COHERENCE DIVERSITY: A NOVEL SOURCE OF GAIN IN WIRELESS MULTIUSER NETWORKS

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This dissertation investigates multiuser networks where the fading links experience unequal coherence conditions as well as dissimilar link CSI availability. It is shown that the disparity in coherence conditions for multiple users leads to a novel gain in the transmission rates compared with techniques that do not explicitly take advantage of this disparity. This gain is denoted *coherence diversity* and is demonstrated by *product superposition* transmission.

First, a frequency-selective broadcast channel is considered, where two users have a disparity in coherence time and coherence bandwidth. This channel is analyzed under three broad scenarios of the disparity between the link qualities: when the disparity is in coherence time, in coherence bandwidth, and in both coherence time and coherence bandwidth. For each scenario, an analysis is provided and coherence diversity gain is demonstrated. The results are obtained in the framework of OFDM transmission covering a variety of pilot transmission schemes and different channel estimation techniques. Numerical simulations are presented to show coherence diversity gains.

Second, coherence diversity is investigated in broadcast and multiple access channels with an arbitrary number of users. The users experience unequal fading block lengths, and CSI is not available. In the broadcast channel, product superposition is employed to find the achievable degrees of freedom. The case of multiple users experiencing fading block lengths

of arbitrary ratio or alignment is studied. Also, in the multiple-access channel with unequal coherence times, achievable and outer bounds on the degrees of freedom are obtained.

Third, a MISO broadcast channel is considered where some receivers experience longer coherence intervals and have CSIR, while some other receivers experience shorter coherence intervals and do not enjoy free CSIR. A variety of CSIT availability models is considered, including no CSIT, delayed CSIT, or hybrid CSIT. For each model, coherence diversity gains are merged with interference alignment and beamforming to achieve degrees of freedom. For several cases, inner and outer bounds are established that either partially meet, or the gap diminishes with increasing coherence times.

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CHAPTER 1

INTRODUCTION

1.1 Background

In a wireless network, variations in node mobility and scattering environment produce unequal delay and Doppler spread. Wireless receivers in the same cell may have widely varying coherence conditions (coherence time and coherence bandwidth) [1, 2, 3]. For instance, in an urban environment, a macro-cell may serve "high-mobility" users moving with speed 120 Km/h and "low-mobility" users moving with speed 5 Km/h. The coherence times of the users are 2.25 ms and 54 ms, respectively [4, 5]. Also, coherence conditions are directly related to the channel state information (CSI) availability at the wireless terminals. The faster the fading, the more often the link needs to be trained and estimated at the receiver, and the more likely that CSI is stale or unavailable at the transmitter [6, 7].

For simplicity, the analysis of multiuser networks in the literature assumes that all fading links have equal coherence conditions [8, 9]. Moreover, most of communication systems, e.g. 3GPP LTE, are designed based on the worst-case link (shortest coherence time and coherence bandwidth) [4, 10].[1] The performance limits of wireless networks under disparity in link coherence conditions and non-uniform CSI availability have been for the most part unknown. Even under identical coherence conditions, understanding the performance limits of many wireless networks under block fading or related models has been far from trivial, with some key results under identical fading intervals being discovered only very recently [11, 12].

This dissertation investigates wireless multiuser networks where the fading links of the users may have unequal coherence conditions. It is shown that the disparity of coherence

[1] In 3GPP LTE, the number of channel estimation pilots (a.k.a. cell-specific reference signals) is designed to support 350 Km/h mobility [4, 10].

conditions leads to gains, denoted *coherence diversity*, that are distinct from those arising from other techniques such as spatial multiplexing [13, 14, 15], multiuser diversity [16, 17], and interference alignment [18, 19, 20, 21]. This class of gains was introduced in [22] where it was demonstrated by designing the transmission signaling using product superposition [22, 23, 11, 24, 25, 26] that allows the signal of the less-selective user to "disappear" into the equivalent channel seen by the more-selective user. Therefore, product superposition transmission occupies all the available degrees of freedom in the more-selective link and in addition transmits simultaneously into the less-selective link without causing interference to the more-selective one. In the dissertation, it is shown that when there is non-identical coherence conditions, coherence diversity applies and leads to gains, and these gains are demonstrated by designing a proper product superposition transmission.

1.2 Literature Survey

A review of the relevant literature is as follows. Under perfect instantaneous CSI at the transmitter and the receivers, the degrees of freedom of a broadcast channel increase with the minimum of the transmit antennas and the total number of receivers antennas [27, 28]. However, due to the time-varying nature of the channel and feedback impairments, perfect instantaneous transmit-side channel state information (CSIT) may not be available, and also receive-side channel state information (CSIR) can be assumed for slow-fading channels only.

The broadcast channel with perfect CSIR has been investigated under a variety of CSIT conditions, including imperfect, delayed, or no CSIT [9, 29, 30, 31, 32, 33]. In the absence of CSIT, Huang *et al.* [9] and Vaze and Varanasi [29] showed that the degrees of freedom collapse to that of the single-receiver, since the receivers are *stochastically equivalent* with respect to the transmitter. For a MISO broadcast channel Lapidoth *et al.* [30] conjectured that as long as the precision of CSIT is finite, the degrees of freedom collapse to unity. This conjecture was recently settled in the positive by Davoodi and Jafar [31, 34, 35]. Moreover, for a MISO broadcast channel under perfect delayed CSIT Maddah-Ali and Tse [32] showed using retrospective interference alignment that the degrees of freedom are $\frac{1}{1+\frac{1}{2}+\ldots+\frac{1}{K}} > 1$, where K is the number of the transmit antennas and also the number of receivers. A scenario of mixed CSIT was investigated [33, 36, 37, 38, 33, 39, 40, 41], where the transmitter has partial knowledge on the current channel in addition to delayed CSI.

The potential variation between the quality of feedback links has led to the model of hybrid CSIT, where the CSIT with respect to different links may not be identical [42, 43, 31, 44]. A MISO broadcast channel with perfect CSIT for some receivers and delayed for the others was studied by Tandon *et al.* [42] and Amuru *et al.* [43]. Davoodi and Jafar [31] showed that for a MISO two-receiver broadcast channel under perfect CSIT for one user and no CSIT for the other, the degrees of freedom collapse to unity. Tandon *et al.* [44] considered a MISO broadcast channel with alternating hybrid CSIT to be perfect, delayed, or no CSIT with respect to different receivers.

In the context of a broadcast channel, frequency-selectivity has been studied extensively [45, 46, 47, 48, 49]. The basic analytical idea is that using OFDM, the time-frequency plane can be divided into blocks facilitating resource allocation and channel training. This is also the underlying motivation of organizing transmission strategies around *resource blocks* [10, 4]. The size of the resource blocks is often taken to be the same for all receivers, reflecting the underlying assumption that the different receivers may be assumed to have roughly the same coherence bandwidth and coherence time. Mismatch in coherence conditions of frequency selective channels has been considered so far only [50, 51, 52] for classifying receivers into low-mobility and high-mobility ones for the purposes of scheduling, but otherwise orthogonal transmission is employed between receivers.

For the block-fading *multiple-access* channel, the capacity in the absence of CSIR is unknown.[2] Shamai and Marzetta [53] conjectured that in the SIMO block-fading multiple

[2] In fact the capacity of point-to-point channel under this condition is also unknown except certain special cases [6].

access without CSIR, the sum capacity can be achieved by activating no more than T transmitters. Also, for a two-receiver SISO multiple access channel with i.i.d. fast fading (all the links have coherence time of length 1), a non-naive TDMA inner bound and a cooperative outer bound on the capacity region were provided [54]. Furthermore, a multiuser multiple access channel with identical coherence times where the transmitters are equipped with single antenna was considered [55] where an inner bound on the network sum capacity was provided based on successive decoding, and an outer bound was obtained based on assuming cooperation between the transmitters.

1.3 Contributions

Chapter 2 analyzes a MIMO multiuser frequency-selective broadcast (a.k.a. downlink) channel where the coherence time and coherence bandwidth of the different receivers are not identical. The results are obtained in the framework of OFDM transmission with pilots [56, 57, 58]. In our studies, a variety of pilot transmission schemes is covered including pilots on all sub-carriers as well as pilot interpolation. Also, different channel estimation techniques is considered including frequency-domain and time-domain estimation [59, 60, 58].

The effects of the disparity of coherence conditions are studied in several distinct cases: when frequency-selective users have different coherence times but the same coherence bandwidth in Section 2.1, when they have different coherence bandwidth but the same coherence time in Section 2.2, and finally when they have different coherence bandwidth and different coherence times in Section 2.3. For each case, a transmission strategy is called for, outlined and analyzed in this chapter. It is shown that proper exploitation of the differences in coherence conditions of the links gives rise to gains that are further demonstrated via numerical simulations in Section 2.4.

Then in Chapter 3, a MIMO block-fading broadcast channel with arbitrary number of users is considered. In Section 3.1, the open problem of the degrees of freedom of the multiuser block-fading broadcast channel with identical fading intervals is settled. It is shown that with CSIR but no CSIT, the degrees of freedom is limited to TDMA. Also in the absence of both CSIR and CSIT, it is shown that once again the degrees of freedom cannot be improved beyond TDMA.

In Section 3.2, a broadcast channel with *unequal* fading intervals is then addressed, where there is no CSIT or for-free CSIR meaning that the cost of acquiring CSIR must be accounted for. For the achievable degrees of freedom, a generalization of the method of product superposition to multiple users with coherence times of arbitrary integer ratios, and without free CSIR is proposed[3]. This is obtained by transmitting a pilot whenever one or more receivers experience a fading transition, and then during each pilot transmission, exactly one (other) receiver who does not need the pilot can simultaneously utilize the channel for data transmission without contaminating the pilot.

It is shown that, if the coherence time is at least twice the number of transmit and receive antennas, the obtained degrees of freedom meet the upper bound in four cases: when the number of transmit antennas is less than or equal to the number of antennas at every receiver, when all the receivers have the same number of antennas, when the coherence times of the receivers are very long compared to one receiver, or when all the receivers have identical coherence times. The development of outer bounds for this problem makes use of the idea of channel enhancement [28], which in our case consists of increasing the coherence time of all receivers to match the coherence time of the slowest channel.

The inner bounds for coherence diversity are further extended in Section 3.4 to the case of multiple receivers experiencing fading block lengths of arbitrary ratio or alignment. Unaligned block fading intervals bring to mind the blind interference alignment of Jafar [62, 63]. A version of bind interference alignment is considered that, unlike [62, 63], takes into

[3]Li and Nosratinia [61, 22] introduced this method for the special case of two-receiver broadcast channel where one receiver has a very long coherence time compared with the other.

account the full cost of CSIR via training; in that framework, the synergies between blind interference alignment and product superposition is explored.

Furthermore, a block-fading *multiple-access* channel is investigated, where the results are as follows. Section 3.5 highlights bounds on the degrees of freedom of the block-fading MIMO multiple access channel with *identical* coherence times in the absence of free CSIR, a result that is not complicated but has been absent from the literature. A conventional pilot-based scheme emitting individual and separate pilots from (a subset of) the antennas of receivers is considered that subsequently allows the receiver to perform zero-forcing. This method is shown to partially meet the cooperative outer bound. In particular, this method always achieves the optimal sum degrees of freedom, and in some cases is optimal throughout the degrees of freedom region. For the case of unequal coherence times in Section 3.6, the same transmission technique is employed with pilots transmitted at the fading transition times of every active transmitter. The outer bound is once again built on the concept of enhancing the channel [28] by increasing the links coherence times so that all the links of the enhanced channel have identical coherence times.

In Chapter 4, a multiuser model under a hybrid CSIR scenario is considered where a set of static receivers are assumed to have CSIR, and another set of dynamic receivers not having free CSIR and with shorter link coherence time. This model under a variety of CSIT conditions is considered, including no CSIT, delayed CSIT, and two hybrid CSIT scenarios. In each of these conditions, the degrees of freedom region is analyzed. A few new tools are introduced, and inner and outer bounds are derived that partially meet in some cases. The results of this chapter are cataloged as follows.

In the absence of CSIT, in Section 4.2, an outer bound on the degrees of freedom region is produced via bounding the rates of a discrete memoryless multilevel broadcast channel [64, 65][4] and then applying the extremal entropy inequality [67, 68]. The achievable degrees of freedom region meets the outer bound in the limiting case where the coherence times of the static and dynamic receivers are the same.

For delayed CSIT, in Section 4.3, we use the outdated CSI model that was used by Maddah-Ali and Tse [32] under i.i.d. fading and assuming global CSIR at all nodes. Noting that the model does not have uniform CSIR, a technique with retrospective interference alignment over super-symbols is produced to utilize outdated CSIT and is merged with product superposition to reuse the pilots of the dynamic receivers for the purpose of transmission to static receivers. Moreover, an outer bound is developped that is suitable for block-fading channels with different coherence times, by appropriately enhancing the channel to a physically-degraded broadcast channel and then applying the extremal entropy inequality [67, 68]. For one static and one dynamic receiver, our achievable degrees of freedom partially meet our outer bound, and furthermore, the gap decreases with the dynamic receiver coherence time T.

Under hybrid CSIT, two conditions are analyzed: First, Section 4.4 considers perfect CSIT for the static receivers and no CSIT with respect to the dynamic receivers. The achievable degrees of freedom in this case are obtained using product superposition with the dynamic receiver's pilots reused and beamforming for the static receivers to avoid interference. Second, Section 4.5 considers perfect CSIT with respect to the static receivers and delayed CSIT with respect to the dynamic receivers. An achievable transmission scheme is proposed via a combination of beamforming, interference alignment, and product superposition methodologies. The outer bounds for the two hybrid-CSIT cases were based on constructing an enhanced physically degraded channel and then applying the extremal en-

[4]Discrete memoryless multilevel three-receiver broadcast channel was considered [64, 65] with degraded message sets. Our outer bound on the multiuser multilevel broadcast channel is an extension of the Körner-Marton outer bound [66, Theorem 5] to more than two receivers.

tropy inequality. For one static receiver with perfect CSIT and one dynamic receiver with delayed CSIT, the gap between the achievable and the outer sum degrees of freedom is $\frac{1}{T}$.

1.4 Notation

The following notations are used in this dissertation: $\text{diag}\{\mathbf{a}\}$ denotes a diagonal matrix whose entries consists of the elements of the vector $\mathbf{a}$. $\det\{\mathbf{A}\}$, $\text{Tr}\{\mathbf{A}\}$, and $\{\mathbf{A}\}^{\dagger}$ denote, respectively, the determinant, trace, and the conjugate transpose of the matrix $\mathbf{A}$. $\mathbf{I}_M$ is the $M \times M$ identity matrix, $\mathbf{0}_{N\times M}$ denotes the all-zero matrix of size $N \times M$, and $\mathbf{1}_N$ denotes the all-one vector of length N. Furthermore, $\otimes$ denotes the Kronecker product. $\mathbb{C}^{N\times M}$ is the set of complex matrices of size $N \times M$ and $\mathbb{C}^N$ is the set of complex vectors of length N. $\mathbb{E}\{\cdot\}$ represents expectation, and log is taken to the base 2. Also, i.i.d. means independent and identically distributed, and $\mathcal{CN}\big(0, 1\big)$ denotes a circularly-symmetric normal distribution with zero mean and unit variance. Additional notation specific to each chapter is introduced as needed.

1.5 Product Superposition

Product superposition was first introduced [61, 22] for one static and one dynamic user and for the antenna setup $M = \max(N_s, N_d)$, where M denotes the number of transmit antennas and N_s and N_d denote the number of receive antennas of the static and dynamic users, respectively. The transmitted space-time code is

$$\mathbf{X} = \mathbf{X}_s\mathbf{X}_d, \tag{1.1}$$

where $\mathbf{X}_s \in \mathbb{C}^{M\times N_d}$ is the matrix containing independent symbols intended for the static user and has i.i.d. $\mathcal{CN}\big(0, 1\big)$, $\mathbf{X}_d \in \mathbb{C}^{N_d\times T_d}$ is the signal matrix for the dynamic user and T_d is the coherence time of the dynamic user. Furthermore, $\mathbf{X}_d$ is given as follows

$$\mathbf{X}_d = [\mathbf{X}_\tau,\ \mathbf{X}_\delta] \tag{1.2}$$

where $\mathbf{X}_\tau \in \mathbb{C}^{N_d \times N_d}$ is a unitary pilot matrix and $\mathbf{X}_\delta \in \mathbb{C}^{N_d \times (T_d - N_d)}$ is the matrix containing symbols intended for the dynamic user whose entries are i.i.d. $\mathcal{CN}\big(0, 1\big)$. Therefore the signal received at the dynamic user is

$$\begin{aligned} \mathbf{Y}_d =& \mathbf{H}_d \mathbf{X}_s \left[\mathbf{X}_\tau,\ \mathbf{X}_\delta\right] + \mathbf{W}_d \\ =& \left[\bar{\mathbf{H}}_d \mathbf{X}_\tau,\ \bar{\mathbf{H}}_d \mathbf{X}_\delta\right] + \mathbf{W}_d, \end{aligned} \tag{1.3}$$

where $\bar{\mathbf{H}}_d = \mathbf{H}_d \mathbf{X}_s$ and $\mathbf{H}_d \in \mathbb{C}^{N_d \times M}$ is the dynamic user channel. The dynamic user estimates the equivalent channel $\bar{\mathbf{H}}_d$ using the pilot matrix and then decodes $\mathbf{X}_\delta$ based on the channel estimate. On the other hand, the received signal at the static user during the first N_d time-slots is

$$\mathbf{Y}_{s1} = \mathbf{H}_s \mathbf{X}_s \mathbf{X}_\tau + \mathbf{W}_{s1}, \tag{1.4}$$

where $\mathbf{H}_s \in \mathbb{C}^{N_s \times M}$ is the static user channel. Since the pilot matrix is known to both the static and dynamic users, and furthermore, the static user knows its channel $\mathbf{H}_s$, it can decode $\mathbf{X}_s$. As a result, the achievable degrees of freedom pair is $\left(N_d\left(1 - \frac{N_d}{T_d}\right), \frac{N_s N_d}{T_d}\right)$ which is strictly greater than TDMA.

The same model was extended [23] to general antenna setup. The achieved degrees of freedom pair is

$$(d_d, d_s) = \big(N_d^*(1 - \frac{N_d^*}{T_d}), \frac{N_s^* N_d^*}{T_d}\big),$$

where $N_d^* = \min(N_d, M)$ and $N_s^* = \min(N_s, M)$. This can be shown as follows. The dimensions of the transmitted signals in (1.1), (1.2) can be modified such that $\mathbf{X}_s \in \mathbb{C}^{M \times N_d^*}$, $\mathbf{X}_\tau \in \mathbb{C}^{N_d^* \times N_d^*}$ and $\mathbf{X}_\delta \in \mathbb{C}^{N_d^* \times \left(T_d - N_d^*\right)}$. Following the same decoding strategy, the dynamic user decodes $\mathbf{X}_\delta$ based on the estimation of the equivalent channel $\bar{\mathbf{H}}_d$ achieving $N_d^*\left(1 - \frac{N_d^*}{T_d}\right)$ degrees of freedom. Furthermore, the static user decodes $\mathbf{X}_s$ achieving $\frac{N_s^* N_d^*}{T_d}$ degrees of freedom.

CHAPTER 2

COHERENCE DIVERSITY IN TIME AND FREQUENCY

Consider an OFDM downlink transmission with an M-antenna transmitter and two receivers equipped with N_1, N_2 antennas and having coherence times T_1 and T_2, respectively. Define $\mathbf{X}_k(t) \in \mathbb{C}^{M\times T_o}$ to be the transmitted space-time block code at subcarrier k, where $t = 1, 2, \ldots$ is the index of the space-time code. The corresponding received signals at user 1 and user 2 are

$$\begin{aligned}\mathbf{Y}_{1,k}(t) &= \sqrt{\frac{\rho}{M}}\mathbf{H}_{1,k}(t)\mathbf{X}_k(t) + \mathbf{W}_{1,k}(t),\\ \mathbf{Y}_{2,k}(t) &= \sqrt{\frac{\rho}{M}}\mathbf{H}_{2,k}(t)\mathbf{X}_k(t) + \mathbf{W}_{2,k}(t), \quad k = 1, \ldots, K,\end{aligned} \tag{2.1}$$

where $\mathbf{W}_{1,k}(t) \in \mathbb{C}^{N_1\times T_o}$ and $\mathbf{W}_{2,k}(t) \in \mathbb{C}^{N_2\times T_o}$ are the additive Gaussian noise matrices whose elements are i.i.d. circularly symmetric with zeros mean and unit variance, ρ represents the received SNR, and K is the number of subcarriers.[1]

The matrices $\mathbf{H}_{1,k}(t) \in \mathbb{C}^{N_1\times M}$ and $\mathbf{H}_{2,k}(t) \in \mathbb{C}^{N_2\times M}$ represent the MIMO channel frequency response at subcarrier k of the two receivers. Assume $T_o = \min\{T_1, T_2\}$; hence the elements of $\mathbf{H}_{1,k}(t)$ and $\mathbf{H}_{2,k}(t)$ stay the same during the period T_o. In the sequel, for notational convenience, we drop the index t. Define $\mathbf{h}_{1,ij} \in \mathbb{C}^{L_1}$ and $\mathbf{h}_{2,ij} \in \mathbb{C}^{L_2}$ to be the impulse response vectors between transmit antenna j and receive antenna i for the channels of user 1 and user 2 with L_1 and L_2 taps, respectively. The elements of $\mathbf{h}_{1,ij}$ and $\mathbf{h}_{2,ij}$ are assumed to be i.i.d. zero-mean unit-variance complex-Gaussian taps to model the scenarios of rich scatterers. The MIMO channel impulse response matrices $\bar{\mathbf{H}}_1 \in \mathbb{C}^{N_1L_1\times M}$ and $\bar{\mathbf{H}}_2 \in \mathbb{C}^{N_2L_2\times M}$ are defined as

$$\bar{\mathbf{H}}_1 = \begin{bmatrix} \mathbf{h}_{1,11} & \cdots & \mathbf{h}_{1,1M} \\ \vdots & \ddots & \vdots \\ \mathbf{h}_{1,N_11} & \cdots & \mathbf{h}_{1,N_1M} \end{bmatrix},$$

[1] For instance, $K = 12$ in each resource block in 3GPP LTE standard.

Table 2.1. Disparity scenarios of a two-user broadcast channel

Selectivity	None	Time	Freq	Both
None		[22]	Section 2.2	Section 2.1
Time	[22]		Section 2.3	Section 2.2
Freq	Section 2.2	Section 2.3		Section 2.1
Both	Section 2.1	Section 2.2	Section 2.1	

$$\bar{\mathbf{H}}_2 = \begin{bmatrix} \mathbf{h}_{2,11} & \cdots & \mathbf{h}_{2,1M} \\ \vdots & \ddots & \vdots \\ \mathbf{h}_{2,N_2 1} & \cdots & \mathbf{h}_{2,N_2 M} \end{bmatrix}. \tag{2.2}$$

Furthermore, define $\bar{\mathbf{F}} \in \mathbb{C}^{K \times K}$ to be the standard DFT matrix, where $\frac{1}{\sqrt{K}} \exp(-i\frac{2\pi kn}{K})$ is the value at row k and column n. The standard DFT matrix, $\bar{\mathbf{F}}$, relates the channel frequency response and the channel impulse response of one antenna for all subcarriers. In order to make our analysis easier, we define $\mathbf{F}_{1,k} \in \mathbb{C}^{N_1 \times N_1 L_1}$ and $\mathbf{F}_{2,k} \in \mathbb{C}^{N_2 \times N_2 L_2}$ as two matrices that relate the channel frequency and impulse response for all the antennas at one subcarrier,

$$\begin{aligned} \mathbf{F}_{1,k} &= \bar{\mathbf{F}}_{k,L_1} \otimes \mathbf{I}_{N_1}, \\ \mathbf{F}_{2,k} &= \bar{\mathbf{F}}_{k,L_2} \otimes \mathbf{I}_{N_2}, \end{aligned} \tag{2.3}$$

where $\bar{\mathbf{F}}_{k,L_1} \in \mathbb{C}^{1 \times L_1}$ denotes the vector consists of the first L_1 elements of row k of $\bar{\mathbf{F}}$. Similarly, $\bar{\mathbf{F}}_{k,L_2} \in \mathbb{C}^{1 \times L_2}$ denotes the vector consists of the first L_2 elements of row k of $\bar{\mathbf{F}}$. Therefore, the MIMO channel frequency response matrices, $\mathbf{H}_{1,k}$ and $\mathbf{H}_{2,k}$ in (2.1), are related to the MIMO channel impulse response matrices $\bar{\mathbf{H}}_1$ and $\bar{\mathbf{H}}_2$ in (2.2) as follows.

$$\begin{aligned} \mathbf{H}_{1,k} &= \mathbf{F}_{1,k} \bar{\mathbf{H}}_1, \\ \mathbf{H}_{2,k} &= \mathbf{F}_{2,k} \bar{\mathbf{H}}_2. \end{aligned} \tag{2.4}$$

The elements $\mathbf{h}_{1,ij}$ and $\mathbf{h}_{2,ij}$ are i.i.d. due to rich scatterers. Thus,

$$\mathbb{E}\left\{\mathbf{H}_{1,k}\mathbf{H}_{1,k}^{\dagger}\right\} = M\mathbf{F}_{1,k}\mathbf{F}_{1,k}^{\dagger} = M\|\mathbf{f}_1\|^2\mathbf{I}_{N_1},$$
$$\mathbb{E}\left\{\mathbf{H}_{2,k}\mathbf{H}_{2,k}^{\dagger}\right\} = M\mathbf{F}_{2,k}\mathbf{F}_{2,k}^{\dagger} = M\|\mathbf{f}_2\|^2\mathbf{I}_{N_2}, \tag{2.5}$$

where $\|\mathbf{f}_1\|^2 = \|\bar{\mathbf{F}}_{k,L_1}\|^2$ and $\|\mathbf{f}_2\|^2 = \|\bar{\mathbf{F}}_{k,L_2}\|^2$. In this chapter, we use a system model without CSIT, where the channel matrices $\mathbf{H}_{1,k}$, $\mathbf{H}_{2,k}$, $\bar{\mathbf{H}}_{1,k}$, and $\bar{\mathbf{H}}_{2,k}$ are unknown at the transmitter. The transmitter only knows the channel statistics including the number of taps L_1, L_2 and the coherence times T_1, T_2 (see [69] and the references therein).

2.1 Disparity in Coherence Time

Here we investigate downlink channels that have a disparity in coherence time, where $T_1 \ll T_2$, meaning that user 1 is *time-dynamic* and user 2 is *time-static*. Hence, the cost of sending pilots to estimate $\mathbf{H}_{2,k}$ can be ignored. We model this scenario by assuming $\mathbf{H}_{2,k}$ to be known at user 2 whereas $\mathbf{H}_{1,k}$ is assumed to be unknown to both users. We consider the users have the same coherence bandwidth since the users channels have the same delay spread ($L_1 = L_2 = L$). The disparity of coherence time under frequency selective channel, that is considered here, can occur in several scenarios in practical multiuser networks as shown in the following practical example.

Example 1. *Consider an urban environment where a macro-cell may serve two users: one time-dynamic user that is driving his car and another time-static user that is sitting on his desk. Based on the WINNER radio channel models [5], the propagation of the time-dynamic user is "C2: typical urban macro-cell" with 120 Km/h mobility and 234 ns r.m.s. delay spread. Furthermore, the propagation of the time-static user is "C4: outdoor to indoor macro-cell" with 5 Km/h mobility and 240 ns r.m.s. delay spread. Hence, the coherence time of the time-dynamic and the time-static user are 2.25 ms and 54 ms, respectively;*

furthermore, the coherence bandwidth of both is approximately 85 KHz. We use the definition of coherence time $T = \frac{1}{2f_d}$, where f_d is the Doppler spread at frequency 2 GHz and that of the coherence bandwidth $B_{c,90\%} = \frac{1}{50\sigma_\tau}$, where σ_τ is the delay spread [4].

2.1.1 Conventional OFDM Downlink Transmission

In conventional OFDM downlink transmission, TDMA is used to serve multiple users in the absence of CSIT [9]. In the sequel, we give the MMSE estimated channel, estimation error, and achievable rate for user 1, where the channel is estimated in the frequency-domain.[2] The transmitted space-time code of length T_1 at subcarrier k is

$$\mathbf{X}_k = \left[\sqrt{M}\mathbf{I}_M,\ \mathbf{U}_{1,k}\right], \quad k = 1, \cdots, K, \tag{2.6}$$

where pilot signals are sent during the first M time instances to estimate $\mathbf{H}_{1,k}$, and $\mathbf{U}_{1,k} \in \mathbb{C}^{M\times(T_1-M)}$ contains the symbols intended for user 1, and the symbols are i.i.d. Gaussian. The corresponding received signal is

$$\mathbf{Y}_{1,k} = \left[\sqrt{\rho_\tau}\mathbf{H}_{1,k},\ \sqrt{\frac{\rho_\delta}{M}}\mathbf{H}_{1,k}\mathbf{U}_{1,k}\right] + \mathbf{W}_{1,k}, \tag{2.7}$$

where ρ_τ and ρ_δ are the received SNR during pilot and symbol transmission, respectively. The MMSE estimate of $\mathbf{H}_{1,k}$ is

$$\hat{\mathbf{H}}_{1,k} = \mathbf{\Sigma}_{HY}\mathbf{\Sigma}_Y^{-1}\left(\sqrt{\rho_\tau}\mathbf{H}_{1,k} + \mathbf{W}_{1,\tau,k}\right), \tag{2.8}$$

where $\mathbf{W}_{\tau,k} \in \mathbb{C}^{N_1\times M}$ is the additive noise during the first M time instances and

$$\begin{aligned}\mathbf{\Sigma}_{HY} =& M\sqrt{\rho_\tau}\|\mathbf{f}_1\|^2\mathbf{I}_{N_1}, \\ \mathbf{\Sigma}_Y =& \left(M\rho_\tau\|\mathbf{f}_1\|^2 + M\right)\mathbf{I}_{N_1}.\end{aligned} \tag{2.9}$$

[2] In TDMA, the transmission time is divided into two portions where each user is served individually during one of them. Here, our proposed method provides gain only when user 1 is served since it is the time-dynamic user. During serving user 2, the achievable rate of the conventional and our proposed transmission is the same.

Hence,

$$\hat{\mathbf{H}}_{1,k} = \gamma \mathbf{H}_{1,k} + \frac{\gamma}{\sqrt{\rho_\tau}} \mathbf{W}_{1,\tau,k}, \tag{2.10}$$

where

$$\gamma = \frac{\rho_\tau \|\mathbf{f}_1\|^2}{\rho_\tau \|\mathbf{f}_1\|^2 + 1}. \tag{2.11}$$

The covariance matrix of the estimated channel is,

$$\hat{\boldsymbol{\Sigma}}_k = \mathbb{E}\left\{\hat{\mathbf{H}}_k \hat{\mathbf{H}}_k^\dagger\right\} = M\left(\gamma^2 \|\mathbf{f}_1\|^2 + \frac{\gamma^2}{\rho_\tau}\right)\mathbf{I}_{N_1}, \tag{2.12}$$

and

$$\hat{\sigma}_k^2 = \frac{1}{N_1 M}\mathrm{Tr}\left\{\hat{\boldsymbol{\Sigma}}_k\right\} = \gamma^2 \|\mathbf{f}_1\|^2 + \frac{\gamma^2}{\rho_\tau} = \gamma \|\mathbf{f}_1\|^2. \tag{2.13}$$

The estimation error is

$$\begin{aligned} \tilde{\mathbf{H}}_{1,k} &= \mathbf{H}_{1,k} - \hat{\mathbf{H}}_{1,k} \\ &= (1-\gamma)\,\mathbf{H}_{1,k} - \frac{\gamma}{\sqrt{\rho_\tau}} \mathbf{W}_{1,\tau,k}, \end{aligned} \tag{2.14}$$

and hence the estimation error covariance matrix is

$$\begin{aligned} \tilde{\boldsymbol{\Sigma}}_k &= \mathbb{E}\left\{\tilde{\mathbf{H}}_{1,k} \tilde{\mathbf{H}}_{1,k}^\dagger\right\} \\ &= M\left(\|\mathbf{f}_1\|^2 (1-\gamma)^2 + \frac{\gamma^2}{\rho_\tau}\right)\mathbf{I}_{N_1} = \frac{M\gamma}{\rho_\tau}\mathbf{I}_{N_1}. \end{aligned} \tag{2.15}$$

The normalized MMSE of $\mathbf{H}_{1,k}$ can be written as

$$\tilde{\sigma}_k^2 = \frac{1}{MN_1}\mathrm{Tr}\left\{\tilde{\boldsymbol{\Sigma}}_k\right\} = \frac{\gamma}{\rho_\tau}. \tag{2.16}$$

After estimating $\mathbf{H}_{1,k}$, user 1 decodes $\mathbf{U}_{1,k}$ coherently during $(T_1 - M)$ time instances. The received signal during $\mathbf{U}_{1,k}$ transmission is

$$\mathbf{Y}_{1,\delta,k} = \sqrt{\frac{\rho_\delta}{M}} \mathbf{H}_{1,k} \mathbf{U}_{1,k} + \mathbf{W}_{1,\delta,k}, \tag{2.17}$$

where $\mathbf{W}_{1,\delta,k} \in \mathbb{C}^{N_1 \times (T_1 - M)}$ is the corresponding noise. From (2.14),

$$\mathbf{Y}_{1,\delta,k} = \sqrt{\frac{\rho_\delta}{M}} \hat{\mathbf{H}}_{1,k} \mathbf{U}_{1,k} + \sqrt{\frac{\rho_\delta}{M}} \tilde{\mathbf{H}}_{1,k} \mathbf{U}_{1,k} + \mathbf{W}_{1,\delta,k}. \tag{2.18}$$

Based on [70], the achievable rate at subcarrier k is

$$\begin{aligned} R_{1,k} &= I\left(\mathbf{U}_{1,k};\mathbf{Y}_{1,\delta,k}|\hat{\mathbf{H}}_{1,k}\right) \\ &\geq \left(1-\frac{M}{T_1}\right)\mathbb{E}\left\{\log\det\left\{\mathbf{I}_{N_1}+\frac{\rho_\delta}{1+\rho_\delta\tilde{\sigma}_k^2}\frac{\hat{\mathbf{H}}_{1,k}\hat{\mathbf{H}}_{1,k}^\dagger}{M}\right\}\right\}, \end{aligned} \tag{2.19}$$

where $\tilde{\sigma}_k^2$ is given in (2.16). Define the normalized estimated channel

$$\check{\mathbf{H}}_{1,k} = \frac{\hat{\mathbf{H}}_{1,k}}{\hat{\sigma}_k}, \tag{2.20}$$

where $\hat{\sigma}_k$ is given in (2.13). Thus, the achievable rate in T_1 time slots and K subcarriers is

$$R_1^{[\text{Conv}]} \geq \frac{1}{K}\left(1-\frac{M}{T_1}\right)\sum_{k=1}^{K}\mathbb{E}\left\{\log\det\left\{\mathbf{I}_{N_1}+\rho_k\frac{\check{\mathbf{H}}_{1,k}\check{\mathbf{H}}_{1,k}^\dagger}{M}\right\}\right\}, \tag{2.21}$$

where

$$\rho_k = \frac{\rho_\delta\hat{\sigma}_k^2}{1+\rho_\delta\tilde{\sigma_k}^2}. \tag{2.22}$$

In the sequel, we find the optimal values of ρ_τ and ρ_δ so that the value of ρ_k is maximized, which in turn maximizes the user 1 achievable rate in (2.21). From the energy constraint,

$$\rho_\delta T_{1,\delta} + \rho_\tau T_{1,\tau} = \rho T_1, \tag{2.23}$$

and hence

$$\begin{aligned} \rho_\delta T_{1,\delta} &= (1-\alpha)\rho T_1, \\ \rho_\tau T_{1,\tau} &= \alpha\rho T_1, \end{aligned} \tag{2.24}$$

where $\alpha \in [0,1]$ is the power allocation factor. Therefore,

$$\begin{aligned} \rho_k =& \frac{\rho_\delta\alpha\|\mathbf{f}_1\|^2}{1+\rho_\delta\frac{\gamma}{\rho_\tau}} \\ =& \frac{\alpha(1-\alpha)\frac{(\rho T)^2}{T_{1,\tau}T_{1,\delta}}\|\mathbf{f}_1\|^4}{1+\alpha\frac{\rho T_1}{T_{1,\tau}}\|\mathbf{f}_1\|^2+(1-\alpha)\frac{\rho T_1}{T_{1,\delta}}\|\mathbf{f}_1\|^2} = \frac{\rho T_1\|\mathbf{f}_1\|^2}{(T_{1,\delta}-T_{1,\tau})}\frac{\alpha(1-\alpha)}{\alpha+\beta}, \end{aligned} \tag{2.25}$$

where

$$\beta = \frac{T_{1,\delta}T_{1,\tau} + \rho T_1 T_{1,\tau}\|\mathbf{f}_1\|^2}{T_{1,\delta}\rho T_1\|\mathbf{f}_1\|^2\left(1 - \frac{T_{1,\tau}}{T_{1,\delta}}\right)}. \tag{2.26}$$

By taking the first and the second derivatives of ρ_k with respect to α, we obtain

$$\frac{\partial \rho_k}{\partial \alpha} = \frac{\rho T_1}{T_{1,\delta} - T_{1,\tau}} \frac{(\alpha+\beta)(1-2\alpha) - \alpha(1-\alpha)}{(\alpha+\beta)^2}, \tag{2.27}$$

and hence

$$\begin{aligned}\frac{\partial^2 \rho_k}{\partial \alpha^2} &= \frac{\rho T_1}{(T_{1,\delta} - T_{1,\tau})(\alpha+\beta)^4}\Big(- 2(\alpha+\beta)^3 \quad - 2(\alpha+\beta)\left((\alpha+\beta)(1-2\alpha) - \alpha(1-\alpha)\right)\Big) \\ &= \frac{-2\rho T_1}{(T_{1,\delta} - T_{1,\tau})(\alpha+\beta)^3}\Big((\alpha+\beta)^2 - \alpha^2 - 2\alpha\beta + \beta\Big) \\ &= \frac{-2\rho T_1(\beta+\beta^2)}{(T_\delta - T_{1,\tau})\,(\alpha+\beta)^3},\end{aligned} \tag{2.28}$$

which shows the concavity of ρ_k. Therefore, from the KKT conditions [71], i.e. $\frac{\partial \rho_k}{\partial \alpha} = 0$, and hence the optimal α is

$$\alpha = -\beta + \sqrt{\beta(\beta+1)}. \tag{2.29}$$

2.1.2 Product Superposition OFDM Transmission

Product superposition was first introduced in the context of flat-fading to provide gains [22]. In the following, we give the product superposition transmission for our downlink system under frequency-selective fading. We investigate the case of frequency-domain channel estimation. The transmitted space-time code is

$$\mathbf{X}_k = \left[\sqrt{M}\mathbf{U}_{2,k},\ \mathbf{U}_{21,k}\right], \tag{2.30}$$

where $\mathbf{U}_{21,k} = \mathbf{U}_{2,k}\mathbf{U}_{1,k}$, and $\mathbf{U}_{2,k} \in \mathbb{C}^{M\times M}$ contains symbol intended for the user 2, where the symbols are i.i.d. Gaussian and $\mathbb{E}\left\{\mathbf{U}_{2,k}\mathbf{U}_{2,k}^\dagger\right\} = \mathbf{I}_M$. Hence, the received signal at user 1 at subcarrier k during the first M time instances is

$$\mathbf{Y}_{1,\tau,k} = \sqrt{\rho_\tau}\mathbf{G}_{1,k} + \mathbf{W}_{1,\tau,k}, \tag{2.31}$$

where $\mathbf{G}_{1,k} = \mathbf{H}_{1,k}\mathbf{U}_{2,k}$ is the equivalent channel seen by user 1 that can be estimated during the first M time instances. Following the analysis of the conventional transmission in Section 2.1.1, the MMSE equivalent channel estimate is

$$\hat{\mathbf{G}}_{1,k} = \gamma \mathbf{G}_{1,k} + \frac{\gamma}{\sqrt{\rho_\tau}} \mathbf{W}_{1,\tau,k}, \tag{2.32}$$

where γ, $\hat{\mathbf{\Sigma}}_k, \hat{\sigma}_k^2, \tilde{\mathbf{\Sigma}}_k, \tilde{\sigma}_k^2$ are given in (2.11), (2.12), (2.13), (2.15), and (2.16), respectively.

During the remaining $(T_1 - M)$ time instances, $\mathbf{U}_{12,k}$ is sent, and hence the received signal is

$$\begin{aligned}\mathbf{Y}_{1,\delta,k} &= \sqrt{\frac{\rho_\delta}{M}} \mathbf{G}_{1,k}\mathbf{U}_{1,k} + \mathbf{W}_{1,\delta,k} \\ &= \sqrt{\frac{\rho_\delta}{M}} \hat{\mathbf{G}}_{1,k}\mathbf{U}_{1,k} + \sqrt{\frac{\rho_\delta}{M}} \tilde{\mathbf{G}}_{1,k}\mathbf{U}_{1,k} + \mathbf{W}_{1,\delta,k}.\end{aligned} \tag{2.33}$$

Therefore, for K subcarriers and T_1 time instances the achievable rate is

$$R_1^{[\mathrm{PS}]} = R_1^{[\mathrm{Conv}]}, \tag{2.34}$$

where $R_1^{[\mathrm{Conv}]}$ is given in (2.21). Furthermore, knowing its channel, user 2 achieves the rate [22]

$$R_2^{[\mathrm{PS}]} \geq \frac{M}{T_1 K} \sum_{k=1}^{K} \mathbb{E}\left\{ \log\det\left\{ \mathbf{I}_{N_2} + \frac{1}{\mathbb{E}\left\{\lambda_i^{-2}\right\}} \mathbf{H}_{2,k}\mathbf{H}_{2,k}^\dagger \right\} \right\}, \tag{2.35}$$

where λ_i^{-2} is an eigenvalue of $\tilde{\mathbf{U}}_{1,k}\tilde{\mathbf{U}}_{1,k}^\dagger$ and $\tilde{\mathbf{U}}_{1,k} = [\rho_\tau \mathbf{I},\ \rho_\delta \mathbf{U}_{1,k}]$. For both conventional and product superposition transmission, user 1 achieves the same rate in (2.21). Therefore, the achievable rate in (2.35) comes from product superposition transmission that demonstrates coherence diversity gain. In other words, the sum-rate of conventional transmission is $R_1^{[\mathrm{Conv}]}$, whereas the sum-rate of product superposition is $R_1^{[\mathrm{PS}]} + R_2^{[\mathrm{PS}]}$.

2.1.3 Channel Estimation with Interpolation

In some practical systems, e.g. 3GPP LTE, pilots are sent at some subset of subcarriers where the other subcarrier channels are estimated via interpolation, utilizing the correlation of the channel across frequency [4, 72, 73, 74, 75, 76, 57, 77, 78, 79]. Channel estimation with interpolation has been investigated in the literature via various interpolation techniques: nearest neighbor, linear, and low-pass interpolations [73, 74]. In particular, [73] considered OFDM systems and [74] considered space-time block coding with spatial modulation, and furthermore it was shown that although linear interpolation has low complexity, low-pass interpolation has the best performance.

Two-dimensional interpolation over frequency and time has been investigated [75, 76, 57, 77, 78], and furthermore a two-dimensional Wiener filter interpolation was shown to be the optimal in terms of mean-squared error [75, 80]. Adaptive channel estimator based on raised cosine frequency-domain interpolation along with time-domain windowing was proposed [79].

Here we introduce pilot-interpolated channel estimation for the proposed scheme and analyze its performance. The location of the pilots with respect to time and frequency has been studied [81, 82], where it was shown that an equidistant arrangement of pilots achieves the MMSE of channel estimation. Define $\{i, j\}$ to be the set of subcarriers containing pilots and $\mathbf{A}_k \in \mathbb{C}^{N_1 \times 2N_1}$ to be the interpolation matrix of $\mathbf{H}_{1,k}$ estimation at subcarriers k. Also, define $\mathbf{U}_{2,i} \in \mathbb{C}^{M \times M}$ and $\mathbf{U}_{2,j} \in \mathbb{C}^{M \times M}$ to be two matrices that contain symbols intended for user 2 and sent at subcarriers i and j, respectively. The transmitted space-time code is

$$\mathbf{X}_k = \begin{cases} \left[\sqrt{M}\mathbf{U}_{2,k},\ \mathbf{U}_{2,k}\mathbf{U}_{1,k}\right], & k = i, j \\ \mathbf{U}_{2,k}\mathbf{U}_{1,k}, & k \neq i, j, \end{cases} \tag{2.36}$$

where

$$\mathbf{U}_{1,k} \in \begin{cases} \mathbb{C}^{M \times (T_1 - M)}, & k = i, j \\ \mathbb{C}^{M \times T_1}, & k \neq i, j, \end{cases}$$

$$\mathbf{U}_{2,k} = \mathbf{B}_k \begin{bmatrix} \mathbf{U}_{2,i} \\ \mathbf{U}_{2,j} \end{bmatrix}, \tag{2.37}$$

$\mathbf{B}_k \in \mathbb{C}^{M\times 2M}$ is the interpolation matrix for the symbols intended for user 2. The difference between the interpolation matrix $\mathbf{B}_k$ and the interpolation matrix $\mathbf{A}_k$ is that the former is used by the transmitter to design the transmitted space-time code, whereas the latter is used at user 1 to estimate $\mathbf{H}_{1,k}$. User 1 estimates the equivalent channel $\mathbf{G}_{1,k}$,

$$\begin{aligned}\hat{\mathbf{G}}_{1,k} &= \mathbf{A}_k \begin{bmatrix} \hat{\mathbf{H}}_{1,i}\mathbf{U}_{2,i} \\ \hat{\mathbf{H}}_{1,j}\mathbf{U}_{2,j} \end{bmatrix} = \mathbf{A}_k \begin{bmatrix} \hat{\mathbf{G}}_{1,i} \\ \hat{\mathbf{G}}_{1,j} \end{bmatrix} \\ &= \gamma \mathbf{A}_k \begin{bmatrix} \mathbf{G}_{1,i} \\ \mathbf{G}_{1,j} \end{bmatrix} + \frac{\gamma}{\sqrt{\rho_\tau}} \mathbf{A}_k \begin{bmatrix} \mathbf{W}_{1,\tau,i} \\ \mathbf{W}_{1,\tau,j} \end{bmatrix},\end{aligned} \tag{2.38}$$

and hence the estimation covariance matrix is

$$\begin{aligned}\hat{\boldsymbol{\Sigma}}_k &= \mathbb{E}\left\{\hat{\mathbf{G}}_{1,k}\hat{\mathbf{G}}_{1,k}^\dagger\right\} \\ &= \gamma^2 \mathbf{A}_k \left(\begin{bmatrix} \mathbb{E}\{\mathbf{G}_{1,i}\mathbf{G}_{1,i}^\dagger\} & \mathbf{0} \\ \mathbf{0} & \mathbb{E}\{\mathbf{G}_{1,j}\mathbf{G}_{1,j}^\dagger\} \end{bmatrix} + \frac{\gamma^2}{\rho_\tau}\mathbf{I} \right) \mathbf{A}_k^\dagger \\ &= M\gamma^2 \left(\|\mathbf{f}_1\|^2 + \frac{\gamma^2}{\rho_\tau} \right) \mathbf{A}_k \mathbf{A}_k^\dagger,\end{aligned} \tag{2.39}$$

and furthermore

$$\hat{\sigma}_k^2 = \frac{1}{N_1 M} \mathrm{Tr}\left\{\hat{\boldsymbol{\Sigma}}_k\right\}. \tag{2.40}$$

The estimation error is

$$\begin{aligned}\tilde{\mathbf{G}}_k &= \mathbf{G}_{1,k} - \hat{\mathbf{G}}_{1,k} \\ &= \mathbf{H}_{1,k}\mathbf{B}_k \begin{bmatrix} \mathbf{U}_{2,i} \\ \mathbf{U}_{2,j} \end{bmatrix} - \gamma \mathbf{A}_k \begin{bmatrix} \mathbf{H}_{1,i}\mathbf{U}_{2,i} \\ \mathbf{H}_{1,j}\mathbf{U}_{2,j} \end{bmatrix} - \frac{\gamma}{\sqrt{\rho_\tau}} \mathbf{A}_k \begin{bmatrix} \mathbf{W}_{1,\tau,i} \\ \mathbf{W}_{1,\tau,j} \end{bmatrix} \\ &= \left(\mathbf{H}_{1,k}\mathbf{B}_k - \gamma \mathbf{A}_k \begin{bmatrix} \mathbf{H}_{1,i} & \mathbf{0} \\ \mathbf{0} & \mathbf{H}_{1,j} \end{bmatrix} \right) \begin{bmatrix} \mathbf{U}_{2,i} \\ \mathbf{U}_{2,j} \end{bmatrix} - \frac{\gamma}{\sqrt{\rho_\tau}} \mathbf{A}_k \begin{bmatrix} \mathbf{W}_{1,\tau,i} \\ \mathbf{W}_{1,\tau,j} \end{bmatrix},\end{aligned} \tag{2.41}$$

and hence the error covariance matrix is

$$\tilde{\boldsymbol{\Sigma}}_k = \mathbb{E}\left\{ \left(\mathbf{H}_{1,k}\mathbf{B}_k - \gamma \boldsymbol{\Gamma}_k \right) \left(\mathbf{H}_{1,k}\mathbf{B}_k - \gamma \boldsymbol{\Gamma}_k \right)^\dagger \right\} + \frac{\gamma^2}{\rho_\tau} M \mathbf{A}_k \mathbf{A}_k^\dagger, \tag{2.42}$$

where

$$\mathbf{\Gamma}_k = \mathbf{A}_k \begin{bmatrix} \mathbf{H}_{1,i} & \mathbf{0} \\ \mathbf{0} & \mathbf{H}_{1,j} \end{bmatrix}, \tag{2.43}$$

and furthermore

$$\tilde{\sigma}_k^2 = \frac{1}{N_1 M} \mathrm{Tr}\left\{\tilde{\mathbf{\Sigma}}_k\right\}. \tag{2.44}$$

Therefore, the achievable rate is

$$R_1^{[\mathrm{PS}]} \geq \frac{1}{K}\left(1 - \frac{M}{T_1}\right) \sum_{k\in\{i,j\}} \mathbb{E}\left\{\log\det\left\{\mathbf{I}_{N_1} + \rho_k \frac{\check{\mathbf{G}}_{1,k}\check{\mathbf{G}}_{1,k}^\dagger}{M}\right\}\right\}$$
$$+ \frac{1}{K}\sum_{k=1, k\notin\{i,j\}}^{K} \mathbb{E}\left\{\log\det\left\{\mathbf{I}_{N_1} + \rho_k \frac{\check{\mathbf{G}}_{1,k}\check{\mathbf{G}}_{1,k}^\dagger}{M}\right\}\right\}, \tag{2.45}$$

where

$$\rho_k = \frac{\rho_\delta \hat{\sigma}_k^2}{1 + \rho_\delta \tilde{\sigma}_k^2}, \tag{2.46}$$

and $\hat{\sigma}_k^2, \tilde{\sigma}_k^2$ are given in (2.40) and (2.44), respectively. On the other hand, user 2 decodes $\mathbf{U}_{2,i}$ and $\mathbf{U}_{2,j}$ sent on subcarriers i and j achieving the rate

$$R_2^{[\mathrm{PS}]} \geq \frac{M}{T_1 K} \sum_{k\in\{i,j\}} \mathbb{E}\left\{\log\det\left\{\mathbf{I}_{N_2} + \frac{1}{\mathbb{E}\left\{\lambda_i^{-2}\right\}}\mathbf{H}_{2,k}\mathbf{H}_{2,k}^\dagger\right\}\right\}. \tag{2.47}$$

Now, we investigate the design of the matrix $\mathbf{B}_k$. The estimation error, $\tilde{\sigma}_k^2$, in (2.41) depends on $\mathbf{B}_k$. Hence, we can maximize the achievable rate by finding the optimal $\mathbf{B}_k$,

$$\begin{aligned} \mathbf{B}_k = \arg\min \quad & \tilde{\sigma}_k^2 \\ \text{subject to} \quad & \mathrm{Tr}\left\{\mathbf{B}_k\mathbf{B}_k^\dagger\right\} \leq M, \end{aligned} \tag{2.48}$$

where the above constraint is used to maintain the transmission power constraint. Using the identity $\mathrm{Tr}\{\mathbf{XY}\} = \mathrm{Tr}\{\mathbf{YX}\}$,

$$\tilde{\sigma}_k^2 = \frac{1}{N_1 M}\mathrm{Tr}\left\{\mathbf{B}_k^\dagger \Psi_k \mathbf{B}_k - \mathbf{B}_k\Phi_k - \Phi_k^\dagger \mathbf{B}_k^\dagger \quad + M\gamma^2\left(\|\mathbf{f}_1\|^2 + \frac{\gamma^2}{\rho_\tau}\right)\mathbf{A}_k\mathbf{A}_k^\dagger\right\}, \tag{2.49}$$

where $\Phi_k = \mathbb{E}\left\{\gamma \mathbf{\Gamma}_k^\dagger \mathbf{H}_{1,k}\right\}$ and $\Psi_k = \mathbb{E}\left\{\mathbf{H}_{1,k}^\dagger \mathbf{H}_{1,k}\right\}$. Hence,

$$\begin{aligned}\mathbf{B}_k = \arg\min \ &\mathrm{Tr}\left\{\mathbf{B}_k^\dagger \Psi_k \mathbf{B}_k\right\} - \mathrm{Tr}\left\{\mathbf{B}_k \Phi_k\right\} - \mathrm{Tr}\left\{\Phi_k^\dagger \mathbf{B}_k^\dagger\right\} \\ &\text{subject to } \mathrm{Tr}\left\{\mathbf{B}_k \mathbf{B}_k^\dagger\right\} \leq M. \end{aligned} \tag{2.50}$$

The above optimization problem is convex, and furthermore can be solved using CVX a package for specifying and solving convex programs [83, 84].

2.1.4 Time-domain Channel Estimation

In Section 2.1.2 and Section 2.1.3, channel frequency response is estimated without and with interpolation, respectively. Here channel impulse response is estimated. For comparison, we first give the achievable rate in conventional transmission when channel impulse response is estimated.

Conventional Transmission with Time-domain Channel Estimation

Define $\tilde{\mathbf{Y}}_{1,i} \in \mathbb{C}^K$ to be the received signal of user 1 at receive antenna $i = 1, \cdots, N_1$. Hence,

$$\tilde{\mathbf{Y}}_{1,i} = \sqrt{\frac{\rho}{M}} \sum_{j=1}^{M} \mathrm{diag}\left\{\bar{\mathbf{F}}[\mathbf{h}_{1,ij}^\dagger, \mathbf{0}_{1\times(K-L_1)}^\dagger]^\dagger\right\} \tilde{\mathbf{X}}_j \quad + \tilde{\mathbf{W}}_{1,i}, \tag{2.51}$$

where $\tilde{\mathbf{X}}_j \in \mathbb{C}^K$ is the transmitted signal at antenna j and $\tilde{\mathbf{W}}_{1,i} \in \mathbb{C}^K$ is the additive Gaussian noise at receive antenna i. Therefore,

$$\tilde{\mathbf{Y}}_{1,i} = \sqrt{\frac{\rho}{M}} \sum_{j=1}^{M} \mathrm{diag}\left\{\tilde{\mathbf{X}}_j\right\} \bar{\mathbf{F}}[\mathbf{h}_{1,ij}^\dagger, \mathbf{0}_{1\times(K-L_1)}^\dagger]^\dagger \quad + \tilde{\mathbf{W}}_{1,i}. \tag{2.52}$$

For estimating $\mathbf{h}_{1,ij}$, at least L pilots are needed for each transmit antenna j.[3] Assume that LM pilots are sent at the first L subcarriers where M pilots are sent during the first M time

[3] Recall that throughout Section 2.1, $L_1 = L_2 = L$.

instances of each subcarrier. Furthermore,

$$\tilde{\mathbf{X}}_{\tau,r} = \begin{cases} \sqrt{M}\mathbf{1}_L, & r = j, \\ \mathbf{0}_{L\times 1}, & r \neq j, \end{cases} \tag{2.53}$$

is the transmitted pilot signal to estimate $\mathbf{h}_{1,ij}$, and hence the received signal at antenna i during pilot transmission can be given by

$$\tilde{\mathbf{Y}}_{1,\tau,i} = \sqrt{\rho_\tau}\bar{\mathbf{F}}\mathbf{h}_{1,ij} + \tilde{\mathbf{W}}_{\tau,i}, \tag{2.54}$$

where $\mathbf{F}_L \in \mathbb{C}^{L\times L}$ is the submatrix consists of the first L columns and the first L rows of $\bar{\mathbf{F}}$, and $\tilde{\mathbf{W}}_{1,\tau,i} \in \mathbb{C}^{L\times 1}$ is the corresponding additive noise.

The MMSE estimated channel is

$$\hat{\mathbf{h}}_{1,ij} = \sqrt{\rho_\tau}\mathbf{F}_L^\dagger \left(\rho_\tau \mathbf{F}_L \mathbf{F}_L^\dagger + \mathbf{I}\right)^{-1} \tilde{\mathbf{Y}}_{1,\tau,i}, \tag{2.55}$$

and hence

$$\begin{aligned} \hat{\mathbf{\Sigma}}_h &= \mathbb{E}\left\{\hat{\mathbf{h}}_{1,ij}\hat{\mathbf{h}}_{1,ij}^\dagger\right\} \\ &= \rho_\tau^2 \mathbf{F}_L^\dagger \left(\rho_\tau \mathbf{F}_L \mathbf{F}_L^\dagger + \mathbf{I}\right)^{-1} \mathbf{F}_L \mathbf{F}_L^\dagger \left(\rho_\tau \mathbf{F}_L \mathbf{F}_L^\dagger + \mathbf{I}\right)^{-1} \mathbf{F}_L + \rho_\tau \mathbf{F}_L^\dagger \left(\rho_\tau \mathbf{F}_L \mathbf{F}_L^\dagger + \mathbf{I}\right)^{-2} \mathbf{F}_L. \end{aligned} \tag{2.56}$$

The estimation error is $\tilde{\mathbf{h}}_{1,ij} = \mathbf{h}_{1,ij} - \hat{\mathbf{h}}_{1,ij}$, and hence

$$\tilde{\mathbf{\Sigma}}_h = \mathbb{E}\left\{\tilde{\mathbf{h}}_{1,ij}\tilde{\mathbf{h}}_{1,ij}^\dagger\right\} = \hat{\mathbf{\Sigma}}_h + \mathbf{I} - 2\rho_\tau \mathbf{F}_L^\dagger \left(\rho_\tau \mathbf{F}_L \mathbf{F}_L^\dagger + \mathbf{I}\right)^{-1} \mathbf{F}_L. \tag{2.57}$$

Based on (2.4), the estimated channel, and the estimation error are

$$\hat{\mathbf{H}}_{1,k} = \mathbf{F}_{1,k}\hat{\bar{\mathbf{H}}}_{1,k}, \quad \tilde{\mathbf{H}}_{1,k} = \mathbf{F}_{1,k}\tilde{\bar{\mathbf{H}}}_{1,k}, \tag{2.58}$$

where

$$\hat{\bar{\mathbf{H}}}_{1,k} = \begin{bmatrix} \hat{\mathbf{h}}_{1,11} & \cdots & \hat{\mathbf{h}}_{1,1M} \\ \vdots & \ddots & \vdots \\ \hat{\mathbf{h}}_{1,N_1 1} & \cdots & \hat{\mathbf{h}}_{1,N_1 M} \end{bmatrix},$$

$$\tilde{\tilde{\mathbf{H}}}_{1,k} = \begin{bmatrix} \tilde{\mathbf{h}}_{1,11} & \cdots & \tilde{\mathbf{h}}_{1,1M} \\ \vdots & \ddots & \vdots \\ \tilde{\mathbf{h}}_{1,N_1 1} & \cdots & \tilde{\mathbf{h}}_{1,N_1 M} \end{bmatrix}. \tag{2.59}$$

Furthermore,

$$\begin{aligned} \hat{\sigma}_k^2 &= \frac{1}{N_1 M} \mathrm{Tr}\left\{ \mathbb{E}\left\{ \mathbf{F}_{1,k} \hat{\hat{\mathbf{H}}}_{1,k} \hat{\hat{\mathbf{H}}}_{1,k}^\dagger \mathbf{F}_{1,k}^\dagger \right\}\right\} \\ &= \frac{1}{N_1} \mathrm{Tr}\left\{ \mathbf{F}_{1,k} \left(\mathbf{I}_{N_1} \otimes \hat{\mathbf{\Sigma}}_{\mathcal{H}} \right) \mathbf{F}_{1,k}^\dagger \right\}, \\ \tilde{\sigma}_k^2 &= \frac{1}{N_1 M} \mathrm{Tr}\left\{ \mathbb{E}\left\{ \mathbf{F}_{1,k} \tilde{\tilde{\mathbf{H}}}_{1,k} \tilde{\tilde{\mathbf{H}}}_{1,k}^\dagger \mathbf{F}_{1,k}^\dagger \right\}\right\} \\ &= \frac{1}{N_1} \mathrm{Tr}\left\{ \mathbf{F}_{1,k} \left(\mathbf{I}_{N_1} \otimes \tilde{\mathbf{\Sigma}}_{\mathcal{H}} \right) \mathbf{F}_{1,k}^\dagger \right\}. \end{aligned} \tag{2.60}$$

Thus, the achievable rate of the dynamic receiver is

$$\begin{aligned} R_1^{[\mathrm{Conv}]} \geq & \frac{1}{K} \left(1 - \frac{M}{T_1} \right) \sum_{k=1}^{L} \mathbb{E}\left\{ \log\det\left\{ \mathbf{I}_{N_1} + \rho_k \frac{\check{\mathbf{H}}_{1,k} \check{\mathbf{H}}_{1,k}^\dagger}{M} \right\}\right\} \\ & + \frac{1}{K} \sum_{k=L+1}^{K} \mathbb{E}\left\{ \log\det\left\{ \mathbf{I}_{N_1} + \rho_k \frac{\check{\mathbf{H}}_{1,k} \check{\mathbf{H}}_{1,k}^\dagger}{M} \right\}\right\}, \end{aligned} \tag{2.61}$$

where $\check{\mathbf{H}}_{1,k} = \frac{\hat{\hat{\mathbf{H}}}_{1,k}}{\hat{\sigma}_k}$,

$$\rho_k = \frac{\rho_\delta \hat{\sigma}_k^2}{1 + \rho_\delta \tilde{\sigma_k}^2}, \tag{2.62}$$

and $\hat{\sigma}_k^2, \tilde{\sigma_k}^2$ are given in (2.60).

Product Superposition Transmission with Time-domain Channel Estimation

In product superposition, the transmitted pilot signal is

$$\tilde{\mathbf{X}}_{\tau,r} = \begin{cases} \sqrt{M} u_{2,j} \mathbf{1}_L, & r = j, \\ \mathbf{0}_{1\times L}, & r \neq j, \end{cases} \tag{2.63}$$

where $u_{2,j} \in \mathbb{C}$ is a symbol intended for user 2 that is sent by transmit antenna j. Hence, the received signal at antenna i of user 1 during pilot transmission can be given by

$$\tilde{\mathbf{Y}}_{1,\tau,i} = \sqrt{\rho_\tau} \mathbf{F}_L \mathbf{h}_{1,ij} u_{2,j} + \tilde{\mathbf{W}}_{1,\tau,i} = \sqrt{\rho_\tau} \mathbf{F}_L \mathbf{g}_{1,ij} + \tilde{\mathbf{W}}_{1,\tau,i} \tag{2.64}$$

where $\mathbf{g}_{1,ij} = \mathbf{h}_{1,ij}u_{2,j}$ is the equivalent channel impulse response seen by user 1. The MMSE channel estimation for the equivalent channel follows the same steps of that in Section 2.1.4. In particular, the estimated channel is

$$\hat{\mathbf{g}}_{1,ij} = \sqrt{\rho_\tau}\mathbf{F}_L^\dagger \left(\rho_\tau \mathbf{F}_L \mathbf{F}_L^\dagger + \mathbf{I}\right)^{-1} \tilde{\mathbf{Y}}_{1,\tau,i}, \tag{2.65}$$

and the estimation error is $\tilde{\mathbf{g}}_{1,ij} = \mathbf{g}_{1,ij} - \hat{\mathbf{g}}_{1,ij}$. Thus, the achievable rate of user 1 is

$$\begin{aligned} R_1^{[\mathrm{PS}]} \geq & \frac{1}{K}\left(1 - \frac{M}{T_1}\right)\sum_{k=1}^{L}\mathbb{E}\left\{\log\det\left\{\mathbf{I}_{N_1} + \rho_k \frac{\check{\mathbf{H}}_{1,k}\check{\mathbf{H}}_{1,k}^\dagger}{M}\right\}\right\} \\ & + \frac{1}{K}\sum_{k=L+1}^{K}\mathbb{E}\left\{\log\det\left\{\mathbf{I}_{N_1} + \rho_k \frac{\check{\mathbf{H}}_{1,k}\check{\mathbf{H}}_{1,k}^\dagger}{M}\right\}\right\}, \end{aligned} \tag{2.66}$$

The received signal of user 2 at the subcarriers with pilots, i.e. $k = 1, \cdots, L$, is

$$\mathbf{Y}_{2,k} = \sqrt{\rho_\tau}\mathbf{H}_{2,k}\mathbf{u}_2 + \mathbf{W}_{2,k}, \tag{2.67}$$

where $\mathbf{u}_2 = [u_{2,1}, \cdots, u_{2,M}]^\dagger$ is the vector containing the symbols intended for user 2. Define the received signal at the L subcarriers containing pilots to be

$$\bar{\bar{\mathbf{Y}}}_2 = \sqrt{\rho_\tau}\bar{\bar{\mathbf{H}}}_2\mathbf{u}_2 + \bar{\bar{\mathbf{W}}}_2, \tag{2.68}$$

where $\bar{\bar{\mathbf{Y}}}_2 = \left[\mathbf{Y}_{2,1}^\dagger, \cdots, \mathbf{Y}_{2,L}^\dagger\right]^\dagger$, $\bar{\bar{\mathbf{H}}}_2 = \left[\mathbf{H}_{2,1}^\dagger, \cdots, \mathbf{H}_{2,L}^\dagger\right]^\dagger$, and $\bar{\bar{\mathbf{W}}}_2 = \left[\mathbf{W}_{2,1}^\dagger, \cdots, \mathbf{W}_{2,L}^H\right]^\dagger$. Thus the achievable rate of user 2 is

$$R_2^{[\mathrm{PS}]} \geq \frac{1}{T_1 K}\mathbb{E}\left\{\log\det\left\{\mathbf{I}_{N_2} + \mathbb{E}\frac{1}{\{\lambda_i^{-2}\}}\bar{\bar{\mathbf{H}}}_2\bar{\bar{\mathbf{H}}}_2^\dagger\right\}\right\}. \tag{2.69}$$

The sum-rate of the conventional transmission is $R_1^{[\mathrm{Conv}]}$ given in (2.61), whereas the sum-rate of product superposition is $R_1^{[\mathrm{PS}]} + R_2^{[\mathrm{PS}]}$. Thus, (2.69) demonstrates the gain of the proposed product superposition.

2.2 Disparity in Coherence Bandwidth

Now we study the downlink transmission under disparity in coherence bandwidth, where user 1 is frequency-static and user 2 is frequency-dynamic, i.e. $L_2 \geq L_1$. The coherence time of both users is the same, i.e., $T_1 = T_2 = T$. The coherence time T could be either short, where both channels are time-dynamic, or long, where both channels are time-static. Here, we consider channel estimation in frequency-domain without interpolation. The disparity in coherence bandwidth can occur in practical systems as shown in the following example.

Example 2. *Consider an urban environment, where a micro-cell is serving two users: a frequency-static user with "B1: typical urban micro-cell" with 76 ns r.m.s. delay spread, and a frequency-dynamic user with "B2: bad urban micro-cell" with 480 ns r.m.s. delay spread [5, 4]. Therefore, the coherence bandwidth of the two users are 263 KHz and 41 KHz. Both users are with 70 Km/h mobility leading to a coherence time of 3.85 ms.*

For simplicity, we consider $\mathbf{H}_{1,k} = \mathbf{H}_1 \forall k$, meaning that the channel is flat-fading, whereas $\mathbf{H}_{2,k}$ is frequency selective. For product superposition transmission, every T time instances, $\mathbf{H}_1$ can be estimated at subcarrier i whereas $\mathbf{H}_{2,k}$ needs channel estimation at all subcarriers. Therefore, at subcarrier i, the pilot signals are sent for estimating $\mathbf{H}_{2,i}$ and $\mathbf{H}_1$, and furthermore at the other subcarriers $k \neq i$, product superposition can be used to provide gain. Hence, at subcarriers $k \neq i$, the transmitted space-time code is

$$\mathbf{X}_k = \left[\sqrt{M}\mathbf{U}_{1,k},\ \mathbf{U}_{12,k}\right], \tag{2.70}$$

where $\mathbf{U}_{12,k} = \mathbf{U}_{1,k}\mathbf{U}_{2,k}$. Hence, user 2 received signal at the first M time instances is

$$\mathbf{Y}_{2,\tau,k} = \sqrt{\rho_\tau}\mathbf{H}_{2,k}\mathbf{U}_{1,k} + \mathbf{W}_{2,\tau,k}, = \sqrt{\rho_\tau}\mathbf{G}_{2,k} + \mathbf{W}_{2,\tau,k}, \tag{2.71}$$

where $\mathbf{G}_{2,k} = \mathbf{H}_{2,k}\mathbf{U}_{1,k}$ is the equivalent channel seen by user 2. Following the same analysis as in Section 2.1, the achievable rate of user 2 over K subcarriers and during T time slots is

$$R_2^{[\text{PS}]} \geq \frac{1}{K}(1 - \frac{M}{T})\sum_{k=1}^{K}\mathbb{E}\Big\{\log\det\Big\{\mathbf{I}_{N_2} + \frac{\rho_\delta\hat{\sigma}_G^2}{1+\rho_\delta\tilde{\sigma}_G^2}\frac{\bar{\mathbf{G}}_{2,k}\bar{\mathbf{G}}_{2,k}^\dagger}{M}\Big\}\Big\}, \tag{2.72}$$

where $\bar{\mathbf{G}}_{2,k} = \frac{\hat{\mathbf{G}}_{2,k}}{\hat{\sigma}_G}$. Furthermore, user 1 achievable rate is

$$R_1^{[\mathrm{PS}]} \geq \frac{M}{TK} \sum_{k=1,k\neq i}^{K} \mathbb{E}\left\{\log\det\left\{\mathbf{I}_{N_1} + \rho_k \frac{\bar{\mathbf{H}}_1 \bar{\mathbf{H}}_1^\dagger}{M}\right\}\right\}, \tag{2.73}$$

where

$$\rho_k = \frac{\hat{\sigma}_k^2}{\mathbb{E}\left\{\epsilon_i^{-2}\right\} + \tilde{\sigma}_k^2}, \tag{2.74}$$

and ϵ_i^{-2} is an eigenvalue of $\tilde{\mathbf{U}}_{2,k}\tilde{\mathbf{U}}_{2,k}^\dagger$, $\tilde{\mathbf{U}}_{2,k} = [\rho_\tau \mathbf{I},\ \rho_\delta \mathbf{U}_{2,k}]$. Since $R_2^{[\mathrm{Conv}]} = R_2^{[\mathrm{PS}]}$, the achievable rate of user 1, i.e., $R_1^{[\mathrm{PS}]}$, demonstrates the gain in the sum-rate due to product superposition.

2.3 Disparity in Both Coherence Time and Coherence Bandwidth

Disparity in both coherence time and coherence bandwidth is when $T_1 \neq T_2$ and $L_1 \neq L_2$. An example of this scenario is as follows.

Example 3. *Consider an urban environment, when a micro-cell is serving two users: user 1 is time-dynamic and frequency-dynamic where it has a propagation of "B4: outdoor to indoor", where user 2 is time-static and frequency-static where it has a propagation of "B2: bad urban micro-cell". User 1 has 5 Km/h mobility and 49 ns r.m.s. delay spread, whereas user 2 has 70 Km/h mobility and 480 ns r.m.s. delay spread.*

If $T_1 \leq T_2$ and $L_1 \geq L_2$, i.e., user 2 is time-static and frequency-static, the proposed product superposition transmission in Section 2.1 and Section 2.2 can be used resulting in the same achievable rates. If $T_1 \leq T_2$ and $L_1 \leq L_2$, user 1 is time-dynamic and frequency-static. In this case, since none of the users is static in both time and frequency, we give two versions of product superposition transmissions: product superposition over coherence time and product superposition over coherence bandwidth. In the sequel, we assume $T_2 = \ell T_1$, where $\ell \in \mathbb{Z}$, and furthermore $L_1 = 1, L_2 \geq 1$.

2.3.1 Product Superposition over Coherence Time

The interval T_2 can be divided into ℓ subintervals, each of length T_1. During the first subinterval, the transmitter sends pilots during the first M time instances at subcarrier i so that $\mathbf{H}_1$ and $\mathbf{H}_{2,i}$ can be estimated. Hence,

$$\begin{aligned}\hat{\mathbf{H}}_1 &= \frac{\rho_\tau}{\rho_\tau+1}\mathbf{H}_1 + \frac{\sqrt{\rho_\tau}}{\rho_\tau+1}\mathbf{W}_{1,\tau,i},\\ \hat{\mathbf{H}}_{2,i} &= \gamma\mathbf{H}_{2,i} + \frac{\gamma}{\sqrt{\rho_\tau}}\mathbf{W}_{2,\tau,i}.\end{aligned} \tag{2.75}$$

During the following $(\ell-1)$ subintervals, $\mathbf{H}_1$ needs pilots to be estimated, whereas $\mathbf{H}_{2,i}$ does not. Therefore, the transmitted space-time code at subcarrier i during the first M time instances is

$$\mathbf{X}_{\tau,i} = \sqrt{M}\mathbf{U}_2, \tag{2.76}$$

where $\mathbf{U}_2 \in \mathbb{C}^{M\times T_1}$ contains symbol intended for user 2. Hence, the received signal of user 1 during the first M time instances is

$$\mathbf{Y}_{1,\tau,i} = \sqrt{\rho_\tau}\mathbf{G}_1 + \mathbf{W}_{1,\tau,i}, \tag{2.77}$$

where $\mathbf{G}_1 = \mathbf{H}_1\mathbf{U}_2$ is the equivalent channel seen by user 1. Therefore, the MMSE estimate of the channel is

$$\hat{\mathbf{G}}_1 = \frac{\rho_\tau}{\rho_\tau+1}\mathbf{G}_1 + \frac{\sqrt{\rho_\tau}}{\rho_\tau+1}\mathbf{W}_{1,\tau,i}. \tag{2.78}$$

and furthermore

$$\hat{\sigma}^2 = \frac{\rho_\tau}{1+\rho_\tau}, \tag{2.79}$$

Hence, the estimation error is

$$\tilde{\mathbf{G}}_1 = \mathbf{G}_1 - \hat{\mathbf{G}}_1 = \frac{1}{1+\rho_\tau}\mathbf{G}_1 - \frac{\sqrt{\rho_\tau}}{1+\rho_\tau}\mathbf{W}_{1,\tau,i}. \tag{2.80}$$

and furthermore

$$\tilde{\sigma}^2 = \frac{1}{1+\rho_\tau}. \tag{2.81}$$

Hence, the achievable rate is

$$\begin{aligned}R_1^{[\mathrm{PS}]} \geq & \frac{1}{\ell K}\left(1-\frac{M}{T_1}\right)\mathbb{E}\left\{\log\det\left\{\mathbf{I}_{N_1}+\rho_i\frac{\bar{\mathbf{G}}_1\bar{\mathbf{G}}_1^\dagger}{M}\right\}\right\} \\ & + \frac{1}{\ell K}\sum_{k=1,k\neq i}^{K}\mathbb{E}\left\{\log\det\left\{\mathbf{I}_{N_1}+\rho_k\frac{\bar{\mathbf{G}}_1\bar{\mathbf{G}}_1^\dagger}{M}\right\}\right\} \\ & + \left(1-\frac{1}{\ell}\right)\left(\frac{T_1-M}{T_1K}\mathbb{E}\left\{\log\det\left\{\mathbf{I}_{N_1}+\check{\rho}_i\frac{\bar{\mathbf{G}}_1\bar{\mathbf{G}}_1^\dagger}{M}\right\}\right\}\right. \\ & \left. + \frac{1}{K}\sum_{k=1,k\neq i}^{K}\mathbb{E}\left\{\log\det\left\{\mathbf{I}_{N_1}+\check{\rho}_k\frac{\bar{\mathbf{G}}_1\bar{\mathbf{G}}_1^\dagger}{M}\right\}\right\}\right),\end{aligned} \tag{2.82}$$

where

$$\check{\rho}_k = \frac{\rho_\delta\hat{\sigma}_k^2}{1+\rho_\delta\tilde{\sigma}_k^{\,2}}, \tag{2.83}$$

$\hat{\sigma}^2$, $\tilde{\sigma}^2$ are given in (2.79) and (2.81), respectively, and ρ_k is defined in (2.22). Furthermore, the achievable rate of user 2 is

$$R_2^{[\mathrm{PS}]} \geq \left(1-\frac{1}{\ell}\right)\frac{M}{T_1K}\mathbb{E}\left\{\log\det\left\{\mathbf{I}_{N_2}+\rho_i\frac{\bar{\mathbf{H}}_{2,i}\bar{\mathbf{H}}_{2,i}^\dagger}{M}\right\}\right\}. \tag{2.84}$$

2.3.2 Product Superposition over Coherence Bandwidth

This case is similar to that discussed in Section 2.2. Hence, the achievable rates $R_1^{[\mathrm{PS}]}$ and $R_2^{[\mathrm{PS}]}$ are given in (2.73) and (2.72), respectively.

2.4 Numerical Results

The numerical results are averaged over 10000 Monte Carlo runs. In each run, the channel coefficients and the noise are independent and randomly generated. In the sequel, the sum-rate for conventional transmission represents $R_1^{[\mathrm{Conv}]}$ for disparity in coherence time and $R_2^{[\mathrm{Conv}]}$ for disparity in coherence bandwidth. For product superposition, the sum-rate represents $R_1^{[\mathrm{PS}]}+R_2^{[\mathrm{PS}]}$.

For disparity in coherence time investigated in Section 2.1.2, the performance of the proposed product superposition is shown in Fig. 2.1 and Fig. 2.2. In particular, Fig. 2.1 shows the sum-rate of the conventional transmission and the proposed product superposition versus SNR for $M = N_1 = N_2 = 4$, $L = 2$, $K = 12$, and $T_1 = 20$. We can notice from the figure that the gain in the sum-rate is 10 b/sec/Hz at 20dB SNR, and the improvement is at least 50% over the range of 0-20 dB SNR. The curves with no power optimization use $\rho_\tau = \rho_\delta = \rho$ and that with power optimization use ρ_τ, ρ_δ in (2.24). Furthermore, Fig. 2.2 demonstrates the sum-rate gain versus the number of transmit and receive antennas for $\rho = 20$ dB, $K = 8$, $N_1 = N_2 = N$, $T_1 = 11$. The sum-rate gain is $\Delta R = R_1^{[\text{PS}]} + R_2^{[\text{PS}]} - R_1^{[\text{Conv}]}$. The figure shows that our proposed methods offer gains even in a SISO configuration, therefore it is distinct from beamforming gains.

Fig. 2.3 demonstrates the gain in the sum-rate for disparity in coherence time when channel estimation is done using interpolation. The interpolation matrix $\mathbf{A}_k$ is piece-wise linear, and the matrix $\mathbf{B}_k$ is the optimizer of (2.50). The figure shows the improvement in the sum-rate is at least 35% over the range of 0-20 dB SNR. The effect of the interpolation matrix $\mathbf{A}_k$ is shown in Fig. 2.4 for $L = 3$, where the gain is shown for piece-wise linear, nearest-neighbor, and Wiener interpolations. Furthermore, Fig. 2.5 and Fig. 2.6 demonstrate the effect of the design of the matrix $\mathbf{B}_k$, where the performance is evaluated for optimal, linear, and random $\mathbf{B}_k$.

For time-domain channel estimation in Fig. 2.7, we consider $M = 4$, $N_1 = 2$, $N_2 = 4$, $L = 5$, $K = 10$, and $T_1 = 8$. The figure shows that for time-domain channel estimation, our proposed method improves the sum-rate of at least 10% over the range of 0-20 dB SNR. Furthermore, Fig. 2.8 shows the product superposition gain when the disparity is in coherence bandwidth discussed in Section 2.2, where we consider a system of $M = 4$, $N_1 = 2$, $N_2 = 2$, $T_1 = T_2 = T = 8$, $L_1 = 1$, $L_2 = 2$, and $K = 8$.

For disparity in both coherence time and coherence bandwidth in Section 2.3, Fig. 2.9 and Fig. 2.10 show the gain of product superposition transmission. The two figures consider the case when user 1 is time-dynamic and frequency-static and user 2 is time-static and frequency-dynamic. The simulation parameters for the two figures are $M = 4, N_1 = N_2 = 2, L_1 = 1, L_2 = 3, T_1 = 8$, and $T_2 = 24$. In Fig. 2.9, product superposition over coherence time proposed in Section 2.3.1 provides gain over the conventional transmission of user 1, i.e., $R_1^{[\text{Conv}]}$ (the solid blue curve). Similarly, product superposition over coherence bandwidth proposed in Section 2.3.2 provides gain over the conventional transmission that serves only user 2, i.e. $R_2^{[\text{Conv}]}$ (the dashed blue curve). Furthermore, Fig. 2.10 shows the achievable rate region of the conventional and the proposed product superposition. The blue curve is the 45-degree line that connects the two rate pairs $\left(0, R_1^{[\text{Conv}]}\right)$ and $\left(R_2^{[\text{Conv}]}, 0\right)$. The achievable rate region of product superposition is the polyhedron that is constructed from the time-sharing between the transmission strategies that achieve the polyhedron vertices (the red circles in the figures). The achieved rate pairs $\left(R_2^{[\text{PS}]}, R_1^{[\text{PS}]}\right)$ given in Section 2.3.1 and Section 2.3.2 lead to the two corner points that increase the product superposition rate region compared to the TDMA region.

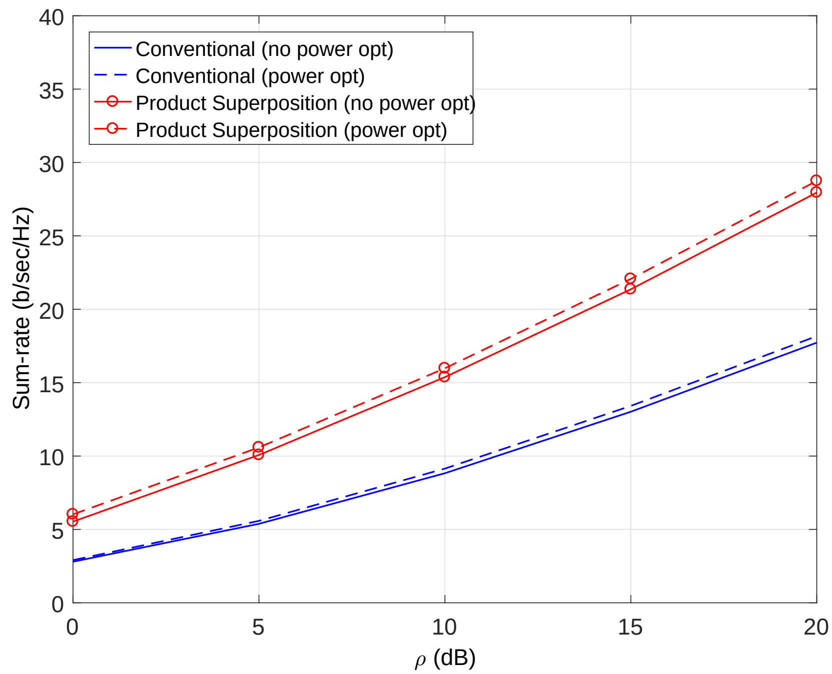

Figure 2.1. Sum-rate versus SNR under disparity in coherence time for frequency-domain channel estimation without interpolation

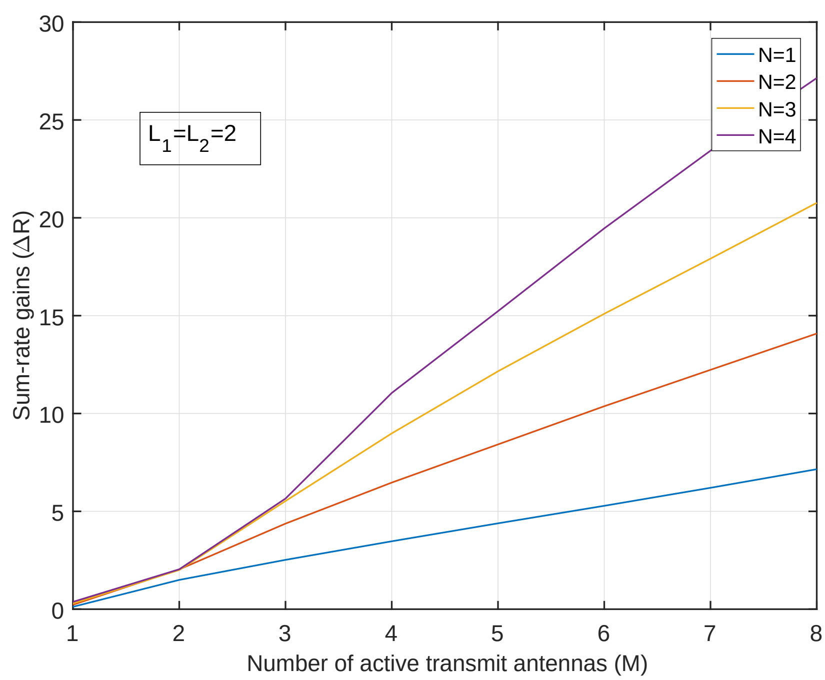

Figure 2.2. Sum-rate gains in a variety of system configurations, under frequency-domain channel estimation

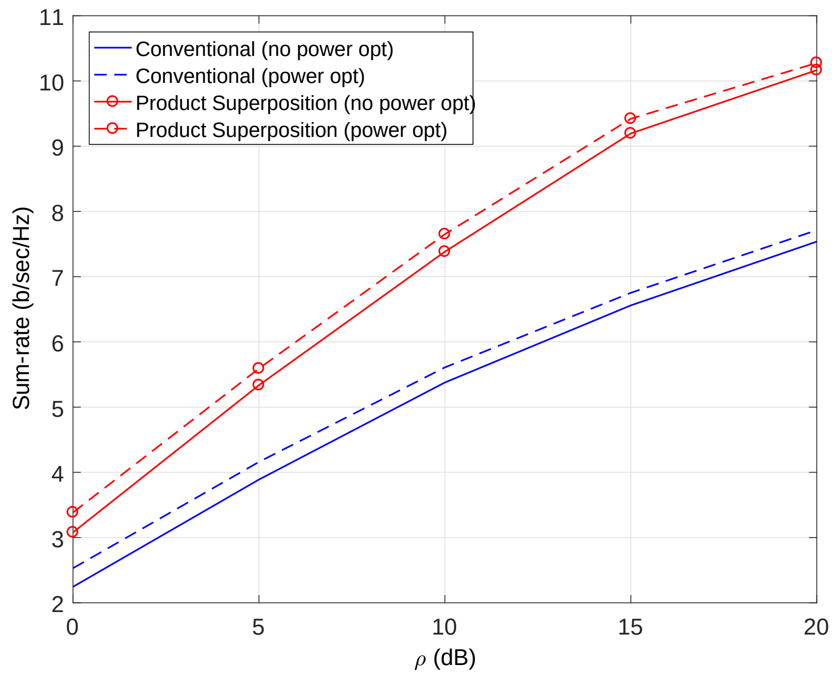

Figure 2.3. Sum-rate versus SNR under disparity in coherence time for frequency-domain channel estimation with interpolation

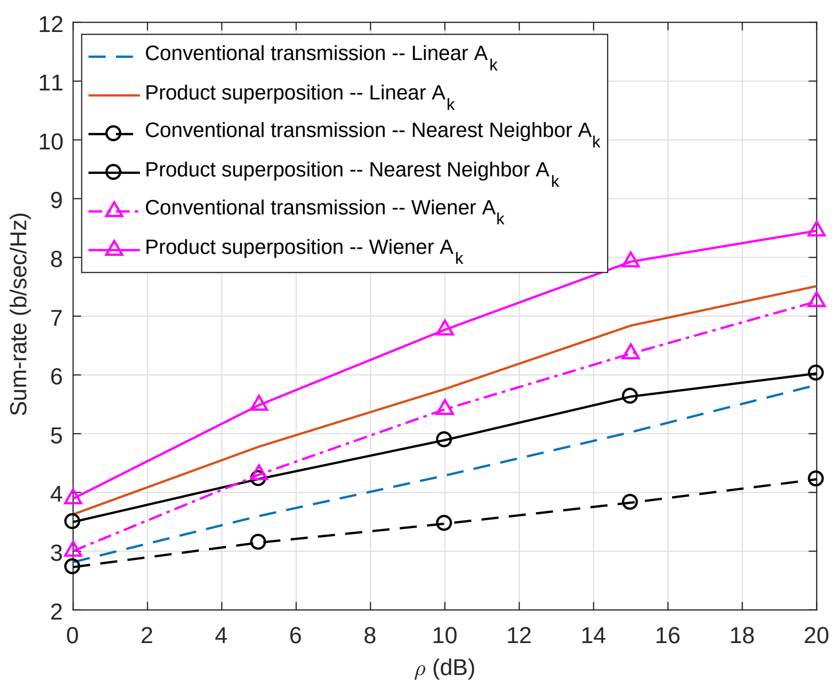

Figure 2.4. Sum-rate versus SNR under disparity in coherence time for different interpolation techniques of $\mathbf{A}_k$

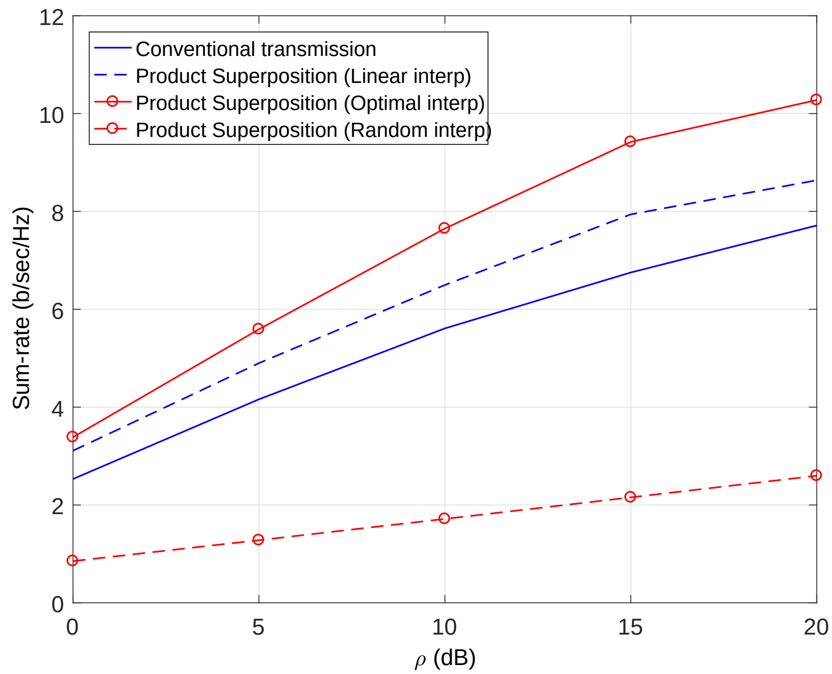

Figure 2.5. The effect of the interpolation matrix $\mathbf{B}_k$ on the sum-rate

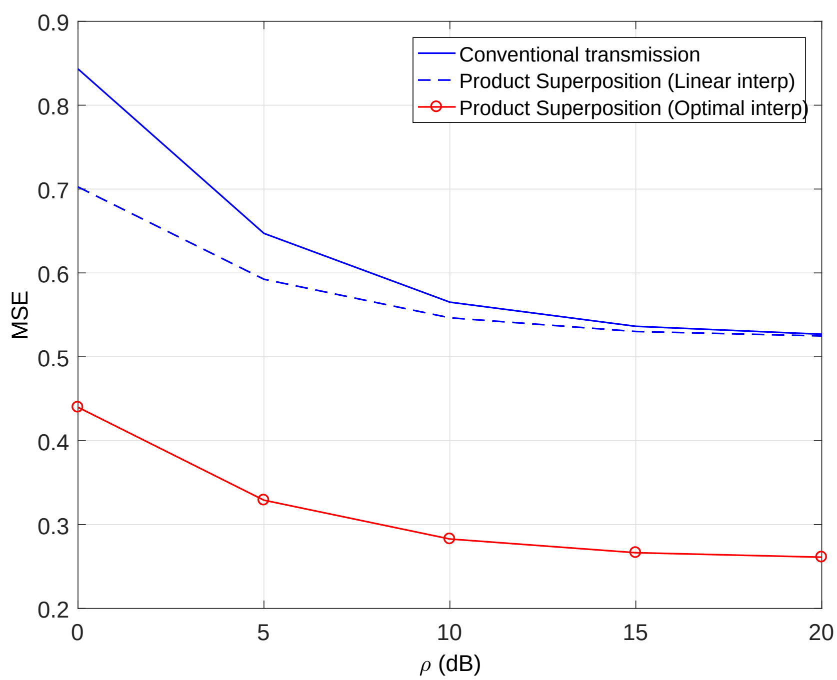

Figure 2.6. The effect of the interpolation matrix $\mathbf{B}_k$ on the MMSE

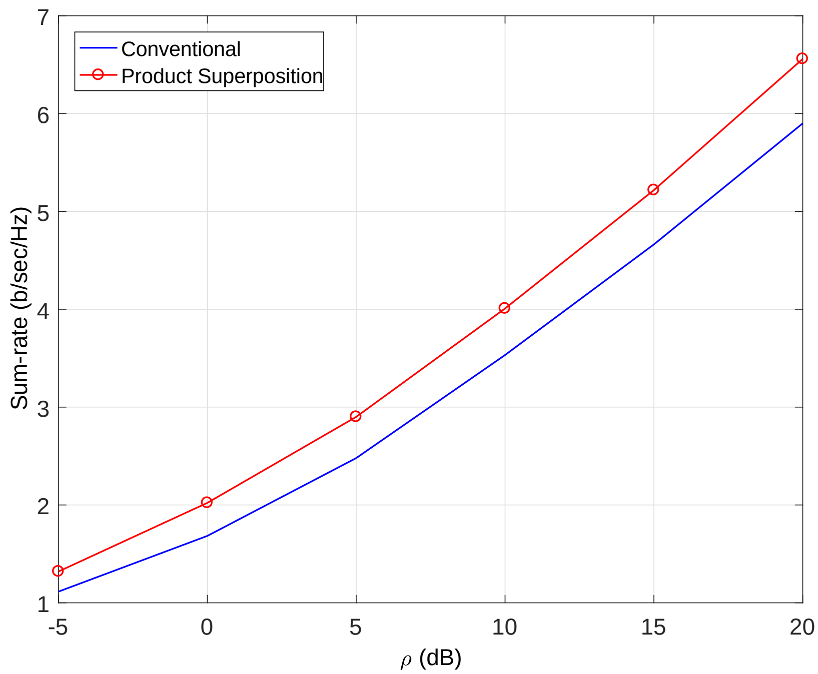

Figure 2.7. Sum-rate versus SNR under disparity in coherence time with time-domain channel estimation

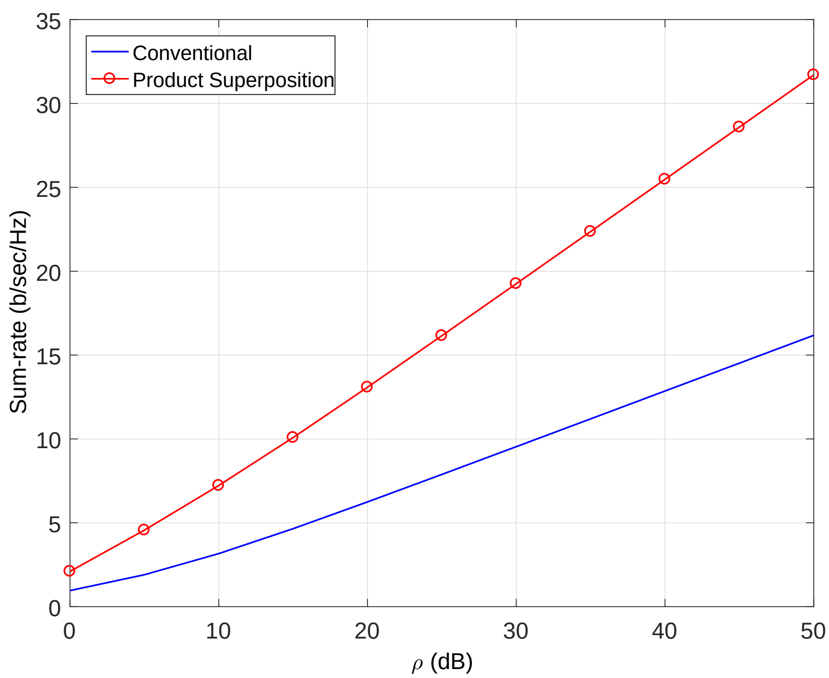

Figure 2.8. Sum-rate versus SNR under disparity in coherence bandwidth

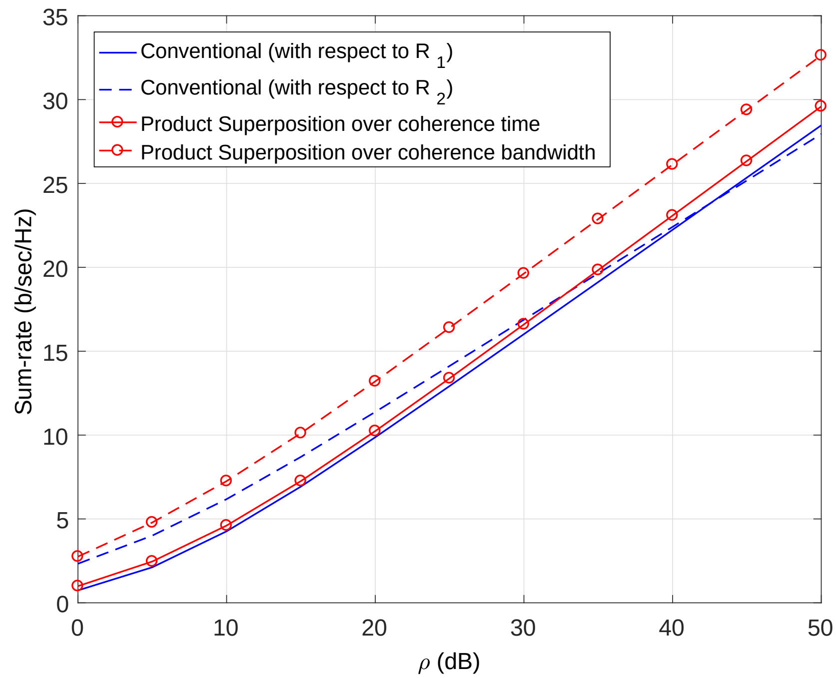

Figure 2.9. Sum-rate versus SNR under disparity in both coherence time and coherence bandwidth

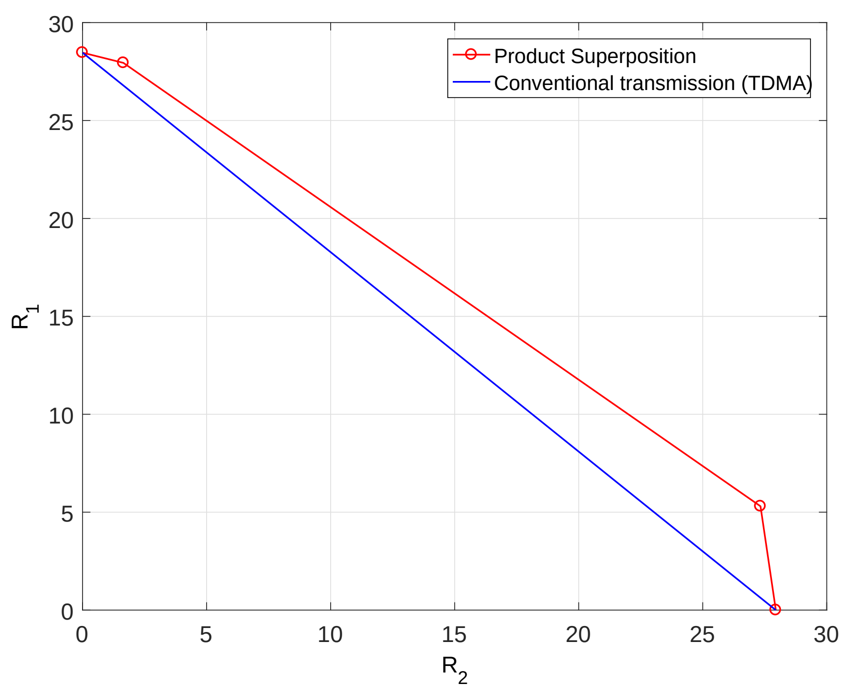

Figure 2.10. Rate region under disparity in both coherence time and coherence bandwidth

CHAPTER 3

COHERENCE DISPARITY IN BROADCAST AND MULTIPLE ACCESS CHANNELS

In this chapter, coherence diversity is investigated for broadcast and multiple access channels with an arbitrary number of users having unequal coherence times [1]. Inner and outer bounds on the degrees of freedom regions are provided. The key results of this chapter are summarized in Table 3.1 for broadcast channel and Table 3.2 for multiple access channel, where $N_i^* = \min\left\{M, N_i, \left\lfloor \frac{T_i}{2} \right\rfloor\right\}$, $N_{\max} = \max_{j\in\mathcal{J}}\left\{N_j\right\}$ and $j_{\min}$ is the receiver with the shortest coherence time in $\mathcal{J}$.

3.1 Broadcast Channel with Identical Coherence Times

Consider a K-receiver MIMO broadcast channel where the transmitter is equipped with M antennas and receiver k is equipped with N_k antennas, $k = 1, \cdots, K$. The received signal at receiver k is

$$\mathbf{y}_k(n) = \overline{\mathbf{H}}_k(n)\mathbf{x}(n) + \mathbf{z}_k(n), \qquad k = 1, \cdots, K, \tag{3.1}$$

where $\mathbf{x}(n) \in \mathbb{C}^{M\times 1}$ is the transmitted signal, $\mathbf{z}_k(n) \in \mathbb{C}^{N_k\times 1}$ is receiver k i.i.d. Gaussian additive noise and $\overline{\mathbf{H}}_k(n) \in \mathbb{C}^{N_k\times M}$ is receiver k Rayleigh block-fading channel matrix with coherence interval of length T_k time slots [6], at the discrete time index n. One time slot is equivalent to a single transmission symbol period, and all T_k are positive integers. We assume no CSIT, meaning the realization of $\overline{\mathbf{H}}_k(n)$ is not known at the transmitter, whereas its distribution (including the length of the coherence time, and its transition) is globally known at the transmitter and at all receivers.

Table 3.1. Degrees of freedom of block-fading broadcast channel with no CSI

Coherence Times	Degrees of freedom
Identical: $T_k = T$	$\sum_{i=1}^{K} \frac{d_i}{N_i^*\left(1-\frac{N_i^*}{T}\right)} \leq 1$
Integer ratio: $\frac{T_k}{T_{k-1}} \in \mathbb{Z},\ T_k \in \mathbb{N}$	Inner bound 1: $d_j = \begin{cases} N_j^*\left(1-\frac{N_j^*}{T_j}-\frac{\min\{M,N_{\max},T_j\}-N_j^*}{T_{j+1}}\right), & j = j_{\min} \\ N_{j_{\min}}^* \min\{M, N_j, T_{j_{\min}}\}\left(\frac{1}{T_{j-1}}-\frac{1}{T_j}\right), & j \neq j_{\min} \end{cases}$ Inner bound 2: $d_j = \begin{cases} N_j^*\left(1-\frac{N_j^*}{T_j}\right), & j = j_{\min} \\ N_{j_{\min}}^* \min\{N_{j_{\min}}^*, N_j^*\}\left(\frac{1}{T_{j-1}}-\frac{1}{T_j}\right), & j \neq j_{\min} \end{cases}$ where $j \in \mathcal{J} \subseteq [1:K]$ Outer bound: $\sum_{j\in\mathcal{J}} \frac{d_j}{N_j^*\left(1-\frac{N_j^*}{T_{j_{\max}}}\right)} \leq 1, \quad \forall \mathcal{J} \subseteq [1:K]$ Cases of tightness: $\begin{cases} 1) M \leq \min_j N_j \\ 2) N_j = N, \quad \forall j \\ 3) T_j \gg T_1, \quad j = 2, \cdots, K \\ 4) T_j = T, \quad \forall j \end{cases}, \quad T_j \geq 2\max\{M, N_j\}$
Arbitrary ratio: $T_k \in \mathbb{N}$	Inner bound: $d_j = \begin{cases} N_j^*\left(1-\frac{N_j^*}{T_j}\right), & j = j_{\min} \\ N_{j_{\min}}^* \min\{N_{j_{\min}}^*, N_j^*\}\left(\frac{1}{T_{j-1}}-\frac{1}{T_j}\right), & j \neq j_{\min} \end{cases}$ where $j \in \mathcal{J} \subseteq [1:K]$

Table 3.2. Degrees of freedom of block-fading multiple access channel with no CSI

Coherence Times	Degrees of freedom
Identical: $T_k = T$	Inner bound: $d_j = M'_j \left(1 - \frac{\sum_{j\in\mathcal{J}} M'_j}{T}\right), \quad j \in \mathcal{J} \subseteq [1:K]$ Outer bound: $\sum_{j\in\mathcal{J}} d_j \leq \min\left\{N, \sum_{j\in\mathcal{J}} M_j\right\}\left(1 - \frac{\min\left\{N, \sum_{j\in\mathcal{J}} M_j\right\}}{T}\right),$ $\forall \mathcal{J} \subseteq [1:K]$ Inner bound is tight against sum degrees of freedom
Integer ratio: $\frac{T_k}{T_{k-1}} \in \mathbb{Z}$	Inner bound: $d_j = M'_j \sum_{m=1}^{J}\left(T_{i_1} - \sum_{n=1}^{m} M'_{i_n}\right)\left(\frac{1}{T_{i_m}} - \frac{1}{T_{i_{m+1}}}\right)$ Outer bound: $\sum_{j\in\mathcal{J}} d_j \leq \min\left\{N, \sum_{j\in\mathcal{J}} M_j\right\}\left(1 - \frac{\min\left\{N, \sum_{j\in\mathcal{J}} M_j\right\}}{T_{i_J}}\right)$ where $\mathcal{J} = \{i_1, \cdots, i_J\} \subseteq [1:K]$

We assume that there are K independent messages associated with rates $R_1(\rho), \cdots, R_K(\rho)$ to be communicated from the transmitter to the K receivers at ρ signal-to-noise ratio. The degrees of freedom at receiver k achieving rate $R_k(\rho)$ can be defined as

$$d_k = \lim_{\rho\to\infty} \frac{R_k(\rho)}{\log(\rho)}. \tag{3.2}$$

The degrees of freedom region of a K-receiver MIMO broadcast is defined as

$$\mathcal{D} = \left\{ \left(d_1, \cdots, d_K\right) \in \mathbb{R}_+^K \middle| \exists \left(R_1(\rho), \cdots, R_K(\rho)\right) \in C(\rho), d_k = \lim_{\rho\to\infty} \frac{R_k(\rho)}{\log(\rho)}, \quad k \in 1, \cdots, K \right\}, \tag{3.3}$$

where $C(\rho)$ is the capacity region at ρ signal-to-noise ratio.

Assume that the receivers have identical coherence times, where the coherence times are perfectly aligned, and furthermore, have the same length, namely T. For the capacity to be determined, it is sufficient to study the capacity of only one coherence time. Define $\mathbf{Y}_k \in \mathbb{C}^{N_k \times T}$, $\mathbf{X} \in \mathbb{C}^{M \times T}$ to be the received signal at receiver $k = 1, \ldots, K$ and the transmitted signal, respectively, during the coherence time T,

$$\mathbf{Y}_k = \mathbf{H}_k \mathbf{X}_k + \mathbf{Z}_k, \quad k = 1, \cdots, K, \tag{3.4}$$

where $\mathbf{H}_k \in \mathbb{C}^{N_k \times M}$ is receiver k channel matrix which remains constant during the interval T.

When there is CSIR, the degrees of freedom optimality of TDMA for two receivers with $T = 1$ was shown [9]. Furthermore, the result was extended to arbitrary number of receivers and for a wider class of fading distribution [29]. Since there is no CSIT, and furthermore, the receivers have identical coherence times, namely T, the receivers are stochastically equivalent (indistinguishable) with respect to the transmitter [30, 8]. As a result, TDMA is enough to achieve the degrees of freedom region of the system, i.e. the degrees of freedom region can be given by,

$$\mathcal{D} = \left\{ (d_1, \cdots, d_K) \in \mathbb{R}_+^K \middle| \sum_{i=1}^{K} \frac{d_i}{\min\{M, N_i\}} \leq 1 \right\}.$$

In Appendix A, we extend this result for $T \geq 1$ showing that TDMA is degrees of freedom optimal.

Now assume that, for a K-receiver broadcast channel, there is no CSIR. As long as the receivers have identical coherence times, the receivers are still stochastically equivalent. In the sequel, we show that TDMA is enough to achieve the degrees of freedom region of the system.

Theorem 1. *Consider a K-receiver broadcast channel with identical coherence times T. When there is no CSIT or CSIR meaning that the channel realization is not known, but the* *channel distribution is globally known, the degrees of freedom region of the channel is given by,*

$$\mathcal{D} = \left\{ (d_1, \cdots, d_K) \in \mathbb{R}_+^K \middle| \sum_{i=1}^{K} \frac{d_i}{N_i^* \left(1 - \frac{N_i^*}{T}\right)} \leq 1 \right\}. \tag{3.5}$$

where $N_i^* = \min\left\{M, N_i, \left\lfloor \frac{T_i}{2} \right\rfloor\right\}$.

Proof. A simple time division multiplexing between the receivers achieves the degrees of freedom region. The remainder of the proof is dedicated to finding a corresponding outer bound. Without loss of generality, assume $N_1 \leq \cdots \leq N_K$. When $M \leq N_1$, the cooperative outer bound [85] for the sum degrees of freedom is

$$\sum_{i=1}^{K} d_i \leq \min\left\{M, \left\lfloor \frac{T}{2} \right\rfloor\right\} \left(1 - \frac{\min\{M, \lfloor \frac{T}{2} \rfloor\}}{T}\right), \tag{3.6}$$

which is tight against the TDMA inner bound. When $M \geq N_1$, to obtain the outer bound we need to introduce the following Lemma.

Lemma 1. *For the above MIMO K-receiver broadcast channel, define the matrix* $\overline{\mathbf{Y}} = \left[\mathbf{Y}_1^\dagger,\ \mathbf{Y}_2^\dagger,\ \cdots,\mathbf{Y}_K^\dagger\right]^\dagger$ *to contain all received signals during T interval, and* $\overline{\mathbf{Y}}_j \in \mathbb{C}^{1\times T}$ *is row j of* $\overline{\mathbf{Y}}$ *and* $\widetilde{\mathbf{Y}}_{\mathcal{S}}$ *is the matrix constructed from excluding the set $\mathcal{S}$ of the rows from the matrix* $\overline{\mathbf{Y}}$. *Then we have*

$$I\left(\mathbf{X}; \overline{\mathbf{Y}}_j | U, \widetilde{\mathbf{Y}}_{\{j,\ell\}}\right) = I\left(\mathbf{X}; \overline{\mathbf{Y}}_\ell | U, \widetilde{\mathbf{Y}}_{\{j,\ell\}}\right), \tag{3.7}$$

and furthermore,

$$I\left(\mathbf{X}; \overline{\mathbf{Y}}_j | U, \widetilde{\mathbf{Y}}_{\{j,\ell\}}\right) \geq I\left(\mathbf{X}; \overline{\mathbf{Y}}_j | U, \widetilde{\mathbf{Y}}_{\{j,\ell\}}, \overline{Y}_\ell\right), \tag{3.8}$$

where $U \to \mathbf{X} \to \overline{\mathbf{Y}}$ *forms a Markov Chain.*

Proof. First, we prove the equality (3.7),

$$\begin{aligned} I\big(\mathbf{X}; \overline{\mathbf{Y}}_j | U, \widetilde{\mathbf{Y}}_{\{j,\ell\}}\big) =& h\left(\overline{\mathbf{Y}}_j | U, \widetilde{\mathbf{Y}}_{\{j,\ell\}}\right) - h\left(\overline{\mathbf{Y}}_j | U, \widetilde{\mathbf{Y}}_{\{j,\ell\}}, \mathbf{X}\right) \\ \overset{(a)}{=}& h\left(\overline{\mathbf{Y}}_\ell | U, \widetilde{\mathbf{Y}}_{\{j,\ell\}}\right) - h\left(\overline{\mathbf{Y}}_\ell | U, \widetilde{\mathbf{Y}}_{\{j,\ell\}}, \mathbf{X}\right) \end{aligned}$$

$$=I\left(\mathbf{X};\overline{\mathbf{Y}}_{\ell}|U,\widetilde{\mathbf{Y}}_{\{j,\ell\}}\right). \tag{3.9}$$

where (a) follows since the random variables are statistically equivalent, and entropies depend only on the statistics. Now we prove the inequality (3.8) as follows,

$$\begin{aligned}
I\big(\mathbf{X};\overline{\mathbf{Y}}_j|U,\widetilde{\mathbf{Y}}_{\{j,\ell\}}\big) =& h\left(\overline{\mathbf{Y}}_j|U,\widetilde{\mathbf{Y}}_{\{j,\ell\}}\right) - h\left(\overline{\mathbf{Y}}_j|U,\widetilde{\mathbf{Y}}_{\{j,\ell\}},\mathbf{X}\right)\\
\overset{(a)}{\geq}& h\left(\overline{\mathbf{Y}}_j|U,\widetilde{\mathbf{Y}}_{\{j,\ell\}},\overline{\mathbf{Y}}_\ell\right) - h\left(\overline{\mathbf{Y}}_j|U,\widetilde{\mathbf{Y}}_{\{j,\ell\}},\mathbf{X}\right)\\
\overset{(b)}{=}& h\left(\overline{\mathbf{Y}}_j|U,\widetilde{\mathbf{Y}}_{\{j,\ell\}},\overline{\mathbf{Y}}_\ell\right) - h\left(\overline{\mathbf{Y}}_j|U,\widetilde{\mathbf{Y}}_{\{j,\ell\}},\mathbf{X},\overline{\mathbf{Y}}_\ell\right)\\
=& I\left(\mathbf{X};\overline{\mathbf{Y}}_j|U,\widetilde{\mathbf{Y}}_{\{j,\ell\}},\overline{\mathbf{Y}}_\ell\right),
\end{aligned} \tag{3.10}$$

where (a) follows since conditioning does not increase the entropy. Furthermore (b) follows by the fact that $\overline{\mathbf{Y}}_j \to \mathbf{X} \to \overline{\mathbf{Y}}_\ell$ forms a Markov chain. The received signal $\overline{\mathbf{Y}}_j, \overline{\mathbf{Y}}_\ell$ are given by, respectively,

$$\begin{aligned}
\overline{\mathbf{Y}}_j =& \mathbf{h}_j^{\dagger}\mathbf{X} + \mathbf{z}_j^{\dagger},\\
\overline{\mathbf{Y}}_\ell =& \mathbf{h}_\ell^{\dagger}\mathbf{X} + \mathbf{z}_\ell^{\dagger},
\end{aligned} \tag{3.11}$$

where $\mathbf{h}_j^{\dagger}, \mathbf{h}_\ell^{\dagger}$ are the channel vectors which are independent from each other whereas $\mathbf{z}_j^{\dagger}, \mathbf{z}_\ell^{\dagger}$ are the independent corresponding noise vectors. Therefore, conditioning on $\mathbf{X}$, $\overline{\mathbf{Y}}_j$ and $\overline{\mathbf{Y}}_\ell$ are independent. □

Now, we are ready to find the outer bound for the case when $M \geq N_1$. Since the receivers have the same noise variance, the system is considered degraded [86, 87], [88, Section 5.7],

$$\begin{aligned}
R_k \leq& I\left(U_k;\mathbf{Y}_k|U^{k-1}\right), \quad k \neq K,\\
R_K \leq& I\left(\mathbf{X};\mathbf{Y}_K|U^{K-1}\right),
\end{aligned} \tag{3.12}$$

where $U_1 \to \cdots \to U_{K-1} \to \mathbf{X} \to (\mathbf{Y}_1, \cdots \mathbf{Y}_K)$ forms a Markov Chain, and U_0 is a trivial random variable. Using the chain rule, we can write (3.12) as

$$R_k \leq I\left(\mathbf{X};\mathbf{Y}_k|U^{k-1}\right) - I\left(\mathbf{X};\mathbf{Y}_k|U^{k}\right), \quad k \neq K,$$

$$R_K \leq I\left(\mathbf{X};\mathbf{Y}_K|U^{K-1}\right).$$

Define r_k to be the degrees of freedom of the term $I\left(\mathbf{X};\mathbf{Y}_k|U^k\right)$, where $0 \leq r_k \leq N_k^*\left(1-\frac{M}{T}\right)$. Furthermore, the degrees of freedom of $I\left(\mathbf{X};\mathbf{Y}_1\right)$ is bounded by the single receiver bound, i.e. $N_1\left(1-\frac{M}{T}\right)$, hence,

$$\begin{aligned}
R_1 &\leq \left(N_1\left(1-\frac{M}{T}\right)-r_1\right)\log\left(\rho\right)+o(\log(\rho)),\\
R_k &\leq I\left(\mathbf{X};\mathbf{Y}_k|U^{k-1}\right)-r_k\log\left(\rho\right)+o(\log(\rho)), \quad k\neq 1,K\\
R_K &\leq I\left(\mathbf{X};\mathbf{Y}_K|U^{K-1}\right).
\end{aligned} \tag{3.13}$$

Furthermore, we have

$$\begin{aligned}
r_k\log(\rho)+o(\log(\rho)) &= I\left(\mathbf{X};\mathbf{Y}_k|U^k\right)\\
&\overset{(a)}{=} I\left(\mathbf{X};\mathbf{Y}_{k,1:N_k^*}|U^k\right)+I\left(\mathbf{X};\mathbf{Y}_{k,N_k^*+1:N_k}|U^k,\mathbf{Y}_{k,1:N_k^*}\right)+o(\log(\rho))\\
&\overset{(b)}{\geq} I\left(\mathbf{X};\mathbf{Y}_{k,1:N_k^*}|U^k\right)+o(\log(\rho))\\
&\overset{(c)}{=} \sum_{i=1}^{N_k^*} I(\mathbf{X};\mathbf{Y}_{k,i}|U^k,\mathbf{Y}_{k,i+1:N_k^*})+o(\log(\rho))\\
&\overset{(d)}{=} \sum_{i=1}^{N_k^*} I(\mathbf{X};\mathbf{Y}_{k,1}|U^k,\mathbf{Y}_{k,i+1:N_k^*})+o(\log(\rho))\\
&\overset{(e)}{\geq} N_k^* I(\mathbf{X};\mathbf{Y}_{k,1}|U^k,\mathbf{Y}_{k,2:N_k^*})+o(\log(\rho)),
\end{aligned} \tag{3.14}$$

where $\mathbf{Y}_{k,i:j}$ denotes the matrix constructed from the rows $i:j$ of the matrix $\mathbf{Y}_k$. (a), and (c) follow from the chain rule, and (b) follows since mutual information is non-negative. Furthermore, (d) follows from Lemma 1 and (e) follows since removing conditioning does not reduce the entropy. Therefore,

$$I\left(\mathbf{X};\mathbf{Y}_{k,1}|U^k,\mathbf{Y}_{k,2:N_k^*}\right) \leq \frac{r_k}{N_k^*}\log\left(\rho\right)+o(\log(\rho)). \tag{3.15}$$

Furthermore,

$$I\left(\mathbf{X};\mathbf{Y}_k|U^{k-1}\right) \overset{(a)}{=} I\left(\mathbf{X};\mathbf{Y}_{k,1:N_k^*}|U^{k-1}\right)+I\left(\mathbf{X};\mathbf{Y}_{k,N_k^*+1:N_k}|U^{k-1},\mathbf{Y}_{k,1:N_k^*}\right)$$

$$
\begin{aligned}
&\overset{(b)}{=} I\left(\mathbf{X};\mathbf{Y}_{k,1:N^*_{k-1}}|U^{k-1}\right)+I\left(\mathbf{X};\mathbf{Y}_{k,N^*_{k-1}+1:N^*_k}|U^{k-1},\mathbf{Y}_{k,1:N^*_{k-1}}\right)+o(\log(\rho))\\
&\overset{(c)}{=} I\left(\mathbf{X};\mathbf{Y}_{k-1,1:N^*_{k-1}}|U^{k-1}\right)+I\left(\mathbf{X};\mathbf{Y}_{k,N^*_{k-1}+1:N^*_k}|U^{k-1},\mathbf{Y}_{k-1,1:N^*_{k-1}}\right)\\
&\quad + o(\log(\rho))\\
&= r_{k-1}\log(\rho)+\sum_{i=N^*_{k-1}+1}^{N^*_k} I\left(\mathbf{X};\mathbf{Y}_{k,i}|U^{k-1},\mathbf{Y}_{k-1,1:N^*_{k-1}},\mathbf{Y}_{k,i+1:N^*_k}\right)+o(\log(\rho))\\
&\overset{(d)}{\leq} r_{k-1}\log(\rho)+\left(N^*_k-N^*_{k-1}\right)I\left(\mathbf{X};\mathbf{Y}_{k-1,1}|U^{k-1},\mathbf{Y}_{k-1,2:N^*_{k-1}}\right)+o(\log(\rho))\\
&\overset{(e)}{\leq} r_{k-1}\log(\rho)+\left(N^*_k-N^*_{k-1}\right)\frac{r_{k-1}}{N^*_{k-1}}\log(\rho)+o(\log(\rho))\\
&\leq \frac{N^*_k}{N^*_{k-1}}r_{k-1}\log(\rho)+o(\log(\rho)),
\end{aligned}
\tag{3.16}
$$

where (a) and (b) follow from applying the chain rule, and $I\left(\mathbf{X};\mathbf{Y}_{k,N^*_k+1:N_k}|U^{k-1},\mathbf{Y}_{k,1:N^*_k}\right)=o(\log(\rho))$ since more receive antennas than N^*_k does not increase the degrees of freedom [7]. Furthermore, (c) follows since $\mathbf{Y}_{k,1:N^*_{k-1}}$ and $\mathbf{Y}_{k-1,1:N^*_{k-1}}$ are statistically the same. (d) follows from applying Lemma 1 and (e) follows from (3.15). Therefore,

$$
\begin{aligned}
d_1 &\leq N_1\left(1-\frac{M}{T}\right)-r_1,\\
d_k &\leq \frac{N^*_k}{N^*_{k-1}}r_{k-1}-r_k, \quad i\neq 1,K,\\
d_K &\leq \frac{N^*_K}{N^*_{K-1}}r_{K-1}.
\end{aligned}
\tag{3.17}
$$

Hence,

$$
\sum_{i=1}^{K}\frac{d_i}{N^*_i\left(1-\frac{M}{T}\right)}\leq 1+\sum_{i=2}^{K}\frac{r_{k-1}}{N^*_{k-1}\left(1-\frac{M}{T}\right)}-\sum_{i=1}^{K-1}\frac{r_k}{N^*_k\left(1-\frac{M}{T}\right)}=1, \tag{3.18}
$$

where the last inequality follows since the two summations on the right hand side cancel each other. Thus, the degrees of freedom region is bounded by TDMA of the single receiver points $N^*_k\left(1-\frac{M}{T}\right)$, which is maximized to be $N^*_k\left(1-\frac{N^*_k}{T}\right)$, completing the proof of Theorem 1. □

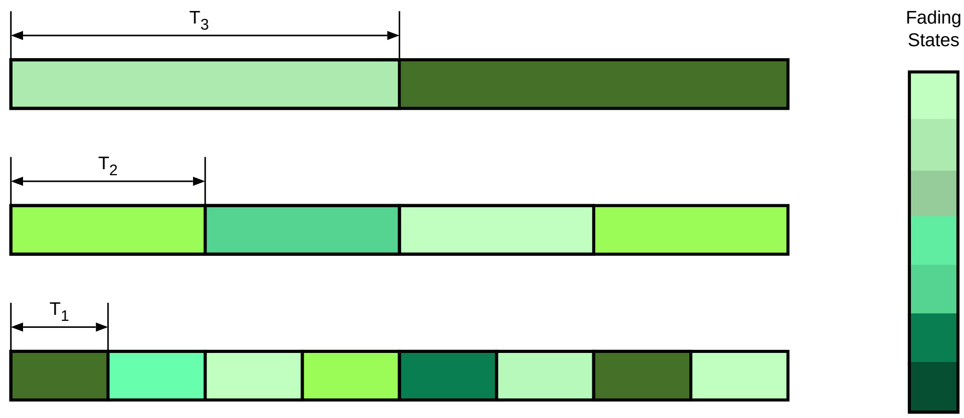

Figure 3.1. Three receivers having aligned coherence times with integer ratio where $T_3 = 2T_2 = 4T_1$.

3.2 Broadcast Channel with Heterogeneous Coherence Times

Consider the K-receiver broadcast channel defined in (3.1) where there is no CSIT or CSIR. Consider the receivers have perfectly aligned coherence times with integer ratio, i.e., $\frac{T_k}{T_{k-1}} \in \mathbb{Z}, \forall k$. Fig. 3.1 denotes three receivers where $T_3 = 2T_2 = 4T_1$. In this system, the receivers are no longer stochastically equivalent, and hence, TDMA inner bound is no longer tight.

3.2.1 Achievability

Theorem 2. *Consider a K-receiver broadcast channel with heterogeneous coherence times and without CSIT or CSIR. The coherence times are perfectly aligned and integer multiples of each other, i.e., $\frac{T_k}{T_{k-1}} \in \mathbb{Z}$. Define $\mathcal{J} \subseteq [1:K]$ to be a set of J receivers ordered ascendingly according to the coherence time length. For $j \in \mathcal{J}$, we can achieve the set of degrees of freedom tuples $\mathcal{D}_1(\mathcal{J})$:*

$$d_j = \begin{cases} N_j^* \left(1 - \frac{N_j^*}{T_j} - \frac{\min\{M, N_{\max}, T_j\} - N_j^*}{T_{j+1}}\right), & j = j_{\min} \\ N_{j_{\min}}^* \min\{M, N_j, T_{j_{\min}}\} \left(\frac{1}{T_{j-1}} - \frac{1}{T_j}\right), & j \neq j_{\min} \end{cases}. \tag{3.19}$$

Furthermore, we can achieve the set of degrees of freedom tuples $\mathcal{D}_2(\mathcal{J})$:

$$d_j = \begin{cases} N_j^*\left(1 - \frac{N_j^*}{T_j}\right), & j = j_{\min} \\ N_{j_{\min}}^* \min\left\{N_j, N_{j_{\min}}^*\right\}\left(\frac{1}{T_{j-1}} - \frac{1}{T_j}\right), & j \neq j_{\min} \end{cases}, \tag{3.20}$$

where $N_j^* = \min\left\{M, N_j, \left\lfloor \frac{T_j}{2} \right\rfloor\right\}$, $N_{\max} = \max_{j\in\mathcal{J}}\{N_j\}$ and $j_{\min}$ is the receiver with the shortest coherence time in $\mathcal{J}$. The achievable degrees of freedom region is the convex hull of the degrees of freedom tuples, $\mathcal{D}_1(\mathcal{J})$ and $\mathcal{D}_2(\mathcal{J})$, over all the possible sets $\mathcal{J} \subseteq [1:K]$, i.e.,

$$\mathcal{D} = \{(d_1, \cdots, d_K) \in Co(\mathcal{D}_1(\mathcal{J}), \mathcal{D}_2(\mathcal{J})), \forall \mathcal{J} \subseteq [1:K]\}. \tag{3.21}$$

Proof. The achievability proof is given in Section 3.3. □

Remark 1. *The receivers of the set $\mathcal{J}$ are ordered ascendingly according to the coherence time length. $J_{\min}$ is the first receiver of $\mathcal{J}$.*

Remark 2. *The two achievable set of degrees of freedom tuples, $\mathcal{D}_1(\mathcal{J})$ and $\mathcal{D}_2(\mathcal{J})$, are achieved by product superposition transmission scheme. The degrees of freedom gains are different in the two sets due to the difference in the number of estimated antennas. In general, none of the two sets includes the other. In the first set, $\mathcal{D}_1(\mathcal{J})$, all the receivers estimate the channel of the maximum number of antennas required for transmission, i.e., receiver j can estimate the channel between N_j^* antennas. In the second set, $\mathcal{D}_2(\mathcal{J})$, the receivers are limited to estimate the channel between N_{i_1} antennas. For more details, the reader can be referred to the achievability proof given in Section 3.3.*

Remark 3. *When the receivers have the same coherence times, the product superposition scheme is not able to achieve degrees of freedom gains. In this case, there is no coherence diversity between the two receivers, hence, the degrees of freedom region is tight against TDMA.*

3.2.2 Outer Bound

Theorem 3. *Consider a K-receiver broadcast channel under heterogeneous coherence times without CSIT or CSIR, meaning that the channel realization is not known, but the channel distribution is globally known. The coherence times are perfectly aligned and integer multiples of each other, i.e., $\frac{T_k}{T_{k-1}} \in \mathbb{Z}$. Define $\mathcal{J} \subseteq [1:K]$ to be a set of J receivers ordered ascendingly according to the coherence time length, if a set of degrees of freedom tuples $(d_1, \cdots, d_K)$ is achievable, then it must satisfy the inequalities*

$$\sum_{j \in \mathcal{J}} \frac{d_j}{N_j^* \left(1 - \frac{N_j^*}{T_{j_{\max}}}\right)} \leq 1, \quad \forall \mathcal{J} \subseteq [1:K], \tag{3.22}$$

where $N_j^* = \min\left\{M, N_j, \left\lfloor \frac{T_j}{2} \right\rfloor\right\}$, and $j_{\max}$ is the receiver with the longest coherence time in $\mathcal{J}$.

Remark 4. *The receivers of the set $\mathcal{J}$ are ordered ascendingly according to the coherence time length, i.e., $\frac{T_k}{T_{k-1}} \in \mathbb{Z}$. $J_{\max}$ is the last receiver of the set, and $T_{j_{\max}}$ is the longest coherence time in the set $\mathcal{J}$.*

Proof. The theorem is proved by showing that for any $\mathcal{J} \subseteq [1:K]$, the degrees of freedom are bounded by the inequality (3.22). First, we show that for the set of receivers $\mathcal{J}$, increasing the coherence time of the receivers to be equal to the longest coherence time, i.e. $T_j = T_{j_{\max}}, \forall j \in \mathcal{J}$ cannot reduce the degrees of freedom. the degrees of freedom region of the enhanced channel includes the original degrees of freedom region. Now, for $\mathcal{J} \subseteq [1:K]$, we show that the degrees of freedom are non-decreasing with the receivers coherence times.

Lemma 2. *For a K-receiver broadcast channel with heterogeneous coherence times and without CSIT or CSIR, define $\mathcal{D}(\mathcal{J})$ to be the degrees of freedom region of a set of receivers $\mathcal{J} \subseteq [1:K]$ where the receivers are ordered ascendingly according to the coherence time length. Define $\overline{\mathcal{D}}(\mathcal{J})$ to be the degrees of freedom region of the same set of* *receivers $\mathcal{J} \subseteq [1:K]$ where the receivers have the coherence time of the longest receiver, i.e., $T_j = T_{j_{\max}}, \forall j \in \mathcal{J}$. Thus, we have*

$$\mathcal{D}(\mathcal{J}) \subseteq \overline{\mathcal{D}}(\mathcal{J}) \tag{3.23}$$

Proof. Consider the set of receivers $\mathcal{J} \subseteq [1:K]$ where the receivers are ordered ascendingly according to the coherence time length, i.e., $T_j \geq T_{j-1}, \forall j \in \mathcal{J}$. The proof consists of two steps. First, we show that the individual degrees of freedom of each receiver is nondecreasing with the increase of the coherence time of this receiver. Second, we show that the degrees of freedom region of the channel is nondecreasing with the increase of the coherence time of the receivers. For the first step of the proof, we introduce the following Lemma.

Lemma 3. *For the broadcast channel considered in Section 3.2, define $\mathcal{J} = \{i_1, \cdots, i_J\} \subseteq [1:K]$ with $T_j \geq T_{j-1}, \forall j \in \mathcal{J}$ and Ψ_j as the message of receiver $j \in \mathcal{J}$. Thus, we have*

$$N_j^* \left(\frac{1}{J} - \frac{N_j^*}{T_j} \right) \leq MG \left\{ \frac{1}{\bar{n}} I\left(\Psi_j; \mathbf{Y}_j^n\right) \right\} \leq N_j^* \left(1 - \frac{N_j^*}{T_j} \right), \tag{3.24}$$

where $MG(x)$ is the multiplexing gain of a function $x(\rho)$ of ρ and defined as

$$MG(x) = \lim_{\rho \to \infty} \sup \frac{x(\rho)}{\log(\rho)}. \tag{3.25}$$

Proof. We first prove the right inequality of (3.24). We have

$$\begin{aligned} \mathrm{MG} \left\{ \frac{1}{\bar{n}} I\left(\Psi_j; \mathbf{Y}_j^n\right) \right\} &\overset{(a)}{\leq} \mathrm{MG} \left\{ \frac{1}{\bar{n}} I\left(\mathbf{X}^n; \mathbf{Y}_j^n\right) \right\} \\ &\overset{(b)}{\leq} N_j^* \left(1 - \frac{N_j^*}{T_j} \right), \end{aligned} \tag{3.26}$$

where (a) follows from the data processing inequality and (b) follows from the single-receiver results [7]. Next, we show the left inequality of (3.24). Assume that we have the following transmitted sequence

$$\overline{\mathbf{X}}^n = \left[\mathbf{U}_{i_1}, \cdots, \mathbf{U}_{i_J}\right], \tag{3.27}$$

where $\mathbf{U}_j \in \mathbb{C}^{N_j \times \frac{\bar{n}}{J}}$ is the matrix containing the signal of receiver $j \in \mathcal{J}$ and the matrix is constructed to be on the form of the optimal input of a non-coherent single receiver [7]. Hence,

$$\mathrm{MG}\left\{\frac{1}{\bar{n}} I\left(\Psi_j; \mathbf{Y}_j^n\right)\right\} \geq \mathrm{MG}\left\{\frac{1}{\bar{n}} I\left(U_j; \mathbf{Y}_j^n\right)\right\} \geq \frac{N_j^*}{J}\left(1-\frac{N_j^*}{T_j}\right) \geq N_j^*\left(\frac{1}{J}-\frac{N_j^*}{T_j}\right). \quad (3.28)$$

Thus, the proof of Lemma 3 is completed. □

By Lemma 3, we have lower and outer bounds which are increasing with T_j. Furthermore, the difference between the two bounds is

$$\Delta = N_j^*\left(1-\frac{N_j^*}{T_j}\right) - N_j^*\left(\frac{1}{J}-\frac{N_j^*}{T_j}\right) = N_j^* \frac{J-1}{J}. \quad (3.29)$$

Therefore, $\mathrm{MG}\left\{\frac{1}{\bar{n}} I\left(\Psi_j; \mathbf{Y}_j^n\right)\right\}$ is nondecreasing with the increase of T_j, which completes the first step of the proof of Lemma 2.

Now, we give the second part of the proof via a contradiction argument. Define $\mathcal{D}$ to be the degrees of freedom region of a set of receivers with unequal coherence times where $\max_j T_j = T_{\max}$, and $\mathbf{Y}_j^n$ denotes the received signal at receiver j. Define $\overline{\mathcal{D}}$ to be the degrees of freedom region of the receivers where the coherence time of all receivers is $T_{\max}$, where $\bar{\mathbf{Y}}_j^n$ denotes the received signal at receiver j of this enhanced channel. Define $\tilde{D} \in \mathcal{D}$ to be a degrees of freedom tuple, and $d_j \in \tilde{D}$ is the degrees of freedom of receiver j. Assume that $\tilde{D} \notin \overline{\mathcal{D}}$. By Fano's inequality,

$$\begin{aligned} d_j \leq & \mathrm{MG}\left\{\frac{1}{\bar{n}} I\left(\Psi_j; \mathbf{Y}_j^n\right)\right\}, \\ \leq & \mathrm{MG}\left\{\frac{1}{\bar{n}} I\left(\Psi_j; \bar{\mathbf{Y}}_j^n\right)\right\}, \end{aligned} \quad (3.30)$$

where Ψ_j is the message of receiver $j \in \mathcal{J}$, and the last inequality follows from Lemma 3. Therefore, $d_j \in \overline{\mathcal{D}}, \forall j$, which contradicts the initial assumption completing the second part of the proof. □

Using Lemma 2, the degrees of freedom region for every set of receivers $\mathcal{J} \subseteq [1:K]$ is included in the degrees of freedom region of an enhanced channel with identical coherence times of length $T_{j_{\max}}$ slots. Furthermore, Theorem 1 shows that the degrees of freedom region of the enhanced channel is tight against TDMA inner bound. Thus, we obtain the region in (3.22), and hence, the proof of Theorem 3 is completed. □

3.2.3 Optimality

For four cases, the achievable degrees of freedom region in Section 3.2.1 and the outer degrees of freedom region obtained in Section 3.2.2 are tight. In the four cases, the coherence time is at least twice the number of transmit and receive antennas, i.e., $T_j \geq 2\max\{M, N_j\}$.

The transmitter has fewer antennas than receivers

When $M \leq \min_j \{N_j\}$, the outer degrees of freedom region given by (3.22) is

$$\sum_{j\in\mathcal{J}} d_j \leq M\left(1 - \frac{M}{T_{j_{\max}}}\right), \quad \forall \mathcal{J} \subseteq [1:K]. \tag{3.31}$$

The achievable degrees of freedom tuples in (3.20) are

$$d_j = \begin{cases} M\left(1 - \frac{M}{T_j}\right), & j = j_{\min} \\ M^2\left(\frac{1}{T_{j-1}} - \frac{1}{T_j}\right), & j \neq j_{\min} \end{cases}, \quad j \in \mathcal{J}. \tag{3.32}$$

Hence,

$$\begin{aligned} \sum_{j\in\mathcal{J}} d_j &= M\left(1 - \frac{M}{T_{j_{\min}}}\right) + \sum_{j\in\mathcal{J}, j\neq j_{\min}} M^2\left(\frac{1}{T_{j-1}} - \frac{1}{T_j}\right) \\ &\overset{(a)}{=} M\left(1 - \frac{M}{T_{j_{\min}}}\right) + M^2\left(\frac{1}{T_{j_{\min}}} - \frac{1}{T_{j_{\max}}}\right) = M\left(1 - \frac{M}{T_{j_{\max}}}\right), \end{aligned} \tag{3.33}$$

where (a) follows from the telescoping sum. Thus, the achievable degrees of freedom tuples are at the boundaries of the outer degrees of freedom region, consequently, the convex hull of the achievable degrees of freedom tuples is tight against the outer degrees of freedom region.

The receivers have the same number of antennas

When $N_k = N, \forall k$, the outer degrees of freedom region given in (3.22) is

$$\sum_{j\in\mathcal{J}} d_j \leq N^* \left(1 - \frac{N^*}{T_{j_{\max}}}\right), \quad \mathcal{J} \subseteq [1:K]\,. \tag{3.34}$$

The achievable degrees of freedom tuples in (3.20) are

$$d_j = \begin{cases} N^* \left(1 - \frac{N^*}{T_j}\right), & j = j_{\min} \\ {N^*}^2 \left(\frac{1}{T_{j-1}} - \frac{1}{T_j}\right), & j \neq j_{\min} \end{cases}, \quad j \in \mathcal{J}. \tag{3.35}$$

Hence,

$$\begin{aligned} \sum_{j\in\mathcal{J}} d_j &= N^* \left(1 - \frac{N^*}{T_{j_{\min}}}\right) + \sum_{j\in\mathcal{J}, j\neq j_{\min}} {N^*}^2 \left(\frac{1}{T_{j-1}} - \frac{1}{T_j}\right) \\ &\overset{(a)}{=} N^* \left(1 - \frac{N^*}{T_{j_{\min}}}\right) + {N^*}^2 \left(\frac{1}{T_{j_{\min}}} - \frac{1}{T_{j_{\max}}}\right) = N^* \left(1 - \frac{N^*}{T_{j_{\max}}}\right). \end{aligned} \tag{3.36}$$

The achievable degrees of freedom tuples are at the boundaries of the outer degrees of freedom region, thus the outer degrees of freedom region is tight.

One receiver has very short coherence time

When $T_j \gg T_1$, where $j = 2, \cdots, K$, the outer region given in (3.22) is

$$\begin{aligned} d_1 &\leq N_1^* \left(1 - \frac{N_1^*}{T_1}\right), \\ \sum_{j\in\mathcal{J}} \frac{d_j}{N_j^*} &\leq 1, \quad \mathcal{J} \subseteq [1:K]\,. \end{aligned} \tag{3.37}$$

The achievable degrees of freedom tuples in (3.19), $\mathcal{D}_1(\mathcal{J})$, are

$$d_j = \begin{cases} N^* \left(1 - \frac{N^*}{T_j}\right), & j = j_{\min} \\ \frac{N_{j_{\min}}^* N_j^*}{T_{j-1}}, & j \neq j_{\min} \end{cases}, \quad j \in \mathcal{J}. \tag{3.38}$$

Therefore,

$$\begin{aligned}\sum_{j\in\mathcal{J}} \frac{d_j}{N_j^*} &\approx 1 - \frac{N_{j_{\min}}^*}{T_{j_{\min}}} + \frac{N_{j_{\min}}^*}{T_{j_{\min}}} \\ &= 1, \end{aligned} \tag{3.39}$$

and the achievable degrees of freedom region is tight.

The receivers have identical coherence time

In the case of identical coherence times, we showed in Section 3.1 that the degrees of freedom region is tight against TDMA. When $T_k = T, \forall k$, the outer region given in (3.22) is

$$\sum_{j\in\mathcal{J}} \frac{d_j}{N_j^*\left(1-\frac{N_j^*}{T}\right)} \leq 1, \quad \forall \mathcal{J} \subseteq [1:K], \tag{3.40}$$

which is the same as the TDMA degrees of freedom region. In this case, the achievable degrees of freedom tuples in (3.20), $\mathcal{D}_2(\mathcal{J})$, are reduced to TDMA.

3.2.4 Numerical Examples

Consider a single-antenna two-receiver broadcast channel, i.e. $M = N_1 = N_2 = 1$ with coherence times $T_1 = 2$ and $T_2 = 4$ slots. Thus, in this case, we have four possibilities of $\mathcal{J}$: $\{\}, \{1\}, \{2\}, \{1,2\}$. According to Theorem 3, the outer degrees of freedom region is given by

$$\begin{aligned} d_1 &\leq \frac{1}{2}, \\ d_1 + d_2 &\leq \frac{3}{4}. \end{aligned}$$

The achievable degrees of freedom tuples

$$\mathcal{D}_1(\mathcal{J}) = \mathcal{D}_2(\mathcal{J}) : (0,0), \left(\frac{1}{2}, 0\right), \left(0, \frac{3}{4}\right) \left(\frac{1}{2}, \frac{1}{4}\right). \tag{3.41}$$

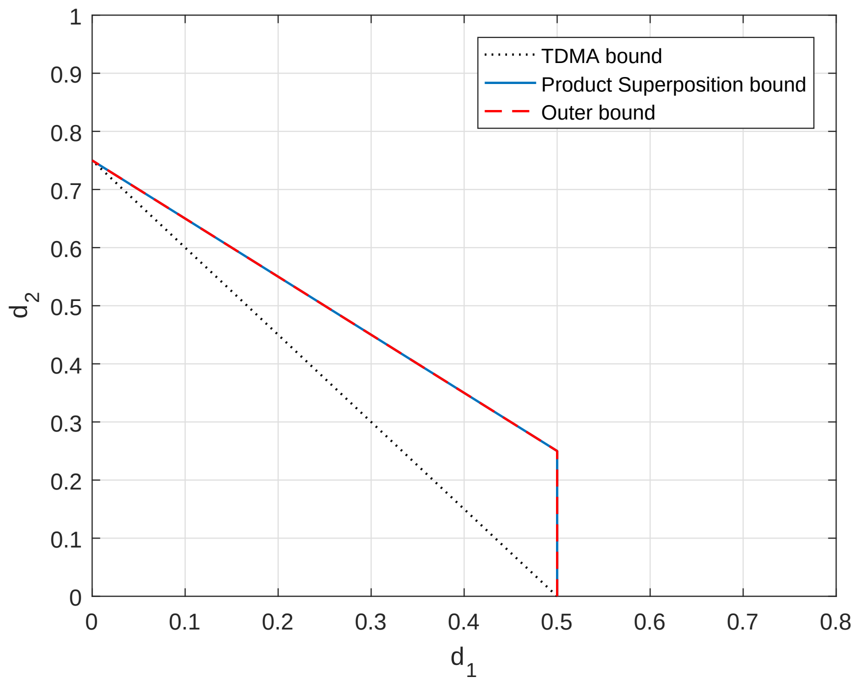

Figure 3.2. Degrees of freedom region of a two-receiver broadcast channel with heterogeneous coherence times where $M = N_1 = N_2 = 1, T_1 = 2, T_2 = 4$.

As shown in Fig. 3.2, the outer and the achievable regions coincide on each other.

For a two-receiver broadcast channel with $M = 2, N_1 = 1, N_2 = 3$, and $T_1 = 4, T_2 = 24$, the outer degrees of freedom is given by

$$d_1 \leq \frac{18}{24},$$
$$\frac{d_1}{23/24} + \frac{d_2}{44/24} \leq 1.$$

Furthermore,

$$\mathcal{D}_1(\mathcal{J}) : (0,0), \left(\frac{18}{24}, 0\right), \left(0, \frac{44}{24}\right), \left(\frac{17}{24}, \frac{10}{24}\right),$$
$$\mathcal{D}_2(\mathcal{J}) : (0,0), \left(\frac{18}{24}, 0\right), \left(0, \frac{44}{24}\right), \left(\frac{18}{24}, \frac{5}{24}\right).$$

Fig. 3.3 shows the gap between the outer and the achievable bounds.

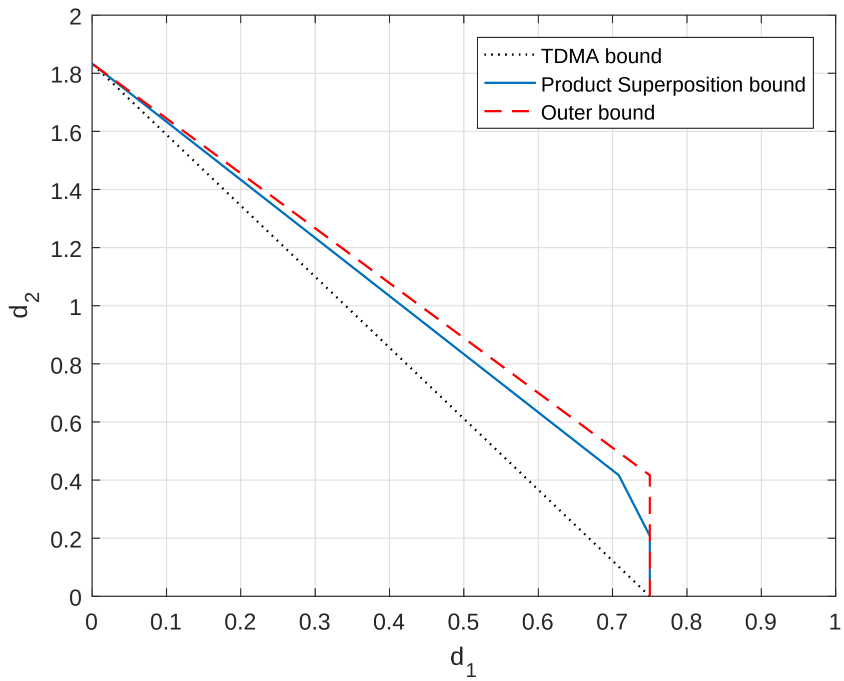

Figure 3.3. Degrees of freedom region of a two-receiver broadcast channel with heterogeneous coherence times where $M = 2, N_1 = 1, N_2 = 3, T_1 = 4, T_2 = 24$.

Furthermore, for a two-receiver broadcast channel with $M = 2, N_1 = 1, N_2 = 3$ and $T_1 = 4$ and $T_2 = 40$, the outer degrees of freedom region is given by

$$d_1 \leq \frac{30}{40},$$
$$\frac{d_1}{39/40} + \frac{d_2}{76/40} \leq 1.$$

For the achievable region in Theorem 2,

$$\mathcal{D}_1(\mathcal{J}) : (0,0), \left(\frac{30}{40}, 0\right), \left(0, \frac{76}{40}\right), \left(\frac{30}{40}, \frac{9}{40}\right)$$
$$\mathcal{D}_2(\mathcal{J}) : \left(\frac{12}{16}, 0\right), \left(0, \frac{28}{16}\right), \left(\frac{29}{40}, \frac{18}{40}\right).$$

Fig. 3.4 shows the gap between the achievable and the outer regions which is decreased with the increase of the ratio between the coherence times, $\frac{T_2}{T_1}$.

Now consider a three-receiver broadcast channel with $M = 4, N_1 = N_2 = N_3 = 2$ and $T_1 = 6, T_2 = 18, T_3 = 54$. When the receivers have the same number of antennas, as discussed

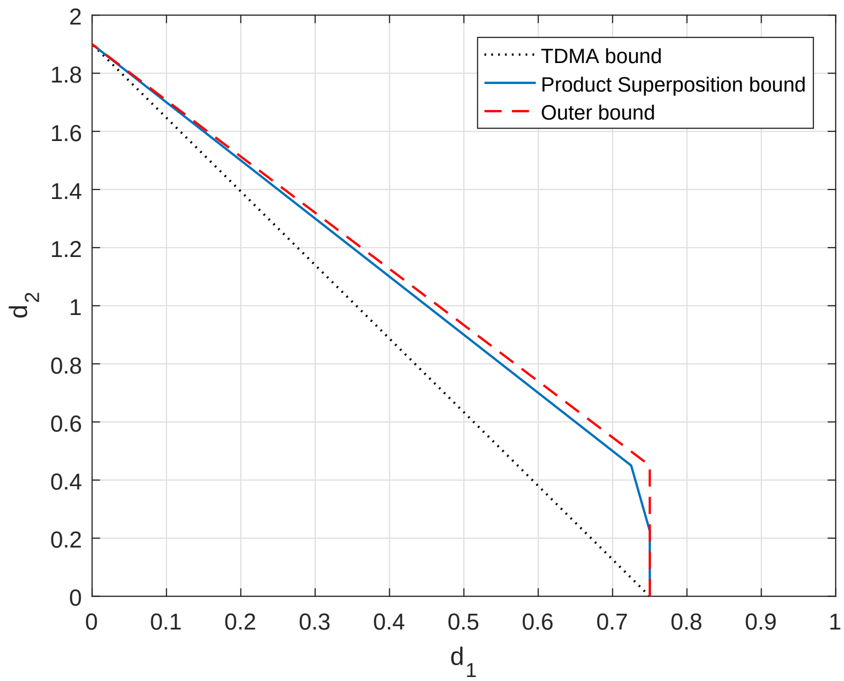

Figure 3.4. Degrees of freedom region of a two-receiver broadcast channel with heterogeneous coherence times where $M = 2, N_1 = 1, N_2 = 3, T_1 = 4, T_2 = 40$.

in Section 3.2.3, the achievable degrees of freedom and outer regions are tight. The outer degrees of freedom region is

$$\begin{aligned} d_1 &\leq \frac{5}{6}, \\ \frac{d_1}{17/18} + \frac{d_2}{32/18} &\leq 1, \\ \frac{d_1}{53/54} + \frac{d_2}{104/54} + \frac{d_3}{153/54} &\leq 1. \end{aligned}$$

For the achievable degrees of freedom region, we have 8 possibilities for $\mathcal{J}$:

$$\{\}, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}.$$

Hence,

$$\mathcal{D}_1(\mathcal{J}) : (0,0,0), \left(\frac{5}{6},0,0\right), \left(0,\frac{32}{18},0\right), \left(0,0,\frac{153}{154}\right), \left(\frac{14}{18},\frac{4}{18},0\right), \left(\frac{43}{54},0,\frac{24}{54}\right),$$
$$\left(0,\frac{94}{54},\frac{12}{54}\right), \left(\frac{13}{18},\frac{4}{18},\frac{6}{54}\right).$$

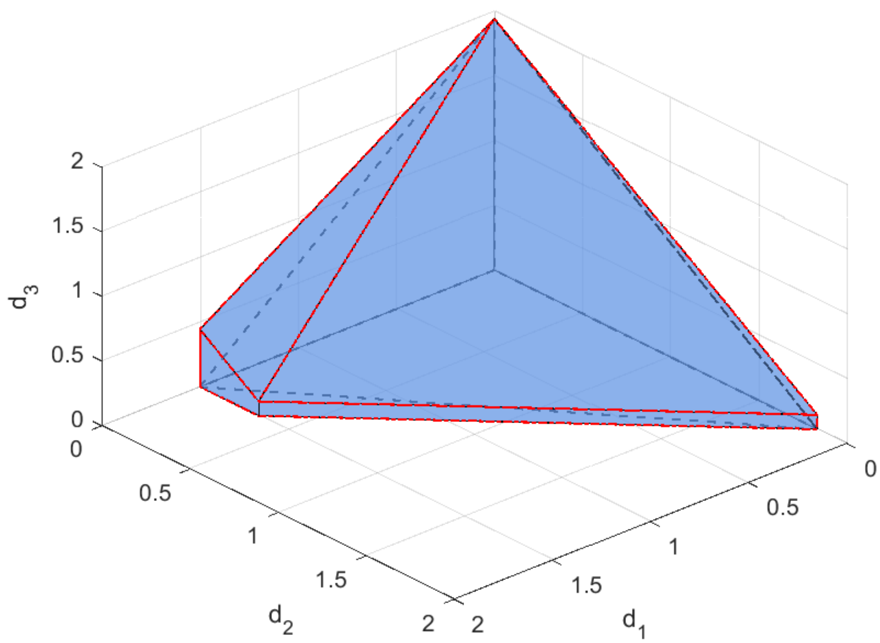

Figure 3.5. Degrees of freedom region of a three-receiver broadcast channel with heterogeneous coherence times where $M = 4, N_1 = N_2 = N_3 = 2, T_1 = 8, T_2 = 24, T_3 = 72$.

$$\mathcal{D}_2\left(\mathcal{J}\right) : (0,0,0)\,, \left(\frac{5}{6},0,0\right), \left(0,\frac{32}{18},0\right), \left(0,0,\frac{153}{154}\right), \left(\frac{5}{6},\frac{2}{18},0\right), \left(\frac{5}{6},0,\frac{8}{54}\right),$$
$$\left(0,\frac{32}{18},\frac{8}{54}\right), \left(\frac{5}{6},\frac{2}{18},\frac{2}{54}\right).$$

Fig. 3.5 shows the achievable degrees of freedom region (denoted by blue), the TDMA achievable region (denoted by black), and furthermore, the tight outer degrees of freedom region (denoted by red).

3.3 Proof for Theorem 2

Achievable rates under coherence diversity for a general K-receiver broadcast channel are attained by finding the best opportunities to re-use certain slots. Because the number of such opportunities blows up with K, the central idea and intuition behind finding such opportunities are not easily visible in the general case of K receivers, where the achievable rates are eventually described via an inductive process. To highlight the ideas and the intuition in the achievable rate methodology, we develop these ideas in the special case of 3 receivers, which is the smallest number of receivers where the full richness of these interactions manifest themselves. We then proceed to describe the K-receiver result in its full generality.

3.3.1 Achievability for Three Receivers

In the case of three receivers we have 8 possible receivers sets $\mathcal{J}$: one empty set $\{\}$ achieving the trivial degrees of freedom tuple $(0,0,0)$, three single receiver sets: $\{1\},\{2\},\{3\}$, three two-receiver sets: $\{1,2\},\{1,3\},\{2,3\}$, and one three-receiver set $\{1,2,3\}$. In the sequel, we first show the achievability of $\mathcal{D}_1(\mathcal{J})$ and after that we give the achievability of $\mathcal{D}_2(\mathcal{J})$.

$\mathcal{D}_1(\mathcal{J})$ achievability

For the three single-receiver sets, we can achieve the three degrees of freedom tuples

$$\left(N_1^*\left(1-\frac{N_1^*}{T_1}\right),0,0\right),\left(0,N_2^*\left(1-\frac{N_2^*}{T_2}\right),0\right),\left(0,0,N_3^*\left(1-\frac{N_3^*}{T_3}\right)\right), \tag{3.42}$$

by serving only one receiver while the other receivers remain unserved. In particular, for receiver $k=1,2,3$, every T_k slots, a training sequence is sent during N_k^* slots and then data for receiver k is sent during the remaining $T_k-N_k^*$ slots. $N_k^*\left(1-\frac{N_k^*}{T_k}\right)$ is achieved for receiver k whereas the other receivers achieve zero degrees of freedom.

For the three two-receiver sets, two receivers are being served while the third receiver remains unserved. Using product superposition for two receivers, the degrees of freedom tuples are

$$\Bigg(N_1^*\left(1-\frac{N_1^*}{T_1}\right)-\frac{N_1^*\left(\min\left\{M,\max\{N_1,N_2\},T_1\right\}-N_1^*\right)}{T_2},N_1^*\min\{M,N_2,T_1\}\left(\frac{1}{T_1}-\frac{1}{T_2}\right),$$
$$0\Bigg), \tag{3.43}$$

$$\left(N_1^*\left(1-\frac{N_1^*}{T_1}\right)-\frac{N_1^*\left(\min\left\{M,\max\{N_1,N_3\},T_1\right\}-N_1^*\right)}{T_3},0,\right.$$

$$\left.N_1^*\min\{M,N_3,T_1\}\left(\frac{1}{T_1}-\frac{1}{T_3}\right)\right), \tag{3.44}$$

$$\left(0,N_2^*\left(1-\frac{N_2^*}{T_2}\right)-\frac{N_2^*\left(\min\left\{M,\max\{N_2,N_3\},T_2\right\}-N_2^*\right)}{T_3},\right.$$

$$\left.N_2^*\min\{M,N_3,T_2\}\left(\frac{1}{T_2}-\frac{1}{T_3}\right)\right). \tag{3.45}$$

To achieve (3.43), product superposition transmission is as follows.

- Every T_2-length interval is divided into T_1-length subintervals.

- During the first subinterval, training is sent during $\min\big\{M,\max\{N_1,N_2\},T_1\big\}$ slots for receiver 1 and receiver 2 channel estimation. After that, data for receiver 1 is sent during the following $T_1-\min\left\{M,\max\{N_1,N_2\},T_1\right\}$ slots. Receiver 1 achieves $N_1^*\left(T_1-\min\left\{M,\max\{N_1,N_2\},T_1\right\}\right)$ degrees of freedom.

- During the remaining subintervals, the transmitter sends, every T_1 slots,

$$\mathbf{X}_i^{(12)}=\left[\mathbf{V}_i,\ \mathbf{V}_i\mathbf{U}_i\right],\quad i=1,\cdots,\frac{T_2}{T_1}-1, \tag{3.46}$$

  where $\mathbf{U}_i\in\mathbb{C}^{N_1^*\times\left(T_1-N_1^*\right)},\mathbf{V}_i\in\mathbb{C}^{M\times N_1^*}$ are data matrices for receiver 1, and receiver 2, respectively. Thus, receiver 1 estimates its equivalent channel $\overline{\mathbf{H}}_{1,i}=\mathbf{H}_{1,i}\mathbf{V}_i$, and decodes $\mathbf{U}_i$ achieving $\left(\frac{T_2}{T_1}-1\right)N_1^*\left(T_1-N_1^*\right)$ degrees of freedom. Furthermore, the channel of receiver 2 remains constant and known, hence, $\mathbf{V}_i$ can be decoded coherently at receiver 2 achieving $\left(\frac{T_2}{T_1}-1\right)N_1^*\min\{M,N_2,T_1\}$ degrees of freedom since when $N_2\geq T_1$, receiver 2 estimates only T_1 antennas during the first subinterval.

Thus, by the above product superposition scheme, for every T_2 slots, receiver 1 achieves

$$\frac{T_2}{T_1}N_1^*\left(T_1-N_1^*\right)-N_1^*\left(\min\left\{M,\max\{N_1,N_2\},T_1\right\}-N_1^*\right)$$

degrees of freedom, and furthermore, receiver 2 achieves $\left(\frac{T_2}{T_1}-1\right)N_1^*\min\{M,N_2,T_1\}$ degrees of freedom obtaining (3.43).

For achieving (3.44), product superposition transmission is as follows.

- Every T_3-length interval is divided into T_1-length subintervals.
- During the first subinterval, training is sent during $\min\big\{M,\max\{N_1,N_3\},T_1\big\}$ slots for receiver 1 and receiver 3 channel estimation. After that, data for receiver 1 is sent during the following $T_1-\min\{M,\max\{N_1,N_3\},T_1\}$ slots. Receiver 1 achieves $N_1^*\left(T_1-\min\{M,\max\{N_1,N_3\},T_1\}\right)$ degrees of freedom.
- During the remaining subintervals, every T_1 slots, the transmitted signal is similar to (3.46), yet, after replacing $\mathbf{V}_i$ with $\mathbf{W}_i$ which contains data for receiver 3. Receiver 1, and receiver 3 achieve $\left(\frac{T_3}{T_1}-1\right)N_1^*\left(T_1-N_1^*\right)$, and $\left(\frac{T_3}{T_1}-1\right)N_1^*\min\{M,N_3,T_1\}$ degrees of freedom, respectively.

Thus, for every T_3 slots, receiver 1 achieves

$$\frac{T_3}{T_1}N_1^*\left(T_1-N_1^*\right)-N_1^*\left(\min\{M,\max\{N_1,N_3\},T_1\}-N_1^*\right)$$

degrees of freedom, and furthermore, receiver 3 achieves $\left(\frac{T_3}{T_1}-1\right)N_1^*\min\{M,N_3,T_1\}$ degrees of freedom obtaining (3.44).

Similar to the degrees of freedom tuples (3.43) and (3.44), we can achieve (3.45) by the same transmission strategy, yet, with respect to T_2 and T_3. Thus, every T_3 slots, receiver 2 achieves

$$\frac{T_3}{T_2}N_2^*\left(T_2-N_2^*\right)-N_2^*\left(\min\{M,\max\{N_2,N_3\},T_2\}-N_2^*\right)$$

degrees of freedom, and furthermore, receiver 3 achieves $\left(\frac{T_3}{T_2}-1\right)N_2^*\min\{M,N_3,T_2\}$ degrees of freedom.

Now the remaining degrees of freedom tuple is the one with the three-receiver set $\{1,2,3\}$. In this case, the achievable degrees of freedom tuples are

$$\left(N_1^*\left(1-\frac{N_1^*}{T_1}\right)-\frac{N_1^*\left(\min\{M,\max\{N_1,N_2,N_3\},T_1\}-N_1^*\right)}{T_2},\right.$$
$$\left.N_1^*\min\{M,N_2,T_1\}\left(\frac{1}{T_1}-\frac{1}{T_2}\right),N_1^*\min\{M,N_3,T_1\}\left(\frac{1}{T_2}-\frac{1}{T_3}\right)\right),$$

which can be achieved by product superposition as follows.

- Every T_3-length interval is divided into T_2-length subintervals.
- During the first subinterval, the transmitted signal is the same as that used to achieve (3.43). Thus, receiver 1 achieves
$$\frac{T_2}{T_1}N_1^*\left(T_1-N_1^*\right)-N_1^*\left(\min\{M,\max\{N_1,N_2\},T_1\}-N_1^*\right) \tag{3.47}$$
degrees of freedom, receiver 2 achieves $\left(\frac{T_2}{T_1}-1\right)N_1^*\min\{N_2^*,T_1\}$ degrees of freedom, and furthermore, receiver 3 estimates its channel.
- During the remaining $\left(\frac{T_3}{T_2}-1\right)$ subintervals, the transmitter sends, every T_2-length subinterval, the same signal that achieves (3.43) after multiplying it from the left by $\mathbf{W}_i$ which contains data for receiver 3. Therefore, during the first T_1 of every T_2-length subinterval, the transmitted signal is
$$\mathbf{X}^{(123)}=\left[\mathbf{W}_i,\ \mathbf{W}_i\mathbf{U}_i\right]. \tag{3.48}$$
After that during $\left(\frac{T_2}{T_1}-1\right)T_1$ slots, the transmitted signal is
$$\widetilde{\mathbf{X}}^{(123)}=\left[\mathbf{W}_i\mathbf{V}_i,\ \mathbf{W}_i\mathbf{V}_i\mathbf{U}_i\right]. \tag{3.49}$$
receiver 1 estimates the equivalent channel $\overline{\overline{\overline{\mathbf{H}}}}_{1,i}=\mathbf{H}_{1,i}\mathbf{W}_i\mathbf{V}_i$, and decodes $\mathbf{U}_i$, receiver 2 estimates $\mathbf{H}_{2,i}=\mathbf{H}_{2,i}\mathbf{W}_i$, and decodes $\mathbf{V}_i$ and receiver 3 decodes $\mathbf{W}_i$. Thus, receiver 1, receiver 2, and receiver 3 achieve $\left(\frac{T_3}{T_2}-1\right)\frac{T_2}{T_1}N_1^*\left(T_1-N_1^*\right)$, $\left(\frac{T_3}{T_2}-1\right)\left(\frac{T_2}{T_1}-1\right)N_1^*\min\{M,N_2,T_1\}$, and $\left(\frac{T_3}{T_2}-1\right)N_1^*\min\{M,N_3,T_1\}$ degrees of freedom, respectively.

Therefore, by using the above product superposition transmission scheme, we can achieve the above degrees of freedom tuple.

$\mathcal{D}_2(\mathcal{J})$ achievability

Similar to $\mathcal{D}_1(\mathcal{J})$, we can achieve the degrees of freedom tuples (3.42) that correspond to the three single-receiver sets by serving only one receiver while the other receivers remain unserved.

For the three two-receiver sets, only two receivers are being served while the third receiver remains unserved. Using product superposition for two receivers, the degrees of freedom tuples

$$\left(N_1^*\left(1-\frac{N_1^*}{T_1}\right), N_1^*\min\left\{N_1^*, N_2\right\}\left(\frac{1}{T_1}-\frac{1}{T_2}\right), 0\right), \tag{3.50}$$

$$\left(N_1^*\left(1-\frac{N_1^*}{T_1}\right), 0, N_1^*\min\left\{N_1^*, N_3\right\}\left(\frac{1}{T_1}-\frac{1}{T_3}\right)\right), \tag{3.51}$$

$$\left(0, N_2^*\left(1-\frac{N_2^*}{T_2}\right), N_2^*\min\left\{N_2^*, N_3\right\}\left(\frac{1}{T_2}-\frac{1}{T_3}\right)\right), \tag{3.52}$$

are achieved as follows. To achieve (3.50), the product superposition transmission strategy is as follows.

- Every T_2-length interval is divided into T_1-length subintervals.
- During the first T_1-length subinterval, a training sequence is sent during N_1^* slots and data for receiver 1 is sent during the following $T_1 - N_1^*$ slots. Receiver 1 achieves $N_1^*(T_1 - N_1^*)$ degrees of freedom, and receiver 2 estimates the channel of $\min\{N_1^*, N_2\}$ transmit antennas
- During the remaining subintervals, every T_1 slots, the transmitter sends

  $$\mathbf{X}_i^{(12)} = [\mathbf{V}_i,\ \mathbf{V}_i\mathbf{U}_i], \quad i = 1, \cdots, \frac{T_2}{T_1} - 1, \tag{3.53}$$

  Thus, receiver 1, and receiver 2 achieve $\left(\frac{T_2}{T_1}-1\right)N_1^*(T_1 - N_1^*)$, and $\left(\frac{T_2}{T_1}-1\right)N_1^*\min\{N_1^*, N_2\}$ degrees of freedom, respectively.

Thus, by the above product superposition scheme, for every T_2 slots, receiver 1, and receiver 2 achieve $\frac{T_2}{T_1} N_1^* \left(T_1 - N_1^*\right)$, and $\left(\frac{T_2}{T_1} - 1\right) N_1^* \min\left\{N_1^*, N_2\right\}$ degrees of freedom, respectively obtaining (3.50).

For achieving (3.51) by product superposition, we use the same transmission scheme of achieving (3.50) with respect to receiver 1 and receiver 3, i.e. replacing $T_2, \min\left\{N_1^*, N_2\right\}$ with $T_3, \min\left\{N_1^*, N_3\right\}$, respectively. Thus, receiver 1, and receiver 3 achieve $\frac{T_3}{T_1} N_1^* \left(T_1 - N_1^*\right)$, and $\left(\frac{T_3}{T_1} - 1\right) N_1^* \min\left\{N_1^*, N_3\right\}$ degrees of freedom, respectively, for every T_3 slots. Similarly, we can achieve (3.52) by the same transmission strategy, yet, with respect to T_2 and T_3.

For the three-receiver set, the achievable degrees of freedom tuples are

$$\left(N_1^* \left(1 - \frac{N_1^*}{T_1}\right), N_1^* \min\left\{N_1^*, N_2\right\} \left(\frac{1}{T_1} - \frac{1}{T_2}\right), N_1^* \min\left\{N_1^*, N_3\right\} \left(\frac{1}{T_2} - \frac{1}{T_3}\right)\right),$$

which can be achieved by product superposition for the three receivers as follows.

- Every T_3-length interval is divided into T_2-length subintervals.

- During the first subinterval, the transmitted signal is the same as that used to achieve (3.50). Thus, receiver 1, and receiver 2 achieve $\frac{T_2}{T_1} N_1^* \left(T_1 - N_1^*\right)$, and $\left(\frac{T_2}{T_1} - 1\right) N_1^* \min\left\{N_1^*, N_2\right\}$ degrees of freedom, respectively, and furthermore, receiver 3 estimates its channel between $\min\left\{N_1^*, N_3\right\}$ transmit antennas.

- During the remaining subintervals, the transmitter sends, every T_2-length subinterval, the same signal that achieves (3.50) after multiplying it from the left by $\mathbf{W}_i$ which contains data for receiver 3. Therefore, during the first T_1 of every T_2-length subinterval, the transmitted signal is

  $$\mathbf{X}^{(123)} = \left[\mathbf{W}_i,\ \mathbf{W}_i \mathbf{U}_i\right]. \tag{3.54}$$

  After that during $\left(\frac{T_2}{T_1} - 1\right) T_1$ slots, the transmitted signal is

  $$\widetilde{\mathbf{X}}^{(123)} = \left[\mathbf{W}_i \mathbf{V}_i,\ \mathbf{W}_i \mathbf{V}_i \mathbf{U}_i\right]. \tag{3.55}$$

Thus receiver 1 can estimate the equivalent channel $\overline{\overline{\overline{\mathbf{H}}}}_{1,i} = \mathbf{H}_{1,i}\mathbf{W}_i\mathbf{V}_i$, and decode $\mathbf{U}_i$. Also, receiver 2 can estimate the equivalent channel $\overline{\overline{\mathbf{H}}}_{2,i} = \mathbf{H}_{2,i}\mathbf{W}_i$, and decode $\mathbf{V}_i$ and furthermore, receiver 3 can decode $\mathbf{W}_i$. Receiver 1, receiver 2, and receiver 3 achieve $\left(\frac{T_3}{T_2} - 1\right)\frac{T_2}{T_1}N_1^*\left(T_1 - N_1^*\right)$, $\left(\frac{T_3}{T_2} - 1\right)\left(\frac{T_2}{T_1} - 1\right)N_1^*\min\left\{N_1^*, N_2\right\}$, and $\left(\frac{T_3}{T_2} - 1\right)N_1^*\min\left\{N_1^*, N_3\right\}$ degrees of freedom, respectively.

Therefore, the above degrees of freedom tuples is obtained.

3.3.2 Achievability for K Receivers

To obtain the achievability for the K-receiver case, we show that for every set of receivers $\mathcal{J} \subseteq [1:K]$, ordered ascendingly according to the coherence time length, the degrees of freedom tuples $\mathcal{D}_1(\mathcal{J})$ and $\mathcal{D}_2(\mathcal{J})$ are achievable. We use an induction argument in our proof as follows. First, the achievability when $\mathcal{J}$ has only three receivers was shown in Section 3.3. In the sequel, the rest of the proof is dedicated to showing that for arbitrary set of receivers, $\mathcal{J} \subset [1:K]$ where the receivers are ordered ascendingly according to the coherence time length, the product superposition achieves the degrees of freedom tuples $\mathcal{D}_1(\mathcal{J})/\mathcal{D}_2(\mathcal{J})$, we can achieve the degrees of freedom tuple $\mathcal{D}_1(\widetilde{\mathcal{J}})/\mathcal{D}_2(\widetilde{\mathcal{J}})$, where $\widetilde{\mathcal{J}} \subseteq [1:K]$ is the set constructed by adding one more receiver to the set $\mathcal{J}$ where the length of the added receiver coherence time is an integer multiple of $j_{\max}$. To complete the proof we need to show that product superposition achieves the degrees of freedom tuples $\mathcal{D}_1(\widetilde{\mathcal{J}})/\mathcal{D}_2(\widetilde{\mathcal{J}})$ for the set $\widetilde{\mathcal{J}}$. The following Lemma addresses this part of the proof.

Lemma 4. *For the broadcast channel considered in Section 3.2, define $\mathbf{X}_o \in \mathbb{C}^{T_\tau \times T_o}$ to be a pilot-based transmitted signal during T_o slots where a training matrix $\mathbf{X}_\tau \in \mathbb{C}^{T_\tau \times T_\tau}$ is sent during T_τ slots and then the data is sent during $T_o - T_\tau$ slots achieving the degrees of freedom tuple $\mathcal{D}^{(o)} = \left(d_1^{(o)}, d_2^{(o)}, \cdots, d_J^{(o)}\right)$ for J receivers. We are able to achieve $\mathcal{D}^{(o)}$ for the J receivers in addition to $\left(\frac{T_\epsilon}{T_o} - 1\right)\frac{T_\tau \min\{T_\tau, N_\epsilon^*\}}{T_\epsilon}$ to a receiver ϵ with T_ϵ-length coherence time and N_ϵ receive antennas, where $\frac{T_\epsilon}{T_o} \in \mathbb{Z}$.*

Proof. This can be achieved by the following product superposition transmission scheme.

- Every T_ϵ slots is divided into T_o-length subintervals.

- During the first T_o subinterval, the transmitted signal is $\mathbf{X}_o$. Thus $\mathcal{D}^{(\mathrm{o})}$ degrees of freedom tuple is achieved for the J receivers and no degrees of freedom for receiver ϵ, yet, it estimates its channel between $\min\{N_\epsilon^*, T_\tau\}$ transmission antennas.

- During the remaining subintervals, product superposition is used. Every T_o slots, the transmitter sends

$$\widetilde{\mathbf{X}}_o = \mathbf{P}\mathbf{X}_o, \tag{3.56}$$

  where $\mathbf{P} \in \mathbb{C}^{M\times T_\tau}$ contains data for receiver ϵ. $\mathbf{X}_o$ contains the training matrix $\mathbf{X}_\tau$, hence, receiver ϵ can decode $\mathbf{P}$, using its channel estimate. Furthermore, the J receivers estimate their equivalent channels and decode their data during $T_o - T_\tau$ slots. Thus, J receivers achieve $\left(\frac{T_\epsilon}{T_o} - 1\right)\mathcal{D}^{(\mathrm{o})}$ degrees of freedom tuple and receiver ϵ achieves $\left(\frac{T_\epsilon}{T_o} - 1\right) T_\tau \min\{N_\epsilon^*, T_\tau\}$ degrees of freedom.

Thus, in T_ϵ slots, J receivers achieve $\frac{T_\epsilon}{T_o}\mathcal{D}^{(\mathrm{o})}$ degrees of freedom, and furthermore, receiver ϵ achieves $\left(\frac{T_\epsilon}{T_o} - 1\right) T_\tau \min\{N_\epsilon^*, T_\tau\}$ degrees of freedom which completes the proof of Lemma 4. □

Using Lemma 4 the second part of the proof is completed, and hence, the proof of Theorem 2 is completed.

3.4 General Coherence Times

Here, we study a K-receiver broadcast channel with general coherence times. An achievable degrees of freedom region is obtained, where the coherence times have arbitrary ratio or alignment.[2]

[2]Coherence times, as is required in a block fading model in a time-sampled domain, continue to take positive integer values.

3.4.1 Unaligned Coherence Times

Now, we relax the assumption on the alignment of coherence intervals. Consider a broadcast channel with K receivers where the coherence times are integer multiple of each other, i.e. $\frac{T_k}{T_{k-1}} \in \mathbb{Z}$. The coherence times have arbitrary alignment, meaning that there could be an offset between the transition times of the coherence intervals of different receivers. Recall that in the case of aligned coherence intervals, product superposition provided the achievable degrees of freedom region in (3.21). The receiver with longer coherence time reuses some of the unneeded pilots and achieves gains in degrees of freedom without affecting the receivers with shorter coherence times. Under unaligned coherence times the same gains in degrees of freedom are available with product superposition. Using the transmitted signal given in Section 3.3, the longer coherence times include the same number of unneeded pilot sequences regardless of the alignment. These unneeded pilot sequences can be reused by product superposition transmission, achieving degrees of freedom gain. For instance, if we have two receivers with $M = 2, N_1 = N_2 = 1, T_1 = 4, T_2 = 8$, with an offset of one transmission symbol as shown in Fig. 3.6. We can achieve the degrees of freedom pair $(\frac{3}{4}, \frac{1}{8})$ via a transmission strategy over pairs of coherence intervals for receiver 1, as follows.

- In odd intervals for receiver 1, one pilot is transmitted during which both receivers estimate their channels. In the 3 remaining times in this interval, data is transmitted for receiver 1.
- In the even intervals for receiver 1, during the first time slot a product superposition is transmitted providing one degree of freedom for receiver 2 (whose channel has not changed) while allowing receiver 1 to renew the estimate of his channel. The three remaining time slots provide 3 further degrees of freedom for receiver 1.

Thus, in 8 time slots, receiver 1 achieved 6 degrees of freedom and receiver 2 achieved 1. This is the same "corner point" that was obtained earlier, noting that the nature of the algorithm

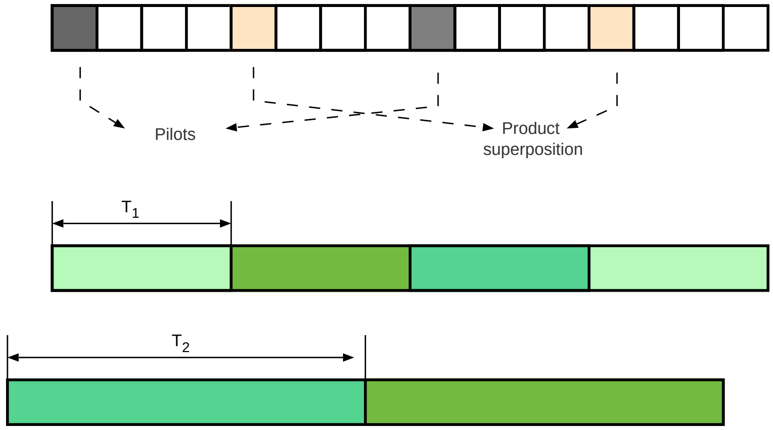

Figure 3.6. Product superposition transmission for unaligned coherence times.

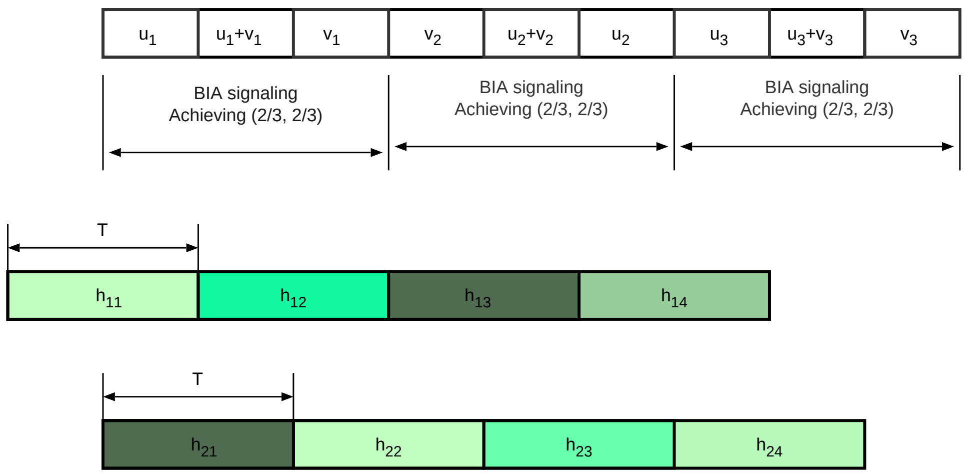

Figure 3.7. Blind interference alignment for staggered coherence times with CSIR. $\left(\frac{2}{3}, \frac{2}{3}\right)$ degrees of freedom pair is achieved.

is not changed, only the position of the pilot transmission must be carefully chosen while keeping in mind the transition points of the block fading.

3.4.2 Unaligned Coherence Times with Perfect Symmetry (Staggered)

We now consider a special case of two-receiver unaligned coherence times where the transition of each coherence interval is exactly in the middle of the other coherence interval. This special case is motivated by blind interference alignment model in Fig. 3.7, and for easy reference, we call this configuration a *staggered* coherence time.

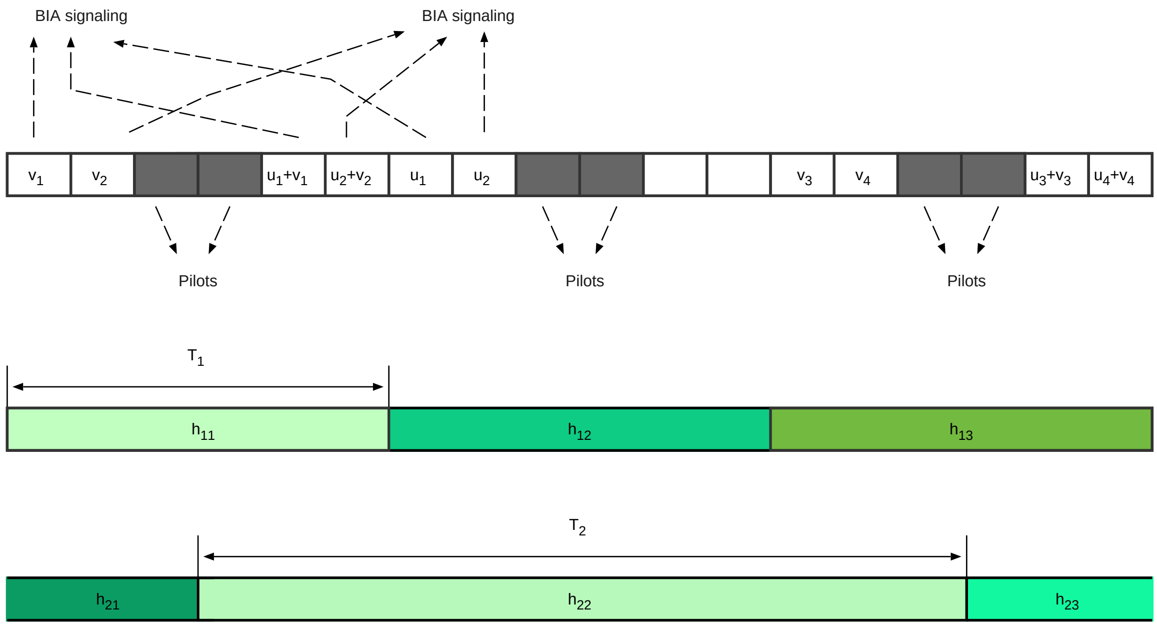

Figure 3.8. Blind interference alignment with pilot transmission. Receiver 1 cancels $\mathbf{h}_{1i}^{\dagger}\mathbf{v}$, and decodes $\mathbf{u}$, whereas receiver 2 cancels $\mathbf{h}_{2i}^{\dagger}\mathbf{u}$, and decodes $\mathbf{v}$.

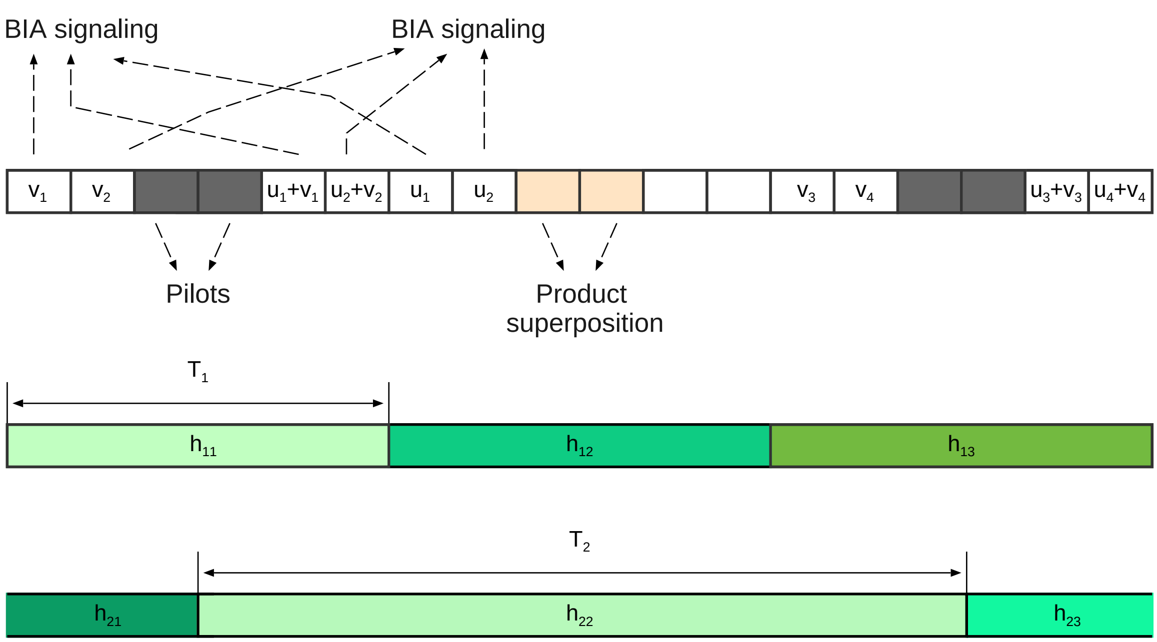

Figure 3.9. Combining blind interference alignment with product superposition.

We follow the example of blind interference alignment [62]: a 2-receiver broadcast channel with $M = 2, N_1 = N_2 = 1$. As shown in Fig. 3.8, the transitions of the longer coherence interval occur at the middle of the shorter coherence interval. Based on the discussion in Section 3.4.1, product superposition can obtain degrees of freedom gain for the staggered scenario. Blind interference alignment [62] achieved degrees of freedom pair $\left(\frac{2}{3}, \frac{2}{3}\right)$ while ignoring the cost of CSIR, which is a key part of our analysis. To allow comparison and synergy, we analyze a version of blind interference alignment with channel estimation shown in Fig. 3.8. The gain of blind interference alignment comes from the staggering of the coherence time, whereas the source of product superposition gain is reusing the unneeded pilots with respect to the longer coherence times. Therefore, we can give a transmission scheme that uses both blind interference alignment and product superposition over $\frac{T_2}{T_1}$ coherence intervals of receiver 1, as shown in Fig. 3.9.

- During the first 2 intervals, two pilots are sent at the middle of each interval for channel estimation. Furthermore, during the first interval and half of the second interval, $\left(\frac{T_1}{2} - 1\right)$ blind interference alignment signaling is sent achieving $\left(2\left(\frac{T_1}{2} - 1\right), 2\left(\frac{T_1}{2} - 1\right)\right)$ degrees of freedom pair. For the remaining half of the second interval, data for receiver 1 is sent achieving $\left(\frac{T_1}{2} - 1\right)$ further degrees of freedom.

- During the remaining $\left(\frac{T_2}{T_1} - 2\right)$ intervals, product superposition transmission is sent achieving $\left(\left(\frac{T_2}{T_1} - 2\right)(T_1 - 1), \left(\frac{T_2}{T_1} - 2\right)\right)$ degrees of freedom pair.

- Receiver 2 estimates its channel during the first and the last time slots of its coherence interval.

Thus, the above transmission scheme obtain the degrees of freedom pair

$$\left(1 - \frac{1}{T_1} - \frac{1}{T_2} - \frac{T_1}{2T_2}, \frac{T_1}{T_2} + \frac{1}{T_1} - \frac{4}{T_2}\right). \tag{3.57}$$

Furthermore, product superposition can achieve the degrees of freedom pair $\left(1-\frac{1}{T_1}, \frac{1}{T_2}-\frac{1}{T_1}\right)$. Hence, the achievable degrees of freedom is the convex hull of the degrees of freedom pairs achieved by blind interference alignment, product superposition, and combining blind interference alignment with product superposition.

3.4.3 Arbitrary Coherence Times

Theorem 4. *Consider a K-receiver broadcast channel without CSIT or CSIR having heterogeneous coherence times, where the coherence times are allowed to take any positive integer value. Product superposition can achieve the degrees of freedom tuple defined in* (3.20).

Remark 5. *Blind interference alignment signaling can be sent at the location of the staggering coherence times achieving degrees of freedom gain. Hence, similar to the case of staggered coherence times with integer ratio in Section 3.4.2, product superposition can be combined with blind interference alignment increasing the achievable degrees of freedom region.*

Proof. For clarity of explanation, we start by giving the achievable scheme for 3 receivers with $N_k = N \leq \min\left\{M, \left\lfloor \frac{T_1}{2} \right\rfloor\right\}, \forall k$ over (T_2T_3) coherence intervals of receiver 1.

- For every interval, a pilot sequence of length N slots, and receiver 1 data of length $T_1 - N$ slots are sent, achieving $N(T_1 - N)$ degrees of freedom for receiver 1.

- The number of pilot sequences of length N slots is T_2T_3. Having coherence time T_2, receiver 2 needs only T_1T_3 pilot sequences for channel estimation. Hence, produced superposition can be sent during $(T_2T_3 - T_1T_3)$ pilot sequences to send data for receiver 2 achieving $NT_3(T_2 - T_1)$ degrees of freedom.

- Furthermore, receiver 3 needs only T_1T_2 pilot sequences for channel estimation, and hence, data signal for receiver 3 can be sent during $(T_2T_3 - T_1T_2)$ pilot sequences via product superposition. Since, data for receiver 2 already is sent via product superposition during $(T_2T_3 - T_1T_3)$ pilot sequences, receiver 3 can only reuse $(T_2T_3 - T_1T_2) - (T_2T_3 - T_1T_3) = (T_3 - T_2)T_1$ pilot sequences achieving $NT_1(T_3 - T_2)$ degrees of freedom.

Thus, we can achieve the degrees of freedom tuple

$$\left(N\left(1 - \frac{N}{T_1}\right), N^2\left(\frac{1}{T_1} - \frac{1}{T_2}\right), N^2\left(\frac{1}{T_2} - \frac{1}{T_3}\right)\right). \tag{3.58}$$

Now, we give the proof for arbitrary number of receivers, and general antenna setup. For a set of receiver $\mathcal{J} \subseteq [1 : K]$ having J receiver where, $\frac{T_j}{T_{j-1}} \in \mathbb{Q}$, $j \in \mathcal{J}$, the degrees of freedom tuple (3.20) can be obtained over $\prod_{i=2}^{J} T_i$ coherence intervals of receiver $j_{\min}$.

- For every interval, a pilot sequence of length $N^*_{j_{\min}}$ slots, and data of length $T_{j_{\min}} - N^*_{j_{\min}}$ for receiver $j_{\min}$ are sent, achieving $N^*_{j_{\min}}\left(T_{j_{\min}} - N^*_{j_{\min}}\right)$ degrees of freedom for receiver $j_{\min}$.

- The number of pilot sequences of length $N^*_{j_{\min}}$ slots is $\prod_{i=2}^{J} T_i$. Receiver $j \neq j_{\min}$, with coherence time T_j, can estimate the channel of $\min\left\{N^*_{j_{\min}}, N_j\right\}$ transmit antennas using $\prod_{i=1, i\neq j}^{J} T_i$ pilot sequences. After excluding the pilots reused by receivers $i = \{j_{\min} + 1, \cdots, j - 1\}$ to send data by product superposition transmission, data for receiver j can be sent via product superposition during $(T_j - T_{j-1}) \prod_{i=1, i\neq j, j-1}^{J} T_i$ pilots obtaining the degrees of freedom $N^*_{j_{\min}} \min\left\{N^*_{j_{\min}}, N_j\right\} (T_j - T_{j-1}) \prod_{i=1, i\neq j, j-1}^{J} T_i$.

Thus, the proof of Theorem 4 is completed. □

3.5 Multiple Access Channel with Identical Coherence Times

Consider a K-transmitter MIMO multiple access channel without CSIT or CSIR, where transmitter k is equipped with M_k antennas, and the receiver is equipped with N antennas. The received signal at the discrete time n can be given by

$$\mathbf{y}(n) = \sum_{k=1}^{K} \overline{\mathbf{H}}_k(n)\mathbf{x}_k(n) + \mathbf{z}(n), \tag{3.59}$$

where $\mathbf{x}_k(n) \in \mathbb{C}^{M_k \times 1}$ is transmitter k transmitted signal, $\mathbf{z}(n) \in \mathbb{C}^{N \times 1}$ is the i.i.d. Gaussian additive noise and $\overline{\mathbf{H}}_k(n) \in \mathbb{C}^{N \times M_k}$ is transmitter k Rayleigh block-fading channel matrix with coherence time T_k [6]. We study the case when $T_k \geq 2N, \forall k$ [7].

Assume that all transmitters have identical coherence time, T. In the sequel, we define a degrees of freedom achievable region based on a pilot-based scheme in Section 3.5.1. Furthermore, an outer degrees of freedom region is given in Section 3.5.2 based on the cooperative bound. Some numerical examples are given in Section 3.5.3 where it is shown that the achievable and the outer degrees of freedom regions coincide at the regions of sum degrees of freedom.

3.5.1 Achievability

Theorem 5. *Consider a K-transmitter MIMO multiple access channel without CSIT or CSIR, meaning that the channel realization is not known, but the channel distribution is globally known. If the transmitters have identical coherence times, namely T, then for every ordered set of transmitters, $\mathcal{J} = \{k_1, k_2, \cdots, k_J\} \subseteq [1:K]$, we can achieve the set of degrees of freedom tuples $\mathcal{D}(\mathcal{J})$:*

$$d_j = M'_j \left(1 - \frac{\sum_{j \in \mathcal{J}} M'_j}{T}\right), \quad j \in \mathcal{J}, \tag{3.60}$$

where $M'_j = \min\left\{M_j, \left[N - \sum_{m=1}^{j-1} M'_{k_m}\right]^+\right\}$, and $T \geq 2N$. The achievable degrees of freedom region is the convex hull of the degrees of freedom tuples, $\mathcal{D}(\mathcal{J})$, over all the $\sum_{i=1}^{K} \frac{K!}{(K-i)!}$ possible ordered sets $\mathcal{J} \subseteq [1:K]$, i.e.,

$$\mathcal{D} = \Big\{ (d_1, \cdots, d_K) \in Co\left(\mathcal{D}(\mathcal{J})\right), \forall \mathcal{J} \subseteq [1:K] \Big\}. \tag{3.61}$$

Proof. We show that a simple pilot-based scheme can achieve the above achievable degrees of freedom region. Assume that we have an ordered set of transmitters $\mathcal{J} = \{k_1, \cdots, k_J\} \subseteq [1:K]$. In order to achieve the degrees of freedom tuple in (3.60), we can use the following transmission scheme.

- At the beginning of every T-length interval, a training sequence of length $\sum_{j\in\mathcal{J}} M'_j$ is sent so that the receiver can estimate the channels where a M'_j-length training sequence is sent from transmitter j.

- During the remaining $T-\sum_{j\in\mathcal{J}} M'_j$ period, simultaneously, $M'_j \times \left(T-\sum_{j\in\mathcal{J}} M'_j\right)$ data matrix is sent from transmitter j. Hence, the receiver, using $\sum_{j\in\mathcal{J}} M'_j$ antennas, can invert the channel (via zero forcing) and decode the transmitted signal during $T-\sum_{j\in\mathcal{J}} M'_j$ slots.

Therefore, every T period, transmitter $j \in \mathcal{J}$ can achieve $M'_j\left(T-\sum_{j\in\mathcal{J}} M'_j\right)$ degrees of freedom, and hence (3.60) is obtained. □

3.5.2 Outer Bound

For the considered K-transmitter multiple access channel with identical coherence times, namely T, the cooperative bound [85] can be given by [88]

$$\sum_{j\in\mathcal{J}} R_j \leq I\left(X\left(\mathcal{J}\right);Y|X\left(\mathcal{J}^c\right)\right), \quad \forall\mathcal{J}\subseteq[1:K]. \tag{3.62}$$

An outer bound on the degrees of freedom region is [7],

$$\sum_{j\in\mathcal{J}} d_j \leq \min\left\{N,\sum_{j\in\mathcal{J}} M_j\right\}\left(1-\frac{\min\left\{N,\sum_{j\in\mathcal{J}} M_j\right\}}{T}\right), \quad \forall\mathcal{J}\subseteq[1:K]. \tag{3.63}$$

3.5.3 Numerical Examples

Consider a two-transmitter multiple access channel where the transmitters are equipped with $M_1 = 3, M_2 = 2$ antennas and the receiver is equipped with $N_2 = 4$ antennas. The coherence time of the two transmitters is $T = 10$ slots. Thus, the outer degrees of freedom region is given by

$$d_1 \leq \frac{21}{10},$$

$$d_2 \leq \frac{16}{10},$$
$$d_1 + d_2 \leq \frac{24}{10}.$$

The achievable degrees of freedom pairs in Theorem 5 can be obtained as follows. For the case of two transmitters, there are 5 ordered sets of transmitters $\mathcal{J}$: $\{\}, \{1\}, \{2\}$, $\{1, 2\}$ and $\{2, 1\}$. For $\{\}$, the trivial degrees of freedom pair $(0, 0)$ can be obtained. For the two sets $\{1\}, \{2\}$, the degrees of freedom pairs $\left(\frac{21}{10}, 0\right)$ and $\left(0, \frac{16}{10}\right)$, respectively, can be obtained. For the two sets $\{1, 2\}$ and $\{2, 1\}$, the degrees of freedom pairs $\left(\frac{18}{10}, \frac{6}{10}\right)$ and $\left(\frac{12}{10}, \frac{12}{10}\right)$, respectively, can be obtained. The convex hull of the achieved degrees of freedom pairs gives the achievable degrees of freedom region which is tight against the sum degrees of freedom as shown in Fig 3.10.

Consider a two-transmitter multiple access channel where the transmitters are equipped with $M_1 = 4, M_2 = 2$ antennas and the receiver is equipped with $N = 3$ antennas. The coherence time of the two transmitters is $T = 10$ slots. As shown in Fig. 3.11, the achievable degrees of freedom regions are tight against the sum degrees of freedom.

3.6 Multiple Access Channel with Heterogeneous Coherence Times

Consider the multiple access channel defined in (3.59) where there is no CSIT or CSIR. Consider the case where the receivers coherence times are perfectly aligned and integer multiples of each other, i.e., $\forall k, \frac{T_k}{T_{k-1}} \in \mathbb{Z}$.

3.6.1 Achievability

Theorem 6. *Consider a K-transmitter MIMO multiple access channel without CSIT or CSIR, meaning that the channel realization is not known, but the channel distribution is globally known. Furthermore, the transmitters coherence times are assumed to be perfectly aligned and integer multiples of each other. Define $\mathcal{J} = \{i_1, \cdots, i_J\} \subseteq [1 : K]$ to be a set of*

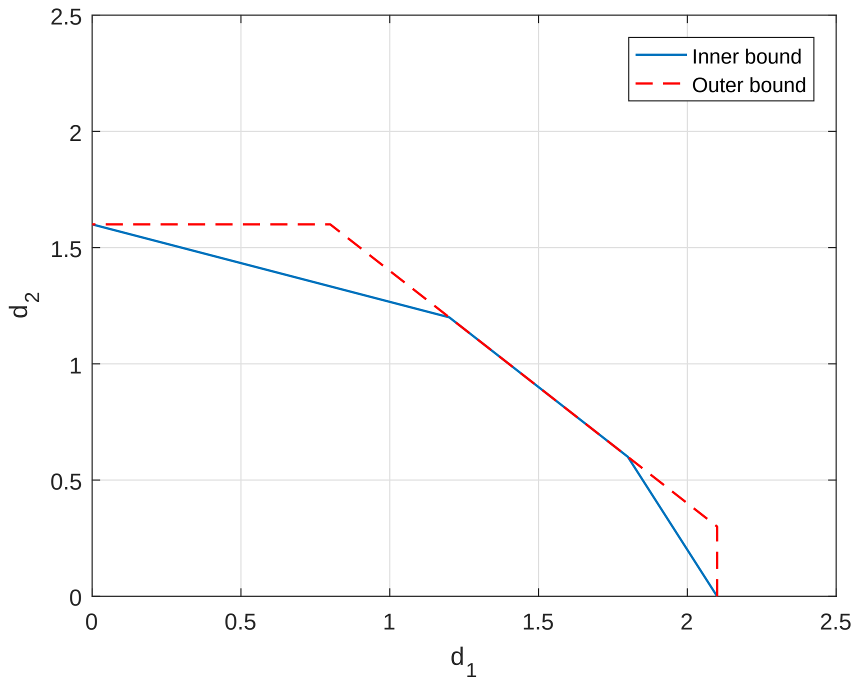

Figure 3.10. Degrees of freedom region of a two-transmitter multiple access channel with identical coherence times where $M_1 = 3, M_2 = 2, N = 4, T = 10$.

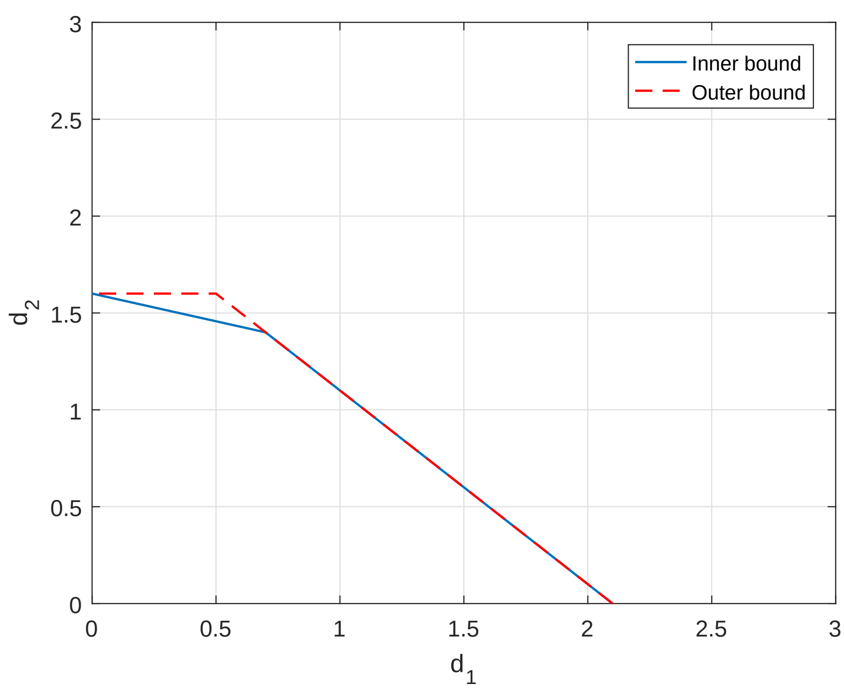

Figure 3.11. Degrees of freedom region of a two-transmitter multiple access channel with identical coherence times where $M_1 = 4, M_2 = 2, N = 3, T = 10$.

J transmitters where $\forall j \in \mathcal{J}, \frac{T_j}{T_{j-1}} \in \mathbb{Z}$. Define $\breve{\mathcal{J}} = \{k_1, \cdots, k_J\}$ to be one of the $J!$ possible ordered sets of $\mathcal{J}$. If $T_k \geq 2N, \forall k$, we can achieve the set of degrees of freedom tuples $\mathcal{D}(\breve{\mathcal{J}})$:

$$d_j = M'_j \sum_{m=1}^{J} \left(T_{i_1} - \sum_{n=1}^{m} M'_{i_n} \right) \left(\frac{1}{T_{i_m}} - \frac{1}{T_{i_{m+1}}} \right), \tag{3.64}$$

where $M'_j = \min \left\{ M_j, \left[N - \sum_{m=1}^{j-1} M'_{k_m} \right]^+ \right\}$, and, for notational convenience, we introduce the trivial random variable $T_{i_{J+1}}$, i.e., $\frac{1}{T_{i_{J+1}}} = 0$. Hence, the achievable degrees of freedom region is the convex hull of the degrees of freedom tuples, $\mathcal{D}(\mathcal{J})$, over all the $\sum_{i=1}^{K} \frac{K!}{(K-i)!}$ possible ordered sets $\breve{\mathcal{J}} \subseteq [1:K]$, i.e.,

$$\mathcal{D} = \left\{ (d_1, \cdots, d_K) \in Co\left(\mathcal{D}(\breve{\mathcal{J}})\right), \forall \breve{\mathcal{J}} \subseteq [1:K] \right\}. \tag{3.65}$$

Proof. By time-sharing between the transmission schemes that achieve the degrees of freedom tuples $\mathcal{D}(\breve{\mathcal{J}})$, we can construct the achievable degrees of freedom region which is the convex hull of the achieved degrees of freedom tuples. The remainder of the proof is dedicated to showing the achievability of the degrees of freedom tuple in (3.64) using the following transmission scheme.

- Every T_{i_J} interval is divided into T_{i_1}-length subintervals.

- During the first subinterval, $\sum_{j\in\mathcal{J}}^{J} M'_j$ pilots are sent to estimate M'_j antennas of transmitter j, and hence, during the following $T_{i_1} - \sum_{j\in\mathcal{J}}^{J} M'_j$ period, the transmitters can communicate coherently achieving $M'_j \left(T_{i_1} - \sum_{j\in\mathcal{J}}^{J} M'_j \right)$ degrees of freedom for transmitters $j \in \mathcal{J}$.

- During the remaining subintervals, when the index of the subinterval is $\ell \frac{T_{i_m}}{T_{i_1}} + 1$, where $m = 2, \cdots J-1$ and $\ell = 1, \cdots, \frac{T_{i_m+1}}{T_{i_m}} - 1, \frac{T_{i_m+1}}{T_{i_m}} + 1, \cdots, 2\frac{T_{i_m+1}}{T_{i_m}} - 1, \frac{T_{i_m+1}}{T_{i_m}} + 1, \cdots, \frac{T_{i_J}}{T_{i_m}} - 1$, the channel of transmitter $j = i_1, \cdots, i_m$ needs to be estimated, whereas the channel of transmitter $j = i_m + 1, \cdots, J$ stays the same. Hence, $\sum_{n=1}^{m} M'_{i_n}$ pilots are sent to estimate M'_j antennas of transmitters $j = i_1, \cdots, i_m$. After that, during the following $T_{i_1} - \sum_{n=1}^{m} M'_{i_n}$ period, the transmitters can communicate coherently achieving $M'_j \left(T_{i_1} - \sum_{n=1}^{m} M'_{i_n}\right)$ degrees of freedom for transmitters $j \in \mathcal{J}$. The number of subintervals of index $k\frac{T_m}{T_{m-1}} + 1$ is $\sum_{m=2}^{J} \left(\frac{T_{i_{m+1}}}{T_{i_m}} - 1\right) \frac{T_{i_J}}{T_{i_{m+1}}}$.

- For the T_{i_1}-length subintervals whose index is not $\ell\frac{T_{i_m}}{T_{i_1}}+1$, the channels of all transmitters remain the same except the channel of transmitter i_1. Hence, M'_{i_1} pilots are sent to estimate the channel of transmitter i_1, after that the transmitters can communicate coherently during the following $T_{i_1} - M'_{i_1}$ period achieving $M'_j \left(T_{i_1} - M'_{i_1}\right)$ degrees of freedom for transmitter $j \in \mathcal{J}$. The number of the subintervals whose index is not equal to $\ell\frac{T_{i_m}}{T_{i_1}} + 1$ is $\left(\frac{T_{i_2}}{T_{i_1}} - 1\right) \frac{T_{i_J}}{T_{i_2}}$

For every T_{i_J} interval transmitter $j \in \mathcal{J}$ achieves

$$M'_j \sum_{m=1}^{J} \left(T_{i_1} - \sum_{n=1}^{m} M'_{i_n}\right) \left(\frac{1}{T_{i_m}} - \frac{1}{T_{i_{m+1}}}\right) T_{i_J} \tag{3.66}$$

degrees of freedom obtaining (3.64) which completes the proof of Theorem 6. □

3.6.2 Outer Bound

Theorem 7. *Consider a K-transmitter MIMO multiple access channel without CSIT or CSIR, meaning that the channel realization is not known, but the channel distribution is globally known. Furthermore, the transmitters coherence times are assumed to be perfectly aligned and integer multiples of each other. Define $\mathcal{J} = \{i_1, \cdots, i_J\} \subseteq [1 : K]$ to be a set of J transmitters where $\frac{T_j}{T_{j-1}} \in \mathbb{Z}, \forall j \in \mathcal{J}$, and $T_k \geq 2\max_k\{M, N_k\}, \forall k = 1, \cdots, K$. For every $\mathcal{J} \subseteq [1 : K]$, if a set of degrees of freedom tuples $(d_{i_1}, \cdots, d_{i_J})$ is achievable, then it must satisfy the inequalities*

$$\sum_{j\in\mathcal{J}} d_j \leq \min\left\{N, \sum_{j\in\mathcal{J}} M_j\right\} \left(1 - \frac{\min\left\{N, \sum_{j\in\mathcal{J}} M_j\right\}}{T_{i_J}}\right). \tag{3.67}$$

Proof. The proof is divided into two parts. First, we enhance the channel by increasing the coherence times of the receivers so that the enhanced channel has identical coherence times.

Lemma 5. *For the considered K-transmitter MIMO multiple access channel, define $\mathcal{D}(\mathcal{J})$ to be the degrees of freedom region of a set of transmitters $\mathcal{J} = \{i_1, \cdots, i_J\} \subseteq [1:K]$ with $\frac{T_j}{T_{j-1}} \in \mathbb{Z}, \forall j \in \mathcal{J}$. Define $\overline{\mathcal{D}}(\mathcal{J})$ to be the degrees of freedom region of the same set of transmitters $\mathcal{J} = \{i_1, \cdots, i_J\} \subseteq [1:K]$ with $T_j = T_{i_J}, \forall j \in \mathcal{J}$, where the transmitters have identical coherence times, namely T_{i_J}. Thus, we have*

$$\mathcal{D}(\mathcal{J}) \subseteq \overline{\mathcal{D}}(\mathcal{J}) \tag{3.68}$$

Proof. Consider the set of transmitters $\mathcal{J} = \{i_1, \cdots, i_J\} \subseteq [1:K]$ where $\forall j \in \mathcal{J}, \frac{T_j}{T_{j-1}} \in \mathbb{Z}$. By Fano's inequality, as $\bar{n} \to \infty$,

$$\sum_{j \in \mathcal{J}} R_j \leq \frac{1}{\bar{n}} I\left(\mathbf{X}_{i_1}^n, \cdots, \mathbf{X}_{i_J}^n; \mathbf{Y}^n\right), \tag{3.69}$$

where $\mathbf{X}_j^n$ is transmitter $j \in \mathcal{J}$ signal and $\mathbf{Y}^n$ is the received signal over the entire transmission time $1:\bar{n}$. In the sequel, we show that the degrees of freedom of $\frac{1}{\bar{n}} I\left(\mathbf{X}_j; \mathbf{Y}^n\right), j \in \mathcal{J}$ is nondecreasing in T_j. We give lower and upper bounds on this term, and furthermore, both bounds are nondecreasing in T_j. We introduce the following Lemma.

Lemma 6. *For the multiple access channel in Section 3.6, define $\mathcal{J} = \{i_1, \cdots, i_J\} \subseteq [1:K]$ and Ψ_j as the message of transmitter $j \in \mathcal{J}$. Thus, we have*

$$\sum_{j \in \mathcal{J}} M_j^* \left(\frac{1}{J} - \frac{M_j^*}{T_j}\right) \leq MG\left\{\frac{1}{\bar{n}} I\left(\mathbf{X}_{i_1}^n, \cdots, \mathbf{X}_{i_J}^n; \mathbf{Y}^n\right)\right\} \leq \sum_{j \in \mathcal{J}} M_j^* \left(1 - \frac{M_j^*}{T_j}\right), \tag{3.70}$$

Proof. We first prove the right inequality of (3.70). We have

$$\mathrm{MG}\left\{\frac{1}{\bar{n}} I\left(\mathbf{X}_{i_1}^n, \cdots, \mathbf{X}_{i_J}^n; \mathbf{Y}^n\right)\right\} \leq \sum_{j \in \mathcal{J}} M_j^* \left(1 - \frac{M_j^*}{T_j}\right), \tag{3.71}$$

where the above inequality follows from the single-transmitter results [7]. Next, we show the left inequality of (3.70). Assume that we have the following transmitted sequence

$$\overline{\mathbf{X}}_j^n = [0, \cdots, 0, \mathbf{U}_j, 0, \cdots, 0], \tag{3.72}$$

where $\mathbf{U}_j \in \mathbb{C}^{M_j \times \frac{\bar{n}}{J}}$ is the matrix containing the signal of transmitter $j \in \mathcal{J}$ and the matrix is constructed to be on the form of the optimal input of a non-coherent single transmitter [7]. Hence,

$$\begin{aligned}
\text{MG}\left\{\frac{1}{\bar{n}} I\left(\mathbf{X}_{i_1}^n, \cdots, \mathbf{X}_{i_J}^n; \mathbf{Y}^n\right)\right\} &\geq \text{MG}\left\{\frac{1}{\bar{n}} \sum_{j\in\mathcal{J}} I\left(U_j; \mathbf{Y}_j^n\right)\right\} \\
&\geq \sum_{j\in\mathcal{J}} \frac{M_j^*}{J}\left(1 - \frac{M_j^*}{T_j}\right) \\
&\geq \sum_{j\in\mathcal{J}} M_j^*\left(\frac{1}{J} - \frac{M_j^*}{T_j}\right).
\end{aligned} \tag{3.73}$$

Thus, the proof of Lemma 6 is completed. □

By Lemma 6, we have lower and outer bounds which are increasing with T_j. Furthermore, the difference between the two bounds is

$$\Delta = \sum_{j\in\mathcal{J}} M_j^*\left(1 - \frac{M_j^*}{T_j}\right) - M_j^*\left(\frac{1}{J} - \frac{M_j^*}{T_j}\right) = M_j^* \frac{J-1}{J}. \tag{3.74}$$

Therefore, $\text{MG}\left\{\frac{1}{\bar{n}} I\left(\mathbf{X}_{i_1}^n, \cdots, \mathbf{X}_{i_J}^n; \mathbf{Y}^n\right)\right\}$ is nondecreasing with the increase of T_j, and hence, the proof of Lemma 5 is completed. □

Now we show the second part of the proof. The enhanced channel has identical coherence times, namely T_{i_J}, hence, the cooperative outer bound [85] bound is [88],

$$\sum_{j\in\mathcal{J}} R_j \leq I\left(\mathbf{X}(\mathcal{J}); \mathbf{Y}|\mathbf{X}(\mathcal{J}^c)\right). \tag{3.75}$$

According to the results of noncoherent communication [7], the bound in (3.67) can be obtained, and the proof of Theorem 7 is completed. □

3.6.3 Numerical Examples

Consider a two-transmitter multiple access channel where the transmitters are equipped with $M_1 = 2, M_2 = 4$ antennas and the receiver is equipped with $N_2 = 4$ antennas. The coherence time of the two transmitters is $T_1 = 8$ and $T_2 = 32$ slots. From Theorem 7, the outer degrees of freedom region is given by

$$\begin{aligned} d_1 &\leq \frac{3}{2}, \\ d_1 + d_2 &\leq \frac{7}{2}. \end{aligned} \tag{3.76}$$

The achievable degrees of freedom pairs in Theorem 6 can be obtained as follows. For the case of two transmitters, there are 5 ordered sets of transmitters $\mathcal{J}$: $\{\}, \{1\}, \{2\}$, $\{1, 2\}$ and $\{2, 1\}$. For $\{\}$, the trivial degrees of freedom pair $(0, 0)$ can be obtained. For the two sets $\{1\}, \{2\}$, the degrees of freedom pairs $\left(\frac{3}{2}, 0\right)$ and $\left(0, \frac{7}{2}\right)$, respectively, can be obtained. For the two sets $\{1, 2\}$ and $\{2, 1\}$, the degrees of freedom pairs $\left(\frac{11}{8}, \frac{11}{8}\right)$ and $\left(0, \frac{7}{2}\right)$, respectively, can be obtained. The convex hull of the achieved degrees of freedom pairs gives the achievable degrees of freedom region which is shown in Fig 3.12.

Next, consider a two-transmitter multiple access channel with $M_1 = 3, M_2 = 2, N = 4$ antennas and coherence times $T_1 = 8, T_2 = 24$ slots. In this case the achievable and the outer degrees of freedom regions are shown in Fig. 3.13.

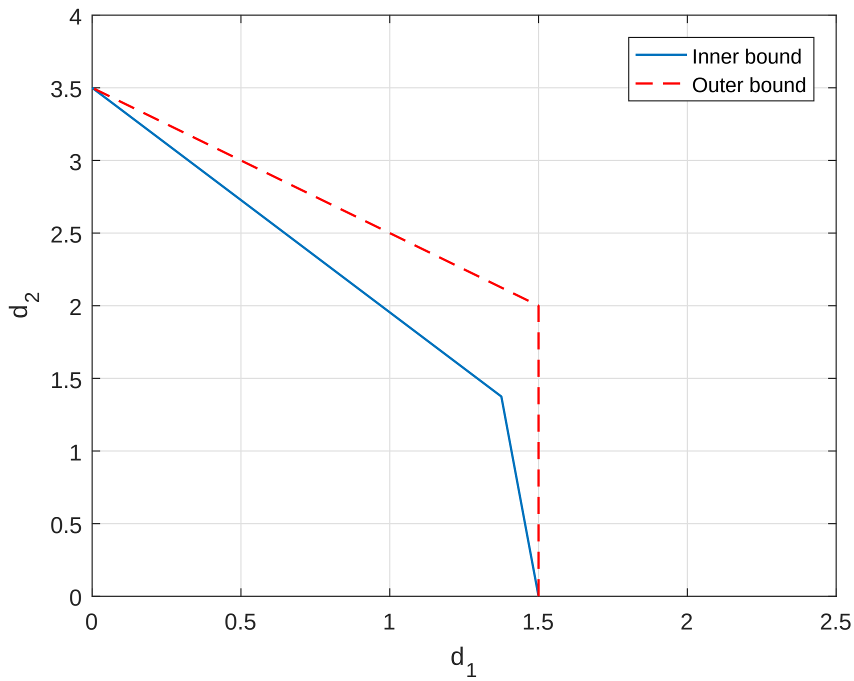

Figure 3.12. Degrees of freedom region of a two-transmitter multiple access channel with heterogeneous coherence times where $M_1 = 2, M_2 = 4, N = 4, T_1 = 8, T_2 = 32$.

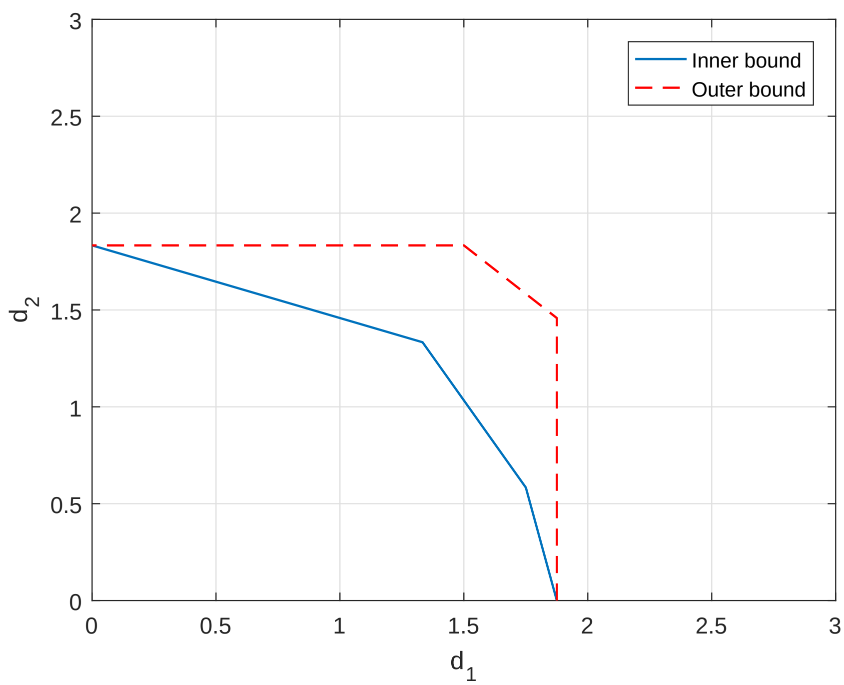

Figure 3.13. Degrees of freedom region of a two-transmitter multiple access channel with heterogeneous coherence times where $M_1 = 3, M_2 = 2, N = 4, T_1 = 8, T_2 = 24$.

CHAPTER 4

DEGREES OF FREEDOM OF THE BROADCAST CHANNEL WITH HYBRID CSI AT TRANSMITTER AND RECEIVERS

In this chapter, a multiuser MISO broadcast channel is investigated, where the users are partitioned into two sets: a set of static users and a set of dynamic users. For various CSIT models: delayed, hybrid and no CSIT, inner and outer bounds on the degrees of freedom are provided demonstrating that coherence diversity gains are merged with interference alignment and beamforming gains. Table 4.1 gives the notations that are used in this chapter.

4.1 System Model

Consider a broadcast channel with multiple single-antenna receivers and the transmitter is equipped with N_t antennas. The channels of the users are modeled as Rayleigh block fading where the channel coefficients remain constant over each block and change independently across blocks [6, 7]. As shown in Fig. 4.1, the users are partitioned into two sets based on channel availability and the length of the coherence interval: one set of dynamic users and another set of static users. The former contains m dynamic users having coherence time T and no free CSIR[1], and the latter contains m' static users having coherence time T' and perfect instantaneous CSIR. We consider the transmitter is equipped with more antennas than the number of dynamic and static users, i.e., $N_t \geq m' + m$.

The received signals $y'_j(t), y_i(t)$ at the static user j, and the dynamic user i, respectively, at time instant t are

$$\begin{aligned} y'_j(t) &= \mathbf{g}_j^{\dagger}(t)\mathbf{x}(t) + z'_j(t), \quad j = 1, \ldots, m', \\ y_i(t) &= \mathbf{h}_i^{\dagger}(t)\mathbf{x}(t) + z_i(t), \quad i = 1, \ldots, m, \end{aligned} \tag{4.1}$$

[1] This means that the cost of knowing CSI at the receiver, e.g., by channel estimation, is not ignored.

Table 4.1. Notation of Chapter 4

	Static Users	**Dynamic Users**
number of users	m'	m
MISO channel gains	$\mathbf{g}_1, \ldots, \mathbf{g}_{m'}$	$\mathbf{h}_1, \ldots, \mathbf{h}_m$
received signals (continuous)	$y'_1, \ldots, y'_{m'}$	$y_1, \ldots, y_m$
DMC receive variables	$Y'_1, \ldots, Y'_{m'}$	$Y_1, \ldots, Y_m$
transmission rates	$R'_1, \ldots, R'_{m'}$	$R_1 \ldots, R_m$
messages	$M'_1, \ldots, M'_{m'}$	$M_1, \ldots, M_m$
degrees of freedom	$d'_1, \ldots, d'_{m'}$	$d_1, \ldots, d_m$
coherence time	T'	T

General Variables

$\mathbf{X}$	transmit signal
ρ	signal-to-noise ratio
U_i, V_j, W	auxiliary random variables
$\mathcal{H}$	set of all channel gains
$\mathcal{D}_x$	vertex of degrees of freedom region
$\mathbf{e}_i$	canonical coordinate vector

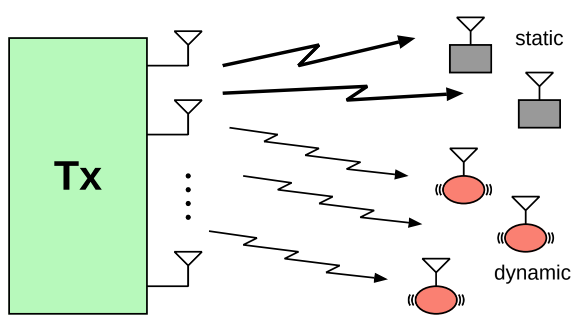

Figure 4.1. A broadcast channel with multiple static and multiple dynamic users

where $\mathbf{x}(t) \in \mathbb{C}^{N_t}$ is the transmitted signal, $z'_j(t), z_i(t)$ denote the corresponding additive i.i.d. Gaussian noise of the users, and $\mathbf{g}_j(t) \in \mathbb{C}^{N_t}, \mathbf{h}_i(t) \in \mathbb{C}^{N_t}$ denote the channels of the static user j and the dynamic user i whose coefficients stay the same over T' and T time instances, respectively. The distributions of $\mathbf{g}_j$ and $\mathbf{h}_i$ are globally known at the transmitter and at the users.[2] Having CSIR, the value of $\mathbf{g}_j(t)$ is available instantaneously and perfectly at the static user j. Furthermore, the static user j obtains an outdated version of the dynamic users channels $\mathbf{h}_i$, and also the dynamic user i obtains an outdated version of the static users channel $\mathbf{g}_i$ (completely stale) [32]. CSIT for each user can take one of the following:

- Perfect CSIT: the channel vectors, $\mathbf{g}_j(t), \mathbf{h}_i(t)$, are available at the transmitter instantaneously and perfectly.

- Delayed CSIT: the channel vectors, $\mathbf{g}_j(t), \mathbf{h}_i(t)$, are available at the transmitter after they change independently in the following block (completely stale [32]).

- No CSIT: the channel vectors, $\mathbf{g}_j(t), \mathbf{h}_i(t)$, cannot not be known at the transmitter.

We consider the broadcast channel with private messages for all users and no common messages. More specifically, we assume that the independent messages $M'_j \in [1 : 2^{nR'_i(\rho)}], M_i \in [1 : 2^{nR_i(\rho)}]$ associated with rates $R'_j(\rho), R_i(\rho)$ are communicated from the transmitter to the static user j and dynamic user i, respectively, at ρ signal-to-noise ratio. The degrees of freedom of the static and dynamic users achieving rates $R'_j(\rho), R_i(\rho)$ can be defined as

$$d'_j = \lim_{\rho\to\infty} \frac{R'_j(\rho)}{\log(\rho)}, \quad j = 1, \dots, m',$$
$$d_i = \lim_{\rho\to\infty} \frac{R_i(\rho)}{\log(\rho)}, \quad i = 1, \dots, m. \tag{4.2}$$

[2] Also, the coherence times of all channels are globally known at the transmitter and at the users.

The degrees of freedom region is defined as

$$\mathcal{D} = \Big\{(d'_1, \ldots, d'_{m'}, d_1, \ldots, d_m) \in \mathbb{R}_+^{m'+m} \Big| \; \exists(R'_1(\rho), \ldots, R'_{m'}(\rho), R_1(\rho), \ldots, R_m(\rho)) \in C(\rho),$$
$$d'_j = \lim_{\rho\to\infty} \frac{R'_j(\rho)}{\log(\rho)}, d_i = \lim_{\rho\to\infty} \frac{R_i(\rho)}{\log(\rho)}, \;\; j = 1, \ldots, m', \;\; i = 1, \ldots, m\Big\}, \tag{4.3}$$

where $C(\rho)$ is the capacity region at ρ signal-to-noise ratio. The sum degrees of freedom is defined as

$$d_{\text{sum}} = \lim_{\rho\to\infty} \frac{C_{\text{sum}}(\rho)}{\log(\rho)}, \tag{4.4}$$

where

$$C_{\text{sum}}(\rho) = \max \sum_{j=1}^{m'} R'_j(\rho) + \sum_{i=1}^{m} R_i(\rho). \tag{4.5}$$

In the sequel, we study the degrees of freedom of the above MISO broadcast channel under different CSIT scenarios that could be perfect, delayed or no CSIT.

4.2 No CSIT for All Users

Here we study the broadcast channel defined in Section 4.1 when there is no CSIT for all users. In particular, we give outer and achievable degrees of freedom regions in Section 4.2.2 and Section 4.2.3, respectively. The outer degrees of freedom region is based on the construction of an outer bound on the rates of a multiuser multilevel discrete memoryless channel that is given in Section 4.2.1.

4.2.1 Multiuser Multilevel Broadcast Channel

The multilevel broadcast channel was introduced by Borade *et al.* [64] as a three-user broadcast discrete memoryless broadcast channel where two of the users are degraded with respect to each other. The capacity of this channel under degraded message sets was established by Nair and El Gamal [65]. Here, we study a multiuser multilevel broadcast channel with two

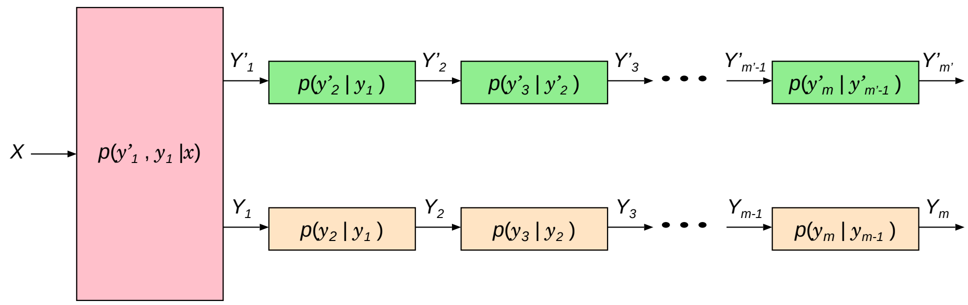

Figure 4.2. Discrete memoryless multiuser multilevel broadcast channel

sets of degraded users (see Fig. 4.2). One set contains m' users with Y'_j received signal at user j, and the other set contains m users with Y_i received signal at user i. Therefore,

$$\begin{aligned} X \rightarrow & Y'_1 \rightarrow Y'_2 \rightarrow \cdots \rightarrow Y'_{m'} \\ X \rightarrow & Y_1 \rightarrow Y_2 \rightarrow \cdots \rightarrow Y_m \end{aligned} \tag{4.6}$$

form two Markov chains. We consider a broadcast channel with $(m' + m)$ private messages and no common message. An outer bound for the above multilevel broadcast channel is given in the following theorem.

Theorem 8. *The rate region of the multilevel broadcast channel with two sets of degraded users (Eq.* (4.6)*) is outer bounded by the intersection of*

$$R_1 \leq I(U_{m'}, W; Y_1|V_1) - I(W; Y'_{m'}|U_{m'}), \tag{4.7}$$

$$R_i \leq I(V_{i-1}; Y_i|V_i), \qquad i = 2, \ldots, m, \tag{4.8}$$

$$R'_j \leq I(U_{j-1}; Y'_j|U_j), \qquad j = 1, \ldots, m' - 1, \tag{4.9}$$

$$R'_{m'} \leq I(W; Y'_{m'}|U_{m'}) + I(X; Y'_{m'}|U_{m'}, W) - I(X; Y'_{m'}|U_{m'-1}), \tag{4.10}$$

and

$$R_i \leq I(\tilde{U}_{i-1}; Y_i|\tilde{U}_i), \qquad i = 1, \ldots, m - 1, \tag{4.11}$$

$$R_m \leq I(\tilde{W}; Y_m|\tilde{U}_m) + I(X; Y_m|\tilde{U}_m, \tilde{W}) - I(X; Y_m|\tilde{U}_{m-1}), \tag{4.12}$$

$$R_1' \leq I(\tilde{U}_m, \tilde{W}; Y_1'|\tilde{V}_1) - I(\tilde{W}; Y_m|\tilde{U}_m), \tag{4.13}$$

$$R_j' \leq I(\tilde{V}_{j-1}; Y_j'|\tilde{V}_j), \qquad j = 2, \ldots, m', \tag{4.14}$$

for some pmf

$$p(u_1, \ldots, u_{m'}, \tilde{u}_1, \ldots, \tilde{u}_m, v_1, \ldots, v_m, \tilde{v}_1, \ldots, \tilde{v}_{m'}, w, \tilde{w}, x), \tag{4.15}$$

where

$$\begin{aligned}
U_{m'} \to \cdots \to U_1 \to & X \to (Y_1, \ldots, Y_m, Y_1', \ldots, Y_{m'}') \\
V_m \to \cdots \to V_1 \to (W, U_{m'}) \to & X \to (Y_1, \ldots, Y_m, Y_1', \ldots Y_{m'}') \\
\tilde{U}_m \to \cdots \to \tilde{U}_1 \to & X \to (Y_1, \ldots Y_m, Y_1', \ldots, Y_{m'}') \\
\tilde{V}_{m'} \to \cdots \to \tilde{V}_1 \to (\tilde{W}, \tilde{U}_m) \to & X \to (Y_1, \ldots Y_m, Y_1', \ldots, Y_{m'}')
\end{aligned} \tag{4.16}$$

forms Markov chains and $U_0 = \tilde{U}_0 \triangleq X$*.*

Proof. Recall, M_j', M_i are the messages of users $j = 1, \ldots, m'$ and $i = 1, \ldots, m$, respectively. We enhance the channel by assuming that user $j = 1, \ldots, m'$ knows the messages $M_{j+1}', \ldots, M_{m'}'$ and $M_1, \ldots, M_m$ and user $i = 1, \ldots, m$ knows the messages $M_{i+1}, \ldots, M_m$. Using Fano's inequality, chain rule, and data processing inequality we can bound the rates of the static user $j = 1, \ldots, m'$,

$$\begin{aligned}
nR_j' &\leq I(M_j'; Y_{j,1}', \ldots, Y_{j,n}'|M_{j+1}', \ldots, M_{m'}', M_1, \ldots, M_m) && (4.17)\\
&= \sum_{k=1}^{n} I(M_j'; Y_{j,k}'|U_{j,k}) && (4.18)\\
&\leq \sum_{k=1}^{n} I(M_j', U_{j,k}, Y_{j-1,1}', \ldots, Y_{j-1,k-1}'; Y_{j,k}'|U_{j,k}) && (4.19)\\
&= \sum_{k=1}^{n} I(U_{j-1,k}; Y_{j,k}'|U_{j,k}) && (4.20)
\end{aligned}$$

where

$$U_{j,k} = \left(M_{j+1}', \ldots, M_{m'}', M_1, \ldots, M_m, Y_{j,1}', \ldots, Y_{j,k-1}'\right),$$

$Y'_{j,k}$ denotes the received signal of user j at time instant k,

$$U_{m'} \to \cdots \to U_1 \to X \to (Y'_1, \ldots, Y'_{m'}, Y_1, \ldots, Y_m)$$

forms a Markov chain, and $U_0 = X$. The rate of static user m' can be bounded as

$$\begin{aligned}
nR'_{m'} &\leq \sum_{k=1}^{n} I(U_{m'-1,k}; Y'_{m',k}|U_{m',k}) && (4.21)\\
&= \sum_{k=1}^{n} I(X_k; Y'_{m',k}|U_{m',k}) - \sum_{k=1}^{n} I(X_k; Y'_{m',k}|U_{m'-1,k}) && (4.22)\\
&\leq \sum_{k=1}^{n} I(X_k, Y_{1,k+1}, \ldots, Y_{1,n}; Y'_{m',k}|U_{m',k}) - \sum_{k=1}^{n} I(X_k; Y'_{m',k}|U_{m'-1,k}) && (4.23)\\
&= \sum_{k=1}^{n} I(Y_{1,k+1}, \ldots, Y_{1,n}; Y'_{m',k}|U_{m',k}) + \sum_{k=1}^{n} I(X_k; Y'_{m',k}|U_{m',k}, Y_{1,k+1}, \ldots, Y_{1,n})\\
&\quad - \sum_{k=1}^{n} I(X_k; Y'_{m',k}|U_{m'-1,k}) && (4.24)\\
&= \sum_{k=1}^{n} I(W_k; Y'_{m',k}|U_{m',k}) + \sum_{k=1}^{n} I(X_k; Y'_{m',k}|U_{m',k}, W_k) - \sum_{k=1}^{n} I(X_k; Y'_{m',k}|U_{m'-1,k}), && (4.25)
\end{aligned}$$

where $W_k = Y^n_{1,k+1}$. Similarly,

$$\begin{aligned}
nR_i &\leq I(M_i; Y_{i,1}, \ldots, Y_{i,n}|M_{i+1}, \ldots, M_m) && (4.26)\\
&= \sum_{k=1}^{n} I(M_i; Y_{i,k}|V_{i,k}) && (4.27)\\
&= \sum_{k=1}^{n} I(M_i, V_{i,k}, Y_{i-1,k+1}, \ldots, Y_{i-1,n}; Y_{i,k}|V_{i,k}) && (4.28)\\
&= \sum_{k=1}^{n} I(V_{i-1,k}; Y_{i,k}|V_{i,k}), && (4.29)
\end{aligned}$$

where we define $V_{i,k} \triangleq (M_{i+1}, \ldots, M_m, Y_{i,k+1}, \ldots, Y_{i,n})$, which leads to the Markov chain $V_m \to \cdots \to V_1 \to (U_{m'}, W) \to X \to (Y'_1, \ldots, Y'_{m'}, Y_1, \ldots, Y_m)$. Using the chain rule and Csiszár sum identity [89], we obtain the bound (4.35).

$$
\begin{aligned}
R_1 &\le \sum_{k=1}^{n} I(M_1,\ldots,M_m;Y_{1,k}|V_{1,k}) &(4.30)\\
&\le \sum_{k=1}^{n} I(M_1,\ldots,M_m,Y_{1,k+1},\ldots,Y_{1,n};Y_{1,k}|V_{1,k}) &(4.31)\\
&= \sum_{k=1}^{n} I(M_1,\ldots,M_m,Y_{1,k+1},\ldots,Y_{1,n},Y'_{m',1},\ldots,Y'_{m',k-1};Y_{1,k}|V_{1,k}) &(4.32)\\
&\quad - \sum_{k=1}^{n} I(Y'_{m',1},\ldots,Y'_{m',k-1};Y_{1,k}|M_1,\ldots,M_m,Y_{1,k+1},\ldots,Y_{1,n}) &(4.33)\\
&= \sum_{k=1}^{n} I(U_{m',k},W_k;Y_{1,k}|V_{1,k}) - \sum_{k=1}^{n} I(Y_{1,k+1},\ldots,Y_{1,n};Y'_{m',k}|U_{m',k}) &(4.34)\\
&= \sum_{k=1}^{n} I(U_{m',k},W_k;Y_{1,k}|V_{1,k}) - \sum_{k=1}^{n} I(W_k;Y'_{m',k}|U_{m',k}). &(4.35)
\end{aligned}
$$

By introducing a time-sharing auxiliary random variable, Q, [88] and defining

$$
\begin{aligned}
X &\triangleq (X_Q,Q), & Y'_j &\triangleq (Y'_{j,Q},Q)\\
Y_i &\triangleq (Y_{i,Q},Q), & U_i &\triangleq (U_{i,Q},Q)\\
V_j &\triangleq (V_{j,Q},Q), & W &\triangleq (W_Q,Q), \qquad (4.36)
\end{aligned}
$$

we establish (4.7)-(4.10). Similarly, we can follow the same steps to prove (4.11)-(4.14) after switching the role of the two sets of variables $Y'_1,\ldots,Y'_{m'}$ and $Y_1,\ldots,Y_m$. This completes the proof of Theorem 8. □

Remark 6. *Theorem 8 is an extension of the Körner-Marton outer bound [66, Theorem 5] to more than two users, and it recovers the Körner-Marton bound when $m = m' = 1$.*

Remark 7. *For the multiuser multilevel broadcast channel characterized by* (4.6)*, we establish the capacity for degraded message sets in Appendix B, where one common message is communicated to all receivers and one further private message is communicated to one receiver.*

4.2.2 Outer Degrees of Freedom Region

In the sequel, we give an outer bound on the degrees of freedom of the broadcast channel defined in Section 4.1 when there is no CSIT for all users. The outer bound development depends on the results of Theorem 8 in Section 4.2.1.

Theorem 9. *An outer bound on the degrees of freedom region of the fading broadcast channel characterized by Eq.* (4.1) *without CSIT is,*

$$\sum_{j=1}^{m'} d'_j \leq 1, \tag{4.37}$$

$$\sum_{i=1}^{m} d_i \leq 1 - \frac{1}{T}, \tag{4.38}$$

$$\sum_{j=1}^{m'} d'_j + \sum_{i=1}^{m} d_i \leq \begin{cases} 1 & T = T',\ \Delta T = 0 \\ \frac{4}{3} & \text{otherwise}, \end{cases} \tag{4.39}$$

where ΔT is the offset between the two coherence intervals.

Proof. Equations (4.37) and (4.38) are outer bounds for a broadcast channel whose users are either all homogeneously static or all homogeneously dynamic [11, 23]. The remainder of the proof is dedicated to establishing (4.39). We enhance the channel by giving all users global CSIR. When $T' = T$ and $\Delta T = 0$, (4.39) follows directly from [23, 11]. When $T' \neq T$ or $\Delta T \neq 0$, having no CSIT, the channel belongs to the class of multiuser multilevel broadcast channels in Section 4.2.1. We then use the two outer bounds developed for the multilevel broadcast channels to generate two degrees of freedom bounds, and merge them to get the desired result.

We begin with the outer bound described in (4.7)-(4.10); we combine these equations to obtain partial sum-rate bounds on the static ($\sum R'_j$) and dynamic ($\sum R_i$) receivers:

$$\sum_{j=1}^{m'} R'_j \leq \sum_{j=1}^{m'-1} I(U_{j-1}; y'_j | U_j, \mathcal{H}) + I(W; y'_{m'} | U_{m'}, \mathcal{H}) + I(\mathbf{x}; y'_{m'} | U_{m'}, W, \mathcal{H})$$

$$
\begin{aligned}
&- I(\mathbf{x}; y'_{m'}|U_{m'-1}, \mathcal{H}) \\
=&\sum_{j=1}^{m'-1} h(y'_j|U_j, \mathcal{H}) - h(y'_j|U_{j-1}, \mathcal{H}) + I(W; y'_{m'}|U_{m'}, \mathcal{H}) + h(y'_{m'}|U_{m'}, W, \mathcal{H}) \\
&- h(y'_{m'}|U_{m'-1}, \mathcal{H}) + o(\log(\rho)) && (4.40) \\
=&I(W; y'_{m'}|U_{m'}, \mathcal{H}) + h(y'_{m'}|U_{m'}, W, \mathcal{H}) + o(\log(\rho)), && (4.41)
\end{aligned}
$$

where $\mathcal{H}$ is the set of all channel vectors, (4.40) follows from the chain rule, $h(y'_j|\mathbf{x}, \mathcal{H}) = o(\log(\rho))$, and (4.41) follows since the received signals of all static users, y'_j, have the same statistics [23, 11]. Also, using Theorem 8,

$$
\begin{aligned}
\sum_{j=1}^{m} R_j \leq& I(U_{m'}, W; y_1|V_1, \mathcal{H}) - I(W; y'_{m'}|U_{m'}, \mathcal{H}) + \sum_{j=2}^{m} I(V_{j-1}; y_j|V_j, \mathcal{H}) \\
=&h(y_1|V_1, \mathcal{H}) - h(y_1|U_{m'}, W, \mathcal{H}) - I(W; y'_{m'}|U_{m'}, \mathcal{H}) + \sum_{j=2}^{m} h(y_j|V_j, \mathcal{H}) \\
&- h(y_j|V_{j-1}, \mathcal{H}) && (4.42) \\
=&- h(y_1|U_{m'}, W, \mathcal{H}) - I(W; y'_{m'}|U_{m'}, \mathcal{H}) + h(y_m|V_m, \mathcal{H}) + o(\log(\rho)) && (4.43) \\
\leq&- h(y_1|U_{m'}, W, \mathcal{H}) - I(W; y'_{m'}|U_{m'}, \mathcal{H}) + \log(\rho) + o(\log(\rho)), && (4.44)
\end{aligned}
$$

where (4.42) follows from the chain rule, (4.43) follows since y_j have the same statistics, and (4.44) follows since $h(y_m|V_m, \mathcal{H}) \leq n\log(\rho) + o(\log(\rho))$. Define $Y'_{j,k}$ to be the received signal of user j at time instance k. From (4.41) and (4.44), we can obtain the bound (4.47) on the rates.

$$
\begin{aligned}
\frac{1}{2}\sum_{j=1}^{m'} R'_j + \sum_{j=1}^{m} R_j \leq& \frac{1}{2} I(W; y'_{m'}|U_{m'}, \mathcal{H}) + \frac{1}{2} h(y'_{m'}|U_{m'}, W, \mathcal{H}) - h(y_1|U_{m'}, W, \mathcal{H}) \\
&- I(W; y'_{m'}|U_{m'}, \mathcal{H}) + \log(\rho) + o(\log(\rho)) \\
=&\frac{1}{2} h(y'_{m'}|U_{m'}, W, \mathcal{H}) - h(y_1|U_{m'}, W, \mathcal{H}) + \log(\rho) + o(\log(\rho)) \\
\leq&\frac{1}{2} h(y'_{m'}, y_1|U_{m'}, W, \mathcal{H}) - h(y_1|U_{m'}, W, \mathcal{H}) + \log(\rho) + o(\log(\rho)) && (4.45)
\end{aligned}
$$

$$
\begin{aligned}
&\leq \sum_{k=1}^{n} \frac{1}{2} h(y'_{m',k}, y_{1,k} | U_{m'}, W, \mathcal{H}, y'_{m',1}, \ldots, y'_{m',k-1}, y_{1,1}, \ldots, y_{1,k-1}) \\
&\quad - h(y_{1,k} | U_{m'}, W, \mathcal{H}, y'_{m',1}, \ldots, y'_{m',k-1}, y_{1,1}, \ldots, y_{1,k-1}) + \log(\rho) \\
&\quad + o(\log(\rho)) \qquad (4.46) \\
&\leq \max_{\mathrm{Tr}\{\boldsymbol{\Sigma}_x\} \leq \rho, \boldsymbol{\Sigma}_x \succcurlyeq 0} \mathbb{E}_H \Big\{ \frac{1}{2} \log |\mathbf{I} + \mathbf{H}\boldsymbol{\Sigma}_x \mathbf{H}^\dagger| - \log(1 + \mathbf{h}_1^\dagger \boldsymbol{\Sigma}_x \mathbf{h}_1) \Big\} + \log(\rho) \\
&\quad + o(\log(\rho)), \qquad (4.47)
\end{aligned}
$$

where (4.45) and (4.46) follow from the chain rule and that conditioning does not increase differential entropy, and (4.47) follows from extremal entropy inequality [67, 68, 90]. In order to bound (4.47), we use a specialization of [39, Lemma 3] as follows.

Lemma 7. *Consider two random matrices $\mathbf{H}_1 \in \mathbb{C}^{N_1 \times N_t}$ and $\mathbf{H}_2 \in \mathbb{C}^{N_2 \times N_t}$, where $N_1 \geq N_2$. For a covariance matrix $\boldsymbol{\Sigma}_x$, where $Tr\{\boldsymbol{\Sigma}_x\} \leq \rho$, we have*

$$
\max_{\boldsymbol{\Sigma}_x} \frac{1}{\min\{N_t, N_1\}} \log |\mathbf{I} + \mathbf{H}_1 \boldsymbol{\Sigma}_x \mathbf{H}_1^\dagger| - \frac{1}{\min\{N_t, N_2\}} \log |\mathbf{I} + \mathbf{H}_2 \boldsymbol{\Sigma}_x \mathbf{H}_2^\dagger| \leq o(\log(\rho)). \quad (4.48)
$$

The proof of Lemma 7 is omitted as it directly follows from [39, Lemma 3]. Lemma 7 yields the following outer bound on the degrees of freedom:

$$
\frac{1}{2} \sum_{j=1}^{m'} d'_j + \sum_{i=1}^{m} d_i \leq 1. \qquad (4.49)
$$

We now repeat the exercise of bounding the sum rates and deriving degrees of freedom, this time starting from (4.11)-(4.14). By following bounding steps parallel to (4.41), (4.44), and (4.47),

$$
\sum_{j=1}^{m'} d'_j + \frac{1}{2} \sum_{i=1}^{m} d_i \leq 1. \qquad (4.50)
$$

Adding (4.49) and (4.50) yields the outer bound (4.39), completing the proof of Theorem 9.

□

4.2.3 Achievable Degrees of Freedom Region

Theorem 10. *The fading broadcast channel described by Eq.* (4.1) *can achieve the following degrees of freedom without CSIT:*

$$\sum_{i=1}^{m} d_i \leq 1 - \frac{1}{T}, \tag{4.51}$$

$$\sum_{j=1}^{m'} d'_j + \sum_{i=1}^{m} d_i \leq 1. \tag{4.52}$$

Proof. The achievable scheme uses product superposition [22, 23], where the transmitter uses one antenna to send the super symbol to two users: one dynamic and one static,

$$\mathbf{x}^\dagger = x_s \mathbf{x}_d^\dagger, \tag{4.53}$$

where $x_s \in \mathbb{C}$ is a symbol intended for the static user,

$$\mathbf{x}_d^\dagger = [x_\tau, \ \ \mathbf{x}_\delta^\dagger] \tag{4.54}$$

where $x_\tau \in \mathbb{C}$ is a pilot and $\mathbf{x}_\delta \in \mathbb{C}^{T-1}$ is a super symbol intended for the dynamic user. Since degrees of freedom analysis is insensitive to the additive noise, we omit the noise component in the following.

$$\begin{aligned} \mathbf{y}^\dagger &= h x_s [x_\tau, \ \ \mathbf{x}_\delta^\dagger] \\ &= [\overline{h} x_\tau, \ \ \overline{h} \mathbf{x}_\delta^\dagger], \end{aligned} \tag{4.55}$$

where $\overline{h} = h x_s$. The dynamic user estimates the equivalent channel $\overline{h}$ during the first time instance and then decodes $\mathbf{x}_\delta$ *coherently* based on the channel estimate. The static receiver only utilizes the received signal during the first time instance:

$$y'_1 = g x_s. \tag{4.56}$$

Knowing its channel gain g, the static receiver can decode x_s. The achievable degrees of freedom of the two users are,

$$(d', d) = \left(\frac{1}{T}, 1 - \frac{1}{T}\right). \tag{4.57}$$

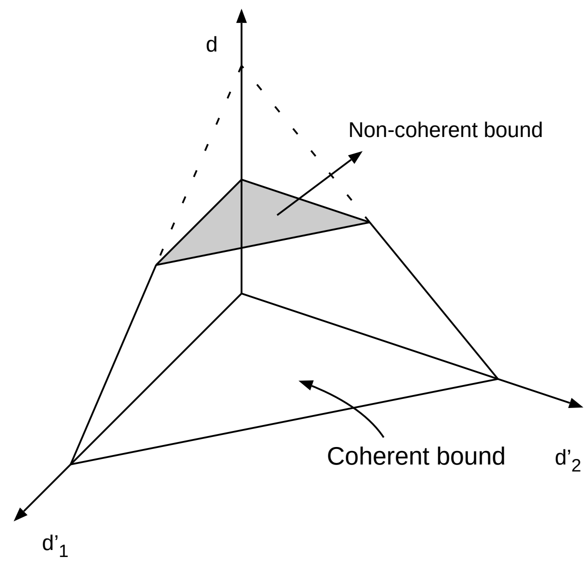

Figure 4.3. Achievable degrees of freedom region of one dynamic and two static users

We now proceed to prove that the degrees of freedom region characterized by (4.51) and (4.52) can be achieved via a combination of two-user product superposition strategies that were outlined above, and single-user strategies. For clarity of exposition we refer to (4.51), which describes the degrees of freedom constraints of the dynamic receivers, as the *non-coherent bound*, and to (4.52) as the *coherent bound*. The non-negativity of degrees of freedom restricts them to the non-negative orthant $\mathbb{R}_+^{m+m'}$. The intersection of the coherent bound and the non-negative orthant is a $(m'+m)$–simplex that has $m+m'+1$ vertices. The non-coherent bound is a hyperplane that partitions the simplex with $m'+1$ vertices on one side of the non-coherent bound and m on the other. Therefore the intersection of the simplex with the non-coherent bound produces a polytope with $(m'+1)(m+1)$ vertices.[3] For illustration, see Fig. 4.3 showing the three-user degrees of freedom with two static users and Fig. 4.4 with one static user.

We now verify that each of the $(m'+1)(m+1)$ vertices can be achieved with either a single-user strategy or via a two-user product superposition strategy:

[3]This can be verified with a simple counting exercise involving the number of edges of the simplex that cross the non-coherent bound.

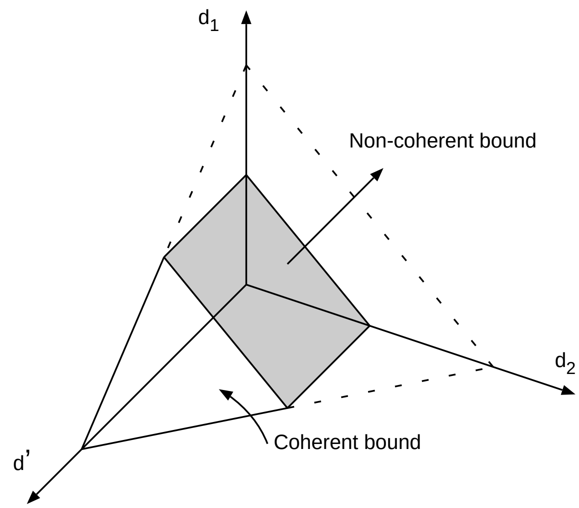

Figure 4.4. Achievable degrees of freedom region of one static and two dynamic users

- m' vertices corresponding to single-user transmission to each static user j achieving one degree of freedom.

- m vertices corresponding to single-user transmission to each dynamic user i achieving $(1 - \frac{1}{T})$ degrees of freedom.

- $m'm$ vertices corresponding to product superposition applied to all possible pairs of static and dynamic users, achieving $\frac{1}{T}$ degrees of freedom for one static user and $(1-\frac{1}{T})$ degrees of freedom for one dynamic user.

- One trivial vertex at the origin, corresponding to no transmission achieving zero degrees of freedom for all users.

Hence, the number of the vertices is $m' + m + m'm + 1 = (m + 1)(m' + 1)$. This completes the achievability proof of Theorem 10. □

Remark 8. *When the static and dynamic users have the same coherence time, the inner and outer bounds on degrees of freedom coincide. In this case it is degrees of freedom optimal to serve two-users at a time (one dynamic and one static).*

4.3 Delayed CSIT for All Users

Under delayed CSIT, the transmitter knows each channel gain only after it is no longer valid. This condition is also known as outdated CSIT. We begin by providing inner and outer bounds when transmitting only to static users, only to dynamic users, and to one static and one dynamic user. We then synthesize this collection of bounds into an overall degrees of freedom region.

4.3.1 Transmission to Static Users

Theorem 11. *The degrees of freedom region of the fading broadcast channel characterized by Eq.* (4.1) *with delayed CSIT and having* m' *static users and no dynamic users is*

$$d'_j \leq \frac{1}{1+\frac{1}{2}+\ldots+\frac{1}{m'}}, \quad j=1,\ldots,m'. \tag{4.58}$$

Proof. The special case of fast fading ($T'=1$) was discussed by Maddah-Ali and Tse [32], where the achievability was established by *retrospective interference alignment* that aligns the interference using the outdated CSIT, and the converse was proved by generating an improved channel without CSIT having a tight degrees of freedom region against TDMA according to the results [9, 29]. For $T' \geq 1$, the achievability is established by employing retrospective interference alignment [32] over super symbols each of length T'. The converse is proved by following the same procedures in [32] to generate a block-fading improved channel without CSIT and with identical coherence intervals of length T'. According to the results of [23, 11], TDMA is tight against the degrees of freedom region of the improved channel. □

4.3.2 Transmission to Dynamic Users

Theorem 12. *The fading broadcast channel characterized by Eq.* (4.1)*, with delayed CSIT and having* m *dynamic users and no static users, can achieve the degrees of freedom*

$$d_i \leq \frac{1}{1+\frac{1}{2}+\ldots+\frac{1}{m}}(1-\frac{m}{T}), \quad i=1,\ldots,m. \tag{4.59}$$

An outer bound on the degrees of freedom region is

$$d_i \leq 1-\frac{1}{T}, \tag{4.60}$$

$$\sum_{i=1}^{m} d_i \leq \frac{m}{1+\frac{1}{2}+\ldots+\frac{1}{m}}. \tag{4.61}$$

Proof. The achievability part can be proved as follows. At the beginning of each super symbol, m pilots are sent for channel estimation. Then retrospective interference alignment [32] over super symbols is employed during the remaining $(T-m)$ instances, to achieve (4.59). For the converse part, (4.61) is proved by giving the users global CSIR, and then applying Theorem 11. Moreover, (4.60) is the single-user bound for each dynamic user that can be proved as follows. For a single user with delayed CSIT, feedback does not increase the capacity [91], and consequently the assumption of delayed CSIT can be removed. Hence, the single-user bound for each dynamic user with delayed CSIT is the same as the single-user bound without CSIT [7]. □

4.3.3 Transmission to One Static and One Dynamic User

Theorem 13. *The fading broadcast channel characterized by Eq.* (4.1)*, with delayed CSIT and having one static and one dynamic user, can achieve the following degrees of freedom*

$$\mathcal{D}_1 : (d', d) = \big(\frac{2}{3}(1+\frac{1}{T}), \frac{2}{3}(1-\frac{2}{T})\big), \tag{4.62}$$

$$\mathcal{D}_2 : (d', d) = (\frac{1}{T}, 1-\frac{1}{T}). \tag{4.63}$$

Furthermore, the achievable degrees of freedom region is the convex hull of the above degrees of freedom pairs.

Proof. From Section 4.2.3, product superposition achieves the pair (4.63) that does not require CSIT for any of the two users. The remainder of the proof is dedicated to the achievability of the pair (4.62). We provide a transmission scheme based on retrospective interference alignment [32] along with product superposition.

1. The transmitter first emits a super-symbol intended for the static user:

$$\mathbf{X}_1 = [\mathbf{X}_{1,1}, \ \ \cdots, \ \ \mathbf{X}_{1,\ell}], \tag{4.64}$$

where $\ell = \frac{T'}{T}$, and each $\mathbf{X}_{1,n} \in \mathbb{C}^{2\times T}$ occupies T time instances and has the following structure:

$$\mathbf{X}_{1,n} = [\bar{\mathbf{U}}_n, \ \ \bar{\mathbf{U}}_n\mathbf{U}_n], \quad n = 1, \ldots, \ell, \tag{4.65}$$

both the diagonal matrix $\bar{\mathbf{U}}_n \in \mathbb{C}^{2\times 2}$ and $\mathbf{U}_n \in \mathbb{C}^{2\times(T-2)}$ contain symbols intended for the static user. The components of $\mathbf{y}_1^{'\dagger} = [\mathbf{y}_{1,1}^{'\dagger}, \ \ \cdots, \ \ \mathbf{y}_{1,\ell}^{'\dagger}]$ are:

$$\begin{aligned}\mathbf{y}_{1,n}^{'\dagger} =& [\mathbf{g}_1^{\dagger}\bar{\mathbf{U}}_n, \ \ \mathbf{g}_1^{\dagger}\bar{\mathbf{U}}_n\mathbf{U}_n], \quad n = 1, \ldots, \ell \\ =& [\tilde{\mathbf{g}}_{1,n}^{\dagger}, \ \ \tilde{\mathbf{g}}_{1,n}^{\dagger}\mathbf{U}_n],\end{aligned} \tag{4.66}$$

where $\tilde{\mathbf{g}}_{1,n}^{\dagger} = \mathbf{g}_1^{\dagger}\bar{\mathbf{U}}_n$. The static user by definition knows $\mathbf{g}_1$ so it can decode $\bar{\mathbf{U}}_n$ which yields $2\frac{T'}{T}$ degrees of freedom. The remaining $\frac{T'}{T}(T-2)$ observations in $\tilde{\mathbf{g}}_{1,n}^{\dagger}\mathbf{U}_n$ involve $2\frac{T'}{T}(T-2)$ unknowns, so they require a further $\frac{T'}{T}(T-2)$ independent observations for reliable decoding.

The components of $\mathbf{y}_1^{\dagger} = [\mathbf{y}_{1,1}^{\dagger}, \ \ \cdots, \ \ \mathbf{y}_{1,\ell}^{\dagger}]$ are

$$\begin{aligned}\mathbf{y}_{1,n}^{\dagger} =& [\mathbf{h}_{1,n}^{\dagger}\bar{\mathbf{U}}_n, \ \ \mathbf{h}_{1,n}^{\dagger}\bar{\mathbf{U}}_n\mathbf{U}_n], \quad n = 1, \ldots, \ell \\ =& [\tilde{\mathbf{h}}_{1,n}^{\dagger}, \ \ \tilde{\mathbf{h}}_{1,n}^{\dagger}\mathbf{U}_n],\end{aligned} \tag{4.67}$$

where $\tilde{\mathbf{h}}_{1,n}^{\dagger} = \mathbf{h}_{1,n}^{\dagger}\bar{\mathbf{U}}_n$ is the equivalent channel estimated by the dynamic user. The dynamic user saves $\tilde{\mathbf{h}}_{1,n}^{\dagger}\mathbf{U}_n$ for interference cancellation in the upcoming steps.

2. The transmitter sends a second super symbol intended for the dynamic user:

$$\mathbf{X}_2 = [\mathbf{X}_{2,1}, \ \cdots, \ \mathbf{X}_{2,\ell}], \tag{4.68}$$

where

$$\mathbf{X}_{2,n} = [\tilde{\mathbf{U}}_n, \ \tilde{\mathbf{U}}_n\mathbf{V}_n], \quad n = 1, \ldots, \ell, \tag{4.69}$$

$\tilde{\mathbf{U}}_n \in \mathbb{C}^{2\times 2}$ is diagonal and includes 2 independent symbols intended for the static user, and $\mathbf{V}_n \in \mathbb{C}^{2\times(T-2)}$ contains independent symbols intended for the dynamic user. The components of $\mathbf{y}_2^\dagger = [\mathbf{y}_{2,1}^\dagger, \ \cdots, \ \mathbf{y}_{2,\ell}^\dagger]$ are

$$\begin{aligned}\mathbf{y}_{2,n}^\dagger =& [\mathbf{h}_{2,n}^\dagger\tilde{\mathbf{U}}_n, \ \mathbf{h}_{2,n}^\dagger\tilde{\mathbf{U}}_n\mathbf{V}_n], \quad n = 1, \ldots, \ell \\ =& [\tilde{\mathbf{h}}_{2,n}^\dagger, \ \tilde{\mathbf{h}}_{2,n}^\dagger\mathbf{V}_n],\end{aligned} \tag{4.70}$$

where $\tilde{\mathbf{h}}_{2,n}^\dagger = \mathbf{h}_{2,n}^\dagger\tilde{\mathbf{U}}_n$ is the equivalent channel estimated by the dynamic user. The dynamic user saves $\tilde{\mathbf{h}}_{2,n}^\dagger\mathbf{V}_n$ which includes $\frac{T'}{T}(T-2)$ independent observations about $2\frac{T'}{T}(T-2)$ unknowns, and hence an additional $\frac{T'}{T}(T-2)$ observations are needed to decode $\mathbf{V}_n$. The components of $\mathbf{y}_2'^\dagger = [\mathbf{y}_{2,1}'^\dagger, \ \cdots, \ \mathbf{y}_{2,\ell}'^\dagger]$ are

$$\begin{aligned}\mathbf{y}_{2,n}' =& [\mathbf{g}_2^\dagger\tilde{\mathbf{U}}_n, \ \mathbf{g}_2^\dagger\tilde{\mathbf{U}}_n\mathbf{V}_n], \quad n = 1, \ldots, \ell \\ =& [\tilde{\mathbf{g}}_{2,n}^\dagger, \ \tilde{\mathbf{g}}_{2,n}^\dagger\mathbf{V}_n],\end{aligned} \tag{4.71}$$

where $\tilde{\mathbf{g}}_{2,n}^\dagger = \mathbf{g}_2^\dagger\tilde{\mathbf{U}}_n$ is the equivalent channel estimated by the static user; the static user saves $\tilde{\mathbf{g}}_{2,n}^\dagger\mathbf{V}_n$ for the upcoming steps. Knowing $\mathbf{g}_2$, the static user achieves $2\frac{T'}{T}$ further degrees of freedom from decoding $\tilde{\mathbf{U}}_n$.

3. The transmitter emits a third super symbol consisting of a linear combination of the signals generated from the first and the second super symbols.

$$\mathbf{X}_3 = [\mathbf{X}_{3,1}, \ \cdots, \ \mathbf{X}_{3,\ell}], \tag{4.72}$$

where

$$\mathbf{X}_{3,n} = [\hat{\mathbf{U}}_n, \quad \hat{\mathbf{U}}_n(\tilde{\mathbf{h}}_{1,n}^\dagger \mathbf{U}_n + \tilde{\mathbf{g}}_{2,n}^\dagger \mathbf{V}_n)], \quad n = 1, \ldots, \ell, \tag{4.73}$$

$\hat{\mathbf{U}}_n \in \mathbb{C}^{2\times 2}$ is diagonal and contains 2 independent symbols intended for the static user, and hence the static user achieves further $2\frac{T'}{T}$ degrees of freedom.

The static user cancels $\tilde{\mathbf{g}}_{2,n}^\dagger \mathbf{V}_n$ saved during the second super symbol and obtains $\tilde{\mathbf{h}}_{1,n}^\dagger \mathbf{U}_n$ that includes the additional independent $\frac{T'}{T}(T-2)$ observations needed for decoding $\mathbf{U}_n$. Therefore, the static user achieves $2\frac{T'}{T}(T-2)$ further degrees of freedom. The dynamic user estimates the equivalent channel $\tilde{\mathbf{h}}_{3,n}^\dagger = \mathbf{h}_{3,n}^\dagger \hat{\mathbf{U}}_n$, cancels $\tilde{\mathbf{h}}_{1,n}^\dagger \mathbf{U}_n$ saved during the first super symbol, and obtains $\tilde{\mathbf{g}}_{2,n}^\dagger \mathbf{V}_n$ that contains the additional observations needed for decoding $\mathbf{V}_n$. Hence, the dynamic user achieves $2\frac{T'}{T}(T-2)$ degrees of freedom.

In aggregate, over $3T'$ time instants, the static and dynamic user achieve the degrees of freedom

$$d' = 6\frac{T'}{T} + 2\frac{T'}{T}(T-2), \qquad d = 2\frac{T'}{T}(T-2). \tag{4.74}$$

This completes the proof of Theorem 13. □

Theorem 14. *An outer bound on the degrees of freedom region of the fading broadcast channel characterized by Eq.* (4.1)*, with one static and one dynamic user having delayed CSIT, is*

$$\frac{d'}{2} + d \leq 1, \tag{4.75}$$

$$d' + \frac{d}{2} \leq 1, \tag{4.76}$$

$$d \leq 1 - \frac{1}{T}. \tag{4.77}$$

Proof. The inequality (4.77) represents the single-user outer bound [7]. We prove the bound (4.75) as follows. We enhance the original channel by giving both users global CSIR.

In addition, the channel output of the dynamic user, $y(t)$, is given to the static user. Therefore, the channel outputs at time instant t are $(y'(t), y(t), \mathcal{H})$ at the static user, and $(y(t), \mathcal{H})$ at the dynamic user. The enhanced channel is physically degraded [86, 87], hence, removing the delayed CSIT does not reduce the capacity [92]. Also,

$$\begin{aligned} R' \leq & I(x(t); y'(t), y(t)|U, \mathcal{H}) = h(y'(t), y(t)|U, \mathcal{H}) - h(y'(t), y(t)|U, x(t), \mathcal{H}) \\ R \leq & I(U; y(t)|\mathcal{H}) = h(y(t)|\mathcal{H}) - h(y(t)|U, \mathcal{H}), \end{aligned} \tag{4.78}$$

where U is an auxiliary random variable, and $U \to x \to (y'(t), y(t))$ forms a Markov chain. Therefore,

$$\begin{aligned} \frac{R'}{2} + R \leq & h(y(t)|\mathcal{H}) + \frac{1}{2} h(y'(t), y(t)|U, \mathcal{H}) - h(y(t)|U, \mathcal{H}) + o(\log(\rho)) \\ \leq & \log(\rho) + \frac{1}{2} h(y'(t), y(t)|U, \mathcal{H}) - h(y(t)|U, \mathcal{H}) + o(\log(\rho)) && (4.79) \\ \leq & \log(\rho) + \max_{\mathrm{Tr}\{\mathbf{\Sigma}_x\} \leq \rho, \mathbf{\Sigma}_x \succcurlyeq 0} \mathbb{E}_H \Big\{ \frac{1}{2} \log |\mathbf{I} + \mathbf{H} \mathbf{\Sigma}_x \mathbf{H}^\dagger| - \log(1 + \mathbf{h}^\dagger(t) \mathbf{\Sigma}_x \mathbf{h}(t)) \Big\} + o(\log(\rho)) && (4.80) \\ \leq & \log(\rho) + o(\log(\rho)), && (4.81) \end{aligned}$$

where (4.79) follows since $h(y(t)|\mathcal{H}) \leq \log(\rho) + o(\log(\rho))$ [13], (4.80) follows from extremal entropy inequality [39, 67, 68], and (4.81) follows from Lemma 7. Hence, the bound (4.75) is proved. A similar argument, with the role of the two users reversed, leads to the bound (4.76).

□

Remark 9. *The inner and outer bounds obtained for the two-user case partially meet, with the gap diminishing with the coherence time of the dynamic user as shown in Fig. 4.5 and Fig. 4.6 for* $T = 15$ *and* $T = 30$*, respectively.*

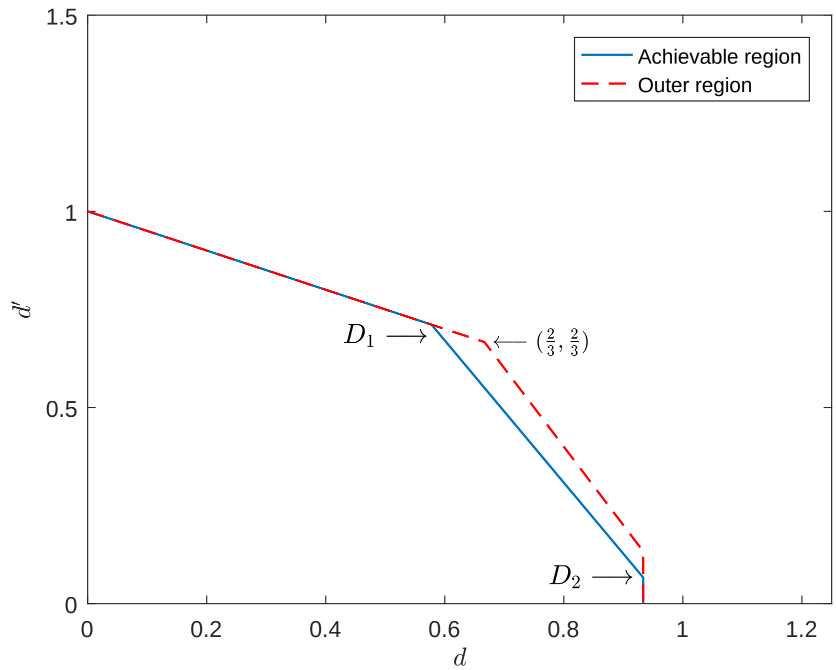

Figure 4.5. One static and one dynamic with delayed CSIT and $T = 15$

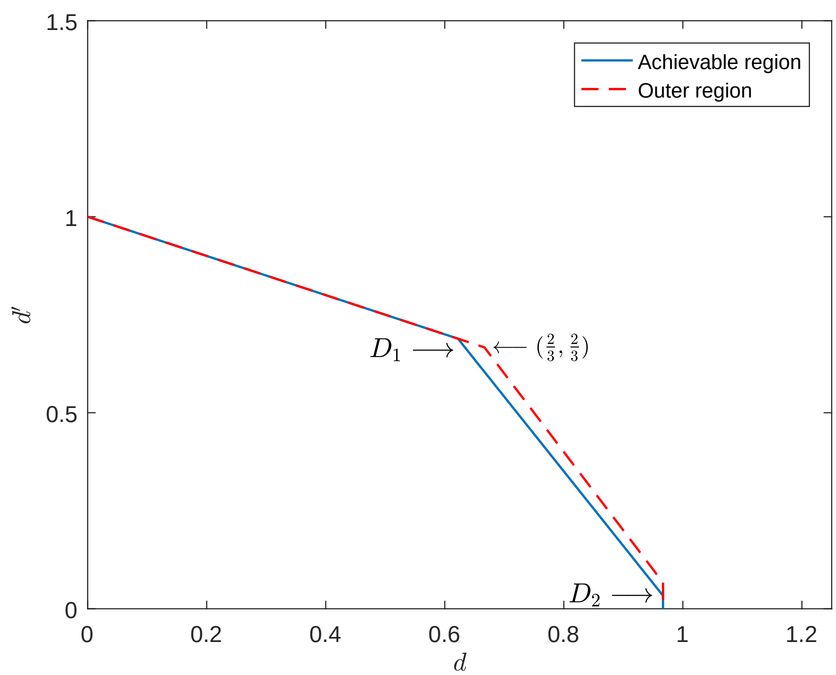

Figure 4.6. One static and one dynamic with delayed CSIT and $T = 30$

4.3.4 Transmission to Arbitrary Number of Static and Dynamic Users

Theorem 15. *The fading broadcast channel characterized by Eq.* (4.1)*, with delayed CSIT, can achieve the multiuser degrees of freedom characterized by vectors* $\mathcal{D}_i$*,*

$$\mathcal{D}_1 : \frac{1}{1+\frac{1}{2}+\ldots+\frac{1}{m'}}\sum_{i=1}^{m'}\mathbf{e}_i^{\dagger}, \tag{4.82}$$

$$\mathcal{D}_2,\ldots,\mathcal{D}_{mm'+1} : \frac{2}{3}(1+\frac{1}{T})\mathbf{e}_j^{\dagger}+\frac{2}{3}(1-\frac{2}{T})\mathbf{e}_{m'+i}^{\dagger},\ j=1,\ldots,m',\quad i=1,\ldots,m, \tag{4.83}$$

$$\mathcal{D}_{mm'+2},\ldots,\mathcal{D}_{mm'+m'+2} : \frac{m}{T}\mathbf{e}_j^{\dagger}+\frac{1}{1+\frac{1}{2}+\ldots+\frac{1}{m}}(1-\frac{m}{T})\sum_{i=1}^{m}\mathbf{e}_i^{\dagger},\quad j=1,\ldots,m', \tag{4.84}$$

where $\mathbf{e}_j$ *is the canonical coordinate vector. Their convex hull characterized an achievable degrees of freedom region.*

Proof. The achievability of (4.82) was proved in Section 4.3.1 via multiuser transmission to static users. The achievability of (4.83) was proved in Section 4.3.3, via a two-user transmission to a dynamic-static pair.

We now show the achievability of (4.84) via retrospective interference alignment [32] along with product superposition. Over a super symbol of length T, consider the following transmission:

$$\mathbf{X} = [\mathbf{U},\ \ \mathbf{UV}], \tag{4.85}$$

where $\mathbf{U} \in \mathbb{C}^{m\times m}$ is diagonal and includes m independent symbols intended for the static user j, and $\mathbf{V} \in \mathbb{C}^{m\times(T-m)}$ is a super symbol containing independent symbols intended for the dynamic users according to retrospective interference alignment [32]. Therefore, the static user decodes $\mathbf{U}$. Thus, over T time instants, the static user achieves m degrees of freedom and the dynamic users achieve $\frac{1}{1+\frac{1}{2}+\ldots+\frac{1}{m}}(T-m)$, hence (4.84) is achieved. □

Theorem 16. *An outer bound on the degrees of freedom of the fading broadcast channel characterized by Eq.* (4.1)*, with delayed CSIT, is*

$$\sum_{j=1}^{m'}\frac{d'_j}{m'+m}+\sum_{i=1}^{m}\frac{d_i}{m}\leq 1, \tag{4.86}$$

$$\sum_{j=1}^{m'} \frac{d'_j}{m'} + \sum_{i=1}^{m} \frac{d_i}{m'+m} \leq 1, \tag{4.87}$$

$$d'_j \leq 1, \quad \forall j = 1, \ldots, m', \tag{4.88}$$

$$d_i \leq 1 - \frac{1}{T}, \quad \forall i = 1, \ldots, m. \tag{4.89}$$

Proof. The inequalities (4.88), (4.89) represent the single-user bounds on the static and the dynamic users, respectively [13, 7]. The remainder of the proof is dedicated to establishing the bounds (4.86) and (4.87).

We enhance the channel by providing global CSIR as well as allowing full cooperation among static users and full cooperation among dynamic users. The enhanced channel is equivalent to a broadcast channel with two users: one static equipped with m' antennas and one dynamic equipped with m antennas. Define $\mathbf{Y}' \in \mathbb{C}^{m'}$ and $\mathbf{Y} \in \mathbb{C}^m$ to be the received signals of the static and the dynamic super-user, respectively, in the enhanced channel. We further enhance the channel by giving $\mathbf{Y}$ to the static user, generating a physically degraded channel since $\mathbf{X} \to (\mathbf{Y}', \mathbf{Y}) \to \mathbf{Y}$ forms a Markov chain. Feedback including delayed CSIT has no effect on capacity [92], therefore we remove it from consideration. Subsequently, we can utilize the Körner-Marton outer bound [66],

$$\begin{aligned} \sum_{j=1}^{m'} R'_j \leq & I(\mathbf{X}; \mathbf{Y}', \mathbf{Y} | U, \mathcal{H}) \\ \sum_{i=1}^{m} R_i \leq & I(U; \mathbf{Y} | \mathcal{H}). \end{aligned} \tag{4.90}$$

Therefore, from applying extremal entropy inequality [39, 67, 93] and Lemma 7,

$$\begin{aligned} \sum_{j=1}^{m'} \frac{R'_j}{m'+m} + \sum_{i=1}^{m} \frac{R_i}{m} \leq & \frac{1}{m'+m} I(\mathbf{X}; \mathbf{Y}', \mathbf{Y} | U, \mathcal{H}) + \frac{1}{m} I(U; \mathbf{Y} | \mathcal{H}) \\ = & \frac{1}{m'+m} h(\mathbf{Y}', \mathbf{Y} | U, \mathcal{H}) + o(\log(\rho)) + \frac{1}{m} h(\mathbf{Y} | \mathcal{H}) - \frac{1}{m} h(\mathbf{Y} | U, \mathcal{H}) \\ \leq & \log(\rho) + o(\log(\rho)). \end{aligned} \tag{4.91}$$

Therefore, the bound (4.86) is proved. Similarly, we can prove the bound (4.87) using the same steps after switching the roles of the two users in the enhanced channel. □

4.4 Hybrid CSIT: Perfect CSIT for the Static Users and No CSIT for the Dynamic Users

Theorem 17. *The fading broadcast channel characterized by Eq.* (4.1)*, with perfect CSIT for the static users and no CSIT for the dynamic users, can achieve the following multiuser degrees of freedom,*

$$\mathcal{D}_1 : \sum_{j=1}^{m'} \mathbf{e}_j^\dagger, \tag{4.92}$$

$$\mathcal{D}_2, \ldots, \mathcal{D}_{m+1} : \frac{1}{T}\sum_{j=1}^{m'} \mathbf{e}_j^\dagger + (1-\frac{1}{T})\mathbf{e}_i^\dagger, \quad i = 1, \ldots, m. \tag{4.93}$$

Therefore, their convex hull is also achievable.

Proof. $\mathcal{D}_1$ is achieved by inverting the channels of the static users at the transmitter and then every static user achieves one degree of freedom. $\mathcal{D}_2, \ldots, \mathcal{D}_{m+1}$ in (4.93) are achieved using product superposition along with channel inversion as follows. The transmitted signal over T instants is,

$$\mathbf{X} = [\mathbf{u}, \;\; \mathbf{u}\mathbf{v}^\dagger], \tag{4.94}$$

where $\mathbf{u} = \sum_{j=1}^{m'} \mathbf{b}_j u_j$, u_j is a symbol intended for the static user j, $\mathbf{g}_j^\dagger \mathbf{b}_j = 0$, and $\mathbf{v} \in \mathbb{C}^{T-1}$ contain independent symbols intended for the dynamic user i. Each of the static users receive an interference-free signal during the first time instant achieving one degrees of freedom. The dynamic user estimates its equivalent channel during the first time instant and decodes $\mathbf{v}$ during the remaining $(T-1)$ time instants. □

Theorem 18. *An outer bound on the degrees of freedom of the fading broadcast channel characterized by Eq.* (4.1)*, with perfect CSIT for the static users and no CSIT for the dynamic* *users, is*

$$\sum_{j=1}^{m'} \frac{d'_j}{m'+1} + \sum_{i=1}^{m} d_i \le 1, \tag{4.95}$$

$$d'_j \le 1, \quad \forall j = 1, \ldots, m', \tag{4.96}$$

$$\sum_{i=1}^{m} d_i \le 1 - \frac{1}{T}. \tag{4.97}$$

Proof. The inequalities (4.96) represent single-user bounds for the static users [13], and (4.97) is a time-sharing outer bound for the dynamic users that was established in [23, 11]. It remains to prove (4.95), as follows.

We enhance the channel by giving global CSIR to all users and allowing full cooperation between the static users. This gives rise to an equivalent static user with m' antennas receiving $\mathbf{Y}'$ over an equivalent channel $\mathbf{G}$ and noise $\mathbf{Z}'$. At this point, we have a multiuser system where CSIT is available with respect to one user, but not others. We then bound the performance of this system with that of another (similar) system that has no CSIT. To do so, we use the *local statistical equivalence property* developed and used in [94, 42, 44]. First, we draw $\tilde{\mathbf{G}}, \tilde{\mathbf{Z}}$ according to the distribution of $\mathbf{G}, \mathbf{Z}'$ and independent of them. We enhance the channel by providing $\tilde{\mathbf{Y}} = \tilde{\mathbf{G}}\mathbf{X} + \tilde{\mathbf{Z}}$ to the static receiver and $\tilde{\mathbf{G}}$ to all receivers. Because we do *not* provide $\tilde{\mathbf{G}}$ to the transmitter, there is no CSIT with respect to $\tilde{\mathbf{Y}}$. According to [94], we have $h(\tilde{\mathbf{Y}}, \mathbf{Y}'|\mathcal{H}) = h(\mathbf{Y}'|\mathcal{H}) + o(\log(\rho))$, where $\mathcal{H} = (\mathbf{G}, \tilde{\mathbf{G}}, \mathbf{h}_1, \ldots, \mathbf{h}_m)$, therefore we can remove $\mathbf{Y}'$ from the enhanced channel without reducing its degrees of freedom. This new equivalent channel has one user with m' antennas receiving $(\tilde{\mathbf{Y}}, \mathcal{H})$, m single-antenna users receiving $(y_i, \mathcal{H})$, and no CSIT.[4] Having no CSIT, the enhanced channel is in the form of a multilevel broadcast channel studied in Section 4.2.1, and hence using Theorem 8,

[4]In the enhanced channel after removal of $\mathbf{Y}'$, the transmitter and receivers still share information about $\mathbf{G}$, but this random variable is now independent of all (remaining) transmit and receive variables.

$$
\begin{aligned}
\sum_{j=1}^{m'} R'_j \leq & I(W;\tilde{\mathbf{Y}}|U,\mathcal{H}) + I(\mathbf{X};\tilde{\mathbf{Y}}|U,W,\mathcal{H}) \\
R_1 \leq & I(U,W;\mathbf{y}_1|V_1,\mathcal{H}) - I(W;\tilde{\mathbf{Y}}|U,\mathcal{H}) \\
R_i \leq & I(V_{i-1};\mathbf{y}_i|V_i,\mathcal{H}), \quad i = 2,\ldots,m.
\end{aligned} \tag{4.98}
$$

The dynamic receiver received signals have the same distribution. By following bounding steps parallel to (4.42), (4.43), (4.44),

$$
\sum_{j=1}^{m} R_i \leq \log(\rho) + o(\log(\rho)) - I(W;\tilde{\mathbf{Y}}|U,\mathcal{H}) - h(\mathbf{y}_1|U,W,\mathcal{H}). \tag{4.99}
$$

Therefore,

$$
\begin{aligned}
\sum_{j=1}^{m'} \frac{R'_j}{m'+1} + \sum_{j=1}^{m} R_i \leq & \log(\rho) + o(\log(\rho)) + (\frac{1}{m'+1} - 1)I(W;\tilde{\mathbf{Y}}|U,\mathcal{H}) + \frac{h(\tilde{\mathbf{Y}}|U,W,\mathcal{H})}{m'+1} \\
& - h(\mathbf{y}_1|U,W,\mathcal{H}), & (4.100) \\
\leq & \log(\rho) + o(\log(\rho)) + \frac{h(\tilde{\mathbf{Y}},\mathbf{y}_1|U,W,\mathcal{H})}{m'+1} - h(\mathbf{y}_1|U,W,\mathcal{H}) & (4.101) \\
\leq & \log(\rho) + o(\log(\rho)), & (4.102)
\end{aligned}
$$

where the last inequality follows from applying the extremal entropy inequality [39, 67, 93] and Lemma 7. This concludes the proof of the bound (4.95). □

4.5 Hybrid CSIT: Perfect CSIT for the Static Users and Delayed CSIT for the Dynamic Users

We begin with inner and outer bounds for one static and one dynamic user, then extend the result to multiple users. The transmitter knows the channel of the static users perfectly and instantaneously, and an outdated version of the channel of the dynamic users.

4.5.1 Transmitting to One Static and One Dynamic User

Theorem 19. *For the fading broadcast channel characterized by Eq.* (4.1) *with one static and one dynamic user, with perfect CSIT for the static user and delayed CSIT for the dynamic user, the achievable degrees of freedom region is the convex hull of the vectors,*

$$\mathcal{D}_1 :(d', d) = (1 - \frac{1}{2T}, \frac{1}{2} - \frac{1}{2T}), \tag{4.103}$$

$$\mathcal{D}_2 :(d', d) = (\frac{1}{T}, 1 - \frac{1}{T}). \tag{4.104}$$

Proof. The degrees of freedom (4.104) can be achieved by product superposition as discussed in Section 4.2, without CSIT. We proceed to prove the achievability of (4.103).

1. Consider $[\mathbf{u}_1, \quad \cdots, \quad \mathbf{u}_{T-1}]$ to be a complex $2 \times (T-1)$ matrix containing symbols intended for the static user, $[v_1, \quad \cdots, \quad v_{T-1}]$ intended for the dynamic user, and $\mathbf{b} \in \mathbb{C}$ is a beamforming vector so that $\mathbf{g}^{\dagger}\mathbf{b} = 0$. In addition we define $\mathbf{u}_0 = \mathbf{0}, v_0 = 1$. Using these components, the transmitter constructs and transmits a super-symbol of length T, whose value at time t is:

   $$\mathbf{x}_1^{\dagger}(t) = \mathbf{u}_t + \mathbf{b}\, v_t. \tag{4.105}$$

   Note that $\mathbf{x}_1(0) = \mathbf{b}$ does not carry any information for either user, and serves as a pilot. The received super symbol at the static user is:

   $$\mathbf{y}_1^{'\dagger} = [0, \quad \mathbf{g}^{\dagger}\mathbf{u}_1, \quad \cdots, \quad \mathbf{g}^{\dagger}\mathbf{u}_{T-1}]. \tag{4.106}$$

   The received super symbol at the dynamic user

   $$y_1^{\dagger} = [\mathbf{h}_1^{\dagger}\mathbf{b}, \quad (\mathbf{h}_1^{\dagger}\mathbf{u}_1 + \mathbf{h}_1^{\dagger}\mathbf{b}v_1), \quad \cdots, \quad (\mathbf{h}_1^{\dagger}\mathbf{u}_{T-1} + \mathbf{h}_1^{\dagger}\mathbf{b}v_{T-1})]. \tag{4.107}$$

   The dynamic user estimates its equivalent channel $\mathbf{h}_1^{\dagger}\mathbf{b}$ from the received value in the first time instant. The remaining terms include symbols intended for the dynamic user plus some interference, whose cancellation is the subject of the next step.

2. The transmitter next sends a second super symbol of length T,

$$x_2 = [\bar{u},\ \ \bar{u}(\mathbf{h}_1^{\dagger}\mathbf{u}_1),\ \ \cdots,\ \ \bar{u}(\mathbf{h}_1^{\dagger}\mathbf{u}_{T-1})], \tag{4.108}$$

where $\bar{u} \in \mathbb{C}$ is a symbol intended for the static user. Hence,

$$y_2^{\dagger} = [h_2\bar{u},\ \ h_2\bar{u}(\mathbf{h}_1^{\dagger}\mathbf{u}_1),\ \ \cdots,\ \ h_2\bar{u}(\mathbf{h}_1^{\dagger}\mathbf{u}_{T-1})]. \tag{4.109}$$

The dynamic user estimates the equivalent channel $h_2\bar{u}$ during the first time instant and then acquires $\mathbf{h}_1^{\dagger}\mathbf{u}_t$, the interference in (4.107). Therefore, using y_1, y_2, the dynamic user solves for v_t achieving $(T-1)$ degrees of freedom. Furthermore,

$$y_2^{\prime\dagger} = [g_1\bar{u},\ \ g_1\bar{u}(\mathbf{h}_1^{\dagger}\mathbf{u}_1),\ \ \cdots,\ \ g_1\bar{u}(\mathbf{h}_1^{\dagger}\mathbf{u}_{T-1})]. \tag{4.110}$$

The static user solves for $\bar{u}$ achieving one degree of freedom and also uses $\mathbf{h}_1^{\dagger}\mathbf{u}_t$ to solve for $\mathbf{u}_t$ achieving further $2\,(T-1)$ degrees of freedom.

In summary, during $2T$ instants, the static user achieves $(2T-1)$ degrees of freedom and the dynamic user achieves $(T-1)$ degrees of freedom. This shows the achievability of (4.103). □

Theorem 20. *For the fading broadcast channel characterized by Eq.* (4.1) *with one static and one dynamic user, where there is perfect CSIT for the static user and delayed CSIT for the dynamic user, an outer bound on the degrees of freedom region is,*

$$\frac{d'}{2} + d \leq 1, \tag{4.111}$$

$$d' \leq 1, \tag{4.112}$$

$$d \leq 1 - \frac{1}{T}. \tag{4.113}$$

Proof. The inequalities (4.112) and (4.113) represent the single-user outer bounds [13, 7]. It only remains to prove the outer bound (4.111), as follows.

1. We enhance the channel by giving global CSIR to both users and also give y to the static user. The enhanced channel is physically degraded having $(\mathbf{Y}', \mathbf{G})$ at the static user and $(y, \mathbf{G})$ at the dynamic user, where $\mathbf{Y}' \triangleq (y', y)$ and $\mathbf{G} \triangleq (\mathbf{h}, \mathbf{g})$. In a physically degraded channel, causal feedback (including delayed CSIT) does not affect capacity [92], so we can remove the delayed CSIT with respect to the dynamic user.

2. We now use another enhancement with the motivation to remove the remaining CSIT (non-causal, with respect to the static user). This is accomplished, similar to Theorem 18, via local statistical equivalence property [94, 42, 44] in the following manner. We create a channel $\tilde{\mathbf{G}}$, and noise $\tilde{\mathbf{Z}}$ with the same distribution but independently of the true channel and noise, and a signal $\tilde{\mathbf{Y}} = \tilde{\mathbf{G}}\mathbf{X} + \tilde{\mathbf{Z}}$. A genie will give $\tilde{\mathbf{Y}}$ to the static receiver and $\tilde{\mathbf{G}}$ to both receivers. It has been shown [94] that $h(\tilde{\mathbf{Y}}, \mathbf{Y}'|\mathcal{H}) = h(\mathbf{Y}'|\mathcal{H}) + o(\log \rho)$, where $\mathcal{H} = (\mathbf{G}, \tilde{\mathbf{G}})$, therefore we can remove $\mathbf{Y}'$ from the enhanced channel without reducing its degrees of freedom.

3. The enhanced channel is still physically degraded, therefore [86, 87]

$$\begin{aligned} R' \leq & I(\mathbf{x}; \tilde{\mathbf{Y}}|U, \mathcal{H}) = h(\tilde{\mathbf{Y}}|U, \mathcal{H}) + o(\log(\rho)) \\ R \leq & I(U; y|\mathcal{H}) = h(y|\mathcal{H}) - h(y|U, \mathcal{H}), \end{aligned} \tag{4.114}$$

where U is an auxiliary random variable, and $U \to x \to (y', y)$ forms a Markov chain. Therefore,

$$\begin{aligned} \frac{1}{2}R' + R \leq & h(y|\mathcal{H}) + \frac{1}{2}h(\tilde{\mathbf{Y}}|U, \mathcal{H}) - h(y|U, \mathcal{H}) + o(\log(\rho)) \\ \leq & \log(\rho) + o(\log(\rho)), \end{aligned} \tag{4.115}$$

where the last inequality follows from extremal entropy inequality and Lemma 7 [39, 67, 93]. This concludes the proof of the bound (4.111).

□

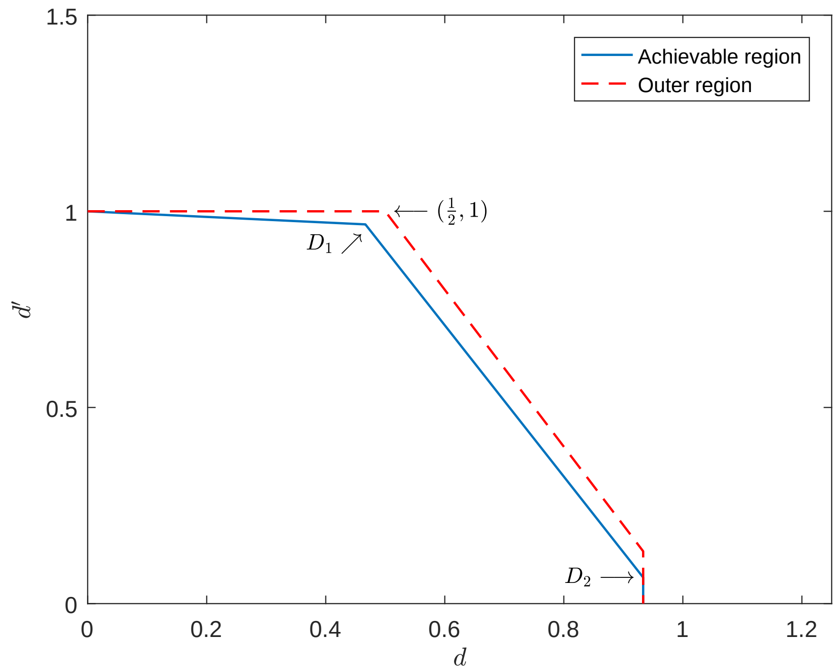

Figure 4.7. One static and one dynamic user with hybrid CSIT and $T = 15$

Remark 10. *For the above broadcast channel with hybrid CSIT, the achievable sum degrees of freedom is* $d_{sum} = \frac{3}{2} - \frac{1}{T}$*, and the outer bound on the sum degrees of freedom is* $d_{sum} \leq \frac{3}{2}$*. The gap decreases with the dynamic user coherence time (see Fig. 4.7 and 4.8).*

4.5.2 Multiple Static and Dynamic Users

Theorem 21. *The fading broadcast channel characterized by Eq.* (4.1)*, with perfect CSIT for the static users and delayed CSIT for the dynamic users, can achieve the following degrees of freedom,*

$$\mathcal{D}_1 : \sum_{j=1}^{m'} \mathbf{e}_j^{\dagger}, \tag{4.116}$$

$$\mathcal{D}_2, \dots, \mathcal{D}_{mm'+1} : (1 - \frac{1}{2T})\mathbf{e}_j^{\dagger} + (\frac{1}{2} - \frac{1}{2T})\mathbf{e}_i^{\dagger}, \quad j = 1, \dots, m', \; i = 1, \dots, m, \tag{4.117}$$

$$\mathcal{D}_{mm'+2}, \dots, \mathcal{D}_{mm'+m+2} : \frac{1}{T} \sum_{j=1}^{m'} \mathbf{e}_j^{\dagger} + (1 - \frac{1}{T})\mathbf{e}_i^{\dagger}, \quad i = 1, \dots, m, \tag{4.118}$$

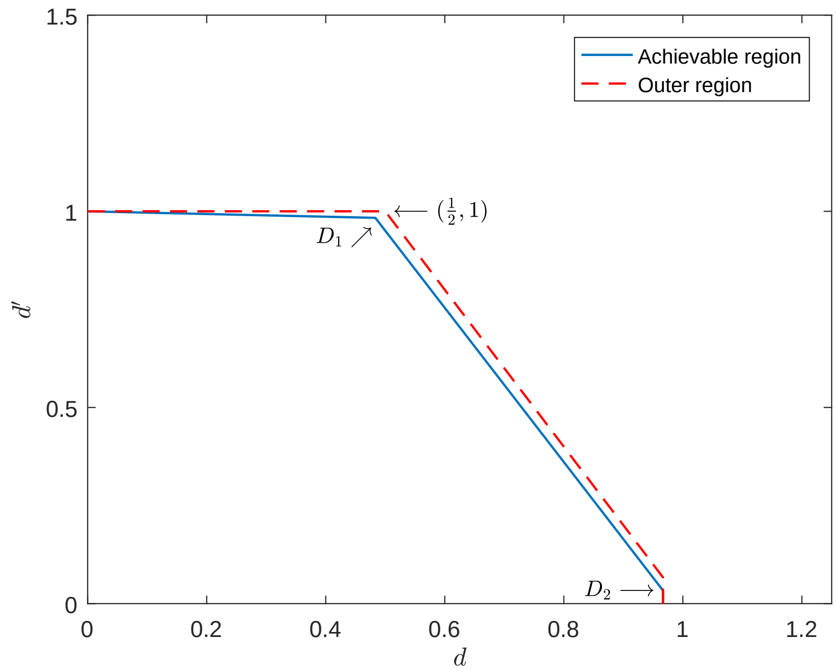

Figure 4.8. One static and one dynamic user with hybrid CSIT and $T = 30$

$$\mathcal{D}_{mm'+m+3} : \frac{m}{T}\sum_{j=1}^{m'}\mathbf{e}_j^{\dagger} + (\frac{1}{1+\frac{1}{2}+\ldots+\frac{1}{m}}(1-\frac{m}{T}))\sum_{i=1}^{m}\mathbf{e}_i^{\dagger}. \tag{4.119}$$

The achievable region consists of the convex hull of the above vectors.

Proof. $\mathcal{D}_1$ is achieved by inverting the channel of the static users at the transmitter, providing one degree of freedom per static user. The achievability of $\mathcal{D}_2, \ldots, \mathcal{D}_{mm'+1}$ was established in Section 4.5.1, and that of $\mathcal{D}_{mm'+2}, \ldots, \mathcal{D}_{mm'+m+2}$ was proved in Section 4.4 without CSIT for the dynamic user, so it remains achievable with delayed CSIT. $\mathcal{D}_{mm'+m+3}$ is achieved by retrospective interference alignment [32] along with product superposition, as follows. The transmitted signal over T instants is

$$\mathbf{X} = [\bar{\mathbf{U}}, \quad \bar{\mathbf{U}}\mathbf{V}], \tag{4.120}$$

where $\bar{\mathbf{U}} \in \mathbb{C}^{m\times m}$ contains independent symbols intended for the static users sent by inverting the channels of the static users. Therefore, during the first m time instants, each static user receives an interference-free signal and achieves m degree of freedom, and furthermore the dynamic users estimate their equivalent channels. During the remaining time instants, each dynamic receiver obtains coherent observations of $(T-m)$ transmit symbols, which are pre-processed, combined and interference-aligned into super-symbols $\mathbf{V}$ according to retrospective interference alignment techniques of [32]. Accordingly, each dynamic receiver achieves $\frac{1}{1+\frac{1}{2}+\ldots+\frac{1}{m}}(1-\frac{m}{T})$ degrees of freedom. $\square$

Theorem 22. *An outer bound on the degrees of freedom region of the fading broadcast channel characterized by Eq.* (4.1)*, with perfect CSIT for the static users and delayed CSIT for the dynamic users, is*

$$\sum_{j=1}^{m'} \frac{d'_j}{m'+m} + \sum_{i=1}^{m} \frac{d_i}{m} \leq 1, \tag{4.121}$$

$$\sum_{i=1}^{m} d_i \leq \frac{m}{1+\frac{1}{2}+\ldots+\frac{1}{m}}, \tag{4.122}$$

$$d'_j \leq 1, \qquad j=1,\ldots,m', \tag{4.123}$$

$$d_i \leq 1-\frac{1}{T}, \qquad i=1,\ldots,m. \tag{4.124}$$

Proof. The inequalities (4.123) and (4.124) represent the single-user outer bounds for the static and dynamic users, respectively [13, 7]. According to Theorem 12, (4.122) represents an outer bound for the dynamic users. It only remains to prove (4.121) as follows.

1. The original channel is enhanced by giving the users global CSIR. Furthermore, we assume full cooperation between the static users and between the dynamic users. The resulting enhanced channel is a broadcast channel with two users: one static user equipped with m' antennas, received signal $\mathbf{Y}'$, channel $\mathbf{G}$, and noise noise $\mathbf{Z}'$, and one dynamic user equipped with m antennas, received signal $\mathbf{Y}$, channel $\mathbf{H}$, and noise $\mathbf{Z}$.

2. We further enhance the channel by giving $\mathbf{Y}$ to the static user, constructing a physically degraded channel. For the enhanced channel, the static receiver is equipped with $m'+m$

antennas and has received signal $\hat{\mathbf{Y}} = [\mathbf{Y}^\dagger,\ \mathbf{Y}'^\dagger]^\dagger$, channel $\hat{\mathbf{G}} = [\mathbf{G}^\dagger,\ \mathbf{H}^\dagger]^\dagger$, and noise $\hat{\mathbf{Z}} = [\mathbf{Z}^\dagger,\ \mathbf{Z}'^\dagger]^\dagger$. Since any causal feedback (including delayed CSIT) does not affect the capacity of a physically degraded channel [92], the delayed CSIT for the dynamic receiver can be removed.

3. We now use another enhancement with the motivation to remove the remaining CSIT (non-causal, with respect to the static user). We create an artificial channel and noise, $\tilde{\mathbf{G}}$, $\tilde{\mathbf{Z}}$, with the same distribution but independent of $\hat{\mathbf{G}}$, $\hat{\mathbf{Z}}$, and a signal $\tilde{\mathbf{Y}} = \tilde{\mathbf{G}}\mathbf{X} + \tilde{\mathbf{Z}}$. A genie will give $\tilde{\mathbf{Y}}$ to the static receiver and $\tilde{\mathbf{G}}$ to both receivers. It has been shown [94] that $h(\tilde{\mathbf{Y}}, \hat{\mathbf{Y}}|\mathcal{H}) = h(\hat{\mathbf{Y}}|\mathcal{H}) + o(\log \rho)$, where $\mathcal{H} = (\hat{\mathbf{G}}, \tilde{\mathbf{G}})$, therefore we can remove $\hat{\mathbf{Y}}$ from the enhanced channel without reducing its degrees of freedom.

4. The enhanced channel is physically degraded without CSIT, therefore [86, 87],

$$\begin{aligned} \sum_{j=1}^{m'} R'_j &\leq I(\mathbf{X}; \tilde{\mathbf{Y}}|U, \mathcal{H}) \\ \sum_{i=1}^{m} R_i &\leq I(U; \mathbf{Y}|\mathcal{H}). \end{aligned} \tag{4.125}$$

Hence,

$$\begin{aligned} \sum_{j=1}^{m'} \frac{R'_j}{m'+m} + \sum_{j=1}^{m} \frac{R_i}{m} \leq & \frac{1}{m'+m} h(\tilde{\mathbf{Y}}|U, \mathcal{H}) + \frac{1}{m} h(\mathbf{Y}|\mathcal{H}) - \frac{1}{m} h(\mathbf{Y}|U, \mathcal{H}) + o(\log(\rho)) \\ \leq & \log(\rho) + o(\log(\rho)), \end{aligned} \tag{4.126}$$

where the last inequality follows from the extremal entropy inequality [39, 67, 93] and Lemma 7 and since $h(\mathbf{Y}|\mathcal{H}) \leq m \log(\rho) + o(\log(\rho))$ [13]. This concludes the proof of the bound (4.121).

□

CHAPTER 5

CONCLUSION

This dissertation investigated multiuser wireless channels when the receivers have disparities in coherence time and bandwidth. The disparity is a source of novel gains that is denoted coherence diversity. Employing product superposition transmission, coherence diversity gains were explored and demonstrated in various multiuser scenarios and in a wide set of CSI availability models.

First, an OFDM downlink transmission for two users was investigated where the fading links experience differences either in coherence time, in coherence bandwidth, or in both. In each scenario a version of product superposition transmission was proposed and gains were demonstrated analytically as well as via numerical simulations.

Second, multiuser networks without CSIT or CSIR were studied. For a broadcast channel where the receivers have identical coherence times, it was shown that the degrees of freedom region is tight against the TDMA inner bound. However, when the receivers have heterogeneous coherence times, TDMA is no longer optimal since the difference of the coherence times can be a source of diversity in the wireless systems. This was shown for users coherence times of arbitrary ratio or alignment. For a multiple access channel with identical coherence times, a pilot-based achievable scheme was shown to be sum degrees of freedom optimal. Furthermore, a multiple access channel with heterogeneous coherence times is considered. When the transmitters coherence times are integer multiples of each other, an achievable pilot-based inner bound and an outer bound were obtained.

Third, a multiuser broadcast channel was studied where some receivers experience longer coherence intervals and have CSIR while other receivers experience a shorter coherence interval and do not enjoy free CSIR. The degrees of freedom were studied under delayed CSIT, hybrid CSIT, and no CSIT. Among the techniques employed were interference alignment and beamforming along with product superposition for the inner bounds. The outer bounds involved a bounding of the rate region of the multiuser (discrete memoryless) multilevel broadcast channel.

Appendices

APPENDIX A

COHERENT BROADCAST CHANNEL WITH IDENTICAL COHERENT TIMES

In the sequel, we show the degrees of freedom optimality of TDMA inner bound when the receivers have identical coherence times and CSI is assumed to be available at the receiver. We enhance the channel by providing global CSI at the receivers. Without loss of generality, assume that $N_1 \leq \cdots \leq N_K$. When $M \leq N_1$, the cooperative outer bound [85] is tight against the TDMA inner bound. When $M > N_1$, the broadcast channel is degraded [86], hence,

$$\begin{aligned} R_i \leq & I\left(U_i; \mathbf{Y}_i | \mathcal{H}, U^{i-1}\right) \\ =& I\left(\mathbf{X}; \mathbf{Y}_i | \mathcal{H}, U^{i-1}\right) - I\left(\mathbf{X}; \mathbf{Y}_i | \mathcal{H}, U^{i}\right), \end{aligned} \tag{A.1}$$

where $U^i = \{U_j\}_{j=1}^{i}$ is a set of auxiliary random variables such that $U_1 \to \cdots \to U_{K-1} \to \mathbf{X} \to (\mathbf{Y}_1, \cdots \mathbf{Y}_K)$ forms a Markov chain and for notational convenience we introduced a trivial random variable U_0 and $U_K = \mathbf{X}$. $\mathcal{H}$ is the set of all channels. Furthermore,

$$\begin{aligned} R_1 \leq & (N_1 - r_1)\log(\rho) + o(\log(\rho)), \\ R_i \leq & I\left(\mathbf{X}; \mathbf{Y}_i | \mathcal{H}, U^{i-1}\right) - r_i \log(\rho) + o(\log(\rho)), \quad i \neq 1, K, \\ R_K \leq & I\left(\mathbf{X}; \mathbf{Y}_K | \mathcal{H}, U^{K-1}\right), \end{aligned} \tag{A.2}$$

since the degrees of freedom of $I(\mathbf{X}; \mathbf{Y}_1 | \mathcal{H})$ is bounded by the single-receiver bound, i.e. N_1, and r_i is defined to be the degrees of freedom of the term $I(\mathbf{X}; \mathbf{Y}_i | \mathcal{H}, U^i)$, where $0 \leq r_i \leq N_i^*$. The extension of [9, Lemma 1] to the K-receiver case is straight forward, and hence, we can write

$$I\left(\mathbf{X}; \mathbf{Y}_{i,1} | \mathcal{H}, U^i, \mathbf{Y}_{i,2:N_i^*}\right) \leq \frac{r_i}{N_i^*}\log(\rho) + o(\log(\rho)), \tag{A.3}$$

where $\mathbf{Y}_{i,1} \in \mathbb{C}^{1\times T}$ is the received signal at antenna 1 of receiver i over the entire T-length coherence time whereas $\mathbf{Y}_{i,2:N_i^*} \in \mathbb{C}^{(N_i^*-1)\times T}$ is the matrix consists of the received signal at antennas $2, 3, \ldots, N_i^*$ of receiver i over the entire T-length coherence time. Furthermore,

$$\begin{aligned}
I\left(\mathbf{X}; \mathbf{Y}_i | \mathcal{H}, U^{i-1}\right) \overset{(a)}{=} & I\left(\mathbf{X}; \mathbf{Y}_{i,1:N_i^*} | \mathcal{H}, U^{i-1}\right) + I\left(\mathbf{X}; \mathbf{Y}_{i,N_i^*+1:N_i} | \mathcal{H}, U^{i-1}, \mathbf{Y}_{i,1:N_i^*}\right) \\
\overset{(b)}{=} & I\left(\mathbf{X}; \mathbf{Y}_{i,1:N_{i-1}^*} | \mathcal{H}, U^{i-1}\right) + I\left(\mathbf{X}; \mathbf{Y}_{i,N_{i-1}^*+1:N_i^*} | \mathcal{H}, U^{i-1}, \mathbf{Y}_{i,1:N_{i-1}^*}\right) \\
& + o(\log(\rho)) \\
\overset{(c)}{=} & I\left(\mathbf{X}; \mathbf{Y}_{i-1,1:N_{i-1}^*} | \mathcal{H}, U^{i-1}\right) + I\left(\mathbf{X}; \mathbf{Y}_{i,N_{i-1}^*+1:N_i^*} | \mathcal{H}, U^{i-1}, \mathbf{Y}_{i-1,1:N_{i-1}^*}\right) \\
& + o(\log(\rho)) \\
= & r_{i-1} \log(\rho) + \sum_{j=N_{i-1}^*+1}^{N_i^*} I\left(\mathbf{X}; \mathbf{Y}_{i,j} | \mathcal{H}, U^{i-1}, \mathbf{Y}_{i-1,1:N_{i-1}^*}, \mathbf{Y}_{i,j+1:N_i^*}\right) + o(\log(\rho)) \\
\overset{(d)}{\leq} & r_{i-1} \log(\rho) + \left(N_i^* - N_{i-1}^*\right) I\left(\mathbf{X}; \mathbf{Y}_{i-1,1} | \mathcal{H}, U^{i-1}, \mathbf{Y}_{i-1,2:N_{i-1}^*}\right) + o(\log(\rho)) \\
\overset{(e)}{\leq} & r_{i-1} \log(\rho) + \left(N_i^* - N_{i-1}^*\right) \frac{r_{i-1}}{N_{i-1}^*} \log(\rho) + o(\log(\rho)) \\
\leq & \frac{N_i^*}{N_{i-1}^*} r_{i-1} \log(\rho) + o(\log(\rho)),
\end{aligned} \tag{A.4}$$

where (a) and (b) follow from applying the chain rule, and $h\left(\mathbf{Y}_{i,N_i^*+1:N_i} | \mathcal{H}, U^{i-1}, \mathbf{Y}_{i,N_i^*}\right) = o(\log(\rho))$. Furthermore, (c) follows since $\mathbf{Y}_{i,1:N_{i-1}^*}$ and $\mathbf{Y}_{i-1,1:N_{i-1}^*}$ are statistically the same. (d) follows from applying the straight forward extension of [9, Lemma 1] and (e) follows from (A.3). Therefore,

$$\begin{aligned}
d_1 \leq & N_1 - r_1, \\
d_i \leq & \frac{N_i^*}{N_{i-1}^*} r_{i-1} - r_i, \quad i \neq 1, K, \\
d_K \leq & \frac{N_K^*}{N_{K-1}^*} r_{K-1},
\end{aligned} \tag{A.5}$$

which gives the region defined in (3.5).

APPENDIX B

MULTILEVEL BROADCAST CHANNEL WITH DEGRADED MESSAGE SETS

Here, we study the capacity of the multiuser multilevel broadcast channel that is characterized by (4.6) with degraded message sets. In particular, $M_0 \in \left[1 : 2^{nR_0}\right]$ is to be communicated to all receivers, and furthermore $M_1 \in \left[1 : 2^{nR_1}\right]$ is to be communicated to receiver Y_1.[1] A three-receiver special case was studied by Nair and El Gamal [65] where the idea of indirect decoding was introduced, and the capacity is the set of rate pairs (R_1, R_0) such that

$$\begin{aligned} R_0 &\leq \min\left\{I(U;Y_2), I(V;Y_1')\right\}, \\ R_1 &\leq I(X;Y_1|U), \\ R_0 + R_1 &\leq I(V;Y_1') + I(X;Y_1|V), \end{aligned} \tag{B.1}$$

for some pmf $p(u,v)p(x|v)$. In the sequel, we give a generalization of Nair and El Gamal for multiuser multilevel broadcast channel.

Theorem 23. *The capacity of multiuser multilevel broadcast channel characterized by* (4.6), *with degraded message sets, is the set of rate pairs* (R_1, R_0) *such that*

$$\begin{aligned} R_0 &\leq \min\left\{I(U;Y_m), I(V;Y_{m'}')\right\}, \\ R_1 &\leq I(X;Y_1|U), \\ R_0 + R_1 &\leq I(V;Y_{m'}') + I(X;Y_1|V), \end{aligned} \tag{B.2}$$

for some pmf $p(u,v)p(x|v)$.

Proof. The converse parallels the proof of the converse of the three-receiver case studied by Nair and El Gamal [65] after replacing Y_2, Y_1' with $Y_m, Y_{m'}'$, respectively. In particular, U

[1] For compactness of expression, here we refer to each receiver by the variable denoting its received signal.

and V are defined as follows.

$$U_k \triangleq (M_0, Y_{1,1}, \ldots, Y_{1,k-1}, Y_{m,k+1}, \ldots, Y_{m,n}),$$
$$V_k \triangleq (M_0, Y_{1,1}, \ldots, Y_{1,k-1}, Y'_{m',k+1}, \ldots, Y'_{m',n}),$$

$k = 1, \ldots, n$, and let Q be a time-sharing random variable uniformly distributed over the set $\{1, \ldots, n\}$ and independent of $X^n, Y_1^n, Y_{m,1}, \ldots, Y_{m,n}, Y'_{m',1}, \ldots, Y'_{m',n}$. We then set $U = (U_Q, Q), V = (V_Q, Q), X = X_Q, Y_1 = Y_{1,Q}, Y_m = Y_{m,Q}$, and $Y'_{m'} = Y'_{m',Q}$. This completes the converse part of the proof.

The achievability part uses superposition coding and indirect decoding as follows.

- **Rate splitting:** divide the private message M_1 into two independent messages M_{10} at rate R_{10} and M_{11} at rate R_{11}, where $R_1 = R_{10} + R_{11}$.

- **Codebook generation:** fix a pmf $p(u,v)p(x|v)$ and randomly and independently generate 2^{nR_0} sequences $u^n(m_0)$, $m_0 \in \left[1 : 2^{nR_0}\right]$, each according to $\prod_{k=1}^n p_U(u_k)$. For each m_0, randomly and conditionally independently generate $2^{nR_{10}}$ sequences $v^n(m_0, m_{10})$, $m_{10} \in [1 : 2^{nR_{10}}]$, each according to $\prod_{k=1}^n p_{V|U}(v_k|u_k(m_0))$. For each pair (m_0, m_{10}), randomly and conditionally independently generate $2^{nR_{11}}$ sequences $x^n(m_0, m_{10}, m_{11})$, $m_{11} \in [1 : 2^{nR_{11}}]$, each according to $\prod_{k=1}^n p_{X|V}(x_k|v_k(m_0, m_{10}))$.

- **Encoding:** to send the message pair $(m_0, m_1) = (m_0, m_{10}, m_{11})$, the encoder transmits $x^n(m_0, m_{10}, m_{11})$.

- **Decoding at the users $Y_2, \ldots, Y_m$:** decoder i declares that $\hat{m}_{0i} \in [1 : 2^{nR_0}]$ is sent if it is the unique message such that $(u^n(\hat{m}_{0i}), y_i^n) \in \mathcal{T}_\epsilon^{(n)}$. Hence, by law of large numbers and the packing lemma [88], the probability of error tends to zero as $n \to \infty$ if

$$\begin{aligned} R_0 < \quad & \min_{2 \le i \le m} \{I(U; Y_i) - \delta(\epsilon)\}, \\ &= I(U; Y_m) - \delta(\epsilon), \end{aligned} \tag{B.3}$$

where the last equality follows from applying data processing inequality on the Markov chain $U \to X \to Y_1 \to Y_2 \to \cdots \to Y_m$.

- **Decoding at Y_1:** decoder 1 declares that $(\hat{m}_{01}, \hat{m}_{10}, \hat{m}_{11})$ is sent if it is the unique message triple such that $\big(u^n(\hat{m}_{01}), v^n(\hat{m}_{01}, \hat{m}_{10}), x^n(\hat{m}_{01}, \hat{m}_{10}, \hat{m}_{11}), y_1^n\big) \in [1 : 2^{nR_0}]$. Hence, by law of large numbers and the packing lemma [88], the probability of error tends to zero as $n \to \infty$ if

$$\begin{aligned} R_{11} &< I(X; Y_1|V) - \delta(\epsilon), \\ R_{10} + R_{11} &< I(X; Y_1|U) - \delta(\epsilon), \\ R_0 + R_{10} + R_{11} &< I(X; Y_1) - \delta(\epsilon). \end{aligned} \tag{B.4}$$

- **Decoding at users $Y'_1, \ldots, Y'_{m'}$:** decoder j decodes m_0 indirectly by declaring $\tilde{m}_{0j}$ is sent if it is the unique message such that $(u^n(\tilde{m}_{0j}), v^n(\tilde{m}_{0j}, m_{10}), z_j^n) \in \mathcal{T}_\epsilon^{(n)}$ for some $m_{10} \in [1 : 2^{nR_0}]$. Hence, by law of large numbers and packing lemma, the probability of error tends to zero as $n \to \infty$ if

$$\begin{aligned} R_0 + R_{10} < & \min_{1 \le j \le m'} \{I(U, V; Y'_j) - \delta(\epsilon)\}, \\ = & \min_{1 \le j \le m'} \{I(V; Y'_j) - \delta(\epsilon)\}, \\ = & \, I(V; Y'_{m'}) - \delta(\epsilon), \end{aligned} \tag{B.5}$$

where the last two equalities follow from applying the chain rule and data processing inequality on the Markov chain $U \to V \to X \to Y'_1 \to Y'_2 \to \cdots \to Y'_{m'}$.

By combining the bounds in (B.3), (B.4), (B.5), substituting $R_{10} + R_{11} = R_1$, and eliminating R_{10} and R_{11} by the Fourier-Motzkin procedure [65], the proof of the achievability is completed. □

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