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UT Dallas Author(s):

Alain Bensoussan

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PARABOLIC BELLMAN EQUATIONS WITH RISK CONTROL*

ALAIN BENSOUSSAN†, DOMINIC BREIT‡, AND JENS FREHSE§

Abstract. We consider stochastic optimal control problems with an additional term representing the variance of the control functions. The latter one may serve as a risk control. We present and treat the problem in a purely analytical way via a Vlasov–McKean functional and Bellman equations with mean field dependence. We obtain global existence and, essentially, optimal global regularity for the solutions of the Bellman equation and the minimizing control. Surprisingly, the risk term simplifies the analysis to a certain extend.

Key words. onlinear parabolic equations, Bellman equations, stochastic differential games, mean field dependence, risk control

AMS subject classifications. 35B50, 35K55, 35B65, 49L20

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1. Introduction.

The present paper is motivated by stochastic control problems where the usual functional to be minimized is augmented by an additional "risk term." The latter penalizes the variance of the controls. In order to explain the problem in detail, it is convenient to use a purely analytical approach based on the minimization of Vlasov–McKean functionals. In the classical case these functionals are defined by

$$\mathscr{J}_0[\mathbf{v}] = \int_0^T \int_\Omega m\, f(\cdot, \mathbf{v})\, dx\, dt + \int_\Omega u_T\, m(T, \cdot)\, dx.$$

Here, the functions $\mathbf{v} \in L^q(0, T; L^q(\Omega, \mathbb{R}^s))$ are the controls, f is the given payoff function, and m is the mean field variable. The latter one satisfies the parabolic equation

$$\partial_t m - \Delta m + \operatorname{div}\, (\mathbf{g}(\mathbf{v})\, m) = 0, \quad m\big|_{t=0} = m_0, \quad \int_\Omega m_0\, dx = 1, \tag{1.1}$$

subject to Neumann boundary conditions and u_T is the end condition for the Bellman equation (see (1.3) below). Since $\int_\Omega m(t, \cdot)\, dx = 1$ we can interpret m as probability distribution. For details we refer to section 2. In the stochastic application $\mathbf{g}$ represents the dynamics and Δm is related to white noise. We restrict ourselves to affine linear functions $\mathbf{g}$.

The remarkable well-known fact is that minimizing controls $\hat{\mathbf{v}}$ of $\mathscr{J}_0$ satisfy the pointwise necessary condition

$$\partial_{\mathbf{v}} f(\cdot, \hat{\mathbf{v}}) + \nabla u \cdot \partial_{\mathbf{v}} \mathbf{g}(\hat{\mathbf{v}}) = 0. \tag{1.2}$$

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†Naveen Jindal School of Management, University of Texas at Dallas, Richardson, TX 75080, and Department SEEM, City University of Hong Kong, Kowloon Tong, Hong Kong (axb046100@utdallas.edu).
‡Department of Mathematics, Heriot-Watt University, Edinburgh EH14 4AS, UK (d.breit@hw.ac.uk).
§Department of Applied Mathematics, University of Bonn, 53115 Bonn, Germany (erdbeere1@justmail.de).

Here, u is the solution to the backward parabolic equation

$$-\partial_t u - \Delta u = f(\cdot, \mathbf{v}) + \nabla u \cdot \mathbf{g}(\mathbf{v}), \quad u\big|_{t=T} = u_T, \tag{1.3}$$

with Neumann boundary conditions. Under convexity assumptions (1.2) implies a relation $\hat{\mathbf{v}}(t,x) = \boldsymbol{\omega}(t, x\nabla u(t,x))$, i.e., a feedback formula. Inserting this formula into (1.3) one obtains Bellman's equation. Equation (1.3) is exactly the Bellman equation derived in the stochastic setting.

Now, according to the scope of this paper, we consider the augmented functional

$$\mathcal{J}[\mathbf{v}] = \mathcal{J}_0[\mathbf{v}] + \sigma(m, \mathbf{v}). \tag{1.4}$$

Here, σ maps pairs of functions $(m, \mathbf{v})$ of appropriate Lebesgue spaces into $\mathbb{R}$. Our motivation arises from the example

$$\sigma(\mathbf{v}, m) = \sqrt{T\int_0^T \int_\Omega m|\mathbf{v}|^4 dx dt - \left(\int_0^T \int_\Omega m|\mathbf{v}|^2\, dx\, dt\right)^2},$$

i.e., the variance of $|\mathbf{v}|^2$. In order to avoid the singularity of the square root at 0 we consider the regular case, where

$$\sigma(\mathbf{v}, m) = \sigma_\rho(\mathbf{v}, m) = \sqrt{\rho^2 + T\int_0^T \int_\Omega m|\mathbf{v}|^4 dx dt - \left(\int_0^T \int_\Omega m|\mathbf{v}|^2\, dx\, dt\right)^2}$$

with $\rho > 0$. Other examples of σ are discussed in section 2. The minimization problem for $\mathcal{J}$ leads to a condition which differs from (1.2) and a Bellman equation which differs from (1.3). The new Bellman equation is field dependent, i.e., there occurs an additional term $\partial_m \sigma(m, \mathbf{v})$ on the right-hand side; cf. (2.10).

Mean field dependent Bellman equations have been studied recently in other situations; see, e.g., recent results by Porretta [10, 21, 22], [15] as well as [20]. In these papers the payoff depends in a pointwise way on the mean field variable. Note that in the case of systems (not treated in this paper), however, a pointwise mean field dependence gives severe technical problems concerning existence and regularity analysis. For example, up to now, strong growth conditions for the nonlinearity with respect to the variable m are needed; see, e.g., [9]. Different from that, we have a functional dependence on the mean field variable in (1.4). For further examples of functionals with this property we refer to section 2.

Surprisingly, the term $\sigma = \sigma_\rho$ has a stabilizing influence in the estimates. The special structure properties of σ allow a uniform L^∞-estimate for the solutions $\mathbf{v}_L$ of approximate problems with bounded control ranges in $B_L(0)$; cf. section 3. For classical problems (without the additional term σ) this is usually more difficult and requires additional assumptions. In fact, a minimizer $\mathbf{v}_L$ exists due to lower semicontinuity properties. From the Bellman variational inequality and the growth and coerciveness properties of σ we obtain a pointwise inequality

$$|\mathbf{v}_L| \leq K\, |\nabla u_L|^{1/3} + K,$$

where u_L is the solution of the Bellman equation; cf. (1.3) with $\mathbf{v} = \mathbf{v}_L$. This implies that u_L solves a subcritical differential inequality

$$|\partial_t u_L - \Delta u_L| \leq K|\nabla u_L|^\beta + K$$

with $\beta = \frac{4}{3}$. Now, a bootstrap argument yields a uniform bound for $u_L \in L^q(W^{2,q})$ and $\partial_t u_L \in L^q(L^q)$ for all $q < \infty$. Hence, we have a uniform L^∞-bound for u_L independent of L. Once having this established, it is simple to obtain the main result in Theorem 2.1. It states the existence of an optimal control $\hat{\mathbf{v}} \in L^\infty(L^\infty)$ and the corresponding $L^q(W^{2,q})$-regularity of the related Bellman equation with mean field dependence. Note that the additional term σ destroys the convexity properties of the Bellman equation. So, we are not able to establish a one-to-one correspondence of $\hat{\mathbf{v}}(t,x)$ and $\nabla u(t,x)$. As a consequence, a feedback formula is not available in the general situation of Theorem 2.1. This lack can be overcome if the risk term is dominated by the payoff. To quantify this effect let us consider instead of (1.4) the minimization problem

$$\mathcal{J}[\mathbf{v}] = \mathcal{J}_0[\mathbf{v}] + \beta\sigma(m, \mathbf{v}).$$

If $\beta > 0$ remains below a given number (see section 4 for details) we reach the convex case and can state the existence of a feedback formula; cf. Theorem 4.1.

Stochastic optimal control problems. Let us finally explain the stochastic setup which motivates minimization problems as in (1.4). Consider a probability space $(\mathcal{O}, \mathcal{F}, \mathbb{P})$ on which a standard n-dimensional Wiener process $(W_t)_{t\geq 0}$ is defined. For simplicity we describe the control problem in $\mathbb{R}^n$ instead of a bounded domain Ω which describes the state space. In Ω the diffusion should be replaced with a reflected diffusion. Formally, the equation for the state $\mathbf{y}$ can be written as

$$d\mathbf{y} = \mathbf{G}(t, \mathbf{y}, \mathbf{v})\, dt + \sqrt{2}\, dW, \quad \mathbf{y}(0) = x. \tag{1.5}$$

Here, $\mathbf{G} : [0,T] \times \mathbb{R}^n \times \mathbb{R}^s \to \mathbb{R}^n$ is the dynamic. The stochastic process $\mathbf{v} = (v_1, \dots, v_s)$, $s \in \mathbb{N}$, is the control variable defined by a feedback on the state. One has

$$\mathbf{v} = \mathbf{v}(y(t)).$$

On account of the initial condition in (1.5) $\mathbf{v}$ also depends on x. Note that the reflected diffusion lies in a bounded domain. Hence, we can assume that $\mathbf{G}$ is bounded in Ω. The Kolmogorov equation, i.e., the equation for the probability density of $\mathbf{y}(t)$ depending on the initial state x, is then given by (1.1). For full details we refer to [8]. In order to minimize the costs we have to minimize the cost-functional

$$\mathfrak{J}[\mathbf{v}] = \mathbb{E}\left[\int_0^T f(t, \mathbf{y}(t), \mathbf{v}(t))\, dt\right] + \sqrt{T\mathbb{E}\int_0^T |\mathbf{v}(t)|^4\, dt - \left(\mathbb{E}\int_0^T |\mathbf{v}(t)|^2\, dt\right)^2}, \tag{1.6}$$

where $f : [0,T] \times \mathbb{R}^n \times \mathbb{R}^s \to \mathbb{R}$ is the payoff.

2. Setup and main result.

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with boundary $\partial\Omega \in C^2$ with unit normal vector field ν, $(0,T)$ the time interval, and $Q = (0,T) \times \Omega$ the parabolic cylinder. We consider control functions $\mathbf{v} : Q \to \mathbb{R}^s$, where $s \in \mathbb{N}$. We suppose the following (for any $\mathbf{v} \in \mathbb{R}^s$, $t \in (0,T)$ and $x \in \Omega$):

- (A1) $f(t, x, \cdot)$ is continuously differentiable and convex.
- (A2) $\partial_{\mathbf{v}} f$ satisfies the Caratheodory conditions; that is, the function $\partial_{\mathbf{v}} f(\cdot, \mathbf{v})$ is measurable for any $\mathbf{v} \in \mathbb{R}^s$ and the function $\partial_{\mathbf{v}} f(t, x, \cdot)$ is continuous for almost every $(t,x) \in Q$.
- (A3) f satisfies some coerciveness-type condition, i.e.,

$$f(\cdot, \mathbf{v}) \geq c_0|\mathbf{v}|^2 - K, \quad \partial_{\mathbf{v}} f(\cdot, \mathbf{v}) \cdot \mathbf{v} \geq c_0|\mathbf{v}|^2 - K,$$

for some $K \geq 0$ and $c_0 > 0$.

(A4) f satisfies some growth conditions, i.e., for some $C_0 > 0$

$$f(\cdot,\mathbf{v}) \le C_0|\mathbf{v}|^2 + K, \quad \partial_{\mathbf{v}} f(\cdot,\mathbf{v})\cdot\mathbf{v} \le C_0|\mathbf{v}|^2 + K.$$

For the mean field equation we consider functions $\mathbf{g} : Q \times \mathbb{R}^n \to \mathbb{R}^n$ with

$$\mathbf{g}(t,x,\mathbf{v}) = \mathbf{A}(t,x)\mathbf{v} + \mathbf{b}_0(t,x).$$

In applications to stochastic optimization or game theory, these functions define the dynamics. We assume that $\mathbf{A} \in L^\infty(Q,\mathbb{R}^{n\times s})$ and $\mathbf{b}_0 \in L^\infty(Q,\mathbb{R}^n)$. For a given $\mathbf{v} \in L^\infty(Q,\mathbb{R}^s)$ the scalar function m is defined as a weak solution of the field equation

$$\partial_t m - \Delta m + \operatorname{div}\,(\mathbf{g}(\mathbf{v})\,m) = 0, \quad \partial_\nu m|_{\partial\Omega} = g(\mathbf{v})\cdot\nu\,m, \quad m\big|_{t=0} = m_0. \tag{2.1}$$

Here and in the following, all PDEs are considered with respect to Neumann boundary conditions.

Definition 2.1. *Let $\mathbf{v} \in L^\infty(Q,\mathbb{R}^s)$ and $m_0 \in L^2(\Omega)$. A weak solution $m = m(t,x,\mathbf{v})$ to (2.1) is a function $m \in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;W^{1,2}(\Omega))$ such that*

$$-\int_Q m\,\partial_t\phi\,dx\,dt + \int_Q \nabla m\cdot\nabla\varphi\,dx\,dt = \int_Q m\mathbf{g}(\mathbf{v})\cdot\nabla\varphi\,dx\,dt - \int_\Omega m_0\,\varphi(0,\cdot)\,dx$$

for all $\varphi \in C^\infty([0,T]\times\overline{\Omega})$ with $\varphi(T,\cdot) = 0$.

As the equation is linear in m, existence, uniqueness, and boundedness for m follow from standard theory. Moreover, positivity of m_0 transfers to m.

Lemma 2.2. *Let $m_0 = \tilde{m}_0|_\Omega$ for some $\tilde{m}_0 \in L^{q_0}(0,T;W^{1,q_0}(\Omega))$ with $q_0 > n+2$. Let $\mathbf{v} \in L^\infty(Q;\mathbb{R}^s)$ and let m be the weak solution to (2.1).*

(a) *There is $\alpha > 0$ such that $m \in C^\alpha(\overline{Q})$ together with*

$$\|m\|_{C^\alpha(\overline{Q})} \le c\left(\|\mathbf{v}\|_\infty, \|\nabla\tilde{m}_0\|_{L^{q_0}(Q)}\right).$$

(b) *Let $q \in (1,\infty)$ and $\tilde{m}_0 \in L^q(0,T;W^{1,q}(\Omega))$. Then we have*

$$\|\nabla m\|_{L^q(Q)} \le c\left(\|\mathbf{v}\|_\infty, \|\nabla\tilde{m}_0\|_{L^q(Q)}\right).$$

(c) *If $\int_\Omega m_0\,dx = 1$, then we have $\int_\Omega m(t,\cdot)\,dx = 1$ for all $t \in [0,T]$.*

(d) *Assume in addition $m_0 > 0$ a.e. in Ω. Then $m > 0$ in Q.*

Parts (a), (c), and (d) are a classical result from [17, Chap. III, sect. 7, Thm. 7.1 and sect. 7, Thm. 10.1]. For (b) we refer to [12, Thm. 1.6] and [19, Chap. XII, sect. 7, Prop. 7.26]. We remark that (a) follows from (b) for $q > n+2$.

We now consider a real-valued functional, denoted by $\sigma(\mathbf{v},m)$, with functional dependence defined on $L^\infty(Q,\mathbb{R}^s)\times L^\infty(Q)$ the prototype of which is given in (2.6). We make the following assumptions:

(B1) The mapping $(\mathbf{w},m)\mapsto\sigma(\mathbf{w},m)$ is lower semicontinuous with respect to the $L^\infty_{w^*}(Q,\mathbb{R}^s)\times C^0(\overline{Q})$-topology.

(B2) σ is bounded from below, i.e.,

$$\sigma(\mathbf{w},m) \ge \rho > 0$$

for some $\rho > 0$.

(B3) The functional σ has a Gâteaux derivative with respect to $\mathbf{w}$: there exists a unique function $\frac{\partial\sigma}{\partial\mathbf{w}}(\mathbf{w},m) \in L^1(Q;\mathbb{R}^s)$ such that

$$\frac{\sigma(\mathbf{w}+\theta\tilde{\mathbf{w}},m)-\sigma(\mathbf{w},m)}{\theta} \longrightarrow \int_Q \frac{\partial\sigma}{\partial\mathbf{w}}(\mathbf{w},m)\cdot\tilde{\mathbf{w}}\,dx\,dt$$

as $\theta \to 0$ for all $\tilde{\mathbf{w}} \in L^\infty(Q,\mathbb{R}^s)$.

(B4) The functional σ has an existence Gâteaux derivative with respect to m: there exists a unique function $\frac{\partial\sigma}{\partial m}(\mathbf{w},m) \in L^1(Q)$ such that

$$\frac{\sigma(\mathbf{w},m+\theta\tilde{m})-\sigma(\mathbf{w},m)}{\theta} \longrightarrow \int_Q \frac{\partial\sigma}{\partial m}(\mathbf{w},m)\,\tilde{m}\,dx\,dt \tag{2.2}$$

as $\theta \to 0$ for all $\tilde{m} \in L^\infty(Q)$.

(B5) The derivatives $\frac{\partial\sigma}{\partial\mathbf{w}}(\mathbf{w},m)$ and $\frac{\partial\sigma}{\partial m}(\mathbf{w},m)$ satisfy the following growth conditions. There are nonnegative functions $\gamma_1,\gamma_2,\beta_1,\beta_2 : \mathbb{R}^s\times\mathbb{R}\to\mathbb{R}$ such that

$$\frac{\partial\sigma}{\partial\mathbf{w}}(\mathbf{w},m)\cdot\mathbf{w} \geq \big(\gamma_1(\mathbf{w},m)|\mathbf{w}|^4-\gamma_2(\mathbf{w},m)\big)\,m, \tag{2.3}$$

$$-\beta_1(\mathbf{w},m) \leq \frac{\partial\sigma}{\partial m}(\mathbf{w},m) \leq \beta_2(\mathbf{w},m)|\mathbf{w}|^4 \tag{2.4}$$

for all $(\mathbf{w},m) \in L^\infty(Q,\mathbb{R}^s)\times L^\infty(Q)$. If $(\mathbf{w},m) \in L^\infty(Q,\mathbb{R}^s)\times L^\infty(Q)$ satisfies

$$\sigma(\mathbf{w},m)+\int_Q |\mathbf{w}|^2 m\,dx\,dt \leq c,$$

then there are constants $c_1, C_1 > 0$ (depending on c) such that

$$\gamma_1(\mathbf{w},m) \geq c_1 > 0, \quad \gamma_2(\mathbf{w},m),\beta_1(\mathbf{w},m),\beta_2(\mathbf{w},m) \leq C_1. \tag{2.5}$$

Remark 2.3. The power 4 in (2.3) and (2.4) can be replaced by some other power, strictly larger than 2. It leads to related calculations.

Remark 2.4. Assumption (B2) is for technical reasons only. Our approach requires dividing by σ. We refer to the computations for the model case below.

The prototype for σ is

$$\sigma(\mathbf{v},m) = \beta\sqrt{\rho^2+T\int_Q m|\mathbf{v}|^4dxdt-\left(\int_Q m|\mathbf{v}|^2\,dx\,dt\right)^2}, \tag{2.6}$$

where $\beta,\rho > 0$. Such a functional is motivated by risk management considerations. We want the energy $|\mathbf{v}|^2$ to be not too far from its average $\int_Q m|\mathbf{v}|^2\,dx\,dt$. In probabilistic terms, we penalize the variance of the energy. We do not know if the mapping $\mathbf{v} \mapsto \sigma(m,\mathbf{v})$ is convex (for a given m). Nevertheless we can use convexity of $\mathbf{v} \mapsto \sigma^2(m,\mathbf{v})$ to conclude the weak lower semicontinuity of $\sigma(m,\cdot)$ in $L^2(Q)$. Hence B1 follows. An easy computation leads to

$$\frac{\partial\sigma}{\partial\mathbf{v}}(\mathbf{v},m) = \frac{2}{\sigma(\mathbf{v},m)}m\mathbf{v}\left(T|\mathbf{v}|^2-\int_Q m|\mathbf{v}|^2\,dx\,dt\right). \tag{2.7}$$

So, we have

$$\frac{\partial\sigma}{\partial\mathbf{v}}(\mathbf{v},m)\cdot\mathbf{v}=\frac{2m}{\sigma(\mathbf{v},m)}\left(T|\mathbf{v}|^4-|\mathbf{v}|^2\int_Q m|\mathbf{v}|^2\,dx\,dt\right)$$
$$\geq\frac{m}{\sigma(\mathbf{v},m)}\left(T|\mathbf{v}|^4-\frac{1}{T}\left(\int_Q m|\mathbf{v}|^2\,dx\,dt\right)^2\right).$$

Hence, (2.3) is satisfied with

$$\gamma_1(\mathbf{v},m)=\frac{T}{\sigma(\mathbf{v},m)},\quad \gamma_2(\mathbf{v},m)=\frac{(\int_Q m|\mathbf{v}|^2\,dx\,dt)^2}{\rho T}.$$

Similarly, we can state

$$\frac{\partial\sigma}{\partial m}(\mathbf{v},m)=\frac{|\mathbf{v}|^2}{2\sigma(\mathbf{v},m)}\left(T|\mathbf{v}|^2-2\int_Q m|\mathbf{v}|^2\,dx\,dt\right)\tag{2.8}$$

and (2.4) is satisfied with

$$\beta_1(\mathbf{v},m)=\frac{\left(\int_Q m|\mathbf{v}|^2\,dx\,dt\right)^2}{2\rho T},\quad \beta_2(\mathbf{v},m)=\frac{T}{2\rho}.$$

Now we are ready to define the Vlasov–McKean functional as

$$\mathcal{J}[\mathbf{v}]=\int_Q mf(\cdot,\mathbf{v})\,dx\,dt+\sigma(\mathbf{v},m)+\int_\Omega m(T,\cdot)\,u_T\,dx\tag{2.9}$$

with given data u_T. The objects of the present paper are existence, regularity, and further properties of minimizers of $\mathcal{J}$. For $\mathbf{v}\in L^\infty(Q,\mathbb{R}^s)$ given the corresponding (pre-)Bellman equation reads as

$$\begin{cases}-\partial_t u-\Delta u=f(\cdot,\mathbf{v})+\nabla u\cdot\mathbf{g}(\mathbf{v})+\dfrac{\partial\sigma}{\partial m}(\mathbf{v},m) & \text{in}\quad Q,\\ \partial_\nu u=0 & \text{on}\quad (0,T)\times\partial\Omega,\\ u(T,\cdot)=u_T & \text{in}\quad \Omega.\end{cases}\tag{2.10}$$

Definition 2.5. *Let $m\in L^\infty(Q)$, $\mathbf{v}\in L^\infty(Q,\mathbb{R}^s)$, and $u_T\in L^2(\Omega)$. A weak solution u to* (2.10) *is a function $u\in L^\infty(0,T;L^2(\Omega))\cap L^2(0,T;W^{1,2}(\Omega))$ such that*

$$\int_Q u\,\partial_t\phi\,dx\,dt+\int_Q\nabla u\cdot\nabla\varphi\,dx\,dt=\int_Q\left(f(\cdot,\mathbf{v})+\nabla u\cdot\mathbf{g}(\mathbf{v})\right)\varphi\,dx\,dt$$
$$+\int_Q\frac{\partial\sigma}{\partial m}(\mathbf{v},m)\,\varphi\,dx\,dt+\int_\Omega u_T\,\varphi(0,\cdot)\,dx$$

for all $\varphi\in C^\infty([0,T]\times\overline{\Omega})$ with $\varphi(0,\cdot)=0$.

We define the Lagrangian L as

$$L(\cdot,\mathbf{v},\mathbf{p}):=f(\cdot,\mathbf{v})+\mathbf{p}\cdot\mathbf{g}(\cdot,\mathbf{v}).\tag{2.11}$$

We can then derive the necessary conditions of optimality. If $\mathbf{v}=\hat{\mathbf{v}}$ is an optimal control to the Vlasov–McKean functional we call (2.10) a generalized Bellman equation. One can prove the variational inequality

$$\left(m\, f_{\mathbf{v}}(\cdot,\hat{\mathbf{v}}) + m\, \mathbf{A}^T \nabla u + \frac{\partial \sigma}{\partial \mathbf{v}}(\hat{\mathbf{v}}, m)\right) \cdot (\mathbf{w} - \mathbf{v}) \geq 0$$

for all $\mathbf{w} \in B_L(0)$; see Lemma 3.2. If L is strictly convex in $\mathbf{v}$ (which is not the case here) we have a unique representation,

$$\hat{\mathbf{v}}(t,x) = \boldsymbol{\omega}(t,x;\nabla u(t,x), m(t,x)).$$

Consequently, we can define the Hamiltonian $\mathcal{H}$ as

$$\mathcal{H}(\cdot, m, \nabla \mathbf{u}) := L(\cdot, \hat{\mathbf{v}}, \nabla u) + \frac{\partial \sigma}{\partial m}(\hat{\mathbf{v}}, m). \tag{2.12}$$

However, in our case there is no one-to-one relation between ∇u and $\hat{\mathbf{v}}$ because of a nonuniqueness. So, the definition of $\mathcal{H}$ is more involved.

Let us state the main result of this paper, which is the global existence and optimal global regularity for the solutions of the Bellman equation and the minimizing control.

THEOREM 2.1. *Assume that* (A1)–(A4) *as well as* (B2)–(B5) *hold. Let* $m_0 = \tilde{m}_0|_\Omega$ *for some* $\tilde{m}_0 \in L^{q_0}(0,T;W^{1,q_0}(\Omega))$ *and* $u(T,\cdot) = u_T$ *for some* $u_T \in L^{q_0}(0,T; W^{2,q_0}(\Omega))$ *with* $\partial_\nu u_T = 0$ *on* $\partial\Omega$ *and* $\partial_t u_T \in L^{q_0}(0,T;L^{q_0}(\Omega))$ *for some* $q_0 > n+2$. *Further assume that* $m_0 > 0$ *a.e. and* $\int_\Omega m_0\, dx = 1$. *Then there is minimizer* $\mathbf{v}$ *of the McKean Vlasov functional in* (2.9), *solution* m *of the field equation* (2.1), *and solution* u *of the Bellman-type equation* (2.10) *such that*

$$\begin{aligned} &\mathbf{v} \in L^\infty(Q,\mathbb{R}^s), \quad m \in L^q(0,T;W^{1,q}(\Omega)) \cap C^\alpha(\overline{Q}),\\ &u \in L^q(0,T;W^{2,q}(\Omega)), \;\; \partial_t u \in L^q(0,T;L^q(\Omega)), \end{aligned}$$

for any $q < \infty$ *and some* $\alpha > 0$.

Remark 2.6. By an additional selection procedure and a smallness condition on σ we obtain a unique minimizer in the class of selected minimizers as well as a feedback formula; see Theorem 4.1.

Remark 2.7. The operator Δ in the Bellman system and the mean field equation only serves as a prototype. Theorem 2.1 extends to elliptic differential operators with Lipschitz coefficients.

3. Proof of the main theorem.

3.1. Necessary condition of optimality. Let $\hat{\mathbf{v}}$ be an optimal control. The key issue is to show that it will be bounded by a constant that we can compute a priori. This is obtained by computing the Gâteaux differential of $\mathcal{J}(\mathbf{v})$ at point $\hat{\mathbf{v}}$ and writing the Euler necessary condition of optimality

$$\frac{d}{d\theta}\mathcal{J}(\hat{\mathbf{v}} + \theta\mathbf{w})\Big|_{\theta=0} = 0 \quad \forall \mathbf{w} \in L^\infty(Q,\mathbb{R}^s). \tag{3.1}$$

We write $\hat{m} = m_{\hat{\mathbf{v}}}$ for the solution to (2.1) with $\mathbf{v} = \hat{\mathbf{v}}$. For any $\mathbf{w}$ bounded, we introduce $\tilde{m}_{\mathbf{w}}$ (also depending on $\hat{\mathbf{v}}$) as the solution of

$$\partial_t \tilde{m} - \Delta\tilde{m} + \operatorname{div}((\mathbf{A}\hat{\mathbf{v}} + \mathbf{b}_0)\tilde{m}) + \operatorname{div}(\mathbf{A}\mathbf{w}\hat{m}) = 0, \quad \tilde{m}(0,\cdot) = 0, \tag{3.2}$$

which is well defined since $\hat{\mathbf{v}}$ and $\mathbf{w}$ are bounded. An easy calculation shows that

$$\frac{d}{d\theta}\mathcal{J}(\hat{\mathbf{v}}+\theta\mathbf{w})\Big|_{\theta=0} = \int_Q \left(\hat{m}\,\partial_{\mathbf{v}} f(\cdot,\hat{\mathbf{v}}) + \frac{\partial\sigma}{\partial\mathbf{v}}(\hat{\mathbf{v}},\hat{m})\right)\cdot\mathbf{w}\,dx\,dt + \int_Q \left(f(\cdot,\hat{\mathbf{v}}) + \frac{\partial\sigma}{\partial m}(\hat{\mathbf{v}},\hat{m})\right)\tilde{m}_{\mathbf{w}}\,dx\,dt + \int_\Omega \tilde{m}_{\mathbf{w}}\,u_T(x)\,dx. \tag{3.3}$$

We then introduce u as the solution of the parabolic problem

$$-\partial_t u - \Delta u = \nabla u\cdot(\mathbf{A}\hat{\mathbf{v}}+\mathbf{b}_0) + f(\cdot,\hat{\mathbf{v}}) + \frac{\partial\sigma}{\partial m}(\hat{\mathbf{v}},\hat{m}), \quad u(\cdot,T) = u_T. \tag{3.4}$$

Then the Gâteaux differential can be expressed as

$$\int_Q \left(\hat{m}(\partial_{\mathbf{v}} f(\cdot,\hat{\mathbf{v}}) + \mathbf{A}^T\nabla u) + \frac{\partial\sigma}{\partial\mathbf{v}}(\hat{\mathbf{v}},\hat{m})\right)\cdot\mathbf{w}\,dx\,dt = 0.$$

Since $\mathbf{w}$ is arbitrary, we immediately obtain the necessary condition

$$\hat{m}(\partial_{\mathbf{v}} f(\cdot,\hat{\mathbf{v}}) + \mathbf{A}^T\nabla u) + \frac{\partial\sigma}{\partial\mathbf{v}}(\hat{\mathbf{v}},\hat{m}) = 0 \quad \text{a.e.} \tag{3.5}$$

3.2. A priori estimates. In this subsection we give a formal proof of the a priori estimates based on the assumption of sufficient smoothness of $\hat{\mathbf{v}}$. In the following is meant subsection we will make it rigorous by an appropriate approximation procedure.

Let $\hat{\mathbf{v}}$ be a minimizer of $\mathcal{J}$. Since $\hat{\mathbf{v}}$ is optimal, we have $\mathcal{J}(\hat{\mathbf{v}}) \le \mathcal{J}(0)$. Therefore, taking into account the assumptions (A3) and (A4), the fact that u_T is bounded, and $\hat{m}$ is a probability density (i.e., $\int_\Omega \hat{m}(t,\cdot)\,dx = 1$ for any t), we deduce the estimates

$$\begin{aligned} c_0\int_Q \hat{m}|\hat{\mathbf{v}}|^2 dxdt - K_0|Q| + \sigma(\hat{\mathbf{v}},\hat{m}) &\le \mathcal{J}(\hat{\mathbf{v}}) \le \mathcal{J}(0) \\ &= \int_Q \hat{m} f(\cdot,0)\,dx\,dt + \sigma(0,\hat{m}) + \int_\Omega \hat{m}(T,\cdot)\,u_T\,dx \\ &\le c(1+\|u_T\|_\infty). \end{aligned} \tag{3.6}$$

Hence $\int_Q \hat{m}|\hat{v}|^2\,dx\,dt + \sigma(\hat{\mathbf{v}},\hat{m}) \le c$. From (2.5) we can assert that $\gamma_1(\hat{\mathbf{v}},\hat{m}) \ge c_1 > 0$, and

$$\gamma_2(\hat{\mathbf{v}},\hat{m}), \beta_1(\hat{\mathbf{v}},\hat{m}), \beta_2(\hat{\mathbf{v}},\hat{m}) \le C_1.$$

From the assumptions (2.3) and (2.4) it follows that

$$\frac{\partial\sigma}{\partial\mathbf{v}}(\hat{\mathbf{v}},\hat{m})\cdot\hat{\mathbf{v}} \ge (c_1|\hat{\mathbf{v}}|^4 - C_1)\hat{m} \tag{3.7}$$

as well as

$$-C_1 \le \frac{\partial\sigma}{\partial m}(\hat{\mathbf{v}},\hat{m}) \le C_1|\hat{\mathbf{v}}|^4. \tag{3.8}$$

From the Euler equation (3.5), we have

$$\hat{m}(\partial_{\mathbf{v}} f(\cdot,\hat{\mathbf{v}})\cdot\hat{\mathbf{v}} + \mathbf{A}^T\nabla u\cdot\hat{\mathbf{v}}) + \frac{\partial\sigma}{\partial\mathbf{v}}(\hat{\mathbf{v}},\hat{m})\cdot\hat{\mathbf{v}} = 0.$$

From (3.7) it follows, dropping $\hat{m}$,

$$\partial_{\mathbf{v}} f(\cdot,\hat{\mathbf{v}})\cdot\hat{\mathbf{v}} + \nabla u\cdot\mathbf{A}\hat{v} + c_1|\hat{\mathbf{v}}|^4 - C_1 \le 0.$$

Using next (A3), we can assert that

$$-K + \nabla u \cdot \mathbf{A}\hat{\mathbf{v}} + c_1|\hat{\mathbf{v}}|^4 - C_1 \leq 0$$

and thus obtain immediately

$$|\hat{\mathbf{v}}|^4 \leq C(1 + |\nabla u|^{\frac{4}{3}}). \tag{3.9}$$

We turn next to the parabolic equation (3.4). Using the right inequalities (3.8) and (A4) we obtain

$$-\partial_t u - \Delta u - c|\nabla u|^{\frac{4}{3}} \leq K. \tag{3.10}$$

Using then the left inequalities (3.8) and (A3) we also get

$$-\partial_t u - \Delta u + c|\nabla u|^{\frac{4}{3}} \geq -K. \tag{3.11}$$

Moreover, we have $|u(x,T)| \leq \|u_T\|_\infty$. From the maximum principle, which holds due to the Neumann boundary condition, we obtain

$$|u(x,t)| \leq C \quad \forall (t,x) \in \overline{Q}. \tag{3.12}$$

Additionally, relations (3.10) and (3.11) yield

$$| - \partial_t u - \Delta u| \leq K(1 + |\nabla u|^{\frac{4}{3}}). \tag{3.13}$$

On account of

$$\int_Q |\nabla u|^2 \, dx\, dt \leq \frac{1}{2}\int_\Omega |u_T(x)|^2 \, dx + \int_Q (-\partial_t u - \Delta u) u \, dx\, dt$$

we obtain, thanks to (3.12) and (3.13), that $\int_Q |\nabla u|^2 \, dx\, dt \leq C$. Therefore, we deduce boundedness of $-\partial_t u - \Delta u$ in $L^{\frac{3}{2}}(Q)$ by (3.13). From the L^p-theory for linear parabolic equations (see [17]), it follows that $\nabla^2 u$ is bounded in $L^{\frac{3}{2}}(Q;\mathbb{R}^{2n})$. Using

$$\mathrm{div}(\nabla u \, |\nabla u|) = \Delta u \, |\nabla u| + \frac{\nabla u^T \nabla^2 u \, \nabla u}{2|\nabla u|}$$

we obtain

$$|\mathrm{div}(\nabla u \, |\nabla u|)| \leq C|\nabla u| \, |\nabla^2 u|.$$

But then, recalling the Neumann boundary condition, we have

$$\begin{aligned} \int_Q |\nabla u|^3 \, dx\, dt &= \int_Q \nabla u \cdot \nabla u |\nabla u| \, dx\, dt \\ &= - \int_Q u \, \mathrm{div}(\nabla u \, |\nabla u|) \, dx\, dt \\ &\leq C \int_Q |\nabla u| \, |\nabla^2 u| \, dx\, dt \\ &\leq \frac{\varepsilon}{3} \int_Q |\nabla u|^3 \, dx\, dt + \frac{2C^{\frac{3}{2}}}{3\varepsilon^{\frac{1}{2}}} \int_Q |\nabla^2 u|^{\frac{3}{2}} \end{aligned} \tag{3.14}$$

for every $\varepsilon > 0$. Choosing ε small enough we gain $\int_Q |\nabla u|^3 \, dx\, dt \leq C$. So we have improved the estimate on ∇u. It follows from (3.13) that $-\partial_t u - \Delta u$ is bounded in $L^{\frac{9}{4}}(Q)$. Using again L^p-theory, and proceeding with a bootstrap argument, we see that u is bounded in $L^p(0,T; W^{2,p}(\Omega))$ and $\partial_t u$ is bounded in $L^p(Q)$, for any finite integer p. In particular, we have $|\nabla u(x,t)| \leq C$ for all (t,x) in $\overline{Q}$. Therefore, relation (3.9) implies $|\hat{\mathbf{v}}(x,t)| \leq C$, where the constant can be evaluated in terms of the various constants of the problem.

3.3. Approximation. In order to obtain an appropriate regularization we minimize the functional $\mathcal{J}$ with respect to a bounded control range. To be precise we seek a function $\mathbf{v}_L$ which solves

$$\mathcal{J}[\mathbf{v}] \to \min \quad \text{in} \quad \mathbb{K}_L := L^\infty(Q; B_L(0)). \tag{3.15}$$

LEMMA 3.1. *Let $L \geq 1$ be fixed. Under the assumptions of Theorem* 2.1 *there is a solution $\mathbf{v}_L$ to* (3.15)*.*

Proof. Let $(\mathbf{v}_L^k)$ be a minimizing sequence. By definitions it is essentially bounded and hence there is a (nonrelabeled) subsequence which converges in $L^\infty(Q, \mathbb{R}^s)$ (in the weak* sense). The corresponding mean field $m_L^k = m(\mathbf{v}_L^k)$ (solution to (2.1)) is uniformly bounded in $C^\alpha(\overline{Q})$ by Lemma 2.2. The Arcela–Ascoli theorem implies that m_L^k converges uniformly. By assumption (B1) we have lower semicontinuity of σ in the corresponding topology and the claim follows. ◻

LEMMA 3.2. *Let the assumptions of Theorem* 2.1 *be satisfied.*

(a) *We have that*

$$\sup_{L\geq 1} \mathcal{J}[\mathbf{v}_L] < \infty. \tag{3.16}$$

(b) *Let u be the solution to* (2.10) *with $\mathbf{v} = \mathbf{v}_L$. Then we have a.e. in Q*

$$\left(m_L f_{\mathbf{v}}(\cdot, \mathbf{v}_L) + m_L \mathbf{A}^T \nabla u_L + \frac{\partial \sigma}{\partial \mathbf{v}}(\mathbf{v}_L, m)\right) \cdot (\mathbf{w} - \mathbf{v}_L) \geq 0 \tag{3.17}$$

for all $\mathbf{w} \in B_L(0)$.

Proof. (a) Inequality (3.16) follows from the fact that the mapping $L \mapsto \inf_{\mathbb{K}_L} \mathcal{J}$ is decreasing.

For (b) we use that

$$\frac{d}{d\alpha}\mathcal{J}(\mathbf{v}_L + \alpha(\mathbf{u} - \mathbf{v}_L))\Big|_{\alpha=0} \geq 0$$

for all $\mathbf{u} \in \mathbb{K}_L$. Using the definition of $\mathcal{J}$ we have for $M_L := \frac{d}{d\alpha} m_L(\mathbf{v}_L + \alpha(\mathbf{u} - \mathbf{v}_L))|_{\alpha=0}$

$$\begin{aligned}
&\int_Q M_L \left(f(\cdot, \mathbf{v}_L) + \frac{\partial \sigma}{\partial m}(\mathbf{v}_L, m)\right) dx\, dt \\
&\quad + \int_Q \left(m_L f_{\mathbf{v}}(\cdot, \mathbf{v}_L) + \frac{\partial \sigma}{\partial \mathbf{v}}(\mathbf{v}_L, m)\right) \cdot (\mathbf{u} - \mathbf{v}_L)\, dx\, dt \geq 0.
\end{aligned}$$

Note that M_L satisfies the equation

$$\begin{aligned}
&\partial_t M_L - \Delta M + \operatorname{div}(M_L\, g(\mathbf{v}_L)) + \operatorname{div}(m_L \mathbf{A}(\mathbf{u} - \mathbf{v}_L)) = 0, \\
&\qquad \partial_\nu M_L|_{\partial\Omega} = 0, \quad M_L\Big|_{t=0} = 0.
\end{aligned} \tag{3.18}$$

We test (2.10) with M_L and (3.18) with u_L. After integration by parts we find that

$$\int_Q M_L f(\cdot, \mathbf{v}_L) + \int_Q \frac{\partial \sigma}{\partial \mathbf{v}}(\mathbf{v}_L, m)\, M_L\, dx\, dt \geq \int_Q m_L \mathbf{A}^T \nabla u_L \cdot (\mathbf{u} - \mathbf{v}_L)\, dx\, dt$$

and hence

$$\int_Q \left(m_L f_{\mathbf{v}}(\cdot, \mathbf{v}_L) + m_L \mathbf{A}^T \nabla u_L + \frac{\partial \sigma}{\partial \mathbf{v}}(\mathbf{v}, m) \right) \cdot (\mathbf{u} - \mathbf{v}_L)\, dx\, dt \geq 0.$$

Now we choose $\mathbf{u} = (1-\theta)\mathbf{v}_L + \theta \mathbf{w}$, where $\theta \in C_0^\infty(Q; [0,1])$ and $\mathbf{w} \in B_L(0)$ are arbitrary. The claim follows by arbitrariness of θ. ∎

The bulk of the proof of Theorem 2.1 is the following maximum estimate for the Bellmann equation.

LEMMA 3.3. *Let the assumptions of Theorem* 2.1 *be satisfied. Let u_L be the weak solution of* (2.10) *with $\mathbf{v} = \mathbf{v}_L$. Then we have*

$$\|u_L\|_\infty \leq c, \tag{3.19}$$

where c does not depend on L.

Proof. For the estimate from above we choose $\mathbf{w} = 0$ in (3.17). Rearranging the terms shows

$$\frac{\partial \sigma}{\partial \mathbf{v}}(\mathbf{v}_L, m_L) \cdot \mathbf{v}_L \leq -m_L f_{\mathbf{v}}(\cdot, \mathbf{v}_L) \cdot \mathbf{v}_L - m_L \mathbf{A}^T \nabla u_L \cdot \mathbf{v}_L.$$

As a consequence of the growth conditions (A3), (A4), and (2.3) we obtain

$$m_L \left(\gamma_1(\mathbf{v}, m)|\mathbf{v}|^4 - \gamma_2(\mathbf{v}, m) \right) \leq c\, m_L (1 + |\mathbf{v}_L|^2 + |\nabla u_L||\mathbf{v}_L|).$$

By positivity of m_L (recall Lemma 2.2) we end up with

$$\left(\gamma_1(\mathbf{v}, m)|\mathbf{v}|^4 - \gamma_2(\mathbf{v}, m) \right) \leq c\, (1 + |\mathbf{v}_L|^2 + |\nabla u_L||\mathbf{v}_L|).$$

Due to the boundedness of $\mathcal{J}(\mathbf{v}_L)$ (recall Lemma 3.2), $\int_\Omega m_L(t, \cdot)\, dx = 1$ for every t, and $u_T \in L^\infty(\Omega)$ we have

$$\sigma(\mathbf{v}, m) + \int_Q |\mathbf{v}|^2 m\, dx\, dt \leq \mathcal{J}[\mathbf{v}_L] - \int_\Omega u_T m_L(T, \cdot)\, dx \leq c.$$

Hence, the assumptions on γ_1 and γ_2 in (2.5) imply that

$$|\mathbf{v}|^4 \leq c\, (1 + |\mathbf{v}_L|^2 + |\nabla u_L||\mathbf{v}_L|).$$

We gain for every $\varepsilon > 0$

$$|\mathbf{v}_L|^4 \leq \varepsilon |\mathbf{v}_L|^4 + c(\varepsilon)(1 + |\nabla u_L|^{4/3})$$

by Young's inequality. Choosing $\varepsilon = 1/2$ yields

$$|\mathbf{v}_L|^4 \leq c(1 + |\nabla u_L|^{4/3}). \tag{3.20}$$

This can be used to estimate the right-hand side of the Bellmann equation. We obtain

$$-\partial_t u_L - \Delta u_L \leq c|\nabla u_L|^{4/3} + K.$$

This can be written as

$$-\partial_t u - \Delta u \leq c|\nabla u_L||\nabla u_L|^{1/3} + K. \tag{3.21}$$

Our aim is to apply Corollary A.2 with $g = c|\nabla u_L|^{1/3}$. So we have to explain why $\nabla u_L \in L^\infty(Q,\mathbb{R}^n)$ (where the norm can be dependent on L). This follows easily from regularity theory for parabolic equations using boundedness of $\mathbf{v}_L$ and the growth assumptions (A1) and (2.4). Consequently, we obtain from Corollary A.2

$$\sup_Q u_L \leq c.$$

Using (A1) as well as (2.4), the right-hand side of (2.10) can be estimated from below such that

$$-\partial_t u_L - \Delta u_L \geq -c|\nabla u_L||\mathbf{v}_L| - K. \tag{3.22}$$

Now we use Corollary A.1 with $g = c|\mathbf{v}_L| \in L^\infty(Q)$ for $-u$ such that

$$\inf_Q u_L \geq -c.$$

The claim follows by combining the estimate from above and below. □

Proof of Theorem 2.1. We follow the computations from subsection 3.2, which are well-defined now. First, we use that the right-hand side of (2.10) is bounded by $|\nabla u_L|^{4/3}$; recall (3.21). Testing with u_L and using (3.19) yield

$$\begin{aligned}\int_Q |\nabla u_L|^2\,dx\,dt &\leq c\int_Q (1+|\nabla u_L|^{4/3})|u_L|\,dx\,dt \leq c\int_Q (1+|\nabla u_L|^{4/3})\,dx\,dt\\ &\leq \frac{1}{2}\int_Q |\nabla u_L|^2\,dx\,dt + c.\end{aligned}$$

Hence, we gain uniform bounds on $\int_Q |\nabla u_L|^2\,dx\,dt$ and eventually on $\int_Q |\mathbf{v}_L|^6\,dx\,dt$ using (3.20) again. Furthermore, (3.19) implies

$$\int_Q |\nabla^2 u_L|^{3/2}\,dx\,dt \leq c \tag{3.23}$$

by parabolic L^p-theory; see [17]. Due to (3.19) and Young's inequality we have as in (3.14)

$$\int_Q |\nabla u|^3\,dx\,dt \leq \frac{1}{2}\int_Q |\nabla u|^3\,dx\,dt + \int_Q |\nabla^2 u|^{3/2}\,dx\,dt$$

on account of $\partial_\nu u_L = 0$. We deduce from (3.23) that

$$\int_Q |\nabla u_L|^3\,dx\,dt \leq c \tag{3.24}$$

uniformly. We can use this information to improve the regularity of $\nabla^2 u_L$ (and of $\partial_t u_L$) by parabolic L^p-theory. Consequently, an iteration of these arguments implies

$$u_L \in L^q(0,T;W^{2,q}(\Omega)),\ \ \partial_t u_L \in L^q(0,T;L^q(\Omega))$$

for all $q < \infty$ uniformly in L. In particular, we have

$$\nabla u_L \in L^\infty(Q, \mathbb{R}^n)$$

uniformly in L. Hence the same is true for $\mathbf{v}_L$ by (3.20). Finally, Lemma 2.2 yields a uniform bound for m_L in $C^\alpha(\overline{Q})$. So we can pass to subsequences with limit functions $\mathbf{v}, m, u$ enjoying the claimed regularity properties. Passing to the limit in the equations for m is obvious. It remains to show that $\mathbf{v}$ is indeed a minimizer of $\mathcal{J}$ in $L^\infty(Q, \mathbb{R}^s)$. Then, the Bellman equation for u can be derived from this as in section 3.1. Due to compactness of m_L and lower semicontinuity of $\mathcal{J}$ (cf. (B1)), we obtain

$$\mathcal{J}(\mathbf{v}) \le \liminf J(\mathbf{v}_L) \le J(\boldsymbol{\phi})$$

for all $\boldsymbol{\phi} \in \mathbb{K}_M = L^\infty(Q; B_M(0))$ and any $M \gg 1$. The claim follows by arbitrariness of M. ∎

4. Feedback controls and uniqueness. In Theorem 2.1 we are not able to establish a feedback formula

$$\hat{\mathbf{v}}(t,x) = \boldsymbol{\omega}(t, x; \nabla u(t,x), m(t,x)). \tag{4.1}$$

This is due to the missing unique correspondence between $\nabla u(t,x)$ and $\hat{\mathbf{v}}(t,x)$ as a consequence of nonconvexity. The term $\frac{\partial \sigma}{\partial \mathbf{v}}(\mathbf{v}, m)$ in (2.12) is nonmonotone. This destroys the unique solvability of (2.12). The lack of a feedback formula also has the consequence that we had to define the (pre-)Bellmann equation (2.10) via the optimal control. It is not, as usual, an equation in ∇u and m. However, in the model problem 2.6, for small parameters β, we may state a "quasi-feedback formula." By this we mean a representation formula of type (4.1), but with an additional dependence of $\boldsymbol{\omega}$ on two numbers $\sigma_0 = \sigma(\hat{\mathbf{v}}, m[\hat{\mathbf{v}}])$ and $v_0 = \int_Q m[\hat{\mathbf{v}}]|\hat{\mathbf{v}}|^2\, dx\, dt$, i.e.,

$$\hat{\mathbf{v}}(t,x) = \boldsymbol{\omega}(t, x; \nabla u(t,x), m(t,x); \sigma_0, v_0). \tag{4.2}$$

To this end we have a look at (3.17) again and write it as

$$\left(m\, f_{\mathbf{v}}(\cdot, \hat{\mathbf{v}}) + m\, \mathbf{A}^T \nabla u + \frac{2\beta}{\sigma_0} m \hat{\mathbf{v}} \left(|\hat{\mathbf{v}}|^2 - v_0 \right) \right) \cdot (\mathbf{w} - \hat{\mathbf{v}}) \ge 0.$$

After cancelling m we gain

$$\left(f_{\mathbf{v}}(\cdot, \hat{\mathbf{v}}) + \mathbf{A}^T \nabla u + \frac{2\beta}{\sigma_0} \hat{\mathbf{v}} \left(|\hat{\mathbf{v}}|^2 - v_0 \right) \right) \cdot (\mathbf{w} - \hat{\mathbf{v}}) \ge 0. \tag{4.3}$$

We claim that $\beta v_0 \to 0$ for $\beta \to 0$. Let us denote by $\mathcal{A}_{\mathcal{J}_\beta}$ the class of minimizers of $\mathcal{J}_\beta$. Among the class $\mathcal{A}_{\mathcal{J}_\beta}$ we can minimize $\sigma(\mathbf{v}, m)$, where $m = m[\mathbf{v}]$. A minimizer exists due to the lower semicontinuity of σ assumed in (B1) and the properties of m; recall Lemma 2.2. Again the minimizer may not be unique due to the missing convexity of σ. We denote the hence created set of lexicographic minimizers by $\mathcal{A}^\sigma_{\mathcal{J}_\beta}$. Indeed, we have

$$v_0 = v_0(\beta) = \inf_{\mathcal{A}^\sigma_{\mathcal{J}_\beta}} \mathcal{J}_0 \le \inf_{\mathcal{A}^\sigma_{\mathcal{J}_\beta}} \mathcal{J}_\beta = \inf_{\mathcal{A}_{\mathcal{J}_\beta}} \mathcal{J}_\beta \le \mathcal{J}_1[0].$$

So, v_0 is bounded in β. Hence we have $\beta v_0 \to 0$ for $\beta \to 0$ as claimed. Consequently, the function

$$\mathbf{v} \mapsto f(\cdot, \mathbf{v}) + \mathbf{A}^T \nabla u \cdot \mathbf{v} + \frac{2\beta}{\sigma_0} |\mathbf{v}|^2 \left(|\mathbf{v}|^2 - v_0 \right)$$

is convex provided β is small enough (to be precise if $\frac{2\beta v_0}{\sigma_0} < c_0$, where c_0 is given in (A3)) and, in addition to (A1)–(A4), f belongs to the class C^2 and satisfies

$$\frac{\partial^2 f(m, \mathbf{v})}{\partial \mathbf{v}^2}(\boldsymbol{\xi}, \boldsymbol{\xi}) \geq c_0 |\boldsymbol{\xi}|^2 \tag{4.4}$$

for all $\boldsymbol{\xi} \in \mathbb{R}^N$. As a consequence $\mathbf{v}^*$ can be uniquely derived from (4.3) for ∇u given. This implies the desired uniqueness. We summarize this in the following theorem.

Theorem 4.1. *Let σ be given by* (2.6). *Assume that the assumptions of Theorem* 2.1 *are satisfied and we have* (4.4). *Assume in addition that $\frac{2\beta v_0}{\sigma_0} < c_0$ holds, where c_0 is given in* (A3). *Then the quasi-feedback formula* (4.2) *holds.*

The disadvantage of the quasi-feedback formula is that, due to nonuniqueness, the quantities σ_0 and v_0 may vary for different optimal controls $\hat{\mathbf{v}}$. There is a possibility to overcome this lack by considering lexicographic minimizers $\hat{\mathbf{v}}$. As before we consider the functional

$$\mathcal{J}_\beta[\mathbf{v}] = \int_Q m[\mathbf{v}] f(\cdot, \mathbf{v}) \, dx \, dt + \sigma(\mathbf{v}, m[\mathbf{v}]) + \int_\Omega m[\mathbf{v}](T, \cdot) \, u_T \, dx, \tag{4.5}$$

where $\sigma = \sigma_\beta$ is given in (2.6). The minimizer $\hat{\mathbf{v}}$ of (4.5) may not be unique in general. So, we minimize

$$\tilde{\mathcal{J}}[\mathbf{v}] = \int_Q m[\mathbf{v}] |\mathbf{v}|^2 \, dx \, dt$$

in $\mathcal{A}^\sigma_{\mathcal{J}_\beta}$, where $\mathcal{A}^\sigma_{\mathcal{J}_\beta}$ is set of lexicographic minimizers introduced above. The existence of a minimizer follows again from lower semicontinuity and the properties of m. The so obtained lexicographic minimizer is still not necessarily unique. But the procedure ensures that two lexicographic minimizer $\hat{\mathbf{v}}_1$ and $\hat{\mathbf{v}}_2$ satisfy $\mathcal{J}[\hat{\mathbf{v}}_1] = \mathcal{J}[\hat{\mathbf{v}}_2]$ and we replace σ_0 and v_0 in (4.2) by

$$v_0' := \tilde{\mathcal{J}}[\hat{\mathbf{v}}_1] = \tilde{\mathcal{J}}[\hat{\mathbf{v}}_2], \quad \sigma_0' := \sigma(\hat{\mathbf{v}}_1, m[\hat{\mathbf{v}}_1]) = \sigma(\hat{\mathbf{v}}_2, m[\hat{\mathbf{v}}_2]).$$

We formulate this in the following corollary.

Corollary 4.1. *Under the assumptions of Theorem* 4.1 *there is a unique lexicographic minimizer of $\mathcal{J}_\beta$ in the class $\mathcal{A}^\sigma_{\mathcal{J}_\beta}$.*

Appendix A. Maximum principles. The following maximum principle concerns solutions to parabolic equations where the solution is not in C^2. We refer to [9, Lemma 4.1] for the proof. This theorem is usually proved in a C^2 setting, where the proof is very simple (see, e.g., [2]). However, it is convenient to have it also in a $W^{2,q}$ setting which avoids clumsy approximations.

Lemma A.1. *Let $U \in L^q(0, T; W^{2,q}(\Omega))$ be a subsolution to*

$$\begin{cases} \partial_t U - \Delta U + \lambda U \leq |\nabla U| g + f & \text{in } Q, \\ \partial_\nu U = 0 & \text{on } (0,T) \times \partial\Omega, \\ U(0, \cdot) = U_0 & \text{in } \Omega, \end{cases} \tag{A.1}$$

where $\lambda > 0$. Assume that $g \in L^q(Q)$ for some $q > n+2$ and $f \in L^\infty(Q)$. Then there holds $U \leq \max\{\|f\|_\infty / \lambda, \|U_0\|_\infty\}$ in Q.

COROLLARY A.1. *Let $V \in L^q(0,T;W^{2,q}(\Omega))$ be a subsolution to*

$$\text{(A.2)}\qquad \begin{cases} \partial_t V - \Delta V \le |\nabla V|\tilde{g} + \tilde{f} & \text{in } Q, \\ \partial_\nu V = 0 & \text{on } (0,T)\times\partial\Omega, \\ V(0,\cdot) = V_0 & \text{in } \Omega. \end{cases}$$

Assume that $\tilde{g} \in L^q(Q)$ for some $q > n+2$ and $\tilde{f} \in L^\infty(Q)$. Then there holds $V \le c\max\{\|\tilde{f}\|_\infty, \|V_0\|_\infty\}$ in Q.

Proof. We set $U = e^{-t}V$ and obtain

$$\partial_t U - \Delta U + U \le |\nabla U|g + f,$$

where $g = e^t\tilde{g}$ and $f = \tilde{f}$. Lemma A.1 with $\lambda = 1$ yields the claim. ∎

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