GEOMETRIC AND COMBINATORIAL PROPERTIES OF NETS IN PLANE AND HIGHER-DIMENSIONS

by

Roger Fidèle Ranomenjanahary

APPROVED BY SUPERVISORY COMMITTEE:

Dr.Vladimir Dragović, Chair

Dr. Mieczyslaw K. Dabkowski

Dr. Dmitry Rachinskiy

Dr. Maxim Arnold

To my parents and my siblings.

GEOMETRIC AND COMBINATORIAL PROPERTIES OF NETS IN PLANE AND HIGHER-DIMENSIONS

by

ROGER FIDÈLE RANOMENJANAHARY, MS

DISSERTATION

Presented to the Faculty of

The University of Texas at Dallas

in Partial Fulfillment

of the Requirements

for the Degree of

DOCTOR OF PHILOSOPHY IN

MATHEMATICS

THE UNIVERSITY OF TEXAS AT DALLAS

August 2019

ACKNOWLEDGMENTS

I would like to express my deepest gratitude to my supervisor Dr. Vladimir Dragović for his guidance, generous support and coordinating the whole dissertation. I also extend my thanks to Dr. Milena Radnović, Dr. Fanja Rakotondrajao and Dr. Abdoul Salam Diallo for their valuable help, encouragement and support. My thanks go to colleagues and staff members of the Department of Mathematical Sciences at The University of Texas at Dallas for providing a good working environment. Last but not least, I appreciate my family and my friends for their continuous encouragement and prayers.

July 2019

GEOMETRIC AND COMBINATORIAL PROPERTIES OF NETS IN PLANE AND HIGHER-DIMENSIONS

Roger Fidèle Ranomenjanahary, PhD
The University of Texas at Dallas, 2019

Supervising Professor: Dr.Vladimir Dragović, Chair

This work belongs to a broad area of geometry of discrete integrable systems. The main objects of our study are nets which are configurations of affine subspaces limited to certain geometric and combinatorial constraints. For instance, confocal nets are characterized by their lines which are tangent to a given conic in plane. Similarly, incircular nets (IC nets) are congruences of straight lines in a plane having the property that every quadrilateral admits an inscribed circle. Checkerboard IC nets are a generalization of the IC nets in planar case and are defined by the property that every second elementary quadrilateral is circumscribed. One of the aims of this dissertation is to provide a new method of constructing confocal IC nets and confocal checkerboard IC nets which is based on integrable billiards.

The second aim of this dissertation is to, following Böhm's work, study of divisions of space into circumscribed cuboids. We give a proof of the following statement: a division of 3-dimensional Euclidean space by planes into circumscribed cuboids consists of three families of planes such that all planes in the same family intersect along a line, and the three lines are coplanar. Then we generalize this statement to 4-dimensional case and prove it.

TABLE OF CONTENTS

LIST OF FIGURES

LIST OF TABLES

CHAPTER 1

INTRODUCTION

Discrete differential geometry is a new area of mathematics that interacts with both differential geometry and discrete geometry. The main objective of this field of Mathematics is to develop discrete equivalents of the geometric notions and method of classical differential geometry. Nowadays, mathematics and physics communities are attracted by the study of discrete differential geometry since this area of mathematics has several applications in computer graphics, architecture, and theoretical physics [3]. One example of the fields of discrete differential geometry is the discrete integrable systems. The later provides an interaction with dynamical systems, algebraic geometry and geometry [28].

For instance, in the nineteenth century, Poncelet's theorem is one of most famous and most beautiful result in algebraic geometry. This theorem concerns closed polygons inscribed in one conic and circumscribed about another [9, 14, 30]. Darboux has developed the subject by extending the sides of the Poncelet polygon and has introduced corresponding Poncelet-Darboux grids. In 2005, Richard Evan Schwartz studied further the Poncelet-Darboux grids and proved that the sets $P_k = \cup_{i-j=k} l_i \cap l_j$ lie on ellipses and they are projectively equivalent to each other. Similarly, the sets $Q_k = \cup_{i+j=k} l_i \cap l_j$ lie on hyperbolas and they are also projectively equivalent, where $l_i, i \in \{1, \cdots, n\}$, are the lines containing the sides of the Poncelet polygon [33]. Another proof of Schwartz's theorem was given by Mark Levi and Serge Tabachnikov the same year using the properties of billiards in ellipses [26, 35]. Few years later, Vladimir Dragović and Milena Radnović generalized Darboux theorem in a plane and also gave a higher-dimensional generalization of that theorem [15].

Lines congruences are among the fundamental objects in discrete differential. These lines congruences play an important role in the field of integrable discrete differential geometry, which are attached to the vertices of a $\mathbb{Z}^2$ lattice. The congruences of straight lines in a plane with the combinatorics of square grid with an additional property, every elementary quadrilateral possesses an inscribed circle define an incircular net, were introduced and studied by Böhm in the framework of Euclidean geometry and spherical geometry [39]. In 2017, Arseniy V. Akopyan and Alexander I. Bobenko investigated further the congruences of straight lines in the plane with the combinatorics of square grid and introduced nets of lines called incircular nets (IC nets), whose quadrilaterals admit inscribed circles, and derived several geometric and combinatorial properties. In addition, they also gave a generalization of the notion of the incircular nets in the planar case, called checkerboard incircular nets (checkerboard IC nets). The checkerboard IC net can be defined as congruences of straight lines in the plane with the combinatorics of square grid such that every second quadrilateral admits an inscribed circle [1]. A subclass of the checkerboard IC nets is confocal checkerboard IC nets [4]. Those kind of checkerboard IC nets are characterized by their lines, meaning that all lines are tangent to a given conic.

A geometric property of an IC net is the 3×3 incircles incidence theorem. An elementary proof of this theorem is given in Chapter 3 that is based on Lemma 27. The main result in Chapter 4 is to provide a new method of constructing confocal IC nets and cofoncal checkerboard IC nets based on integrable billiards. Double reflection nets are discrete systems arising from the dynamics of billiards within confocal quadrics. Our main new result on such systems is a new geometric configuration of twelve focal nets and eight lines. These focal nets and these lines form a cuboid in a dual projective space (Theorem 47). In Chapter 5, we give a proof of Böhm' s theorem in 3-dimensional space and generalize that theorem in 4-dimensional case and prove it.

A quad-graph is a cellular decomposition of an oriented surface whose cells are quadrilaterals. A part of this dissertation describes the connection of the work of V. E. Adler, A. I. Bobenko and Y. B. Suris on the theory of integrable systems of quad-graphs [3] and some results obtained from the billiard algebra [15].

This dissertation is made up of five chapters. The main goal of Chapter 2 is to present basic definitions and elementary properties of conics in the Euclidean plane $\mathbb{E}^2$ and in the projective plane $\mathbb{P}^2$. One remarkable property of confocal conics that we will be presented in this chapter is the Graves-Chasles theorem. This theorem is the key tool of the construction of a confocal incircular net and a confocal checkerboard IC net presented in Chapter 3 and also plays a crucial role in the proof of our main results in Chapter 4. In the second part of this chapter, we will define what is called a quadric and recall its properties in the Euclidean space $\mathbb{E}^3$ and in the projective space $\mathbb{P}^3$ such as Jacobi's theorem, Chasles' theorem, one reflection theorem, double reflection theorem.

We begin Chapter 3 by a review of Böhm's work on the circumscribed quadrilaterals nets. Further study of the circumscribed quadrilaterals nets leads us to the subject of incircular nets in the plane. This chapter also includes the generalization of the IC nets in the planar case. Our presentation on the extended analysis of the IC nets and the checkerboard IC nets follows the exposition found in [1]. However, most of the proofs of the geometric and combinatorial properties of those nets are slightly different and more detailed than those presented in [1].

Chapter 4 consists of the connection of mathematical billiards and confocal IC nets and confocal checkerboard IC nets. The billiard system describes the motion of a free point inside domain such that the point moves with a constant speed along a straight line until it hits the boundary, and at the bouncing point, an incoming segment and an outgoing segment satisfy the geometric law of optics [8, 9, 24, 33, 38]. We suppose that the domain and the caustic of the billiards are smooth and convex. The generalization of the Darboux theorems in [15] leads us to the new method of a construction of the confocal IC net and of the confocal checkerbord IC net [18]. This construction starts from two different billiard trajectories within the same conic and sharing the same caustic. These trajectories are not periodic and they are either winding in the same direction or in the opposite directions.

Chapter 5 is split into two parts. The first part focuses on basic ideas of the theory of quad-graph equations and the study of some classes of discrete configurations that arise from the dynamics of billiards within confocal quadrics. An example of such discrete configurations is double reflection nets. Double reflection nets are defined on cubic lattice, that is, maps that assigns a line to each vertex of the lattice $\mathbb{Z}^3$ satifying that neighboring lines intersect [15, 17, 18]. In other words, the double reflection nets are discrete line congruences. Configurations which are defined on non-cubic lattice consisting of cuboctahedra and octahedra will be discussed. The latter configurations assign hyperplanes to the vertices of the lattice of a given projective space [28]. The second part is motivated by Wolfgang Böhm's work in 1965 concerning division of a space by planes into circumscribed cuboids. The division of an 3-dimensional Euclidean space by planes into circumscribed cuboids necessarily consists of three families of planes such that all planes in the same family intersect along a line and the three lines are coplanar [39]. We also generalize that result in 4-dimensional Euclidean space. Finally, we present and study a natural three-dimensional version of the checkerboard IC nets, called checkerboard inspherical nets (checkerboard IS nets) [1].

The division of a Euclidean space into circumscribed cuboids that we present in this dissertation focuses only on the case where the dimension of space is 3 and 4. It would be interesting to prove the natural generalization of a division of the n-dimensional space by hyperplanes into circumscribed n-cuboid and generalize the following statement in an n-dimensional space: each cuboid, which can be divided by three planes from different families into eight circumscribed cuboids, is itself circumscribed.

CHAPTER 2

ELEMENTARY PROPERTIES OF CONICS AND QUADRICS

2.1 Elementary properties of conics

As we know, there are several definitions of conics depending on the framework of geometry. For instance, as planar section of cones of revolution, conics can be defined by their focal properties or by their quadratic equation. From the point of view of projective geometry, conics can also be defined as the set of intersection points of projective pencils. So, this section contains definitions and some basic properties of conics in affine and projective planes. All results presented can be found in different books on geometry of conics and in several articles, see for instance [2, 11, 12, 19, 23, 24, 25] and [27].

2.1.1 Classification of conics in the Euclidean plane

Definition 1. *A curve of second degree is a set of points whose coordinates in Cartesian coordinate system satisfy a second order equation*

$$a_{11}x_1^2 + 2a_{12}x_1x_2 + a_{22}x_2^2 + 2b_1x_1 + 2b_2x_2 + c = 0. \tag{2.1}$$

An ellipse, a parabola and a hyperbola are examples of curves of degree two. These conics can be defined based on the focal properties.

Definition 2.

- *An ellipse is the set of points P for which the sum of its distances from the two fixed points called foci is constant, that is,*

$$|PF_1| + |PF_2| = 2a_1 \quad \textit{with} \quad 2a_1 > |F_1F_2| \geq 0.$$

An equation of the ellipse can be written as follows:

$$\frac{x_1^2}{a_1^2} + \frac{x_2^2}{a_2^2} = 1, \quad a_1 \geq a_2 > 0.$$

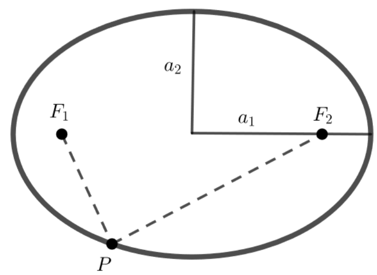


Figure 2.1: An ellipse.

The ellipse in Figure (2.1) *is centered at the origin and has the foci at* $(\pm\sqrt{a_1^2 - a_2^2}, 0)$, *where* a_1 *and* a_2 *are the lengths of major and minor axes, respectively.*

- *A hyperbola is the set of points* Q *for which the absolute value of the difference between the distances to two fixed points, called the foci, is constant, that is,*

$$||QF_1| - |QF_2|| = 2a_1 > 0.$$

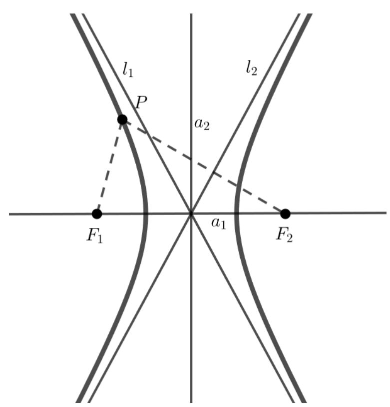


Figure 2.2: A hyperbola, l_1 and l_2 are called asymptotes.

An equation of the hyperbola is expressed as follows:

$$\frac{x_1^2}{a_1^2} - \frac{x_2^2}{a_2^2} = 1, \quad a_1 > 0, \quad a_2 > 0.$$

The hyperbola in Figure 2.2 has the foci at $(\pm\sqrt{a_1^2 + a_2^2}, 0)$. *Moreover, the hyperbola and its real axis intersect in two different points and the distance of these points is equal to* $2a_1$. *The quantities* a_1 *and* a_2 *are called respectively real semi-axis and imaginary semi-axis.*

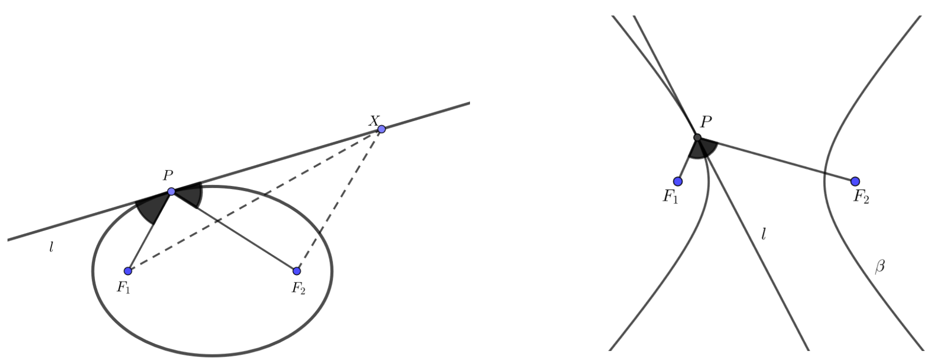
- *A parabola is the set of points* R *whose distances to some fixed point and a line are constant. The point is called focus and the line is called directrix of the parabola. Note that if the quantities* a_1 *and* a_2 *in the equation of hyperbola above are equal, then we have a parabola and its equation can be written as*

$$x_2^2 = 2px_1, \quad p > 0.$$

The axis of the parabola is the same as the x_1*-axis. The parabola has focus at the point* $(\frac{p}{2}, 0)$ *and the directrix is* $x_1 = -\frac{p}{2}$.

2.1.2 Optical and isogonal property of conics

Theorem 1. *Let* C *be a conic and* l *be a tangent line to* C *at a point* P. *Then* l *is the bisector of the angle* $\angle F_1PF_2$, *where* F_1 *and* F_2 *are foci of the conic* C.



(a) Optical property of an ellipse α.

(b) Optical property of a hyperbola β.

Figure 2.3: Optical property of a conic.

Proof. We will prove the theorem only for the case of an ellipse. Similar proof can be done for the other type of conics [2]. Let C be an ellipse. In the Figure (2.3a), α plays the role of C and let $\mathsf{X} \neq \mathsf{P}$ be a point that lies on the line $\mathfrak{l}$. It is clear that the distance $|\mathsf{XF}_1| + |\mathsf{XF}_2|$ is greater than the distance $|\mathsf{PF}_1| + |\mathsf{PF}_2|$. In other words, for all points on the line $\mathfrak{l}$, only the point P has the smallest sum of the distances to the foci F_1 and F_2. This means that the angles formed by the lines (PF_1) and (PF_2) with the tangent line $\mathfrak{l}$ at the point P are equal. ■

Theorem 2. *Let* C *be an ellipse. Let* P *be a point lying outside of* C *and draw the two tangent lines from the point* P *to* C*, and denote by* X *and* Y *their tangency points. Then,* $\angle \mathsf{F}_1\mathsf{PX} = \angle \mathsf{F}_2\mathsf{PY}$*, where* F_1 *and* F_2 *are foci of* C*.*

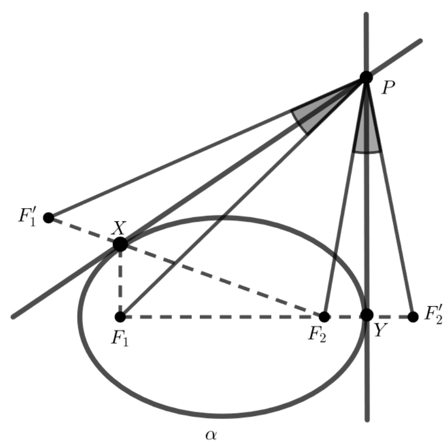


Figure 2.4: Isogonal property of an ellipse.

Proof. In Figure 2.4, let F_1' and F_2' be the reflections of F_1 and F_2 with respect to the respective lines (PX) and (PY). Then,

$$|\mathsf{F}_1'\mathsf{P}| = |\mathsf{F}_1\mathsf{P}| \quad \text{and} \quad |\mathsf{F}_2'\mathsf{P}| = |\mathsf{F}_2\mathsf{P}|. \tag{2.2}$$

According to the Theorem 1, the points F_1, Y, and F_2' are collinear. By the same argument, we can conclude that the points F_2, F_1' and X are also collinear. Then, we obtain

$$|F_2F_1'| = |F_2X| + |XF_1| = |F_2Y| + |YF_1| = |F_1F_2'|. \tag{2.3}$$

The combination of the equation (2.2) and the equation (2.3) proves that the two triangles PF_2F_1' and PF_1F_2' are equal. That implies that

$$\angle F_1PF_2 + 2\angle F_1PX = \angle F_1'PF_2 = \angle F_1PF_2' = \angle F_1PF_2 + 2\angle F_2PY.$$

Therefore, $\angle F_1PX = \angle F_2PY$. ■

The property in the Theorem 2 holds for the hyperbola, see for example [2].

Corollary 3. *Using the same assumptions and terminology of the Theorem 2, the line* (F_1P) *is the bisector of the angle* $\angle XF_1Y$*.*

2.1.3 Properties of conics in the projective plane

Definition 3. *A projective conic in* $\mathbb{RP}^2$ *is a set of points whose homegenuous coordinates satisfy the quadratic equation*

$$C(x) = a_{11}x_1^2 + 2a_{12}x_1x_2 + 2a_{13}x_1x_3 + a_{22}x_2^2 + 2a_{23}x_2x_3 + a_{33}x_3^2 = 0. \tag{2.4}$$

We can express $C(x)$ *in matrix form as follows*

$$C(x) = x'Ax = 0, \tag{2.5}$$

where $x = (x_1, x_2, x_3)$, x' *is the row vector and* A *is a* 3×3 *nonzero symmetric matrix.*

Definition 4. *If the matrix* A *in the Definition 3 is non-singular, then the conic defined by* $C(x) = 0$ *is called a non-degenerate. Otherwise, we call it a degenerate conic.*

Proposition 4. *Let* p *be a projective transformation and* C *be a non-degenerate conic. Then, the image of the conic* C *under the projective transformation* p *is a non-degenerate conic. If* C *is degenerate conic, then* $\mathsf{p}(\mathsf{C})$ *is also a degenerate conic.*

Proof. If the conic C is degenerate, then C can be a pair of lines or a single line or a point. Any projective transformation can be written as $y = Px$, where P is a 3×3 matrix associated with p. The set of the projective transformations forms a group under the operation of composition of functions, then the map p has an inverse p^{-1}. The map p^{-1} is also a projective transformation. Therefore, p^{-1} maps points to points and lines to lines. It follows that the image of the degenerate conic C under a projective transformation is again a degenerate conic.

Now, suppose that the conic C is non-degenerate. By Definition 3, C has the equation of the form $x'Ax = 0$. The map p^{-1} is given by $x = P^{-1}y$. Therefore, we obtain $y'(P^{-1})'AP^{-1}y = 0$. Define $B := (P^{-1})'AP^{-1}$. Notice that the matrix B is a 3×3 symmetric matrix and $\det(B) = \dfrac{\det(A)}{(\det(P))^2}$. Thus, $y'By = 0$, which means that the image of C under p is a conic. It remains to verify that this image cannot be a degenerate conic. The projective map p^{-1} must map the image $p(C)$ back to the original non-degenerate conic C, then the image of C under p must be a non-degenerate conic. ■

Definition 5. *Let* C *be a non-degenerate conic. A line* l_y *is said to be a tangent line to the conic* C *if the line* l_y *meets* C *exactly at only one point, and at no other point. The set of the tangent lines to* C *is called the dual to* C *and is denoted by* C^**.*

Theorem 5. *Let* C *be a non-degenerate conic with equation* $C(x) = x'Ax = 0$ *and* C^* *be the dual of* C*. Then,* $y'A^{-1}y = 0$ *is the equation of* C^**. Moreover, the conic* C^* *is also a non-degenerate in dual projective space* $\mathbb{RP}^{2*}$*.*

Proof. Since the conic C is non-degenerate, it consists of simple points. It follows that the equation of tangent to C at a point x is $y = Ax$. Then, we have $x = A^{-1}y$. By substituting the latest expression into the equation of C to get the equation for C^*, we have $y'(A^{-1})'AA^{-1}y = y'(A^{-1})'y = y'(A')^{-1}y = y'A^{-1}y = 0$ since the matrix A is symmetric, that is, $A' = A$. ■

Theorem 6. *Given five generic points such that no three of them are collinear. Then, a conic C is defined by these points.*

Proof. See the proof of the Theorem 6.1.2 in [24]. ■

2.1.4 Pascal, Pappus and Brianchon theorem

A famous mathematician Blaise Pascal discovered the so called Pascal's theorem in 1640. This theorem is an undeniable generalization of the Pappus's theorem because if the conic in the Pascal's theorem is a degenerate conic consisting of a pair of lines, then we immediately obtain the Pappus's theorem. Theorem 7 is also a simple criterion for six points to lie on a conic.

Theorem 7 (Pascal's theorem)**.** *Let X_1, X_2, X_3, X_4, X_5 and X_6 be six points such that no three of them are collinear. The six points lie on a conic if and only if the three points*

$$P = (X_1X_2) \cap (X_4X_5), \quad Q = (X_2X_3) \cap (X_5X_6), \quad R = (X_3X_4) \cap (X_1X_6)$$

are collinear.

Proof. Suppose that the six points X_1, X_2, X_3, X_4, X_5 and X_6 lie on a conic C (see Figure 2.5). We want to show that the points P, Q and R are collinear. Let $A = (X_1X_6) \cap (X_2X_3)$ and $B = (X_1X_2) \cap (X_5X_6)$. We have the following equalities between cross ratio

$$\begin{aligned} cr(R, A, X_1, X_6) &= cr(X_3R, X_3A, X_3X_1, X_3X_6) \\ &= cr(X_3X_4, X_3X_2, X_3X_1, X_3X_6) \\ &= cr(X_5X_4, X_5X_2, X_5X_1, X_5X_6) \\ &= cr(P, X_2, X_1, B). \end{aligned}$$

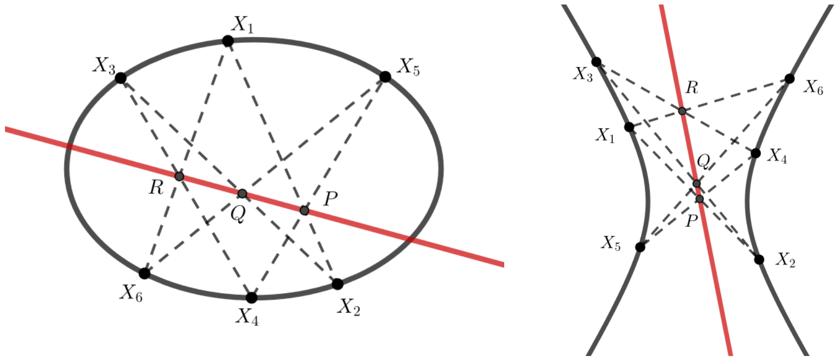


Figure 2.5: Pascal's theorem.

The point Q is the intersection of the lines (RP) and (X_2X_3). Since the perspectivity of center at Q maps R to P, A to X_2 and X_1 to X_1, and preserves the cross ratio, it must map X_6 to B. If follows that the points X_6, Q and B are collinear. Since Q belongs to the lines (X_2X_3) and (RP), it must lie on the line (X_5X_6).

Suppose that the points P, Q and R are collinear. We need to prove that the six points X_1, X_2, X_3, X_4, X_5 and X_6 lie on a conic. Since the five points X_1, X_2, X_3, X_4 and X_5 are in general position, by Theorem 6 there is a conic passing through these points, say C. Note that the points P, Q, and R are the intersections of the pairs of lines (X_1X_2) and (X_4X_5), (X_2X_3) and (X_5X_6), and (X_3X_4) and (X_5X_6) respectively. Let Y be the intersection point of C and (QX_5) different from X_5. By assumption, the intersection of (X_3X_4) and (X_1Y) lies on the line (PQ), which means that their point of intersection coincides with the point R. It follows that the points Y and X_6 must coincide. ■

The dual of the Pascal's theorem is called Brianchon's theorem and is depicted in Figure 2.6, and it was discovered by Chasles Julien Brianchon in 1804.

Theorem 8 (Brianchon's theorem). *Given any six lines* l_1, l_2, l_3, l_4, l_5 *and* l_6 *and let* $X_{12} := l_1 \cap l_2$, $X_{23} := l_2 \cap l_3$, $X_{34} := l_3 \cap l_4$, $X_{45} := l_4 \cap l_5$, $X_{56} := l_5 \cap l_6$ *and* $X_{16} := l_1 \cap l_6$. *The lines*

$(X_{12}X_{45})$, $(X_{23}X_{56})$ and $(X_{34}X_{16})$ intersect at a point if and only if there is a conic tangent to all the lines l_1, l_2, l_3, l_4, l_5 and l_6.

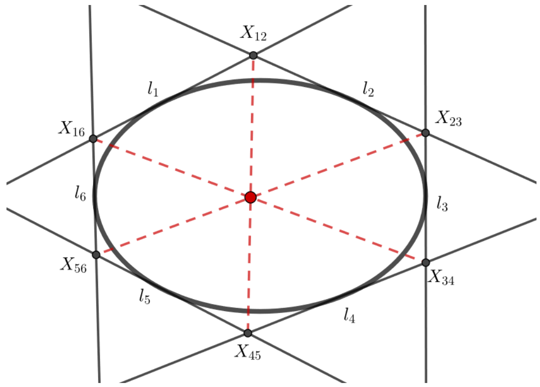


Figure 2.6: Brianchon's theorem.

Proof. The proof of this theorem is similar to the proof of the Theorem 7. ■

2.1.5 Confocal conics, Graves-Chasles theorem and Ivory theorem

Definition 6. *A family of conics are called confocal if all conics in the family have the same foci. An equation of a confocal family can be written as follows*

$$\frac{x_1^2}{a_1^2 + \lambda} + \frac{x_2^2}{a_2^2 + \lambda} = 1, \tag{2.6}$$

where λ is a real parameter.

The proof of the Graves-Chasles Theorem 9 is based on the theorems in the section 2.1.2.

Theorem 9 (Graves-Chasles theorem)**.** *Let $(abcd)$ be a complete quadrilateral such that lines contain its sides are tangent to a conic α. Then the four following conditions are equivalent:*

1. *The quadrilateral $(abcd)$ is circumscribed.*

2. *Points a and c lie on a conic confocal with α.*

3. *Points* b *and* d *lie on a conic confocal with* α.

4. *Points* e *and* f *lie on a conic confocal with* α.

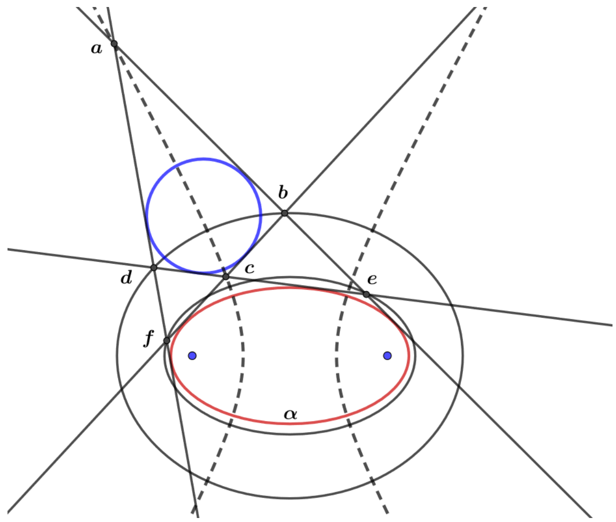


Figure 2.7: Graves-Chasles theorem.

Proof. This proof follows the exposition from [1]. Let γ, β and δ be conics passing through the pairs of points a and c, b and d, and e and f, respectively (see Figure 2.7). Since these conics are confocal with α, they form a dual pencil. By assumption, the lines of all sides of the quadrilateral (abcd) are tangent to the conic α. We then obtain two coordinate systems which are formed by the dual pencil of conics and the family of tangent lines to α. One proves that these two coordinate systems are diagonal-connected. This proves the equivalence of the conditions (2), (3) and (4).

Suppose that the quadrilateral (abcd) satisfies the three last conditions. According to second part of the Lemma 2.4 in [1], the lines (ad), (bc), (ab) and (cd) tangent to the conic α intersect at one point. We also know that the bisectors of the angles formed by the tangent lines from a point, say p, to the conic α coincide with the bisectors of the angles $\angle \mathsf{f}_1\mathsf{p}\mathsf{f}_2$, where f_1 and f_2 are foci of conic α. Let l_p be the bisector of the angle $\angle \mathsf{f}_1\mathsf{p}\mathsf{f}_2$.

The point $\mathfrak{p}$ is an arbitrary point, so we can choose $\mathfrak{p} = \mathfrak{a}$. Now, we will prove that the bisector $\mathfrak{l}_\mathfrak{a}$ is tangent to a conic γ confocal with α at the point $\mathfrak{a}$. The Theorem 1 says that the angles formed by lines $(\mathfrak{a}\mathfrak{f}_1)$ and $(\mathfrak{a}\mathfrak{f}_2)$ with a tangent line to γ at $\mathfrak{a}$, say $\mathfrak{l}$, are equal. Moreover, the lines $(\mathfrak{a}\mathfrak{f}_1)$ and $(\mathfrak{a}\mathfrak{f}_2)$ form equal angles with the bisector $\mathfrak{l}_\mathfrak{a}$, which implies that the lines $\mathfrak{l}$ and $\mathfrak{l}_\mathfrak{a}$ are orthogonal. Thus, $\mathfrak{l}_\mathfrak{a}$ must be tangent to γ. By repeating the foregoing arguments, we can show that the bisectors passing through the other vertices are tangent to some conics confocal with α. Thus, the point of intersection of these bisectors is the center of the quadrilateral $(\mathfrak{a}\mathfrak{b}\mathfrak{c}\mathfrak{d})$.

Assume that the quadrilateral $(\mathfrak{a}\mathfrak{b}\mathfrak{c}\mathfrak{d})$ is circumscribed. We need to show that there is a conic γ confocal with α such that the points $\mathfrak{a}$ and $\mathfrak{c}$ lie on γ. Let us choose two points $\mathfrak{c}'$ and $\mathfrak{d}'$ such that these points lie on the lines $(\mathfrak{b}\mathfrak{c})$ and $(\mathfrak{a}\mathfrak{d})$ respectively, the points $\mathfrak{a}$ and $\mathfrak{c}'$ lie on a conic confocal with α, and the line $(\mathfrak{c}'\mathfrak{d}')$ is tangent to α. Then, the quadrilateral $(\mathfrak{a}\mathfrak{b}\mathfrak{c}'\mathfrak{d}')$ is also circumscribed. Since the inscribed circles of the quadrilaterals $(\mathfrak{a}\mathfrak{b}\mathfrak{c}\mathfrak{d})$ and $(\mathfrak{a}\mathfrak{b}\mathfrak{c}'\mathfrak{d}')$ are uniquely determined by the three lines $(\mathfrak{a}\mathfrak{b})$, $(\mathfrak{a}\mathfrak{d})$ and $(\mathfrak{b}\mathfrak{c})$, these inscribed circles coincide. Notice that the conic α and the inscribed circle have only four tangent lines in common. Since they have already three common tangent lines $(\mathfrak{a}\mathfrak{b})$, $(\mathfrak{a}\mathfrak{d})$ and $(\mathfrak{b}\mathfrak{c})$, this implies that the tangent lines $(\mathfrak{c}\mathfrak{d})$ and $(\mathfrak{c}'\mathfrak{d}')$ must coincide. ■

The Corollary 10 follows immediately from the second part of the proof the Graves-Chasles theorem.

Corollary 10. *Suppose that the three sides* $(\mathfrak{a}\mathfrak{b}), (\mathfrak{b}\mathfrak{c})$ *and* $(\mathfrak{c}\mathfrak{d})$ *of a circumscribed quadrilateral* $(\mathfrak{a}\mathfrak{b}\mathfrak{c}\mathfrak{d})$ *are tangent to a conic* α *and the two opposite vertices* $\mathfrak{a}$ *and* $\mathfrak{c}$ *belong to the same conic confocal with* α*. Then, the forth side* $(\mathfrak{a}\mathfrak{d})$ *of the quadrilateral* $(\mathfrak{a}\mathfrak{b}\mathfrak{c}\mathfrak{d})$ *also touches the conic* α*.*

Confocal conics constitute an orthogonal net. The net of confocal conics in the Euclidean plane satisfies the Ivory theorem.

Theorem 11 (Ivory's theorem). *Let β_1, β_2, γ_1 and γ_2 be confocal conics. Suppose that β_1, β_2 are ellipses, and γ_1, γ_2 are hyperbolas. Denote by a, b, c and d the intersection points of these conics. Then, $|ac| = |bd|$.*

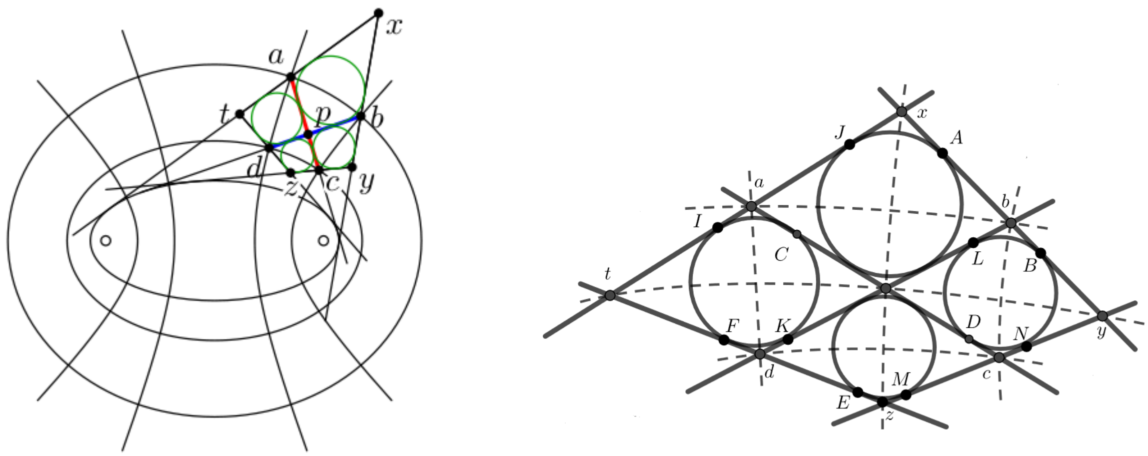


Figure 2.8: Ivory's theorem.

Proof. Note that the ellipses β_1, β_2 and the hyperbolas γ_1, γ_2 are from the confocal family, and a, b, c and d are their intersection points. Then, one proves that there exists a conic α from the confocal family such that the lines (ac) and (bd) are tangent to it. Let us draw four tangent lines to α such that each tangent passes through one of the points a, b, c and d, and denote by x, y, z and t the points of intersection of these tangent lines. We thus obtain a quadrilateral $(xyzt)$. As we see in Figure 2.8, the four small quadrilaterals $(apbx)$, $(apdt)$, $(pdzc)$ and $(bpyc)$ form the big quadrilateral $(xyzt)$. Notice also that all sides of all quadrilaterals are tangent to α and two opposite vertices of each small quadrilateral lie on the same conic confocal with α. According to the Grave-Chasles Theorem 9, these quadrilaterals are circumscribed. Indeed, we obtain the following equations

$$|ap| + |dt| = |at| + |dp|, \tag{2.7a}$$

$$|bp| + |cy| = |cp| + |by|, \tag{2.7b}$$

$$|ax| + |bp| = |ap| + |bx|, \tag{2.7c}$$

$$|dp| + |cz| = |dz| + |cp|. \tag{2.7d}$$

By adding the equations (2.7a), (2.7b), (2.7c) and (2.7d) and by using the facts that $|tF| = |tI|$, $|xA| = |xJ|$, $|zE| = |zM|$, $|yN| = |yB|$, we have the following equality

$$2|ac| + |EF| + |AB| = 2|bd| + |IJ| + |MN|. \tag{2.8}$$

Since $|EF| = |CD| = |AB|$, $|IJ| = |KL| = |MN|$, the equation (2.8) can be expressed as follows

$$2|ac| + 2|AB| = 2|bd| + 2|IJ|$$

$$2|ac| + 2|CD| = 2|bd| + 2|KL|$$

$$2|ac| + 2(|pC| + |pD|) = 2|bd| + 2(|pK| + |pL|).$$

Finally, since $|pC| = |pK|$ and $|pL| = |pD|$, we obtain $|ac| = |bd|$. ■

A different formulation of the Ivory's theorem employs the fact that any two confocal conics C_λ and C_μ of the same types define an affine transformation. This affine transformation is given by the relation (2.11). Then any conic confocal with C_λ, whose type is different than C_λ, intersects orthogonally the conics C_λ and C_μ at the corresponding points $P \in C_\lambda$ and $Q = A_{\lambda,\mu}(P) \in C_\mu$. Therefore, Ivory's theorem can be stated as follows:

$$|P_1Q_2| = |Q_1P_1| \quad \text{for all } P_1, P_2 \in C_\lambda \text{ and } Q_i = A_{\lambda,\mu}(P_i), \text{ i=1, 2.}$$

2.1.6 Darboux coordinates generated by a conic and Elliptic coordinates

We will start with the notion of Darboux coordinates generated by a conic. Let α be a given ellipse. This is a rational curve, so we fix one of its rational parameterizations and denote by x the parameter in this parameterization. Consider a point X outside of α (see Figure 2.9). The Darboux coordinates x_1 and x_2 of the point X are the values of the parameter x at the touching points A and B of the two tangent lines from the point X to the ellipse α [22, 26].

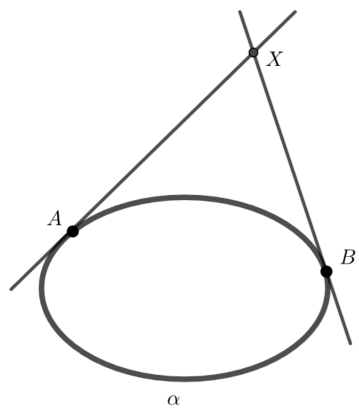


Figure 2.9: Darboux coordinates.

It follows that the equations of confocal ellipses and the confocal hyperbolas are given respectively by the following equations

$$\begin{aligned} x_2 - x_1 &= \text{constant}, \\ x_1 + x_2 &= \text{constant}. \end{aligned} \tag{2.9}$$

We will use the Darboux coordinates to give a simple proof of the 3×3 incircles incidence theorem of the IC net in Chapter 3.

Definition 7. *Elliptic coordinates (or Jacobi coordinates) of a point* $x = (x_1, x_2)$ *in a plane are the values* (λ_1, λ_2) *that satisfy the equation* (2.6).

Now, we will express the Cartesian coordinates of the point $x = (x_1, x_2)$ in term of the elliptic coordinates. From the equation (2.6), the values of λ for the confocal conics through the point $x = (x_1, x_2)$ are given by the quadratic equation

$$\psi(\lambda) := (a_1^2 + \lambda)(a_2^2 + \lambda) - (a_2^2 + \lambda)x_1^2 - (a_1^2 + \lambda)x_2^2 = 0.$$

Let us denote by λ_1 and λ_2 the roots of the equation $\psi(\lambda) = 0$. It is clear from the expression of $\psi(\lambda)$ that the leading coefficient is equal to 1, we thus rewrite the above equation as follows

$$(\lambda - \lambda_1)(\lambda - \lambda_2) = (a_1^2 + \lambda)(a_2^2 + \lambda) - (a_2^2 + \lambda)x_1^2 - (a_1^2 + \lambda)x_2^2.$$

If we substitute the value $-a_1^2$ and $-a_2^2$ in succession in this identity, we obtain

$$\begin{aligned} x_1^2 &= \frac{(a_1^2 + \lambda_1)(a_1^2 + \lambda_2)}{a_1^2 - a_2^2}, \\ x_1^2 &= \frac{(a_2^2 + \lambda_1)(a_2^2 + \lambda_2)}{a_2^2 - a_1^2}. \end{aligned} \tag{2.10}$$

The equation (2.10) shows us the relationship between the Cartesian coordinates of the point (x_1, x_2) and the elliptic coordinates (λ_1, λ_2) [26]. As is known, there exists a linear map which maps a conic to another conic of the same type. Let C_λ, C_μ be two confocal conics of the same type. Then the linear map that maps C_λ to C_μ is given by the diagonal matrix $A_{\lambda,\mu}$ defined by the equation (2.11).

$$A_{\lambda,\mu} = \mathrm{Diag}\left(\sqrt{\frac{a_1^2 + \mu}{a_1^2 + \lambda}}, \sqrt{\frac{a_2^2 + \mu}{a_2^2 + \lambda}}\right). \tag{2.11}$$

Theorem 12 ([26], Lemma 5.1). *Let C_λ and C_μ be two conics of the same type. Suppose that a point P lies on the conic C_λ. Then, the points P and $Q := A_{\lambda,\mu}(P)$ lie on the same conic confocal with C_λ, whose type is different than C_λ and C_μ.*

Proof. We will focus on the case where C_λ and C_μ are ellipses (see Figure 2.10). For the other case, the proof is similar. Let P be a point on C_λ whose elliptic coordinates are (λ_1, λ_2)

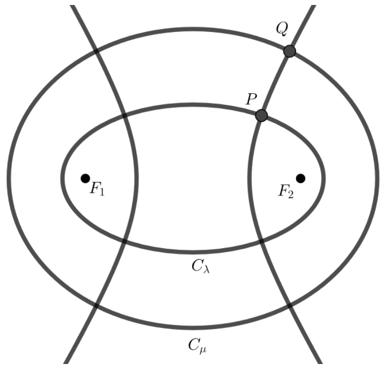


Figure 2.10: Affine transformation between confocal conics.

and Q be a point on C_μ having the elliptic coordinates (μ_1, μ_2). Then $\lambda_2 = \lambda$ and $\mu_2 = \mu$. It remains to verify that $\lambda_1 = \mu_1$. To prove this latest equality, let us denote by (x_1, x_2) and (y_1, y_2) the Cartesian coordinates of the points P and Q, respectively. According to the equations (2.10), we have the following relations

$$\begin{aligned} y_1^2 &= \frac{(a_1^2+\mu_1)(a_1^2+\mu_2)}{a_1^2-a_2^2}, \\ y_2^2 &= \frac{(a_2^2+\mu_1)(a_2^2+\mu_2)}{a_2^2-a_1^2}. \end{aligned} \tag{2.12}$$

Since $Q = A_{\lambda,\mu}(P)$, and using the equation (2.11), we obtain

$$\begin{aligned} y_1^2 &= \frac{a_1^2+\mu}{a_1^2+\lambda}x_1^2 = \frac{(a_1^2+\mu)}{a_1^2+\lambda}\frac{(a_1^2+\lambda_1)(a_1^2+\lambda)}{a_1^2-a_2^2} = \frac{(a_1^2+\mu_2)(a_1^2+\lambda_1)}{a_1^2-a_2^2} \quad \text{since } \lambda_2 = \lambda \text{ and } \mu = \mu_2, \\ y_2^2 &= \frac{a_2^2+\mu}{a_2^2+\lambda}x_2^2 = \frac{(a_2^2+\mu)}{a_2^2+\lambda}\frac{(a_2^2+\lambda_1)(a_2^2+\lambda)}{a_2^2-a_1^2} = \frac{(a_2^2+\mu_2)(a_2^2+\lambda_1)}{a_2^2-a_1^2}. \end{aligned} \tag{2.13}$$

By comparing the equations (2.12) and (2.13), we obtain $\lambda_1 = \mu_1$. ■

Proposition 13 ([1], Lemma 2.11). *Given two confocal conics α and α', let denote by α" the conic dual of α with respect to α'. Then there is an affine transformation which maps α' to α and α" to α.*

Proof. We suppose that the two confocal conics α and α' are ellipses and their equations are the following (we prove the other cases in the same way)

$$\alpha: \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \qquad\qquad \alpha': \quad \frac{x^2}{c^2} + \frac{y^2}{d^2} = 1.$$

Let (x_0, y_0) be a point on α'. Then the equation of a tangent line passing through the point (x_0, y_0) is given by

$$\frac{x_0}{c^2}x + \frac{y_0}{d^2}y = 1.$$

This tangent line intersects the conic α" in two different points and those points are dual to the lines tangent to α with respect to α'. Therefore, we obtain

$$\frac{x_0}{c^2}x + \frac{y_0}{d^2}y = \frac{x^2}{a^2} + \frac{y^2}{b^2} \qquad \Leftrightarrow \qquad \frac{x}{a^2}\left(x - \frac{a^2x_0}{c^2}\right) + \frac{y}{b^2}\left(y - \frac{b^2y_0}{d^2}\right) = 0.$$

The conic α is non-degenerate, then $\dfrac{x}{a^2} \neq 0$ and $\dfrac{y}{b^2} \neq 0$. Therefore, one solution of the equation above is

$$x = \frac{a^2x_0}{c^2}, \qquad y = \frac{b^2x_0}{d^2}.$$

Indeed, the equation of the conic α" can be expressed as follows

$$\frac{\left(\frac{a^2x_0}{c^2}\right)^2}{a^2} + \frac{\left(\frac{b^2x_0}{d^2}\right)^2}{b^2} = 1 \qquad \Leftrightarrow \qquad \frac{a^2x_0^2}{c^4} + \frac{b^2y_0^2}{d^4} = 1.$$

According to the equation (2.11), the affine transformation T which maps α' to α and α" to α is defined by

$$T :=_{\alpha',\alpha} \mathtt{Diag}\left(\sqrt{\frac{a^2}{c^2}}, \sqrt{\frac{b^2}{d^2}}\right) =_{\alpha",\alpha'} \mathtt{Diag}\left(\sqrt{\frac{a^2c^2}{c^4}}, \sqrt{\frac{b^2d^2}{d^4}}\right) = \mathtt{Diag}\left(\frac{a}{c}, \frac{b}{d}\right).$$

$\blacksquare$

2.2 Elementary properties of quadrics

Some fundamental notions and facts on conics have been mentionned in the previous section. Quadrics are generalization of conics. This present section deals with some elementary properties of quadrics. Confocal families of quadrics, Jacobi's theorem, Chasles's theorem, and reflection laws such as one reflection theorem, double reflection theorem and virtual reflection are the main topics that we will present here. All results can be found in [5, 11, 16, 19, 25] and [38]. Let $\mathtt{A}$ be an affine or projective space over a field k of characteristic unequal to 2.

Definition 8. *A quadric in* $\mathtt{A}$ *is the set of points of* $\mathtt{A}$ *such that their coordinates are the zeros of a quadratic form* $F(x) = 0$. *If the space* $\mathtt{A}$ *is projective, then* F *is homogeneous and the quadric is projective quadric. Likewise, if* $\mathtt{A}$ *is affine space, then we obtain an affine quadric.*

2.2.1 Pole and polar plane

Let V be a vector space over the field k. As we mentioned in the Definition 8, a quadratic form F defines a quadric Q in the projective space $\mathtt{A} = \mathbb{P}(V) := \mathbb{P}^3$. There exists a bilinear form $\mathcal{B}$ associated with the quadratic form F. We thus have the following relation

$$\mathcal{B}(x, x) = F(x).$$

If the quadric consists of simple points, then the bilinear form $\mathcal{B}$ is non-degenerate, that is, $\det(B) \neq 0$. The converse is also true. In this case, there exists an isomorphism χ between the vector space V and its dual V^*:

$$\chi(x)(y) := \mathcal{B}(x, y). \tag{2.14}$$

The projective space of hyperplanes in $\mathbb{P}^3$ is denoted by $(\mathbb{P}^3)^*$. Note that the isomorphism χ between the vector spaces V and V^* induces an isomorphism $\mathbf{P}(\chi)$ from $\mathbb{P}^3$ and $(\mathbb{P}^3)^*$.

Definition 9. *The hyperplane $\mathbf{P}(\chi)(x_0)$ is called the polar to the point x_0 with respect to the quadric Q, and its equation is expressed as follows*

$$\mathcal{B}(x_0, x) = 0.$$

Definition 10. *A pole of a hyperplane with respect to the quadric Q is a point such that the point is the conjugate of the hyperplane with respect to the quadric Q.*

We know that with respect to a quadric Q there exists a pole A for any plane $\mathcal{P}_A$. The converse is also true, that is, for any point B, there is a polar plane $\mathcal{P}_B$ whose corresponding pole is the point B. The following proposition states the basic properties of poles and their polar planes.

Proposition 14 ([7]).

1. *If a plane $\mathcal{P}$ is tangent to the quadric Q at a point A, then its pole with respect to Q is the point A.*

2. *If a plane $\mathcal{P}$ contains a point A, then the pole of the plane $\mathcal{P}$ is in the polar plane of the point A.*

3. *A set of polar planes are in pencil if and only if their poles are collinear.*

4. *If a line l contains the pole A of the plane $\mathcal{P}$, and l intersects the plane $\mathcal{P}$ at a point B and intersects the quadric Q at C and D, then the four points form a harmonic set. Any line pencil $(XC), (XD), (XA)$, (XB) from a point X also form a harmonic set.*

Notice that if three points (resp. lines) of a harmonic set are given, then for an assigned order, the forth point (resp. line) is uniquely determined.

2.2.2 Confocal families of quadrics

In this section, we will define families of quadrics in the 3-dimensional Euclidean space $\mathbb{E}^3$ and summarize their elementary properties.

Definition 11. *A family of confocal quadrics in the* 3*-dimensional Euclidean space* $\mathbb{E}^3$ *is a family defined by the equation*

$$Q_\lambda : \frac{x_1^2}{a_1^2+\lambda} + \frac{x_2^2}{a_2^2+\lambda} + \frac{x_3^2}{a_3^2+\lambda} = 1, \qquad (\lambda \in \mathbb{R}) \tag{2.15}$$

and where a_1, a_2 *and* a_3 *are real constants (see Figure 2.11).*

Note that a family of confocal quadrics in the Euclidean space defined by the equation (2.15) is determined exactly by only one quadric. The type of quadric Q_λ changes as the value of the parameter λ passes the values $-a_2^2$ and $-a_1^2$. Then we obtain the following classifications.

- For $-a_3^2 < \lambda < -a_2^2$, the quadric Q_λ is a hyperboloid of two sheets.
- For $-a_2^2 < \lambda < -a_1^2$, the quadric Q_λ is a hyperboloid of one sheet.
- For $-a_1^2 < \lambda$, the quadric Q_λ is an ellipsoid.

Theorem 15 (Jacobi's theorem)**.** *Any generic point* $P = (x_1, x_2, x_3)$ *of the* 3*-dimensional Euclidean space is the intersection of exactly three quadrics of the confocal family* (2.15)*. The quadrics are pairwise orthogonal at* P*.*

Proof. The equation of a quadric, from the confacal family, passing through a given point P satisfies the equation (2.15). Then, we can express the equation of that quadric in the following form

$$(a_1^2+\lambda)(a_2^2+\lambda)(a_3^2+\lambda) - (a_2^2+\lambda)(a_3^2+\lambda)x_1^2 - (a_1^2+\lambda)(a_3^2+\lambda)x_2^2 - (a_1^2+\lambda)(a_2^2+\lambda) = 0.$$

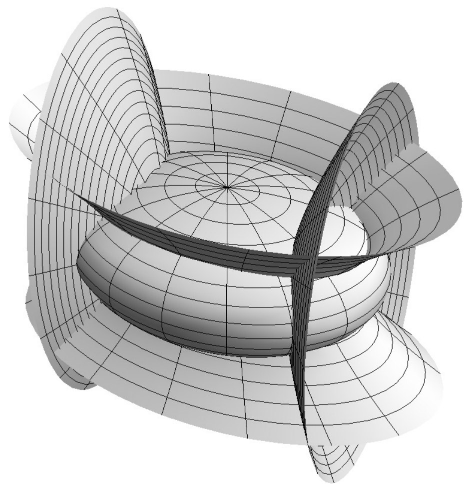


Figure 2.11: Confocal quadrics in the 3-dimensional space.

This is a cubic equation and it has a leading coefficient equal to 1. Let $(\lambda_1, \lambda_2, \lambda_3)$ be the roots of this cubic equation. Now, we will show that the quadrics are pairwise orthogonal at the point P. Let Q_{λ_1}, Q_{λ_2} and Q_{λ_3} be these quadrics. Let us prove the orthogonality of the quadrics Q_{λ_1} and Q_{λ_2}. For the other cases, the proof is similar. The equations of the quadrics Q_{λ_1} and Q_{λ_2} are given by the equation (2.15). By subtracting these equations, we have

$$\frac{x_1^2}{(a_1^2+\lambda_1)(a_1^2+\lambda_2)} + \frac{x_2^2}{(a_2^2+\lambda_1)(a_2^2+\lambda_2)} + \frac{x_3^2}{(a_3^2+\lambda_1)(a_3^2+\lambda_2)} = 0. \tag{2.16}$$

A normal vector to the quadric Q_{λ_1} at the point P is the gradient of the equation (2.15) on the left hand side. Then we have

$$N_{Q_{\lambda_1}} = \begin{pmatrix} \dfrac{x_1}{a_1^2+\lambda_1} \\ \dfrac{x_2}{a_2^2+\lambda_1} \\ \dfrac{x_3}{a_3^2+\lambda_1} \end{pmatrix}.$$

Similarly, a normal vector to the quadric Q_{λ_2} at the point P is

$$N_{Q_{\lambda_2}} = \begin{pmatrix} \dfrac{x_1}{a_1^2+\lambda_2} \\ \dfrac{x_2}{a_2^2+\lambda_2} \\ \dfrac{x_3}{a_3^2+\lambda_2} \end{pmatrix}.$$

Therefore, we obtain

$$\begin{aligned} N_{Q_{\lambda_1}} \cdot N_{Q_{\lambda_2}} &= \begin{pmatrix} \dfrac{x_1}{a_1^2+\lambda_1} \\ \dfrac{x_2}{a_2^2+\lambda_1} \\ \dfrac{x_3}{a_3^2+\lambda_1} \end{pmatrix} \cdot \begin{pmatrix} \dfrac{x_1}{a_1^2+\lambda_2} \\ \dfrac{x_2}{a_2^2+\lambda_2} \\ \dfrac{x_3}{a_3^2+\lambda_2} \end{pmatrix} \\ &= \frac{x_1^2}{(a_1^2+\lambda_1)(a_1^2+\lambda_2)} + \frac{x_2^2}{(a_2^2+\lambda_1)(a_2^2+\lambda_2)} + \frac{x_3^2}{(a_3^2+\lambda_1)(a_3^2+\lambda_2)} \\ &= 0 \quad \text{by (2.16)}. \end{aligned}$$

This proves that the two quadrics Q_{λ_1} and Q_{λ_2} are orthogonal at the point P. ∎

Theorem 16 (Chasles' theorem). *Any generic line in $\mathbb{E}^3$ is tangent to two distinct quadrics from a given confocal family. The tangent planes to these quadrics at the points of tangency with the line are orthogonal to each other.*

Proof. Let l be a line in $\mathbb{E}^3$. Note that the projection of the family of confocal quadrics along l to the orthogonal plane is one-parameter family of apparent contours[1]. This apparent contours is a family of confocal conics. Moreover, the projection of the line is a point. We know that there exits an ellipse and a hyperbola from a confocal family passing through this point, and the ellipse and the hyperbola also are orthogonal to each other at that point. Each of these two curves is the apparent contour of a quadric from the given confocal family, then the two quadrics are tangent to the line l and are also orthogonal at the tangency point. ∎

[1]A generic orthogonal projection of a surface on the plane is a domain bounded by a curve, the apparent contour of the surface.

Let Q_1 and Q_2 be two quadrics in the projective space. These quadrics determine a confocal family. Denote by u the tangent plane to Q_1 at a point x and by z the pole of u with respect to Q_2. Suppose lines l_1 and l_2 intersect at x, and the plane containing the lines l_1 and l_2 intersect u at a line l. We have the following definition.

Definition 12 ([7], [15] Definition 5). *We say that lines l_1 and l_2 obey the reflection law at the point x on the quadric Q_1 with respect to the confocal system containing Q_1 and Q_2 if the four lines l_1, l_2, (xz) and l are coplanar and form a harmonic set.*

There is a coincidence between the reflection law defined in the Definition 12 and the standard reflection if we introduce a coordinate system in which the quadrics Q_1 and Q_2 are confocal in the Euclidean sense.

Theorem 17 ([7],[15], One reflection theorem). *Let lines l_1 and l_2 obey the reflection law on the quadric Q_1 at a point x with respect to the confocal system determined by the quadrics Q_1 and Q_2. Assume that the line l_1 intersects Q_2 at y_1' and y_1, and u is tangent plane to Q_1 at x, and z its pole with respect to Q_2. Then the lines $(y_1'z)$ and (y_1z) respectively contain the intersecting points y_2' and y_2 of the line l_2 with Q_2. The converse of statement is also true.*

Corollary 18. *Assume that l_1 and l_2 satisfy reflection law on Q_1 at x with respect to the confocal system determined by Q_1 and Q_2. Then, the following properties hold:*

1. *The line l_1 is tangent to Q_2 if and only if l_2 is tangent to Q_2.*
2. *The line l_1 intersects Q_2 at two points if and only if l_2 intersects Q_2 at two points.*

According to the billiard law at some confocal quadrics, there exist four lines reflect to each other and these lines form the so-called Double reflection configuration.

Theorem 19 ([7, 15], Double reflection theorem). *Let* Q_1 *and* Q_2 *be two quadrics and* l_1 *be a line which intersects* Q_1 *at* x_1 *and intersects* Q_2 *at* y_1. *Denote by* u_1 *and by* v_1 *the tangent planes to* Q_1 *and to* Q_2 *respectively, and* z_1, w_1 *their poles with respect to* Q_2 *and* Q_1. *Assume that the line* (w_1x_1) *intersects* Q_1 *at another point* x_2 *and the line* (y_1z_1) *intersects* Q_2 *at another point* y_2. *Denote by* $l_1 = (x_1y_2)$, $l_1' = (y_1x_2)$, $l_2' = (x_2y_2)$. *Then, the pairs of lines* l_1, l_2; l_1, l_1'; l_2, l_2'; l_1', l_2' *satisfy the reflection law at* x_1 *on* Q_1, *at* y_1 *on* Q_2, *at* y_2 *on* Q_2, *at* x_2 *on* Q_1 *respectively.*

Proof. Our proof is inspired by the exposition from [7]. Let u_2 be the tangent plane to Q_1 at x_2 and v_2 be the tangent plane to Q_2 at y_2. Suppose that z_2 is the pole of the tangent plane u_2 with respect to Q_2 and w_2 is the pole of the tangent plane v_2 with respect to Q_1. By assumption, the line (w_1x_1) intersects the quadric Q_1 at the point x_2. This means that the points x_1, x_2, and w_1 are collinear. Due to the Proposition 14, their polar planes with respect to Q_1 are in a pencil, that is, the polar planes u_1, u_2 and v_1 are in the same pencil. Thus, the poles of these polar planes with respect to Q_2 are z_1, z_2, and y_1 and these poles are on a straight line. The point x_2 is also contained in this line. It follows that the polar planes of the points z_1, z_2, y_1 and y_2 with respect to Q_2 are u_1, u_2, v_1, v_2, and they are collinear according to the Proposition 14. The poles of the polar planes u_1, u_2, v_1, v_2 with respect to Q_1 are x_1, x_2, w_1, w_2, respectively, and they are on a straight line. Applying the Theorem 17, it is straightforward to verify that the pairs of lines l_1, l_2; l_1, l_1'; l_2, l_2 and l_1', l_2' satisfy the reflection law at the intersection points. Moreover, the reflection of a line at a given point upon a quadric is unique. The double reflection theorem follows immediately. ■

Corollary 20. *Let* l_1, l_2, l_1' *and* l_2' *be lines from the Theorem 19. If the line* l_1 *is tangent to a quadric* Q' *confocal with the quadrics* Q_1 *and* Q_2, *then the lines* l_2, l_1' *and* l_2' *are also tangent to* Q'.

The forth line of a double reflection configuration can be uniquely determined by the three given lines for such configuration and this claim is formulated and proved in the next proposition.

Proposition 21 ([17], Proposition 3.9). *Let* Q_1 *and* Q_2 *be quadrics in the projective space* $\mathbb{P}^3$. *Let* l, l_1 *and* l_3 *be lines such that the lines* l *and* l_1 *reflect to each other off the quadric* Q_1 *and the lines* l *and* l_2 *reflect to each other off* Q_2 *with respect to the confocal system determined by these two quadrics. Then there exists exactly only one line* l_{12} *such that the lines* $\mathsf{l}, \mathsf{l}_1, \mathsf{l}_2$ *and* l_{12} *form a double reflection configuration.*

Proof. Let $\mathcal{P}_1$ be tangent hyperplane to Q_1 at the point of intersection of the lines l and l_1 and $\mathcal{P}_2$ be tangent hyperplane to Q_2 at the point intersection of the lines l, l_2. Then, the hyperplanes $\mathcal{P}_1$ and $\mathcal{P}_2$ determine a pencil of hyperplanes. Choose two hyperplanes $\mathcal{P}'_1$, $\mathcal{P}'_2$ from the pencil determined by $\mathcal{P}_1$ and $\mathcal{P}_2$ such that $\mathcal{P}'_1$ is tangent to Q_1 and $\mathcal{P}_1 \neq \mathcal{P}'_1$, and $\mathcal{P}'_2$ is tangent to Q_2 and $\mathcal{P}_2 \neq \mathcal{P}'_2$. The four hyperplanes belong to the same pencil. Therefore, the four lines form a double reflection configuration. ■

CHAPTER 3

EXTENDED ANALYSIS OF INCIRCULAR NETS AND CHECKERBOARD INCIRCULAR NETS IN PLANE

The present chapter deals with the extended analysis of IC nets and checkerboard IC nets. The main topics discussed are the method of constructing those nets in planar case and their geometric and combinatorial properties. This chapter is devoted to the exposition of the work of W. Böhm [39], Arseny and Bobenko [1]. Lemma 27 is a new result in this chapter. From this lemma, we are also able to find an elementary proof of the Theorem 28. Some proofs of the theorems that we will present here are more detailed and totally different from those presented in the original articles.

3.1 Homothety and Monge's theorem

Definition 13. *An external center of similitude is a point from which at least two geometrically similar figures can be seen as dilation or contraction of one another.*

A homothety (or a central similarity) is a transformation that carries each point P in the plane into a point P' on OP, where O is a fixed point, such that the ratio $\dfrac{\mathsf{OP}'}{\mathsf{OP}} = \mathsf{k}$ is the same for all P different from O (see Figure 3.1). The point O is called the center of the homothety and the number k is called the homothety coefficient. The homothety coefficient can be either negative or positive. For more details on the central similarity or the homothety and the detailed proof of the Theorem 22, see for instance [14].

Theorem 22 ([14], page 29)**.** *Consider three similar geometric figures $\mathcal{F}_1$, $\mathcal{F}$ and $\mathcal{F}'$ such that the figures $\mathcal{F}_1$ and $\mathcal{F}$ are centrally similar with similarity center O_1, and $\mathcal{F}_1$ and $\mathcal{F}'$ are centrally similar with similarity center O_2 (see Figure 3.2). Then we have the following:*

1. *If the points O_1 and O_2 do not coincide, then the line $(\mathsf{O}_1\mathsf{O}_2)$ passes through the center of similarity O of the figures $\mathcal{F}$ and $\mathcal{F}'$.*

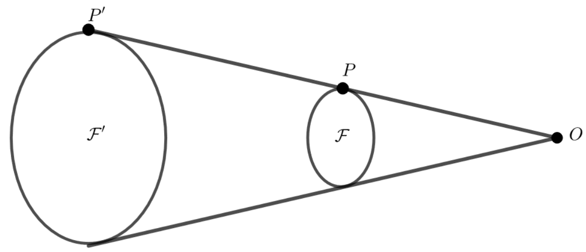


Figure 3.1: Two similar geometric figures related by a homothetic transformation with respect to a homothetic center O.

2. *If the points* O_1 *and* O_2 *coincide, then the point* O_1 *is the center of similarity for the figures* $\mathcal{F}$ *and* $\mathcal{F}'$.

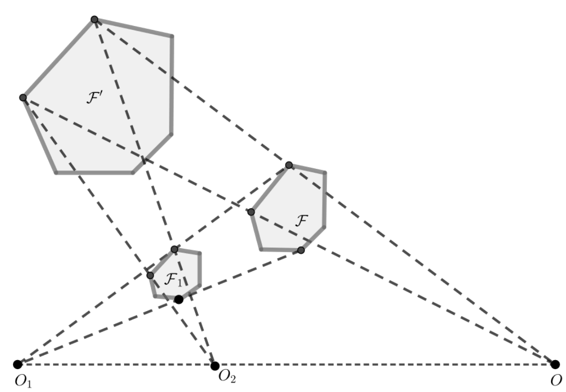


Figure 3.2: Three similar geometric figures.

Theorem 23 (Monge's theorem of three circles). *Given three disjoint circles of different radii in the plane such that they lie completely outside each other. The six external tangents to two of the three circles, taken pairwise, intersect at three points. Then the three points lie on a straight line.*

Proof. In Figure 3.3, let denote by C_1 the circle with center at O_1 and radius $|O_1A_1| = r_1$, by C_2 the circle with center at O_2 and radius $|O_2B_1| = r_1$, and by C_3 the circle with center at O_3 and radius $|O_3C_1| = r_1$. Consider the triangle PO_1A_1. Since the lines (O_1A_1) and (O_2B_1) are

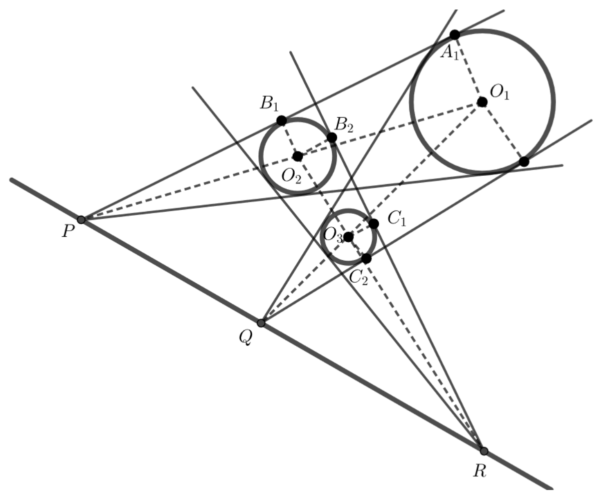


Figure 3.3: Monge's Theorem of three circles.

perpendicular to the line (A_1P), these two lines are parallel. Then, we obtain the following relation

$$\frac{|PO_1|}{|PO_2|} = \frac{|O_1A_1|}{|O_2B_1|} = \frac{r_1}{r_2}. \tag{3.1}$$

Applying that argument to the triangles QO_1A_2 and RO_2B_2, we have

$$\frac{|QO_3|}{|QO_1|} = \frac{|O_3C_2|}{|O_1A_2|} = \frac{r_3}{r_1}, \tag{3.2}$$

$$\frac{|RO_2|}{|RO_3|} = \frac{|O_2B_2|}{|O_3C_1|} = \frac{r_2}{r_3}. \tag{3.3}$$

Hence,

$$\frac{r_1}{r_2}.\frac{r_3}{r_1}.\frac{r_2}{r_3} = 1.$$

By Menelaus' theorem, the points P, Q and R are on the same straight line. ■

3.2 Extended analysis of incircular nets in plane

Circumscribed quadrilaterals nets have been studied by W. Böhm in the setting of Euclidean and spherical geometries [39]. Recently, Akopyan and Bobenko gave new results regarding checkerboard incircular nets in a plane [1]. These are congruences of straight lines in the plane with the combinatorics of a square grid such that every second elementary quadrilateral possesses an inscribed circle.

3.2.1 Circumscribed quadrilaterals nets

We know that a quadrilateral is circumscribed if the sum of the lengths of a pair of opposite edges is equal to the sum of the lengths of the other pair of opposite sides. The following theorems concerns the circumscribed quadrilaterals whose sides are tangent lines to a given conic.

Theorem 24 ([39], Böhm). *A division of a plane by straight lines into circumscribed quadrilaterals necessarily consists of the tangent lines to a given conic.*

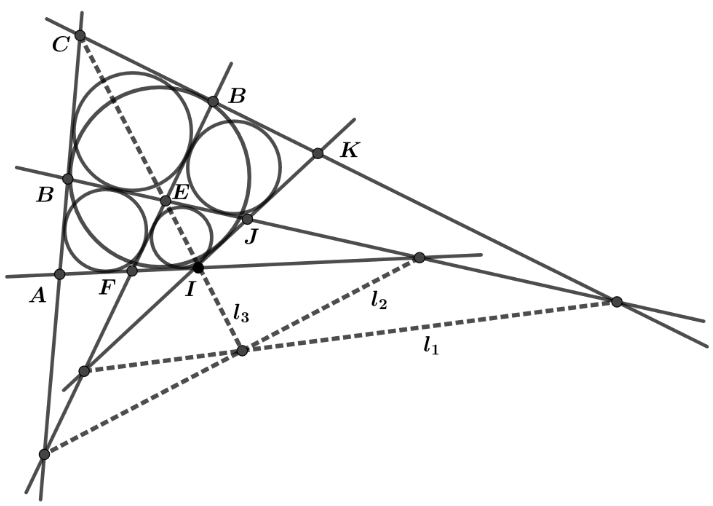


Figure 3.4: Quadrilaterals net.

Proof. In the Figure 3.4, let denote C_1, C_2, C_3, C_2, C_3, C_4, and C_5 the inscribed circles in the quadrilaterals $(ABEF)$, $(BCDE)$, $(EFIJ)$, $(DEJK)$, and $(ACKI)$ respectively. We observe that the circumscribed quadrilateral $(ACKI)$ is divided into four small circumscribed quadrilaterals. Let us consider the three triplets of circles C_1, C_2 and C_3, C_2, C_3 and C_4, and C_2, C_3 and C_5. According to Monge's theorem, the three external centers of similitude for each triplet of circles lie on a straight line. Then lines l_1, l_2 and l_3 contain the external centers of similitude of the circles C_2, C_3 and C_4, C_1, C_2 and C_3, and C_2, C_3 and C_5 respectively.

The lines l_1, l_2 and l_3 also connect the opposite vertices of the hexagon made by the lines $(\mathsf{AC}), (\mathsf{CD}), (\mathsf{BJ}), (\mathsf{AI}), (\mathsf{IJ})$ and (EF). Moreover, these lines intersect with external center of similitude of the circles C_2 and C_3. Applying Theorem 8, the six sides of the hexagon are tangent to a conic. ∎

Definition 14. *A net in a plane is configurations of points and lines limited to certain geometric and combinatorial constraints.*

Definition 15. *A rectilinear net is a net whose quadrilaterals made by straight lines.*

Theorem 25 ([39], Böhm)**.** *Two adjacent circumscribed quadrilaterals determine a circumscribed quadrilaterals net.*

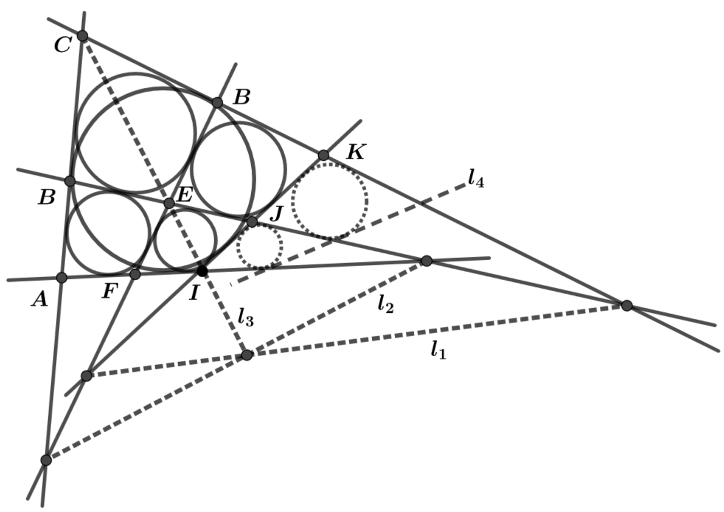


Figure 3.5: Two adjacent circumscribed quadrilaterals determine a net.

Proof. In the Figure 3.5, let denote C_1, C_2, C_3, C_2, C_3, C_4, and C_5 the inscribed circles in the quadrilaterals (ABEF), (BCDE), (EFIJ), (DEJK), and (ACKI) respectively. Since any five lines in general position such that no three of them are concurrent uniquely define a conic, it is sufficient to prove that the line l_4 is tangent to the conic determined by the lines $(AC), (CD), (BJ), (AI), (IJ)$ and (EF). By Theorem 24, the line (IJ) touches the conic determined by the lines $(AC), (CD), (BJ), (AI)$ and (EF), say α, and the line l_4 touches the conic determined by the lines $(AC), (BJ), (AI), (EF)$ and (IJ), say β. However, the two conic α and β share five tangent lines, thus they must coincide. Therefore, the line l_4 is tangent to the conic determined by the lines $(AC), (CD), (BJ), (AI), (IJ)$ and (EF). By continuing this procedure, we will construct the entire circumscribed quadrilaterals net. ∎

Theorem 26 ([39], Böhm). *Every quadrilateral that can be divided into four circumscribed quadrilaterals, by two lines, is itself circumscribed.*

Proof. The proof is straightforward by using the fact that the four small quadrilaterals are circumscribed. ∎

3.2.2 Definition, construction and geometric properties of IC nets

Let us introduce some useful notations that we need throughout this section. Let $f : \mathbb{Z}^2 \longrightarrow \mathbb{R}^2$ be a map of a square grid to a plane. Then we have the following:

- the vertices of a net are denoted by $f_{i,j} = f(i, j)$,
- $\square^c_{i,j}$ is a quadrilateral with vertices $f_{i,j}$, $f_{i+c,j}$, $f_{i+c,j+c}$, and $f_{i,j+c}$. This quadrilateral $\square^c_{i,j}$ is called a *net-square*,
- if $c = 1$, the quadrilateral $\square^1_{i,j}$ is called a *unit net-square* and denoted by $\square_{i,j}$.

Definition 16 ([1], Definition 2.1). *A map $f : \mathbb{Z}^2 \longrightarrow \mathbb{R}^2$ is called an incircular net (IC net) if the following conditions hold:*

1. *For any integer i, the points $\{f_{i,j} | j \in \mathbb{Z}\}$ lie on a straight line. The order of the points on this straight line is preserved, that is, the point $f_{i,j}$ lies between the points $f_{i,j-1}$ and $f_{i,j+1}$. The same holds for the points $\{f_{i,j} | i \in \mathbb{Z}^2\}$. These lines form the lines of the IC net.*

2. *All the unit net-squares* $\square_{i,j}$ *are circumscribed. The inscribed circle in the unit net square* $\square_{i,j}$ *is denoted by* $\omega_{i,j}$ *and by* $o_{i,j}$ *its center.*

From the Theorem 9, we observe that if the lines m_j, m_{j+1}, l_i and l_{i+1} are tangent to the conic α, then the points $m_j \cap m_{j+1}$ and $l_i \cap l_{i+1}$ belong to the same conic confocal with α. Having this kind of property, we are able to give a global definition of the IC net.

Definition 17 ([1], Definition 2.3)**.** *Let* α *and* α' *be two confocal conics. Let* m_j *and* l_i *be tangent lines to* α *such that the intersection points* $m_j \cap m_{j+1}$ *and* $l_i \cap l_{i+1}$ *belong to* α'*. The lines* m_j *and* l_i*,* $i, j \in \mathbb{Z}$*, are called the lines of an IC net and its vertices are the points of the form* $f_{i,j} = l_i \cap m_j$*.*

Lemma 27. *Consider a quadrilateral formed by four tangent lines to a given conic* α *such that such quadrilateral is divided into four small quadrilaterals by two other tangent lines to* α*. Suppose that three of these four quadrilaterals are circumscribed (see Figure 3.6). Then the fourth quadrilateral is also circumscribed.*

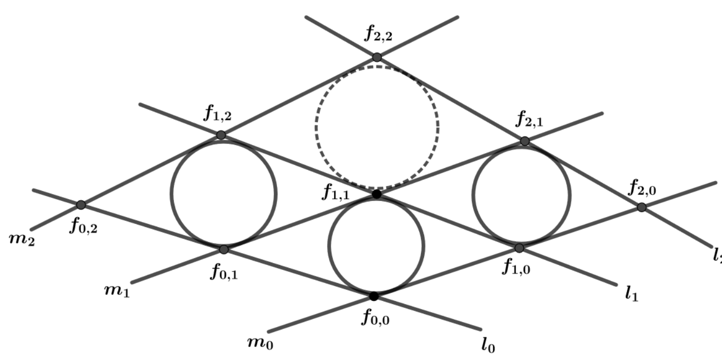


Figure 3.6: The quadrilateral $(f_{1,1}f_{2,1}f_{2,2}f_{1,2})$ is also circumscribed.

Proof. Let denote by l_0, l_1, l_2, m_0, m_1 and m_2 the tangent lines to the conic α. Then the big quadrilateral formed by tangent lines l_0, l_2, m_0 and m_2 has the vertices $f_{0,0}$, $f_{2,0}$,

$f_{2,2}$, $f_{0,2}$. Let the coordinates of tangency points on α be x_0, x_1, x_2, y_0, y_1, y_2 so that $f_{1,0}(x_1, y_0)$, $f_{2,1}(x_2, y_1)$, $f_{1,2}(x_1, y_2)$, $f_{0,1}(x_0, y_1)$. Notice that the quadrilaterals $(f_{0,0}f_{1,0}f_{1,1}f_{0,1})$, $(f_{1,0}f_{2,0}f_{2,1}f_{1,1})$ and $(f_{0,1}f_{1,1}f_{1,2}f_{0,2})$ are circumscribed. By the Graves-Chasles theorem , the pairs of points $f_{1,2}$ and $f_{0,1}$, $f_{0,1}$ and $f_{1,0}$, and $f_{1,0}$ and $f_{2,1}$ lie on conics confocal with α. Assume that the conics passing through these pairs of points are ellipse, hyperbola, and ellipse, respectively. We utilize the same arguments to demonstrate the other cases. Since the pairs of points $f_{0,1}$ and $f_{1,2}$, and $f_{1,0}$ and $f_{2,1}$ lie on ellipses and the points $f_{0,1}$ and $f_{1,0}$ lie on same hyperbola, we have the following equations respecting the order of these pairs of points

$$y_1 - x_0 = y_2 - x_1, \tag{3.4a}$$

$$y_1 - x_2 = y_0 - x_1, \tag{3.4b}$$

$$y_1 + x_0 = y_0 + x_1. \tag{3.4c}$$

Substracting the equations (3.4a) and (3.4b) and then adding that result to the equation (3.4c), we obtain

$$x_1 + y_2 = x_2 + y_1.$$

The last equation shows that the points $f_{1,2}$ and $f_{2,1}$ lie on the same hyperbola. Due to the Graves-Chasles Theorem 9, the forth quadrilateral is also circumscribed. ■

Theorem 28 (3×3 **incircles incidence theorem**, [1]). *Consider a quadrilateral such that such quadrilateral is split into nine quadrilaterals by two pairs of lines l_1 and l_2, and m_1 and m_2. Assume that all quadrilaterals are circumscribed, except one at a corner (see Figure 3.7). Then it turns out that such quadrilateral at the corner is also circumscribed.*

Proof. We know that a conic is defined by any five lines such that no three of which are concurent. Let denote by α the conic determined by the lines l_0, l_1, l_2, m_0 and m_1. First, we need to show that all the lines in the 3×3 incircles incidence theorem are tangent to

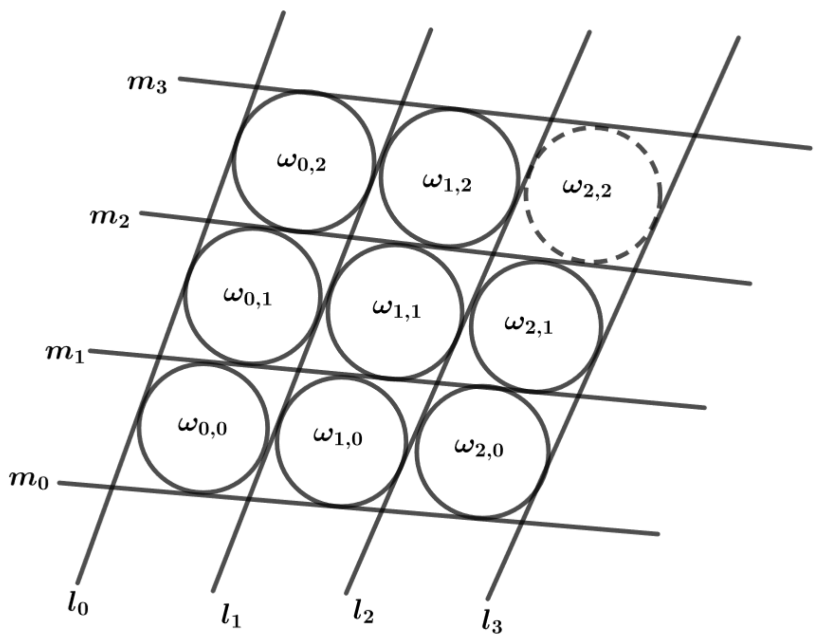


Figure 3.7: 3×3 incircles incidence theorem.

α. Note that the lines $\mathfrak{m}_0$, $\mathfrak{m}_1$ and $\mathfrak{l}_2$ are sides of the quadrilateral $(\mathfrak{f}_{2,0}\mathfrak{f}_{3,0}\mathfrak{f}_{3,1}\mathfrak{f}_{2,1})$ and these lines are tangent to α. Applying Corollary 10, the line $\mathfrak{l}_3 = (\mathfrak{f}_{3,0}\mathfrak{f}_{3,1})$ is the forth side of the quadrilateral $(\mathfrak{f}_{2,0}\mathfrak{f}_{3,0}\mathfrak{f}_{3,1}\mathfrak{f}_{2,1})$. Repeating this argument several times, we will prove that all lines of the 3×3 incircles theorem are tangent to α. It remains to show that the quadrilateral formed by the lines $\mathfrak{l}_2$, $\mathfrak{l}_3$, $\mathfrak{m}_2$ and $\mathfrak{m}_3$ is circumscribed. Consider the quadrilateral $(\mathfrak{f}_{1,1}\mathfrak{f}_{3,1}\mathfrak{f}_{3,3}\mathfrak{f}_{1,3})$. It is clear that this quadrilateral is cut by the lines $\mathfrak{l}_2$ and $\mathfrak{m}_2$ into four small quadrilaterals such that three of them are circumscribed. Then, it follows directly from Lemma 27 that the quadrilateral $(\mathfrak{f}_{2,2}\mathfrak{f}_{3,2}\mathfrak{f}_{3,3}\mathfrak{f}_{2,3})$ is circumscribed. ■

The Theorem 29 is a new formulation of the Theorem 26 and shows us a natural way to construct an IC net.

Theorem 29 ([1], Corollary 2.10)**.** *Two neighboring circles $\omega_{0,0}$, $\omega_{1,0}$ and their five tangent lines $\mathfrak{l}_0$, $\mathfrak{l}_1$, $\mathfrak{l}_2$, $\mathfrak{m}_0$, and $\mathfrak{m}_1$ uniquely determine an IC net (see Figure 3.8).*

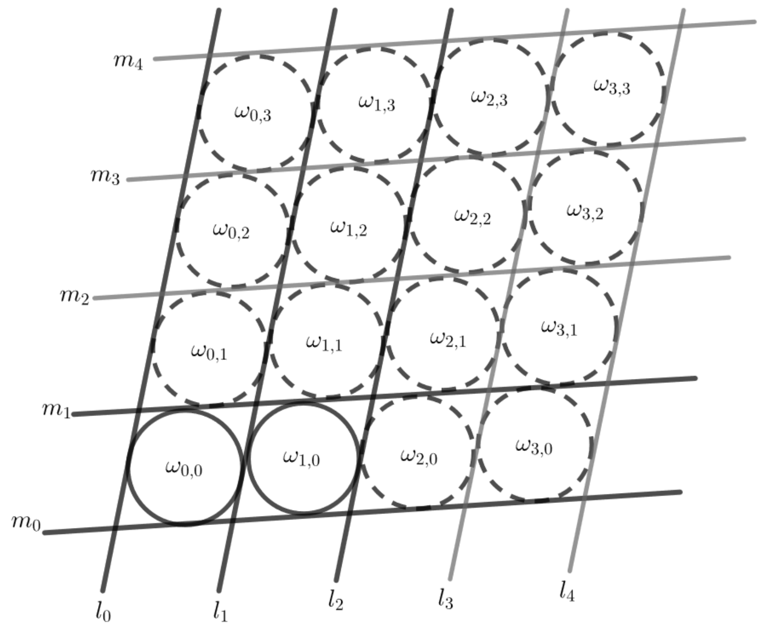


Figure 3.8: Construction of an IC net.

Now, let us list the geometric and combinatorial properties of the IC nets.

Theorem 30 ([1], Theorem 2.1). *Let f be an IC net. Then f satisfies the following properties:*

1. *All lines of the IC net f touch some conic α (the conic α can be degenerate).*

2. *The points $\mathsf{f}_{i,j}$, where $i+j$=constant, belong to a conic confocal with α. The point $\mathsf{f}_{i,j}$, where $i-j$=constant, also belong to a conic confocal with α.*

3. *All net-squares of f are circumscribed.*

4. *In any net-square with even sides, the midlines have equal lengths*

$$|\mathsf{f}_{i-c,j}\mathsf{f}_{i+c,j}| = |\mathsf{f}_{i,j-c}\mathsf{f}_{i,j+c}|.$$

5. *The cross ratios*

$$\mathrm{cr}(\mathsf{f}_{i,j_1}, \mathsf{f}_{i,j_2}, \mathsf{f}_{i,j_3}, \mathsf{f}_{i,j_4}) = \frac{(\mathsf{f}_{i,j_1} - \mathsf{f}_{i,j_2})(\mathsf{f}_{i,j_3} - \mathsf{f}_{i,j_4})}{(\mathsf{f}_{i,j_2} - \mathsf{f}_{i,j_3})(\mathsf{f}_{i,j_4} - \mathsf{f}_{i,j_1})}$$

   is independent of i. The cross ratios $\mathrm{cr}(\mathsf{f}_{i_1,j}, \mathsf{f}_{i_2,j}, \mathsf{f}_{i_3,j}, \mathsf{f}_{i_4,j})$ is also independent of j.

Proof. 1. The statement is proven in the same way as the 3×3 incircles incidence theorem. 2. For the second property of the IC net, we will only show that the points $f_{i,j}$, with $i+j=5$, lie on the same conic confocal with α. The remaining cases can be proven similarly. Here in particular, we need to show that the three points $f_{1,4}$, $f_{2,3}$ and $f_{3,2}$ lie on the same conic confocal with α as we see in Figure 3.9. Without loss of generality, we assume that the pairs

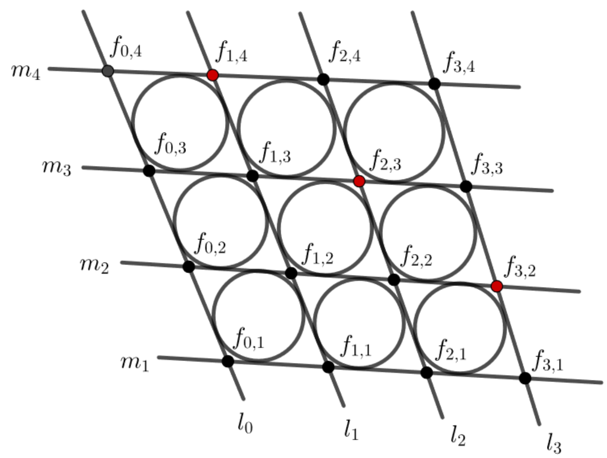


Figure 3.9: Three points belong to the same conic.

of points $f_{1,2}$ and $f_{2,3}$, and $f_{2,1}$ and $f_{3,2}$ lie on the ellipses confocal with α and the pairs points $f_{1,2}$ and $f_{2,1}$, and $f_{2,3}$ and $f_{3,2}$ lie on the hyperbolas confocal with α. Since the quadrilateral $(f_{1,3}f_{2,3}f_{2,4}f_{1,4})$ is circumscribed, there exists a conic confocal with α passing through the vertices $f_{2,3}$ and $f_{1,4}$. This conic is either an ellipse or a hyperbola. Suppose that this conic is a hyperbola. Let denote by γ_1 and γ_2 the hyperbolas which contain the pairs of points $f_{3,2}$ and $f_{2,3}$, and $f_{2,3}$ and $f_{1,4}$, respectively. Clearly, γ_1 and γ_2 intersect at the point $f_{2,3}$. However, they are confocal, therefore they must coincide. Suppose now that the conic is an ellipse. Denote by β_1 and β_2 the respective ellipses that pass through the pairs of points $f_{1,2}$ and $f_{2,3}$, and $f_{2,3}$ and $f_{1,4}$. Then the point $f_{2,3}$ is a common point of these ellipses. These ellipses must be different since they are confocal ellipses and the confocal ellipses cannot have a common point. Let x_0, x_1, x_2, y_0, y_1 and y_2 be the canonical coordinates of the tangency points of the lines l_0, l_1, l_2, m_2, m_3, m_4 with α respectively. Since the points $f_{1,2}$, $f_{2,3}$ and $f_{1,4}$ belong to the same ellipse, we obtain the following equations

$$y_0 - x_1 = y_1 - x_2, \tag{3.5a}$$

$$y_2 - x_1 = y_1 - x_2. \tag{3.5b}$$

The equations (3.5a) and (3.5b) imply that $y_0 = y_2$, so the points $f_{1,4}$ and $f_{1,2}$ coincide, which is a contradiction.

3. We have already proved that all lines of the IC net f touch the conic α and all the points $f_{i,j}$ lie on some conics confocal with α. It follows that all net-squares of f are circumscribed according to the Graves-Chasles Theorem 9.

4. One of the properties of the IC net f is that all of its points $f_{i,j}$ belong to the conics confocal with α. Apart from that, each of these points is the intersection point of an ellipse and a hyperbola. Indeed, the points $f_{i-c,j}$, $f_{i,j-c}$, $f_{f_{i+c,j}}$ and $f_{i,j+c}$ are intersection points of pairs of confocal ellipses and hyperbolas. Due to Ivory's theorem, we obtain

$$|f_{i-c,j} f_{i+c,j}| = |f_{i,j-c} f_{i,j+c}|.$$

5. Let A, B, C, D and E be points on the conic α (see Figure 3.10). Let l_{i_1} be a tangent line to α at the point E. Then the tangent lines at the points A, B, C and D intersect the tangent line l_{i_1} at the points f_{i_1,j_1}, f_{i_1,j_2}, f_{i_1,j_3} and f_{i_1,j_4}, respectively. One has already proved that

$$cr(f_{i_1,j_1}, f_{i_1,j_2}, f_{i_1,j_3}, f_{i_1,j_4}) = cr(A, B, C, D).$$

This equality means that there exists a projective transformation p_1 which maps the point A to f_{i_1,j_1}, B to f_{i_1,j_2}, C to f_{i_1,j_3} and D to f_{i_1,j_4}. Let us draw a tangent line l_{i_2} to the conic α at a point F. Using again the previous argument, we find another projective transformation p_2 which sends the point A to f_{i_2,j_1}, B to f_{i_2,j_2}, C to f_{i_2,j_3}, D to f_{i_2,j_4}. Since the set of projective

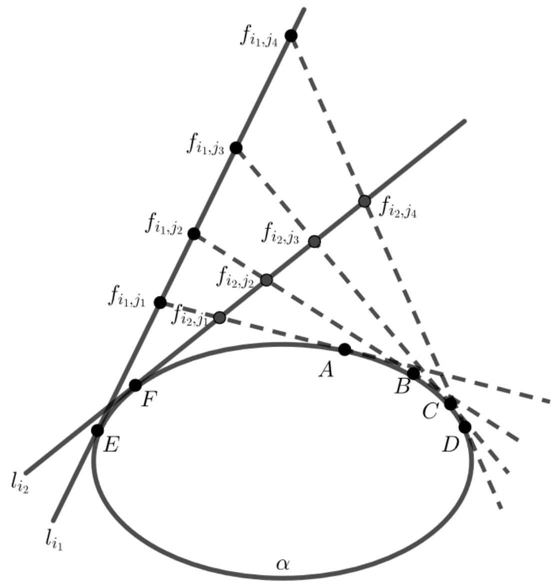


Figure 3.10: Projective map between two tangent lines to the conic α.

transformations forms a group under the operation of composition of functions, the map $p := p_2 \circ p_1^{-1}$ is a projective transformation and it can be written as follows

$$p : (f_{i_1,j_1}, f_{i_1,j_2}, f_{i_1,j_3}, f_{i_1,j_4}) \longmapsto (f_{i_2,j_1}, f_{i_2,j_2}, f_{i_2,j_3}, f_{i_2,j_4}).$$

Therefore, the cross ratio is preserved under the projective transformation p. This completes the proof of the first statement of the property 5. The second statement can be proved in a similar way. ■

Theorem 31 ([1], Theorem 2.1). *1. Let C_k be conics such that they contain the points $f_{i,j}$, with $i+j=k$. Then, for any $l \in \mathbb{Z}$, there is an affine transformation $A_{k,l} : C_k \longrightarrow C_{k+2l}$ such that $A_{k,l}(f_{i,j}) = f_{i+l,j+l}$. Furthermore, let $C_{k'}$ be conics such that they also contain the points $f_{i,j}$, with $i-j=k'$. Then, for any $l' \in \mathbb{Z}$, there exists an affine transformation $A_{k',l'} : C_{k'} \longrightarrow C_{k'-2l'}$ such that $A_{k',l'}(f_{i,j}) = f_{i-l',j+l'}$.*

2. The net-squares $\square^{c}_{i,j}$ and $\square^{c+2l}_{i-l,j-l}$ are perspective.

3. *Let* $\omega_{i,j}$ *be the inscribed circle of the unit net-square* $\square_{i,j}$ *with center* $o_{i,j}$. *Then all points* $o_{i,j}$, *with* $i+j=$ *constant, lie on a conic. The points* $o_{i,j}$, *with* $i-j$=*constant, also lie on a conic.*

4. *The centers of the circles inscribed in the unit net-squares of an IC net build a projective image of an IC net.*

Proof. 1. We know that there is a linear map that maps a point to a point between two confocal conics of the same type, and a point belonging to one of these conics and its image under the linear map lie on the same conic. Note that the type of the later conic is different than the former conics and it is confocal with them. Moreover, each point of the IC net is the intersection point of one ellipse and one hyperbola. Then, the statement of the theorem follows directly from the Theorem 12.

2. Suppose that l is positive. The case for l negative can be proven the same way. The net-square $\square^{c+2l}_{i-l,j-l}$ is split by the net-square $\square^{c}_{i,j}$ into nine cells. Denote by $\square_t$, $t=1,2,3,4$, the four cells in at corners. Let a_t, b_t be the respective vertices of $\square^{c}_{i,j}$ and $\square^{c+2l}_{i-l,j-l}$. Let ω_t, Ω and Ω' be the inscribed circles of $\square_t$, $\square^{c}_{i,j}$ and $\square^{c+2l}_{i-l,j-l}$ respectively. The circles ω_t and Ω have two common tangent lines and these tangent lines intersect at the point a_t. Then a_t is the homothetic center of ω_t and Ω. In addition, the circles ω_t and Ω' have also two common tangent lines and the point b_t is the point of intersection of these lines. Thus, the point b_t is the homothetic center of ω_t and Ω'. According to the theorem on the three centers of similarity, the line $(a_t b_t)$ passes through the homothety center of Ω and Ω'. Therefore, all lines $(a_t b_t)$ intersect at one point. This means that the net-squares $\square^{c}_{i,j}$ and $\square^{c+2l}_{i-l,j-l}$ are perspective.

3. and 4. Let l_i, l_{i+1}, m_j and m_{j+1} be four lines of an IC net f. The Definition 17 says that the points $l_i \cap l_{i+1}$ and $m_j \cap m_{j+1}$ lie on the conic α' confocal with α. Denote by l'_i and m'_j be the respective bisectors of the pairs of lines l_i and l_{i+1}, and m_j and m_{j+1}. The proof

of the statement has two steps. First, we need to prove that the lines l'_i and m'_j are tangent to the conic α'. The lines l_i and l_{i+1} are tangent to the conic α and let C and D be their

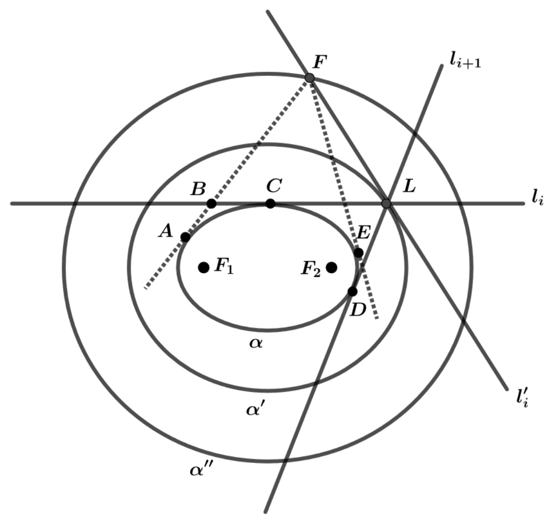


Figure 3.11: The bisector l'_i of l_i and l_{i+1} is tangent to α'.

tangency points to α respectively. We then have $\angle F_1LC = \angle F_2LD$, where F_1 and F_2 are foci of the conic α. It follows that the exterior angles $\angle CLD$ and $\angle F_1LF_2$ have the same bisector. This bisector is the line l'_i. Now, choose an arbitrary point F on the line l'_i and draw two tangent lines from the point F to α. Let us denote by A and E their tangency points to α. Note that the lines l_i and (AF) intersect at the point B as we see in the Figure 3.11. Then, we have the following inequalities

$$|FD| < |EF| + |\overset{\frown}{ED}| \quad \text{and} \quad |\overset{\frown}{AC}| < |AB| + |BC|,$$

where $|\overset{\frown}{ED}|$ represents the length of the arc joining the points E and D. We also have $|BL| + |LD| < |BF| + |FD|$ since l'_i is the bisector of the exterior angle $\angle BLD$. Therefore, we obtain

$$|CL| + |LD| + |\overset{\frown}{DC}| < |CL| + |LD| + |\overset{\frown}{DA}| + |AB| + |BC| = |BL| + |LD| + |\overset{\frown}{DA}| + |AB|$$

$$< |BF| + |FD| + |\overset{\frown}{DA}| + |AB| = |AF| + |FD| + |\overset{\frown}{DA}|$$

$$< |AF| + |FE| + |\overset{\frown}{ED}| + |\overset{\frown}{DA}| = |AF| + |FE| + |\overset{\frown}{EA}| \,.$$

This inequality proves that the point F is outside of the conic α'. In other words, the only point on l_i' which lies on the conic α' is the point L, that is, l_i' is tangent to α' at L. By using an argument similar, we can also prove that the line m_j' is tangent to the conic α'. Next, we need to show that the points $l_i' \cap l_{i+1}'$ and $m_j' \cap m_{j+1}'$ belong to the conic α''. From the Proposition 13, the conics α and α' are confocal conics and the conic α'' is a dual of α with respect to α', then the points $l_i' \cap l_{i+1}'$ and $m_j' \cap m_{j+1}'$ lie on conic α''.

■

3.3 Circular and conical nets

The two-dimensional nets can be regarded as discrete analogs of curvature lines parametrized surfaces. This section gives a quick review of two-dimensional nets called circular nets and their main property.

Definition 18 ([3], Definition 3.1). *A two-dimensional circular net is a map $f : \mathbb{Z}^2 \longrightarrow \mathbb{R}^2$ such that for every $n_0 \in \mathbb{Z}^2$ and for every $i, j \in \{1, 2\}$, $i \neq j$, the four points $f = f(n_0)$, $f_i = f(n_0 + e_i)$, $f_j = f(n_0 + e_j)$ and $f_{i,j} = f(n_0 + e_i + e_j)$ are concircular (see Figure 3.12).*

Definition 19 ([3], Definition 3.20). *A conical net is a map $P : \mathbb{Z}^2 \to \{$oriented planes in $\mathbb{R}^3\}$ such that if for every $n_0 \in \mathbb{Z}^2$ and each pair of indices $i, j \in \{1, 2\}$ with $i \neq j$, the four planes P, P_i, P_{ij} and P_j intersect at a common point and are in oriented contact with a cone of revolution.*

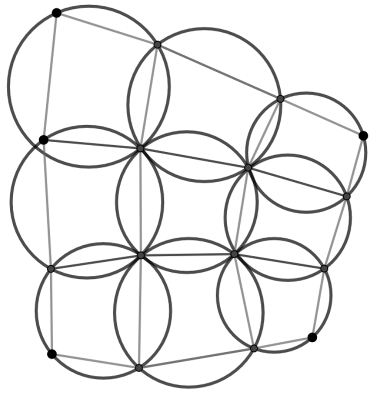


Figure 3.12: A circular net.

In generic case, let us consider a vertex of valence four, such that the vertex star is not contained in a plane. Denote by $\mathfrak{a}_1$, $\mathfrak{a}_2$, $\mathfrak{a}_3$ and $\mathfrak{a}_4$ the measure of the angles of the vertex star in cyclic order, depicted in Figure 3.13.

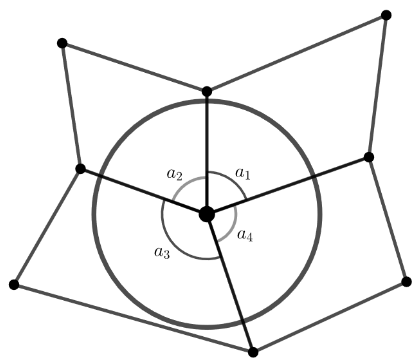


Figure 3.13: A conical net.

The following theorem states the conical property at a vertex. That property can be expressed in terms of the successive four angles $\mathfrak{a}_1, \mathfrak{a}_2, \mathfrak{a}_3$ and $\mathfrak{a}_4$ enclosed by the four edges emanating from that vertex.

Theorem 32 ([21] Proposition 1). *Consider a quadrilateral mesh and let* $\mathfrak{a}_1, \mathfrak{a}_2, \mathfrak{a}_3$ *and* $\mathfrak{a}_4$ *be the measure the of angles between the edges enclosed by the edges emanating from a vertex in cyclic order. The quadrilateral mesh is conical if and only if for all vertices, the sums of two opposite angles at that vertex obey* $\mathfrak{a}_1 + \mathfrak{a}_3 = \mathfrak{a}_2 + \mathfrak{a}_4$.

The complete proof of the Theorem 32 can be found in [37]. In the case where the four adjacent planes are coplanar, we have additional condition: the sum of all angles at one vertex is equal to 2π, that is, $a_1 + a_2 + a_3 + a_4 = 2\pi$. Then, in the planar case, a net is called conical net if $a_1 + a_3 = a_2 + a_4 = \pi$.

3.4 Extended analysis of checkerboard incircular nets

A checkerboard IC net is a generalization of the IC net that we have discussed in the previous section. The lines of the checkerboard IC net have the combinatorics of the square grid such that every second quadrilateral is circumscribed. The main objective of this current section is to present the way how to construct a checkerboard IC net and its geometric properties.

3.4.1 Definition and construction of a checkerboard IC net

Definition 20 ([1], Definition 3.1). *A map $f : \mathbb{Z}^2 \longrightarrow \mathbb{R}^2$ is called a checkerboard IC net if it satisfies the following conditions:*

1. *For any integer i, the points $\{f_{i,j} | j \in \mathbb{Z}\}$ lie on a straight line. The order of the points $f_{i,j}$ on this line is preserved, meaning that the point $f_{i,j}$ lies between the points $f_{i,j-1}$ and $f_{i,j+1}$. This condition also holds for the points $\{f_{i,j} | i \in \mathbb{Z}\}$. These lines are called lines of the checkerboard IC net.*

2. *For any integers i and j of the same parity, the quadrilateral, whose vertices are the points $f_{i,j}$, $f_{i+1,j}$, $f_{i+1,j+1}$, $f_{i,j+1}$, is circumscribed.*

Lemma 33 ([1], Lemma 3.2). *Let $(ABCD)$ be a quadrilateral such that this quadrilateral is cut by two pairs of lines into nine quadrilaterals. Assume that the quadrilateral at the center and all the quadrilaterals at the corners are circumscribed, (Figure 3.14). Then the quadrilateral $(ABCD)$ is also circumscribed.*

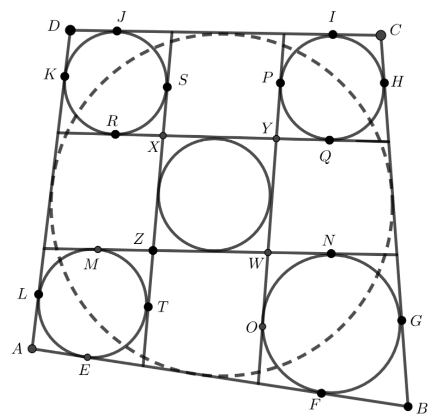


Figure 3.14: Six inscribed circles lemma.

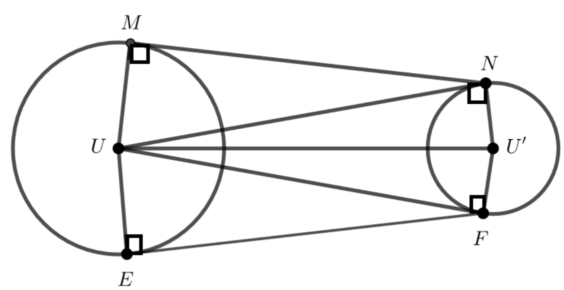


Figure 3.15: One piece of the Figure 3.14.

Proof. First, we will prove that $|MN| = |EF|$. The Figure 3.15 is just a piece of the Figure 3.14.

We observe that the triangles NUU' and FUU' are right triangles and $|U'N| = |U'F|$. By Pythagorean theorem, we obtain $|UN| = |UF|$. It follows that $|MN| = |EF|$. Repeating this argument, we can show that $|IJ| = |QR|$, $|KL| = |ST|$ and $|PO| = |HG|$. Since all quadrilaterals in all corners are circumscribed, we get

$$|RX| = |XS|, \quad |PY| = |YQ|,$$

$$|MZ| = |ZT|, \quad |OW| = |WN|.$$

The circumscribability of the quadrilateral at the center combined with the above equalities leads us to the following identity

$$|MN| + |RQ| = |ST| + |PO|.$$

Therefore,

$$|EF| + |IJ| = |KL| + |HG|. \tag{3.6}$$

By using the fact that all quadrilaterals at the corners are circumscribed, we thus have the following identities

$$|AE| = |AL|, \quad |FB| = |BG|,$$
$$|HC| = |CI|, \quad |DJ| = |DK|,$$

which imply that

$$|AE| + |FB| + |DJ| + |IC| = |AL| + |BG| + |HC| + |DK|. \tag{3.7}$$

Adding the equations (3.6) and (3.7), we get $|AB| + |CD| = |AD| + |BC|$. The later equality means the sum of the lengths of a pair of opposite sides of the quadrilateral $(ABCD)$ is equal to the sum of the lengths of the other pair of opposite sides. In other words, the quadrilateral $(ABCD)$ is circumscribed. ■

We have mentioned in the beginning of this section that the generalization of an IC net in planar case is a checkerboard IC net and the 3×3 incircles incidence theorem is one of the geometric properties of the IC net and can be generalized as follows:

Theorem 34 ([1], Checkerboard incircles incidence theorem). *Consider a quadrilateral which is divided by two sets of four lines into* 25 *quadrilaterals. Suppose that all quadrilaterals in a checkerboard pattern are circumscribed, except the one at a corner (see Figure 3.16). Then the last quadrilateral at the corner is also circumscribed.*

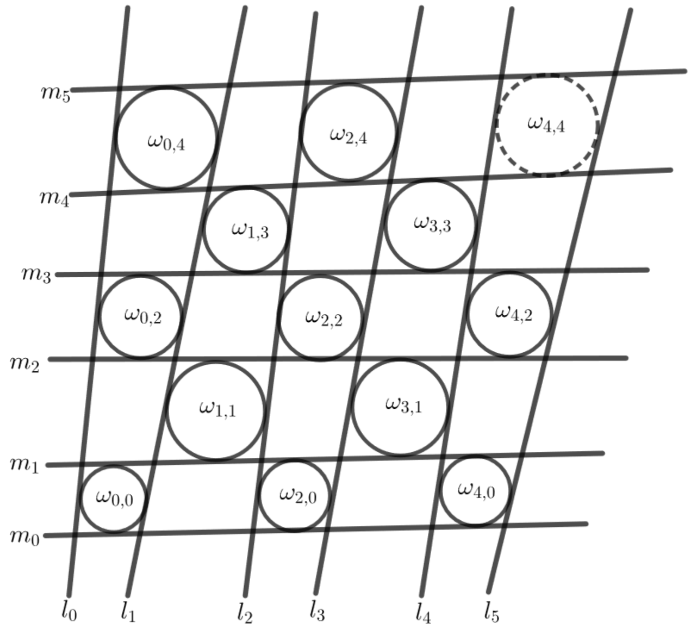


Figure 3.16: Checkerboard incircles incidence theorem.

Theorem 35 ([1], Corollary 3.6). *Five neighboring circles* $\omega_{0,0}, \omega_{2,0}, \omega_{0,2}, \omega_{2,2}, \omega_{1,1}$ *and a circle* $\omega_{3,3}$*, and their four tangent lines* l_1, l_2, m_1 *and* m_2 *uniquely determine a checkerboard IC net (see Figure 3.17).*

3.4.2 Geometric properties of checkerboard IC nets

Theorem 36 ([1], Theorem 3.1). *The checkerboard IC net* f *has the following properties:*

1. *All net-squares are circumscribed.*

2. *Net-squares* $\square^{c}_{i,j}$ *and* $\square^{c+2l}_{i-l,j-l}$ *are perpective, where* l *is an odd number.*

Proof. 1. We will prove the first property by induction on c.

- For $c = 1$. The second condition in the definition of the checkerboard IC net shows that all unit net-squares are circumscribed. Then the statement is true for $c = 1$.

- Assume now that the circumscribability is known for all net-squares of size c. We need to prove that the net-squares of size $c + 2$ are circumscribed. Let divide the

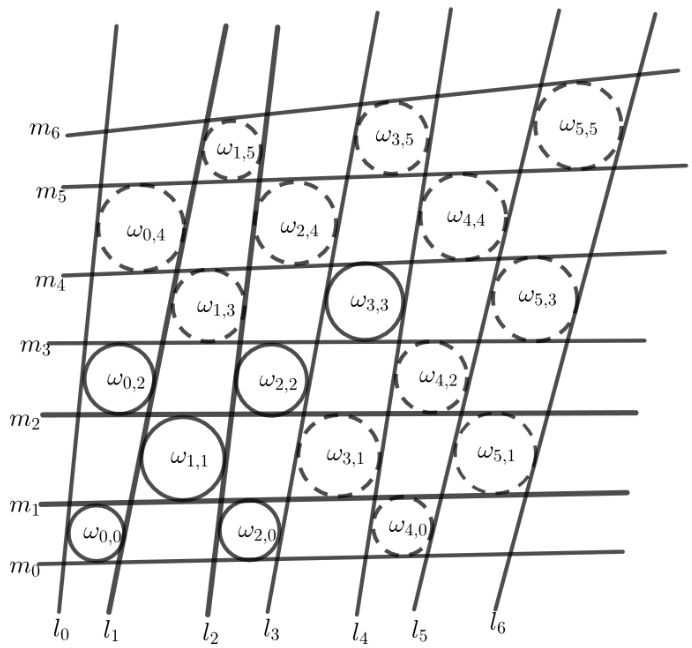


Figure 3.17: Construction of a checkerboard IC net.

quadrilateral $\square_{i,j}^{c+2}$ by two pairs of lines so that we can have nine pieces of quadrilaterals. $\square_{i,j}$, $\square_{i,j+c+1}$, $\square_{i+c+1,j+c+1}$ and $\square_{i+c+1,j}$ are the quadrilaterals at the corners. Since they are unit net-squares, they are circumscribed. The quadrilateral at the center is $\square_{i+1,j+1}^{c}$. This quadrilateral is also circumscribed by the induction hypothesis. Applying Lemma 33, the quadrilateral $\square_{i,j}^{c+2}$ is thus circumscribed. Therefore, all net-squares are circumscribed.

2. The second property of the checkerboard IC net is proved in the same way as the property of perspective of two net-squares of the IC net that we have seen in the Theorem 31. ■

Before stating and proving the Ivory-type theorem of the checkerboard IC net, let us define first the distance between two unit net-squares of the checkerboard IC net.

Definition 21 ([1], Theorem 3.1 (iv)). *The distance between two unit net-squares $\square_{a,b}$ and $\square_{c,d}$ of a checkerboard IC net, denoted by $d_C(\square_{a,b}, \square_{c,d})$, is the distance between the tangency points on a common exterior tangent line to the respective inscribed circles $\omega_{a,b}$, $\omega_{c,d}$ in $\square_{a,b}$ and in $\square_{c,d}$. If $a = c$ or $b = d$, then these lines are the lines of the checkerboard IC net.*

Theorem 37 ([1], Theorem 3.1 (iv)). *For any* $(i,j) \in \mathbb{Z}^2$*, with* $i+j$ *is even, and any even integer* c*, the checkerboard IC net* f *satisfies*

$$d_C(\Box_{i-c,j}, \Box_{i+c,j}) = d_C(\Box_{i,j-c}, \Box_{i,j+c}). \tag{3.8}$$

Proof. Consider a net-square $\Box_{i,j}^{c+1}$. Let us divide this net-square by two pairs of lines into nine quadrilaterals such that the four quadrilaterals at corners are unit net-squares and the one at the center is net-square. According to the definition of the checkerboard IC net and the first property in the Theorem 36, these five quadrilaterals are circumscribed. Due to Lemma 33, the net-square $\Box_{i,j}^{c+1}$ is circumscribed. In other words, the sums of the lengths of its opposite sides are equal. Indeed, the sum of the lengths of two opposite sides of $\Box_{i,j}^{c+1}$ is equal to $d_C(\Box_{i,j}, \Box_{i+c,j}) + d_C(\Box_{i,j+c}, \Box_{i+c,j+c})$ plus the sum of the lengths of the intervals from the corners of $\Box_{i,j}^{c+1}$ to the touching points with the inscribed circles of the corresponding corner unit net-squares. Then, we have

$$d_C(\Box_{i,j}, \Box_{i+c,j}) + d_C(\Box_{i,j+c}, \Box_{i+c,j+c}) = d_C(\Box_{i,j}, \Box_{i,j+c}) + d_C(\Box_{i+c,j}, \Box_{i+c,j+c}).$$

Applying this equality to the circumscribed net-squares $\Box_{i-c,j-c}^{c+1}$, $\Box_{i-c,j}^{c+1}$, $\Box_{i,j-c}^{c+1}$, $\Box_{i,j}^{c+1}$ and $\Box_{i-c,j-c}^{2c+1}$, we have the following equations

$$d_C(\Box_{i-c,j-c}, \Box_{i,j-c}) + d_C(\Box_{i-c,j}, \Box_{i,j}) = d_C(\Box_{i-c,j-c}, \Box_{i-c,j}) + d_C(\Box_{i,j-c}, \Box_{i,j}) \tag{3.9}$$

$$d_C(\Box_{i,j-c}, \Box_{i+c,j-c}) + d_C(\Box_{i,j}, \Box_{i+c,j}) = d_C(\Box_{i,j-c}, \Box_{i,j}) + d_C(\Box_{i+c,j-c}, \Box_{i+c,j}) \tag{3.10}$$

$$d_C(\Box_{i,j-c}, \Box_{i+c,j-c}) + d_C(\Box_{i,j}, \Box_{i+c,j}) = d_C(\Box_{i,j-c}, \Box_{i,j}) + d_C(\Box_{i+c,j-c}, \Box_{i+c,j}) \tag{3.11}$$

$$d_C(\Box_{i,j}, \Box_{i+c,j}) + d_C(\Box_{i,j+c}, \Box_{i+c,j+c}) = d_C(\Box_{i,j}, \Box_{i,j+c}) + d_C(\Box_{i+c,j}, \Box_{i+c,j+c}) \tag{3.12}$$

$$d_C(\Box_{i-c,j-c}, \Box_{i+c,j-c}) + d_C(\Box_{i-c,j+c}, \Box_{i+c,j+c}) = d_C(\Box_{i-c,j-c}, \Box_{i-c,j+c}) + d_C(\Box_{i+c,j-c}, \Box_{i+c,j+c}) \tag{3.13}$$

Therefore, we obtain

$$\begin{aligned}
d_C(\Box_{i-c,j}, \Box_{i+c,j}) &= d_C(\Box_{i-c,j}, \Box_{i,j}) + d_C(\Box_{i,j}, \Box_{i+c,j}) \\
&= d_C(\Box_{i-c,j-c}, \Box_{i-c,j}) + d_C(\Box_{i,j-c}, \Box_{i,j}) - d_C(\Box_{i-c,j-c}, \Box_{i,j-c}) \\
&+ d_C(\Box_{i+c,j-c}, \Box_{i+c,j}) - d_C(\Box_{i,j-c}, \Box_{i+c,j-c}) \\
&\text{by (3.9) and (3.10)} \\
&= d_C(\Box_{i-c,j-c}, \Box_{i-c,j}) + d_C(\Box_{i,j-c}, \Box_{i,j}) - d_C(\Box_{i-c,j-c}, \Box_{i,j-c}) \\
&+ d_C(\Box_{i,j-c}, \Box_{i+c,j-c}) + d_C(\Box_{i,j}, \Box_{i+c,j}) - d_C(\Box_{i+c,j-c}, \Box_{i+c,j}) \\
&+ d_C(\Box_{i+c,j-c}, \Box_{i+c,j}) - d_C(\Box_{i,j-c}, \Box_{i+c,j-c}) \text{ by (3.11)} \\
&= d_C(\Box_{i-c,j-c}, \Box_{i-c,j}) + d_C(\Box_{i,j-c}, \Box_{i,j}) - d_C(\Box_{i-c,j-c}, \Box_{i,j-c}) \\
&+ d_C(\Box_{i,j-c}, \Box_{i+c,j-c}) + d_C(\Box_{i,j}, \Box_{i,j+c}) + d_C(\Box_{i+c,j}, \Box_{i+c,j+c}) \\
&- d_C(\Box_{i,j+c}, \Box_{i+c,j+c}) - d_C(\Box_{i+c,j-c}, \Box_{i+c,j}) + d_C(\Box_{i+c,j-c}, \Box_{i+c,j}) \\
&- d_C(\Box_{i,j-c}, \Box_{i+c,j-c}) \text{ by (3.12)} \\
&= d_C(\Box_{i,j-c}, \Box_{i,j+c}) + d_C(\Box_{i-c,j-c}, \Box_{i-c,j}) - d_C(\Box_{i-c,j-c}, \Box_{i,j-c}) \\
&+ d_C(\Box_{i,j-c}, \Box_{i+c,j-c}) + d_C(\Box_{i+c,j}, \Box_{i+c,j+c}) - d_C(\Box_{i,j+c}, \Box_{i+c,j+c}) \\
&- d_C(\Box_{i+c,j-c}, \Box_{i+c,j}) + d_C(\Box_{i+c,j-c}, \Box_{i+c,j}) - d_C(\Box_{i,j-c}, \Box_{i+c,j-c}) \text{ by (3.13)} \\
&= d_C(\Box_{i,j-c}, \Box_{i,j+c}).
\end{aligned}$$

■

Theorem 38 ([1], Theorem 3.1). *Let f be a checkerboard. The following properties hold:*

1. *The points $f_{i,j}$, where the integers i and j satisfy the condition that $i + j$ is an odd constant, lie on a conic. This result is valid for the points $f_{i,j}$ with $i - j$ is an even constant.*

2. *The centers $o_{i,j}$ of the inscribed circles of a checkerboard IC net build a circle-conical net.*

Proof. 1. We will show only the case where $i-j=0$, since the proof of other cases is similar. For simplicity, we will show that any successive points lie on some conic. Due to Theorem 7, it is sufficient to prove that the intersection of the lines $(f_{0,0}f_{1,1})$ and $(f_{3,3}f_{4,4})$, $(f_{1,1}f_{2,2})$ and $(f_{4,4}f_{5,5})$, and $(f_{2,2}f_{3,3})$ and $(f_{5,5}f_{0,0})$ are collinear.

i. (a) Consider the net-squares $\square_{0,0}$ and $\square^3_{0,0}$. Then, the point $f_{0,0}$ is the center of homethety of the inscribed circles in these net-squares.

(b) The point $f_{1,1}$ is the homothety center of the inscribed circles of the unit-net-square $\square_{0,0}$ and the net-square $\square^3_{1,1}$.

Since the homethety centers in the case (a) and in the case (b) do not coincide, due to Theorem 22, the line $(f_{0,0}f_{1,1})$ passes through the homothety center of the inscribed circles of the net-squares $\square^3_{0,0}$ and $\square^3_{1,1}$.

ii. (c) The point $f_{4,4}$ is the center of the homothety of the inscribed circles of the net-squares $\square_{3,3}$ and $\square^3_{1,1}$.

(d) The point $f_{3,3}$ is the center of the homothety of the inscribed circles of the net-squares $\square_{3,3}$ and $\square^3_{0,0}$.

Applying again the theorem of three centers of similarity, we obtain that the line $(f_{3,3}f_{4,4})$ passes through the homothety center of the net-squares $\square^3_{1,1}$ and $\square^3_{0,0}$.

The combination of the results that we got in (i) and (ii) above proves that the point $(f_{0,0}f_{1,1}) \cap (f_{3,3}f_{4,4})$ is the homothety center of the net-squares $\square^3_{1,1}$ and $\square^3_{0,0}$. Using the foregoing arguments, we can also prove that the points $(f_{1,1}f_{2,2}) \cap (f_{4,4}f_{5,5})$ and $(f_{2,2}f_{3,3}) \cap (f_{5,5}f_{0,0})$ are the respective homothety centers of the inscribed circles of the pairs of net-squares $\square^3_{1,1}$ and $\square^3_{2,2}$, and $\square^3_{0,0}$ and $\square^3_{2,2}$. Finally, the three centers of the homothety lie on a straight line according to Monge's theorem. This shows the converse of Pascal's theorem.

2. It is straightforward to verify that $d_C(\square_{i-1,j+1}, \square_{i+1,j+1}) = d_C(\square_{i,j}, \square_{i,j+2})$. This implies that the distance between the points $o_{i-1,j+1}$ and $o_{i+1,j+1}$ is equal to the distance of the points

$o_{i,j}$ and $o_{i,j+2}$. Then, there exists a circle passing through these points. Therefore, the center $o_{i,j}$ of the inscribed circles of a checkerboard IC net build a circular net.

We have just proved that the centers $o_{i,j}$ of the inscribed circles of a checkerboard IC net form a circular quadrilateral. The sum of all angles at one vertex $o_{i,j}$ is equal 2π. Therefore, if α_1, α_2, α_3 and α_4 are the measure of the angles around the vertex $o_{i,j}$ in the cyclic order, then $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = 2\pi$. This implies that $\alpha_1 + \alpha_3 = \alpha_2 + \alpha_4 = \pi$, which means that the net formed by the centers $o_{i,j}$ of the incircles of a checkerboard IC net is conical. ■

CHAPTER 4

CONFOCAL IC NETS AND CONFOCAL CHECKERBOARD IC NETS BASED ON INTEGRABLE BILLIARDS

The main objective of this chapter is to show that a confocal IC net and a checkerboard IC net can be constructed by using two different billiard trajectories within the same boundary and with the same caustic. In our context, these billiard trajectories are winding either in the same direction or in the opposite directions. A brief description of a mathematical billiard in a plane will be discussed in the first section of this chapter and will be used in the construction of the confocal IC net and the checkerboard IC net [18].

4.1 Billiards inside conics and the Poncelet grid

Mathematical billiard consists of a domain and a point mass that moves freely inside of the domain. The domain and the point mass of the mathematical billiard are called a billiard table and a billiard ball respectively [8, 9, 24, 33] and [35].

Definition 22 ([13], page 377)**.** *The motion of a point that moves freely with a constant speed along a line segment inside of a closed region and reflects elastically at the boundary is called billiard system. At the bouncing point, the incoming line segment and the outgoing line segment obey the geometrical law of optics, that is, the angle of incidence and the angle of reflection are equal, depicted in Figure 4.1 and in Figure 4.2.*

Definition 23 ([13], page 381)**.** *A caustic of a billiard trajectory is a curve inside a billiard table such that if an incoming segment of a billiard trajectory is tangent to such curve, then each reflected segment is also tangent to it.*

We assume that the billiard table and the caustic are smooth and convex.

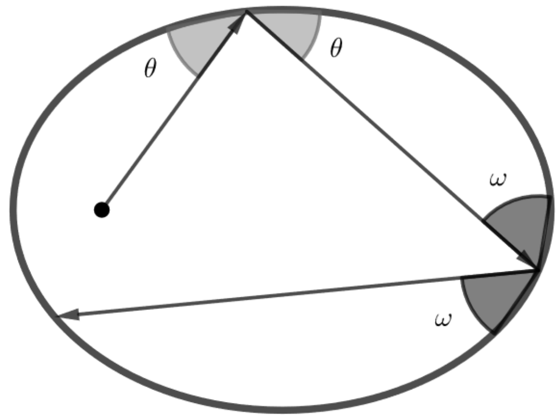


Figure 4.1: Billiard reflection.

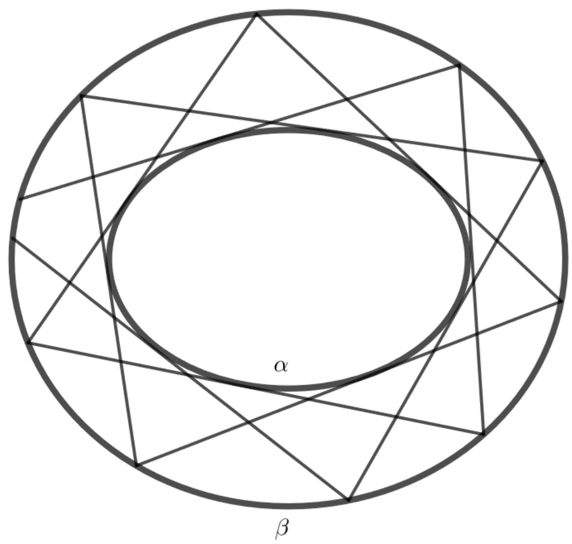


Figure 4.2: The billiard table β and the caustic α are confocal ellipses.

Theorem 39 ([35], Theorem 4.4)**.** *A billiard trajectory within an ellipse forever remains tangent to a fixed conic which is confocal with the ellipse (see Figure 4.3). In other words, if a segment of a billiard trajectory does not intersect the segment* $\mathsf{F}_1\mathsf{F}_2$*, then all segments in this trajectory do not intersect the segment* $\mathsf{F}_1\mathsf{F}_2$ *and are all tangent to the same caustic, which is an ellipse with foci* F_1 *and* F_2*. Likewise, if a segment of a billiard trajectory intersects the segment* $\mathsf{F}_1\mathsf{F}_2$*, then all segments of this trajectory intersect the segment* $\mathsf{F}_1\mathsf{F}_2$ *and are all tangent to the same hyperbola with foci* F_1 *and* F_2*.*

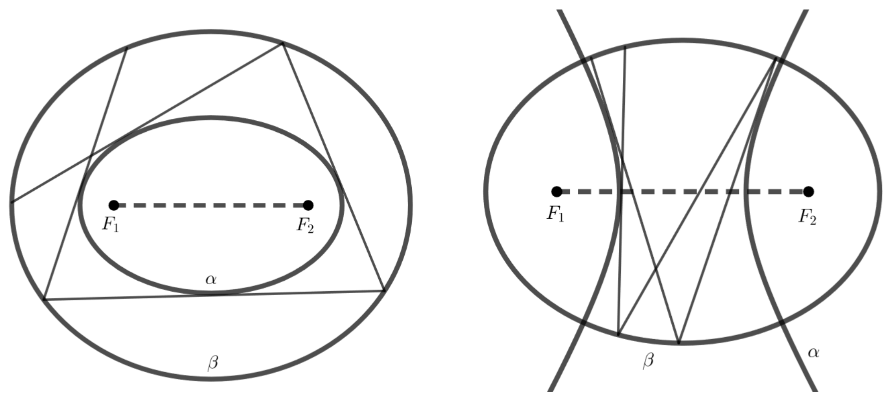


Figure 4.3: Billiard trajectory within ellipse β and the caustic is α.

Proof. The proof follows the exposition from [13]. Let AA_1 and A_1A_2 be two consecutive segments of a trajectory such that they do not intersect the segment F_1F_2. According to

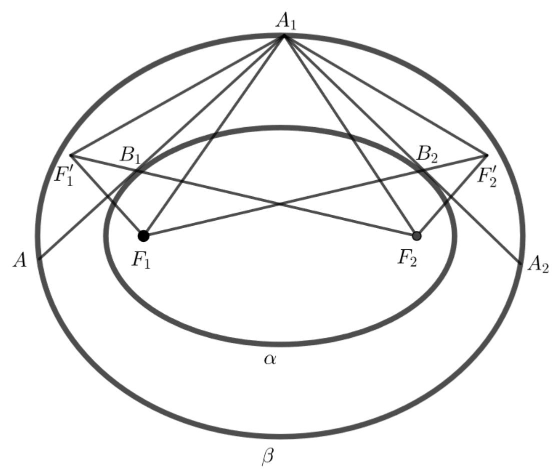


Figure 4.4: Proof of the theorem

Theorem 1, the angles made by the segments F_1A_1 and F_2A_1 with ellipse β are equal. Using again Theorem 1, we can conclude that the segments AA_1 and A_2A_1 make equal angles with β. Then, $\angle AA_1F_1 = \angle F_2A_1A_2$. Let F_1' be the reflection of F_1 in AA_1 and F_2' be the reflection of F_2 in A_1A_2. Denote $B_1 = (AA_1) \cap (F_1'F_2)$ and $B_2 = (A_1A_2) \cap (F_1F_2')$. Let α_1 (resp. α_2) be the ellipse, with foci F_1 and F_2, that is tangent to the segment AA_1 (resp. A_1A_2) at the point

B_1 (resp. B_2). Since $\angle F_2B_1A_1 = \angle AB_1F_1'$ and $\angle AB_1F_1' = \angle AB_1F_1$, $\angle AB_1F_1 = \angle F_2B_1A_1$. This shows that the line (AA_1) meets the ellipse α_1 at the point B_1. Similarly, the line (A_1A_2) meets the ellipse α_2 at the point B_2. It remains to prove that the ellipses α_1 and α_2 coincide. By Theorem 1, $|F_1'F_2| = |F_1F_2'|$. It follows that α_1 and α_2 coincide, which is denoted by α in the Figure 4.4. ■

Definition 24 ([33]). *A polygon $\mathcal{P}$ is called Poncelet polygon if the polygon $\mathcal{P}$ is inscribed in one ellipse and circumscribed about another ellipse.*

Note that all the edges of the Poncelet polygon $\mathcal{P}$ are tangent to the inner ellipse and its vertices lie on the outer ellipse.

Definition 25 ([26, 33]). *Given a Poncelet polygon $\mathcal{P}$ and let $l_1, l_2, \cdots, l_n$ be lines that contain the edges of $\mathcal{P}$. A set of points of the form $l_i \cap l_j$ is called Poncelet grid. When $i = j$, we define $l_i \cap l_j$ to be the tangency point of the line l_i with the caustic.*

Define sets

$$P_k = \bigcup_{i-j=k} l_i \cap l_j, \quad Q_k = \bigcup_{i+j=k} l_i \cap l_j. \tag{4.1}$$

Theorem 40 (Schwartz theorem). *The sets P_k and Q_k have the following properties:*

1. *The sets P_k lie on the same nested ellipses, and the sets Q_k lie on disjoint hyperbolas.*

2. *The complexified version of these ellipses and hyperbolas have four common complex tangent lines.*

3. *All the sets P_k are projectively equivalent to each other, and likewise, all the sets Q_k are projectively to each other.*

Proof. See for instance [24, 26, 33]. ■

4.2 Poncelet-Darboux grids in Euclidean plane

Theorem 41 and Theorem 42 are generalizations of Darboux theorems related to the periodic billiards trajectories within a conic.

Theorem 41 (V. Dragović, M. Radnović). *Let α be an ellipse in the Euclidean plane. Let $(l_m)_{m\in\mathbb{Z}}$ and $(t_m)_{m\in\mathbb{Z}}$ be two sequences of segments of billiard trajectories within α and they also share the same caustic κ. The following results hold:*

1. *All the points $l_m \cap t_m$ belong to one conic β confocal with α.*

2. *If the conic κ is an ellipse and both trajectories are winding in one direction about κ, then β is an ellipse.*

3. *If the conic κ is an ellipse and both trajectories are winding in opposite directions about κ, then β is an hyperbola.*

4. *If the caustic κ is a hyperbola and the segments l_m and t_m intersect the major axis of α in the same direction, then β is a hyperbola.*

5. *If the caustic κ is a hyperbola and the segments l_m and t_m intersect the major axis of α in the opposite directions, then β is a an ellipse.*

Proof. We will prove the theorem by induction on m. Since the lines l_0 and t_0 intersect at a point, this point belongs exactly to one ellipse and one hyperbola from the confocal family. Knowing the orientation of the billiard motion along the lines, those lines satisfy the reflection law on one of these two conics. Applying the Theorem 19, the lines l_1 and t_1 obey the reflection law on the same conic. By the induction hypothesis, the lines l_{m-1} and t_{m-1} satisfy the reflection law on a conic. Due to the Theorem 19, the lines l_m and t_m satisfy also the reflection law one the same conic. This proves (1).

For the remaining statements, it suffices to observe the winding direction of both trajectories. Note that the winding direction about an ellipse is changed by the reflections on the hyperbolas, and preserved by the reflections on the ellipses from the confocal family. If an oriented line is placed between the foci of conic, then this line intersects the axis containing the foci of the conic. The direction of this trajectory is changed by the reflections on the ellipses and preserved by the reflections of the hyperbolas. ■

The Theorem 42 is more general than the Theorem 40 because in Theorem 42, the billiard trajectory is supposed not to be closed.

Theorem 42 (V. Dragović, M. Radnović). *Let $(a_m)_{m\in\mathbb{Z}}$ be the sequence of segments of a billiard trajectory within the ellipse α. Define the sets*

$$P_k = \bigcup_{i-j=k} a_i \cap a_j \quad \textit{and} \quad Q_k = \bigcup_{i+j=k} a_i \cap a_j \quad \textit{with} \quad k \in \mathbb{Z}.$$

Then, the following results hold:

1. *Each of the sets P_k and Q_k belongs to a single conic confocal with α.*
2. *If the caustic κ of the trajectory a_m is an ellipse, the sets P_k are situated on ellipses and Q_k on hyperbolas.*
3. *If the caustic κ of the trajectory a_m is an hyperbola, then the sets P_k and Q_k are situated on ellipses for k even and on hyperbolas for k odd.*

Proof. For the sets P_k, choose $l_m = a_m$ and $t_m = a_{m+k}$ and apply Theorem 41. Similarly, for the sets Q_k, choose $l_m = a_m$ and $t_m = a_{m-k}$ and apply again Theorem 41. ■

4.3 Confocal IC nets and checkerboard IC nets related to billiard trajectories

The aim of this current section is to present a new method of constructing a confocal IC net starting from two different billiard trajectories within the same conic and sharing the same caustic (see Figure 4.5). We do not assume that these trajectories are periodic. They are also either winding in the same direction or in the opposite directions. And then, we will show how to obtain a confocal checkerboard IC net from the confocal IC net [1, 18].

Definition 26 ([1], Definition 3.2)**.** *An IC net (a checkerboard IC net) is called confocal if all lines of it are tangent to the caustic.*

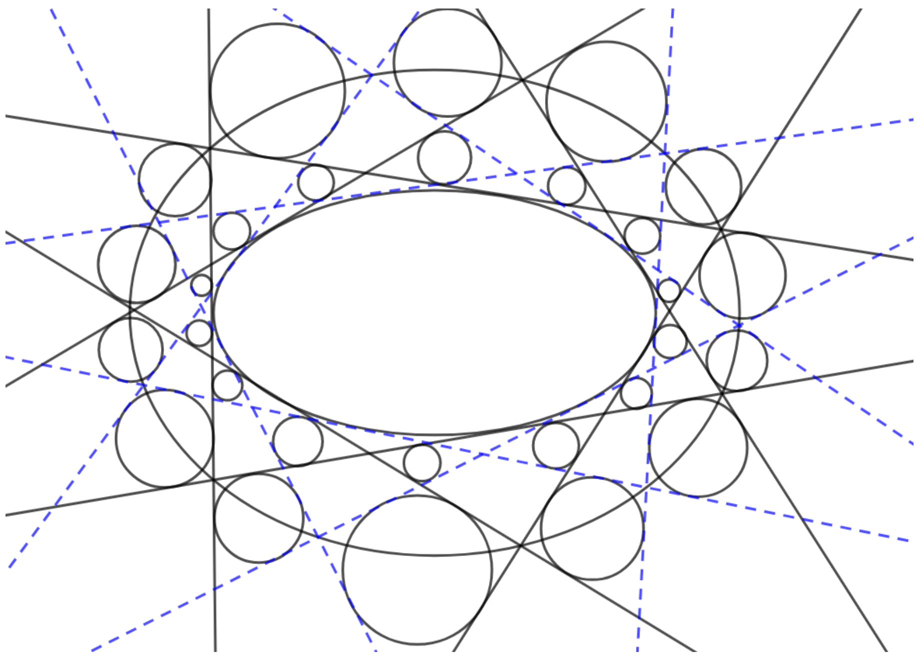


Figure 4.5: Confocal checkerboard IC net.

Theorem 43. *Let α be an ellipse in $\mathbb{E}^2$, and $(l_m)_{m\in\mathbb{Z}}$ and $(t_m)_{m\in\mathbb{Z}}$ be two sequences of the segments of billiard trajectories within α, sharing the same caustic κ. Consider a quadrilateral $(ABCD)$ made by two billiard trajectories (see Figure 4.6). This quadrilateral is divided into four small quadrilaterals. If the small quadrilaterals at the opposite corners are circumscribed, then the quadrilateral $(ABCD)$ is also circumscribe*

Proof. Since both billiard trajectories $(l_m)_{m\in\mathbb{Z}}$ and $(t_m)_{m\in\mathbb{Z}}$ share the same caustic and have the same boundary, the points of the form $l_m \cap l_{m+1}$ and $t_m \cap t_{m+1}$ lie on the boundary. It follows that the quadrilateral formed by the lines l_m, l_{m+1}, t_m and t_{m+1} is circumscribed

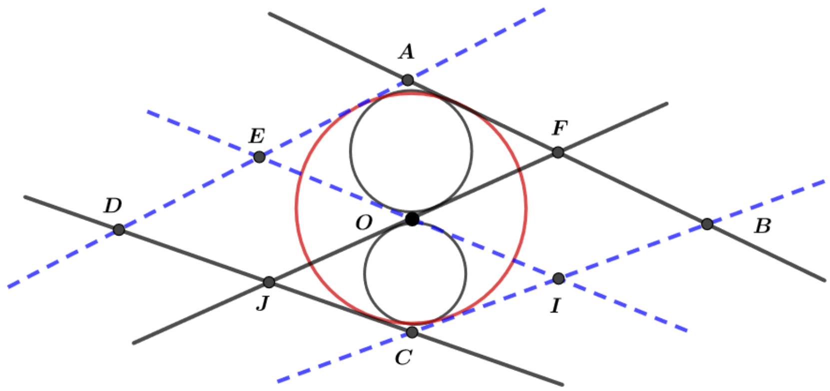


Figure 4.6: If two small quadrilaterals at the corner are circumscribed, then big quadrilateral is circumscribed.

according to the Grave-Chasles theorem. According to the Darboux theorem on grids, the intersection of n^{th} and m^{th} sides of all such trajectories belong to an ellipse. Applying again the Graves Chasles theorem, the other quadrilateral is also circumscribed. In the Figure 4.6, all lines are tangent to the same caustic κ since they are lines from the the billiard trajectories. Now, let us denote by x_1, x_2, x_3, y_1, y_2 and y_3 the coordinates of the points tangency of the tangent lines $(BC), (FJ), (AD), (AB), (EI)$ and (CD) respectively on the caustic κ so that $A(x_3, y_1)$, $B(x_1, y_1)$, $C(x_1, y_3)$, $D(x_3, y_3)$, $E(x_3, y_2)$, $F(x_2, y_1)$, $I(x_1, y_2)$, $J(x_2, y_3)$ and $O(x_2, y_2)$. Suppose that the quadrilaterals $(AEOF)$ and $(OJCI)$ are circumscribed. Since the quadrilateral $(AEOF)$ is circumscribed, due to the Graves- Chasles theorem, the points E and F lie on the same conic confocal with κ, say β. Suppose that both billiard trajectories are winding in the same direction. Then, β is an ellipse according to the Theorem 41. It implies that the points A and O lie on the same hyperbola. Therefore, we obtain the following equations

$$y_2 - x_3 = y_1 - x_2, \tag{4.2a}$$

$$y_1 + x_3 = y_2 + x_2. \tag{4.2b}$$

Notice that the equations (4.2a) and (4.2b) are identical. Using the argument above to the circumscribed quadrilateral $(OJCI)$, we also have

$$y_2 - x_1 = y_3 - x_2, \tag{4.3a}$$

$$y_2 + x_2 = y_3 + x_1. \tag{4.3b}$$

Subtracting the equations (4.2a) and (4.3a), we have $y_1 + x_3 = y_3 - x_1$. This equation shows us that the points A and C lie on the same hyperbola. Thus, the quadrilateral $(ABCD)$ is circumscribed according to the Graves-Chasles Theorem. ∎

Remark 44. The result of the Theorem 43 does not change if we add more additional assumptions such as fixing the direction of both trajectories and specifying type of the caustic. For instance, if the caustic is an ellipse, then the conic β is either an ellipse or a hyperbola depending on the direction of the billiard trajectories. Likewise, if the caustic is a hyperbola and the both segments l_m and t_m intersect along the major axis of the boundary α in the same direction, then β is a hyperbola; otherwise it is an ellipse. Moreover, the type of the caustic and the direction of billiard trajectories determine the position of small circles in the Figure 4.6.

Theorem 45. *We use the same assumptions and terminology of the Theorem 43. The billiard trajectories $(l_m)_{m\in\mathbb{Z}}$ and $(t_m)_{m\in\mathbb{Z}}$ form a confocal IC net. The same trajectories also form a confocal checkerboard IC net.*

Proof. First of all, notice that all lines l_m and t_n are tangent to the caustic κ. The first condition in the definition of IC net is clear because the orders of the points on the lines of confocal IC net $\mathtt{l}_\mathtt{m}$ and $\mathtt{t}_\mathtt{m}$ are preserved (this result still holds for a confocal checkerboard IC net). The circumscribability of the unit net-squares of those confocal nets follows from the Theorem 43. Therefore, the billiard trajectories $(\mathtt{l}_\mathtt{m})_{\mathtt{m}\in\mathbb{Z}}$ and $(\mathtt{t}_\mathtt{m})_{\mathtt{m}\in\mathbb{Z}}$ form a confocal checkerboard IC net and also a confocal checkerboard IC net. ■

CHAPTER 5

INTEGRABLE LINES CONGRUENCES AND NETS IN HIGHER - DIMENSIONAL SPACE

We are concerned in this chapter with describing and investigating discrete systems which arise from dynamics of billiards within confocal quadrics [16]. The dynamics of billiards within confocal quadrics are linked into the concept of integrable quad-graphs [17]. The theory of integrable systems on quad-graphs can be traced back to [3]. A quad-graph interpretation of some results obtained from billiard algebra presented here leads us to discrete systems defined on a non-cubic lattice [18, 28]. We also present generalizations of the IC nets and the checkerboard IC nets in a space of dimension 3, inspired from the work of Böhm [39] and the work of Akopyan and Bobenko [1]. Our new result is to give a proof of the following statement: a division of 3-dimensional Euclidean space by planes into circumscribed cuboids consists of three families of planes such that all planes in the same family intersect along a line, and the three lines are coplanar (Theorem 54). Then we generalize this statement to 4-dimensional case and prove it.

5.1 Discrete line congruences and focal net

Let us begin with the basic results about the theory of discrete integrable systems on the quad-graphs, which has been developed by Adler, Bobenko and Suris [3]. Equations on quadrilaterals of the form

$$Q(x_1, x_2, x_3, x_4) = 0, \tag{5.1}$$

where Q is a polynomial of degree one in each variable, are called *quad-equations*. The field variables x_1, x_2, x_3 and x_4 are assigned to the vertices of a quadrilateral, as depicted in the Figure 5.1.

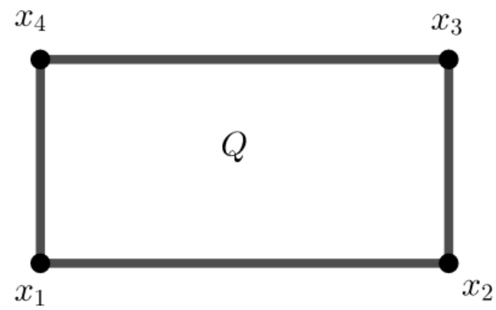


Figure 5.1: Quad-equation $Q(x_1, x_2, x_3, x_4) = 0$.

The function Q satisfies the linearity assumption and is an irreductible polynomial of degree one in each variable, then the equation 5.1 can be solved for each variable. The solution of this equation is also a rational function of the other three variables. Let (a, b, c, d) be a solution of the equation 5.1. This solution is called *singular* with respect to x_i if the following equation holds

$$\frac{\partial Q}{\partial x_i}(a, b, c, d) = 0. \tag{5.2}$$

Now, let us present the notion of 3D-consistency. In this case, six different quad-equations are assigned to the faces of coordinate cube (see Figure 5.2).

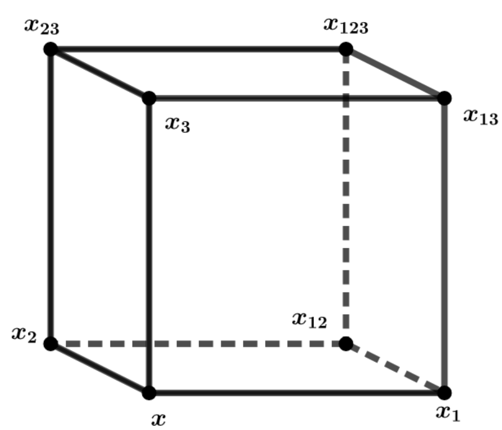


Figure 5.2: 3D-consistency

Given x, x_1, x_2 and x_3, we can determine the values of x_{12}, x_{13}, and x_{23} from the three following quad-equations

$$Q(x, x_1, x_{12}, x_2) = 0, \quad Q(x, x_1, x_{13}, x_3) = 0, \quad Q(x, x_2, x_{23}, x_3) = 0.$$

And then, the value of x_{123} can be determined by the remaining of quad-equations, which are

$$Q(x_1, x_{12}, x_{123}, x_{13}) = 0, \quad Q(x_2, x_{12}, x_{123}, x_{23}) = 0, \quad Q(x_3, x_{12}, x_{123}, x_{23}) = 0.$$

Notice that the value of x_{123} obtained for each quad-equation may be different. For given data x, x_1, x_2 and x_3, we say that the system is *3D-consistent* if the value of x_{123} obtained from last three quad-equations above coincide.

From now, our interest focuses on the geometric version of the quad-graphs. Lines in $\mathbb{P}^3$ play the role of the vertex fields. Let us denote by $\mathcal{L}^3$ the space of lines in $\mathbb{RP}^3$.

Definition 27 ([3], Definition 2.9). *A 3-dimensional discrete line congruence in* $\mathbb{RP}^3$ *is a map* $l : \mathbb{Z}^3 \longrightarrow \mathcal{L}^3$ *satisfying that every two neighboring lines* $l(n_0)$ *and* $l(n_0 + e_i)$ *intersect for every* $n_0 \in \mathbb{Z}^3$ *and for each* $i \in \{1, 2, 3\}$*, equivalently, they are coplanar.*

Notice that discrete line congruence in two-dimensional is called *generic* if any pairs of lines such as l and l_{12}, and l_1 and l_2 do not intersect. This implies that each of these pairs span a three-dimensional space such that the lines l, l_1, l_2 and l_{12} lie in that space. This result is still true in higher-dimensional space. The generalization of the lines congruences and the way it is constructed in m-dimensional space, with $m \geq 2$, can be found in [3].

Any two neighboring lines $l(n_0)$ and $l(n_0 + e_i)$ intersect exactly at one point in $\mathbb{RP}^3$. We denote the point of intersection of two neighboring lines by $F^{(i)} = F(n_0, n_0 + e_i) = l(n_0) \cap l(n_0 + e_i)$.

Definition 28 (Focal net). *Given a discrete line congruence* $l : \mathbb{Z}^3 \longrightarrow \mathcal{L}^3$*, the map* $F^{(i)} : \mathbb{Z}^3 \longrightarrow \mathcal{L}^3$ *defined by* $F^{(i)} = l(n_0) \cap l(n_0 + e_i)$ *is called its i-th focal net.*

The focal net has the following property.

Theorem 46 ([3], Theorem 2.14). *Given a discrete line congruence* $\mathfrak{l} : \mathbb{Z}^3 \longrightarrow \mathcal{L}^3$*, all its focal nets* $F^{(k)} : \mathbb{Z}^3 \longrightarrow \mathcal{L}^3$*, with* $k \in \{1, 2, 3\}$*, are* Q*-nets.*

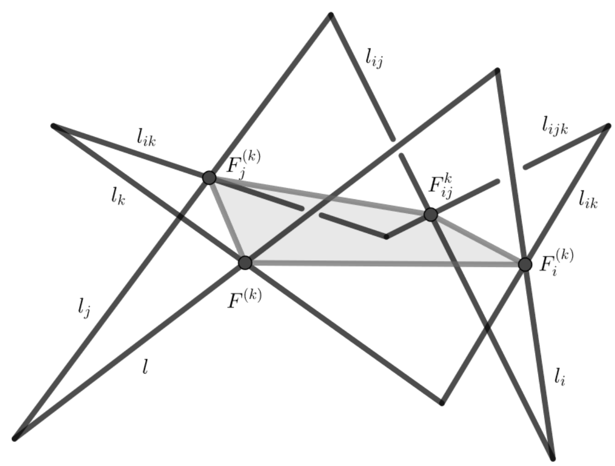


Figure 5.3: Quadrilateral of k-th focal net.

Proof. For the k-th focal net $F^{(k)}$, we first need to prove that all quadrilaterals of the form $(F^{(k)}F_i^{(k)}F_{ik}^{(k)}F_k^{(k)})$, with $i \neq k$, are planar (see Figure 5.3). By definition, we have

$$F^{(k)} = \mathfrak{l}(n_0) \cap \mathfrak{l}(n_0 + e_k), \quad F_k^{(k)} = \mathfrak{l}(n_0 + e_k) \cap \mathfrak{l}(n_0 + 2e_k),$$
$$F_i^{(k)} = \mathfrak{l}(n_0 + e_i) \cap \mathfrak{l}(n_0 + e_i + e_k), \quad F_{ik}^{(k)} = \mathfrak{l}(n_0 + e_i + e_k) \cap \mathfrak{l}(n_0 + e_i + 2e_k).$$

Then, the pairs of points $F^{(k)}$ and $F_k^{(k)}$, and $F_i^{(k)}$ and $F_{ik}^{(k)}$ lie respectively on the lines $\mathfrak{l}_k$ and $\mathfrak{l}_{ik}$. Since the lines $\mathfrak{l}_k$ and $\mathfrak{l}_{ik}$ intersect, they span a plane. Therefore, all the four points lie in the same plane spanned by the lines $\mathfrak{l}_k$ and $\mathfrak{l}_{ik}$.

Next, for $i \neq j$ and $j \neq k$, we want to show that all quadrilaterals of the form $(F^{(k)}F_i^{(k)}F_{ij}^{(k)}F_j^{(k)})$ are planar. We suppose that the line congruence $\mathfrak{l}$ is generic. Then, all four points lie in each of the three-dimensional spaces

$$S_{ij} = span(\mathfrak{l}, \mathfrak{l}_i, \mathfrak{l}_j, \mathfrak{l}_{ij}) \quad \text{and} \quad \tau_k S_{ij} = span(\mathfrak{l}_k, \mathfrak{l}_{ik}, \mathfrak{l}_{jk}, \mathfrak{l}_{ijk}),$$

where τ_k is a translation in a standard manner. Notice that both three-dimensional spaces above lie in the four-dimensional space $S_{ijk} = \mathrm{span}(l, l_i, l_j, l_k)$. Thus, the intersection of these spaces is a plane. Therefore, the four points lie in such plane. ■

5.2 Double reflection nets

We assume that a family of quadrics is given in $\mathbb{P}^3$. The Theorem 16 states that any line in $\mathbb{P}^3$ touches two quadrics from the family. Let $\mathcal{L}$ be the set of all lines touching the two fixed quadrics from the pencil.

Definition 29 ([17], Definition 5.1). *A map $\varphi : \mathbb{Z}^3 \to \mathcal{L}$ is called a double reflection net, if there exist three quadrics Q_1, Q_2, Q_3 from the confocal pencil, and φ satisfies the following conditions:*

1. *a billiard trajectory within Q_j is represented by sequence $\{\varphi(n_0+ie_j)\}_{i\in\mathbb{Z}}$ for $j \in \{1,2,3\}$ and $n_0 \in \mathbb{Z}^3$,*
2. *lines $\varphi(n_0)$, $\varphi(n_0+e_i)$, $\varphi(n_0+e_j)$ and $\varphi(n_0+e_i+e_j)$ constitute a double reflection configuration, for all $i, j \in \{1,2,3\}$ with $i \neq j$ and $n_0 \in \mathbb{Z}^3$.*

The detailed construction of this double reflection nets can be found in [17]. Now, consider a double reflection net $\varphi : \mathbb{Z}^3 \to \mathcal{L}$, for given $n_0 \in \mathbb{Z}^3$ and three distinct indices $i, j, k \in \{1,2,3\}$, there exist in total twelve focal nets and they are expressed as follows

$$F_1 = \varphi(n_0) \cap \varphi(n_0+e_1), \quad F_{12} = \varphi(n_0+e_2) \cap \varphi(n_0+e_2+e_1),$$

$$F_{13} = \varphi(n_0+e_3) \cap \varphi(n_0+e_3+e_1), \quad F_{123} = \varphi(n_0+e_2+e_3) \cap \varphi(n_0+e_2+e_3+e_1),$$

$$F_2 = \varphi(n_0) \cap \varphi(n_0+e_2), \quad F_{21} = \varphi(n_0+e_1) \cap \varphi(n_0+e_1+e_2),$$

$$F_{23} = \varphi(n_0+e_3) \cap \varphi(n_0+e_3+e_2), \quad F_{213} = \varphi(n_0+e_1+e_3) \cap \varphi(n_0+e_1+e_3+e_2),$$

$$F_3 = \varphi(n_0) \cap \varphi(n_0+e_3), \quad F_{31} = \varphi(n_0+e_1) \cap \varphi(n_0+e_1+e_3),$$

$$F_{32} = \varphi(n_0+e_2) \cap \varphi(n_0+e_2+e_3), \quad F_{321} = \varphi(n_0+e_2+e_1) \cap \varphi(n_0+e_2+e_1+e_3).$$

These focal nets can be organized in three quadruplets of points in the following manner:

$$(F_1, F_{12}, F_{13}, F_{123}), \quad (F_2, F_{21}, F_{23}, F_{213}), \quad (F_3, F_{31}, F_{32}, F_{321}).$$

Notice that the points in each quadruplet are coplanar according to the Theorem 46 and also belong to the same quadric Q_1, Q_2 and Q_3 respectively. The second condition in the definition of the double reflection allows to express the twelve points above in the following form

$$
\begin{aligned}
F_1 &= \varphi(n_0) \cap \varphi(n_0 + e_1), \quad F_{12} = \varphi(n_0 + e_2) \cap \varphi(n_0 + e_1 + e_2), \\
F_{13} &= \varphi(n_0 + e_3) \cap \varphi(n_0 + e_1 + e_3), \quad F_{123} = \varphi(n_0 + e_2 + e_3) \cap \varphi(n_0 + e_1 + e_2 + e_3), \\
F_2 &= \varphi(n_0) \cap \varphi(n_0 + e_2), \quad F_{21} = \varphi(n_0 + e_1) \cap \varphi(n_0 + e_1 + e_2), \\
F_{23} &= \varphi(n_0 + e_3) \cap \varphi(n_0 + e_2 + e_3), \quad F_{213} = \varphi(n_0 + e_1 + e_3) \cap \varphi(n_0 + e_1 + e_2 + e_3), \\
F_3 &= \varphi(n_0) \cap \varphi(n_0 + e_3), \quad F_{31} = \varphi(n_0 + e_1) \cap \varphi(n_0 + e_1 + e_3), \\
F_{32} &= \varphi(n_0 + e_2) \cap \varphi(n_0 + e_2 + e_3), \quad F_{321} = \varphi(n_0 + e_1 + e_2) \cap \varphi(n_0 + e_1 + e_2 + e_3).
\end{aligned}
$$

It is straightforward from the construction of these points to see that

1. there are eight triplets of points such that the points in each triplets are collinear

$$
\begin{gathered}
(F_1, F_2, F_3), \quad (F_1, F_{21}, F_{31}), \quad (F_3, F_{13}, F_{23}), \quad (F_{21}, F_{12}, F_{321}), \\
(F_{321}, F_{213}, F_{123}), \quad (F23, F_{32}, F_{123}), \quad (F_2, F_{12}, F_{32}), \quad (F_{13}, F_{31}, F_{213}).
\end{gathered}
$$

2. there are eight lines that can be organized in six quadruplets and the lines in each quadruplet satisfy the double reflection configuration

$$\{\varphi(n_0), \varphi(n_0+e_1), \varphi(n_0+e_2), \varphi(n_0+e_1+e_2)\},$$
$$\{\varphi(n_0), \varphi(n_0+e_1), \varphi(n_0+e_3), \varphi(n_0+e_1+e_3)\},$$
$$\{\varphi(n_0+e_1), \varphi(n_0+e_1+e_2), \varphi(n_0+e_1+e_3), \varphi(n_0+e_1+e_2+e_3)\},$$
$$\{\varphi(n_0+e_2), \varphi(n_0+e_1+e_2), \varphi(n_0+e_2+e_3), \varphi(n_0+e_1+e_2+e_3)\},$$
$$\{\varphi(n_0), \varphi(n_0+e_2), \varphi(n_0+e_3), \varphi(n_0+e_2+e_3)\},$$
$$\{\varphi(n_0+e_3), \varphi(n_0+e_1+e_3), \varphi(n_0+e_2+e_3), \varphi(n_0+e_1+e_2+e_3)\}.$$

Theorem 47. *Let $\varphi : \mathbb{Z}^3 \to \mathcal{L}$ be a double reflection net, $n_0 \in \mathbb{Z}^3$ and three distinct indices $i, j, k \in \{1, 2, 3\}$. The points F_1, F_2, F_3, F_{12}, F_{13}, F_{21}, F_{23}, F_{31}, F_{32}, F_{123}, F_{213} and F_{321}, and the lines $\varphi(n_0), \varphi(n_0+e_1)$, $\varphi(n_0+e_2)$, $\varphi(n_0+e_1+e_2)$, $\varphi(n_0+e_3)$, $\varphi(n_0+e_1+e_3)$, $\varphi(n_0+e_2+e_3)$ and $\varphi(n_0+e_1+e_2+e_3)$ form a cuboid in dual projective space.*

Proof. A cuboid is composed of six faces that meet each other and has eight vertices and twelve edges. In our context, the eight lines represent the vertices and the twelve points correspond to the edges of the cuboid. The six quadruplets of lines represent the faces of the cuboid since the four lines in each quadruplet form a double reflection reflection. ■

A group structure on a set of lines in a 3-dimensional Euclidean space such that all lines in such set are simultaneously tangent to two quadrics from a given confocal family 2.15 is called *billiard algebra*. This billiard algebra can be constructed by utilizing the Theorem 19 and some other billiard constructions. Moreover, a fundamental property of such set of lines is that any two lines in that set can be obtained from each other by at most two billiard reflections at some quadrics from the confocal family [15]. The next theorem, called *Six-pointed star theorem*, is a quad-graph interpretation of some results obtained using billiard algebra. This theorem can be found in [16, 17, 18] and the references therein.

Theorem 48 (**Six-pointed star theorem**). *Let $\mathcal{F}$ be a family of confocal quadrics in projective space $\mathbb{P}^3$. Then there are configurations of twelve planes in projective space $\mathbb{P}^3$ having the following properties:*

1. *These planes are organized in eight triplets, such that each plane in a triplet is tangent to a different quadric from $\mathcal{F}$ and the three touching points are collinear. In addition, every plane in the configuration belongs to two different triplets.*

2. *These planes are also organized in six quadruplets: all planes in each quadruplets belong to the same pencil and are tangent to two different quadrics from $\mathcal{F}$. Each plane in the configuration is member of two different quadruplets.*

This configuration is determined by three planes which are tangent to three different quadrics from $\mathcal{F}$, with collinear touching points.

Proof. Let Q_1, Q_2 and Q_3 be three quadrics from $\mathcal{F}$, and $\mathcal{P}_1, \mathcal{P}_2$ and $\mathcal{P}_3$ be the respective tangent planes such that the touching points A_1, A_2 and A_3 are collinear. Let x be the line that contains those points. Then lines x_1, x_2 and x_3 are obtained from x by the billiard reflection off Q_1, Q_2 and Q_3 at the points A_1, A_2 and A_3 respectively. Due to the Proposition 21, the lines x_{12}, x_{13}, x_{23} and x_{123} are well determined. The twelve planes of the configuration are thus tangent to corresponding quadrics at the points of intersection of the lines. ■

In the dual projective space $(\mathbb{P}^3)^*$, the configuration of the planes in the six-pointed theorem is depicted in Figure 5.4.

The Figure 5.4 consists of twelve vertices, six edges and eight triangles, which are described in following manner.

- **Twelve vertices:** Each plane that is tangent to one of the quadrics Q_1, Q_2 and Q_3 corresponds to a vertex of the polygonal line in the configuration of planes from the six-pointed star theorem, and a pair of lines are assigned to it. The lines in each pair

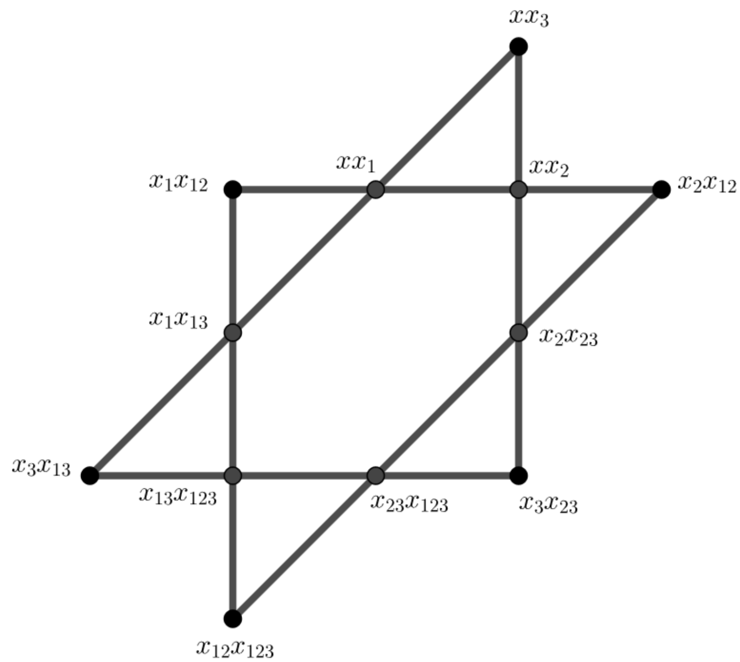


Figure 5.4: Configuration of planes of the six-pointed star theorem.

are reflected to each other off the quadric at the tangency point with the assigned plane.

- **Eight triangles:** The three planes assigned to the vertices of any triangle are touching the corresponding quadrics at three collinear points. Thus the line which contains these points is naturally assigned.

- **Six edges:** Each edge contains four vertices. The four planes assigned to the four vertices of one edge belong to the same pencil. Therefore, they satisfy a double reflection configuration.

The next theorem is the 3D-consistency of the quad-relation by use of the double reflection configuration, meaning that the reflections off the three quadrics $\mathrm{Q}_1, \mathrm{Q}_2$ and Q_3 commute.

Theorem 49 ([17], Theorem 4.2). *Let* $\mathrm{x}, \mathrm{x}_1, \mathrm{x}_2$ *and* x_3 *be lines in the projective space* $\mathbb{P}^3$ *such that* $\mathrm{x}_1, \mathrm{x}_2$ *and* x_3 *are obtained from* x *by reflections off confocal quadrics* $\mathrm{Q}_1, \mathrm{Q}_2$

and Q_3 *respectively. Let introduce lines* x_{12}, x_{13}, x_{23} *and* x_{123} *such that the four quadruplets* $\{x, x_1, x_{12}, x_2\}$*,* $\{x, x_1, x_{13}, x_3\}$*,* $\{x, x_2, x_{23}, x_3\}$ *and* $\{x_1, x_{12}, x_{123}, x_{13}\}$ *are double reflection configurations. Then the two quadruplets of lines* $\{x_2, x_{12}, x_{123}, x_{23}\}$ *and* $\{x_3, x_{13}, x_{123}, x_{23}\}$ *are also double reflection configurations.*

Proof. It follows immediately from the Theorem 48. ■

Assume that each plane in the Theorem 48 is assigned to the mid-point of the corresponding edge of a cube, then each plane represents a vertex of a cuboctahedron (see Figure 5.5). Notice that the cuboctahedron is one of the Archimedian solids and it is also circumscribed.

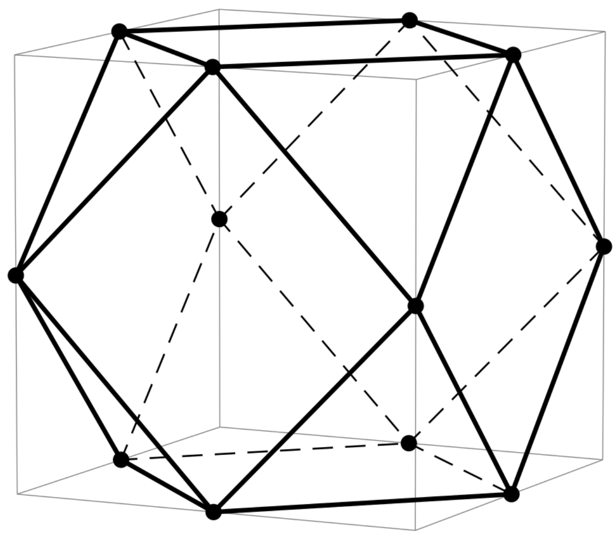


Figure 5.5: A different configuration of the planes from the six-pointed star theorem.

This latest configuration of planes from the six-pointed star theorem leads us to some analysis of the lattice of hyper-planes because the cuboctahedra will represent the building blocks for such lattice.

5.3 Hyper-plane billiard nets

According to [20], in a three-dimensional space, an infinite set of polyhedra that fit together to fill the whole space just once such that every face of each polyhedron belongs to one other polyhedron is called *honeycomb*. If the cells of a honeycomb are regular and equal, then the honeycomb is called *regular*. Given a regular honeycomb. A polyhedron is called *vertex figure* of the honeycomb if its vertices are the mid-points of all the edges that emanate from a given vertex and its faces are the vertex figure of the cells which surround the given vertex. On the other hand, a honeycomb is said to be quasi-regular if its cells are regular and its vertex figures are quasi-regular. For the quasi-regular honeycomb, the vertex figures are all alike and it has two kinds of cells that are arranged alternately.

Following [28], let us consider a lattice $\mathbb{Z}^3$ in $\mathbb{R}^3$. This lattice generates a regular honeycomb consisting of 3-cubes. Then the set of all mid-points of the edges of the cubes is expressed as follows

$$\mathcal{M}^3 = \bigcup_{i=1}^{3} \left(\mathbb{Z}^3 + \frac{1}{2}e_i\right).$$

A honeycomb determined by the lattice $\mathcal{M}^3$ is characterized by two type of polytopes: *cuboctahedra* and *octahedra*. The vertices of each cuboctahedron are mid-points of the edges of a 3-cube in the lattice $\mathbb{Z}^3$ and the vertices of each octahedron are mid-points of all edges with a common endpoint in $\mathbb{Z}^3$. This honeycomb is shown in Figure 5.6.

Suppose that we have a double reflection net and we introduce the following maps

$$\mathcal{H} : \mathcal{M}^3 \longrightarrow (\mathbb{P}^3)^*, \qquad \mathcal{P} : \mathcal{M}^3 \longrightarrow (\mathbb{P}^3)^*.$$

The map $\mathcal{H}$ assigns to the mid-point of the edge $(n_0, n_0 + e_i)$ the tangent plane to the quadric Q_i at the point of reflection, while the map $\mathcal{P}$ assigns the intersection point $\varphi(n_0) \cap \varphi(n_0 + e_i)$ to the mid-point of the edge $(n_0, n_0 + e_i)$.

Proposition 50 ([28], Proposition 3.1). *The two maps $\mathcal{H}$ and $\mathcal{P}$ satisfy the following properties:*

1. *For each square 2-face of any cuboctahedron, $\mathcal{H}$ assigns to its vertices hyperplanes that belong to same pencil and form a harmonic quadruple. The hyperplanes corresponding to the opposite vertices are tangent to the same quadrics from the confocal family.*

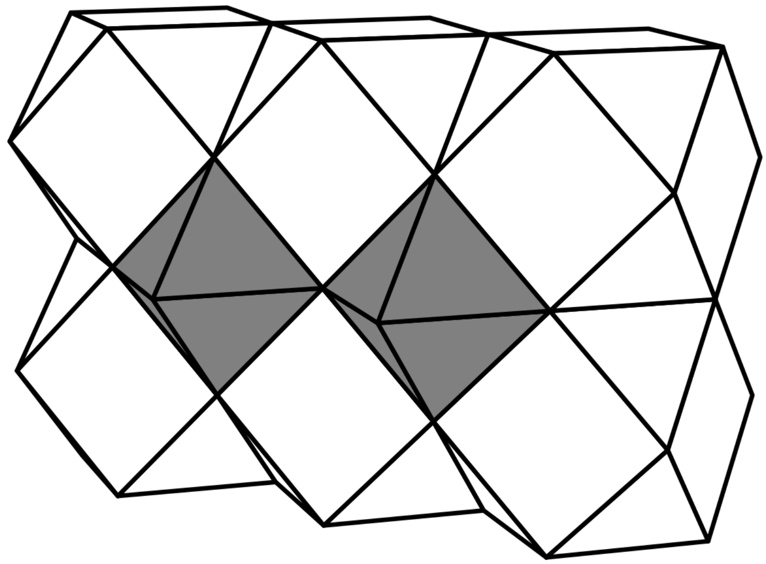


Figure 5.6: Honeycomb consisting of cuboctahedra and actahedra.

2. *The hyperplanes assigned to any two adjacent vertices of a square* 2*-face uniquely determine the hyperplanes assigned to the other two vertices.*

3. *For each octohedron of the honeycomb, $\mathcal{P}$ assigns to all its vertices collinear points. The points assigned to the opposite vertices are on the same quadric from the confocal family.*

Proof. The proof of this proposition also follows from the Theorem 48. ■

5.4 Circumscribed cuboids net in space

We know that a polyhedron can be defined as a finite connected set of polygons satisfying that every side of each polygon belongs to just one other. A polyhedron is called a *cuboid* if it is combinatorial equivalent to a cube [36]. If the cuboid possesses an inscribed sphere, we call it circumscribed cuboid. The subject of the circumscribed cuboids nets was introduced by Wolfgang Böhm in 1965 [39], which is a way of a generalization of circumscribed quadrilaterals nets. Our main focus of the present section is to investigate the geometric properties of such nets. Before reformulating and proving these properties, let us begin with two well-known theorems, Monge's theorem in space and Daniel Sleator's theorem, which play an important role in the proof of our main results.

Theorem 51 (Monge's theorem)**.** *Consider three spheres of different radii lying completely outside each other and their centers are not collinear. Then the three external centers of similitude are collinear.*

The complete proof of this theorem can found in many different books of geometry and in different articles. One version of proof can be seen in [32]. The next theorem is a consequence of the previous theorem. It concerns about centers of similitude for four spheres in three-dimensional space.

Theorem 52 (D. Sleator)**.** *If four spheres are given in an Euclidean space such that they lie completely outside each other, then the six external centers of similitude of pairs of spheres are coplanar.*

Proof. By following the exposition in [32], we first note that a sphere in three-dimensional space depends upon four parameters. These parameters are the coordinates of the center of the sphere in the Euclidean space $\mathbb{E}^3$ and its radius. The radius of a given sphere is either positive or negative. Then, we can map spheres in $\mathbb{E}^3$ onto points of $\mathbb{E}^4$. This means that any sphere in $\mathbb{E}^3$ with center $(\mathsf{a}, \mathsf{b}, \mathsf{c})$ and radius r can be mapped onto a point $(\mathsf{a}, \mathsf{b}, \mathsf{c}, \mathsf{r})$ in $\mathbb{E}^4$. If the sphere has radius zero, then we call it *point-sphere*. Thus, the points-spheres are represented by points in $\mathbb{E}^3$ and their representations coincide with themselves. The map which maps spheres in $\mathbb{E}^3$ onto points of $\mathbb{E}^4$ can be constructed as follows.

1. We set up an orthagonal Cartesian axes $\mathsf{O}x, \mathsf{O}y, \mathsf{O}z, \mathsf{O}w$. The spheres that we consider lie in the $\mathsf{O}xyzw$ hyperplane.

2. A pair of spheres with centers (a_1, b_1, c_1) and (a_2, b_2, c_2) and radii r_1 and r_2 defines a system of spheres with centers $(\lambda a_1 + \mu a_2, \lambda b_1 + \mu b_2, \lambda c_1 + \mu c_2)$ and radii $(\lambda r_1 + \mu r_2)$, where $\lambda + \mu = 1$. Then, two spheres determine a system and the point-sphere is a center of the similitude. Also, the system of spheres in $\mathbb{E}^4$ is represented by a line passing through the points which represent the two spheres. Note that four spheres in $\mathbb{E}^3$ determine a system with three degrees of freedom and the representation of theses spheres in $\mathbb{E}^4$ is a hyperplane containing the lines which represent two spheres systems and hence the centers of similitude. These centers of similitude are point-spheres and are contained in $Oxyzw$ hyperplane. Therefore, the intersection of the two hyperplanes is a plane. This complete the proof of the theorem.

■

Definition 30. *A family of planes is a collection of planes satisfying the condition that all planes intersect along a line (see Figure 5.7).*

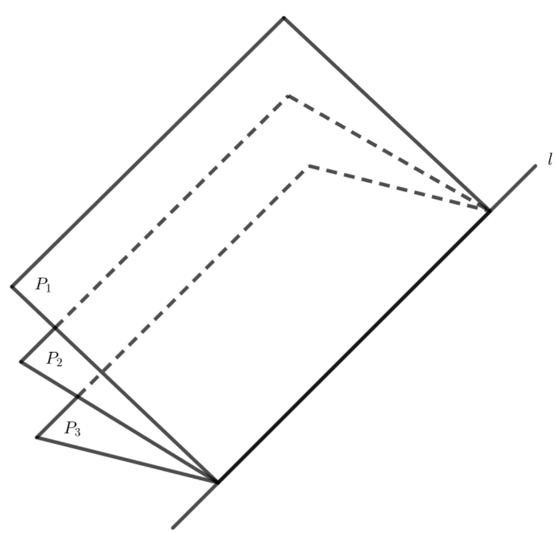


Figure 5.7: One family of planes.

Note that all net-cubes are projective images of the standard cubes. A cube in three-dimensional space has three infinity points.

Lemma 53. *One face plane of a circumscribed cuboid (resp. a projective cube) is uniquely determined by its five given face planes.*

Theorem 54. *A division of an Euclidean space by planes into circumscribed cuboids necessarily consists of three families satisfying the conditions:*

1. *all planes for each family intersect along a line,*

2. *the three lines are coplanar.*

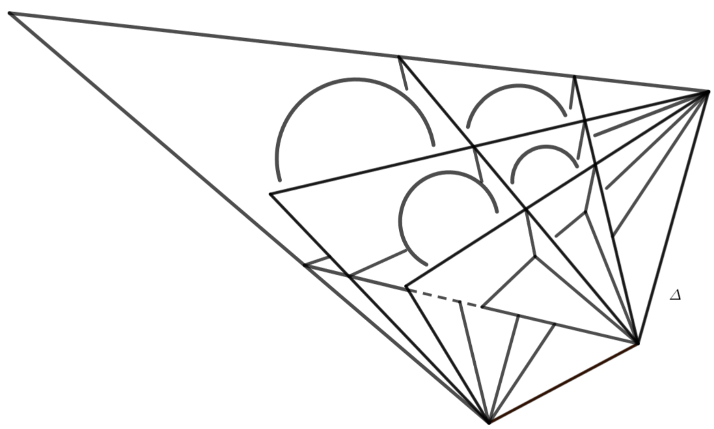


Figure 5.8: Circumscribed cuboids net.

The Figure 5.8 represents a circumscribed cuboids net in 3-dimensional Euclidean space.

Proof. 1. Let M_1, M_2, N_1, N_3, P_1 and P_3 be face planes of a cuboid, and M_2, N_2 and P_2 be three different planes which divide it into eight circumscribed cuboids. Let define

$$m_{12} = M_1 \cap M_2, \quad m_{23} = M_2 \cap M_3,$$

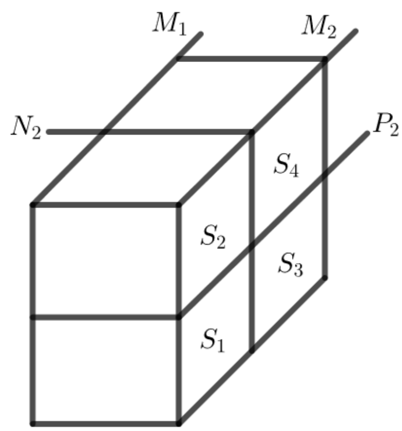
$$n_{12} = N_1 \cap N_2, \quad n_{23} = N_2 \cap N_3,$$

$$p_{12} = P_1 \cap P_2, \quad p_{23} = P_2 \cap P_3.$$

All possible external centers of similitude for each pair of spheres are shown in the Table 5.1.

Table 5.1: External centers of similitude of all pairs of spheres.

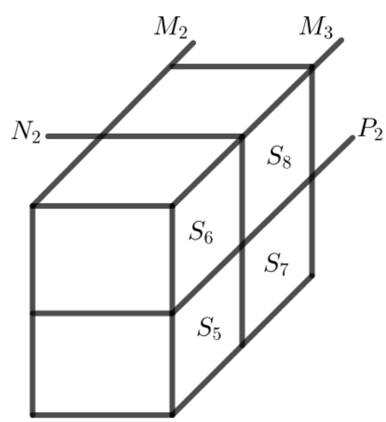
Spheres	Centers of similitude	Spheres	Centers of similitude
S_1, S_2	A	S_3, S_5	S
S_1, S_3	B	S_3, S_6	X
S_1, S_4	C	S_3, S_7	Y
S_1, S_5	D	S_3, S_8	Z
S_1, S_6	E	S_4, S_5	V
S_1, S_7	F	S_4, S_6	W
S_1, S_8	I	S_4, S_7	U
S_2, S_3	J	S_4, S_8	A'
S_2, S_4	K	S_5, S_6	B'
S_2, S_5	L	S_5, S_7	C'
S_2, S_6	O	S_5, S_8	D'
S_2, S_7	Q	S_6, S_7	E'
S_2, S_8	R	S_6, S_8	F'
S_3, S_4	T	S_7, S_8	I'



Step 1: 1. Consider the cuboid whose face planes are M_1, M_2, N_1, N_3, P_1 and P_3. This kind of cuboid is split by the planes N_2 and P_2 into four circumscribed cuboids. Since the four inscribed spheres lie completely outside each other, by Theorem 52, their six external centers of similitude are coplanar. The six external centers of similitude of pairs of spheres are A, B, C, J, K and T due to the Table 5.1. The planes M_1 and M_2 are also common external tangent planes for all inscribed spheres, these external centers of similitude thus lie on a

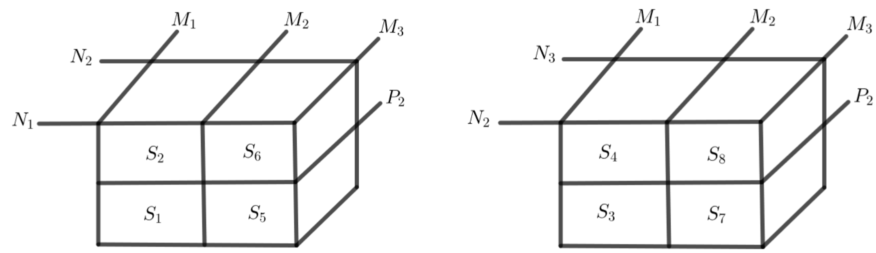
straight line. This line is the line m_{12}. However, the plane P_2 is an external tangent plane of the pairs S_1 and S_3, and S_2 and S_4. This implies that the external centers of similitude B and K lie in three different planes M_1, M_2 and P_2. Notice that the lines $M_1 \cap P_2$ and $M_2 \cap P_2$ are different. Then, $m_{12} \notin P_2$. Therefore, the points B and K must coincide. The plane N_2 is also a common external tangent plane of the pairs of spheres S_1 and S_2, and S_3 and S_4, and A and T are their respective external centers of similitude. We know that $A, T \in m_{12}$ and they also lie in the planes M_1, M_2 and N_2, with $M_1 \cap N_2 \neq M_2 \cap N_2$, then $m_{12} \notin N_2$. Therefore, A must coincide with the T.

2. Consider the cuboid formed by the planes M_2, M_3, N_1, N_3, P_1 and P_3. This cuboid is divided by the planes N_2 and P_2 into four circumscribed cuboids. By using the foregoing



arguments, we can prove that the line m_{23} contains the six external centers of similitude B', C', D', E', F' and I' of the pairs of spheres inscribed in that cuboid. Using the fact that the plane P_2 is common external tangent plane of pairs of spheres S_5 and S_6, and S_7 and S_8, we can prove that the external centers of similitude B' and I' coincide. Similarly, the plane N_2 is a common external tangent plane of the pairs of spheres S_5 and S_7, and S_6 and S_8, then the points C' and F' also coincide. Thus, there are four external centers of similitude B', C', D' and E' lying on the line m_{23}.

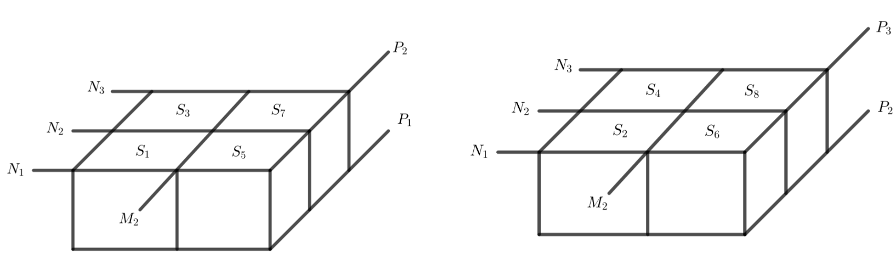
Step 2: 1. Consider the cuboid formed by the planes N_1, N_2, M_1, M_3, P_1 and P_3. This cuboid is divided by the planes M_2, and P_2 into four circumscribed cuboids. Then, the



six external centers of similitude A, D, E, L, O and B' lie on the line n_{12}. Using the same arguments as in step 1, we prove that the pairs of external centers of similitude D and O, and A and B' coincide. Indeed, the line n_{12} contains four external centers of similitude A, D, E and L.

2. Repeating the preceding arguments to the cuboid formed by the planes N_2, N_3, M_1, M_3, P_1 and P_3 and using the fact that the planes M_2 and P_2 divide it into four circumscribed cuboids, we can demonstrate that the line n_{23} contains the four external centers of similitude T, A', U and Z of the inscribed spheres in such cuboid since the external centers of similitude T and I' coincide, and the external centers of similitude A' and Y coincide as well.

Step 3: Let us consider the two following cuboids. Following all steps in the step 1 and



using a similar argument, we can show that the external centers of similitude D, Y, O and A' coincide. Therefore, the external centers of similitude B, D, S and F lie on the line p_{12}

and the external centers of similitude K, O, W and R lie on the line p_{23}. By combining all results obtained in the three steps above, we have the coincidence of the following external centers of similitude

$$B \equiv K \equiv C' \equiv F',$$

$$A \equiv T \equiv B' \equiv I',$$

$$D \equiv O \equiv A' \equiv Y.$$

Indeed,

$$A, B, C, J \in m_{12}, \quad \text{and} \quad A(\equiv B'), B(\equiv C'), D', E' \in m_{23}, \tag{5.3}$$

$$A, D, E, L \in n_{12}, \quad \text{and} \quad A(\equiv T), D(\equiv Y), U, Z \in n_{23}, \tag{5.4}$$

$$B, D, S, F \in p_{12}, \quad \text{and} \quad B(\equiv K), D(\equiv O), R, W \in p_{23}. \tag{5.5}$$

To ensure that the lines m_{12} and m_{23} are identical, we need to verify that the points A and B do not coincide. Assume that they coincide. This implies that $A \equiv B$ lies in the planes M_1, M_2, N_2 and P_2. This means that N_2 and P_2 becomes respectively an external tangent plane for each pair of spheres $S_1$and S_2, and S_1 and S_3. Then we get a contradiction because these planes are by assumptions the respective internal tangent plane for those pairs. Therefore, A does not coincide with B. For the rest, we use the same argument. This proves that the planes M_1, M_2 and M_3 intersect along a line, that means that they belong to the same family of planes. That result is still true for the two triplets of planes N_1, N_2 and N_3, and P_1, P_2 and P_3. The coplanarity of these three lines immediately follows from (5.3), (5.4) and (5.5). ■

Theorem 55 (Böhm, [39]).

1. *The circumscribed cuboids net is uniquely determined by one of its circumscribed cuboids.*

2. *Each cuboid, which can be divided by three planes from different families into eight circumscribed cuboids, is itself circumscribed.*

5.5 Circumscribed hyper-cuboids in 4-dimensional space

Definition 31. *A family of hyperplanes in a* 4*-dimensional space is called nice if there is a* 2*-dimensional plane that is the intersection of all of them.*

Theorem 56. *A division of a* 4*-dimensional Euclidean space by hyperplanes into circumscribed* 4*-cuboids consists of four families of hyperplanes satisfying the following conditions:*

1. *All four families of hyperplanes are nice.*

2. *The four* 2*-dimensional planes lie in a space of dimension* 3*.*

Before demonstrating the above theorem, let us state and prove the following propositions.

Proposition 57. *Consider a* 4*-cuboid in a* 4*-dimensional space formed by the hyperplanes* A_1, A_2, B_1, B_3, C_1, C_3, D_1 *and* D_3. *Let denote by* B_2, C_2 *and* D_2 *the hyperplanes which divide this* 4*-cuboid into eight circumscribed* 4*-cuboids. Then, among the* 28 *external centers of similitude, there are three quadruplets of external centers of similitude such that all external centers of similitude in each quadruplet coincide.*

Proof. The inscribed spheres $\mathbb{S}^3$ and the 3-faces of the 4-cuboids in the Proposition 57 are indicated in the Table 5.2.

All hyper-spheres lie completely outside each other, then one proves that the 28 external centers of similitude of these hyper-spheres lie in a space of dimension 3. By using the fact that the hyperplanes A_1 and A_2 are common tangent to all the eight hyper-spheres S_i, $i \in \{1, 2, \cdots 8\}$, we can conclude that the 28 external centers of similitude must lie in an 2-dimensional plane. Moreover, the pairs of hyper-spheres in the Table 5.3 may be organized in the following ways:

Table 5.2: The 3-faces of the eight circumscribed 4-cuboids in the Proposition 57.

Inscribed spheres $\mathbb{S}^3$	3-faces of the 4-cuboids
S_1	$A_1, A_2, B_1, B_2, C_1, C_2, D_1, D_2$
S_2	$A_1, A_2, B_2, B_3, C_1, C_2, D_1, D_2$
S_3	$A_1, A_2, B_1, B_2, C_2, C_3, D_1, D_2$
S_4	$A_1, A_2, B_1, B_2, C_2, C_3, D_2, D_3$
S_5	$A_1, A_2, B_1, B_2, C_1, C_2, D_2, D_3$
S_6	$A_1, A_2, B_2, B_3, C_1, C_2, D_2, D_3$
S_7	$A_1, A_2, B_2, B_3, C_2, C_3, D_1, D_2$
S_8	$A_1, A_2, B_2, B_3, C_2, C_3, D_2, D_3$

Table 5.3: External centers of all pairs of spheres in the Proposition 57.

Pairs of spheres	External centers	Pairs of spheres	External centers
S_1, S_2	E_1	S_3, S_5	E_{15}
S_1, S_3	E_2	S_3, S_6	E_{16}
S_1, S_4	E_3	S_3, S_7	E_{17}
S_1, S_5	E_4	S_3, S_8	E_{18}
S_1, S_6	E_5	S_4, S_5	E_{19}
S_1, S_7	E_6	S_4, S_6	E_{20}
S_1, S_8	E_7	S_4, S_7	E_{21}
S_2, S_3	E_8	S_4, S_8	E_{22}
S_2, S_4	E_9	S_5, S_6	E_{23}
S_2, S_5	E_{10}	S_5, S_7	E_{24}
S_2, S_6	E_{11}	S_5, S_8	E_{25}
S_2, S_7	E_{12}	S_6, S_7	E_{26}
S_2, S_8	E_{13}	S_6, S_8	E_{27}
S_3, S_4	E_{14}	S_7, S_8	E_{28}

$$\mathfrak{T}_1 = \{S_1, S_2; S_3, S_7; S_4, S_8; S_5, S_6\},$$

$$\mathfrak{T}_2 = \{S_1, S_3; S_2, S_7; S_4, S_5; S_6, S_8\},$$

$$\mathfrak{T}_3 = \{S_1, S_5; S_2, S_6; S_3, S_4; S_7, S_8\}.$$

We claim that the external centers of the four pairs in each set $\mathfrak{T}_i$, with $i \in \{1, 2, 3\}$, coincide and the external centers of similitude obtained from two different sets $\mathfrak{T}_i$ and $\mathfrak{T}_j$ are different. Notice that the hyperplanes D_1 and D_2 are common external tangent of the pairs of hyperspheres S_1 and S_2, and S_3 and S_7. Then, their external centers of similitude lie in a 2-

dimensional plane which is the intersection of the hyperplanes D_1 and D_2. Therefore, the external centers of similitude E_1 and E_{17} belong to the intersection of four hyperplanes. This implies that these two external centers of similitude coincide. The proofs for the remaining pairs are exactly similar. Combining all together, we obtain

$$\begin{aligned} E^0 &:= E_1 \equiv E_{17} \equiv E_{22} \equiv E_{23}, \\ E^1 &:= E_2 \equiv E_{12} \equiv E_{19} \equiv E_{27}, \\ E^2 &:= E_4 \equiv E_{11} \equiv E_{14} \equiv E_{28}. \end{aligned} \tag{5.6}$$

We need to verify that the external centers of similitude E^0, E^1 and E^2 do not coincide and are not collinear. Suppose that E^0 and E^1 coincide. Then, B_2 becomes an external tangent hyperplane of S_1 and S_2. By assumption however, the hyperplane B_2 is the internal tangent of the hyper-spheres S_1 and S_2. This is a contradiction. To see the non-collinearity of E^0, E^1 and E^2, let us choose the hyper-spheres S_1, S_2, S_3 and S_5. According to the Table 5.3, the external centers of similitude of the pairs of spheres S_1 and S_2, S_1 and S_3, S_1 and S_5, S_2 and S_3, S_2 and S_5, and S_3 and S_5 are E^0, E^1, E_8, E^2, E_{10} and E_{15} respectively. One proves that these external centers of similitude are vertices of a complete quadrilateral. Due to the Monge's theorem, the points E^0, E^1 and E_8 are collinear, and the points E^2, E_{10} and E_{15} are collinear as well. This implies that the external centers of similitude E^0, E^1 and E^2 lie in two different lines, meaning that they are not collinear. ■

Proposition 58. *Consider a* 4*-cuboid in a* 4*-dimensional space formed by the pairs of hyperplanes* A_2, A_3*,* B_1, B_3*,* C_1, C_3*,* D_1 *and* D_3*. Suppose that this* 4*-cuboid is divided by the hyperplanes* B_2*,* C_2 *and* D_2 *into eight circumscribed* 4*-cuboids. Then, among the* 28 *external centers of similitude, there exist three quadruplets of external centers of similitude such that all external centers of similitude in each quadruplet coincide.*

The Table 5.4 and Table 5.5 show the 3-faces of the circumscribed 4-cuboids and the external centers of all pairs of spheres in the Proposition 58, respectively.

Table 5.4: The 3-faces of the eight circumscribed 4-cuboids in the Proposition 58.

Inscribed spheres S^3	3-faces of the 4-cuboids
S_1'	$A_2, A_3, B_1, B_2, C_1, C_2, D_1, D_2$
S_2'	$A_2, A_3, B_2, B_3, C_1, C_2, D_1, D_2$
S_3'	$A_2, A_3, B_1, B_2, C_2, C_3, D_1, D_2$
S_4'	$A_2, A_3, B_1, B_2, C_2, C_3, D_2, D_3$
S_5'	$A_2, A_3, B_1, B_2, C_1, C_2, D_2, D_3$
S_6'	$A_2, A_3, B_2, B_3, C_1, C_2, D_2, D_3$
S_7'	$A_2, A_3, B_2, B_3, C_2, C_3, D_1, D_2$
S_8'	$A_2, A_3, B_2, B_3, C_2, C_3, D_2, D_3$

Table 5.5: External centers of all pairs of spheres $\mathbb{S}^3$ in the Proposition 58.

Pairs of spheres	External centers	Pairs of spheres	External centers
S_1', S_2'	E_1'	S_3', S_5'	E_{15}'
S_1', S_3'	E_2'	S_3', S_6'	E_{16}'
S_1', S_4'	E_3'	S_3', S_7'	E_{17}'
S_1', S_5'	E_4'	S_3', S_8'	E_{18}'
S_1', S_6'	E_5'	S_4', S_5'	E_{19}'
S_1', S_7'	E_6'	S_4', S_6'	E_{20}'
S_1', S_8'	E_7'	S_4', S_7'	E_{21}'
S_2', S_3'	E_8'	S_4', S_8'	E_{22}'
S_2', S_4'	E_9'	S_5', S_6'	E_{23}'
S_2', S_5'	E_{10}'	S_5', S_7'	E_{24}'
S_2', S_6'	E_{11}'	S_5', S_8'	E_{25}'
S_2', S_7'	E_{12}'	S_6', S_7'	E_{26}'
S_2', S_8'	E_{13}'	S_6', S_8'	E_{27}'
S_3', S_4'	E_{14}'	S_7', S_8'	E_{28}'

Proof. The proof of the Proposition 58 is similar to the proof of Proposition 57. We thus obtain:

$$\begin{aligned} E_*^0 &:= E_1' \equiv E_{17}' \equiv E_{22}' \equiv E_{23}', \\ E_*^1 &:= E_2' \equiv E_{12}' \equiv E_{19}' \equiv E_{27}', \\ E_*^2 &:= E_4' \equiv E_{11}' \equiv E_{14}' \equiv E_{28}'. \end{aligned} \tag{5.7}$$

■

Proposition 59. *Consider a 4-cuboid in a 4-dimensional space whose 3-faces are A_1, A_3, B_1, B_3, C_1, C_3, D_1 and D_3. Suppose that this 4-cuboid is split by four hyperplanes A_2, B_2, C_2 and D_2 into sixteen circumscribed 4-cuboids. Then, the four triplets of hyperplanes A_1, A_2 and A_3, B_1, B_2 and B_3, C_1, C_2 and C_3, and D_1, D_2 and D_3 are nice.*

Proof. Let us consider the two 4-cuboids mentioned in the Propositions 57 and 58. The last thing that we need to prove here is that the pairs of external centers of similitude E^i and E^i_*, for $i = 0, 1, 2$, coincide. We need only to prove the coincidence of the external centers of similitude E^0 and E^0_* and the proofs of the rest are similar. Choose the pairs of spheres S_1 and S_2, and S'_1 and S'_2. We have proved in the Propositions 57 and 58 that the points E^0 and E^0_* are the external centers of similitude of those pairs spheres respectively. Moreover, these external centers of similitude are the points of intersection of four hyperplanes C_1, C_2, D_1 and D_2. Therefore, E^0 and E^0_* must coincide. It follows that the hyperplanes A_1, A_2 and A_3 intersect in a plane spanned by the lines (E^0E^1) and (E^0E^2). In other words, the hyperplanes A_1, A_2 and A_3 are nice. The application of the Propositions 57 and 58 to the circumscribed 4-cuboids shown in the first column of the Table 5.6, combined with the previous argument, leads us to the conclusion of the second column of the Table 5.6. Therefore, the hyperplanes B_1, B_2 and B_3, C_1, C_2 and C_3, and D_1, D_2 and D_3 are also nice.

Table 5.6: Pairs of circumscribed 4-cuboids have three common external centers of similitude.

3-faces of the 4-cuboids	Common external centers of similitude of pairs of circumscribed 4-cuboids
$A_1, A_2; B_1, B_3; C_1, C_3; D_1, D_3$	E^0, E^1, E^2
$A_2, A_3; B_1, B_3; C_1, C_3; D_1, D_3$	
$A_1, A_3; B_1, B_2; C_1, C_3; D_1, D_3$	E^1, E^2, F^0
$A_1, A_3; B_2, B_3; C_1, C_3; D_1, D_3$	
$A_1, A_3; B_1, B_3; C_1, C_2; D_1, D_3$	E^0, E^2, F^0
$A_1, A_3; B_1, B_3; C_2, C_3; D_1, D_3$	
$A_1, A_3; B_1, B_3; C_1, C_3; D_1, D_2$	E^0, E^1, F^0
$A_1, A_3; B_1, B_3; C_1, C_3; D_2, D_3$	

■

Proof. (Theorem 56). The Proposition 59 has proved the first condition of the theorem. It remains to show that the four 2-dimensional planes lie in a 3-dimensional space. It is sufficient to prove that any pair of plane intersects along a line. We observe from the Table 5.6 that the four triplets of hyperplanes A_1, A_2 and A_3, B_1, B_2 and B_3, C_1, C_2 and C_3, and D_1, D_2 and D_3 intersect at the planes spanned by the lines (E^0E^1) and (E^0E^2), (E^1E^2) and (E^1F^0), (E^0F^0) and (E^0E^2), and (E^0E^1) and (E^0F^0), respectively, where F^0 is one the external centers of similitude of the inscribed hyper-spheres in the 4-cuboid formed by the hyperplane A_1, A_3, B_1, B_2, C_1, C_3, D_1 and D_3. These lines form a tetrahedron with vertices E^0, E^1, E^2 and F^0. This proves the last condition of the Theorem 56. ■

5.6 Checkerboard inscribed spherical nets in $\mathbb{R}^3$

This last section contains an overview of a checkerboard inscribed sphere in $\mathbb{R}^3$, which is a generalization of a circumscribed cuboids in spatial case. Here, we simply recall the definition of the checkerboard inscribed spherical nets and their main geometric and combinatorial properties. We will omit the proofs of some theorems since they require additional concepts. Also, we follow the exposition presented in [1]. Let us start with some notation that we will use in the sequel. Consider images of the integer lattice $f : \mathbb{Z}^3 \longrightarrow \mathbb{R}^3$. A cube is represented by $\square^{c}_{i,j,k}$ and its vertices are $f_{i,j,k}$, $f_{i+c,j,k}$, $f_{i,j+c,k}$, $f_{i,j,k+c}$, $f_{i+c,j+c,k}$, $f_{i,j+c,k+c}$, $f_{i+c,j,k+c}$ and $f_{i+c,j+c,k+c}$. If the indices i, j and k are either all even or all odd, and c is odd, then we call this cube a *net-cube*. It is an *unit net-cube* if $c = 1$ and denoted by $\square_{i,j,k}$.

Definition 32 ([1], Definition 4.1). *A map $f : \mathbb{Z}^3 \longrightarrow \mathbb{R}^3$ is called a checkerboard inscribed spherical net (or a checkerboard IS net) if the following conditions hold:*

1. *For any integers i and j, the points of the form $\{f_{i,j,k} | k \in \mathbb{Z}\}$ lie on a straight line and three consecutive points $f_{i,j,k-1}$, $f_{i,j,k}$ and $f_{i,j,k+1}$ on such line satisfy the condition that the point $f_{i,j,k}$ lies between the points $f_{i,j,k-1}$ and $f_{i,j,k+1}$. The points of the form $\{f_{i,j,k} | i \in \mathbb{Z}\}$ and $\{f_{i,j,k} | j \in \mathbb{Z}\}$ also satisfy these conditions.*

2. All unit net-cubes $\boxcube_{i,j,k}$ are circumscribed.

Notice that lines $n_{i,j}$, $m_{i,k}$ and $l_{j,k}$ are lines of the checkerboard IS net and they contain respectively the points $\{f_{i,j,k}|\forall k\}$, $\{f_{i,j,k}|\forall j\}$ and $\{f_{i,j,k}|\forall i\}$. The planes of the checkerboard IS net are

$$L_i \supset \{f_{i,j,k}|\forall j,k\}, \quad N_j \supset \{f_{i,j,k}|\forall i,k\} \quad \text{and} \quad N_k \supset \{f_{i,j,k}|\forall i,j\}.$$

Proposition 60 ([1], Lemma 4.3)**.** *Consider a cubical polytope $\boxcube$ in $\mathbb{R}^3$ such that this cubical polytope is divided by three sets of two planes into 27 combinatorial cubes. Suppose that the central cube $\boxcube_{1,1,1}$ is circumscribed and the cubes $\boxcube_{0,0,0}$, $\boxcube_{2,0,0}$,$\boxcube_{0,2,0}$ and $\boxcube_{0,0,2}$ are projective cubes and are circumscribed. Then, the unit net-cubes $\boxcube_{2,2,0}$, $\boxcube_{2,0,2}$, $\boxcube_{0,2,2}$ and $\boxcube_{2,2,2}$ are also circumscribed and they are projective cubes as well. In other words, cubical polytope $\boxcube$ is an IS net.*

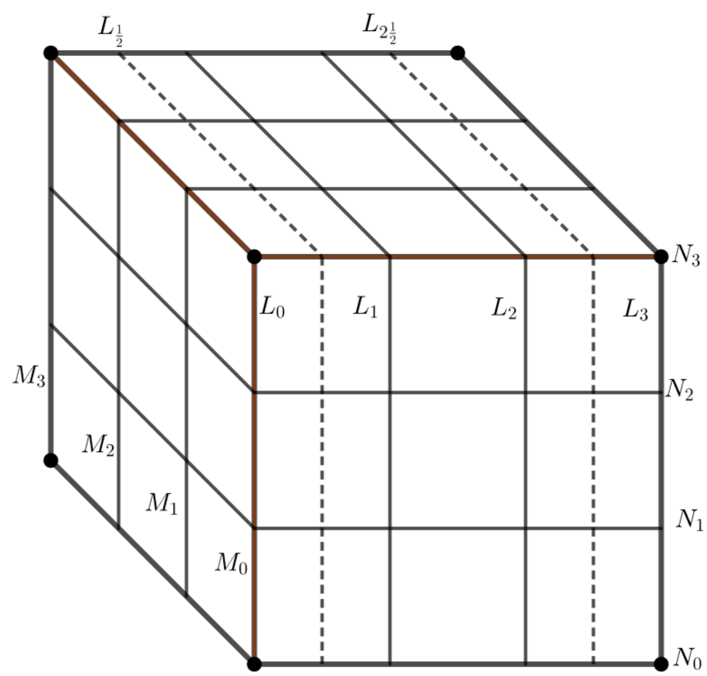


Figure 5.9: An IS-net $\boxcube^3_{0,0,0}$.

Proof. If an unit net-cube $\boxcube_{i,j,k}$ is circumscribed, then we denote by $o_{i,j,k}$ the center of its inscribed sphere. The vertices of the central unit net-cube are $f_{1,1,1}$, $f_{2,1,1}$, $f_{1,2,1}$, $f_{2,2,1}$, $f_{1,1,2}$,

$f_{2,1,2}$, $f_{1,2,2}$ and $f_{2,2,2}$, and the center of its inscribed sphere is $o_{1,1,1}$. Let us define a projective map σ as follows

$$\sigma(f_{1,1,1}) = o_{0,0,0}, \quad \sigma(f_{2,1,1}) = o_{2,0,0},$$
$$\sigma(f_{1,2,1}) = o_{0,2,0}, \quad \sigma(f_{1,1,2}) = o_{0,0,2}.$$

Since the four straight lines $(f_{1,1,1}o_{0,0,0})$, $(f_{2,1,1}o_{2,0,0})$, $(f_{1,2,1}o_{0,2,0})$ and $(f_{1,1,2}o_{0,0,2})$ pass through the point $o_{1,1,1}$ and they are preserved by the projective map σ, all lines passing through the point $o_{1,1,1}$ are also preserved by the projective map σ. Notice that the unit net-cubes $\square_{2,0,0}$ and $\square_{0,2,0}$ are circumscribed projective cubes and their face planes coincide with the face planes of the cell $\square_{2,2,0}$. We are going to show that the cell $\square_{2,2,0}$ is circumscribed and the center of the inscribed sphere is $\sigma(f_{2,2,1}) = o_{2,2,0}$ (see Figure 5.9). The image of the plane L_1 under σ is the plane $L_{\frac{1}{2}}$. The plane $L_{\frac{1}{2}}$ passes through the points $o_{0,0,0}$, $o_{0,2,0}$ and $o_{0,0,2}$, and it is the bisector of the planes L_0 and L_1. The point $\sigma(f_{1,2,2}) = L_{\frac{1}{2}} \cap (f_{1,2,2}o_{1,1,1}) = o_{2,2,0}$ is equidistant from the planes L_1, M_2 and N_2 as a point on $(f_{1,2,2}o_{1,1,1})$ and belongs to the plane $L_{\frac{1}{2}}$, then this point is exactly the center of inscribed sphere in $\square_{2,2,0}$. The remaining cells $\square_{2,0,2}$ and $\square_{0,2,2}$ are also circumscribed by using the foregoing arguments and the respective corresponding centers of the inscribed spheres are $\sigma(f_{2,1,2}) = o_{2,0,2}$ and $\sigma(f_{1,2,2}) = o_{0,2,2}$. The cell $\square_{2,2,2}$ possesses an inscribed sphere and the center of the sphere is the point $o_{2,2,2} = L_{2\frac{1}{2}} \cap M_{2\frac{1}{2}} \cap N_{2\frac{1}{2}}$, where the planes $L_{2\frac{1}{2}}, M_{2\frac{1}{2}}$ and $N_{2\frac{1}{2}}$ are bisectors of the pairs of planes L_2 and L_3, M_2 and M_3, and N_2 and N_3 respectively. The point $o_{2,2,2}$ lies on the line $(o_{1,1,1}f_{2,2,2})$ and is equidistant from all face planes of the cell $\square_{2,2,2}$. This completes the proof. ∎

Corollary 61 (9 inspheres incidence theorem). *Let $\square$ be a cubical polytope in $\mathbb{R}^3$ such that this cubical polytope is split by three sets of two planes into 27 combinatorial cubes. Assume that the cells $\square_{0,0,0}$, $\square_{2,0,0}$, $\square_{0,2,0}$, $\square_{0,0,2}$, $\square_{2,2,0}$, $\square_{2,0,2}$, $\square_{0,2,2}$ and $\square_{1,1,1}$ are circumscribed. Then, the last cell $\square_{2,2,2}$ is also circumscribed.*

Theorem 62 ([1], Theorem 4.7). *Let $\square$ be a cubical polytope in $\mathbb{R}^3$. Suppose that this cubical polytope is divided by three sets of two planes into 27 combinatorial cubes and the cubes $\square_{0,0,0}$, $\square_{2,0,0}$, $\square_{0,2,0}$, $\square_{0,0,2}$, $\square_{2,2,0}$, $\square_{2,0,2}$, $\square_{0,2,2}$, $\square_{1,1,1}$ and $\square_{2,2,2}$ are circumscribed. Then the cubical polytope $\square$ is circumscribed as well.*

Theorem 63 (Construction of a checkerboard IS net, [1], Theorem 4.5). *A checkerboard IS net is determined by its 8 cubic cells with vertices $f_{i,j,k}$, $i, j, k \in \{0, 1, 2\}$, such that the cells $\square_{0,0,0}$ and $\square_{1,1,1}$ are circumscribed projective cubes.*

Proof. Notice that the 8 cubic cells are formed by the planes M_0, M_1, M_2, N_0, N_1, N_2, L_0, L_1 and L_2. The existence of a sphere touching the planes M_0, M_1, N_0, N_1 and L_2 follows from the fact that the cubic cell $\square_{0,0,0}$ is circumscribed projective cube. Then, the plane L_3 is uniquely determined according to the Lemma 53. In other words, the cubic cell $\square_{2,0,0}$ is circumscribed projective cube. Applying this argument several times, we can prove that the cubic cell $\square_{0,0,2}$, $\square_{0,2,0}$, $\square_{2,2,0}$, $\square_{2,0,2}$, $\square_{0,2,2}$ and $\square_{2,2,2}$ are circumscribed projective cubes. Further, in a checkerboard IS net pattern there is a sphere touching the planes M_1, M_2, N_1, N_2 and L_3. The use of the Lemma 53 determines the plane L_4. By continuing this process of construction, we can generate the whole checkerboard IS net. ■

Theorem 64 ([1], Theorem 4.1). *Let f be a checkerboard IS net. Then, the following properties hold:*

1. *All net-cubes of f are circumscribed.*

2. *The net-cubes $\square^{c}_{i,j,k}$ and $\square^{c+4s+2}_{i-2s-1,j-2s-1,k-2s-1}$ are perspective.*

3. *The line $l_{i,j}$, with $i+j =$ constant, lie on a one-sheeted hyperboloid. This result is still valid for $l_{i,j}$, with $i-j =$ constant. Both statements are still true for the lines $n_{i,j}$ and $m_{i,k}$.*

4. The centers $o_{i,j,k}$ *of the spheres inscribed in the net-cubes* $\square_{i,j,k}$*, with all indices* i, j, k *are either even or odd, build a grid projectively equivalent to an orthogonal grid.*

Proof. 1. Let us prove the first statement by induction on c. For $c = 1$, all net-cubes are unit net-cubes. Due to the second condition in the Definition 32, all unit net-cubes are circumscribed. Suppose that the net-cube $\square^{c}_{i+1,j+1,k+1}$ is circumscribed. We need to show that all net-cubes $\square^{c+2}_{i,j,k}$ are also circumscribed. We observe that the net-cube $\square^{c+2}_{i,j,k}$ is made up of the following net-cubes

$$\square_{i,j,k},\quad \square_{i+c+1,j,k},\quad \square_{i,j+c+1,k},\quad \square_{i,j,k+c+1},\quad \square^{c}_{i+1,j+1,k+1},$$
$$\square_{i+c+1,j+c+1,k},\quad \square_{i+c+1,j,k+c+1},\quad \square_{i,j+c+1,k+c+1},\quad \square_{i+c+1,j+c+1,k+c+1}.$$

All the net-cubes above are unit net-cubes except the net cube $\square^{c}_{i+1,j+1,k+1}$, then they are circumscribed. By induction hypothesis, the net-cube $\square^{c}_{i+1,j+1,k+1}$ is also circumscribed. Thus, due to Theorem 62 the net-cube $\square^{c+2}_{i,j,k}$ is also circumscribed. Hence, all net-cubes of IS net are circumscribed.

2. The proof of the statement focuses only on the case where the value of s is positive. The proof is similar for the other case. Notice that the six faces of the net-cube $\square^{c}_{i,j,k}$ divide the net-cube $\square^{c+2}_{i-2s-1,j-2s-1,k-2s-1}$ into twenty seven cells such that the eight cells at the corners are net-cubes. Let denote these net-cubes by $\square_{l}$, $l = 1, 2, \cdots, 8$. The vertices of the net-cubes $\square^{c}_{i,j,k}$ and $\square^{c+4s+2}_{i-2s-1,j-2s-1,k-2s-1}$ are denoted by a_l and b_l respectively. Let us call by ω_l the inscribed spheres of the net-cubes $\square_{l}$, and by Ω and Ω' the respective inscribed spheres of $\square^{c}_{i,j,k}$ and $\square^{c+4s+2}_{i-2s-1,j-2s-1,k-2s-1}$. Then, the center of similitude of ω_l and Ω' is the point a_l, and b_l is the center of similitude of ω_l and Ω. It follows from the Theorem 22 that the line $(a_l b_l)$ passes through the center of similitude of Ω and Ω'. Thus, all lines $(a_l b_l)$ intersect at one point, this proves that the net-cubes $\square^{c}_{i,j,k}$ and $\square^{c+4s+2}_{i-2s-1,j-2s-1,k-2s-1}$ are perspective.

3. Let l_{i_1,j_1} and l_{i_2,j_2} be two lines of the checkerboard IS net such that the integers i_1 and i_2 are of different parities and $i_1 + j_1 = i_2 + j_2$. Then, there exists a net-cube having the edges l_{i_1,j_1} and l_{i_2,j_2}. As mentioned earlier, the net-cubes are projective cubes. Therefore these lines must intersect. From the previous argument, we can separate the lines $l_{i,j}$ into different families of lines depending on the parity of the integer i. It follows that any two lines from these families intersect. Thus, they lie on hyperboloid of one sheet.

4. It has been mentioned in the proof of the Theorem 62 that the point $o_{i+1\pm1,j+1\pm1,k+1\pm1}$ are vertices of a cube, projectively equivalent to the unit net-cube $\square_{i+1,j+1,k+1}$. The latter unit net-cube is a projective cube. All elementary cells of the grid $o_{i,j,k}$, $i, j, k \in 2\mathbb{Z}$, are also projective cubes. The coincidence of the infinite points of all elementary cells is straightforward to verify. This completes the proof the statement. $\blacksquare$

REFERENCES

[1] Akopyan Arseniy and Alexander I. Bobenko, *Incircular nets and confocal conics*, Trans. Amer. Math.Soc. 370 (2018), no. 4, 2825-2854.

[2] Akopyan Arseniy and Alexy A. Zaslavsky, *Geometry of conics*, American Mathematical Society, 2007.

[3] Alexander I. Bobenko and Yuri B. Suris, *Discrete differential geometry: Integrable structure*, Graduate Studies in Mathematics, vol 98, American Mathematical Society, Providence, RI, 2008.

[4] Alexander I. Bobenko, Wolfgang K. Schief and Jan Techter, *Checkerboard incircular nets. Laguerre geometry and parametrisation*, arXiv preprint arXiv:1808.07254, 2018.

[5] Barry Spain, *Analytical quadrics*, International Series of Monographs on Pure and Applied Mathematics, Vol. 14 Pergamon Press, Oxford-London-New York-Paris 1960 ix+135 pp.

[6] Bancroft Huntington Brown, *Centers of similitude and their N-dimensional analogies*, Amer. Math. Monthly 23 (1916), no. 5, 155–159.

[7] Chang Shau-Jin, Crespi Bruno and Shi Kang-Jie, *Elliptical billiard systems and the full Poncelet's theorem in* $\mathfrak{n}$ *dimensions*, J. Math. Phys. 34 (1993), no. 6, 2242-2256.

[8] Chang Shau-Jin and Shi Kang-Jie, *Billiard systms on quadric surfaces and the Poncelet theorem*, J. Math. Phys. 30 (1989), no. 4, 798-804.

[9] Chang Shau-Jin and Richard Friedberg, *Elliptical billiards and Poncelet's theorem*, J. Math. Phys. 29 (1988), no. 7, 1537-1550.

[10] Crespi Bruno, Chang Shau-Jin and Shi Kang-Jie, *Elliptical billiards and hyperelliptic functions*, J. of Math. Phys. 34 (1993), no. 6, 2257-2289.

[11] Dan Pedoe, *Geometry: A comprehensive course*, Courier Coporation, 2013.

[12] David A. Brannan, Matthew F. Esplen and Jeremy J. Gray, *Geometry*, Cambridge University press, 1999.

[13] Dmitry Fuchs and Serge Tabachnikov, *Mathematical omnibus: thirty lectures on classic mathematics.* Providence: American Mathematical Society, 2007.

[14] Dragović Vladimir and Milena Radnović, *On periodical trajectories of the billiard systems within an ellipsoid in* R^{d} *and generalized Cayley's condition*, J. Math. Phys 39 (1998), no 11, 5866-5869.

[15] Dragović Vladimir and Milena Radnović, *Hyperelliptic Jacobians as billiard algebra of pencils of quadrics: beyond Poncelet porisms*, Adv. Math. 219 (2008), no. 5, 1577-1607.

[16] Dragović Vladimir and Milena Radnović *Poncelet porisms and beyond. Integrable billiards, hyperelliptic Jacobians and pencils of quadrics*, Frontiers in Mathematics. Birkhäuser/Springer Basel AG, Basel, 2011. viii+293 pp.

[17] Dragović Vladimir and Milena Radnović, *Billiard algebra, integrable line congruences, and double reflection nets*, J. Nonlinear Math. Phys. 19 (2012), no. 3 1250019, 18 pp.

[18] Dragović Vladimir and Milena Radnović, and Roger Fidèle Ranomenjanahary, *Pencils of quadrics, billiard double-reflection and confocal incircular nets*, Integrable Systems: A recognition of E. Previato's work. Vol. 1, Cambridge University Press: LMS Lecture Notes Series, 2019.

[19] Georg Glaeser, Hellmuth Stachel and Boris Odehnal, *The Universe of Conics: From the ancient Greeks to 21st century developments*, Springer, 2016.

[20] Harold Scott Macdonald Coxeter, *Regular polytopes*, Courier Corporation, 1973.

[21] Helmut Pottmann and Johannes Wallner, *The focal geometry of circular and conical meshes*, Adv. Comput. Math. 29 (2008), no. 3, 249-268.

[22] Izmestiev Ivan and Serge Tabachnikov, *Ivory's theorem revisited*, J. Integrable Syst. 2 (2017), no. 1, xyx006, 36 pp.

[23] Jürgen Richter-Gebert, *Perspectives on projective geometry. A guided tour through real and complex geometry*. Springer, Heidelberg, 2011. xxii+571 pp.

[24] Leopold Flatto, *Poncelet's theorem*, American Mathematical Soc., 2009. 8.

[25] Marcel Berger, *Geometry II*, Springer-Verlag (Berlin Heidelberg), 1987.

[26] Mark Levi and Serge Tabachnikov, *The Poncelet grid and billiards in ellipses*, Amer. Math. Monthly 114 (2007), no. 10, 895-90

[27] Michèle Audin, *Geometry*, Springer, 2002.

[28] Milena Radnović, *Integrable lattices of hyperplanes related to billiards within confocal quadrics*, Publ. Inst. Math. (Beograd) (N.S.) 98(112) (2015), 25–30.

[29] Müller Christian, *Planar discrete isothermic nets of conical type*, Beitr. Algebra Geom. 57(2016), no. 2, 459-482.

[30] Phillip Griffiths and Joe Harris, *A Poncelet theorem in space*, Comment. Math. Helv. 52 (1977), no. 2, 145-160.

[31] Raymond Clare Archibald, *Centers of similitude of circles and certain theorem attributed to Monge. Were they known to the Greeks?* Amer. Math. Monthly 22 (1915), no. 1, 6–12.

[32] Richard Walker, *Monge's theorem in many dimensions*, Math. Gaz. 60 (1976), no. 413, 185-188.

[33] Richard Evan Schwartz, *The Poncelet grid*, Adv. Geom 7 (2007), no. 2, 157-175.

[34] Roger Fidèle Ranomenjanahary, *Confocal incircular nets in plane*, (presentation, GSCC, University of Texas at Dallas, April 6, 2018).

[35] Serge Tabachnikov, *Geometry and Billiards*, Student Mathematical Library, 30. American Mathematical Society, Providence, RI, Mathematics Advanced Study Semesters, University Park, PA, 2005. xii+176 pp.

[36] Stewart A. Robertson, *Polytopes and Symmetry*, Cambridge University Press, Vol. 90, 1984.

[37] Wenping Wang, Johannes Wallner and Yang Liu, *An angle criterion for conical mesh vertices*, J. Geom. Graph. 11 (2007), no. 2, 199-208.

[38] Wolfgang Böhm, *Ein geometrischer Beweis des Satzes von Ivory*, Archiv der Mathematik, 16 (1965), no. 1, 135–137, Springer.

[39] Wolfgang Böhm, *Verwandte Sätze über Kreisvierseitnetze*, Arch. Math. (Basel) 21(1970), 326-330 (German).

BIOGRAPHICAL SKETCH

Roger Fidèle Ranomenjanahary was born at Mahazoarivo Fandriana in Madagascar. He started the undergraduate studies in 2007 at the University of Antananarivo, Madagascar. He obtained his Bachelor of Science in Mathematics in 2010. During his Advanced Studies of the academic year 2012-2013 in Madagascar, he got a full scholarship from the African Institute for Mathematical Sciences (AIMS), Mbour-Sénégal and he got his Master of Science in Mathematics. Few months after, he completed his advanced studies that he began before he went to the African Institute for Mathematical Sciences and obtained his Diploma of Advanced Studies in Mathematics (Diplôme d'Études Approfondies, in French). He joined The University of Texas at Dallas to pursue the PhD program in Fall 2014. He was working under the supervision of Professor Vladimir Dragović. His research focuses on the discrete differential geometry.

CURRICULUM VITAE

Roger Fidèle Ranomenjanahary

August 2019

Contact Information:

Department of Mathematical Science
The University of Texas at Dallas
800 W. Campbell Rd.
Richardson, TX 75080-3021, U.S.A.

Voice: (972) 883-4724
Fax: (972) 883-2349
Email: `roger@utdallas.edu`

Educational History:

B.S. in Mathematical Sciences, University of Antananarivo Madagascar, 2010.
M.S. in Mathematics, African Institute for Mathematical Sciences Mbour-Senegal, 2013.
Diploma of Advanced Studies in Mathematics, University of Antananarivo Madagascar, 2013.
Ph.D in Mathematics, University of Texas at Dallas, 2019.

Geometric and combinatorial properties of nets in plane and higher-dimensions.
Ph.D. Dissertation
Department of Mathematical Science, University of Texas at Dallas
Advisors: Prof. Vladimir Dragović

Employment History:

Research Assistant, The University of Texas at Dallas, Fall 2017 – present.
Teaching Assistant, The University of Texas at Dallas, Fall 2014 – Summer 2017.
Mathematics Teacher, Private High School, Antananarivo Madagascar, 2010-2012.

Scholarship and Fellowship:

African Institute for Mathematical Sciences (Scholarship), July 2012-June 2013.