Left-orderability for surgeries on twisted torus knots

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Abstract: We show that the fundamental group of the 3-manifold obtained by $\frac{p}{q}$-surgery along the $(n-2)$-twisted $(3, 3m+2)$-torus knot, with $n, m \geq 1$, is not left-orderable if $\frac{p}{q} \geq 2n + 6m - 3$ and is left-orderable if $\frac{p}{q}$ is sufficiently close to 0.

Key words: Dehn surgery; left-orderable; L-space; twisted torus knot.

1. Introduction. The motivation of this paper is the L-space conjecture of Boyer, Gordon and Watson [BGW] which states that an irreducible rational homology 3-sphere is an L-space if and only if its fundamental group is not left-orderable. Here a rational homology 3-sphere Y is an L-space if its Heegaard Floer homology $\widehat{HF}(Y)$ has rank equal to the order of $H_1(Y; \mathbf{Z})$, and a nontrivial group G is left-orderable if it admits a total ordering $<$ such that $g < h$ implies $fg < fh$ for all elements f, g, h in G.

Many hyperbolic L-spaces can be obtained via Dehn surgery. A knot K in S^3 is called an L-space knot if it admits a positive Dehn surgery yielding an L-space. For an L-space knot K, Ozsvath and Szabo [OS] proved that the $\frac{p}{q}$-surgery of K is an L-space if and only if $\frac{p}{q} \geq 2g(K) - 1$, where $g(K)$ is the genus of K. In view of the L-space conjecture, one would expect that the fundamental group of the $\frac{p}{q}$-surgery of an L-space knot K is not left-orderable if and only if $\frac{p}{q} \geq 2g(K) - 1$.

By [BM] among the set of all Montesinos knots, the $(-2, 3, 2n+1)$-pretzel knots, with $n \geq 3$, and their mirror images are the only hyperbolic L-space knots. Nie [Ni] has recently proved that the fundamental group of the 3-manifold obtained by $\frac{p}{q}$-surgery along the $(-2, 3, 2n+1)$-pretzel knot, with $n \geq 3$, is not left-orderable if $\frac{p}{q} \geq 2n + 3$ and is left-orderable if $\frac{p}{q}$ is sufficiently close to 0. This result extends previous ones by Jun [Ju], Nakae [Na], and Clay and Watson [CW]. Note that the genus of the $(-2, 3, 2n+1)$-pretzel knot, with $n \geq 3$, is equal to $n + 2$.

In this paper, we study the left-orderability for surgeries on the twisted torus knots. Some results about non left-orderable surgeries of twisted torus knots were obtained by Clay and Watson [CW], Ichihara and Temma [IT1,IT2], and Christianson, Goluboff, Hamann, and Varadaraj [CGHV]. We will focus our study on the $(n-2)$-twisted $(3, 3m+2)$-torus knots, which are the knots obtained from the $(3, 3m+2)$-torus knot by adding $(n-2)$ full twists along an adjacent pair of strands. For $n, m \geq 1$, these knots are known to be L-space knots, see [Va]. Moreover, the $(n-2)$-twisted $(3, 5)$-torus knots are exactly the $(-2, 3, 2n+1)$-pretzel knots. Note that the genus of the $(n-2)$-twisted $(3, 3m+2)$-torus knot, with $n, m \geq 1$, is equal to $n + 3m - 1$.

The following result generalizes the one in [Ni].

Theorem 1. *Suppose* $n, m \geq 1$. *Then the fundamental group of the* 3*-manifold obtained by* $\frac{p}{q}$*-surgery along the* $(n-2)$*-twisted* $(3, 3m+2)$*-torus knot is*

(i) *not left-orderable if* $\frac{p}{q} \geq 2n + 6m - 3$,

(ii) *left-orderable if* $\frac{p}{q}$ *is sufficiently close to* 0.

The rest of this paper is devoted to the proof of Theorem 1. In Section 2 we prove part (i). To do so, we follow the method of Jun [Ju], Nakae [Na] and Nie [Ni] which was developed for studying the non left-orderable surgeries of the $(-2, 3, 2n+1)$-pretzel knots. In Section 3 we prove part (ii). To this end, we apply a criterion for the existence of left-orderable surgeries of knots which was first developed by Culler and Dunfield [CD], and then improved by Herald and Zhang [HZ].

2. Non left-orderable surgeries. Let $K_{n,m}$ denote the $(n-2)$-twisted $(3, 3m+2)$-torus knot. By [IT2] (see also [IT1], [CW]), the knot group of

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$K_{n,m}$ has a presentation with two generators a, b and one relation

$$w^n(aw)^m a^{-1}(aw)^{-m} = (wa)^{-m}a(wa)^m w^{n-1},$$

where a is a meridian. Moreover, the preferred longitude corresponding to $\mu = a$ is

$$\lambda = a^{-(4n+9m-2)}[(wa)^m w^n](aw)^{m-1}a[w^n(aw)^m]. \tag{2.1}$$

Note that the first homology class of w is twice that of the meridian a.

Remark 2.1. (i) It is known that $K_{n,1}$ is the pretzel knot of type $(-2, 3, 2n+1)$. The above presentation of the knot group of $K_{n,1}$ was first derived in [LT] and [Na].

(ii) The formula (2.1) for the longtitude of $K_{n,m}$ in [IT1], [IT2] contains a small error: $a^{-(4n+9m-2)}$ was written as $a^{-(2n+9m+2)}$.

Let $M_{\frac{p}{q}}$ be the 3-manifold obtained by $\frac{p}{q}$-surgery along the $(n-2)$-twisted $(3, 3m+2)$-torus knot $K_{n,m}$. Then the fundamental group of $M_{\frac{p}{q}}$ has a presentation with two generators a, b and two relations

$$w^n(aw)^m a^{-1}(aw)^{-m} = (wa)^{-m}a(wa)^m w^{n-1},$$
$$a^p\lambda^q = 1.$$

Since $a^p\lambda^q = 1$ in $\pi_1(M)$ and $a\lambda = \lambda a$, there exists an element $k \in \pi_1(M)$ such that $a = k^q$ and $\lambda = k^{-p}$, see e.g. [Na, Lemma 3.1].

Suppose $m, n \geq 1$. Assume the fundamental group of $M_{\frac{p}{q}}$ is left-orderable for some $\frac{p}{q} \geq 2n + 6m - 3$, where $q > 0$. Then there exists a monomorphism $\rho : \pi_1(M_{\frac{p}{q}}) \to \mathrm{Homeo}^+(\mathbf{R})$ such that there is no $x \in \mathbf{R}$ satisfying $\rho(g)(x) = x$ for all $g \in \pi_1(M)$, see e.g. [CR, Problem 2.25].

From now on we write gx for $\rho(g)(x)$.

Lemma 2.2. *We have $kx \neq x$ for any $x \in \mathbf{R}$.*

Proof. Assume $kx = x$ for some $x \in \mathbf{R}$. Then $x = k^q x = ax$. If $x = wx$ then $gx = x$ for all $g \in \pi_1(M)$, a contradiction. Otherwise, without loss of generality, we assume that $x < wx$. Then we have

$$\begin{aligned} x &= a^{(4n+9m-2)}k^{-p}x \\ &= a^{(4n+9m-2)}\lambda x \\ &= [(wa)^m w^n](aw)^{m-1}a[w^n(aw)^m]x \\ &> x, \end{aligned}$$

which is also a contradiction. □

Since $kx \neq x$ for any $x \in \mathbf{R}$ and kx is a continuous function of x, without loss of generality, we may assume $x < kx$ for any $x \in \mathbf{R}$. Then $x < k^q x = ax$.

Lemma 2.3. *We have $(aw)^m ax < w(aw)^m x$ for any $x \in \mathbf{R}$.*

Proof. Since

$$w^n(aw)^m a^{-1}(aw)^{-m} = (wa)^{-m}a(wa)^m w^{n-1}$$

in $\pi_1(M_{\frac{p}{q}})$, we have

$$\begin{aligned} &w(aw)^m x \\ &\quad = [(aw)^m a(aw)^{-m}w^{-n}(wa)^{-m}a(wa)^m w^{n-1}] \\ &\qquad \times w(aw)^m x \\ &\quad = (aw)^m a[(wa)^m w^n(aw)^m]^{-1}a[(wa)^m w^n(aw)^m]x. \end{aligned}$$

Writing g for $(wa)^m w^n(aw)^m$, we then obtain

$$w(aw)^m x = (aw)^m ag^{-1}agx > (aw)^m ax,$$

since $g^{-1}agx > g^{-1}gx = x$. □

Lemma 2.3 implies that $(aw)^m x < (aw)^m ax < w(aw)^m x$. Hence $x < wx$ for any $x \in \mathbf{R}$.

Lemma 2.4. *For any $x \in \mathbf{R}$ and $k \geq 1$ we have*

$$(aw)^m a^k x < w^k(aw)^m x,$$
$$a^k(wa)^m x < (wa)^m w^k x.$$

Proof. We prove the lemma by induction on $k \geq 1$. The base case ($k = 1$) is Lemma 2.3. Assume $(aw)^m a^k x < w^k(aw)^m x$ for any $x \in \mathbf{R}$. Then

$$\begin{aligned} (aw)^m a^{k+1}x &= (aw)^m a^k(ax) \\ &< w^k(aw)^m ax \\ &< w^k(wa)^m wx \\ &= w^{k+1}(aw)^m x. \end{aligned}$$

Similarly, assuming $a^k(wa)^m x < (wa)^m w^k x$ for any $x \in \mathbf{R}$ then

$$\begin{aligned} a^{k+1}(wa)^m x &< a(wa)^m w^k x \\ &= (aw)^m aw^k x \\ &< w(aw)^m w^k x \\ &= (wa)^m w^{k+1}x. \end{aligned}$$

This completes the proof of Lemma 2.4. □

Lemma 2.5. *With $\frac{p}{q} \geq 2n + 6m - 3$ we have $wx < ax$ for any $x \in \mathbf{R}$.*

Proof. With $\frac{p}{q} \geq 2n + 6m - 3$ and $q > 0$, we have $-p + (2n + 6m - 3)q \leq 0$. Since $a = k^q$, $\lambda = k^{-p}$ and $x < kx$ for any $x \in \mathbf{R}$, we have

$$\begin{aligned} ax &\geq k^{-p+(2n+6m-3)q}ax \\ &= a^{2n+6m-2}\lambda x \end{aligned}$$

$$= a^{-n}[(wa)^m w^n](aw)^{m-1}a[w^n(aw)^m]a^{-(n+3m)}x.$$

Then, by Lemma 2.4, we obtain

$$\begin{aligned} ax &> a^{-n}[a^n(wa)^m](aw)^{m-1}a[(aw)^m a^n]a^{-(n+3m)}x \\ &= w(aw)^{m-1}a(aw)^{m-1}a(aw)^m a^{-3m}x \\ &> wa^{m-1}aa^{m-1}aa^m a^{-3m}x \\ &= wx. \end{aligned}$$

Here, in the last inequality, we use the fact that $x < wx$ for any $x \in \mathbf{R}$. □

With $\frac{p}{q} \geq 2n + 6m - 3$, by Lemmas 2.4 and 2.5 we have

$$\begin{aligned} (aw)^m x &= [(aw)^m a]a^{-1}x \\ &< [w(aw)^m]a^{-1}x = (wa)^m w(a^{-1}x) \\ &< (wa)^m a(a^{-1}x) = a^{-1}[(aw)^m a]x \\ &< a^{-1}[w(aw)^m]x = a^{-1}w[(aw)^m x] \\ &< a^{-1}a[(aw)^m x] = (aw)^m x, \end{aligned}$$

a contradiction. This proves Theorem 1(i).

3. Left-orderable surgeries. To prove Theorem 1(ii) we apply the following result. It was first stated and proved by Culler and Dunfield [CD] under an additional condition on K.

Theorem 3.1 ([HZ]). *For a knot K in S^3, if its Alexander polynomial $\Delta_K(t)$ has a simple root on the unit circle, then the fundamental group of the manifold obtained by $\frac{p}{q}$-surgery along K is left-orderable if $\frac{p}{q}$ is sufficiently close to 0.*

In view of Theorem 3.1, to prove Theorem 1(ii) it suffices to show that the Alexander polynomial of the twisted torus knot $K_{n,m}$ has a simple root on the unit circle. The rest of the paper is devoted to the proof of this fact. We start with a formula for the Alexander polynomial of a knot via Fox's free calculus.

3.1. The Alexander polynomial. Let K be a knot in S^3 and $E_K = S^3 \setminus K$ its complement. We choose a deficiency one presentation for the knot group of K:

$$\pi_1(E_K) = \langle a_1, \ldots, a_l \mid r_1, \ldots, r_{l-1} \rangle.$$

Note that this does not need to be a Wirtinger presentation. Consider the abelianization

$$\alpha : \pi_1(E_K) \to H_1(E_K; \mathbf{Z}) \cong \mathbf{Z} = \langle t \rangle.$$

The map α naturally induces a ring homomorphism $\tilde{\alpha} : \mathbf{Z}[\pi_1(E_K)] \to \mathbf{Z}[t^{\pm 1}]$, where $\mathbf{Z}[\pi_1(E_K)]$ is the group ring of $\pi_1(E_K)$. Consider the $(l-1) \times l$ matrix A whose (i,j)-entry is $\tilde{\alpha}(\frac{\partial r_i}{\partial a_j}) \in \mathbf{Z}[t^{\pm 1}]$, where $\frac{\partial}{\partial a}$ denotes the Fox's free differential. For $1 \leq j \leq l$, denote by A_j the $(l-1) \times (l-1)$ matrix obtained from A by removing the jth column. Then it is known that the rational function

$$\frac{\det A_j}{\det \tilde{\alpha}(a_j - 1)}$$

is an invariant of K, see e.g. [Wa]. It is well-defined up to a factor $\pm t^k$ ($k \in \mathbf{Z}$) and is related to the Alexander polynomial $\Delta_K(t)$ of K by the following formula

$$\frac{\det A_j}{\det \tilde{\alpha}(a_j - 1)} = \frac{\Delta_K(t)}{t-1}.$$

3.2. Proof of Theorem 1(ii). Let

$$\begin{aligned} r_1 &= w^n(aw)^m a^{-1}(aw)^{-m}, \\ r_2 &= (wa)^{-m}a(wa)^m w^{n-1}. \end{aligned}$$

Then we can write $\pi_1(E_{K_{n,m}}) = \langle a, w \mid r_1 r_2^{-1} = 1 \rangle$. In $\pi_1(E_{K_{n,m}})$ we have

$$\begin{aligned} \frac{\partial r_1 r_2^{-1}}{\partial a} &= \frac{\partial r_1}{\partial a} + r_1 \frac{\partial r_2^{-1}}{\partial a} \\ &= \frac{\partial r_1}{\partial a} - r_1 r_2^{-1} \frac{\partial r_2}{\partial a} \\ &= \frac{\partial r_1}{\partial a} - \frac{\partial r_2}{\partial a}. \end{aligned}$$

Let $\delta_k(g) = 1 + g + \cdots + g^k$. Then

$$\begin{aligned} &\frac{\partial r_1 r_2^{-1}}{\partial a} \\ &= w^n[\delta_{m-1}(aw) - (aw)^m a^{-1}(aw)^{-m} \\ &\quad \times (\delta_{m-1}(aw) + (aw)^m)] \\ &\quad - [-(wa)^{-m}\delta_{m-1}(wa)w + (wa)^{-m} \\ &\quad \times (1 + a\delta_{m-1}(wa)w)] \\ &= -w^n(aw)^m a^{-1}[1 - (a-1)(aw)^{-m}\delta_{m-1}(aw)] \\ &\quad - (wa)^{-m}[1 + (a-1)w\delta_{m-1}(aw)]. \end{aligned}$$

The Alexander polynomial $\Delta_{K_{n,m}}(t)$ of $K_{n,m}$ satisfies

$$\frac{\Delta_{K_{n,m}}(t)}{t-1} = \frac{\tilde{\alpha}(\frac{\partial r_1 r_2^{-1}}{\partial a})}{\tilde{\alpha}(w) - 1}.$$

Hence, since $\tilde{\alpha}(a) = t$ and $\tilde{\alpha}(w) = t^2$, we have

$$\begin{aligned} &-(t+1)\Delta_{K_{n,m}}(t) \\ &= t^{2n+3m-1}[1 - (t-1)t^{-3m}\delta_{m-1}(t^3)] \\ &\quad + t^{-3m}[1 + (t-1)t^2\delta_{m-1}(t^3)] \end{aligned}$$

$$= t^{2n+3m-1} + t^{-3m} - (t^{2n-1} - t^{2-3m})(t-1)\delta_{m-1}(t^3)$$
$$= t^{2n+3m-1} + t^{-3m} - (t^{2n-1} - t^{2-3m})\frac{t^{3m}-1}{t^2+t+1}$$
$$= t^{-3m}\frac{1+t+t^{3m+2}+t^{2n+3m-1}+t^{2n+6m}+t^{2n+6m+1}}{t^2+t+1}.$$

Let $f(t) = t^{n+3m+1/2} + t^{-(n+3m+1/2)} + t^{n+3m-1/2} + t^{-(n+3m-1/2)} + t^{n-3/2} + t^{-(n-3/2)}$. Then

$$\Delta_{K_{n,m}}(t) = -\frac{t^{n-1}f(t)}{(t^{1/2}+t^{-1/2})(t+t^{-1}+1)}.$$

Hence $\Delta_{K_{n,m}}(e^{i\theta}) = -\frac{e^{i(n-1)\theta}f(e^{i\theta})}{2\cos(\theta/2)(2\cos\theta+1)}$.

Let $g(\theta) = f(e^{i\theta})/2$. To show that $\Delta_{K_{n,m}}(t)$ has a simple root on the unit circle, it suffices to show that $g(\theta)$ has a simple root on $(0, 2\pi/3)$. We have

$$\begin{aligned} g(\theta) &= \cos(n+3m+1/2)\theta + \cos(n+3m-1/2)\theta \\ &\quad + \cos(n-3/2)\theta \\ &= 2\cos(\theta/2)\cos(n+3m)\theta + \cos(n-3/2)\theta. \end{aligned}$$

If $n = 1$ then $g(\theta) = \cos(\theta/2)(2\cos(n+3m)\theta + 1)$. It is clear that $\theta = \frac{2\pi/3}{n+3m}$ is a simple root of $g(\theta)$ on $(0, \pi/6]$.

Suppose $n \geq 2$. We claim that $g(\theta)$ has a simple root on (θ_0, θ_1) where $\theta_0 = \frac{\pi/2}{n+3m}$ and $\theta_1 = \frac{\pi/2}{n+3m/2-3/4}$. Note that $0 < \theta_0 < \theta_1 \leq \frac{\pi/2}{7/4} = \frac{2\pi}{7}$. We have

$$g(\theta_0) = \cos(n-3/2)\theta_0 = \cos\left(\frac{\pi}{2}\frac{n-3/2}{n+3m}\right) > 0,$$

since $0 < \frac{\pi}{2}\frac{n-3/2}{n+3m} < \frac{\pi}{2}$.

At $\theta = \theta_1 = \frac{\pi/2}{n+3m/2-3/4}$ we have

$$\begin{aligned} &\cos(n+3m)\theta + \cos(n-3/2)\theta \\ &\quad = 2\cos(n+3m/2-3/4)\theta\cos(3m/2+3/4)\theta \\ &\quad = 0. \end{aligned}$$

Hence

$$\begin{aligned} g(\theta_1) &= (1-2\cos(\theta_1/2))\cos(n-3/2)\theta_1 \\ &= (1-2\cos(\theta_1/2))\cos\left(\frac{\pi}{2}\frac{n-3/2}{n+3m/2-3/4}\right) \\ &< 0, \end{aligned}$$

since $1 - 2\cos(\theta_1/2) < 0 < \cos(\frac{\pi}{2}\frac{n-3/2}{n+3m/2-3/4})$.

We show that $g(\theta)$ is a strictly decreasing function on (θ_0, θ_1). Indeed, we have

$$\begin{aligned} -g'(\theta) &= \sin(\theta/2)\cos(n+3m)\theta \\ &\quad + 2(n+3m)\cos(\theta/2)\sin(n+3m)\theta \\ &\quad + (n-3/2)\sin(n-3/2)\theta. \end{aligned}$$

Since

$$0 < (n-3/2)\theta < \frac{\pi}{2}\frac{n-3/2}{n+3m/2-3/4} < \frac{\pi}{2},$$

we have $(n-3/2)\sin(n-3/2)\theta > 0$. Since $\frac{\pi/2}{n+3m} < \theta < \frac{\pi/2}{n+3m/2-3/4}$ we have

$$\pi/2 < (n+3m)\theta < \frac{\pi}{2}\frac{n+3m}{n+3m/2-3/4} < \pi,$$

which implies that $\cos(n+3m)\theta < 0 < \sin(n+3m)\theta$. Hence

$$\begin{aligned} -g'(\theta) &> \sin(\theta/2)\cos(n+3m)\theta \\ &\quad + \cos(\theta/2)\sin(n+3m)\theta \\ &= \sin(n+3m+1/2)\theta. \end{aligned}$$

Since $0 < (n+3m+1/2)\theta < \frac{\pi}{2}\frac{n+3m+1/2}{n+3m/2-3/4} \leq \pi$, we have $\sin(n+3m+1/2)\theta \geq 0$. Hence $-g'(\theta) > 0$ on (θ_0, θ_1). This, together with $g(\theta_0) > 0 > g(\theta_1)$, implies that $g(\theta)$ has a simple root on (θ_0, θ_1). The proof of Theorem 1(ii) is complete.

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