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Threshold-Type Policies for Real Options Using Regime-Switching Models*

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Abstract. To investigate the impact of macroeconomic conditions on irreversible investments under a regime-switching model, our main effort in this work is to rigorously justify the existence and uniqueness of optimal threshold-type policies. The underlying cash flow process is modeled as a geometric Brownian motion with return rate and volatility depending on a continuous-time Markov chain. The problem is similar to the American style of call options. When dealing either with American options in a financial market or with real options, a common practice in the literature is to postulate threshold-type strategies and to find the optimal threshold levels as solutions of systems of nonlinear algebraic equations. Although from a computational standpoint, this seems to be a reasonable approach, the issue of existence and uniqueness of solutions has never been addressed to date. Instead of assuming the threshold-type policies, this paper establishes that indeed the threshold-type policies are the right choice. Variational inequalities are used to characterize the optimal strategy by an *abstract*, nonconstructive reasoning. In addition, numerical simulations are also provided to demonstrate quantitative properties and properties of the systems.

Key words. variational inequality, irreversible investment, real option, regime shift, macroeconomic condition, optimal stopping problem

AMS subject classifications. 91G80, 91B70, 93E20, 35K87, 49J40

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1. Introduction.

Treating investment decisions as options is the result of more than a couple of decades of research by many economists. While the economic concept of real options has been rapidly developing, the research of the mathematical basis serving as the foundation of the economic theory has received increasing attention recently. The traditional approach paid little attention to the irreversibility of investments. As pointed out in [6], the investment decisions share three important features: (1) The investment is at least partially irreversible. The initial cost of investment is at least partially sunk, and there is no way to recover it completely. (2) The future rewards of the investment are uncertain. (3) For each investor,

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there is the concept of the right timing of the investment corresponding to the need for flexibility to mitigate uncertainties and risk. These issues justify the need to study investment policies that can make use of flexibility, particularly in the context of irreversible markets, which connect the topics of investment decisions in the financial theory of American options. A variety of methods have been introduced to investigate such investments leading to the real option theory. It treats the opportunity to invest as holding an option similar to a financial American call option. This method has been fully developed in the work of Dixit and Pindyck [6]. Real options are an important improvement of standard practices like net present value (NPV) or discounted cash flow (DCF) since they allow the possibility of flexibility. In the literature, it is commonly assumed that the rates of return and volatility of the underlying resource are constants; see [16], [17], and [18]. However, empirical evidence shows that the payoffs of investment are affected by the macroeconomic conditions, and market parameters may vary in bullish and bearish modes [8] and may change subject to additional economic conditions. The use of regime-switching models facilitates the consideration of uncertain factors and interactions of continuous dynamics and discrete events. A finite-state continuous-time Markov chain is often used in the investment models to delineate the random shifts due to market changes and other economic conditions, which can grant a more accurate representation of the market. However, it complicates considerably the mathematical treatment and the characterization of the optimal strategy.

Recently, Hackbarth, Miao, and Morellec [13] studied credit risk and capital structure in a regime-switching framework. Chen [5] introduced regime switching into firm financing decision making to explain the credit spread puzzle and underleverage puzzle. Optimal strategies for perpetual American put options were treated by Guo and Zhang in [9] (see also [20]); optimal selling rules were studied in [10]; portfolio optimization was investigated by Zhou and Yin [22]; entry and exit investment decisions were given in [7]. As shown in many theoretical studies and managerial and investment practice, macroeconomic conditions are crucial for firm decisions. The ever-changing economic conditions affect the future cash flow and have a significant impact on investment timing and consumption.

In this paper, we treat real options and analyze the investment timing aspect of an investor's activities. The problem is similar to the American style of call options. The classical result is that the optimal stopping is obtained by a so-called threshold strategy. That is, there is a threshold for the underlying state such that the optimal strategy is the time when the state attains the threshold (hitting time). When dealing either with American-style options in a financial market or with real options, a common practice in the literature is to postulate threshold-type strategies and to find the optimal threshold levels. Obtaining the optimal threshold can be done by using smooth pasting conditions. Although it is intuitively plausible, it has not been proved or justified to date that threshold policies are indeed optimal to the best of our knowledge.

Considering a regime-switching model is not the main contribution of this paper. Previously, Guo and Zhang [9] treated perpetual American put options of a regime-switching model. The extensions to the switching model naturally try to follow the same approach (see related work [11]). The conjecture is that the optimal stopping is still of threshold type. However, there are two states—the underlying state and the state of the Markov chain that modulates the dynamics of the underlying state; there are also two thresholds to be found. As a result, we have to solve a system of six nonlinear equations with six unknowns. The main issue is to prove that the system has one and only one solution. This turns out to be a virtually impossible task that has not been achieved in the literature. Assuming this system has a solution, Guo and Zhang [9] proved that the hitting time strategy defined by these two thresholds is optimal. The assumption that the system has a solution is a very strong one. Although this assumption appears to be satisfied numerically, a proof is not provided. Instead of proving directly that the system of six nonlinear equations has a unique solution, we use an abstract and nonconstructive approach. It is well known that an optimal stopping problem can be solved by using the approach of variational inequalities (VIs) developed in Bensoussan and Lions [4]. We study directly the VIs by PDE techniques. Our contribution is to prove that the optimal strategy derived from the VI (the free boundary) is indeed a threshold strategy, and the solution to the VI is unique. Thus, we prove that the condition imposed in [9] is valid. Then we proceed to further identify the optimal threshold levels. As a demonstration of our results, we carry out numerical experiments and report on some simulation studies providing us with further insight. It is important to emphasize that our approach is different from what is common in the literature. Our approach is abstract and solves the two issues (existence and uniqueness of solutions and optimality) simultaneously. In the literature, variational inequalities have been used in a wide variety of situations and many applications in systems control and optimization. For instance, for some related applications, we refer the reader to [12] and [19] for options in the financial market and to [2] and [3] for real options. It is likely that this method can be extended to more complex situations such as jump diffusions with regime switching. However, this is an objective beyond the scope of the current paper.

The rest of the paper is arranged as follows. Section 2 presents the model of the problem. Section 3 concentrates on the existence and uniqueness of solution. Section 4 derives an explicit solution. Section 5 presents simulation and numerical experimental results. Section 6 issues some further remarks. Finally, an appendix containing some technical developments is provided at the end of this paper.

2. Problem formulation. Let $(\Omega, \mathcal{F}, P)$ be a probability space and W_t a standard Brownian motion defined on it. We also have $\theta(t)$, a continuous-time Markov chain, to represent the state of the macroeconomic conditions. Assume that $\theta(t)$ has state space $\mathcal{M}$ and generator Q and that $\theta(t)$ is independent of W_t. For simplicity, in this paper, we consider the case that the Markov chain has only two states. This is mainly for mathematical tractability and enables us to get insight on the solution of the problems under consideration. Denote the state space of $\theta(t)$ by $\mathcal{M} = \{0, 1\}$ and the generator of the Markov chain $\theta(t)$ by

$$Q = (q_{ij}) = \begin{pmatrix} -\lambda_0 & \lambda_0 \\ \lambda_1 & -\lambda_1 \end{pmatrix};$$

see [21, p. 17]. Suppose that the cash flow process $Y(t)$ from the investment operation evolves according to the following dynamics:

$$\mathrm{d}Y(t) = \alpha(\theta(t))Y(t)\mathrm{d}t + \sigma(\theta(t))Y(t)\mathrm{d}W(t), \quad Y(0) = y, \quad \theta(0) = i. \tag{2.1}$$

That is, the firm's cash flow follows a geometric Brownian motion model, whose return and volatility rates are modulated by the continuous-time Markov chain. Such Markov modulated models have become popular in recent years and are commonly referred to as regime-switching geometric Brownian motion models. The addition of the Markov chain makes the model more realistic and provides an alternative to the so-called stochastic volatility model. Compared to the stochastic volatility model, the formulation is much simpler.

Suppose that the firm makes an investment at time 0. For each $i \in \mathcal{M}$, the project value $V_i(y)$ from the operation is the expected discounted cash flow stream given by

$$V_i(y) = \delta E\left[\int_0^\infty e^{-\mu s} Y(s)\mathrm{d}s | Y(0) = y, \theta(0) = i\right], \tag{2.2}$$

where $\delta \in (0, 1]$ represents market share of the investor. In this equation, the discount factor μ is exogenous as a part of the objective function.

Using properties of the conditional expectation and applying Dynkin's formula, similar to the arguments in stochastic control theory, we deduce that $V_i(y)$ satisfies the system of ordinary differential equations (ODEs)

$$\begin{aligned} \mu V_0(y) &= \alpha_0 y \frac{\partial}{\partial y} V_0(y) + \frac{1}{2} y^2 \sigma_0^2 \frac{\partial}{\partial y^2} V_0(y) + \lambda_0 (V_1(y) - V_0(y)) + \delta y, \\ \mu V_1(y) &= \alpha_1 y \frac{\partial}{\partial y} V_0(y) + \frac{1}{2} y^2 \sigma_1^2 \frac{\partial}{\partial y^2} V_0(y) + \lambda_1 (V_0(y) - V_1(y)) + \delta y, \end{aligned} \tag{2.3}$$

where we denote $\alpha(i)$ and $\sigma(i)$ by α_i and σ_i, respectively. Solving the system of ODEs subject to the conditions

$$\begin{aligned} &\lim_{y \to \infty} \frac{V_i(y)}{y} < \infty, \\ &\lim_{y \to 0} V_i(y) = 0, \end{aligned} \tag{2.4}$$

we obtain

$$\begin{aligned} &V_i(y) = a_i y, \;\; i = 0, 1, \quad \text{where} \\ &a_i = \frac{(\mu - \alpha_{1-i} + \lambda_i + \lambda_{1-i})\delta}{(\mu - \alpha_{1-i})(\mu - \alpha_i) + \lambda_i(\mu - \alpha_{1-i}) + \lambda_{1-i}(\mu - \alpha_i)}. \end{aligned} \tag{2.5}$$

We also have

$$a_i(\mu + \lambda_i - \alpha_i) - \lambda_i a_{1-i} = \delta, \;\; i = 0, 1. \tag{2.6}$$

At a stopping time τ (a random time), the investor pays the investment cost $K > 0$ and obtains the perpetual stream of payoff $\{\delta Y(t) : t \geq \tau\}$. The expected discounted payoff from the capital investment project undertaken at time τ is

$$J_{y,i}(\tau) = E_{y,i}[e^{-\mu\tau}(a_{\theta(\tau)} Y(\tau) - K) I_{\tau<\infty}], \tag{2.7}$$

where $E_{y,i}(\cdot) = E(\cdot | Y(0) = y, \theta(0) = i)$. For this problem to make sense, we must assume that $\mu > \alpha_i$; otherwise the value in (2.7) could be made arbitrarily large by choosing a large τ, which corresponds to holding the option for a very long time. Thus the investment would not happen. The firm's objective is to find an optimal stopping time to maximize the expected discounted payoff

$$g_i(y) = \sup_{\tau \geq 0} J_{y,i}(\tau). \tag{2.8}$$

Using a dynamic programming approach, we write the VI in the strong sense (that is, $g_i \in C^1(0, L)$, $g_i' \in L^\infty(0, L)$, and $g_i'' \in L^\infty(0, L)$ with compact support) such that for $i \in \mathcal{M}$, $g_i(y)$ satisfies

$$\begin{cases} -\dfrac{1}{2}y^2\sigma_i^2 g_i'' - \alpha_i y g_i' + (\lambda_i + \mu)g_i - \lambda_i g_{1-i} \geq 0, \\ g_i(y) \geq a_i y - K, \\ (g_i - a_i y + K)\Big(-\dfrac{1}{2}y^2\sigma_i^2 g_i'' - \alpha_i y g_i' + (\lambda_i + \mu)g_i - \lambda_i g_{1-i}\Big) = 0, \\ g_i(0) = 0. \end{cases} \tag{2.9}$$

Letting $u_i(y) = g_i(y) - a_i y + K$ and making use of (2.6), we get

$$\begin{cases} -\dfrac{1}{2}y^2\sigma_i^2 u_i'' - \alpha_i y u_i' + (\lambda_i + \mu)u_i - \lambda_i u_{1-i} + \delta y - \mu K \geq 0, \\ u_i \geq 0, \\ u_i\Big(-\dfrac{1}{2}y^2\sigma_i^2 u_i'' - \alpha_i y u_i' + (\lambda_i + \mu)u_i - \lambda_i u_{1-i} + \delta y - \mu K\Big) = 0, \\ u_i(0) = K. \end{cases} \tag{2.10}$$

We can check that $u_i(y) \leq K$. Indeed, let $v_i(y) = K$; then it satisfies

$$\begin{cases} -\dfrac{1}{2}y^2\sigma_i^2 v_i''(y) - \alpha_i y v_i'(y) + (\lambda_i + \mu)v_i(y) - \lambda_i v_{1-i}(y) + \delta y - \mu K \geq 0, \\ v_i \geq 0, \\ v_i(0) = K. \end{cases}$$

Hence

$$u_i(y) \leq v_i(y) = K. \tag{2.11}$$

To help the reader, a proof of (2.11) is provided in the appendix,

Remark 2.1. This remark is concerned with the verification theorem. Considering the Hamilton–Jacobi–Bellman equation for stochastic control, once we can obtain a sufficiently smooth solution (C^1 plus the L^∞_{loc} estimate for the 2nd derivative), the value function and the optimal feedback control policy can be derived. This is the classical verification property. For optimal stopping problems, the VI plays a role identical to the Hamilton–Jacobi–Bellman equation. When a similar type of smoothness is available (one states the VI in a strong sense), the solution of the VI is the value function. Besides, a free boundary is defined by equating the solution of the VI (the value function) to the obstacle (the constant function). Then the optimal stopping is the first hitting time of the free boundary. This is a classical result given in Bensoussan and Lions [4]. This result holds independently of the shape of the thresholds. The free boundary being a threshold is an additional property, which may or may not hold, but does not prevent the optimal stopping time strategy from being defined. For the convenience of the reader, we place a short proof of the verification theorem in the appendix.

Now the problem set-up is complete. We are in a position to obtain the existence and uniqueness of the solution of (2.10), characterize the optimal strategy of stopping, and prove that it is defined by a threshold.

3. Existence and uniqueness of solution.

In this section, we will present a theorem that establishes that the system of variational inequalities has a unique solution. The assertion is given next. The proof is carried out in three steps by first establishing some technical lemmas.

Theorem 3.1. *For each $i \in \mathcal{M}$, there exists a unique function $u_i \in C^1(0,\infty)$ satisfying $u_i'' \in L^\infty(0,\infty)$, which is the solution of* (2.10)*. Moreover, there exist numbers y_i uniquely defined such that*

$$\begin{cases} -\dfrac{1}{2}y^2\sigma_i^2 u_i'' - \alpha_i y u_i' + (\lambda_i+\mu)u_i - \lambda_i u_{1-i} + \delta y - \mu K = 0, \quad 0 < y < y_i, \\ u_i(0) = K, \quad u_i(y) = 0, \quad y \geq y_i. \end{cases} \tag{3.1}$$

Proof. The proof is rather technical; we divide the whole proof into three parts.

Part 1. We begin by considering (2.10) on a finite interval and work with a positive but finite L. The result is stated as a lemma.

Lemma 3.2. *For $L > 0$, consider the system of variational inequalities*

$$\begin{cases} -\dfrac{1}{2}y^2\sigma_i^2 u_i'' - \alpha_i y u_i' + (\lambda_i+\mu)u_i - \lambda_i u_{1-i} + \delta y - \mu K \geq 0, \quad 0 < y < L, \\ u_i \geq 0, \\ u_i\Big(-\dfrac{1}{2}y^2\sigma_i^2 u_i'' - \alpha_i y u_i' + (\lambda_i+\mu)(u_i - \lambda_i u_{1-i} + \delta y - \mu K\Big) = 0, \\ u_i(0) = K, \quad u_i(L) = 0, \end{cases} \tag{3.2}$$

where $u_i \in C^1(0,L)$ and $u_i'' \in L^\infty(0,L)$. Then, there exists a unique solution of (3.2) *such that $0 \leq u_i(y) \leq K$ for $y \in [0,L]$.*

Proof of Lemma 3.2. *Existence.* We use the penalization method given in Bensoussan and Lions [4]; see also Kinderlehrer and Stampacchia [15]. Note that one of the standard approximation methods to study the existence of a solution to the VI is the penalization method (see Bensoussan and Lions [4]). In particular, it is extremely useful when the obstacle is simple. Here the obstacle is 0, which was the main purpose of going from g_i to u_i. The penalization allows us to approximate the VI by an equation in which there are no more constraints.

For $\varepsilon > 0$, we look for $u_i^\varepsilon(y) \in C^2(0, L)$ such that

$$\begin{cases} -\frac{1}{2}y^2\sigma_i^2\, u_i^{\varepsilon\,\prime\prime} - \alpha_i y\, u_i^{\varepsilon\,\prime} + (\lambda_i + \mu)u_i^\varepsilon - \lambda_i u_{1-i}^\varepsilon + \delta y - \mu K = \frac{1}{\varepsilon}(u_i^\varepsilon)^-, \\ u_i^\varepsilon(0) = K, \quad u_i^\varepsilon(L) = 0. \end{cases} \tag{3.3}$$

Equation (3.3) is the penalized problem associated with the VI (3.2).

Problem (3.3) is a Hamilton–Jacobi–Bellman equation with a Dirichlet condition. It admits a unique solution. In the following part, when $\varepsilon \to 0$, we will show that the limit of u_i^ε is the solution of (3.2).

Let y^ε be a point such that

$$\max_{0\le y\le L} \max_{i=0,1} u_i^\varepsilon(y) = \max_{i=0,1} u_i^\varepsilon(y^\varepsilon) = u_{i^\varepsilon}^\varepsilon(y^\varepsilon) > 0. \tag{3.4}$$

Clearly, $y^\varepsilon < L$. Suppose $y^\varepsilon > 0$; then

$$u_{i^\varepsilon}^{\varepsilon\,\prime}(y^\varepsilon) = 0, \quad u_{i^\varepsilon}^{\varepsilon\,\prime\prime}(y^\varepsilon) < 0.$$

Hence we can write

$$\begin{aligned} -\frac{1}{2}(y^\varepsilon)^2\sigma_{i^\varepsilon}^2\, u_{i^\varepsilon}^{\varepsilon\,\prime\prime}(y^\varepsilon) - \alpha_{i^\varepsilon} y^\varepsilon\, u_{i^\varepsilon}^{\varepsilon\,\prime}(y^\varepsilon) + (\lambda_{i^\varepsilon} + \mu)u_{i^\varepsilon}^\varepsilon(y^\varepsilon) \\ - \lambda_{i^\varepsilon} u_{1-i^\varepsilon}^\varepsilon(y^\varepsilon) + \delta y^\varepsilon - \mu K = \frac{1}{\varepsilon}(u_{i^\varepsilon}^\varepsilon)^-(y^\varepsilon) = 0. \end{aligned} \tag{3.5}$$

By means of (3.4), we have

$$u_{i^\varepsilon}^\varepsilon(y^\varepsilon) \ge u_{1-i^\varepsilon}^\varepsilon(y^\varepsilon).$$

Hence, using the above result in (3.5), we obtain

$$u_{i^\varepsilon}^\varepsilon(y^\varepsilon) \le K.$$

This inequality is clearly true if $y^\varepsilon = 0$. As a result,

$$u_i^\varepsilon(y) \le K. \tag{3.6}$$

Similarly, consider a point of negative minimum; we use the same notation y^ε as in the positive maximum case. We have

$$\min_{0\le y\le L} \min_{i=0,1} u_i^\varepsilon(y) = \min_{i=0,1} u_i^\varepsilon(y^\varepsilon) = u_{i^\varepsilon}^\varepsilon(y^\varepsilon) < 0.$$

Then

$$u_{i^\varepsilon}^{\varepsilon\,\prime}(y^\varepsilon) = 0, \quad u_{i^\varepsilon}^{\varepsilon\,\prime\prime}(y^\varepsilon) > 0.$$

It follows that

$$\mu u_{i^\varepsilon}^\varepsilon(y^\varepsilon) + \delta y^\varepsilon - \mu K \ge -\frac{1}{\varepsilon}(u_{i^\varepsilon}^\varepsilon)(y^\varepsilon).$$

Therefore,

$$\frac{1}{\varepsilon}(u^{\varepsilon}_{i^{\varepsilon}})^-(y^{\varepsilon}) \le (\delta L - \mu K)^+,$$

and

$$\frac{1}{\varepsilon}(u^{\varepsilon}_{i})^-(y) \le (\delta L - \mu K)^+. \tag{3.7}$$

From the estimates (3.6), (3.7), and (3.3), it can be seen that

$$\int_0^L y^2({u^{\varepsilon}_i}')^2 \mathrm{d}y \le C, \quad \int_0^L y^4({u^{\varepsilon}_i}'')^2 \mathrm{d}y \le C. \tag{3.8}$$

Consider u_i, the weak limit of u^{ε}_i. From

$$\begin{array}{llll} u^{\varepsilon}_i \to u_i & \text{in} & L^{\infty}(0,L) & \text{weak star,} \\ y\,(u^{\varepsilon}_i)' \to y u_i' & \text{in} & L^2(0,L) & \text{weakly,} \\ y^2\,(u^{\varepsilon}_i)'' \to y^2 u_i'' & \text{in} & L^2(0,L) & \text{weakly} \end{array}$$

and the compactness of $H^1(0,L)$ in $L^2(0,L)$ (see the Rellich theorem in [1] or [14]), we can claim that

$$\begin{array}{llll} y u^{\varepsilon}_i \to y u_i & \text{in} & L^2(0,L) & \text{strongly,} \\ y^2\,(u^{\varepsilon}_i)' \to y^2 u_i' & \text{in} & L^2(0,L) & \text{strongly.} \end{array}$$

As a result, we obtain

$$\begin{array}{llll} y^{\frac{3}{2}} u^{\varepsilon}_i \to y^{\frac{3}{2}} u_i & \text{pointwise,} & & \\ u^{\varepsilon}_i \to u_i & \text{in} & L^2(0,L) & \text{strongly.} \end{array}$$

From (3.3), we have

$$-\frac{1}{2}\sigma_i^2 y^2\,(u^{\varepsilon}_i)'' - \alpha_i y\,(u^{\varepsilon}_i)' + (\lambda_i + \mu)u^{\varepsilon}_i - \lambda_i u^{\varepsilon}_{1-i} + \delta y - \mu K \ge 0.$$

Using the weak limit, we get the first inequality in (3.2). By (3.7), we get the second inequality in (3.2), Also

$$\begin{aligned} &\bigg(-\frac{1}{2}\sigma_i^2 y^2 (u^{\varepsilon}_i)'' - \alpha_i y (u^{\varepsilon}_i)' + (\lambda_i + \mu)u^{\varepsilon}_i - \lambda_i u^{\varepsilon}_{3-i} + \delta y - \mu K \bigg) u^{\varepsilon}_i \\ &= -\frac{1}{\varepsilon}\Big((u^{\varepsilon}_i)^-\Big)^2 = -\varepsilon\bigg(\frac{(u^{\varepsilon}_i)^-}{\varepsilon}\bigg)^2. \end{aligned}$$

So

$$\int_0^L \bigg(-\frac{1}{2}\sigma_i^2 y^2 (u^{\varepsilon}_i)'' - \alpha_i y (u^{\varepsilon}_i)' + (\lambda_i + \mu)u^{\varepsilon}_i - \lambda_i u^{\varepsilon}_{1-i} + \delta y - \mu K \bigg) u^{\varepsilon}_i \mathrm{d}y \to 0; \tag{3.9}$$

passing to the limit in (3.9), we deduce

$$\int_0^L \bigg(-\frac{1}{2}\sigma_i^2 y^2 u_i'' - \alpha_i y u_i' + (\lambda_i + \mu)u_i - \lambda_i u_{1-i} + \delta y - \mu K \bigg) u_i \mathrm{d}y \to 0.$$

But because the integrand is positive, it must be 0; hence we have the equality in (3.2).

Now the three relations in (3.2) hold a.e. so the value $u_i(0)$ can be defined arbitrarily. By taking $u_i(0) = K$, we get

$$u_i^\varepsilon(y) \to u_i(y) \quad \forall y \in [0, L].$$

So we have obtained the existence of a solution of (3.2) with the desired properties.

Uniqueness. In what follows, we prove the uniqueness of the solution of (3.2). Let $v_i \in C^1$, $0 \le v_i \le K$, $v_i(0) = K$, $v_i(L) = 0$. If u_i is a solution of (3.2), then

$$u_i + \theta(v_i - u_i) \ge 0, \quad 0 < \theta < 1,$$

and

$$u_i(0) + \theta(v_i(0) - u_i(0)) = K,$$
$$u_i(L) + \theta(v_i(L) - u_i(L)) = 0.$$

Thus,

$$\left(-\frac{1}{2} y^2 \sigma_i^2 u_i'' - \alpha_i y u_i' + (\lambda_i + \mu) u_i - \lambda_i u_{1-i} + \delta y - \mu K \right) (u_i + \theta(v_i - u_i)) \ge 0.$$

Therefore, from the equality (3.2),

$$\left(-\frac{1}{2} y^2 \sigma_i^2 u_i'' - \alpha_i y u_i' + (\lambda_i + \mu) u_i - \lambda_i u_{1-i} + \delta y - \mu K \right) (v_i - u_i) \ge 0. \tag{3.10}$$

Suppose that there are two solutions of (3.2), u_i^1 and u_i^2. We can write (3.10) with $u_i = u_i^1$, $v_i = u_i^2$, and $u_i = u_i^2$, $v_i = u_i^1$. We obtain

$$\left(-\frac{1}{2} y^2 \sigma_i^2 (u_i^1)'' - \alpha_i y (u_i^1)' + (\lambda_i + \mu) u_i^1 - \lambda_i u_{1-i}^1 + \delta y - \mu K \right) (u_i^2 - u_i^1) \ge 0,$$

$$\left(-\frac{1}{2} y^2 \sigma_i^2 (u_i^2)'' - \alpha_i y (u_i^2)' + (\lambda_i + \mu) u_i^2 - \lambda_i u_{1-i}^2 + \delta y - \mu K \right) (u_i^1 - u_i^2) \ge 0.$$

By adding up the two equations, we get

$$\begin{aligned} -\tfrac{1}{2} y^2 \sigma_i^2 ((u_i^1)'' - (u_i^2)'')(u_i^1 - u_i^2) - \alpha_i y ((u_i^1)' - (u_i^2)')(u_i^1 - u_i^2) \\ + (\lambda_i + \mu)(u_i^1 - u_i^2)^2 - \lambda_i (u_{1-i}^1 - u_{1-i}^2)(u_i^1 - u_i^2) \le 0. \end{aligned} \tag{3.11}$$

Setting $w_i = u_i^1 - u_i^2$, (3.11) can be written as

$$-\frac{1}{2} \sigma_i^2 y^2 w_i'' w_i - \alpha_i y w_i' w_i + (\lambda_i + \mu) w_i^2 - \lambda_i w_{1-i} w_i \le 0 \tag{3.12}$$

with $w_i(0) = w_i(L) = 0$.

Consider a point y^* such that

$$\max_{0 \le y \le L} \max_{i=0,1} w_i(y) = \max_{i=0,1} w_i(y^*) = w_{i^*}(y^*).$$

Suppose $w_{i^*}(y^*) > 0$; then write (3.12) with $i = i^*$, $y = y^*$, and note that $y^* \neq 0, L$. We have

$$w_{i^*}(y^*) \geq w_{1-i^*}(y^*), \quad w'_{i^*}(y^*) = 0, \quad w''_{i^*}(y^*) < 0,$$

which leads to a contradiction to (3.12). Therefore, $w_i(y) \leq 0$. Hence $u_i^1 - u_i^2 \leq 0$.

Similarly, we write (3.10) with $u_i = u_i^2$, $v_i = u_i^1$ and $u_i = u_i^1$, $v_i = u_i^2$; then $u_i^2 - u_i^1 \leq 0$, which implies $u_i^1 = u_i^2$, and the uniqueness is obtained. ■

Part 2. We continue to work with a finite L satisfying $L > L_0$, where L_0 will be specified in what follows.

Lemma 3.3. *There exist numbers $y_i < L$ uniquely defined such that*

$$\begin{cases} -\frac{1}{2}y^2\sigma_i^2 u_i'' - \alpha_i y u_i' + (\lambda_i + \mu)u_i - \lambda_i u_{1-i} + \delta y - \mu K = 0, \quad 0 < y < y_i, \\ u_i = 0, \quad y_i \leq y \leq L, \\ u_i(0) = K. \end{cases} \tag{3.13}$$

Proof of Lemma 3.3. We cannot have for L sufficiently large the following properties:

$$\begin{cases} -\frac{1}{2}y^2\sigma_i^2 u_i'' - \alpha_i y u_i' + (\lambda_i + \mu)u_i - \lambda_i u_{1-i} + \delta y - \mu K = 0, \quad 0 < y < L, \\ u_i > 0, \quad 0 \leq y < L, \\ u_i(0) = K, \quad u_i(L) = 0, \\ u_i \in C^1(0, L). \end{cases} \tag{3.14}$$

Indeed, the first equality of (3.14) yields that

$$-\frac{1}{2}y^2\sigma_i^2 u_i'' - \alpha_i y u_i' + (\lambda_i + \mu)u_i \leq -\delta y + (\mu + \lambda_i)K.$$

So $u_i \leq v_i$, where v_i satisfy

$$\begin{cases} -\dfrac{1}{2}y^2\sigma_i^2 v_i'' - \alpha_i y v_i' + (\lambda_i + \mu)v_i = -\delta y + (\mu + \lambda_i)K, \\ v_i(0) = K, \quad v_i(L) = 0. \end{cases} \tag{3.15}$$

Then the solution of (3.15) can be written as

$$v_i(y) = -\frac{\delta y}{\lambda_i + \mu - \alpha_i} + K + \left(\frac{\delta L}{\lambda_i + \mu - \alpha_i} - K\right)\left(\frac{y}{L}\right)^{\beta_i},$$

where $\beta_i > 1$ is the solution of

$$-\frac{1}{2}\sigma_i^2\beta_i(\beta_i - 1) - \alpha_i\beta_i + \lambda_i + \mu = 0.$$

Hence

$$\begin{aligned} v_i'(y) &= -\frac{\delta}{\lambda_i + \mu - \alpha_i} + \beta_i\left(\frac{\delta L}{\lambda_i + \mu - \alpha_i} - K\right)\frac{y^{\beta_i - 1}}{L^{\beta_i}}, \\ v_i'(L) &= -\frac{\delta}{\lambda_i + \mu - \alpha_i} + \beta_i\left(\frac{\delta}{\lambda_i + \mu - \alpha_i} - \frac{K}{L}\right) \\ &= (\beta_i - 1)\frac{\delta}{\lambda_i + \mu - \alpha_i} - \frac{\beta_i K}{L}. \end{aligned}$$

If

$$L > \frac{\beta_i K(\lambda_i + \mu - \alpha_i)}{(\beta_i - 1)\delta}, \tag{3.16}$$

we get $v_i'(L) > 0$; hence $v_i(y) < 0$ for y close to L, which is impossible since $u_i(y) > 0$. Therefore, there exists $0 < y_i < L$ such that $u_i(y_i) = 0$. We can consider that y_i is the smallest such point. This point is a minimum of u_i; then

$$u_i'(y_i) = 0, \quad u_i''(y_i) > 0.$$

Hence

$$-\lambda_i u_{1-i}(y_i) + \delta y_i - \mu K \geq 0, \tag{3.17}$$

which implies

$$y_i \geq \frac{\mu K}{\delta}.$$

If $y_i = y_{1-i}$, we extend $u_i(y)$ beyond y_i by 0; then it is also C^1 and is the solution of the VI (3.2). By the uniqueness, this is the solution, and thus (3.13) is obtained.

Suppose $y_i \neq y_{1-i}$, and we can assume $y_i < y_{1-i}$. Therefore, $u_i(y_i) = 0$, $u_{1-i}(y_i) > 0$ with (3.17). Consider a maximum y^* of $\max_{i=0,1} u_i(y)$ on (y_i, L). Let us check that $y^* = y_i$. Let i^* be such that

$$u_{i^*}(y^*) = \max_{i=0,1} u_i(y^*) \quad \text{on} \quad (y_i, L).$$

Since $y_i < y^* < L$, we have

$$u_{i^*}'(y^*) = 0, \quad u_{i^*}''(y^*) < 0, \quad u_{i^*}(y^*) \geq u_{1-i^*}(y^*).$$

Note that $u_{i^*}(y^*) > 0$. We have

$$-\frac{1}{2}(y^*)^2 \sigma_{i_*}^2 u_{i^*}''(y^*) - \alpha_{i^*} y^* u_{i^*}'(y^*) + (\lambda_i + \mu) u_{i^*}(y^*) - \lambda_i u_{1-i^*}(y^*) + \delta y^* - \mu K = 0.$$

Therefore, $\delta y^* - \mu K \leq 0$, which is impossible since $y^* > y_i$. It follows that

$$u_i(y), \quad u_{1-i}(y) \leq u_{1-i}(y_i) \quad \forall y \in [y_i, L].$$

We necessarily have

$$u_i(y) = 0 \quad \text{on} \quad [y_i, L].$$

Otherwise, there would exist a positive maximum of $u_i(y)$ on (y_i, L). At such a point $\widehat{y}_i$, we have

$$-\frac{1}{2}(\widehat{y}_i)^2 \sigma_i^2 u_i''(\widehat{y}_i) - \alpha_i \widehat{y}_i u_i'(y_i) + (\lambda_i + \mu) u_i(\widehat{y}_i) - \lambda_i u_{1-i}(\widehat{y}_i) + \delta \widehat{y}_i - \mu K = 0,$$

and

$$u_{i^*}'(y^*) = 0, \quad u_{i^*}''(y^*) < 0,$$

$$-\lambda_i u_{1-i}(\widehat{y}_i) + \delta \widehat{y}_i - \mu K \geq -\lambda_i u_{1-i}(y_i) + \delta y_i - \mu K \geq 0,$$

which yields a contradiction, and the last inequality comes from (3.17).

We then have $u_i(y) = 0$, $y \geq y_i$. It follows that on (y_i, L), $u_{1-i}(y)$ solves the standard VI

$$\begin{cases} -\frac{1}{2}y^2\sigma_i^2 u''_{1-i} - \alpha_i y u'_{1-i} + (\lambda_{1-i} + \mu)u_{1-i} + \delta y - \mu K \geq 0, \\ u_{1-i} \geq 0, \\ \ u_{1-i}(-\frac{1}{2}y^2\sigma_i^2 u''_{1-i} - \alpha_i y u'_{1-i} + (\lambda_{1-i} + \mu)u_{1-i} \\ \ + \delta y - \mu K) = 0, \quad y_i < y < L, \\ u_{1-i}(y_i) > 0, \quad u_{1-i}(L) = 0. \end{cases} \tag{3.18}$$

The proof is straightforward. Since $u_{1-i}(y) > 0$, the equation holds near y_i:

$$-\frac{1}{2}y^2\sigma_i^2 u''_{1-i} - \alpha_i y u'_{1-i} + (\lambda_{1-i} + \mu)u_{1-i} + \delta y - \mu K = 0.$$

There cannot be any local maximum since

$$\delta y - \mu K \geq \delta y_i - \mu K > 0.$$

Therefore, the function u_{1-i} decreases after y_i and becomes 0 at y_{1-i}. Since y_{1-i} is a minimum and $u'_{1-i}(y_{1-i}) = 0$, we can consider the extension of $u_{1-i}(y)$ by 0 after y_{1-i}. It is also C^1 and hence the solution of the VI (3.18).

Define

$$L_0 = \frac{K}{\delta} \max\left(\mu, \max_{i=0,1} \frac{\beta_i}{\beta_i - 1}(\lambda_i + \mu - \alpha_i)\right). \tag{3.19}$$

Then the statement of part 2 has been proven. The uniqueness of y_i is a consequence of the uniqueness of u_i. ■

Part 3. We work with L satisfying $L \in (L_0, \infty)$. The solution vanishes for L large enough, and the infinite horizon problem becomes the same as the finite horizon problem. So the solution of (3.2) coincides with the solution of (2.10). The proof of the theorem is thus concluded. ■

The above theorem leads to the conclusion that the optimal stopping rule is of threshold type. This result is the verification theorem, whose proof is placed in the appendix.

Proposition 3.4. *The optimal stopping rule that achieves the supremum in* (2.8) *can be written as*

$$\widehat{\tau} = \inf\{t : Y(t) \geq y_{\theta(t)}\}. \tag{3.20}$$

Remark 3.5. The verification theorem above states that the investor undertakes the investment as soon as the cash flow reaches the threshold y_i from below. Since the investment policy is affected by a return factor besides volatility and macroeconomic conditions, it is interesting to see how these parameters affect the solution. In the following section, we obtain an explicit optimal stopping rule and value function for this regime-switching model.

4. Derivation of solution. Depending on parameter values, the two thresholds may be ordered differently. For any y_0 and y_1, there are only three possibilities—$y_0 < y_1$, $y_0 > y_1$, and $y_0 = y_1$. In the following theorem, we give the optimal stopping rule and the corresponding value functions using the technique in [9].

Theorem 4.1. *If $y_0 < y_1$, then the two investment thresholds satisfy*

$$\begin{pmatrix} y_1^{-\gamma_1} & 0 \\ 0 & y_1^{-\gamma_2} \end{pmatrix} F_1(y_1) = \begin{pmatrix} y_0^{-\gamma_1} & 0 \\ 0 & y_0^{-\gamma_2} \end{pmatrix} F_0(y_0),$$

where

$$F_1(y_1) = c_1 + c_2 y_1,$$
$$F_0(y_0) = b_1 + b_2 y_0,$$

and the value function is given by

$$g_0(y) = \begin{cases} B_1 y^{\beta_1} + B_2 y^{\beta_2} & \text{if} \quad y < y_0, \\ a_0 y - K & \text{if} \quad y \geq y_0, \end{cases}$$
$$g_1(y) = \begin{cases} A_1 y^{\beta_1} + A_2 y^{\beta_2} & \text{if} \quad y < y_0, \\ C_1 y^{\gamma_1} + C_2 y^{\gamma_2} + \phi(y) & \text{if} \quad y_0 \leq y < y_1, \\ a_1 y - K & \text{if} \quad y \geq y_1. \end{cases}$$

If $y_0 > y_1$, then the two investment thresholds satisfy

$$\begin{pmatrix} y_1^{-\tilde{\gamma}_1} & 0 \\ 0 & y_1^{-\tilde{\gamma}_2} \end{pmatrix} \widetilde{F}_1(y_1) = \begin{pmatrix} y_0^{-\tilde{\gamma}_1} & 0 \\ 0 & y_0^{-\tilde{\gamma}_2} \end{pmatrix} \widetilde{F}_0(y_0),$$

where

$$\widetilde{F}_1(y_1) = \tilde{c}_1 + \tilde{c}_2 y_1,$$
$$\widetilde{F}_0(y_0) = \tilde{b}_1 + \tilde{b}_2 y_0,$$

and the value functions are given by

$$g_0(y) = \begin{cases} \widetilde{B}_1 y^{\beta_1} + \widetilde{B}_2 y^{\beta_2} & \text{if} \quad y < y_1, \\ \widetilde{C}_1 y^{\gamma_1} + \widetilde{C}_2 y^{\gamma_2} + \tilde{\phi}(y) & \text{if} \quad y_1 \leq y < y_0, \\ a_0 y - K & \text{if} \quad y \geq y_0, \end{cases}$$
$$g_1(y) = \begin{cases} \widetilde{A}_1 y^{\beta_1} + \widetilde{A}_2 y^{\beta_2} & \text{if} \quad y \leq y_1, \\ a_1 y - K & \text{if } y \geq y_1. \end{cases}$$

If $y_0 = y_1 = y_*$, then y_* satisfy

$$y_* = \frac{l_1 K - K}{a_1(l_1 - l_2)},$$

and the value function is given by

$$g_0(y) = \begin{cases} B_1 y^{\beta_1} + B_2 y^{\beta_2} & \text{if} \quad y < y_*, \\ a_0 y - K & \text{if} \quad y \geq y_*, \end{cases}$$

$$g_1(y) = \begin{cases} A_1 y^{\beta_1} + A_2 y^{\beta_2} & \text{if} \quad y \leq y_*, \\ a_1 y - K & \text{if} \quad y \geq y_*. \end{cases}$$

The unknown coefficients can be found in Appendix A.

Proof. See Appendix A. ∎

For the purpose of comparison, we also review the traditional real option model. The assumption of cash flow without regime switching corresponds to the following equation:

$$\mathrm{d}Y_t = \alpha Y_t \mathrm{d}t + \sigma Y_t \mathrm{d}W_t, \quad Y(0) = y.$$

Then the optimal investment threshold is given by

$$y_* = \frac{\beta_1}{\beta_1 - 1}(\mu - \alpha)K,$$

where β_1 is the positive root of the quadratic equation

$$\frac{1}{2}\sigma^2\beta^2 + \left(\alpha - \frac{1}{2}\sigma^2\right)\beta - \mu = 0.$$

The value function is derived as

$$g(y) = \begin{cases} Ay^{\beta_1} & \text{if} \quad y < y_*, \\ \dfrac{y}{\mu - \alpha} - K & \text{if} \quad y \geq y_*, \end{cases}$$

where $A = \frac{1}{y_*^{\beta_1}}(\frac{y_*}{\mu-\alpha} - K)$.

5. Numerical simulation.

To further demonstrate the quantitative properties of the model, we consider a numerical example. For the benchmark case, we use

$$\begin{aligned} &\mu = 0.06, \quad \alpha_0 = 0.04, \quad \alpha_1 = 0.01, \\ &\sigma_0 = 0.2, \quad \sigma_1 = 0.3, \quad \lambda_0 = 0.15, \quad \lambda_1 = 0.1, \text{ and} \\ &\delta = 1, \quad K = 5. \end{aligned}$$

In regime 0, the return rate is higher, while the volatility rate is lower than that of regime 1. So we call regime 0 and regime 1 the "good state" and "bad state," respectively.

First, consider the impact of λ_i on the option value and investment threshold. We set $\lambda_0 = 0.05$, $\lambda_0 = 0.25$, $\lambda_1 = 0.05$, and $\lambda_1 = 0.25$. From Figure 1, an increase in λ_0 results in a decrease in option value and an increase in the threshold. The reason is that as λ_0 becomes larger, the time staying at the good regime 0 decreases, and it becomes better to wait for a higher threshold. Meanwhile, an increase of λ_1 leads to an increase of the option value and a decrease of the investment threshold, because the time staying at the bad regime reduces. Thus, it encourages early investment.

In what follows, we explain why the option value and threshold are negatively correlated. From the value-matching condition $g_i(y_i) = a_i y_i - K$, we could equivalently write the equation as $a_i y_i = g_i(y_i) + K$, setting the value of the project equal to the full cost (direct cost plus opportunity cost). When the option value decreases (resp., increases), the firm value also decreases (resp., increases), and the total effect gives an increase (resp., decrease) in threshold.

Figure 2 shows how threshold in regime i depends on the drift parameters. Similar to the traditional real option model, the higher the expected return rate, the smaller the investment

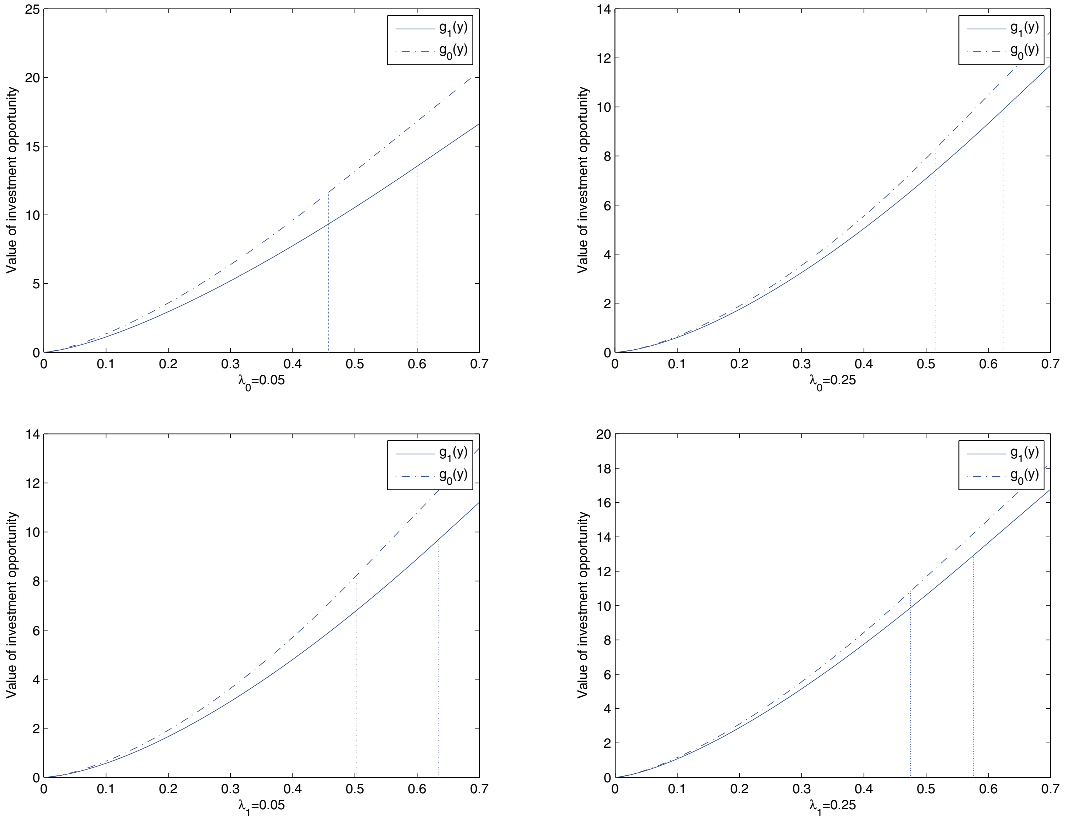

Figure 1. *The option value $g_0(y)$ of regime 0 and the option value $g_1(y)$ of regime 1. We plot the results for $\lambda_0 = 0.05$, $\lambda_0 = 0.25$, $\lambda_1 = 0.05$, and $\lambda_1 = 0.25$.*

threshold, and hence it encourages early investment. In both regimes, the option values increase as the expected return rate increases.

The dependence of threshold on σ_i is shown directly in Figure 3. Observe that threshold increases with volatility σ_i and hence postpones investment in both regimes. Figure 3 also plots investment threshold as a function of volatility when $\alpha = 0.04$ (resp., $\alpha = 0.01$) in traditional real options. We can see the threshold is smaller (resp., larger) than the regime-shift model in the good (resp., bad) state. This is not surprising because there is a possibility of changing to the bad (resp., good) state, so there is an incentive to wait longer (resp., less) in the case of regime shift. This figure also illustrates that the two investment thresholds may switch order depending on parameter values. In this model, an increase in σ_i results in an increase in option value that is the same as in the traditional real option models without regime switching.

As revealed in Figures 2 and 3, changes in drift and volatility parameters in regime i affect the option value and investment threshold in both regimes, but the impact on threshold in regime $1 - i$ is not larger than that of regime i.

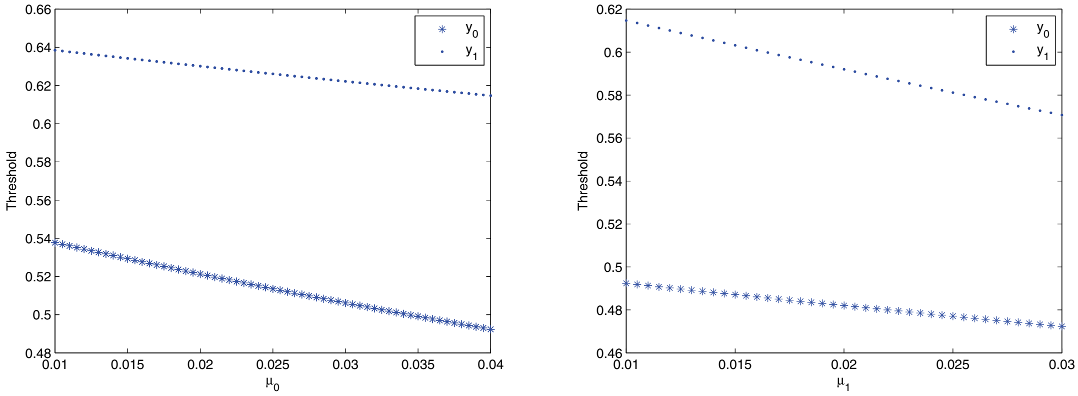

Figure 2. *Thresholds as functions of* μ_0 *and* μ_1.

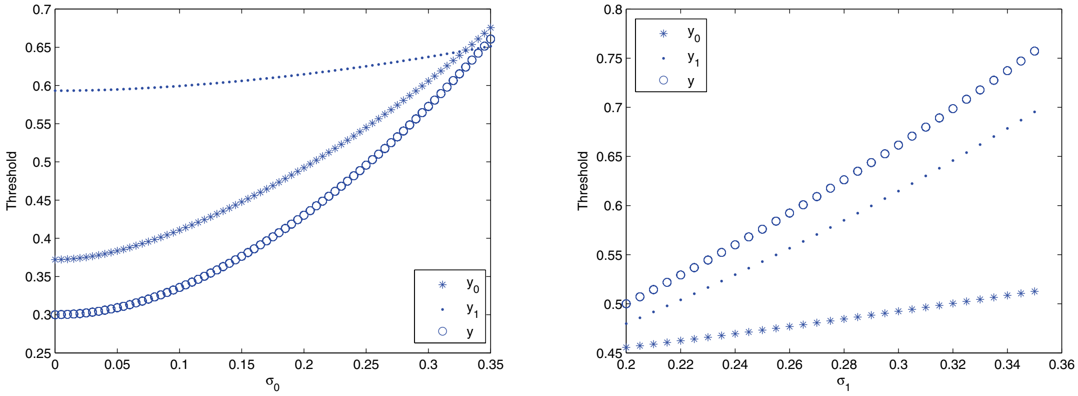

Figure 3. *Thresholds with two regimes as functions of* σ_0 *and* σ_1. *The figures display the investment threshold* y *as a function of the volatility parameter when there is no regime shift. The drift parameters are set to be* $\alpha = 0.04$ *and* $\alpha = 0.01$.

6. Further remarks. This work has proved that threshold-type policies are optimal for real options of regime-switching models. The underlying problem belongs to the class of optimal stopping problems. We used VI techniques developed by Bensoussan and Lions [4] to rigorously justify the existence and uniqueness of the solution of optimal thresholds and to establish the optimality. The results obtained in this paper will be useful in solving many real option problems.

A main premise of the current paper assumes the market to be complete. To investigate real option strategies in an incomplete market is both practically useful and mathematically interesting. Further study in this direction is a worthwhile effort.

Appendix A. Technical complements.

Proof of (2.11). The result is clear when $u_i = 0$. So we consider a domain in which $u_i > 0$.

Therefore, the following equation holds:

$$\text{(A.1)}\qquad \begin{aligned}&-\frac{1}{2}y^2\sigma_i^2 u_i'' - \alpha_i y u_i' + (\lambda_i+\mu)u_i - \lambda_i u_{1-i} + \delta y - \mu K = 0,\\ &u_i > 0.\end{aligned}$$

Take the largest interval in which (A.1) holds, say, $(\widetilde{a}, \widetilde{b})$. Necessarily

$$\text{(A.2)}\qquad u_i(\widetilde{a}) = u_i(\widetilde{b}) = 0.$$

But on the interval $(\widetilde{a}, \widetilde{b})$, v_i satisfies

$$\text{(A.3)}\qquad \begin{aligned}&-\frac{1}{2}y^2\sigma_i^2 v_i'' - \alpha_i y v_i' + (\lambda_i+\mu)v_i - \lambda_i v_{1-i} + \delta y - \mu K \geq 0,\\ &v_i(\widetilde{a}) \geq 0, \quad v_i(\widetilde{b}) \geq 0.\end{aligned}$$

Comparing (A.1) and (A.2) with (A.3), and using the maximum principle, we get $v_i(y) \geq u_i(y)$ in $(\widetilde{a}, \widetilde{b})$. The proof will extend to any such interval $(\widetilde{a}, \widetilde{b})$ and to the whole domain on which $u_i > 0$. ■

Proof of Proposition 3.4. Considering (2.9), we know that g_i is C^1, g_i'' is L^∞, and there exist thresholds y_i. Note that the pair $(Y(t), \theta(t))$ is a Markov process. Take a function $\varphi(y,i) = \varphi_i(y)$. The generator of the Markov process is given by

$$(A\varphi)_i(y) = a_i y \varphi_i'(y) + \frac{1}{2}\sigma_i^2 y^2 \varphi_i''(y) + \lambda_i \varphi_{1-i}(y) - \lambda_i \varphi_i(y).$$

We know that

$$\varphi_{\theta(t)}(Y(t)) - \int_0^t (A\varphi)_{\theta(s)}(Y(s))ds$$

is an $\mathcal{F}_t$ martingale, where $\mathcal{F}_t = \sigma\{W(s), \theta(s) : s \leq t\}$. As a result,

$$e^{-\mu t}\varphi_{\theta(t)}(Y(t)) - \int_0^t e^{-\mu s}[(A\varphi)_{\theta(s)}(Y(s)) - \mu\varphi_{\theta(s)}(Y(s))]ds$$

is an $\mathcal{F}_t$ martingale.

Using $\varphi_i(y) = g_i(y)$, we obtain that

$$e^{-\mu t}g_{\theta(t)}(Y(t)) - \int_0^t e^{-\mu s}[(Ag)_{\theta(s)}(Y(s)) - \mu g_{\theta(s)}(Y(s))]ds$$

is an $\mathcal{F}_t$ martingale. Let τ be any $\mathcal{F}_t$ stopping time.

Since $(Y(t), \theta(t))$ is a strong Markov process,

$$\text{(A.4)}\qquad \begin{aligned}&E_{y,i}\exp(-\mu\tau)g_{\theta(\tau)}(Y(\tau))1_{\{\tau<\infty\}}\\ &\quad -E_{y,i}1_{\{\tau<\infty\}}\int_0^\tau \exp(-\mu s)[(Ag)_{\theta(s)}(Y(s)) - \mu g_{\theta(s)}(Y(s))ds] = g_i(y).\end{aligned}$$

Note that $(Ag)_i(y) - \mu g_i(y) \leq 0$; hence

$$\begin{aligned}g_i(y) &\geq E_{y,i}\exp(-\mu\tau)g_{\theta(\tau)}(Y(\tau))1_{\{\tau<\infty\}}\\ &\geq E_{y,i}\exp(-\mu\tau)(a_{\theta(\tau)}Y(\tau) - K)1_{\{\tau<\infty\}} = J_{y,i}(\tau).\end{aligned}$$

Therefore, $g_i(y) \geq \sup_{\tau\geq 0} J_{y,i}(\tau)$.

Let $\widehat{\tau}$ be the optimal stopping defined by the threshold with $\widehat{\tau} = \inf\{t : Y(t) \geq y_{\theta(t)}\}$. We note that $(Ag)_{\theta(s)}(Y(s)) - \mu g_{\theta(s)}(Y(s)) = 0$ for $s < \widehat{\tau}$ since $Y(s) < y_{\theta(s)}$. Applying (A.4) with $\tau = \widehat{\tau}$ yields

$$g_i(y) = E_{y,i} \exp(-\mu\widehat{\tau}) g_{\theta(\widehat{\tau})}(Y(\widehat{\tau})) 1_{\{\widehat{\tau}<\infty\}}.$$

But if $\widehat{\tau} < \infty$, we have $Y(\widehat{\tau}) = y_{\theta(\widehat{\tau})}$ and

$$g_{\theta(\widehat{\tau})}(y_{\theta(\widehat{\tau})}) = a_{\theta(\widehat{\tau})} y_{\theta(\widehat{\tau})} - K = a_{\theta(\widehat{\tau})} Y(\widehat{\tau}) - K,$$
$$g_i(y) = E_{y,i} \exp(-\mu\widehat{\tau})(a_{\theta(\widehat{\tau})} Y(\widehat{\tau}) - K) 1_{\{\widehat{\tau}<\infty\}} = J_{y,i}(\widehat{\tau}).$$

Therefore, $g_i(y) = \sup_{\tau\geq 0} J_{y,i}(\tau)$. The proof is completed. ∎

Proof of Theorem 4.1.

Case 1. $y_0 < y_1$. For $y \in (0, y_0]$,

(A.5)
$$\begin{cases} (\lambda_1 + \mu) g_1(y) = \dfrac{1}{2}\sigma_1^2 y^2 g_1''(y) + \alpha_1 y g_1'(y) + \lambda_1 g_0(y), \\ (\lambda_0 + \mu) g_0(y) = \dfrac{1}{2}\sigma_0^2 y^2 g_0''(y) + \alpha_1 y g_0'(y) + \lambda_0 g_1(y). \end{cases}$$

For $y \in [y_0, y_1)$,

(A.6)
$$\begin{cases} (\lambda_1 + \mu) g_1(y) = \dfrac{1}{2}\sigma_1^2 y^2 g_1''(y) + \alpha_1 y g_1'(y) + \lambda_1 g_0(y), \\ g_0(y) = a_0 y - K. \end{cases}$$

For $y \in [y_1, \infty)$,

(A.7)
$$g_i(y) = a_i y - K.$$

Then the associated characteristic function of (A.5) is

(A.8)
$$G_0(\beta) G_1(\beta) = \lambda_0 \lambda_1,$$

where

$$G_0(\beta) = \lambda_0 + \mu - \left(\alpha_0 - \frac{1}{2}\sigma_0^2\right)\beta - \frac{1}{2}\sigma_0^2 \beta^2,$$
$$G_1(\beta) = \lambda_1 + \mu - \left(\alpha_1 - \frac{1}{2}\sigma_1^2\right)\beta - \frac{1}{2}\sigma_1^2 \beta^2.$$

Clearly, this characteristic function has four distinct roots $\beta_1 > \beta_2 > 0 > \beta_3 > \beta_4$. Then the general solution to (A.5) is given by

$$g_1(y) = \sum_{j=1}^{4} A_j y^{\beta_j},$$
$$g_0(y) = \sum_{j=1}^{4} B_j y^{\beta_j},$$

with $B_i = l_i A_i$ and $l_i = \frac{G_1(\beta_i)}{\lambda_1} = \frac{\lambda_0}{G_0(\beta_i)}$.

From the boundary condition $g_i(0) = 0$, the negative power of x should be eliminated. Thus the solution of (A.5) has the form

$$g_1(y) = A_1 y^{\beta_1} + A_2 y^{\beta_2},$$
$$g_0(y) = B_1 y^{\beta_1} + B_2 y^{\beta_2}.$$

The general solution of (A.6) takes the following form. For $y \in [y_0, y_1]$,

$$g_1(y) = C_1 y^{\gamma_1} + C_2 y^{\gamma_2} + \phi(y),$$

where $\phi(y)$ is a particular solution to the nonhomogeneous equation and γ_1, γ_2 are the two real roots of

$$\lambda_1 + \mu - \alpha_1 \gamma - \frac{1}{2}\sigma_1^2 \gamma(\gamma - 1) = 0.$$

From the assumption $\mu > \alpha_i > 0$ so that $\mu + \lambda_1 - \alpha_1 \neq 0$, we have

$$\phi(y) = \frac{a_0 \lambda_1 y}{\mu + \lambda_1 - \alpha_1} - \frac{\lambda_1 K}{\mu + \lambda_1}.$$

In what follows, we solve A_3, A_4, C_1, C_2, y_1, and y_2. Applying the value matching and smooth fit condition to $g_0(y)$ at the investment threshold y_0, we obtain $g_0(y_0+) = g_0(y_0-)$ and $g_0'(y_0+) = g_0'(y_0-)$; that is,

$$\text{(A.9)} \qquad \begin{cases} l_1 A_1 y_0^{\beta_1} + l_2 A_2 y_0^{\beta_2} = a_0 y_0 - K, \\ \beta_1 l_1 A_1 y_0^{\beta_1} + \beta_2 l_2 A_2 y_0^{\beta_2} = a_0 y_0. \end{cases}$$

Similarly, we obtain the following for $g_1(y)$ at y_0 and y_1:

$$\text{(A.10)} \qquad \begin{cases} A_1 y_0^{\beta_1} + A_2 y_0^{\beta_2} = C_1 y_0^{\gamma_1} + C_2 y_0^{\gamma_2} + \phi(y_0), \\ \beta_1 A_1 y_0^{\beta_1} + \beta_2 A_2 y_0^{\beta_2} = \gamma_1 C_1 y_0^{\gamma_1} + \gamma_2 C_2 y_0^{\gamma_2} + y_0 \phi'(y_0), \end{cases}$$

$$\text{(A.11)} \qquad \begin{cases} C_1 y_1^{\gamma_1} + C_2 y_1^{\gamma_2} + \phi(y_1) = a_1 y_1 - K, \\ \gamma_1 C_1 y_1^{\gamma_1} + \gamma_2 C_2 y_1^{\gamma_2} + y_1 \phi(y_1) = a_1 y_1. \end{cases}$$

After some simple algebra, we have

$$\begin{pmatrix} y_1^{-\gamma_1} & 0 \\ 0 & y_1^{-\gamma_2} \end{pmatrix} F_1(y_1) = \begin{pmatrix} y_0^{-\gamma_1} & 0 \\ 0 & y_0^{-\gamma_2} \end{pmatrix} F_0(y_0),$$

where

$$F_1(y_1) = c_1 + c_2 y_1,$$
$$F_0(y_0) = b_1 + b_2 y_0,$$

with

$$c_1 = \begin{pmatrix} 1 & 1 \\ \gamma_1 & \gamma_2 \end{pmatrix}^{-1} \begin{pmatrix} \frac{\mu K}{\mu+\lambda_1} \\ 0 \end{pmatrix},$$

$$c_2 = \begin{pmatrix} 1 & 1 \\ \gamma_1 & \gamma_2 \end{pmatrix}^{-1} \begin{pmatrix} a_1 - \frac{a_0\lambda_1}{\mu+\lambda_1-\alpha_1} \\ a_1 - \frac{a_0\lambda_1}{\mu+\lambda_1-\alpha_1} \end{pmatrix},$$

$$b_1 = \begin{pmatrix} 1 & 1 \\ \gamma_1 & \gamma_2 \end{pmatrix}^{-1} \left[\begin{pmatrix} 1 & 1 \\ \beta_1 & \beta_2 \end{pmatrix} \begin{pmatrix} l_1 & l_2 \\ \beta_1 l_1 & \beta_2 l_2 \end{pmatrix}^{-1} \begin{pmatrix} -K \\ 0 \end{pmatrix} + \begin{pmatrix} \frac{\lambda_1 K}{\mu+\lambda_1} \\ 0 \end{pmatrix} \right],$$

$$b_2 = \begin{pmatrix} 1 & 1 \\ \gamma_1 & \gamma_2 \end{pmatrix}^{-1} \left[\begin{pmatrix} 1 & 1 \\ \beta_1 & \beta_2 \end{pmatrix} \begin{pmatrix} l_1 & l_2 \\ \beta_1 l_1 & \beta_2 l_2 \end{pmatrix}^{-1} \begin{pmatrix} a_0 \\ a_0 \end{pmatrix} - \begin{pmatrix} \frac{a_0\lambda_1}{\mu+\lambda_1-\alpha_1} \\ \frac{a_0\lambda_1}{\mu+\lambda_1-\alpha_1} \end{pmatrix} \right].$$

Moreover, we obtain

$$\begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = \begin{pmatrix} l_1 y_0^{\beta_1} & l_2 y_0^{\beta_2} \\ \beta_1 l_1 y_0^{\beta_1} & \beta_2 l_2 y_0^{\beta_2} \end{pmatrix}^{-1} \begin{pmatrix} a_0 y_0 - K \\ a_0 y_0 \end{pmatrix},$$

$$\begin{pmatrix} B_1 \\ B_2 \end{pmatrix} = \begin{pmatrix} l_1 A_1 \\ l_2 A_2 \end{pmatrix},$$

$$\begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = \begin{pmatrix} y_1^{\gamma_1} & y_1^{\gamma_2} \\ \gamma_1 y_1^{\gamma_1} & \gamma_2 y_1^{\gamma_2} \end{pmatrix}^{-1} \begin{pmatrix} a_1 y_1 - K - \phi(y_1) \\ a_1 y_1 - y_1 \phi'(y_1) \end{pmatrix}.$$

We obtain the value functions as follows:

$$g_0(y) = \begin{cases} B_1 y^{\beta_1} + B_2 y^{\beta_2} & \text{if} \quad y \le y_0, \\ a_0 y - K & \text{if} \quad y \ge y_0, \end{cases}$$

$$g_1(y) = \begin{cases} A_1 y^{\beta_1} + A_2 y^{\beta_2} & \text{if} \quad y \le y_0, \\ C_1 y^{\gamma_1} + C_2 y^{\gamma_2} + \phi(y) & \text{if} \quad y_0 \le y \le y_1, \\ a_1 y - K & \text{if} \quad y \ge y_1. \end{cases}$$

Case 2. $y_1 < y_0$. We merely state the results. They are easily verified using the same method as in the case $y_0 < y_1$.

Let $\widetilde{\gamma}_1$,$\widetilde{\gamma}_2$ be the two real roots of

$$\lambda_0 + \mu - \alpha_0\gamma - \frac{1}{2}\sigma_0^2\gamma(\gamma-1) = 0,$$

and

$$\widetilde{\phi}(y) = \frac{a_1\lambda_0 y}{\mu+\lambda_0-\alpha_0} - \frac{\lambda_0 K}{\mu+\lambda_0}.$$

Then y_0, y_1 satisfy

$$\begin{pmatrix} y_1^{-\widetilde{\gamma}_1} & 0 \\ 0 & y_1^{-\widetilde{\gamma}_2} \end{pmatrix} \widetilde{F}_1(y_1) = \begin{pmatrix} y_0^{-\widetilde{\gamma}_1} & 0 \\ 0 & y_0^{-\widetilde{\gamma}_2} \end{pmatrix} \widetilde{F}_0(y_0),$$

where

$$\widetilde{F}_1(y_1) = \widetilde{c}_1 + \widetilde{c}_2 y_1,$$
$$\widetilde{F}_0(y_0) = \widetilde{b}_1 + \widetilde{b}_2 y_0,$$

with

$$\widetilde{c}_1 = \begin{pmatrix} 1 & 1 \\ \widetilde{\gamma}_1 & \widetilde{\gamma}_2 \end{pmatrix}^{-1} \left[\begin{pmatrix} 1 & 1 \\ \beta_1 & \beta_2 \end{pmatrix} \begin{pmatrix} \widetilde{l}_1 & \widetilde{l}_2 \\ \beta_1 \widetilde{l}_1 & \beta_2 \widetilde{l}_2 \end{pmatrix}^{-1} \begin{pmatrix} -K \\ 0 \end{pmatrix} + \begin{pmatrix} \frac{\lambda_0 K}{\mu+\lambda_0} \\ 0 \end{pmatrix} \right],$$

$$\widetilde{c}_2 = \begin{pmatrix} 1 & 1 \\ \widetilde{\gamma}_1 & \widetilde{\gamma}_2 \end{pmatrix}^{-1} \left[\begin{pmatrix} 1 & 1 \\ \beta_1 & \beta_2 \end{pmatrix} \begin{pmatrix} \widetilde{l}_1 & \widetilde{l}_2 \\ \beta_1 \widetilde{l}_1 & \beta_2 \widetilde{l}_2 \end{pmatrix}^{-1} \begin{pmatrix} a_1 \\ a_1 \end{pmatrix} - \begin{pmatrix} \frac{a_1 \lambda_0}{\mu+\lambda_0-\alpha_0} \\ \frac{a_1 \lambda_0}{\mu+\lambda_0-\alpha_0} \end{pmatrix} \right],$$

$$\widetilde{b}_1 = \begin{pmatrix} 1 & 1 \\ \widetilde{\gamma}_1 & \widetilde{\gamma}_2 \end{pmatrix}^{-1} \begin{pmatrix} \frac{\mu K}{\mu+\lambda_0} \\ 0 \end{pmatrix},$$

$$\widetilde{b}_2 = \begin{pmatrix} 1 & 1 \\ \widetilde{\gamma}_1 & \widetilde{\gamma}_2 \end{pmatrix}^{-1} \begin{pmatrix} a_0 - \frac{a_1 \lambda_0}{\mu+\lambda_0-\alpha_0} \\ a_0 - \frac{a_1 \lambda_0}{\mu+\lambda_0-\alpha_0} \end{pmatrix},$$

$$\widetilde{l}_i = \frac{1}{l_i}.$$

Moreover,

$$\begin{pmatrix} \widetilde{A}_1 \\ \widetilde{A}_2 \end{pmatrix} = \begin{pmatrix} y_1^{\beta_1} & y_1^{\beta_2} \\ \beta_1 y_1^{\beta_1} & \beta_2 y_1^{\beta_2} \end{pmatrix}^{-1} \begin{pmatrix} a_1 y_1 - K \\ a_1 y_1 \end{pmatrix},$$

$$\begin{pmatrix} \widetilde{B}_1 \\ \widetilde{B}_2 \end{pmatrix} = \begin{pmatrix} l_1 \widetilde{A}_1 \\ l_2 \widetilde{A}_2 \end{pmatrix},$$

$$\begin{pmatrix} \widetilde{C}_1 \\ \widetilde{C}_2 \end{pmatrix} = \begin{pmatrix} y_0^{\widetilde{\gamma}_1} & y_0^{\widetilde{\gamma}_2} \\ \widetilde{\gamma}_1 y_0^{\widetilde{\gamma}_1} & \gamma_2 y_0^{\widetilde{\gamma}_2} \end{pmatrix}^{-1} \begin{pmatrix} a_0 y_0 - K - \widetilde{\phi}(y_0) \\ a_0 y_0 - y_0 \widetilde{\phi}'(y_0) \end{pmatrix}.$$

We obtain the value functions

$$g_0(y) = \begin{cases} \widetilde{B}_1 y^{\beta_1} + \widetilde{B}_2 y^{\beta_2} & \text{if} \quad y \le y_1, \\ \widetilde{C}_1 y^{\gamma_1} + \widetilde{C}_2 y^{\gamma_2} + \widetilde{\phi}(y) & \text{if} \quad y_1 \le y \le y_0, \\ a_0 y - K & \text{if} \quad y \ge y_0, \end{cases}$$
$$g_1(y) = \begin{cases} \widetilde{A}_1 y^{\beta_1} + \widetilde{A}_2 y^{\beta_2} & \text{if} \quad y \le y_1, \\ a_1 y - K & \text{if} \quad y \ge y_1. \end{cases}$$

Case 3. $y_1 = y_0 = y_*$. When $y < y_*$, we have

$$g_0(y) = B_1 y^{\beta_1} + B_2 y^{\beta_2},$$
$$g_1(y) = A_1 y^{\beta_1} + A_2 y^{\beta_2}.$$

When $y > y_*$,

$$g_0(y) = a_0 y - K,$$
$$g_1(y) = a_1 y - K.$$

By using the value matching condition and the smooth pasting condition, we have

$$\begin{cases} B_1 y_*^{\beta_1} + B_2 y_*^{\beta_2} = a_0 y_* - K & \text{if} \quad y \le y_*, \\ \beta_1 B_1 y_*^{\beta_1} + \beta_2 B_2 y_*^{\beta_2} = a_0 y_* & \text{if} \quad y \ge y_*, \end{cases}$$

$$\begin{cases} A_1 y_*^{\beta_1} + A_2 y_*^{\beta_2} = a_1 y_* - K & \text{if} \quad y \le y_*, \\ \beta_1 A_1 y_*^{\beta_1} + \beta_2 A_2 y_*^{\beta_2} = a_1 y_* & \text{if} \quad y \ge y_*. \end{cases}$$

This implies

$$\begin{pmatrix} a_0 y_* - K \\ a_0 y_* \end{pmatrix} = \begin{pmatrix} l_1 & 0 \\ 0 & l_2 \end{pmatrix} \begin{pmatrix} a_1 y_* - K \\ a_1 y_* \end{pmatrix}.$$

Hence

$$y_* = \frac{\beta_1(1-l_1) + \beta_2(l_2-1)}{a_0(\beta_1-\beta_2) + a_1(l_1(\beta_2-1) + l_2(1-\beta_1))} K$$

and

$$\begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = \begin{pmatrix} y_*^{\beta_1} & y_*^{\beta_2} \\ \beta_1 y_*^{\beta_1} & \beta_2 y_*^{\beta_2} \end{pmatrix}^{-1} \begin{pmatrix} a_1 y_* - K \\ a_1 y_* \end{pmatrix},$$

$$\begin{pmatrix} B_1 \\ B_2 \end{pmatrix} = \begin{pmatrix} l_1 A_1 \\ l_2 A_2 \end{pmatrix}.$$

The value functions are given as follows:

$$g_0(y) = \begin{cases} B_1 y^{\beta_1} + B_2 y^{\beta_2} & \text{if} \quad y \le y_*, \\ a_0 y - K & \text{if} \quad y \ge y_*, \end{cases}$$

$$g_1(y) = \begin{cases} A_1 y^{\beta_1} + A_2 y^{\beta_2} & \text{if} \quad y \le y_*, \\ a_1 y - K & \text{if} \quad y \ge y_*. \end{cases}$$

The proof is complete. ■

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