}_o}$ for the group $\widetilde{G} := \Gamma \times S^1$.*

3.3 Fullerene Model

3.3.1 Equations for Carbons

The 5 conjugacy classes in A_5 are listed in Table 3.3. The C_{60} molecule is arranged in 12 unconnected pentagons of atoms. We implement the following notation for the indices of the 60 atoms, see Figure 1:

- $\tau \in \mathcal{C}_4$ is used to denote each of the 12 pentagonal faces.
- $k \in \{1, ..., 5\} =: \mathbb{Z}[1, 5]$ is used to denote each of the 5 vertices in the 12 pentagonal faces.

We define the set of indices as

$$\Lambda = \mathcal{C}_4 \times : \mathbb{Z}[1,5].$$

With this notations each index $(\tau, k) \in \Lambda$ represents a face and the vertex in the face of the truncated icosahedron, Figure 3.1. The vectors that represent the positions of the carbon atoms are

$$u_{\tau,k} \in \mathbb{R}^3,$$

and the vector for the 60 positions of carbons is

$$u = (u_{\tau,k})_{(\tau,k)\in\Lambda} \in \left(\mathbb{R}^3\right)^{60}.$$

The space $(\mathbb{R}^3)^{60}$ is a representation of the group $I \times O(3)$, where the icosahedral group of symmetries correspond to the group $I = A_5 \times \mathbb{Z}_2$, where A_5 is the alternating group of permutations of five elements $\{1,2,3,4,5\}$. With this notation, the action of $I \times O(3)$ on V has a simple definition: the action of $\sigma \in A_5$ and $-1 \in \mathbb{Z}_2$ in u is defined in each component by

$$\rho(\sigma) u_{\tau,k} = u_{\sigma^{-1}\tau\sigma, \sigma^{-1}(k)}, \qquad \rho(-1) u_{\tau,k} := u_{\tau^{-1},k}. \tag{3.34}$$

And the action of the group $A \in O(3)$ is defined by

$$Au = (Au_{\tau,k})_{(\tau,k)\in\Lambda}.$$

3.3.2 Force Field

The equations of motion is

$$m\ddot{u} = -\nabla V(u) \ , \tag{3.35}$$

where $u(t)$ represents the positions of carbon atoms in space, m is the carbon mass and $V(u)$ is the energy given by a force field. In order to define the force field, we define the following:

- The 60 edges in the 12 pentagonal faces represent single bonds. For these single bonds we define the function $S : \Lambda \to \Lambda$,

$$S(\tau, k) = (\tau, \tau(k)) \ , \quad \tau \in \mathcal{C}_4, \ k \in \mathbb{Z}[1,5]$$

- The 30 remaining edges in the hexagon, which connect the different pentagonal faces, represent double bounds. For these double bonds we define the function $D : \Lambda \to \Lambda$,

$$D(\tau, k) = (\sigma, k) \text{ with } \sigma = \big(k, \tau^2(k), \tau(k), \tau^4(k), \tau^3(k)\big) \ .$$

Using the previous notation, the force field energy is elegantly expressed by

$$V(u) = \sum_{(\tau,k)\in\Lambda} \left(U(\big|u_{\tau,k} - u_{S(\tau,k)}\big|) + \frac{1}{2} U(\big|u_{\tau,k} - u_{D(\tau,k)}\big|) + U_{(\tau,k)}(u) \right) ,$$

the coefficient $\frac{1}{2}$ standing before the second term is due to our double count of the same double bonds, and the term $U_{(\tau,k)}(u)$ will include bending bonds and torsion forces.

The bond stretching is represent by the potential

$$U(x) = E_0 \left(\left(1 - e^{-\beta(x-r_0)}\right)^2 - 1 \right) ,$$

where r_0 is the equilibrium bond length, E_0 is the bond energy and β^{-1} is the width of the energy. The term $U_{(\tau,k)}(u)$ includes bending bonds and torsion forces given by

$$U_{(\tau,k)}(u) = E_B(\theta_1) + E_B(\theta_2) + E_B(\theta_3) + E_T(\phi_1) + E_T(\phi_2) + E_T(\phi_3),$$

where these terms are given as follows:

The bond bending $E_B(\theta)$ around each atom in a molecule is governed by the hybridization of orbitals around which is given by

$$E_B(\theta) = \frac{1}{2} k_0 (\cos\theta - \cos\theta_0)^2 = \frac{1}{2} k_\theta (\cos\theta + 1/2)^2,$$

where $\theta_0 = 2\pi/3$ is the equilibrium bond angle and k_0 is the bending force constant. Each carbon $(\tau, k) \in \Lambda$ has three angles,

$$\cos\theta_1 = \frac{u_{\tau,k} - u_{S(\tau,k)}}{\left|u_{\tau,k} - u_{S(\tau,k)}\right|} \cdot \frac{u_{\tau,k} - u_{S^{-1}(\tau,k)}}{\left|u_{\tau,k} - u_{S^{-1}(\tau,k)}\right|},$$
$$\cos\theta_2 = \frac{u_{\tau,k} - u_{D(\tau,k)}}{\left|u_{\tau,k} - u_{D(\tau,k)}\right|} \cdot \frac{u_{\tau,k} - u_{S(\tau,k)}}{\left|u_{\tau,k} - u_{S(\tau,k)}\right|},$$
$$\cos\theta_3 = \frac{u_{\tau,k} - u_{D(\tau,k)}}{\left|u_{\tau,k} - u_{D(\tau,k)}\right|} \cdot \frac{u_{\tau,k} - u_{S^{-1}(\tau,k)}}{\left|u_{\tau,k} - u_{S^{-1}(\tau,k)}\right|},$$

then we have that the bond bending at each atom $(\tau, k) \in \Lambda$ is $E_B(\theta_1) + E_B(\theta_2) + E_B(\theta_3)$.

The torsion energy $E_T(\phi)$ describes the energy change associated with rotation around a bond with a four-atom sequence, which is given by

$$E_T(\phi) = \frac{1}{2} k_\phi \left(1 - \cos 2\phi\right) = k_\phi \left(1 - \cos^2 \phi\right).$$

The torsion energy takes a maximum value at the angles $\phi = \pm\pi/2$. Let

$$n = \frac{u_{D(\tau,k)} - u_{S(\tau,k)}}{\left|u_{D(\tau,k)} - u_{S(\tau,k)}\right|} \times \frac{u_{D(\tau,k)} - u_{S^{-1}(\tau,k)}}{\left|u_{D(\tau,k)} - u_{S^{-1}(\tau,k)}\right|}$$

be the unitary norm to the plane passing by $u_{D(\tau,k)}$, $u_{S(\tau,k)}$ and $u_{S^{-1}(\tau,k)}$. Each carbon $(\tau, k) \in \Lambda$ has three torsion energies given by

$$\cos\phi_1 = n \cdot n_1 , \qquad n_1 = \frac{u_{(\tau,k)} - u_{S(\tau,k)}}{\left|u_{(\tau,k)} - u_{S(\tau,k)}\right|} \times \frac{u_{(\tau,k)} - u_{S^{-1}(\tau,k)}}{\left|u_{(\tau,k)} - u_{S^{-1}(\tau,k)}\right|},$$
$$\cos\phi_2 = n \cdot n_2 , \qquad n_2 = \frac{u_{(\tau,k)} - u_{D(\tau,k)}}{\left|u_{(\tau,k)} - u_{D(\tau,k)}\right|} \times \frac{u_{(\tau,k)} - u_{S^{-1}(\tau,k)}}{\left|u_{(\tau,k)} - u_{S^{-1}(\tau,k)}\right|},$$
$$\cos\phi_3 = n \cdot n_3 , \qquad n_3 = \frac{u_{(\tau,k)} - u_{D(\tau,k)}}{\left|u_{(\tau,k)} - u_{D(\tau,k)}\right|} \times \frac{u_{(\tau,k)} - u_{S(\tau,k)}}{\left|u_{(\tau,k)} - u_{S(\tau,k)}\right|}.$$

Then, the bond bending at each atom $(\tau, k) \in \Lambda$ is $E_T(\phi_1) + E_T(\phi_2) + E_T(\phi_3)$.

The quantity value of the force field in [20] are $E_0 = 6.1322\ eV$, $\beta = 1.8502\ A^{-1}$, $r_0 = 01.4322\ A$, $k_\theta = 10\ eV$, $k_\phi = 0.346\ eV$.

3.3.3 Icosahedral Symmetries

Due to the definition of the potential energy V, it is invariant with respect to the action of $c \in \mathbb{R}^3$ in $(\mathbb{R}^3)^{60}$ by shifting, $V(u+c) = V(u)$. Therefore in order to make the system 3.37 reference point-depended, we define the subspace

$$\mathscr{V} := \{u \in \left(\mathbb{R}^3\right)^{60} : \sum_{(\sigma,k)\in\Lambda} u_{\sigma,k} = 0\} \tag{3.36}$$

and

$$\Omega_o = \{u \in \mathscr{V} : u_{\sigma,k} \neq u_{\tau,j}\}.$$

We have that $\mathscr{V} = \{u \in \mathbb{V} : u \perp (v, ...v)\}$ and Ω_o are flow-invariant for (3.37).

By the properties of the functions S and D, the potential

$$V : \Omega_o \to \mathbb{R} \tag{3.37}$$

is well defined and I-invariant. Moreover, the potential V is invariant by rotations and reflections of the group $O(3)$ because the bonding, bending and torsion forces depend only on the norm of the distances among pairs of atoms. Therefore, the potential V is $I \times O(3)$-invariant,

$$V((\sigma, A)u) = V(u), \qquad (\sigma, A) \in I \times O(3)\ .$$

Let

$$J_1 := \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}, \quad J_2 := \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad J_3 := \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

be the three infinitesimal generator of the rotations in $O(3)$, i.e.,

$$R_x = e^{\phi J_1},\ R_y = e^{\theta J_2},\ R_z = e^{J_3 \psi}\ ,$$

where ϕ, θ and ψ are the Euler angles. The angle among the center of two adjacent pentagons in a dodecahedron is $\theta_0 = \arctan 2$. Then, the rotation by π that fixes a pair of edges is

$$A = e^{-(\theta_0/2)J_2} e^{\pi J_3} e^{(\theta_0/2)J_2} = \begin{bmatrix} -\frac{1}{\sqrt{5}} & 0 & \frac{2}{\sqrt{5}} \\ 0 & -1 & 0 \\ \frac{2}{\sqrt{5}} & 0 & \frac{1}{\sqrt{5}} \end{bmatrix}. \tag{3.38}$$

The rotation over the upper pentagonal face of a dodecahedron is

$$B = e^{\frac{2\pi}{5}J_3} = \begin{bmatrix} \frac{-1+\sqrt{5}}{4} & -\sqrt{\frac{5+\sqrt{5}}{8}} & 0 \\ \sqrt{\frac{5+\sqrt{5}}{8}} & \frac{-1+\sqrt{5}}{4} & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{3.39}$$

We have that the subgroup of $O(3)$ generated by the matrices A and B is isomorphic to the icosahedral group, because it has the presentation

$$\{A, B : A^2 = B^5 = (AB)^3 = I\}\ .$$

The group A_5 is generated by

$$a = (23)(45), \qquad b = (12345)\,. \tag{3.40}$$

and has a similar presentation given by

$$\{a, b : a^2 = b^5 = (ab)^3 = I\}.$$

Therefore, the explicit homomorphism $\rho : A_5 \to \mathrm{SO}(3)$ defined by the formulas

$$a \to A, \qquad b \to B, \tag{3.41}$$

is well defined by the equivalent presentations in the generators. The icosahedral group $\rho(A_5) \subset O(3)$ given by contains rotation by $2\pi/5$, rotations by $2\pi/3$ and rotations by π. We extend

$$\rho : A_5 \times \mathbb{Z}_2 \to O(3)$$

with the image of the generator $-1 \in \mathbb{Z}_2$ given by $\rho(-1) = -I \in O(3)$.

Table 3.4. Factorization of elements in A_5 by generators $a = (23)(45)$ and $b = (12345)$

Element	Factorization	Element	Factorization	Element	Factorization
(12)(34)	$b^{-1}ab$	(13)(24)	$bab^{-1}abab^{-1}$	(14)(23)	$ab^{-2}ab^2a$
(12)(35)	$ab^2ab^{-2}ab$	(13)(25)	$ab^{-1}aba$	(15)(23)	$b^{-2}ab^2$
(12)(45)	b^2ab^{-2}	(14)(25)	$abab^{-1}a$	(15)(24)	$ab^{-2}ab^2ab^{-1}$
(13)(45)	$ab^2ab^{-2}a$	(14)(35)	$b^{-1}abab^{-1}ab$	(15)(34)	bab^{-1}
(23)(45)	a	(24)(35)	$b^2ab^{-2}ab^2$	(25)(34)	$ab^2ab^{-2}ab^2$
(123)	$b^2ab^{-2}a$	(132)	ab^2ab^{-2}	(124)	ba
(142)	ab^{-1}	(125)	$bab^{-2}ab$	(152)	$b^{-1}ab^2ab^{-1}$
(134)	$b^{-1}abab^{-1}$	(143)	$bab^{-1}ab$	(135)	ab
(153)	$b^{-1}a$	(145)	$ab^{-2}ab^2$	(154)	$b^{-2}ab^2a$
(234)	$b^{-1}ab^2a$	(243)	$ab^{-2}ab$	(235)	b^2ab^{-1}
(253)	bab^{-2}	(245)	$b^{-1}ab^2$	(254)	$b^{-2}ab$
(345)	ab^2ab^{-1}	(354)	$bab^{-2}a$		
(12345)	b	(12453)	$b^{-1}aba$	(12534)	ab^2
(13254)	aba	(13542)	$ab^{-1}ab$	(13425)	b^2a
(14235)	$abab^{-1}$	(14352)	$b^{-2}a$	(14523)	$ab^{-1}a$
(15243)	ab^{-2}	(15432)	b^{-1}	(15324)	$bab^{-1}a$
(12354)	$ab^{-1}abab^{-1}$	(12435)	ab^2a	(12543)	$b^{-1}ab^2ab^{-2}$
(13245)	$b^{-1}abab^{-1}a$	(13524)	b^2	(13452)	$b^2ab^{-2}ab$
(14253)	b^{-2}	(14325)	$b^{-2}ab^2ab^{-1}$	(14532)	$bab^{-1}aba$
(15234)	$bab^{-2}ab^2$	(15423)	$abab^{-1}ab$	(15342)	$ab^{-2}a$

3.3.4 Icosahedral Minimizer

Let $\tilde{I}$ be the icosahedral subgroup of $I \times O(3)$ given by

$$\tilde{I} = \{(\sigma, \rho(\sigma)) \in I \times O(3) : \sigma \in I\}.$$

The fixed point space of $\tilde{I}$ consist of all the truncated icosahedral configurations. The Fullerene Molecule is a minimizer among these configurations. That is, since V is $I \times O(3)$-invariant, by the Palais criticality principle, the minimizer of the potential V in the fixed point space of $\tilde{I}$,

$$\Omega_0^{\tilde{I}} = \mathscr{V}^{\tilde{I}} \cap \Omega_o = \left\{ (a_{\tau,k})_{(\tau,k)\in\Lambda} \in \Omega_0 : a_{\tau,k} = (\sigma, \rho(\sigma)) a_{\tau,k} \right\},$$

is a critical point of V.

To find the minimizer among the configurations with symmetries $\tilde{I}$, we parameterize the position of the carbons by one of them. Let us chose

$$u_{b,1} = (x, 0, z),$$

since we have

$$u_{b,1} = (\sigma, \rho(\sigma))u_{b,1} = \rho(\sigma)u_{\sigma^{-1}b\sigma,\sigma^{-1}(1)} \ ,$$

then the relations

$$u_{\sigma^{-1}b\sigma,\sigma^{-1}(1)} = \rho(\sigma)^{-1}u_{b,1} = \rho(\sigma)^{-1}(x, 0, z)^T, \tag{3.42}$$

for $\sigma \in A_5$ determine the vector of positions $u(x, z)$.

Therefore, the representation $u(x, z)$ given the relations (3.42) maps

$$\mathcal{C} = \{(x, z) \in \mathbb{R}^2 : 0 < z, \ x < Cz\}$$

in a connected component of $\Omega_0^{\tilde{I}}/O(3)$. We define the restriction of V to $\mathcal{C}$ as

$$\phi(x, z) = V(u(x, z)).$$

Since $\phi(x, z)$ is exactly the restriction of V to the fixed-point subspace $\Omega_0^{\tilde{I}}/O(3)$, then by symmetric criticality condition, a critical point of $\phi : \mathcal{C} \to \mathbb{R}$ is also a critical point of V.

We implement numerically minimizing procedure using Mathematica to find the minimizer (x_o, z_o) of $\phi(x, z)$. We denoted the truncated icosahedral minimizer corresponding to the Fullerene C_{60} as

$$u_o = u\left(x_o, z_o\right) \in \mathscr{V}.$$

The lengths of single and double bonds in this minimizer are given by

$$d_S = |a_{b,1} - a_{b,2}| = 1.438084 \text{ and } d_D = 1.420845,$$

respectively. This results are close to the distances measured in the paper [55].

An advantage of the notation $u = (u_{\tau,k})$ is that it is easy to visualize the elements associated to the rotations $\rho(\sigma)$ in the truncated icosahedron (Figure 1). In these configurations we have $\rho(\sigma)u_{\tau,k} = \sigma^{-1}u_{\tau,k} = u_{\sigma\tau\sigma^{-1},\sigma(k)}$, then $\rho(\sigma)$ is identified with the rotation that sends the face τ into $\sigma\tau\sigma^{-1}$ and the particle identified by k into $\sigma(k)$; for example, the face $b = (12345)$ goes into the face $aba^{-1} = (13254)$ under the π-rotation given by $\rho(a) = A$ and the element 1 into $a(1) = 1$. In this manner, we conclude that the elements of the conjugacy classes $\mathcal{C}_2$, $\mathcal{C}_3$, $\mathcal{C}_4$ and $\mathcal{C}_5$ correspond to the 15 rotation by π, the 20 rotation by $2\pi/3$, the 12 rotation by $2\pi/5$ and the 12 rotation by $\pi/5$, respectively.

3.3.5 Isotypical Decomposition

Since the subspace $\{(v,...,v) \in V : v \in \mathbb{R}^3\}$ is $\tilde{I}$-invariant, we have that $\mathscr{V} = \{u \in \mathbb{V} : u \perp (v,...v)\}$ has the $\tilde{I}$-isotypical decomposition. Since the system (3.37) is symmetric respect the group action of $I \times O(3)$, we have that the orbit of equilibria u_o is a 3-dimensional submanifold in $\mathscr{V}$. To describe the tangent space, we define

$$\mathcal{J}_j u = (J_j u_{\sigma,k}),$$

where J_j are the three infinitesimal generator of the rotation defined in (x.x). Then, the slice S_o to the orbit of u_o is

$$S_o := \{x \in \mathscr{V} : x \bullet \mathcal{J}_j u_0 = 0, \quad j = 1,2,3\}.$$

Since the u_o has isotropy group $\tilde{I}$, then S_o is an orthogonal $\tilde{I}$ representation.

In this subsection, we identify the $\tilde{I}$-isotypical components of S_o. In order to simplify the notation, hereafter we identify $\tilde{I}$ with the group $A_5 \times \mathbb{Z}_2$,

$$\tilde{I} = A_5 \times \mathbb{Z}_2 \ .$$

Set $\varphi_\pm = \frac{1}{2}(1 \pm \sqrt{5})$, where φ_+ is the golden ratio. Let

$$c = (123),$$

Table 3.5. Character table of A_5

Rep.	Character	(1)	a	c	b	b^2
$\mathcal{V}_1$	χ_1	1	1	1	1	1
$\mathcal{V}_2$	χ_2	4	0	1	-1	-1
$\mathcal{V}_3$	χ_3	5	1	-1	0	0
$\mathcal{V}_4$	χ_4	3	-1	0	φ_+	φ_-
$\mathcal{V}_5$	χ_5	3	-1	0	φ_-	φ_+

and as before $a = (23)(45)$ and $b = (12345)$. The character table of A_5 is:

The character table of $\tilde{I} \simeq A_5 \times \mathbb{Z}_2$ is obtained from the table of A_5. We denote the irreducible representations of I by $\mathcal{V}_{\pm n}$ for $j = 1, ..., 5$, where the element $-1 \in \mathbb{Z}_2$ acts as $\pm I$ in $\mathcal{V}_{\pm n}$ and by $\gamma \in A_5$ acts as in $\mathcal{V}_n$. Therefore, the character table has an extension for the elements κ as follows:

Table 3.6. Charater table of $\tilde{I} \simeq A_5 \times \mathbb{Z}_2$

	(1)	a	c	b	b^2	κ	κa	κc	κb	κb^2
$\chi(\mathcal{V}_1)$	1	1	1	1	1	1	1	1	1	1
$\chi(\mathcal{V}_2)$	4	0	1	-1	-1	4	0	1	-1	-1
$\chi(\mathcal{V}_3)$	5	1	-1	0	0	5	1	-1	0	0
$\chi(\mathcal{V}_4)$	3	-1	0	φ_+	φ_-	3	-1	0	φ_+	φ_-
$\chi(\mathcal{V}_5)$	3	-1	0	φ_-	φ_+	3	-1	0	φ_-	φ_+
$\chi(\mathcal{V}_{-1})$	1	1	1	1	1	-1	-1	-1	-1	-1
$\chi(\mathcal{V}_{-2})$	4	0	1	-1	-1	-4	0	-1	1	1
$\chi(\mathcal{V}_{-3})$	5	1	-1	0	0	-5	-1	1	0	0
$\chi(\mathcal{V}_{-4})$	3	-1	0	φ_+	φ_-	-3	1	0	$-\varphi_+$	$-\varphi_-$
$\chi(\mathcal{V}_{-5})$	3	-1	0	φ_-	φ_+	-3	1	0	$-\varphi_-$	$-\varphi_+$

We computed numerically the spectrum μ_j of the Hessian $D^2V(u_o)$ at this minimizer u_o. Since $D^2V(u_o) : \mathscr{V} \to \mathscr{V}$ is $\tilde{I}$-equivariant, we have for the irreducible representations $\mathcal{V}_n$ that

$$D^2V(u_o)|_{\mathcal{V}_n} = \mu_j \,\mathrm{Id} : \mathcal{V}_n \to \mathcal{V}_n.$$

In order to determine the absolutely irreducible representation $\mathcal{V}_n$ corresponding to the eigenvalue μ_j, we apply the isotypical projections

$$P_{\mathcal{V}_n} v = \frac{\dim(\mathcal{V}_n)}{120} \sum_{g \in \tilde{I}} \chi_{\mathcal{V}_n}(g)\ gv, \quad v \in V,$$

where $\mathcal{V}$ is the irreducible representation identified by (Table 3.6).

We identify the representation numerically using the eigenvectors and the projections $P_{\mathcal{V}_n} : \mathscr{V} \to \mathcal{V}_n$. We find that the number of irreducible representations is 48. In the slice S_o, we have that $\sigma(D^2 V(u_o)|_{S_o}) = \{\mu_1, ..., \mu_{46}\}$ with $\mu_j > 0$ and the $\tilde{I}$-irreducible representations decomposition is

$$S_o = \bigoplus_{j=1}^{46} \mathcal{V}_{n_j}. \tag{3.43}$$

In the Table 3.7, we show the number $n_j \in \{-5, ..., -1, 1, ..., 5\}$ that identifies the irreducible representation corresponding to the eigenvalue μ_j for $j = 1, .., 46$, i.e.,

$$D^2 V(u_o)|_{\mathcal{V}_{n_j}} = \mu_j \ \mathrm{Id}\,.$$

Remark 3.3.1. The model [107] (or [20]) consider van der Waals forces among carbon atoms, which is given by the potential

$$W(x) = \varepsilon \left(\frac{\sigma^{12}}{x^{12}} - 2\frac{\sigma^6}{x^6} \right) \ ,$$

where $\sigma = 3.4681\ A^{-1}$ is the minimum energy distance and $\epsilon = 0.0115\ eV$ the depth of this minimum. However, our numerical computation using van der Waals forces do not produce acceptable bond lengths among atoms given in [55] not the spectra given in most papers such as [33]. Actually the models in [112] and [108] do not consider van der Waals forces. Using the models [107] and [20] without van der Waals forces give us results that approximate well the measurements in [55] and [33] for the bond lengths d_S and d_D and the frequencies $\sqrt{\mu_j}$ which are within the range 100 to 1800 cm^{-1}.

Table 3.7. Irreducible representations and corresponding eigenvalues

j	Mult.	μ_j	n_j
1	5	176.536	-3
2	5	176.366	3
3	4	164.083	2
4	4	160.292	-2
5	3	159.290	-5
6	5	148.597	3
7	3	141.071	-4
8	3	140.573	5
9	1	135.632	1
10	4	134.935	-2
11	4	129.544	2
12	5	125.431	-3
13	3	107.719	4
14	5	98.5525	3
15	3	93.4648	-5
16	5	87.7541	-3
17	3	83.9718	-4
18	4	71.6288	2
19	5	67.1181	3
20	1	59.3865	-1
21	3	50.4797	-5
22	4	47.5646	-2
23	3	42.2947	4

j	Mult.	μ_j	n_j
24	3	41.3918	5
25	4	33.9885	-2
26	5	28.8031	-3
27	5	27.4795	3
28	3	27.3153	5
29	4	25.5388	-2
30	4	22.7212	2
31	5	19.4536	-3
32	5	19.3377	3
33	3	19.2379	-5
34	4	16.5356	2
35	3	16.5255	5
36	3	15.1033	-4
37	3	10.3908	4
38	1	10.2520	1
39	5	10.1098	-3
40	3	9.03077	-4
41	4	9.02666	2
42	5	6.99929	3
43	5	6.95354	-3
44	3	5.42311	-5
45	4	5.26429	-2
46	5	3.04384	3

3.4 Equivariant Bifurcation

In what follows, we are interested in finding non-trivial T-periodic solutions to (3.37), bifurcating from the orbit equilibrium points of u_o. By normalizing the period, i.e., by making the substitution $v(t) := u(\lambda t)$ in (3.37), we obtain the following system

$$\begin{cases} \ddot{v} = -\lambda^2 \nabla V(v), \\ v(0) = v(2\pi), \quad \dot{v}(0) = \dot{v}(2\pi), \end{cases} \tag{3.44}$$

where $\lambda^{-1} = 2\pi/T$ is the frequency.

3.4.1 Equivariant Gradient Map

Since $\mathscr{V}$ is an orthogonal $I \times O(3)$- representation, we can consider the the first Sobolev space of 2π-periodic functions from $\mathbb{R}$ to $\mathscr{V}$, i.e.,

$$H^1_{2\pi}(\mathbb{R}, \mathscr{V}) := \{u : \mathbb{R} \to \mathscr{V} \ : \ u(0) = u(2\pi),\ u|_{[0,2\pi]} \in H^1([0, 2\pi]; \mathscr{V})\},$$

equipped with the inner product

$$\langle u, v\rangle := \int_0^{2\pi} (\dot{u}(t) \bullet \dot{v}(t) + u(t) \bullet v(t))dt \ .$$

Let $O(2) = SO(2) \cup \kappa SO(2)$ denote the group of 2×2-orthogonal matrices, where

$$\kappa = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \qquad \begin{bmatrix} \cos\tau & -\sin\tau \\ \sin\tau & \cos\tau \end{bmatrix} \in SO(2) \ .$$

It is convenient to identify a rotation with $e^{i\tau} \in S^1 \subset \mathbb{C}$. Notice that $\kappa e^{i\tau} = e^{-i\tau}\kappa$, i.e., κ as a linear transformation of $\mathbb{C}$ into itself, acts as complex conjugation.

Clearly, the space $H^1_{2\pi}(\mathbb{R}, \mathscr{V})$ is an orthogonal Hilbert representation of

$$G := I \times O(3) \times O(2).$$

Indeed, we have for $u \in H^1_{2\pi}(\mathbb{R}, \mathscr{V})$ and $(\sigma, A) \in I \times O(3)$ (see (3.34))

$$\begin{aligned} (\sigma, A)\, u(t) &= (\sigma, A)u(t), \\ e^{i\tau}u(t) &= u(t + \tau), \\ \kappa u(t) &= u(-t). \end{aligned} \tag{3.45}$$

It is useful to identify a 2π-periodic function $u : \mathbb{R} \to V$ with a function $\widetilde{u} : S^1 \to \mathscr{V}$ via the map $\mathfrak{e}(\tau) = e^{i\tau} : \mathbb{R} \to S^1$. Using this identification, we will write $H^1(S^1, \mathscr{V})$ instead of $H^1_{2\pi}(\mathbb{R}, \mathscr{V})$.

Put

$$\Omega := \{u \in H^1(S^1, \mathscr{V}) : u(t) \in \Omega_o\}.$$

Then, the system (3.44) can be written as the following variational equation

$$\nabla_u J(u, \lambda) = 0, \quad (\lambda, u) \in \mathbb{R} \times \Omega, \tag{3.46}$$

where $J : \Omega \times \mathbb{R} \to \mathbb{R}$ is defined by

$$J(u, \lambda) := \int_0^{2\pi} \left[\frac{1}{2} |\dot{u}(t)|^2 - \lambda^2 V(u(t)) \right] dt. \tag{3.47}$$

Assume that $u_o \in \mathscr{V}$ is the equilibrium point of (3.37) described in previous section. Then clearly, u_o is a critical point of J. We are interested in finding non-stationary 2π-periodic solutions bifurcating from u_o, i.e., non-constant solutions to system (3.46).

We consider the G-orbit of u_o in the space $H^1(S^1, \mathscr{V})$. We denote by $\mathcal{S}_o$ the slice to $G(u_o)$ in $H^1(S^1, \mathscr{V})$. We will also denote by

$$\mathscr{J} : \mathbb{R} \times (\mathcal{S}_o \cap \Omega) \to \mathbb{R},$$

the restriction of J to the set $\mathcal{S}_o \cap \Omega$. Then clearly, $\mathscr{J}$ is G_{u_o}-invariant. Then, by Slice Criticality Principle (see Theorem 3.2.13), critical points of $\mathscr{J}$ are critical points of J and consequently, they are solutions to system (3.46).

Consider the operator $L : H^2(S^1; \mathscr{V}) \to L^2(S^1; \mathscr{V})$, given by $Lu = -\ddot{u} + u$ for $u \in H^2(S^1, \mathscr{V})$. Then the inverse operator L^{-1} exists and is bounded. Put $j : H^2(S^1; \mathscr{V}) \to H^1(S^1, \mathscr{V})$ be the natural embedding operator. Clearly, j is a compact operator. Then, one can easily verify that

$$\nabla_u J(\lambda, u) = u - j \circ L^{-1}(\lambda^2 \nabla V(u) + u), \tag{3.48}$$

where $u \in H^1(S^1, \mathscr{V})$. Consequently, the bifurcation problem (3.46) can be written as $u - j \circ L^{-1}(\lambda^2 \nabla V(u) + u) = 0$. Moreover, we have

$$\nabla_u^2 J(\lambda, u_o) v = v - j \circ L^{-1}(\lambda^2 \nabla^2 V(u_o) v + v) \ , \tag{3.49}$$

where $v \in H^1(S^1, \mathscr{V})$.

Consider the operator

$$\mathscr{A}(\lambda) := \nabla_u^2 J(\lambda, u_o)|_{\mathcal{S}_o} : \mathcal{S}_o \to \mathcal{S}_o. \tag{3.50}$$

Notice that

$$\nabla_u^2 \mathscr{J}(\lambda, u_o) = \mathscr{A}(\lambda),$$

thus, by implicit function theorem, $G(u_o)$ is an isolated orbit of critical points of J, whenever $\mathscr{A}(\lambda)$ is an isomorphism. Therefore, if a point (λ_o, u_o) is a bifurcation point for (3.46), then $\mathscr{A}(\lambda_o)$ cannot be an isomorphism. In such a case, we put

$$\Lambda := \{\lambda > 0 : \mathscr{A}(\lambda_o) \text{ is not an isomorphism}\} ,$$

and will call the set Λ the *critical set* for the trivial solution u_o.

3.4.2 Bifurcation Theorem

Consider the S^1-action on $H^1(S^1, \mathscr{V})$, where S^1 acts on functions by shifting the argument (see (3.45)). Then, $(H^1(S^1, \mathscr{V}))^{S^1}$ is the space of constant functions, which can be identified with the space $\mathscr{V}$, i.e.,

$$H^1(S^1, \mathscr{V}) = \mathscr{V} \oplus \mathscr{W}, \quad \mathscr{W} := \mathscr{V}^{\perp}.$$

Then, the slice $\mathcal{S}_o$ in $H^1(S^1, \mathscr{V})$ to the orbit $G(u_o)$ at u_o is exactly

$$\mathcal{S}_o = S_o \oplus \mathscr{W}.$$

Any $\lambda_o \in \Lambda$ satisfies the condition that $\mathscr{A}(\lambda_o)|_{S_o} : S_o \to S_o$ is an isomorphism, since the eigenvalues are $\mu_j \neq 0$ for $j = 1, ..., 46$, under this condition we have:

Theorem 3.4.1. *Consider the bifurcation system* (3.46) *and assume that* $\lambda_o \in \Lambda$ *is isolated in the critical critical set* Λ*, i.e., there exists* $\lambda_- < \lambda_o < \lambda_+$ *such that* $[\lambda_-, \lambda_+] \cap \Lambda = \{\lambda_o\}$*. Define*

$$\omega_G(\lambda_o) := \Theta \left[\nabla_\Gamma\text{-}deg\Big(\mathscr{A}(\lambda_-), B_1(0)\Big) - \nabla_\Gamma\text{-}deg\Big(\mathscr{A}(\lambda_+), B_1(0)\Big) \right],$$

where $\Gamma = G_{u_0}$ *and* $B_1(0)$ *stands for the open unit ball in* $\mathscr{H}$. *If*

$$\omega_G(\lambda_o) = n_1(H_1) + n_2(H_2) + \cdots + n_m(H_m)$$

is non-zero, i.e., $n_j \neq 0$ *for some* $j = 1, 2, \ldots, m$, *then there exists a bifurcating branch of nontrivial solutions to* (3.46) *from the orbit* $\{\lambda_o\} \times G(u_o)$ *with symmetries at least* (H_j).

Consider the S^1-isotypical decomposition of $\mathscr{W}$, i.e.,

$$\mathscr{W} = \overline{\bigoplus_{l=1}^{\infty} \mathscr{W}_l}, \quad \mathscr{W}_l := \{\cos(l\cdot)\mathfrak{a} + \sin(l\cdot)\mathfrak{b} : \mathfrak{a},\, \mathfrak{b} \in \mathscr{V}\}$$

In a standard way, the space $\mathscr{W}_l$, $l = 1, 2, \ldots$, can be naturally identified with the space $\mathscr{V}^{\mathbb{C}}$ on which S^1 acts by l-folding,

$$\mathscr{W}_l = \{e^{il\cdot} z : z \in \mathscr{V}^{\mathbb{C}}\}.$$

Since the operator $\mathscr{A}(\lambda)$ is G_{u_o}-equivariant with

$$G_{u_o} = \tilde{I} \times O(2),$$

it is also S^1-equivariant and thus $\mathscr{A}(\lambda)(\mathscr{W}_l) \subset \mathscr{W}_l$. Using the $\tilde{I}$-isotypical decomposition of $\mathscr{V}^{\mathbb{C}}$, we have the G_{u_o}-isotypical decomposition is

$$\mathscr{W}_l = \bigoplus_{j=1}^{46} \mathcal{W}_{n_j,l} \ .$$

We conclude that in the $\tilde{I} \times O(2)$-representation $\mathcal{W}_{n_j,l}$, where the group $\tilde{I}$ acts as in the irreducible representation $\mathcal{V}_{n_j}$ and $O(2)$ acts as the l-fold representation, we have

$$\mathscr{A}(\lambda)|_{\mathcal{W}_{n_j,l}} = \left(1 - \frac{\lambda^2 \mu_j + 1}{l^2 + 1}\right) I \ ,$$

This implies that $A(\lambda_o)|_{\mathcal{W}_{n_j,l}} = 0$ if and only if $\lambda_o^2 = l^2/\mu_j$ for some $l = 1, 2, 3, \ldots$ and $j = 0, 1, 2$.

Then the critical set for the equilibrium u_o of the system (3.37) is

$$\Lambda = \left\{ \frac{l}{\sqrt{\mu_j}} : j = 0, ..., 46, \quad l = 1, 2, 3, \ldots \right\}.$$

We can identify the critical numbers writing

$$\lambda_{j,l} = \frac{l}{\sqrt{\mu_j}}$$

The critical numbers are not uniquely identified by the indices (j, l) if there are resonances. The first and last critical numbers for $l = 1$ are $\lambda_{1,1} = .07526\,3$ and $\lambda_{46,1} = 0.573\,18$, respectively. We compute numerically different values $\lambda_{j,l}$ from $\lambda_{1,1}$ to $\lambda_{46,1}$. We obtain to precision 10^{-5} that there is no-resonance with harmonic critical values from $\lambda_{1,1}$ to $\lambda_{46,1}$, i.e.,

$$\lambda_{1,1} < \lambda_{2,1} < \lambda_{3,1} < \lambda_{4,1} < \lambda_{\ 5,1} < ... < \lambda_{5,7} < \ \lambda_{26,3} < \lambda_{21,4} < \lambda_{27,3} < \lambda_{46,1}.$$

Definition 3.4.2. For simplicity, hereafter we denote by I to the isotropy group $\tilde{I}$, i.e., with this notation we have that

$$G_{u_0} = I \times O(2)\ .$$

From the computation of the gradient degree in (3.26) with G_{u_0}, we obtain for $\lambda \notin \Lambda$ that

$$\nabla_{G_{u_0}}\text{–deg}\Big(\mathscr{A}(\lambda_o), B_1(0)\Big) = \prod_{\left\{(j,l)\in\mathbb{N}^2 : \lambda_{j,l}<\lambda_o\right\}} \text{Deg}_{\mathcal{W}_{n_j,l}}\ . \tag{3.51}$$

For each critical number $\lambda_{j,l}$ we choose two numbers $\lambda_- < \lambda_{j,l} < \lambda_+$ such that $[\lambda_-, \lambda_+] \cap \Lambda = \{\lambda_{j,l}\}$. Calculating the difference of the gradient degree at λ_+ and λ_- using (3.51), the topological invariants $\omega_G(\lambda_{j,1})$ can be computed in using the table for the eigenvalues and the order of $\lambda_{j,l}$. For example, the first equivariant invariants are given by

$$\begin{aligned}
&\omega_G(\lambda_{1,1}) = I \times O(2) - \text{Deg}_{\mathcal{W}_{3,1}}\\
&\omega_G(\lambda_{2,1}) = \text{Deg}_{\mathcal{W}_{3,1}} * \Big(I \times O(2) - \text{Deg}_{\mathcal{W}_{-3,1}}\Big)\\
&\omega_G(\lambda_{3,1}) = \text{Deg}_{\mathcal{W}_{3,1}} * \text{Deg}_{\mathcal{W}_{-3,1}} * \Big(I \times O(2) - \text{Deg}_{\mathcal{W}_{2,1}}\Big)\\
&\omega_G(\lambda_{4,1}) = \text{Deg}_{\mathcal{W}_{3,1}} * \text{Deg}_{\mathcal{W}_{-3,1}} * \text{Deg}_{\mathcal{W}_{2,1}} * \Big(I \times O(2) - \text{Deg}_{\mathcal{W}_{-2,1}}\Big)\\
&\omega_G(\lambda_{4,1}) = \text{Deg}_{\mathcal{W}_{3,1}} * \text{Deg}_{\mathcal{W}_{-3,1}} * \text{Deg}_{\mathcal{W}_{2,1}} * \text{Deg}_{\mathcal{W}_{-2,1}} * \Big(I \times O(2) - \text{Deg}_{\mathcal{W}_{-5,1}}\Big)
\end{aligned}$$

Given that the eigenvalues $\lambda_{j,1}$ are non-resonant for $j = 1, ..., 46$, then we have

$$\omega_G(\lambda_{j,1}) = \left(\prod_{\{(k,l):\lambda_{k,l}<\lambda_{j,1}\}} \mathrm{Deg}_{\mathcal{W}_{n_k,l}} \right) \left(I \times O(2) - \mathrm{Deg}_{\mathcal{W}_{n_j,1}} \right) .$$

Therefore, the existence of families of periodic solutions from $\lambda_{j,1}$ are determined by the topological invariant $I \times O(2){-}\mathrm{Deg}_{\mathcal{W}_{n_j,1}}$.

3.4.3 Computation of the Gradient Degree

We consider the product group $G_1 \times G_2$ given two groups G_1 and G_2. The well-known result (see [36, 53]) provides a description of the product group $G_1 \times G_2$. Namely, for any subgroup $\mathscr{H}$ of the product group $G_1 \times G_2$ there exist subgroups $H \leq G_1$ and $K \leq G_2$ a group L and two epimorphisms $\varphi : H \to L$ and $\psi : K \to L$ such that

$$\mathscr{H} = \{(h, k) \in H \times K : \varphi(h) = \psi(k)\}, \tag{3.52}$$

The group $\mathscr{H}$ will be called an *amalgamated* subgroup of $G_1 \times G_2$.

Therefore, any closed subgroup $\mathscr{H}$ of $I \times O(2)$ is an amalgamated subgroup $\mathscr{H}$, where $H \leq I$ and $K \leq O(2)$. In order to make amalgamated subgroup notation simpler and self-contained, we will assume that $L = K/\ker(\psi)$, so $\psi : K \to L$ is evidently the natural projection. On the other hand the group L can be naturally identified with a finite subgroup of $O(2)$ being either D_n or $\mathbb{Z}_n$, $n \geq 1$. Since we are interested in describing conjugacy classes of $\mathscr{H}$, we can identify the epimorphism $\varphi : H \to L$ by indicating

$$Z = \ker(\varphi) \quad \text{ and } \quad R = \varphi^{-1}(\langle r \rangle)$$

where r is the rotation generator in L and $\langle r \rangle$ is the cyclic subgroup generated by r. Then, to identify $\mathscr{H}$ we will write

$$\mathscr{H} =: H \,{}^{Z}_{R}\!\times_L K \tag{3.53}$$

where H, Z and R are subgroups of I.

In the case when all the epimorphisms φ with the kernel Z are conjugate, there is no need to use the symbol R in (3.53), so we will simply write $\mathscr{H} = H\,^{Z}\times_L K$. In addition, in the case all epimorphisms φ from H to L are conjugate, we can also omit the symbol Z, i.e., we will write $\mathscr{H} = H \times_L K$.

The notation in this section is useful to obtain the classification of the all conjugacy classes $(\mathscr{H})$ of closed subgroups in $I \times O(2)$. Let H be a subgroup of A_5, the group H^p denotes the product $H^p = H \times \mathbb{Z}_2$.

We use GAP programming (see [111]) to classification all the conjugacy classes of closed subgroups. In this manner, we compute the following basic gradient degrees,

$$
\begin{aligned}
\deg_{\mathcal{W}_{1,1}} =& -(A_5^p \times D_1) + (A_5^p \times O(2)),\\
\deg_{\mathcal{W}_{2,1}} =& -(D_3^p\,{}^{\mathbb{Z}_3^p}\!\times_{\mathbb{Z}_2} D_2) - (V_4^p\,{}^{\mathbb{Z}_2^p}\!\times_{\mathbb{Z}_2} D_2) + 2(\mathbb{Z}_2^p\,{}^{\mathbb{Z}_1^p}\!\times_{\mathbb{Z}_2} D_2) - (A_4^p \times D_1)\\
& -(D_3^p \times D_1) + 2(\mathbb{Z}_3^p \times D_1) + 2(\mathbb{Z}_2^p \times D_1) - 2(\mathbb{Z}_1^p \times D_1)\\
& -(D_5^p\,{}^{\mathbb{Z}_1^p}\!\times_{D_5} D_5)_1 - (D_5^p\,{}^{\mathbb{Z}_1^p}\!\times_{D_5} D_5)_2 - (D_3^p\,{}^{\mathbb{Z}_1^p}\!\times_{D_3} D_3) + (V_4^p\,{}^{\mathbb{Z}_1^p}\!\times_{D_2} D_2)\\
& +(D_3^p\,{}^{\mathbb{Z}_3^p}\!\times_{D_1} D_1) + (V_4^p\,{}^{\mathbb{Z}_2^p}\!\times_{D_1} D_1) + (A_5^p \times O(2)),\\
\deg_{\mathcal{W}_{3,1}} =& -(V_4^p\,{}^{\mathbb{Z}_2^p}\!\times_{\mathbb{Z}_2} D_2) + (\mathbb{Z}_2^p\,{}^{\mathbb{Z}_1^p}\!\times_{\mathbb{Z}_2} D_2) - (D_5^p \times D_1) - (D_3^p \times D_1)\\
& +2(\mathbb{Z}_2^p \times D_1) - (\mathbb{Z}_1^p \times D_1) - (D_5^p\,{}^{\mathbb{Z}_1^p}\!\times_{D_5} D_5)_1 - (D_5^p\,{}^{\mathbb{Z}_1^p}\!\times_{D_5} D_5)_2\\
& +(V_4^p\,{}^{\mathbb{Z}_1^p}\!\times_{D_2} D_2) + (\mathbb{Z}_2^p\,{}^{\mathbb{Z}_1^p}\!\times_{D_1} D_1) + (A_5^p \times O(2)),\\
\deg_{\mathcal{W}_{4,1}} =& -(D_5^p\,{}^{\mathbb{Z}_5^p}\!\times_{\mathbb{Z}_2} D_2) - (D_3^p\,{}^{\mathbb{Z}_3^p}\!\times_{\mathbb{Z}_2} D_2) - (V_4^p\,{}^{\mathbb{Z}_2^p}\!\times_{\mathbb{Z}_2} D_2) + 3(\mathbb{Z}_2^p\,{}^{\mathbb{Z}_1^p}\!\times_{\mathbb{Z}_2} D_2)\\
& -(\mathbb{Z}_1^p \times D_1) - (D_5^p\,{}^{\mathbb{Z}_1^p}\!\times_{D_5} D_5)_1 - (D_3^p\,{}^{\mathbb{Z}_1^p}\!\times_{D_3} D_3) + (V_4^p\,{}^{\mathbb{Z}_1^p}\!\times_{D_2} D_2)\\
& +(\mathbb{Z}_2^p\,{}^{\mathbb{Z}_1^p}\!\times_{D_1} D_1) + (A_5^p \times O(2)),\\
\deg_{\mathcal{W}_{5,1}} =& -(D_5^p\,{}^{\mathbb{Z}_5^p}\!\times_{\mathbb{Z}_2} D_2) - (D_3^p\,{}^{\mathbb{Z}_3^p}\!\times_{\mathbb{Z}_2} D_2) - (V_4^p\,{}^{\mathbb{Z}_2^p}\!\times_{\mathbb{Z}_2} D_2) + 3(\mathbb{Z}_2^p\,{}^{\mathbb{Z}_1^p}\!\times_{\mathbb{Z}_2} D_2)\\
& -(\mathbb{Z}_1^p \times D_1) - (D_5^p\,{}^{\mathbb{Z}_1^p}\!\times_{D_5} D_5)_2 - (D_3^p\,{}^{\mathbb{Z}_1^p}\!\times_{D_3} D_3) + (V_4^p\,{}^{\mathbb{Z}_1^p}\!\times_{D_2} D_2)\\
& +(\mathbb{Z}_2^p\,{}^{\mathbb{Z}_1^p}\!\times_{D_1} D_1) + (A_5^p \times O(2)),
\end{aligned}
$$

$$
\begin{aligned}
\deg_{\mathcal{W}_{-1,1}} =& -(A_5^p{}^{A_5}\times_{\mathbb{Z}_2}D_2) + (A_5^p\times O(2)),\\
\deg_{\mathcal{W}_{-2,1}} =& -(A_4^p{}^{A_4}\times_{\mathbb{Z}_2}D_2) - (D_3^p{}^{D_3^d}\times_{\mathbb{Z}_2}D_2) - (D_3^p{}^{D_3}\times_{\mathbb{Z}_2}D_2) - (V_4^p{}^{V_4^z}\times_{\mathbb{Z}_2}D_2)\\
&+ 2(\mathbb{Z}_3^p{}^{\mathbb{Z}_3}\times_{\mathbb{Z}_2}D_2) + 2(\mathbb{Z}_2^p{}^{\mathbb{Z}_2^z}\times_{\mathbb{Z}_2}D_2) + 2(\mathbb{Z}_2^p{}^{\mathbb{Z}_2}\times_{\mathbb{Z}_2}D_2) - 2(\mathbb{Z}_1^p{}^{\mathbb{Z}_1}\times_{\mathbb{Z}_2}D_2)\\
&- (D_5^p{}^{\mathbb{Z}_1}\times_{D_{10}}D_{10})_1 - (D_5^p{}^{\mathbb{Z}_1}\times_{D_{10}}D_{10})_2 - (D_3^p{}^{\mathbb{Z}_1}\times_{D_6}D_6) + (D_3^p{}^{\mathbb{Z}_3}_{\mathbb{Z}_3^p}\times_{D_2}D_2)\\
&+ (V_4^p{}^{\mathbb{Z}_2^z}_{\mathbb{Z}_2^p}\times_{D_2}D_2) + (V_4^p{}^{\mathbb{Z}_2}_{\mathbb{Z}_2^p}\times_{D_2}D_2) + (A_5^p\times O(2)),\\
\deg_{\mathcal{W}_{-3,1}} =& -(D_5^p{}^{D_5}\times_{\mathbb{Z}_2}D_2) - (D_3^p{}^{D_3}\times_{\mathbb{Z}_2}D_2) - (V_4^p{}^{V_4^z}\times_{\mathbb{Z}_2}D_2) + (\mathbb{Z}_2^p{}^{\mathbb{Z}_2^z}\times_{\mathbb{Z}_2}D_2)\\
&+ 2(\mathbb{Z}_2^p{}^{\mathbb{Z}_2}\times_{\mathbb{Z}_2}D_2) - (\mathbb{Z}_1^p{}^{\mathbb{Z}_1}\times_{\mathbb{Z}_2}D_2) - (D_5^p{}^{\mathbb{Z}_1}\times_{D_{10}}D_{10})_1 - (D_5^p{}^{\mathbb{Z}_1}\times_{D_{10}}D_{10})_2\\
&+ (V_4^p{}^{\mathbb{Z}_2^z}_{\mathbb{Z}_2^p}\times_{D_2}D_2) + (\mathbb{Z}_2^p{}^{\mathbb{Z}_1}_{\mathbb{Z}_1^p}\times_{D_2}D_2) + (A_5^p\times O(2)),\\
\deg_{\mathcal{W}_{-4,1}} =& -(D_5^p{}^{D_5^d}\times_{\mathbb{Z}_2}D_2) - (D_3^p{}^{D_3^d}\times_{\mathbb{Z}_2}D_2) - (V_4^p{}^{V_4^z}\times_{\mathbb{Z}_2}D_2) + 3(\mathbb{Z}_2^p{}^{\mathbb{Z}_2^z}\times_{\mathbb{Z}_2}D_2)\\
&- (\mathbb{Z}_1^p{}^{\mathbb{Z}_1}\times_{\mathbb{Z}_2}D_2) - (D_5^p{}_{)}\times_{\mathbb{Z}_1}D_{10}D_{101} - (D_3^p{}^{\mathbb{Z}_1}\times_{D_6}D_6) + (V_4^p{}^{\mathbb{Z}_2^z}_{\mathbb{Z}_2^p}\times_{D_2}D_2)\\
&+ (\mathbb{Z}_2^p{}^{\mathbb{Z}_1}_{\mathbb{Z}_1^p}\times_{D_2}D_2) + (A_5^p\times O(2)),\\
\deg_{\mathcal{W}_{-5,1}} =& -(D_5^p{}^{D_5^d}\times_{\mathbb{Z}_2}D_2) - (D_3^p{}^{D_3^d}\times_{\mathbb{Z}_2}D_2) - (V_4^p{}^{V_4^z}\times_{\mathbb{Z}_2}D_2) + 3(\mathbb{Z}_2^p{}^{\mathbb{Z}_2^z}\times_{\mathbb{Z}_2}D_2)\\
&- (\mathbb{Z}_1^p{}^{\mathbb{Z}_1}\times_{\mathbb{Z}_2}D_2) - (D_5^p{}_{)}\times_{\mathbb{Z}_1}D_{10}D_{102} - (D_3^p{}^{\mathbb{Z}_1}\times_{D_6}D_6) + (V_4^p{}^{\mathbb{Z}_2^z}_{\mathbb{Z}_2^p}\times_{D_2}D_2)\\
&+ (\mathbb{Z}_2^p{}^{\mathbb{Z}_1}_{\mathbb{Z}_1^p}\times_{D_2}D_2) + (A_5^p\times O(2)).
\end{aligned}
$$

We can use GAP programming (see [111]) and the product $*$ of Euler ring $U(I\times O(2))$ to compute the full equivariant invariants.

3.5 Description of the Symmetries

Since $\lambda_{j,1}$ are non-resonant, the existence of families of periodic solutions from $\lambda_{j,1}$ are determined by the maximal isotropy groups in the topological invariant $\mathrm{Deg}_{\mathcal{W}_{n_j,1}} - I\times O(2)$. Therefore, the families bifurcating from the frequency $\sqrt{\mu_j}$, with representations $\mathcal{V}_{n_j}$ for

$n \in \{1, \ldots, 5\}$, have symmetries given by:

$$\mathcal{V}_1 : (A_5^p \times D_1), \tag{3.54}$$

$$\mathcal{V}_2 : (A_4^p \times D_1), (D_3^p \times D_1), (D_3^{p\ \mathbb{Z}_3^p} \times_{\mathbb{Z}_2} D_2), (D_2^{p\ \mathbb{Z}_2^p} \times_{\mathbb{Z}_2} D_2) \tag{3.55}$$

$$(D_5^{p\ D_5} \times_{D_5} \mathbb{Z}_1^p)_1, (D_5^{p\ D_5} \times_{D_5} \mathbb{Z}_1^p)_2, (D_3^{p\ D_3} \times_{D_3} \mathbb{Z}_1^p) \tag{3.56}$$

$$\mathcal{V}_3 : (D_5^p \times D_1), (D_3^p \times D_1), (D_2^{p\ \mathbb{Z}_2^p} \times_{\mathbb{Z}_2} D_2) \tag{3.57}$$

$$(D_5^{p\ \mathbb{Z}_1^p} \times_{D_5} D_5)_1, (D_5^{p\ \mathbb{Z}_1^p} \times_{D_5} D_5)_2, \tag{3.58}$$

$$\mathcal{V}_4 : (D_5^{p\ \mathbb{Z}_5^p} \times_{\mathbb{Z}_2} D_2), (D_3^{p\ \mathbb{Z}_3^p} \times_{\mathbb{Z}_2} D_2), (D_2^{p\ \mathbb{Z}_2^p} \times_{\mathbb{Z}_2} D_2) \tag{3.59}$$

$$(D_5^{p\ \mathbb{Z}_1^p} \times_{D_5} D_5)_1, (D_3^{p\ \mathbb{Z}_1^p} \times_{D_3} D_3) \tag{3.60}$$

$$\mathcal{V}_5 : (D_5^{p\ \mathbb{Z}_5^p} \times_{\mathbb{Z}_2} D_2), (D_3^{p\ \mathbb{Z}_3^p} \times_{\mathbb{Z}_2} D_2), (D_2^{p\ \mathbb{Z}_2^p} \times_{\mathbb{Z}_2} D_2) \tag{3.61}$$

$$(D_5^{p\ \mathbb{Z}_1^p} \times_{D_5} D_5)_2, (D_3^{p\ \mathbb{Z}_1^p} \times_{D_3} D_3) \tag{3.62}$$

The symmetries in the representations $\mathcal{V}_{-1}, \ldots, \mathcal{V}_{-5}$ are exactly as $\mathcal{V}_1, \ldots, \mathcal{V}_5$, except that instead of the element $-1 \in \mathbb{Z}_2 < \tilde{I}$ which gives a symmetry by inversion

$$u_{\tau,k}(t) = -u_{\tau^{-1},k}(t),$$

the representations $\mathcal{V}_{-1}, \ldots, \mathcal{V}_{-5}$ have the element $(-1, -1) \in \mathbb{Z}_2 \times S^1$ in the maximal groups, which gives the symmetry

$$u_{\tau,k}(t) = -u_{\tau^{-1},k}(t + \pi).$$

Therefore, we only describe the symmetries of the maximal groups for the representation $\mathcal{V}_1, \ldots, \mathcal{V}_5$.

The existence of the symmetry $\kappa \in O(2)$ in the maximal groups implies that the solutions are brake orbits,

$$u_{\tau,k}(t) = u_{\tau,k}(-t),$$

i.e., the velocity $\dot{u}$ of all the molecules are zero at the times $t = 0, \pi$, $\dot{u}(0) = \dot{u}(\pi) = 0$. We classify the maximal groups in two classes: the groups that have the element $\kappa \in O(2)$ and the groups that have the element κ coupled with a rotation of $\tilde{I}$. That is, there is an element $\gamma \in \mathcal{C}_2$ such that (γ, κ) is in the second class of groups, i.e., their solutions have the symmetry

$$u_{\tau,k}(t) = \rho(\gamma) u_{\gamma\tau\gamma^{-1},\gamma^{-1}(k)}(-t).$$

3.5.1 Standing Waves (Brake Orbits)

In this category, we consider the groups that have the element $\kappa \in O(2)$, which generate the group $D_1 < O(2)$.

For the groups

$$(A_5^p \times D_1), (A_4^p \times D_1), (D_5^p \times D_1), (D_3^p \times D_1)$$

the solutions have icosahedral symmetries for all times for the group A_5, tetrahedral symmetries for the group A_4, pentagonal symmetries for the group D_5 and triangular symmetries for group D_3.

For the group

$$(D_3^p {}^{\mathbb{Z}_3^p}\!\!\times_{\mathbb{Z}_2} D_2)$$

the solutions are symmetric by the $2\pi/3$-rotations of $\mathbb{Z}_3 < D_3 < I$, while the reflection of $D_3 < I$ is coupled with the π-time shift of $-1 \in \mathbb{Z}_2 < S^1$. Therefore, in the solutions of three faces have the exact dynamics, but these faces are not symmetric by reflection such as in the symmetries of the group $(D_3^p \times D_1)$.

For the group

$$(V_4^p {}^{\mathbb{Z}_2^p}\!\!\times_{\mathbb{Z}_2} D_2),$$

the solutions are symmetric by the π-rotations of $V_4 < I$, while the other π-rotations of $\mathbb{Z}_2 < V_4$ is coupled with the π-time shift of $-1 \in D_2 < S^1$.

These seven symmetries give solutions which are standing waves in the sense that each symmetric face have the exact dynamic repeated for all times.

3.5.2 Traveling Waves

In the groups

$$(D_5^p\,{}^{\mathbb{Z}_1^p}\!\times_{D_5} D_5)_1, (D_5^p\,{}^{\mathbb{Z}_1^p}\!\times_{D_5} D_5)_2,$$

the spatial dihedral group $D_5 < I$ is coupled with the temporal group $D_5 < O(2)$. Therefore, in these solutions 5 faces have the exact dynamics, but there is a $2\pi/5$-time shift in time between consecutive faces. In this sense, the solutions have the appearance of a discrete traveling wave in 5 faces with a $2\pi/5$-time shift. There are two groups because there are two different classes $\mathcal{C}_4$ and $\mathcal{C}_5$ of A_5.

Similarly, these solutions of the group

$$(D_3^p\,{}^{\mathbb{Z}_1^p}\!\times_{D_3} D_3),$$

have 3 faces with the exact dynamics, but with a $2\pi/3$-time shift in time, i.e., the solutions have the appearance of a discrete traveling wave in 3 faces with a $2\pi/3$-time shift.

CHAPTER 4

ON SOME APPLICATIONS OF GROUP REPRESENTATION THEORY TO ALGEBRAIC PROBLEMS RELATED TO THE CONGRUENCE PRINCIPLE FOR EQUIVARIANT MAPS[1,2]

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4.1 Introduction

4.1.1 Topological Motivation

The methods based on the usage of Brouwer degree and its infinite dimensional generalizations are unavoidable in many mathematical areas which, at first glance, have nothing in common: (i) qualitative investigation of differential and integral equations arising in mathematical physics (existence, uniqueness, stability, bifurcation of solutions (see [69, 88, 39])), (ii) combinatorics (equipartition of mass (see [22, 102])), geometry (harmonic maps between surfaces (see [43, 63])), to mention a few. In short, given a continuous map $\Phi : M \to N$ of manifolds of the same dimension, the Brouwer degree $\deg(\Phi)$ is an integer which can be considered as an algebraic count of solutions to equation $\Phi(x) = y$ for a given $y \in N$ (for continuous functions from $\mathbb{R}$ to $\mathbb{R}$, the Brouwer degree theory can be traced to Bolzano-Cauchy Intermediate Value Theorem).

In general, practical computation of the Brouwer degree is a problem of formidable complexity. However, if Φ respects some group symmetries of M and N (expressed in terms of the so-called equivariance, see Section 4.2.1), then the computation of $\deg(\Phi)$ lies in the interplay between topology and group representation theory. Essentially, symmetries lead to restrictions on possible values of the degree. These restrictions (typically formulated in a form of *congruencies*) have been studied by many authors using different techniques (see, for example, [66, 75, 60, 23, 16, 6, 101, 65, 37, 81] and references therein (see also the survey [99])). The following statement (which is a particular case of the so-called *congruence principle* established in [75], Theorems 2.1 and 3.1) is the starting point for our discussion.

Congruence principle: *Let M be a compact, connected, oriented, smooth n-dimensional manifold on which a finite group G acts smoothly, and let W be an orthogonal $(n+1)$-dimensional G-representation. Denote by $\alpha(M)$ the greatest common divisor of lengths of G-orbits occurring in M. Assume that there exists an equivariant map $\Phi : M \to W \setminus \{0\}$.*

Then, for **any** *equivariant map* $\Psi : M \to W \setminus \{0\}$, *one has*

$$\deg(\Psi) \equiv \deg(\Phi) \pmod{\alpha(M)}. \tag{4.1}$$

Clearly, the congruence principle contains a non-trivial information only if $\alpha(M) > 1$ (for example, if a non-trivial group G acts freely on M, then $\alpha(M) = |G| > 1$). Also, the congruence principle can be effectively applied only if there exists a "canonical" equivariant map $\Phi : M \to W \setminus \{0\}$ with $\deg(\Phi)$ easy to calculate (for example, if M coincides (as a G-space) with the unit sphere $S(W)$ in W, then one can take $\Phi := \mathrm{Id}$, in which case, for *any* equivariant map $\Psi : S(W) \to W \setminus \{0\}$), one has: $\deg(\Psi) \equiv \deg(\mathrm{Id}) = 1 \pmod{\alpha(S(W))}$; in particular, $\deg(\Psi) \neq 0$ provided $\alpha(S(W)) > 1$). This way, we arrive at the following two problems:

Problem 4.1.1. Under which conditions on M, is $\alpha(M)$ greater than 1?

Problem 4.1.2. Under which conditions on M and W, does there exist an equivariant map $\Phi : M \to W \setminus \{0\}$ with $\deg(\Phi)$ easy to calculate?

Assume, in addition, that V is an orthogonal G-representation and $M = S(V)$ (recall that $S(V)$ is called a G-representation sphere). Then: (i) Problem A can be traced to the classical result of J. Wolf [109] on classification of finite groups acting freely on a finite-dimensional sphere, (ii) both Problems A and B are intimately related to a classification of G-representations up to a certain (non-linear) equivalence (see [6, 101, 1, 79]).

A study of numerical properties of orbit lengths of finite linear groups has a long history and can be traced back to H. Zassenhaus [102]. A special attention was paid to studying regular orbits, orbits of coprime lengths, etc., in the case of the ground field of positive characteristic (see [49] for a comprehensive account about the current research in this area). To the best of our knowledge, the case of zero characteristic was not as well studied as the one of positive characteristic. It seems that the invariant introduced in our paper (the α-characteristic of a linear representation) has not been studied in detail before.

The *goal* of this paper is to develop some algebraic techniques allowing one to study Problems 4.1.1 and 4.1.2 for finite solvable and 2-transitive groups. We are focused on the situation when V and W are complex unitary G-representations of the same dimension and $M = S(V)$ is a G-representation sphere (in this case, we set $\alpha(V) = \alpha(S(V))$ and call it α-characteristic of V). However, some of our results (see Corollary 4.7.3) are formulated for equivariant maps of G-manifolds.

4.1.2 Main Results and Overview

(A) If V and U are (complex unitary) G-representations, then $\alpha(U \oplus V) = \gcd\{\alpha(U), \alpha(V)\}$. This simple observation suggests to study Problem 4.1.1 first for $S(V)$, where V is an irreducible representation. By combining the main result from [64] with several group theoretical arguments, we obtain the following result: G is solvable if and only if $\alpha(V) > 1$ for *any* non-trivial irreducible G-representation (see Theorem 4.3.11). Among many known characterizations of the class of (finite) solvable groups, we would like to refer to Theorem 3.7 from [16] (where a concept of admissible representations is used) as the result close in spirit to ours. Also, if G is nilpotent, we show that for any non-trivial irreducible G-representation, there exists an orbit $G(x)$ in $S(V)$ such that $|G/G_x| = \alpha(V)$ (see Proposition 4.3.22).

On the other hand, we discovered that a sporadic group (the Janko Group J_1 (see [62])) satisfies the following property: *all* irreducible J_1-representations have the α-characteristic equals 1 (recall that J_1 is of order 175560 and admits 15 irreducible representations). With these results in hand, we arrived at the following question: Given a (finite) non-solvable group G different from J_1, does there exist an easy way to point out an irreducible G-representation V with $\alpha(V) > 1$? In this paper, we focus on the following setting: Given $H < G$, take the G-action on G/H by left translations and denote by V the augmentation submodule of the associated permutation G-representation $\mathbb{C} \oplus V$. It turns out that $\alpha(V) > 1$ if and only if $|G/H| = q^k$, where q is a prime (see Lemma 4.4.4). Combining this observation with the classification of 2-transitive groups (see [31], for example) allows us to completely describe faithful augmented modules V associated with 2-transitive group G-actions on G/H such that $\alpha(V) > 1$ (see Theorem 4.4.3).

Finally, it is possible to show that if $H \trianglelefteq G$, V is an H-representation and W is a G-representation induced from V, then, $\alpha(V)$ divides $\alpha(W)$. This observation suggests the following question: under which conditions, does $\alpha(V) = 1$ imply $\alpha(W) = 1$? We answer this question affirmatively assuming that V is irreducible and G/H is solvable (see Proposition 4.5.5).

(B) In general, Problem 4.1.2 is a subject of the equivariant obstruction theory (see [101, 16] and references therein) and is far away from being settled even in relatively simple cases. On the other hand, if W is a subrepresentation of the m-th symmetric power of V, then one can look for a required map in the form of a G-equivariant *m-homogeneous* map $\Phi : S(V) \to W \setminus \{0\}$, in which case $\deg(\Phi) = m^n$. In particular case when $m = 2$, Problem 4.1.2 reduces to the existence of a commutative (in general, non-associative) bilinear multiplication $* : V \times V \to V \subset \mathrm{Sym}^2(V)$ satisfying two properties: (i) $*$ commutes with the G-actions, and (ii) the complex algebra $(V, *)$ is free from 2-nilpotents. Combining this idea with the techniques related to the so-called Norton algebra (see [32]), we establish the existence of an equivariant quadratic map between two non-equivalent $(n-1)$-dimensional S_n-representations (having the same symmetric square) taking non-zero vectors to non-zero ones, provided that n is odd (see Theorem 4.6.9). For $n = 5$, we give an explicit formula of such a map.

After the Introduction, the paper is organized as follows. Section 4.2 contains preliminaries related to groups and their representations. In Section 4.3, we first consider functorial properties of α-characteristic (see Proposition 4.3.2). Next, we focus on solvable and nilpotent groups and prove Theorem 4.3.11 and Proposition 4.3.22. 2-transitive actions are considered in Section 4.4, while induced representations with trivial α-characteristic are considered in Section 4.5. Section 4.6 is devoted to the existence of quadratic maps relevant to Problem 4.1.2. In the concluding Section 4.7, we consider applications of the obtained results to the congruence principle.

4.2 Preliminaries

4.2.1 Groups and Their Actions

This subsection collects some basic facts about finite groups and their actions that are used in our paper. Although the material given here is well-known to any group theorist, we decided to include it here, because we expect that the paper could be of interest for mathematicians working outside the group theory.

Throughout the paper, we consider only finite groups if no otherwise is stated, and by G, we always mean a finite group.

For any G, denote by $\mathrm{Aut}(G)$ (resp. $\mathrm{Inn}(G)$) the group of automorphisms (resp. inner automorphisms) of G, by e the identity of G and by $\mathbf{1}$ the trivial group or the trivial subgroup of G.

Given $H, K < G$, set $HK := \{hk \in G : h \in H,\, k \in K\}$. Given a prime p, denote by $\mathrm{Syl}_p(G)$ the collection of Sylow p-subgroups of G. Recall an important characterization of solvable groups from [64]:

Theorem 4.2.1. *Let $p_1, p_2, \cdots, p_k$ be a sequence of all distinct prime factors of $|G|$. Then, G is solvable if and only if $G = P_1P_2 \cdots P_k$ for* any *choice of $P_j \in \mathrm{Syl}_{p_j}(G)$, $j = 1, \ldots, k$.*

Recall that $N \trianglelefteq G$ is called a *minimal normal subgroup* if N is non-trivial and contains no other non-trivial normal subgroups of G. The *socle* of G is the subgroup generated by all minimal normal subgroups of G. The following result is well-known (see[59]).

Proposition 4.2.2. *A minimal normal subgroup of a solvable group is elementary abelian.*

Let X be a G-space. For any $x \in X$, denote by G_x the *isotropy* (stabilizer) of x and by $G(x)$ the G-orbit of x in X. We call the conjugacy class of G_x the *orbit type* of x and denote by $\Phi(G; S)$ the collection of orbit types of points in $S \subset V$. For any $H < G$, denote by $X^H := \{x \in X : hx = x \text{ for all } h \in H\}$ the set of H-fixed points in X.

If $|X| \geq 2$, we say that G acts 2-*transitively* on X if for any $a, b, c, d \in X$, $a \neq b$, $c \neq d$, there exists $g \in G$ such that $ga = c$ and $gb = d$. Since any transitive (in particular, 2-transitive) action is equivalent to the G-action on the coset space G/H by left translation for some $H < G$, the existence of 2-transitive G-action is actually an intrinsic property of G. Therefore, we adopt the following definition.

Definition 4.2.3. G is called a 2-*transitive group* if it admits a faithful 2-transitive action, or equivalently, G acts 2-transitively on G/H (by left translation) for some $H < G$.

Suppose X and Y are (topological) G-spaces. A continuous map $f : X \to Y$ is called G-*equivariant* if $f(gx) = gf(x)$ for any $g \in G$ and $x \in X$. Note that in this case, f takes H-fixed points in X to H-fixed points in Y (i.e., $f(X^H) \subset Y^H$) for any $H < G$). If, in addition, X and Y are *linear* G-spaces, a G-equivariant map $f : X \to Y$ is called *admissible* if $f^{-1}(0) = \{0\}$. We refer to [25] and [75] for the equivariant topology background.

4.2.2 Group Representations

Throughout the paper, we consider only *finite-dimensional complex unitary* representations, and by ρ (resp. V and χ), we always mean a G-representation (resp. the associated vector space and the affording character) if no otherwise is stated.

Let K be an arbitrary field. Denote by $K[G]$ the group algebra of G over K. For any ρ, we will simply denote by the same symbol the extension of ρ to $K[G]$ (i.e., depending on the context, it is possible that $\rho : G \to \mathrm{GL}(V)$ or $\rho : K[G] \to \mathrm{End}(V)$).

For any G, denote by $\mathbb{1}_G$ the *trivial representation* or *trivial character* of G (depending on the context) and by $\mathrm{Irr}(G)$ (resp. $\mathrm{Irr}^*(G)$) the collection of *irreducible* (resp. *non-trivial irreducible*) G-representations.

For any representation $\rho : G \to GL(V)$, denote by $\rho(G)(x)$ or $G(x)$ the G-orbit of x for any $x \in V$. In addition, set $\Phi(\rho) := \Phi(G; S(V))$, where $S(V)$ stands for the unit sphere in V.

If ρ and σ are G-representations, then $[\rho, \sigma]$ will stand for the *scalar product* of their characters.

Let $H < G$. For any G-representation ρ with character χ, denote by ρ_H and χ_H the *restriction* of ρ and χ to H, respectively. On the other hand, for any H-representation ψ with character ω, denote by ψ^G and ω^G the *induced representation* and the *induced character* of ψ to G, respectively.

Let σ be an automorphism of G. Denote by ρ^σ (resp. χ^σ) the composition $\rho \circ \sigma$ (resp. $\chi \circ \sigma$). It is clear that: (i) ρ^σ is a G-representation affording character χ^σ, and (ii) if ρ is irreducible, so is ρ^σ. If, in particular, $\sigma : g \mapsto ugu^{-1}$ for some $u \in U \trianglerighteq G$, denote by $\rho^{(u)}$ (resp. $\chi^{(u)}$) the composition $\rho \circ \sigma$ (resp. $\chi \circ \sigma$) instead of ρ^σ (resp. χ^σ). In such a case, $\rho^{(u)}$ (resp. $\chi^{(u)}$) is said to be *U-conjugate* to ρ (resp. χ).

Recall the following result for permutation representations associated to 2-transitive actions (see [58]).

Proposition 4.2.4. *Let G act transitively on X. Then, the permutation representation associated to this action is equivalent to $\mathbb{1}_G \oplus \rho$, where all irreducible components of ρ are non-trivial. If, in addition, G acts 2-transitively on X, then ρ is irreducible.*

The G-representation ρ in Proposition 4.2.4 will play an essential role in our consideration. We adopt the following definition.

Definition 4.2.5. Following [54], we call the representation ρ from Proposition 4.2.4 the *augmentation representation* associated to the transitive G-action on X (resp. G/H by left translation) and denote it by $\rho^a_{(G;X)}$ (resp. $\rho^a_{[G;H]}$)[5]. In particular, denote by $\varrho_2(G)$ the collection of all its non-isomorphic augmentation representations arise from 2-transitive actions of G.

We refer to [96], [27] and [58] for the representation theory background and notation frequently used in this paper.

4.3 α-Characteristic of G-Representations

The following definition is crucial for our discussion.

Definition 4.3.1. For a G-representation $\rho : G \to \mathrm{GL}(V)$, we call

$$\begin{aligned}\alpha(\rho) = \alpha(G, S(V)) :=& \ \gcd\{|G(x)| : x \in S(V)\}\\ =& \ \gcd\{|G/H| : (H) \in \Phi(\rho)\}\end{aligned}$$

the *α-characteristic* of ρ. We will call the α-characteristic of a representation *trivial* if it takes value 1.

Note that α-characteristic admits the following functorial properties.

Proposition 4.3.2. *Suppose ρ is a G-representation.*

(a) *Let $H < G$. Then, $\alpha(\rho_H)$ divides $\alpha(\rho)$.*

(b) *Let $H \trianglelefteq G$ and θ be an H-representation. Then, $\alpha(\theta)$ divides $\alpha(\theta^G)$.*

(c) *Let σ be an automorphism of G. Then, $\alpha(\rho^\sigma) = \alpha(\rho)$.*

[5] The corresponding module will be called the augmentation module. We do not use a special notation for it.

(d) *Let $\mathbb{F}$ be a splitting field of the group algebra $\mathbb{Q}[G]$ and σ an automorphism of $\mathbb{F}$. Then, $\alpha(\rho^\sigma) = \alpha(\rho)$ for $\rho \in \mathrm{Irr}(G)$.*

(e) *Let ψ be another G-representation. Then, $\alpha(\rho \oplus \psi) = \gcd\{\alpha(\rho), \alpha(\psi)\}$.*

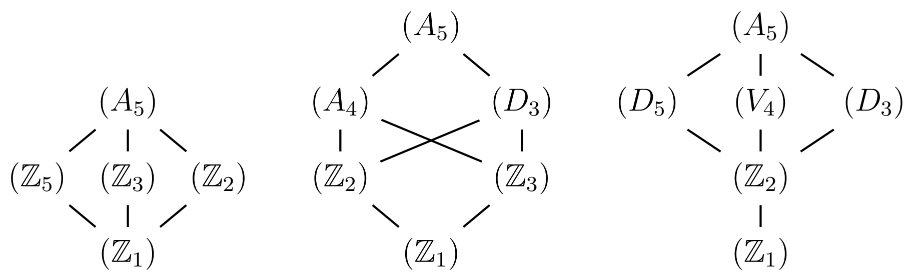
Proof. Here we prove part (b) only since other properties are quite straightforward from Definition 4.3.1. Denote by V and W the representation spaces of θ^G and θ, respectively. Take an arbitrary non-zero $v \in V$. It suffices to show that $\alpha(\theta)$ divides $|N(v)|$. Since V is induced by W, one has $v = \sum g_i w_i$, where $\{g_i\}$ is the complete set of representatives of N-cosets in G and $w_i \in W$. Without loss of generality, assume that $w_1 \neq 0$. Take $n \in N_v$. Since $nv = \sum g_i(g_i^{-1} n g_i) w_i$ and N is normal, we conclude that $n \in N_v$ if and only if $n \in g_i N_{w_i} g_i^{-1}$ for every i. In particular, $n \in g_1 N_{w_1} g_1^{-1}$ implying $N_v < g_1 N_{w_1} g_1^{-1} < N$. Therefore, $|N(w_1)| = |N : N_{w_1}|$ divides $|N(v)| = |N : N_v|$ and the result follows from the fact that $\alpha(\theta)$ divides $|N(w_1)|$. $\square$

Remark 4.3.3. (i) According to Proposition 4.3.2(e), given a G-representation ρ (possibly reducible), one can evaluate $\alpha(\rho)$ by computing α-characteristics of all irreducible components of ρ.

(ii) The conclusion of Proposition 4.3.2(b) is not true if H is not normal in G. The simplest example is provided by the group $G = S_3$ with H to be an order two subgroup. If θ is a non-trivial representation of H, then $\theta^G = \rho_1 \oplus \rho_2$ where $\dim(\rho_1) = 1, \dim(\rho_2) = 2$. In this case, $\alpha(\theta) = 2$ while $\alpha(\rho_1) = 2$ and $\alpha(\rho_2) = 3$, so that $\alpha(\theta^G) = 1$.

(iii) One could think that there always exists an irreducible constituent ρ of θ^G with $\alpha(\theta) = \alpha(\rho)$. But this is not true as the following example shows. Take $G = \mathcal{Q}_8$, a quaternion group of order eight, and let H be its cyclic subgroup of order 4. If θ is a faithful irreducible representation of H, then θ^G is an irreducible 2-dimensional G-representation. In this case, $\alpha(\theta) = 4$ while $\alpha(\theta^G) = 8$.

Example 4.3.4. Computation of $\alpha(\rho)$ for $\rho \in \mathrm{Irr}^*(G)$ involves finding maximal orbit types $(H) \lneq (G)$ of ρ. For example, the group A_5 admits four non-trivial irreducible representations with the lattices of orbit types shown in Figure 4.1. Then, for each $\rho \in \mathrm{Irr}^*(A_5)$, $\alpha(\rho)$ is the greatest common divisor of indices of proper subgroups which appear in the lattice. The result is shown in Table 4.1.



(a) ρ_1 and ρ_2 (b) ρ_3 (c) ρ_4

Figure 4.1. Lattices of orbit types of $\rho \in \mathrm{Irr}^*(A_5)$

Table 4.1. Character table of $G = A_5$

ρ	$\alpha(\rho)$	(2-transitivity) H	$\lvert G : H\rvert$	character				
ψ_0	1			1	1	1	1	1
ψ_1	2			3	-1	.	$\frac{1-\sqrt{5}}{2}$	$\frac{1+\sqrt{5}}{2}$
ψ_2	2			3	-1	.	$\frac{1+\sqrt{5}}{2}$	$\frac{1-\sqrt{5}}{2}$
ψ_3	5	A_4	5	4	.	1	-1	-1
ψ_4	1	D_5	6	5	1	-1	.	.

Remark 4.3.5. Note that the character table of a group does not determine the α-characteristic of its irreducible representations. For example, D_8 and Q_8 have the same character table while the α-characteristic of their unique 2-dimensional irreducible representations are distinct (4 for D_8 and 8 for Q_8).

4.3.1 α-Characteristic of Solvable Group Representations

Problem 4.1.1 together with Remark 4.3.3 give rise to the following questions.

Question 4.3.6. Does there exist a non-trivial group G such that $\alpha(\rho) = 1$ for *any* $\rho \in \operatorname{Irr}(G)$?

Question 4.3.7. Does there exist a reasonable class of groups $\mathcal{A}$ such that for *any* $G \in \mathcal{A}$, one has

$$\alpha(\rho) > 1 \text{ for } \textit{any } \rho \in \operatorname{Irr}^*(G)? \tag{†}$$

Question 4.3.8. Given a group G which is neither in the case of Question 4.3.6 nor Question 4.3.7, how can one find a G-representation ρ with $\alpha(\rho) > 1$?

An affirmative answer to Question 4.3.6 is given by the following example.

Example 4.3.9. The Janko Group J_1 has 15 irreducible representations—all of them admit trivial α-characteristics.

We give a complete answer to Question 4.3.7 in the rest of this subsection and address Question 4.3.8 in Sections 4.4 and 4.5. The following example is the starting point for our discussion.

Example 4.3.10. If G is abelian or a p-group, then (†) is true.

We will show that the following statement is true.

Theorem 4.3.11. *G is solvable if and only if $\alpha(\rho) > 1$ for any $\rho \in \operatorname{Irr}^*(G)$.*

Remark 4.3.12. As it will follow from the proof, the conclusion of the Theorem 4.3.11 remains true if one replace the complex field by an algebraically closed field of a characteristic coprime to $|G|$.

Let us first present two lemmas required for the proof of necessity in Theorem 4.3.11.

Lemma 4.3.13. *Let $\rho \in \mathrm{Irr}^*(G)$ and $P \in \mathrm{Syl}_p(G)$. Then $\alpha(\rho_P) = \alpha(\rho)_p$, where $\alpha(\rho)_p$ is the highest p-power that divides $\alpha(\rho)$. In addition, the following statements are equivalent.*

(i) p divides $\alpha(\rho)$.

(ii) $P_x \lneq P$ for any $x \in S(V)$.

(iii) $[\chi_P, \mathbf{1}_P] = 0$.

Proof. By Proposition 4.3.2(a), $\alpha(\rho_P)$ divides $\alpha(\rho)$. Since $\alpha(\rho_P)$ is a p-power, we conclude that $\alpha(\rho_P) \mid \alpha(\rho)_p$.

Let us show that $\alpha(\rho)_p$ divides the cardinality of every G-orbit in $S(V)$. Let $O \subseteq S(V)$ be a G-orbit. By Exercise 1.4.17 in [41], the length of every P-orbit in O is divisible by $|O|_p$. Therefore, $\alpha(\rho)_p$ divides length of every P-orbit in O. Hence, $\alpha(\rho)_p$ divides the length of every P-orbit in $S(V)$. Hence, $\alpha(\rho)_p \mid \alpha(\rho_P)$.

(i) $\Longrightarrow$ (ii). Since $\alpha(\rho_P) = \alpha(\rho)_p \geq p$, each P-orbit in $S(V)$ is non-trivial, i.e., $[P : P_x] \geq p$ for each $x \in S(V)$.

(ii) $\Longrightarrow$ (i). Suppose (ii) is true. Then, p divides $|P/P_x| = |P(x)|$, which divides $|G(x)|$, for any $x \in S(V)$. It follows that p divides $\alpha(\rho)$.

(ii) $\Longleftrightarrow$ (iii). Both (ii) and (iii) are equivalent to $\dim V^P = 0$.

□

Remark 4.3.14. Notice that in Lemma 4.3.13, for (ii) to imply (i), it is enough to assume that $P < G$ is a p-subgroup.

Remark 4.3.15. In what follows, denote $\underline{H} := \sum_{g \in H} g \in \mathbb{Z}[G]$ and $\hat{\underline{H}} := (1/|H|)\underline{H} \in \mathbb{Q}[G]$ for any $H < G$. Under this notation, Lemma 4.3.13 (iii) reads $\chi(\underline{P}) = 0$ or $\chi(\hat{\underline{P}}) = 0$. In addition, note that Lemma 4.3.13 (iii) is equivalent to saying that ρ is not a constituent of $\mathbf{1}_P^G$.

Proposition 4.3.16. *Let V be an non-trivial irreducible G-representation and $N \trianglelefteq G$. Then, N acts non-trivially on V if and only if $N_x \lneq N$ for any $x \in S(V)$.*

Proof. Since N is normal in G, the subspace V^N is G-invariant. Therefore, either $V^N = \{\mathbf{0}\}$ or $V^N = V$ from which the claim follows. □

The next result immediately follows from Lemma 4.3.13 and Proposition 4.3.16 (see also Remark 4.3.14).

Corollary 4.3.17. *Let $N \trianglelefteq G$ be a p-subgroup and let V be a non-trivial irreducible G-representation where N acts non-trivially. Then, p divides $\alpha(\rho)$.*

As for sufficiency in Theorem 4.3.11, we will need the following lemma.

Lemma 4.3.18. *Let ρ be a G-representation with character χ and $H \leq G$. Then,*

(i) *$\rho(\hat{\underline{H}})$ is an idempotent.*

(ii) *If, in addition, $\chi(\hat{\underline{H}}) = 0$, then both $\rho(\hat{\underline{H}})$ and $\rho(\underline{H})$ are zero matrices.*

Proof. Direct computation shows that $\hat{\underline{H}} \in \mathbb{Q}[G]$ is an idempotent, therefore, so is $\rho(\hat{\underline{H}})$. If, in addition, $\chi(\hat{\underline{H}}) = 0$, i.e., $\rho(\hat{\underline{H}})$ is an idempotent matrix with zero trace, then $\rho(\hat{\underline{H}})$ is a zero matrix. In this case, $\rho(\underline{H}) = |H|\, \rho(\hat{\underline{H}})$ is also a zero matrix. □

The next result follows immediately from Lemmas 4.3.13 and 4.3.18 (see also Remark 4.3.15).

Corollary 4.3.19. *Let $\rho \in \mathrm{Irr}^*(G)$ with $\alpha(\rho) > 1$. Then, there exists a prime factor p of $|G|$ such that $\rho(\underline{P})$ is a zero matrix for any $P \in \mathrm{Syl}_p(G)$.*

The following elementary statement is an immediate consequence of the injectivity of a regular representation of a finite group.

Proposition 4.3.20. *Let ρ be the regular G-representation. Given two elements x, y of the group algebra $\mathbb{Q}[G]$, if $\rho(x) = \rho(y)$, then $x = y$.*

We are now in a position to prove Theorem 4.3.11.

Proof of Theorem 4.3.11. **Necessity.** We will prove the necessity by induction. Clearly, (†) is true for $|G| = 1$. For the inductive step, assume that (†) is true for solvable groups of order less than m. Suppose $|G| = m$. Let N be a minimal normal subgroup of G. Then, N is a p-subgroup (see Proposition 4.2.2). If $N = G$, then the result follows (see Example 4.3.10). Otherwise, if $N \neq G$, consider an arbitrary $\rho \in \mathrm{Irr}^*(G)$. If N is not contained in the kernel of ρ, then p divides $\alpha(\rho)$ (see Corollary 4.3.17) and hence, $\alpha(\rho) > 1$. If N is contained in the kernel of ρ, then ρ can be viewed as a non-trivial irreducible (G/N)-representation. Since G/N is solvable and $|G/N| < m$, by inductive assumption, $\alpha(\rho) > 1$.

Sufficiency. Assume (†) is true. Then, for any $\rho \in \mathrm{Irr}^*(G)$, there exists a prime divisor p (depending on ρ) such that $\rho(\underline{P})$ is a zero matrix for any $P \in \mathrm{Syl}_p(G)$ (see Corollary 4.3.19). Let $(p_1, \ldots, p_k)$ be a sequence of all distinct prime divisors of $|G|$ (no matter what the order is). Take an arbitrary collection of Sylow subgroups $\{P_i : P_i \in \mathrm{Syl}_{p_i}(G)\}_{i=1}^{k}$. We claim that $\rho(\underline{G}) = \rho(\mathcal{P})$, where $\mathcal{P} = \underline{P_1} \cdots \underline{P_k}$, for any $\rho \in \mathrm{Irr}(G)$. Indeed,

(a) if ρ is trivial, then $\rho(\mathcal{P}) = |G| = \rho(\underline{G})$;

(b) if ρ is non-trivial, then since $\rho(\underline{P_j})$ is a zero matrix for some $1 \leq j \leq k$ (see Lemma 4.3.13, Remark 4.3.15 and Lemma 4.3.18), so is $\rho(\mathcal{P})$. On the other hand, since $\rho \in \mathrm{Irr}^*(G)$, it follows that $\chi(\hat{\underline{G}}) = [\chi, \mathbf{1}_G] = 0$ and therefore, $\rho(\underline{G})$ is also a zero matrix (see Lemma 4.3.18).

Then, $\underline{G} = \mathcal{P}$ (see Proposition 4.3.20), from which it follows $G = P_1 \cdots P_k$. Since P_j is arbitrarily taken from $\mathrm{Syl}_{p_j}(G)$, it follows from Theorem 4.2.1 that G is solvable.

□

Example 4.3.21. Since A_5 is not solvable, there exists $\rho \in \mathrm{Irr}^*(A_5)$ such that $\alpha(\rho) = 1$. According to Table 4.1, this is the only non-trivial irreducible representation of A_5 with trivial α-characteristic.

4.3.2 α-Characteristic of Nilpotent Group Representations

If G is a nilpotent group, then one can strengthen the necessity part of Theorem 4.3.11 as follows (see [61]).

Proposition 4.3.22. *If G is a nilpotent group, then for any $\rho \in \mathrm{Irr}(G)$,*

$$\textit{there exists } v \in S(V) \textit{ such that } \alpha(\rho) = |G(v)|. \tag{‡}$$

We say that $\alpha(\rho)$ is *realized by the orbit* $G(v)$ or simply *realizable* if it satisfies (‡).

Remark 4.3.23. If the orders $|A|$ and $|B|$ are coprime, then the numbers $\alpha(\rho), \alpha(\psi)$ are coprime too, and, therefore, $\mathrm{lcm}(\alpha(\rho), \alpha(\psi)) = \alpha(\rho)\alpha(\psi)$ implying $\alpha(\rho\otimes\psi) = \alpha(\rho)\alpha(\psi)$. If the group orders are not coprime, then it could happen that $\alpha(\rho\otimes\psi)$ satisfy $\mathrm{lcm}(\alpha(\rho), \alpha(\psi)) < \alpha(\rho\otimes\psi) < \alpha(\rho)\alpha(\psi)$. As an example, one could take $A = B$ to be an extra special group of order p^3, p is an odd prime. This group has $p-1$ Galois conjugate representations of dimension p. Each of these representations is induced from a one-dimensional representation of $\mathbb{Z}_p \times \mathbb{Z}_p$, from which it follows that the α-characteristic of each representation is equal to p^2. Let ρ be one of these representations. Then, the irreducible representation $\rho\otimes\rho$ of $A \times A$ has α-characteristic equal to p^3.

Example 4.3.24. If G is a p-group, then $\alpha(\rho)$ is realizable for *any* G-representation ρ, which is not the case if G is a nilpotent group but not a p-group. In fact, consider the (reducible) $\mathbb{Z}_6$-representation $\rho := \psi_1 \otimes \mathbf{1}_{\mathbb{Z}_3} \oplus \mathbf{1}_{\mathbb{Z}_2} \otimes \psi_2$, where ψ_1 and ψ_2 are arbitrary non-trivial irreducible representations of $\mathbb{Z}_2$ and $\mathbb{Z}_3$, respectively. Then, $\Phi(\rho) = \{(\mathbb{Z}_1), (\mathbb{Z}_2), (\mathbb{Z}_3)\}$, and $\alpha(\rho) = 1$ is not realizable.

4.4 α-Characteristic of an Augmentation Module Related to 2-transitive Group Actions

The following example is the starting point of our discussion.

Example 4.4.1. Note that A_5, the smallest *non-solvable* group, is a 2-transitive group (see Definitions 4.2.3 and 4.2.5). To be more explicit, A_5 admits two non-equivalent irreducible augmentation representations $\psi_3 = \rho^a_{[A_5;A_4]}$ and $\psi_4 = \rho^a_{[A_5;D_5]}$ (see Table 4.1). Since $\alpha(\psi_3) = 5$ while $\alpha(\psi_4) = 1$, we arrive at the following question: given an augmentation submodule $\rho \in \varrho_2(G)$ associated to the 2-transitive G-action on G/H by left translation, under which conditions does one have $\alpha(\rho) > 1$?

4.4.1 2-Transitive Groups

Let G be a 2-transitive group acting faithfully on a set X. According to Burnside Theorem (see [41, Theorem 4.1B]), the socle S of G is either a non-abelian simple group or an elementary abelian group which acts regularly on X. Thus, 2-transitive groups are naturally divided into two classes

- *Almost simple groups.* G is called *almost simple* if $S \leq G \leq \mathrm{Aut}(S)$ for some non-abelian simple group S.

- *Affine groups.* If S is elementary abelian, then G admits the following description.

  Let V be a d-dimensional vector space over a finite field $\mathbb{F}$. A group G is called *affine* if $V \leq G \leq \mathrm{AGL}(V)$, where V is considered as an additive group and $\mathrm{AGL}(V)$ is the group of all invertible affine transformations of V. A group G admits a decomposition

$G = VG_o$ where $G_o = G \cap \mathrm{GL}(V)$ is a zero stabilizer in G. Thus, $G \cong V \rtimes G_o$ ($\rtimes$ stands for the semidirect product). The group G acts 2-transitively on V if and only if G_o acts transitively on the set of non-zero vectors of V. In this case, V is the socle of G.

Remark 4.4.2. Note that a solvable 2-transitive group is always affine. However, the converse is not true: for example, the full affine group $\mathrm{AGL}(V)$ of the vector space V is solvable if and only if $\mathrm{GL}(V)$ is. The latter happens only when $d = 1$ or $d = 2$ and $|\mathbb{F}| \in \{2, 3\}$.

4.4.2 Main Result

Our main result provides a complete description of all augmentation modules related to 2-transitive group actions with non-trivial α-characteristic.

Theorem 4.4.3. *Let $(G; X)$ be a 2-transitive group action.*

(i) If G is affine and acts faithfully on X, then $\alpha(\rho^a_{(G;X)}) > 1$.

(ii) If G is almost simple, then all $\rho \in \varrho_2(G)$ satisfying $\alpha(\rho) > 1$ are described in Table 4.2 provided that $|X|$ is a prime power.

Table 4.2. Almost simple 2-transitive groups with $\rho \in \varrho_2(G)$ satisfying $\alpha(\rho) > 1$

$\|X\|$	condition	N	$\max \|G/N\|$	# of non-equivalent actions
2	$n \geq 5$	A_n	2	1
n	$n \geq 5$	A_n	2	4 if $n = 6$; 2 otherwise
$(q^d - 1)/(q - 1)$	$d \geq 2$, $(d, q) \neq (2, 2), (2, 3)$	$\mathrm{PSL}(d, q)$	$\gcd\{d, q - 1\} \cdot e$	2 if $d > 2$; 1 otherwise
11		$\mathrm{PSL}(2, 11)$	1	2
11		M_{11}	1	1
23		M_{23}	1	1

$q = p^e$ for some prime p and $e \in \mathbb{N}$; N is the socle of G; M_{11} and M_{23} are Mathieu groups

The proof of Theorem 4.4.3 is based on the classification of finite 2-transitive groups (see [31]) and the following lemma.

Lemma 4.4.4. *Let H be a proper subgroup of G. Then, $\alpha(\rho^a_{[G;H]}) > 1$ if and only if $|G : H|$ is a prime power. In the latter case, p divides $\alpha(\rho^a_{[G;H]})$ where p is the unique prime divisor of $|G : H|$.*

This lemma was first proved in [61]. Here, we provided a more conceptual proof by using *Frobenius reciprocity.*

Proof. If $[G : H]$ is not a prime power, then for each prime divisor p of $|G|$ a Sylow subgroup $P \in \mathrm{Syl}_p(G)$ has at least two orbits on G/H. Therefore, $[(1^G_H)_P, \mathbf{1}_P] \geq 2$ implying $[\rho^a_{[G;H]}, \mathbf{1}_P] > 0$. By Lemma 4.3.13, p does not divide $\alpha(\rho^a_{[G;H]})$. Since this holds for any prime divisor of $|G|$, we conclude that $\alpha(\rho^a_{[G;H]}) = 1$.

Conversely, suppose $|G : H| = p^k$ for some prime p. Then, a Sylow p-subgroup $P \in \mathrm{Syl}(G)$ acts transitively on the coset space G/H implying that $[\mathbf{1}^G_H, \mathbf{1}_P] = 1$. Therefore $[\rho^a_{[G;H]}, \mathbf{1}_P] = 0$ and we are done by Lemma 4.3.13. □

Remark 4.4.5. Note that although any non-trivial irreducible representation of a solvable group admits a non-trivial α-characteristic (see Theorem 4.3.11), it may not be true for their direct sum (see Proposition 4.3.2(e)). However, by Lemma 4.4.4, an augmentation submodule associated to a transitive G-set of order prime power would admit non-trivial α-characteristic. In particular, the α-characteristic of every non-trivial irreducible constituent of the augmentation submodule is non-trivial.

Now we can prove the aforementioned Theorem 4.4.3.

Proof of Theorem 4.4.3. If G is an affine 2-transitive group, then its socle S is an elementary abelian group which acts faithfully on X. By the Burnside Theorem, S acts regularly on X. Therefore, $|X| = |S|$ is a prime power and we are done by Lemma 4.4.4.

If G is an almost simple 2-transitive group, then all 2-transitive G-sets of power prime order are obtained by the inspection of Table 7.4 in [31], which yields Table 4.2. □

Remark 4.4.6. For the complete description of 2-transitive faithful actions of affine groups, we refer to Table 7.3 in [31].

4.4.3 Examples

In this subsection, we will give some concrete examples of 2-transitive groups illustrating Theorem 4.4.3 and Remark 4.4.5.

Example 4.4.7. The group $G := \mathrm{AGL}_3(2) = \mathbb{Z}_2^3 \rtimes \mathrm{GL}(3,2)$ is a non-solvable 2-transitive affine group (see Remark 4.4.2) with four augmentation representations (see Table 4.3) arising from 2-transitive actions. By Theorem 4.4.3 (i), $\alpha(\rho) > 1$ for all $\rho \in \varrho_2(G)$.

Table 4.3. Irreducible AGL$(3,2)$-representations associated to 2-transitive actions

$\rho_i = \rho^a_{(G;X_i)}$	$\alpha(\rho_i)$	$\lvert X_i\rvert$	character										
ρ_1	7	7	6	6	2	2	2	.	.	.	.	-1	-1
ρ_2	2	2^3	7	-1	3	-1	-1	1	-1	1	-1	.	.
ρ_3	2	2^3	7	7	-1	-1	-1	-1	-1	1	1	.	.
ρ_4	2	2^3	7	-1	-1	3	-1	-1	1	1	-1	.	.

Example 4.4.8. The group S_5 is an almost simple group with three 2-transitive actions: S_5/A_5, S_5/S_4 and $S_5/\mathrm{AGL}_1(5)$. The corresponding augmentation representations are denoted by ξ_1, ξ_3 and ξ_6 in Table 4.4. According to Lemma 4.4.4, only $\xi_1 = \rho^a_{[S_5;A_5]}$ and $\xi_6 = \rho^a_{[S_5;S_4]}$ admit non-trivial α-characteristics.

Remark 4.4.9. In some cases, Theorem 4.4.3 can still help one to determine whether $\alpha(\rho)$ is trivial even when ρ is not an augmentation representation. For example, S_5 admits two 4-dimensional irreducible representations ξ_2 and ξ_6 (see Table 4.4). Note that ξ_6 is an augmentation representation related to a 2-transitive action while ξ_2 is not. Let V and V^- be S_5-modules corresponding to ζ_6 and ζ_2, respectively. In Section 4.6, we will show

Table 4.4. Character table of $G = S_5$

ρ	$\alpha(\rho)$	(2-transitivity) H	$\lvert G:H\rvert$	character						
ξ_1	2	A_5	2	1	-1	1	1	-1	-1	1
ξ_2	10			4	-2	.	1	1	.	-1
ξ_3	1	$\mathrm{AGL}(\mathbb{Z}_5)$	6	5	-1	1	-1	-1	1	.
ξ_4	2			6	.	-2	.	.	.	1
ξ_5	1			5	1	1	-1	1	-1	.
ξ_6	5	S_4	5	4	2	.	1	-1	.	-1
ξ_0	1			1	1	1	1	1	1	1

that there exists an admissible equivariant map from V^- to V from which it follows that $\alpha(\xi_2) \geq \alpha(\xi_6) > 1$ (see Example 4.4.8)—this agrees with Table 4.4. One can apply similar argument to $(n-1)$-dimensional irreducible S_n-representations with $n > 5$ being a prime power.

Our last example illustrates Remark 4.4.5.

Example 4.4.10. Consider the solvable group $G := SL_2(\mathbb{Z}_3)$ acting transitively (but not 2-transitively, in particular, the augmentation representation is reducible, see Definition 4.2.5) on the set X of eight non-zero vectors of $(\mathbb{Z}_3)^2$. It follows from Lemma 4.4.4 that $2 \mid \alpha(\rho^a_{[G;X]})$. Therefore, 2 divides α-characteristic of every non-trivial constituent of $\rho^a_{[G;X]}$, and there are three of those: two 2-dimensional and one 3-dimensional.

4.5 Irreducible Representations with Trivial α-Characteristic

As we already know, a finite group G admitting an irreducible complex representation ρ with trivial α-characteristic is non-solvable.

In this section, given $N \triangleleft G$ and an irreducible G-representation (resp. irreducible N-representation) with trivial α-characteristics, we study the α-characteristics of its restriction to N (resp. induction to G).

4.5.1 Motivating Examples

Keeping in mind Proposition 4.3.2, consider the following example.

Example 4.5.1. Consider $N := A_5 \trianglelefteq S_5 =: G$ (N is simple and $G \simeq \operatorname{Aut}(N)$). The isotypical decomposition of ξ_N for $\xi \in \operatorname{Irr}(G)$ are as follows (see also Tables 4.1 and 4.4):

$$\begin{aligned} (\xi_0)_N = (\xi_1)_N &= \psi_0, \\ (\xi_2)_N = (\xi_6)_N &= \psi_3, \\ (\xi_3)_N = (\xi_5)_N &= \psi_4, \\ (\xi_4)_N &= \psi_1 \oplus \psi_2. \end{aligned}$$

Observe that

(i) $\alpha(\psi_0) = 1$ divides both $\alpha(\xi_0) = 1$ and $\alpha(\xi_1) = 2$;

(ii) $\alpha(\psi_3) = 5$ divides both $\alpha(\xi_2) = 10$ and $\alpha(\xi_6) = 5$;

(iii) $\alpha(\psi_4) = 1$ divides both $\alpha(\xi_3) = 1$ and $\alpha(\xi_5) = 1$;

(iv) both $\alpha(\psi_1) = 2$ and $\alpha(\psi_2) = 2$ divide $\alpha(\xi_4) = 2$.

Remark 4.5.2. Clearly, Example 4.5.1 is in the complete agreement with Proposition 4.3.2 (b). On the other hand, it also gives rise to the following question: under which condition, does $\alpha(\theta) = 1$ imply $\alpha(\theta^G) = 1$ for $\theta \in \operatorname{Irr}(N)$ and $N \trianglelefteq G$? The answer is given in the nest subsection.

4.5.2 Induction and Restriction of Representations with Trivial α-Characteristic

Let N be a non-trivial proper subgroup of G. Pick an arbitrary $\rho \in \operatorname{Irr}(G)$. By Proposition 4.3.2 (a), $\alpha(\rho_N)$ divides $\alpha(\rho)$. Therefore, if $\alpha(\rho) = 1$, then $\alpha(\rho_N) = 1$. Decomposing ρ_N into a direct sum of N-irreducible representations $\rho_N = \sum_{i=1}^{k} \theta_i$, we obtain

$\gcd(\alpha(\theta_1), ..., \alpha(\theta_k)) = 1$ (see Proposition 4.3.2 (e)). If N is normal in G, then all constituents θ_i are G-conjugate by Clifford's theorem (see [58]), and have the same α-characteristic (see Proposition 4.3.2 (c)). Hence, $\alpha(\theta_i) = 1$ for each $i = 1, ..., k$. In other words, trivial α-characteristic of an irreducible G-representation ρ is inherited by all constituents of its restriction ρ_N. But this does not happen for the induction. More precisely, if $\theta \in \mathrm{Irr}^*(N)$ has trivial α-characteristic, then some of the constituents of θ^G may have non-trivial α-characteristic even in the case of N being normal. For example, take $G = A_5 \times \mathbb{Z}_7, N = A_5$ and $\theta = \rho^a_{A_5:D_5}$. Then $\theta^G = \sum_{i=0}^{6} \theta \otimes \lambda^i$ where $\lambda \in \mathrm{Irr}^*(\mathbb{Z}_7)$. Clearly, $\alpha(\theta \otimes 1_{\mathbb{Z}_7}) = \alpha(\theta) = 1$. By Lemma 3.5.2 in [61], $\alpha(\theta \otimes \lambda^i) = 7$ if $i \neq 0$. Thus, θ^G contains only one irreducible constituent with trivial α-characteristic.

Proposition 4.5.3. *Let $N \trianglelefteq G$ and $\theta \in \mathrm{Irr}(N)$ with $\alpha(\theta) = 1$. Then, $\alpha(\theta^G) = 1$.*

Proof. Let W (resp. V) be the N-representation (resp. G-representation) corresponding to θ (resp. θ^G). It suffices to show that $\alpha(\theta^G)_p = 1$ for each prime divisor p of $|G|$.

Pick a Sylow p-subgroup $P \leq G$. Then, $P \cap N$ is a Sylow p-subgroup of N. It follows from $\alpha(\theta) = 1$ that the subspace $W_1 := W^{P \cap N}$ is non-trivial (see Lemma 4.3.13). Pick an arbitrary non-zero $w \in W_1$. Then, the vector $v := \sum_{g \in P} gw$ is fixed by any element of P, that is, $Pv = v$. We claim that $v \neq 0$. Let T_1 be a transversal of $P/(P \cap N)$. By isomorphism $P/(P \cap N) \cong PN/N$, the set T_1 is a transversal of PN/N. Now we complete T_1 to a transversal T of G/N and set $V = \oplus_{t \in T} tW$.

Now $P = T_1(P \cap N)$ implies $v = |P \cap N| \sum_{t \in T_1} tw \neq 0$. Thus, V^P is non-trivial, and, consequently, $\alpha(\theta^G)_p = 1$. □

In general, it is not clear whether $\alpha(\theta) = 1$ implies that θ^G contains an irreducible constituent with trivial α-characteristic. Proposition 4.5.5 below provides sufficient conditions for that. Its proof is based on the following lemma (see [58]).

Lemma 4.5.4. *Suppose $N \trianglelefteq G$. Let χ and ω be irreducible characters of G and N, respectively, such that $[\chi_N, \omega] > 0$. If $p = |G : N|$ is a prime, then for the decompositions of χ_N and ω^G, one of the following two statements takes place:*

(a) $\chi_N = \sum_{i=0}^{p-1} \omega^{(g_i)}$ *and* $\omega^G = \chi$.

(b) $\chi_N = \omega$ *and* $\omega^G = \sum_{i=0}^{p-1} \chi\phi_i$*, where* $\{\phi_i\}_{i=0}^{p-1}$ *is the set of all irreducible characters of* $G/N \simeq \mathbb{Z}_p$*, which can be viewed as irreducible characters of G as well.*

Proposition 4.5.5. *Let $N \trianglelefteq G$ and $\omega \in \mathrm{Irr}(N)$ with $\alpha(\omega) = 1$. If G/N is solvable then, ω^G admits an irreducible component ρ satisfying $\alpha(\rho) = 1$.*

Proof. We use induction over $|G/N|$. Pick a maximal normal subgroup M of G which contains N. Then, M/N is a maximal normal subgroup of G/N, and, by solvability of G/N, M is of prime index, say p, in G. If $M \neq N$, then, by induction hypothesis, the representation ω^M has an irreducible component, say σ, with $\alpha(\sigma) = 1$. Applying induction hypothesis to the pair $M \trianglelefteq G$ and σ we conclude that σ^G contains an irreducible component $\rho \in \mathrm{Irr}(G)$ with $\alpha(\rho) = 1$. Now it follows from $\omega^G = (\omega^M)^G$ that ρ is a constituent of ω^G.

Assume now that $M = N$, that is $[G : N] = p$ is prime. By Lemma 4.5.5 either ω^G is irreducible or $\omega^G = \sum_{i=0}^{p-1} \chi\xi_i$ (hereafter, ω, χ, ζ_j stand for characters rather than for representations). In the first case we are done by Proposition 4.5.3. Consider now the second case:

$\omega^G = \sum_{i=0}^{p-1} \chi\xi_i$. In this case, in suffices to show that there exists $0 \le j \le p-1$ such that $\chi\xi_j(\underline{R}) > 0$ for any prime factor r of $|G|$ and $R \in \mathrm{Syl}_r(G)$. Indeed, note that $\sum_{i=0}^{p-1} \xi_i(g) = p$ if $g \in N$ and 0 otherwise. Therefore,

$$\omega^G(\underline{R}) = \sum_{i=0}^{p-1} \chi\xi_i(\underline{R}) = \sum_{g \in R} \sum_{i=0}^{p-1} \chi(g)\xi_i(g) = \sum_{g \in R} \chi(g) \left(\sum_{i=0}^{p-1} \xi_i(g) \right)$$
$$= p \sum_{g \in R \cap N} \chi(g) = p \sum_{g \in R \cap N} \omega(g) = p\,\omega(\underline{R \cap N}).$$

Since $R \cap N$ is either trivial or a Sylow r-subgroup of N, one always has

$$\sum_{i=0}^{p-1} \chi\xi_i(\underline{R}) = p\,\omega(\underline{R \cap N}) > 0.$$

If $r = p$, then $\chi\xi_j(\underline{R}) > 0$ for some $0 \leq j \leq p-1$. If $r \neq p$, then $R \leq N$ and it follows that $\chi\xi_j(\underline{R}) = \chi(\underline{R}) > 0$ for the same j as well. The result follows. $\square$

The following example illustrates Proposition 4.5.5.

Example 4.5.6. Consider $N := \mathrm{PSL}(2,8) \trianglelefteq \mathrm{P\Gamma L}(2,8) =: G$ (N is simple, $G \simeq \mathrm{Aut}(N)$ and $|G:N| = 3$ is a prime). The relation between $\mathrm{Irr}(G)$ and $\mathrm{Irr}(N)$ with respect to restriction and induction are described as follows (see also Tables 4.5 and 4.6):

$$\begin{aligned} {\xi_0}_N &= {\xi_1}_N = {\xi_2}_N = \psi_0, & \psi_0^G &= \xi_0 \oplus \xi_1 \oplus \xi_2, \\ {\xi_3}_N &= {\xi_4}_N = {\xi_5}_N = \psi_1, & \psi_1^G &= \xi_3 \oplus \xi_4 \oplus \xi_5, \\ {\xi_6}_N &= {\xi_7}_N = {\xi_8}_N = \psi_5, \qquad \textit{and} & \psi_5^G &= \xi_6 \oplus \xi_7 \oplus \xi_8, \\ {\xi_9}_N &= \psi_2 \oplus \psi_3 \oplus \psi_4, & \psi_2^G &= \psi_3^G = \psi_4^G = \xi_9, \\ {\xi_{10}}_N &= \psi_6 \oplus \psi_7 \oplus \psi_8. & \psi_6^G &= \psi_7^G = \psi_8^G = \xi_{10}. \end{aligned}$$

One can observe that $\alpha(\psi_6) = \alpha(\psi_7) = \alpha(\psi_8) = \alpha(\xi_{10}) = 1$, which agrees with Proposition 4.5.5.

Table 4.5. Character table of PSL(2, 8)

ρ	$\alpha(\rho)$	character								
ψ_0	1	1	1	1	1	1	1	1	1	1
ψ_1	2	7	-1	-2	1	1	1	.	.	.
ψ_2	2	7	-1	1	A	C	B	.	.	.
ψ_3	2	7	-1	1	B	A	C	.	.	.
ψ_4	2	7	-1	1	C	B	A	.	.	.
ψ_5	3	8	.	-1	-1	-1	-1	1	1	1
ψ_6	1	9	1	.	.	.	.	D	F	E
ψ_7	1	9	1	.	.	.	.	E	D	F
ψ_8	1	9	1	.	.	.	.	F	E	D

Table 4.6. Character table of $\mathrm{P\Gamma L}(2,8)$

ρ	$\alpha(\rho)$	character										
ξ_0	1	1	1	1	1	1	1	1	1	1	1	1
ξ_1	3	1	1	1	1	A	A^*	A^*	A	1	A	A^*
ξ_2	3	1	1	1	1	A^*	A	A	A^*	1	A^*	A
ξ_3	2	7	-2	1	-1	1	1	-1	-1	.	1	1
ξ_4	6	7	-2	1	-1	A^*	A	$-A$	$-A^*$	.	A^*	A
ξ_5	6	7	-2	1	-1	A	A^*	$-A^*$	$-A$	.	A	A^*
ξ_6	3	8	-1	-1	.	2	2	.	.	1	-1	-1
ξ_7	3	8	-1	-1	.	B	B^*	.	.	1	$-A$	$-A^*$
ξ_8	3	8	-1	-1	.	B^*	B	.	.	1	$-A^*$	$-A$
ξ_9	2	21	3	.	-3	.	.	.	.	.	.	.
ξ_{10}	1	27	.	.	3	.	.	.	.	-1	.	.

4.5.3 Groups with Totally Trivial α-Characteristic

This section is devoted to the groups which have no irreducible representation with non-trivial α-characteristic. In what follows, we denote this class of groups by $\mathfrak{T}$. We know that this class is non-empty, since $J_1 \in \mathfrak{T}$. We also know that all groups in $\mathfrak{T}$ are non-solvable. Below, we collect elementary properties of $\mathfrak{T}$.

Proposition 4.5.7. *Let $G \in \mathfrak{T}$. Then:*

(a) if N is a normal subgroup of G, then $N, G/N \in \mathfrak{T}$;

(b) all composition factors of G belong to $\mathfrak{T}$;

(c) if $H \in \mathfrak{T}$, then $G \times H \in \mathfrak{T}$.

Proof. Part (a). The inclusion $G/N \in \mathfrak{T}$ follows from the fact that every irreducible representation of G/N may be considered as an irreducible representation of G. The inclusion $N \in \mathfrak{T}$ follows from Proposition 4.3.2(b).

Part (b) is a direct consequence of (a).

Part (c) is a direct consequence of Lemma 3.5.2 in [61], since each irreducible representation of $G \times H$ is a tensor product $\psi \otimes \phi$, where $\psi \in \mathrm{Irr}(G), \phi \in \mathrm{Irr}(H)$. $\square$

Remark 4.5.8. A non-abelian simple group $G := PSL_2(q)$ ($q = p^e$ is a prime power) has irreducible representations with trivial α-characteristic. More precisely, a direct check of the character table of G shows that any character of G of degree $q-1$ does not appear in the decomposition of 1_P^G where P is a Sylow p-subgroup of G. Thus, the α-characteristic of any G-representation of dimension $q-1$ is divisible by p.

Our computations in GAP suggest the following conjecture.

Conjecture 4.5.9. If $G \in \mathfrak{T}$ is a simple group, then it is one of the sporadic groups.

4.6 Existence of Quadratic Equivariant Maps

4.6.1 General Construction

In general, the problem of existence of equivariant maps between G-manifolds is rather complicated. We will study Problem 4.1.2 in the following setting: V is a *faithful irreducible* G-representation and W is another G-representation of the same dimension. It is well-known that if V is faithful, then there exists a positive integer k such that the symmetric tensor power $\mathrm{Sym}^k(V)$ contains W (see, for example, [58]). Thus, assume $W \subset \mathrm{Sym}^k(V)$ and let

$$\begin{cases} \triangle : V \to V^{\otimes k}, \\ \triangle(v) = \underbrace{v \otimes v \otimes \cdots \otimes v}_{k \text{ times}}. \end{cases}$$

be the corresponding *diagonal* map. Observe that $\triangle$ is G-equivariant and $\triangle(V)$ spans $\mathrm{Sym}^k(V)$. Let $A : \mathrm{Sym}^k(V) \to W$ be a G-equivariant linear operator (e.g., orthogonal projection). Then, $\phi = A \circ \triangle$ is a k-homogeneous G-equivariant map from V to W, which admits the following criterion of admissibility (see, for example, [75]).

Proposition 4.6.1. *ϕ is admissible if and only if* $\ker A \cap \triangle(V) = \{\mathbf{0}\}$.

In practice, given characters of V and W, constructing an admissible homogeneous equivariant map $\phi : V \to W$ involves the following steps:

(S1) Finding k satisfying $W \subset \mathrm{Sym}^k(V)$.

(S2) Computing matrices representing the G-actions on V and $\mathrm{Sym}^k(V)$, and finding isotypical basis of $W \subset \mathrm{Sym}^k(V)$.

(S3) Verification of the admissibility of $\phi = A \circ \triangle$ by the Weak Nullstellensatz (see [35]):

Proposition 4.6.2. *Let $Q := \{q_i\}_{i=1}^m \subset R := \mathbb{C}[x_1, \dots, x_N]$ be a collection of polynomials and $I := \{\sum_{i=1}^m r_i q_i : r_i \in R\}$ the ideal generated by Q. Then, the following statements are equivalent:*

- *Q admits no common zeros.*
- *The Gröbner basis of I contains the constant polynomial* 1.

Remark 4.6.3.

(i) Steps (S1) and (S2) are related to the classical *Clebsch-Gordan problem* of an isotypical decomposition of tensor product of representations (see, for example, [34]).

(ii) For Step (S3), one can use Mathematica [110] to compute the Gröbner basis.

(iii) If ϕ is admissible, then $\deg(\phi)$ is well-defined and equal to k^n, where $n = \dim(V)$.

To illustrate Proposition 4.6.1 and also give a brief idea about Steps (S1)–(S3), consider the following example (see also [8] and [12]); the detail will be provided later (see Section 4.6.2). In what follows, denote by (V, G) a faithful G-representation and by $(V, G/H)$ a non-faithful G-representation with kernel $H \trianglelefteq G$.

Example 4.6.4. Consider the symmetric tensor square $(\mathrm{Sym}^2_{\mathbb{C}}(\mathbb{H}), \mathcal{Q}_8/\mathbb{Z}_2)$ of the complex representation $(\mathbb{H}, \mathcal{Q}_8)$. One can easily check that

$$e_1 = \frac{1 \otimes 1 + j \otimes j}{2}, \quad e_2 = \frac{1 \otimes 1 - j \otimes j}{2}, \quad e_3 = \frac{j \otimes 1 + j \otimes 1}{2}$$

form an isotypical basis of $\mathrm{Sym}^2_{\mathbb{C}}(\mathbb{H})$ and

$$\triangle(z_1 + jz_2) = e_1({z_1}^2 + {z_2}^2) + e_2({z_1}^2 - {z_2}^2) + e_3(2z_1z_2).$$

Let P_1, P_2 and P_3 denote the natural $\mathcal{Q}_8$-equivariant projections onto the subspaces of $\mathrm{Sym}^2_{\mathbb{C}}(\mathbb{H})$ spanned by $\{e_1, e_2\}$ $\{e_2, e_3\}$ and $\{e_1, e_3\}$, respectively. A direct computation shows that $\ker P_i \cap \triangle(\mathbb{H}) = \{0\}$ for $i = 1, 2, 3$. Consequently, $f_i = P_i \circ \triangle$, $i = 1, 2, 3$, are admissible $\mathcal{Q}_8$-equivariant maps.

Remark 4.6.5. Example 4.6.4 shows the existence of an admissible 2-homogeneous $\mathcal{Q}_8$-equivariant map $f_1 : (\mathbb{H}, \mathcal{Q}_8) \to (\mathbb{C}^2, \mathcal{Q}_8/\mathbb{Z}_2)$, where $\mathbb{C}^2 \subset \mathrm{Sym}^2_{\mathbb{C}}(\mathbb{H})$ is the subrepresentation spanned by $\{e_1, e_2\}$. In addition, $\alpha(\mathcal{Q}_8, \mathbb{H}) = 8$ since $\mathcal{Q}_8$ acts freely on $S(\mathbb{H})$. Therefore, it follows from the congruence principle that for any admissible $\mathcal{Q}_8$-equivariant map f : $(\mathbb{H}, \mathcal{Q}_8) \to (\mathbb{C}^2, \mathbb{H}/\mathbb{Z}_2)$,

$$\deg(f) \equiv \deg(f_1) = 2^2 = 4 \pmod 8.$$

In particular, $\deg(f)$ is different from 0.

In addition to the congruence principle, one can also analyze Example 4.6.4 by the following result (see [6]).

Theorem 4.6.6 (Atiyah-Tall)**.** *Let G be a finite p-group and, V and W two G-representations. There exists an admissible equivariant map $f : V \to W$ with $\deg(f) \not\equiv 0 \pmod p$ if and only if the irreducible components of V and W are Galois conjugate in pairs.*

Remark 4.6.7. Since $(\mathbb{H}, \mathcal{Q}_8)$ is irreducible while $(\mathbb{C}^2, \mathcal{Q}_8/\mathbb{Z}_2)$ is not, the irreducible components of $(\mathbb{H}, \mathcal{Q}_8)$ and $(\mathbb{C}^2, \mathcal{Q}_8/\mathbb{Z}_2)$ are not Galois conjugate in pairs. It follows from Theorem 4.6.6 that

$$\deg(f) \equiv 0 \pmod 2$$

for any admissible $\mathcal{Q}_8$-equivariant map $f : (\mathbb{H}, \mathcal{Q}_8) \to (\mathbb{C}^2, \mathcal{Q}_8/\mathbb{Z}_2)$.

A comparison between Remark 4.6.5 and Remark 4.6.7 shows that the result for Example 4.6.4 obtained from the congruence principle is more informative.

Possible extensions of Example 4.6.4 to arbitrary p-groups were suggested by A. Kushkuley (see [12]). On the other hand, notice that $(\mathbb{H}, \mathcal{Q}_8)$ (resp. $(\mathbb{C}^2, \mathcal{Q}_8/\mathbb{Z}_2)$) is induced by the one-dimensional representation $(\mathbb{C}, \mathbb{Z}_4)$ (resp. $(\mathbb{C}, \mathbb{Z}_4/\mathbb{Z}_2)$). Furthermore, $\psi(z) = z^2$ is an admissible 2-homogeneous Z_4-equivariant map from $(\mathbb{C}, \mathbb{Z}_4)$ to $(\mathbb{C}, \mathbb{Z}_4/\mathbb{Z}_2)$ and f_1 (see Example 4.6.4) is in fact the $\mathcal{Q}_8$-equivariant extension of ψ (see Figure 4.2).

$$\begin{tikzcd}
(\mathbb{H}, \mathcal{Q}_8) \ni gz \arrow[r, "f_1"'] & gz^2 \in (\mathbb{C}^2, \mathcal{Q}_8/\mathbb{Z}_2) \\
(\mathbb{C}, \mathbb{Z}_4) \ni z \arrow[u, "\text{induced representation}"] \arrow[r, "\psi"'] & z^2 \in (\mathbb{C}, \mathbb{Z}_4/\mathbb{Z}_2) \arrow[u, "\text{induced representation}"']
\end{tikzcd}$$

Figure 4.2. f_1 as an extension of ψ

The above example shows that if a G-representation V is induced from an H-representation U $(H < G)$, and f is an H-equivariant homogeneous admissible map defined on U, then f can be canonically extended to a G-equivariant homogeneous admissible map defined on V. However, the construction of an admissible homogeneous G-equivariant map becomes more involved if we are given a representation which is not induced from a subgroup. The example considered in the next subsection suggests a method to deal with this problem in several cases.

4.6.2 Example: S_5-Representations

In this subsection, we will construct an admissible 2-homogeneous equivariant map from V^- to V (see Remark 4.4.9) following Steps (S1)–(S3), which will be illustrated in detail. In what follows, for an S_5-representation X and $\sigma \in S_5$, denote by $\rho_X(\sigma)$ and $\chi_X(\sigma)$ the corresponding matrix representation and character, respectively.

(S1) Denote $U := \mathrm{Sym}^2(V^-)$. Recall that $\chi_U(\sigma) = (1/2)(\chi_{V^-}(\sigma)^2 + \chi_{V^-}(\sigma^2))$ (see, for example, [96]) and using Table 4.4, one has $U = \mathbf{1}_{S_5} \oplus V \oplus V_5$.

(S2) One can take the linear equivariant map $A : U \to V$ to be the orthogonal projection, which is also given by

$$A = \frac{\dim(V)}{|S_5|} \sum_{\sigma \in S_5} \overline{\chi_V(\sigma)}\, \rho_U(\sigma). \tag{4.2}$$

Take basis $\mathcal{B}_{V^-} := \{e_i\}_{i=1}^4$ of V^- and $\mathcal{B}_U := \{e_i \otimes e_j\}_{1 \le i \le j \le 4}$ of U. To obtain $\rho_U(\sigma)$ corresponding to $\mathcal{B}_U$, $\sigma \in S_5$, it suffices to let

$$\rho_{V^-}((12)) := \begin{bmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}, \ \rho_{V^-}((12345)) := \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & -1 & -1 & -1 \end{bmatrix}$$

be the matrices corresponding to $\mathcal{B}_{V^-}$, and use formula

$$\rho_U(\sigma)(e_i \otimes e_j) = \rho_{V^-}(\sigma)(e_i) \otimes \rho_{V^-}(\sigma)(e_j).$$

Substitution of $\rho_U(\sigma)$, $\sigma \in S_5$, in (4.2) yields

$$A = \begin{bmatrix}
\frac{3}{5} & -\frac{2}{5} & -\frac{2}{5} & -\frac{2}{5} & -\frac{2}{5} & -\frac{2}{5} & -\frac{2}{5} & -\frac{2}{5} & -\frac{2}{5} & -\frac{2}{5} \\
-\frac{1}{15} & \frac{4}{15} & \frac{4}{15} & \frac{4}{15} & -\frac{1}{15} & \frac{4}{15} & \frac{4}{15} & \frac{4}{15} & \frac{4}{15} & \frac{4}{15} \\
-\frac{1}{15} & \frac{4}{15} & \frac{4}{15} & \frac{4}{15} & \frac{4}{15} & \frac{4}{15} & \frac{4}{15} & -\frac{1}{15} & \frac{4}{15} & \frac{4}{15} \\
-\frac{1}{15} & \frac{4}{15} & \frac{4}{15} & \frac{4}{15} & \frac{4}{15} & \frac{4}{15} & \frac{4}{15} & \frac{4}{15} & \frac{4}{15} & -\frac{1}{15} \\
-\frac{2}{5} & -\frac{2}{5} & -\frac{2}{5} & -\frac{2}{5} & \frac{3}{5} & -\frac{2}{5} & -\frac{2}{5} & -\frac{2}{5} & -\frac{2}{5} & -\frac{2}{5} \\
\frac{4}{15} & \frac{4}{15} & \frac{4}{15} & \frac{4}{15} & -\frac{1}{15} & \frac{4}{15} & \frac{4}{15} & -\frac{1}{15} & \frac{4}{15} & \frac{4}{15} \\
\frac{4}{15} & \frac{4}{15} & \frac{4}{15} & \frac{4}{15} & -\frac{1}{15} & \frac{4}{15} & \frac{4}{15} & \frac{4}{15} & \frac{4}{15} & -\frac{1}{15} \\
-\frac{2}{5} & -\frac{2}{5} & -\frac{2}{5} & -\frac{2}{5} & -\frac{2}{5} & -\frac{2}{5} & -\frac{2}{5} & \frac{3}{5} & -\frac{2}{5} & -\frac{2}{5} \\
\frac{4}{15} & \frac{4}{15} & \frac{4}{15} & \frac{4}{15} & \frac{4}{15} & \frac{4}{15} & \frac{4}{15} & -\frac{1}{15} & \frac{4}{15} & -\frac{1}{15} \\
-\frac{2}{5} & -\frac{2}{5} & -\frac{2}{5} & -\frac{2}{5} & -\frac{2}{5} & -\frac{2}{5} & -\frac{2}{5} & -\frac{2}{5} & -\frac{2}{5} & \frac{3}{5}
\end{bmatrix}.$$

(S3) The column vectors of the projection matrix A span V and one can obtain a basis $\mathcal{B}_V$ of $V \subset U$ using the *Gram-Schmidt process*. With $\mathcal{B}_{V^-}$ and $\mathcal{B}_V$, $\phi := A \circ \triangle$ can be viewed as a map from $\mathbb{C}^4$ to $\mathbb{C}^4$. To be more explicit, $\phi = [\phi_1, \dots, \phi_4]$, where $\phi_i = \phi_i(x_1, x_2, x_3, x_4)$, $i = 1, \dots, 4$, is a 2-homogeneous polynomial. Denote $\mathcal{P} := \{\phi_i\}_{i=1}^4$ and $\mathcal{P}_k := \{\phi_i|_{x_k=1}\}_{i=1}^4$. Then, ϕ admits no non-trivial zero, i.e., $\mathcal{P}$ admits no non-trivial common zeros, if and only if $\mathcal{P}_k$ admits no common zeros, i.e., the Gröbner basis of the ideal generated by $\mathcal{P}_k$ contains only 1, for $k = 1, 2, 3, 4$ (see Proposition 4.6.2). One can use Mathematica to show that it is indeed the case and thereby ϕ is admissible.

We have the following result for S_5-representations V^- and V.

Proposition 4.6.8. *There exists an admissible 2-homogeneous S_5-equivariant map $\phi : V^- \to V$.*

The approach for constructing an admissible 2-homogeneous equivariant map ϕ in Proposition 4.6.8, which involves computing the orthogonal projection matrix and verifying the criterion provided by Proposition 4.6.1, being "universal" from the theoretical viewpoint, can meet difficulties for its practical implementation in the case when the dimension of the original representation is high. In the next two subsections, we will employ an alternative approach which is more "algebraic" and allows extension of Proposition 4.6.1 to S_n for arbitrary odd n.

4.6.3 Extension of Proposition 4.6.8

Let us describe explicitly the setting to which we want to extend Proposition 4.6.8.

Theorem 4.6.9. *Let $(S_n;[n])$ be the natural action of the symmetric group S_n on the set $[n]=\{1,\dots,n\}$ and V, V^- be the modules corresponding to the irreducible representations $\rho^a_{(S_n,[n])}, \rho^a_{(S_n,[n])} \otimes 1^-_{S_n}$, respectively ($\mathbf{1}^-_{S_n}$ is the sign representation). Assume that n is odd. Then, there exists an admissible 2-homogeneous equivariant map from V^- to V.*

The following statement is a starting point for proving Theorem 4.6.9.

Proposition 4.6.10. *Let V, V^- be as in Theorem 4.6.9 and W an arbitrary S_n-representation. Then, there exists an admissible 2-homogeneous equivariant map from V^- to W if and only if there exists an admissible 2-homogeneous equivariant map from V to W.*

Proof. By taking the standard basis in V (resp. V^-), any map defined on V (resp. V^-) can be identified with a map on $\mathbb{C}^{n-1}$. Let $\rho_V(\sigma)$ ($\sigma \in S_n$) be matrices representing the S_n-action on V. Since $V^- = V \otimes \mathbf{1}^-_{S_n}$, one can use the simple character argument to show that the formula

$$\rho_{V^-}(\sigma) := \begin{cases} \rho_V(\sigma), & \text{if } \sigma \in A_n, \\ -\rho_V(\sigma), & \text{if } \sigma \in S_n \setminus A_n, \end{cases} \tag{4.3}$$

defines matrices representing the S_n-action on V^-.

Assume that $\phi : V \to W$ is an admissible 2-homogeneous equivariant map. Then,

$$\phi(\rho_{V^-}(\sigma)v) = \begin{cases} \phi(\rho_V(\sigma)v), & \sigma \in A_n \\ \phi(-\rho_V(\sigma)v), & \sigma \in S_n \setminus A_n \end{cases} = \phi(\rho_V(\sigma)v) = \rho_W(\sigma)\phi(v).$$

Therefore, ϕ can be viewed as an admissible 2-homogeneous equivariant map from V^- to W as well. Similarly, one can show that if $\psi : V^- \to W$ is an admissible 2-homogeneous equivariant map, then ψ can be viewed as an admissible 2-homogeneous equivariant map from V to W as well. The result follows. $\square$

Proposition 4.6.10 reduces the proof of Theorem 4.6.9 to providing an admissible 2-homogeneous equivariant map from V to V. Clearly, the later problem is equivalent to the existence of a bi-linear commutative (not necessarily associative) multiplication $* : V \times V \to V$ commuting with the G-action on V such that the algebra $(V, *)$ does not have 2-nilpotents. In the next subsection, the existence of such multiplication will be studied using the Norton algebra techniques.

4.6.4 Norton Algebras without 2-Nilpotents

In this subsection, we will recall the construction of the Norton algebra (see also [32]) and apply related techniques to prove Theorem 4.6.9.

Definition 4.6.11. Let Ω be a finite G-set ($|\Omega| = k$) and U the associated permutation representation. With the standard basis $\{e_\omega\}_{\omega\in\Omega}$, $u \in U$ can be viewed as a vector in $\mathbb{C}^k$ and, hence, U is endowed with the natural componentwise multiplication $u \cdot v := [u_1v_1, \dots, u_kv_k]$ and the scalar product $\langle u, v\rangle := \sum_{i=1}^k u_i\overline{v_i}$ for $u = [u_1, \dots, u_k]$ and $v = [v_1, \dots, v_k]$. Then, $(U, \cdot)$ is a commutative and associative algebra with the G-action commuting with the multiplication "$\cdot$", i.e., G is a group of automorphisms of $(U, \cdot)$. Let $W \subset U$ be a non-trivial G-invariant subspace. Denote by $P : U \to W$ the orthogonal projection with respect to $\langle \cdot, \cdot\rangle$

and define the Norton algebra $(W,*)$ as follows: $w_1 * w_2 := P(w_1 \cdot w_2)$ for any $w_1, w_2 \in W$. It is clear that the Norton algebra is a commutative but not necessarily associative complex algebra with the G-action commuting with the multiplication "$*$". In particular, the quadratic map $w \mapsto w * w$ is G-equivariant on W.

Example 4.6.12. Consider $G := S_n$ acting naturally on $\mathbb{N}_n := \{1, \dots n\}$. Let Ω be the set of all two-element subsets of $\mathbb{N}_n$, i.e., $\Omega := \{\{i,j\} : 1 \le i < j \le n\}$, on which S_n acts by $\sigma(\{i,j\}) = \{\sigma(i), \sigma(j)\}$ for any $\sigma \in S_n$ and $\{i,j\} \in \Omega$. In this case, the permutation representation U associated to Ω (see Definition 4.6.11) is equivalent to $\mathrm{Sym}^2(V)$, where V is the same as in Theorem 4.6.9. Moreover, U admits the irreducible decomposition

$$U \simeq \mathrm{Sym}^2(V) = \mathbf{1}_{S_n} \oplus W \oplus W', \tag{4.4}$$

where W is equivalent to V and W' is the irreducible representation of dimension $n(n-3)/2$. Applying Definition 4.6.11 to $W \simeq V$, one obtains the Norton algebra $(V,*)$. Clearly, $f : V \to V$ given by $f(v) := v * v$ is a 2-homogeneous equivariant map which is admissible if and only if $(V,*)$ is free from 2-nilpotents. Moreover, (i) f can serve as an equivariant map from V^- to V (see Proposition 4.6.10), and (ii) for $n = 5$, the map f essentially coincides with ϕ constructed in Section 4.6.2.

Although Example 4.6.12 allows one to link general considerations of Section 4.6.2 to the concept of Norton algebra, it neither provides a condition for $(V,*)$ to be free from 2-nilpotents nor gives an *explicit* formula for f. Keeping this in mind, below we adopt a slight modification of the Norton algebra approach utilized in Example 4.6.12. The following observation, which is a simple consequence of the fact that V is irreducible and appears in (4.4) with multiplicity one, is essentially used in the sequel.

Remark 4.6.13. If $\Phi, \Psi : V \to V$ are 2-homogeneous (polynomial) equivariant maps, then there exists $c \in \mathbb{C}$ such that $\Psi = c\,\Phi$.

Following Definition 4.6.11, consider S_n acting naturally on $\mathbb{N}_n$. Then, the permutation representation U associated to this action is equivalent to $\mathbb{1}_{S_n} \oplus V$. Let $(V, \star)$ be the Norton algebra associated with U (see Remark 4.6.13). We have the following result.

Proposition 4.6.14. *The Norton algebra $(V, \star)$ admits 2-nilpotents if and only if n is even. In such a case, $v \star v = 0$ if and only if $v = \alpha(\sum_{i \in I} e_i - \sum_{i \notin I} e_i)$ for some $\alpha \in \mathbb{C}$ and $I \subset \mathbb{N}_n$ with $|I| = n/2$.*

Proof. Let $P : U \to V$ be the orthogonal projection. We have to solve the equation $P(v \cdot v) = 0$ for $v \in V$. By the choice of U, $\mathbb{1}_{S_n} = \operatorname{span}\{\sum_{i=1}^{n} e_i\}$ and $V = \operatorname{span}\{e_i - e_n : 1 \leq i \leq n-1\}$. Therefore, $P(v \cdot v) = 0$ if and only if $\langle v \cdot v, e_i - e_n \rangle = 0$ for any $i \in \mathbb{N}_{n-1}$ or, equivalently,

$$\langle v \cdot v, e_i \rangle = c \tag{4.5}$$

for any $i \in \mathbb{N}_n$ and some $c \in \mathbb{C}$. For any $v = \sum_{i=1}^{n} z_i e_i \in V$, one has

$$v \cdot v = \sum_{i=1}^{n} (z_i)^2 e_i$$

and $\sum_{i=1}^{n} z_i = 0$. Combining this with (4.5) implies ${z_i}^2 = c$ for any $i \in \mathbb{N}_n$. Then, $z_k = \pm\alpha$ for α being a root of $z^2 = c$. Denote by $I \subset \mathbb{N}_n$ the set of indices k such that $z_k = \alpha$. Since $\sum_{i=1}^{n} z_i = 0$, one has either

- $\alpha = 0$, or
- $\alpha \neq 0$, n is even and $|I| = n/2$,

and the result follows. □

Proof of Theorem 4.6.9. It simply follows from Proposition 4.6.10 and Proposition 4.6.14. □

Next, for n odd, we are now in a position to derive the explicit formula of an admissible (polynomial) 2-homogeneous equivariant map from V (resp. V^-) to V:

Let $\mathcal{B} := \left\{x_i := e_i - (1/n)\sum_{j=1}^n e_j : 1 \le i \le n-1\right\}$ be a (non-orthogonal) basis of $V \subset U$; the matrix representation of S_n on V with respect to $\mathcal{B}$ is given by

$$\rho_V : (12) \mapsto \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}, \quad (1 \cdots n) \mapsto \begin{bmatrix} 0 & 0 & \cdots & 0 & -1 \\ 1 & 0 & \cdots & 0 & -1 \\ 0 & 1 & \cdots & 0 & -1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -1 \end{bmatrix}. \tag{4.6}$$

On the other hand, for any $v = \sum_{i=1}^{n-1} a_i x_i \in V$, one has

$$v \cdot v = \sum_{i=1}^{n-1} (a_i - \frac{1}{n}s)^2 e_i + \frac{1}{n^2} s^2 e_n,$$

where $s := \sum_{j=1}^{n-1} a_j$. Combining this with

$$P(e_i) = \begin{cases} x_i, & \text{if } 1 \le i < n \\ -\sum_{i=1}^{n-1} x_i, & \text{if } i = n \end{cases}$$

yields

$$\begin{aligned} v \star v &= P(v \cdot v) \\ &= \sum_{i=1}^{n-1} (a_i - \frac{1}{n}s)^2 P(e_i) + \frac{1}{n^2} s^2 P(e_n) \\ &= \sum_{i=1}^{n-1} a_i(a_i - \frac{2}{n}s) x_i \end{aligned}$$

As the result,

$$(a_1, \ldots, a_{n-1}) \mapsto (a_1(a_1 - \frac{2}{n}s), \ldots, a_{n-1}(a_{n-1} - \frac{2}{n}s)) \tag{4.7}$$

defines an admissible 2-homogeneous (polynomial) equivariant map, where the S_n-action on V is given by (4.6); also, formula (4.7) defines an admissible 2-homogeneous (polynomial) equivariant map from V^- to V (see Proposition 4.6.10), where the S_n-action on V^- is given by

$$\rho_V : (12) \mapsto \begin{bmatrix} 0 & -1 & 0 & \cdots & 0 \\ -1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & -1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & -1 \end{bmatrix}, \quad (1 \cdots n) \mapsto \begin{bmatrix} 0 & 0 & \cdots & 0 & -1 \\ 1 & 0 & \cdots & 0 & -1 \\ 0 & 1 & \cdots & 0 & -1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -1 \end{bmatrix}.$$

4.7 Applications to Congruence Principle

4.7.1 The Brouwer Degree

Recall the construction of the Brouwer degree. Let M and N be compact, connected, oriented n-dimensional manifolds (without boundary), and let $f : M \to N$ be a smooth map. Let $y \in N$ be a regular value of f. Then, $f^{-1}(y)$ is either empty (in which case, define the Brouwer degree of f to be zero), or consists of finitely many points, say $x_1, \ldots, x_k$. In the latter case, for each $i = 1, \ldots, k$, take the tangent spaces $T_{x_i}M$ and T_yN with the corresponding orientations. Then, the derivative $D_{x_i}f : T_{x_i}M \to T_yN$ is an isomorphism. Define the Brouwer degree by the formula

$$\deg(f) = \deg(f, y) := \sum_{i=1}^{k} \operatorname{sign}(\det(D_{x_i}f)). \tag{4.8}$$

It is possible to show that $\deg(f, y)$ is independent of the choice of a regular value $y \in N$ (see, for example, [88, 39, 84]). If $f : M \to N$ is continuous, then one can approximate f by a smooth map $g : M \to N$ and take $\deg(g)$ to be the Brouwer degree of f (denoted $\deg(f)$). Again, $\deg(f)$ is independent of a close approximation.

Finally, let M be as above and let W be the oriented Euclidean space such that $\dim M = \dim W - 1$. Given a continuous map $f : M \to W \setminus \{0\}$, define the map $\widetilde{f} : M \to S(W)$ by $\widetilde{f}(x) = \frac{f(x)}{\|f(x)\|}$ $(x \in M)$. Then, $\deg(\widetilde{f})$ is correctly defined. Set $\deg(f) := \deg(\widetilde{f})$ and call it the Brouwer degree of f. In particular, if V and W are oriented Euclidean spaces of the same dimension and $f : V \to W$ is admissible, then f takes $S(V)$ to $W \setminus \{0\}$. Define the Brouwer degree of f by $\deg(f) := \deg(f|_{S(V)})$.

4.7.2 Congruence Principle for Solvable Groups

Combining Theorem 4.3.11 and the congruence principle, one immediately obtains the following result.

Corollary 4.7.1. *Let G be a solvable group and let V and W be two n-dimensional representations. Assume, in addition, that V is non-trivial and irreducible, and suppose that there exists an equivariant map $\Phi : S(V) \to W \setminus \{0\}$. Then, $\alpha(V) > 1$ and for any equivariant map $\Psi : S(V) \to W \setminus \{0\}$, one has*

$$\deg(\Psi) \equiv \deg(\Phi) \pmod{\alpha(V)} \tag{4.9}$$

In addition, one has the following

Corollary 4.7.2. *Let G be a solvable group and $V, W \in \mathrm{Irr}^*(G)$. If V and W are Galois-equivalent, then $\alpha(V) > 1$ and $\deg(f) \not\equiv 0 \pmod{\alpha(V)}$ for any G-equivariant map $f : S(V) \to S(W)$.*

Proof. Take a G-equivariant map $f : S(V) \to S(W)$. Since V and W are Galois-equivalent, one has $\dim V^H = \dim W^H$ for any $H < G$. So, there exists a G-equivariant map $g : S(W) \to S(V)$ (see, for example, [16, 101, 75]). In this case, $g \circ f : S(V) \to S(V)$ is a G-equivariant map and, by the congruence principle, $\deg(g \circ f) \equiv 1 \pmod{\alpha(V)}$. Since $\alpha(V) > 1$ (see Theorem 4.3.11), the result follows. $\square$

Corollary 4.7.3. *Let G be a solvable group, W an n-dimensional irreducible (complex) G-representation and M a (real) compact, connected, oriented smooth $2n-1$-dimensional G-manifold. Assume, in addition, that*

$$\dim_{\mathbb{R}} M^H \leq \dim_{\mathbb{R}} W^H - 1 \text{ for any } (H) \in \Phi(G, M). \tag{4.10}$$

Then:

(i) there exists an equivariant map $f : M \to W \setminus \{0\}$;

(ii) $\alpha(M) := \gcd\{|G(x)| : x \in M\} > 1$;

(iii) for any equivariant map $g : M \to W \setminus \{0\}$, one has

$$\deg(g) \equiv \deg(f) \pmod{\alpha(M)}. \tag{4.11}$$

Proof. By condition (4.10), there exists an equivariant map $f : M \to W \setminus \{0\}$ (see, for example, [101, 75]). Therefore, $G_{f(x)} \geq G_x$ so that $|G(f(x))|$ divides $|G(x)|$ for any $x \in M$. Since $\alpha(W) > 1$ (see Theorem 4.3.11), it follows that $\alpha(M) > 1$. Finally, the congruence principle shows that (4.11) is true. □

Remark 4.7.4. By combining Theorem 4.3.11 with other versions of the congruence principle given in [75], one can easily obtain many other results on degrees of equivariant maps for solvable groups. We leave this task to a reader.

4.7.3 Congruence Principle for S_n-representations

In this subsection, we will study the Brouwer degree of equivariant maps from $S(V^-)$ to $S(V)$, where V and V^- are as in Theorem 4.6.9. To this end, we need the following proposition.

Proposition 4.7.5. *Let V and V^- be S_n-modules as in Theorem 4.6.9 with $n = p^k > 3$ being an odd prime power. Then:*

(i) $\alpha(V) = p$;

(ii) $\alpha(V^-) = 2p$.

Proof. (i) According to Lemma 4.4.4, one has that p divides $\alpha(V)$. Hence, it suffices to show that $V \setminus \{0\}$ admits two S_n-orbits, say O_1 and O_2, such that $\gcd\{|O_1|, |O_2|\} = p$. Indeed, let $O_1 = S_n(x)$ and $O_2 = S_n(y)$ be two S_n-orbits in $V \setminus \{0\}$ with

$$x = (n-1, \underbrace{-1, \ldots, -1}_{n-1}), \tag{4.12}$$

$$y = (\underbrace{p-1, \ldots, p-1}_{p^k}, \underbrace{-1, \ldots, -1}_{(p-1)p^{k-1}}). \tag{4.13}$$

Then, $|O_1| = p^k$, $|O_2| = \binom{p^k}{p^{k-1}}$, from which it follows that $\gcd\{|O_1|, |O_2|\} = p$.

(ii) Since there exists an admissible equivariant map from V^- to V (see Theorem 4.6.9), one has that p divides $\alpha(V^-)$. Hence, it suffices to show that

(a) any S_n-orbit in $V^- \setminus \{0\}$ is of even length;

(b) $V^- \setminus \{0\}$ admits two S_n-orbits, say O_1 and O_2, such that $\gcd\{|O_1|, |O_2|\} = 2p$.

For (a), take an S_n-orbit $O \subset V^- \setminus \{0\}$. If the transposition (12) acts on O without fixed points, then $|O|$ is even. Otherwise, let x be a vector fixed by (12). Then, $x = (a, -a, 0, \ldots, 0)$ for some $a \neq 0$ (see (4.3)). Since $n > 3$, one has $-x \in O$, from which it follows that $-O = O$. Thus, the involution $x \mapsto -x$ acting on O is without fixed points, which implies that $|O|$ is even (note that this argument does not work when $n = 3$, in which case $|O| = 3$).

For (b), let $O_1 = S_n(x)$ and $O_2 = S_n(y)$ be two S_n-orbits in $V^- \setminus \{0\}$ with x and y given in (4.12) and (4.13), respectively. Observe that $-x \in O_1$ and $-y \in O_2$ (by transposing two -1 components), from which it follows that $|O_1| = 2p$ and $|O_2| = 2\binom{p^k}{p^{k-1}}$. Therefore, $\gcd\{|O_1|, |O_2|\} = 2p$. $\square$

Combining the congruence principle and Proposition 4.7.5 yields:

Corollary 4.7.6. *Suppose that $n = p^k > 3$ is an odd prime power. For any S_n-equivariant map $\Psi : S(V^-) \to V \setminus \{0\}$,*

$$\deg(\Psi) \equiv 2^{n-1} \pmod{2p}$$

(in particular, $\deg(\Psi) \neq 0$*).*

Proof. Let $\Phi : S(V^-) \to V \setminus \{0\}$ be the 2-homogeneous equivariant map provided by Theorem 4.6.9. Since $\alpha(S_n, V^-) = 2p$ (see Proposition 4.7.5) and $\deg(\Phi) = 2^{n-1}$, it follows from the congruence principle that

$$\deg(\Psi) \equiv \deg(\Phi) = 2^{n-1} \pmod{2p}.$$

□

CHAPTER 5

COMPUTATIONAL ASPECTS

5.1 Overview

In this chapter, we will discuss computational aspects about equivariant degree.

In Section 5.2, we define basic terminologies about group, group actions and Burnside ring. In Section 5.3, we present a characterization of subgroups in the direct product of two groups and propose a general algorithm to find all conjugacy classes of subgroups (CCSs). The specific algorithm for finding all CCSs in $\Gamma \times O(2)$ and $\Gamma \times S^1$ for some finite group Γ and computing the order of the Weyl groups will be explained in Sections 5.4 and 5.5, respectively. As a concrete example, we give the complete description of CCSs in $S_4 \times O(2)$ in Section 5.6. Finally, we demonstrate how the computation in a Burnside ring can be performed.

The computational aspects described in this chapter is implemented in the GAP system ([111]).

5.2 Preliminaries

Group and Group Actions. Let G be a compact Lie group. We assume that all the considered subgroups $H \le G$ are *closed*. For $H \le G$, denote by $N_G(H)$ the *normalizer* of H in G, by $W_G(H) := N_G(H)/H$ the *Weyl group* of H in G and by (H) the *conjugacy class* of H in G

The set $\Phi(G)$ of all conjugacy classes of subgroups (CCSs) in G can be naturally equipped with the partial order: $(H) \le (K)$ if and only if $gHg^{-1} \le K$ for some $g \in G$. For any integer $n \ge 0$, put $\Phi_n(G) := \{(H) \in \Phi(G) : \dim W(H) = n\}$.

Suppose X is a G-space and $x \in X$. Denote by $G_x := \{g \in G : gx = x\}$ the *isotropy* of x, by $G(x) := \{gx : g \in G\}$ the *orbit* of x and by X/G the *orbit space*. For any isotropy G_x, call (G_x) the *orbit type* of x and put $\Phi(G;X) := \{(H) \in \Phi(G) : H = G_x \text{ for some } x \in X\}$ and $\Phi_n(G;X) := \Phi(G;X) \cap \Phi_n(G)$.

Given two subgroups $H \leq K \leq G$, define $N_G(H,K) := \{g \in G : gHg^{-1} \leq K\}$. Recall that the number $n(H,K) := |N_G(H,K)/N_G(K)|$ coincides with the number of conjugate copies of K in G which contains H ([57, 75]).

In what follows, we will omit the subscript "G" when the group G is clear from the context.

Burnside Ring. In the G-equivariant degree, the ring $\mathbb{Z}$ is replaced by the so-called *Burnside ring* $A(G)$. To be more specific, $A(G) := \mathbb{Z}[\Phi_0(G)]$, i.e., it is the free $\mathbb{Z}$-module generated by $(H) \in \Phi_0(G)$. Notice that elements of $A(G)$ can be written as the sum

$$\sum_{(H)\in\Phi_0(G)} n_H(H), \quad n_H \in \mathbb{Z},$$

with finitely many $n_H \neq 0$. To define the ring multiplication "$\cdot$" in $A(G)$, take (H), $(K) \in \Phi_0(G)$ and observe that G acts diagonally on $G/H \times G/K$ with finitely many orbit types. Consider such an orbit type $(L) \in \Phi_0(G)$. As is well-known, $(G/H \times G/K)_{(L)}/G$ is finite ([26, 12]). Put

$$(H) \cdot (K) := \sum_{(L)\in\Phi_0(G)} m_L(H,K)(L), \tag{5.1}$$

where $m_L(H,K) := \left|(G/H \times G/K)_{(L)}/G\right|$. Then, the $\mathbb{Z}$-module $A(G)$ equipped with the above multiplication (extended from generators by distributivity) becomes a ring. Notice that $A(G)$ is a ring with the unity (G), i.e., $(H) \cdot (G) = (H)$ for every $(H) \in \Phi_0(G)$.

5.3 Conjugacy Classes of Subgroups in the Direct Product of Two Groups

In this section, denote by $\mathcal{G}_1$ and $\mathcal{G}_2$ two groups, by $\mathcal{G} := \mathcal{G}_1 \times \mathcal{G}_2$ the group of their direct product and by

$$\begin{aligned} \pi_1 :&\mathcal{G}_1 \times \mathcal{G}_2 \to \mathcal{G}_1, \quad \pi_1(g_1, g_2) = g_1, \\ \pi_2 :&\mathcal{G}_1 \times \mathcal{G}_2 \to \mathcal{G}_2, \quad \pi_2(g_1, g_2) = g_2. \end{aligned}$$

the projection homomorphisms.

The following result (see [36] for more details), which is rooted in Goursat's Lemma (see [53]), characterizes subgroups of $\mathcal{G}$.

Theorem 5.3.1. *Let $\mathcal{H}$ be a subgroup of $\mathcal{G}$. Put $H := \pi_1(\mathcal{H})$ and $K := \pi_2(\mathcal{H})$. Then, there exist a group L and two epimorphisms $\varphi : H \to L$ and $\psi : K \to L$, such that*

$$\mathcal{H} = \{(h, k) \in H \times K : \varphi(h) = \psi(k)\}$$

(see Figure 5.1). In this case, denote $\mathcal{H} =: H\,{}^{\varphi}\!\times_L^{\psi} K$.

$$\begin{tikzcd}
 & \mathcal{G} & \\
 & \mathcal{H} \arrow[u, phantom, sloped, "\leq"] \arrow[dl, "\pi_1"] \arrow[dr, "\pi_2"'] & \\
\mathcal{G}_1 \geq H \arrow[r, "\varphi"'] & L & K \leq \mathcal{G}_2 \arrow[l, "\psi"]
\end{tikzcd}$$

Figure 5.1. Diagram characterizing a subgroup $\mathcal{H} \leq \mathcal{G}$.

With Theorem 5.3.1, one can describe conjugate subgroups in $\mathcal{G}$ as follows (see also [36]).

Proposition 5.3.2. *Two subgroups $H\,{}^{\varphi}\!\times_L^{\psi} K, H'\,{}^{\phi'}\!\times_L^{\psi'} K'$ of $\mathcal{G}$ are conjugate if and only if there exists $(a, b) \in \mathcal{G}$ such that the inner automorphisms $a. : \mathcal{G}_1 \to \mathcal{G}_1$ and $b. : \mathcal{G}_2 \to \mathcal{G}_2$ given by*

$$a.g_1 = ag_1a^{-1}, \quad b.g_2 = bg_2b^{-1},$$

satisfy the properties: $H' = a.H$, $K' = b.K$ *and* $\varphi = \varphi' \circ a.$, $\psi = \psi' \circ b.$. *In other words, the diagram shown in Figure* 5.2 *commutes.*

$$\begin{tikzcd}
H \arrow[dr, "\varphi"] \arrow[dd, "a."'] & & K \arrow[dl, "\psi"'] \arrow[dd, "b."] \\
& L & \\
H' \arrow[ur, "\varphi'"'] & & K' \arrow[ul, "\psi'"]
\end{tikzcd}$$

Figure 5.2. Diagram characterizing conjugate subgroups in $\mathcal{G}$

The next result, which follows immediately from Proposition 5.3.2, is crucial for finding all conjugacy classes of subgroups in $\mathcal{G}$.

Corollary 5.3.3. *Two subgroups* $S_1 = H\,{}^{\varphi_1}\!\times_L^{\psi_1} K$ *and* $S_2 = H\,{}^{\varphi_2}\!\times_L^{\psi_2} K$ *of* $\mathcal{G}$ *are conjugate if and only if there exists* $(a, b) \in \mathcal{G}$ *such that the diagrams shown in Figure* 5.3 *commute.*

Remark 5.3.4. In Figure 5.3(b), χ_i is an isomorphism induced by φ_i and ψ_i, $i = 1, 2$. In addition, if (a, b) in Corollary 5.3.3 exists, then $a \in N_{\mathcal{H}}(H)$ and $b \in N_{\mathcal{K}}(K)$.

Inspired by Figure 5.3(c), we propose the following algorithm for finding $\Phi(\mathcal{G})$ (assuming that $\Phi(\mathcal{G}_1)$ and $\Phi(\mathcal{G}_2)$ are known).

Algorithm 1. Identify $\Phi(\mathcal{G})$

Require: $\Phi(\mathcal{G}_1)$, $\Phi(\mathcal{G}_2)$;
  $\Phi(\mathcal{G}) \leftarrow \{\}$;
  for $(H) \in \Phi(\mathcal{G}_1)$ **do**
    for $(K) \in \Phi(\mathcal{G}_2)$ **do**
      $\mathcal{A} = \mathcal{A}(H, K) \leftarrow \Big\{S = H\,{}^{\varphi}\!\times_L^{\psi} K$ for all possible φ, ψ and $L\Big\}$;
      Classify CCSs of $\mathcal{G}$ in $\mathcal{A}$;
      $\Phi(\mathcal{G}) \leftarrow \Phi(\mathcal{G}) \cup \{$CCSs in $\mathcal{A}\}$;
    end for
  end for

$$\begin{tikzcd}
H \arrow[dr, "\varphi_1"] \arrow[dd, "a."'] & & K \arrow[dl, "\psi_1"'] \arrow[dd, "b."] \\
& L & \\
H \arrow[ur, "\varphi_2"'] & & K \arrow[ul, "\psi_2"]
\end{tikzcd}$$

(a)

$$\begin{tikzcd}
H/Z_H^1 \arrow[r, "\chi_1"] \arrow[d, "a."'] & K/Z_K^1 \arrow[d, "b."] \\
H/Z_H^2 \arrow[r, "\chi_2"] & K/Z_K^2
\end{tikzcd}$$

(b)

$$\begin{tikzcd}
H \arrow[r, "\varphi_1"] \arrow[d, "a."'] & K/Z_K^1 \arrow[d, "b."] \\
H \arrow[r, "\varphi_2"] & K/Z_K^2
\end{tikzcd}$$

(c)

Figure 5.3. Diagrams of conjugate subgroups in $\mathcal{G} = \mathcal{H} \times \mathcal{K}$

5.4 Conjugacy Classes of Subgroups in $\Gamma \times O(2)$ and $\Gamma \times S^1$

In this section, we propose an algorithm, which is based on Algorithm 1, for finding conjugacy classes of subgroups in $\Gamma \times O(2)$ and $\Gamma \times S^1$ for some finite group Γ.

Let us first consider $\mathcal{G} := \Gamma \times O(2)$. Note that $\Phi(\Gamma)$ (resp. $\Phi(O(2))$) can be obtained by computer (resp. by hand), which allows the following algorithm for finding all CCSs in $\mathcal{G}$. Note that when finding K/Z_K to which H is is epimorphic, it suffices to consider K/Z_K such that $|K/Z_K|$ divides $|K|$, which contains only finite possible cases (see Table 5.1 for all possible combinations of K/Z_K in $O(2)$).

Next, one can use GAP system to find all epimorphisms from H to K/Z_K (see [100]). Since there is a one-to-one correspondence between $\mathcal{A}$ and $\mathcal{I}$, if one defines conjugacy in $\mathcal{I}$ which is compatible with the conjugacy in $\mathcal{A}$ (i.e., $\varphi_1, \varphi_2 : H \to K/Z_K$ are conjugate if there

Algorithm 2. Identify $\Phi(\Gamma \times O(2))$

Require: $\Phi(\Gamma)$, $\Phi(O(2))$;
  $\Phi(G) \leftarrow \{\}$;
  for $(H) \in \Phi(\Gamma)$ **do**
    for K/Z_K where $Z_K \triangleleft K$ and $(K) \in \Phi(O(2))$ **do**
      $\mathcal{A} = \mathcal{A}(H, K, Z_K) \leftarrow \left\{H \,{}^{\phi}\!\times^{\psi}_{K/Z_K} K < G : \ker(\psi) = Z_K\right\}$;
      $\mathcal{I} = \mathcal{I}(H, K, Z_K) \leftarrow \{\text{epimorhisms from } H \text{ to } K/Z_K\}$;
      Classify $\mathcal{I}$ into conjugacy classes (it is equivalent to the classification of $\mathcal{A}$);
      $\Phi(\mathcal{G}) \leftarrow \Phi(\mathcal{G}) \cup \{\text{CCSs in } \mathcal{A}\}$;
    end for
  end for

Table 5.1. Types of K/Z_K in $O(2)$.

K/Z_K	K	Z_K
$\mathbb{Z}_1$	$O(2)$	$O(2)$
	$SO(2)$	$SO(2)$
	D_n	D_n
	$\mathbb{Z}_n$	$\mathbb{Z}_n$
$\mathbb{Z}_2$	D_{2n}	D_n
	$\mathbb{Z}_{2n}$	$\mathbb{Z}_n$

K/Z_K	K	Z_K
D_1	$O(2)$	$SO(2)$
	D_n	$\mathbb{Z}_n$
$\mathbb{Z}_m$ $(m > 2)$	$\mathbb{Z}_{mn}$	$\mathbb{Z}_n$
D_m $(m > 1)$	D_{mn}	$\mathbb{Z}_n$

exists $(a, b) \in \mathcal{G}$ such that the commutative diagram in Figure 5.3 is satisfied), (conjugacy) classification of $\mathcal{A}$ is equivalent to that of $\mathcal{I}$, which can be done with the help of GAP system.

Similar algorithm applies to finding conjugacy classes of subgroups in $\Gamma \times S^1$.

5.5 Order of Weyl Group

Our next goal is to effectively compute the order of the Weyl group $W_{\mathcal{G}}(S)$ for given $S \leq \mathcal{G}$ when $\mathcal{G} = \Gamma \times O(2)$ or $\mathcal{G} = \Gamma \times S^1$.

Let us first consider $\mathcal{G} = \Gamma \times O(2)$. Assume that $S \in \mathcal{A} := \mathcal{A}(H, K, Z_K)$ (see Algorithm 2). Consider the subgroup $C := N_\Gamma(H) \times (N_{O(2)}(K) \cap N_{O(2)}(Z_K)) < \mathcal{G}$. Note that $N_{\mathcal{G}}(S) \leq C$ and if C acts on $\mathcal{A}$ by conjugation, then $C_S = N_{\mathcal{G}}(S)$. On the other hand, since there is a one-to-one correspondence between $\mathcal{A}$ and $\mathcal{I} := \mathcal{I}(H, K, Z_K)$, if one defines the C-action on

$\mathcal{I}$ by

$$(a,b).\chi := b. \circ \chi \circ a^{-1}. \quad (a \in N_\Gamma(H),\ b \in N_{O(2)}(K) \cap N_{O(2)}(Z_K),\ \chi \in \mathcal{I}),$$

then $\mathcal{A}$ and $\mathcal{I}$ are C-isomorphic. Hence, one has the following result by the Orbit-Stabilizer Theorem.

Proposition 5.5.1. *Let $H < \Gamma$, $Z_K \lhd K < O(2)$, $\mathcal{A}$, $\mathcal{I}$ and $C < \mathcal{G}$ be as above. Take $S \in \mathcal{A}$ with the corresponding epimorphism $\chi := \mu(S) \in \mathcal{I}$. Then,*

$$|W_{\mathcal{G}}(S)| = \frac{|W_\Gamma(H)| \left| \left(N_{O(2)}(K) \cap N_{O(2)}(Z_K)\right)/K \right| |K/Z_K|}{|C_\chi|}.$$

Remark 5.5.2. It should be pointed out that, given a subgroup S, the above proposition allows to effectively compute the order of $W_{\mathcal{G}}(S)$ without computing the normalizer of S.

Similar procedure works when $\mathcal{G} = \Gamma \times S^1$.

5.6 Example: CCSs in $S_4 \times O(2)$

In this section, we will give the complete description of CCSs in $S_4 \times O(2)$. Note that CCSs in S_4 and $O(2)$ are described in Table 5.2. For convenience, we adopt the modified notation for subgroups in $S_4 \times O(2)$:

$$S = H {}^{Z_H}_{\ R}\!\times_L K,$$

where $Z_H = \ker \varphi$, $L = K/Z_K$ and R is $\varphi^{-1}(K \cap SO(2))/(Z_K \cap SO(2))$ ($\varphi : K \to L$ is the epimorphism). R appears only when H, K, Z and L are not enough to determine a unique conjugacy class of subgroups in $S_4 \times O(2)$. Using Algorithm 2, we obtain the complete list of CCSs in $S_4 \times O(2)$ (see Table 5.3, and also [36]).

Table 5.2. CCS tables in S_4 and $O(2)$

(a) CCSs in S_4

ID	(H)
1	$(\mathbb{Z}_1)$
2	$(\mathbb{Z}_2)$
3	(D_1)
4	$(\mathbb{Z}_3)$
5	(V_4)
6	(D_2)
7	$(\mathbb{Z}_4)$
8	(D_3)
9	(D_4)
10	(A_4)
11	(S_4)

(b) CCSs in $O(2)$

ID	(K)
1	$(\mathbb{Z}_n)$
2	(D_n)
3	$(SO(2))$
4	$(O(2))$

Table 5.3. Conjugacy Classes of Subgroups in $S_4 \times O(2)$

ID	(S)	$\|W(S)\|$	ID	(S)	$\|W(S)\|$	ID	(S)	$\|W(S)\|$
1	$(\mathbb{Z}_1{}^{\mathbb{Z}_1}\times_{\mathbb{Z}_1}\mathbb{Z}_n)$	∞	35	$(D_4{}^{\mathbb{Z}_4}\times_{D_1}D_n)$	4	69	$(\mathbb{Z}_2{}^{\mathbb{Z}_2}\times_{\mathbb{Z}_1}SO(2))$	8
2	$(\mathbb{Z}_2{}^{\mathbb{Z}_2}\times_{\mathbb{Z}_1}\mathbb{Z}_n)$	∞	36	$(S_4{}^{A_4}\times_{D_1}D_n)$	4	70	$(D_1{}^{D_1}\times_{\mathbb{Z}_1}SO(2))$	4
3	$(D_1{}^{D_1}\times_{\mathbb{Z}_1}\mathbb{Z}_n)$	∞	37	$(V_4{}^{\mathbb{Z}_1}\times_{D_2}D_{2n})$	8	71	$(\mathbb{Z}_3{}^{\mathbb{Z}_3}\times_{\mathbb{Z}_1}SO(2))$	4
4	$(\mathbb{Z}_3{}^{\mathbb{Z}_3}\times_{\mathbb{Z}_1}\mathbb{Z}_n)$	∞	38	$(D_2{}^{\mathbb{Z}_1}_{\mathbb{Z}_2}\times_{D_2}D_{2n})$	8	72	$(V_4{}^{V_4}\times_{\mathbb{Z}_1}SO(2))$	12
5	$(V_4{}^{V_4}\times_{\mathbb{Z}_1}\mathbb{Z}_n)$	∞	39	$(D_2{}^{\mathbb{Z}_1}_{D_1}\times_{D_2}D_{2n})$	4	73	$(D_2{}^{D_2}\times_{\mathbb{Z}_1}SO(2))$	4
6	$(D_2{}^{D_2}\times_{\mathbb{Z}_1}\mathbb{Z}_n)$	∞	40	$(D_4{}^{\mathbb{Z}_2}_{V_4}\times_{D_2}D_{2n})$	4	74	$(\mathbb{Z}_4{}^{\mathbb{Z}_4}\times_{\mathbb{Z}_1}SO(2))$	4
7	$(\mathbb{Z}_4{}^{\mathbb{Z}_4}\times_{\mathbb{Z}_1}\mathbb{Z}_n)$	∞	41	$(D_4{}^{\mathbb{Z}_2}_{D_2}\times_{D_2}D_{2n})$	4	75	$(D_3{}^{D_3}\times_{\mathbb{Z}_1}SO(2))$	2
8	$(D_3{}^{D_3}\times_{\mathbb{Z}_1}\mathbb{Z}_n)$	∞	42	$(D_4{}^{\mathbb{Z}_2}_{\mathbb{Z}_4}\times_{D_2}D_{2n})$	4	76	$(D_4{}^{D_4}\times_{\mathbb{Z}_1}SO(2))$	2
9	$(D_4{}^{D_4}\times_{\mathbb{Z}_1}\mathbb{Z}_n)$	∞	43	$(D_3{}^{\mathbb{Z}_1}\times_{D_3}D_{3n})$	2	77	$(A_4{}^{A_4}\times_{\mathbb{Z}_1}SO(2))$	4
10	$(A_4{}^{A_4}\times_{\mathbb{Z}_1}\mathbb{Z}_n)$	∞	44*	$(S_4{}^{V_4}\times_{D_3}D_{3n})$	2	78	$(S_4{}^{S_4}\times_{\mathbb{Z}_1}SO(2))$	2
11	$(S_4{}^{S_4}\times_{\mathbb{Z}_1}\mathbb{Z}_n)$	∞	45*	$(D_4{}^{\mathbb{Z}_1}\times_{D_4}D_{4n})$	2	79	$(\mathbb{Z}_2{}^{\mathbb{Z}_1}\times_{D_1}O(2))$	8
12	$(\mathbb{Z}_2{}^{\mathbb{Z}_1}\times_{\mathbb{Z}_2}\mathbb{Z}_{2n})$	∞	46	$(\mathbb{Z}_1{}^{\mathbb{Z}_1}\times_{\mathbb{Z}_1}D_n)$	48	80	$(D_1{}^{\mathbb{Z}_1}\times_{D_1}O(2))$	4
13	$(D_1{}^{\mathbb{Z}_1}\times_{\mathbb{Z}_2}\mathbb{Z}_{2n})$	∞	47	$(\mathbb{Z}_2{}^{\mathbb{Z}_2}\times_{\mathbb{Z}_1}D_n)$	8	81	$(V_4{}^{\mathbb{Z}_2}\times_{D_1}O(2))$	4
14	$(V_4{}^{\mathbb{Z}_2}\times_{\mathbb{Z}_2}\mathbb{Z}_{2n})$	∞	48	$(D_1{}^{D_1}\times_{\mathbb{Z}_1}D_n)$	4	82	$(D_2{}^{\mathbb{Z}_2}\times_{D_1}O(2))$	4
15	$(D_2{}^{\mathbb{Z}_2}\times_{\mathbb{Z}_2}\mathbb{Z}_{2n})$	∞	49	$(\mathbb{Z}_3{}^{\mathbb{Z}_3}\times_{\mathbb{Z}_1}D_n)$	4	83	$(D_2{}^{D_1}\times_{D_1}O(2))$	2
16	$(D_2{}^{D_1}\times_{\mathbb{Z}_2}\mathbb{Z}_{2n})$	∞	50	$(V_4{}^{V_4}\times_{\mathbb{Z}_1}D_n)$	12	84	$(\mathbb{Z}_4{}^{\mathbb{Z}_2}\times_{D_1}O(2))$	4
17	$(\mathbb{Z}_4{}^{\mathbb{Z}_2}\times_{\mathbb{Z}_2}\mathbb{Z}_{2n})$	∞	51	$(D_2{}^{D_2}\times_{\mathbb{Z}_1}D_n)$	4	85	$(D_3{}^{\mathbb{Z}_3}\times_{D_1}O(2))$	2
18	$(D_3{}^{\mathbb{Z}_3}\times_{\mathbb{Z}_2}\mathbb{Z}_{2n})$	∞	52	$(\mathbb{Z}_4{}^{\mathbb{Z}_4}\times_{\mathbb{Z}_1}D_n)$	4	86	$(D_4{}^{V_4}\times_{D_1}O(2))$	2
19	$(D_4{}^{V_4}\times_{\mathbb{Z}_2}\mathbb{Z}_{2n})$	∞	53	$(D_3{}^{D_3}\times_{\mathbb{Z}_1}D_n)$	2	87	$(D_4{}^{D_2}\times_{D_1}O(2))$	2
20	$(D_4{}^{D_2}\times_{\mathbb{Z}_2}\mathbb{Z}_{2n})$	∞	54	$(D_4{}^{D_4}\times_{\mathbb{Z}_1}D_n)$	2	88	$(D_4{}^{\mathbb{Z}_4}\times_{D_1}O(2))$	2
21	$(D_4{}^{\mathbb{Z}_4}\times_{\mathbb{Z}_2}\mathbb{Z}_{2n})$	∞	55	$(A_4{}^{A_4}\times_{\mathbb{Z}_1}D_n)$	4	89	$(S_4{}^{A_4}\times_{D_1}O(2))$	2
22	$(S_4{}^{A_4}\times_{\mathbb{Z}_2}\mathbb{Z}_{2n})$	∞	56*	$(S_4{}^{S_4}\times_{\mathbb{Z}_1}D_n)$	2	90	$(\mathbb{Z}_1{}^{\mathbb{Z}_1}\times_{\mathbb{Z}_1}O(2))$	24
23	$(\mathbb{Z}_3{}^{\mathbb{Z}_1}\times_{\mathbb{Z}_3}\mathbb{Z}_{3n})$	∞	57	$(\mathbb{Z}_2{}^{\mathbb{Z}_1}\times_{\mathbb{Z}_2}D_{2n})$	8	91	$(\mathbb{Z}_2{}^{\mathbb{Z}_2}\times_{\mathbb{Z}_1}O(2))$	4
24	$(A_4{}^{V_4}\times_{\mathbb{Z}_3}\mathbb{Z}_{3n})$	∞	58	$(D_1{}^{\mathbb{Z}_1}\times_{\mathbb{Z}_2}D_{2n})$	4	92	$(D_1{}^{D_1}\times_{\mathbb{Z}_1}O(2))$	2
25	$(\mathbb{Z}_4{}^{\mathbb{Z}_1}\times_{\mathbb{Z}_4}\mathbb{Z}_{4n})$	∞	59	$(V_4{}^{\mathbb{Z}_2}\times_{\mathbb{Z}_2}D_{2n})$	4	93	$(\mathbb{Z}_3{}^{\mathbb{Z}_3}\times_{\mathbb{Z}_1}O(2))$	2
26	$(\mathbb{Z}_2{}^{\mathbb{Z}_1}\times_{D_1}D_n)$	16	60	$(D_2{}^{\mathbb{Z}_2}\times_{\mathbb{Z}_2}D_{2n})$	4	94	$(V_4{}^{V_4}\times_{\mathbb{Z}_1}O(2))$	6
27	$(D_1{}^{\mathbb{Z}_1}\times_{D_1}D_n)$	8	61*	$(D_2{}^{D_1}\times_{\mathbb{Z}_2}D_{2n})$	2	95	$(D_2{}^{D_2}\times_{\mathbb{Z}_1}O(2))$	2
28	$(V_4{}^{\mathbb{Z}_2}\times_{D_1}D_n)$	8	62	$(\mathbb{Z}_4{}^{\mathbb{Z}_2}\times_{\mathbb{Z}_2}D_{2n})$	4	96	$(\mathbb{Z}_4{}^{\mathbb{Z}_4}\times_{\mathbb{Z}_1}O(2))$	2
29	$(D_2{}^{\mathbb{Z}_2}\times_{D_1}D_n)$	8	63	$(D_3{}^{\mathbb{Z}_3}\times_{\mathbb{Z}_2}D_{2n})$	2	97	$(D_3{}^{D_3}\times_{\mathbb{Z}_1}O(2))$	1
30	$(D_2{}^{D_1}\times_{D_1}D_n)$	4	64	$(D_4{}^{V_4}\times_{\mathbb{Z}_2}D_{2n})$	2	98	$(D_4{}^{D_4}\times_{\mathbb{Z}_1}O(2))$	1
31	$(\mathbb{Z}_4{}^{\mathbb{Z}_2}\times_{D_1}D_n)$	8	65*	$(D_4{}^{D_2}\times_{\mathbb{Z}_2}D_{2n})$	2	99	$(A_4{}^{A_4}\times_{\mathbb{Z}_1}O(2))$	2
32	$(D_3{}^{\mathbb{Z}_3}\times_{D_1}D_n)$	4	66*	$(D_4{}^{\mathbb{Z}_4}\times_{\mathbb{Z}_2}D_{2n})$	2	100	$(S_4{}^{S_4}\times_{\mathbb{Z}_1}O(2))$	1
33	$(D_4{}^{V_4}\times_{D_1}D_n)$	4	67*	$(S_4{}^{A_4}\times_{\mathbb{Z}_2}D_{2n})$	2			
34	$(D_4{}^{D_2}\times_{D_1}D_n)$	4	68	$(\mathbb{Z}_1{}^{\mathbb{Z}_1}\times_{\mathbb{Z}_1}SO(2))$	48			

5.7 Computation in the Burnside Ring $A(\Gamma \times O(2))$

Given an (infinite) group $\mathcal{G} = \Gamma \times O(2)$, it is a difficult task, in general, to perform the multiplication in $A(G)$ according to formula (5.1). In practice, one can use the following *Recurrence Formula*: given generators $(S_1), (S_2) \in A(G)$,

$$(S_1) \cdot (S_2) = \sum_{(S) \in \Phi_0(G)} n_S (S),$$

where

$$n_S = \frac{n_G(S, S_1) \left|W_G(S_1)\right| n_G(S, S_2) \left|W_G(S_2)\right| - \sum_{(\tilde{S})>(S)} n_G(S, \tilde{S}) n_{\tilde{S}} \left|W_G(\tilde{S})\right|}{\left|W_G(S)\right|} \tag{5.2}$$

(see Section 5.2 and [12]).

Remark 5.7.1. In (5.2), $n_S = 0$ for $(S) \notin \Psi_0(S_1, S_2)$, where

$$\Psi_0(S_1, S_2) = \left\{(S) \in \Phi_0(G) : (S) < (S_1), (S_2) \wedge \dim S = \min \left\{\dim S_1, \dim S_2\right\}\right\}.$$

Despite the algebraic nature of formula (5.2), the computation of the numbers $n(S, S_1), n(S, S_2)$ and $n(S, \tilde{S})$ cannot be implemented in GAP directly due to the infiniteness of G. To circumvent this obstacle, one can replace $O(2)$ by D_m with large enough m. To this end, given $(S_1), (S_2) \in \Phi_0(G)$, one may consider a map $\eta : \Psi_1(S_1, S_2) \to \Phi_0(G')$, where $G' = \Gamma \times D_m$ and

$$\Psi_1(S_1, S_2) = \Psi_0(S_1, S_2) \cup \{(S_1), (S_2)\}$$

(see Remark 5.7.1) such that

$$n_G(P, Q) = n_{G'}(\eta(P), \eta(Q)).$$

Observe that the choice of D_m and η essentially depends on (S_1) and (S_2). To be more specific, suppose $S_1, S_2 \in G$ are given by

$$S_1 = H_1 {}^{\varphi_1}\times_{L_1}^{\psi_1} K_1$$

$$S_2 = H_2 {}^{\varphi_2}\times_{L_2}^{\psi_2} K_2,$$

and consider three different cases:

(i) $|K_1| < \infty$ and $\{K_2\} < \infty$. In this case, $K_1 = D_j$ and $K_2 = D_k$ for some j and k. Take $m = 2\gcd(j,k)$. Then, there is a natural embedding $\tau : S_1 \cup S_2 \to G'$. Given $(S) \in \Psi_1(S_1, S_2)$, one may assume that $S \subset S_1 \cup S_2$ and define $\eta((S)) = (\tau(S))$.

(ii) $|K_1| < \infty$ and $|K_2| = \infty$. In this case, $K_1 = D_j$ and $K_2 = O(2)$ or $SO(2)$. Take $m = 2j$ and put

$$\begin{cases} \rho(O(2)) = D_m \\ \rho(SO(2)) = \mathbb{Z}_m \end{cases} . \tag{5.3}$$

Also, there is a natural embedding $\tau : S_1 \to G'$. Then, given $(S) \in \Psi_1(S_1, S_2)$, define

$$\eta((S)) = \begin{cases} (H_2 \,{}^{\varphi_2}\!\times_L^{\tilde{\psi}_2} \rho(K_2)), & \text{if } (S) = (S_2), \quad (\ker(\tilde{\psi}_2) = \rho(\ker(\psi_2))) \\ (\tau(T)), & \text{otherwise.} \end{cases}$$

(iii) $|K_1| = \infty$ and $|K_2| = \infty$. In this case, K_1 and K_2 are $O(2)$ or $SO(2)$. Take $m = 1$. Given $(S) = (H \,{}^{\varphi}\!\times_L^{\psi} K) \in \Psi_1(S_1, S_2)$, define

$$\eta((S)) = (H \,{}^{\varphi}\!\times_L^{\tilde{\psi}} \rho(K)) \quad (\ker(\tilde{\psi}) = \rho(\ker(\psi)))$$

(see (5.3)).

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BIOGRAPHICAL SKETCH

Haopin Wu was born in Taipei city, Taiwan. He was admitted in the National Taiwan University and obtained his Bachelor of Science in Mathematics in 2008 and Master of Science in Mathematics in 2011. Later in 2013, he started working towards his PhD in Mathematics at the University of Texas at Dallas under the guidance of Professor Balanov and Professor Krawcewicz. His research focuses on the degree theory and related applications.

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EDUCATION

2013 – 2018 (anticipated)	Ph.D. in Mathematics University of Texas at Dallas, USA
2008 – 2011	M.S. in Mathematics National Taiwan University, Taiwan
2004 – 2008	B.S. in Mathematics National Taiwan University, Taiwan

EXPERIENCE IN HIGHER EDUCATION

University of Texas at Dallas, USA	
2018 – present	Teaching Assistant
2017	Research Assistant (supported by NSF through grant DMS-1413223)
2015 – 2016	Teaching Associate
2014 – 2015	Teaching Assistant
Academia Sinica, Taiwan	
2012 – 2013	Research Assistant
National Taiwan University, Taiwan	
2008 – 2010	Teaching Assistant

RESEARCH INTERESTS

- Topological and variational methods of equivariant nonlinear analysis
  (especially, equivariant degree theory)
- Bifurcation theory of functional differetial equations with symmetries
  (especially, reversible systems, mixed delay differential equations)
- Representation theory of finite groups
  (especially, solvable and 2-transitive groups)
- Computational algebra
  (especially, GAP (Groups, Algorithms, Programming—a system for computational discrete algebra) and its application to Burnside ring, group theory and representation theory)

PUBLICATION/PREPRINTS

Published/Accepted

[1] Z. Balanov and H.-P. Wu. *Bifurcation of space periodic solutions in symmetric reversible FDEs*. Differential Integral Equations **30** (2017), 289–328.

[2] M. Dabkowski, W. Krawcewicz, Y. Lv, and H.-P. Wu. *Multiple periodic solutions for Γ-symmetric Newtonian systems*. J. Differential Equations **263** (2017), 6684–6730.

[3] W. Krawcewicz, H.-P. Wu, and S. Yu. *Periodic solutions in reversible second order autonomous systems with symmetries*. Journal of Nonlinear and Convex Analysis (2017). (accepted).

Submitted

[4] Z. Balanov, M. Muzychuk, and H.-P. Wu. *On some applications of group representation theory to algebraic problems related to the congruence principle for equivariant maps*. Journal of Algebra (2017). (submitted). arXiv: `1602.00045 [math.DS]`.

In preparation

[5] C. Garcia-Azpeitia, W. Krawcewicz, M. Tejada-Wriedt, and H.-P. Wu. *Symmetries of nonlinear vibrations in C_{60} fullerene*.

[6] W. Krawcewicz, S. Rybicki, and H.-P. Wu. *Bifurcation of central configurations in n-body problem with dihedral symmetry*.

Programs

[7] H.-P. Wu. *A program for the computations of Burnside ring $A(\Gamma \times O(2))$*. Developed at University of Texas at Dallas. 2016. URL: `http://bitbucket.org/psistwu/gammao2`.

PRESENTATIONS

Conferences

[1] *Bifurcation of space periodic solutions in symmetric reversible FDEs*. International Conference on Topological Nonlinear Analysis. (Guangzhou University). (devoted to Norman Dancer's 70th birthday). 2017.

Seminars

[2] *Bifurcation of space periodic solutions in symmetric reversible FDEs*. Nonlinear Analysis and Dynamical Systems. (The University of Texas at Dallas). 2016.

[3] *Brouwer degrees of equivariant maps for solvable and 2-transitive groups revisited*. Nonlinear Analysis and Dynamical Systems. (The University of Texas at Dallas). 2016.

PROGRAMMING SKILLS

MATLAB, Mathematica, Java, Python, GAP, Git.