ESSAYS ON COMMERCIALIZATION OF INFORMATION TECHNOLOGY PRODUCTS

by

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A little learning is a dangerous thing;

drink deep, or taste not the Pierian spring:

there shallow draughts intoxicate the brain,

and drinking largely sobers us again.

Taken from 'An Essay on Criticism' by Alexander Pope (1688 - 1744)

In loving memory of my dear sister Laboni Ray who inspired me to keep learning.

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This dissertation addresses two important issues in software demonstrations — (i) the design of a feature-limited software demonstration, and (ii) the role of data demonstration when firms bargain over proprietary data. These aspects of software demonstrations are presented in three separate essays provided in chapters 2, 3, and 4 with an introduction in chapter 1.

In the first essay we focus on feature-limited demonstration versions where some of the features are provided free of charge while the remaining features are charged by the vendor. As the specific features included in the feature-limited version influence whether the full product is purchased or not, it is essential that the features included in the feature-limited version be selected judiciously. This study fills the gap in the literature by providing an objective approach to the design of demonstration software.

The second essay investigates the data monetization issue through a negotiation process between a seller and a buyer and also considers the role of a data demonstration by the seller to mitigate the buyer side uncertainty in valuation. Both data sellers and buyers are often unclear about the true potential of the proprietary data, which in turn affects their pricing decision. This raises a fundamental question — how should the price of a unique and proprietary data be decided and is there a way to mitigate the uncertainty in valuation

of the data? We model the pricing decision as a Nash bargaining problem where price of the data is mutually decided. The seller has an outside option and will negotiate with the buyer if they expect to gain from the negotiation. The seller also has an option to offer a data demonstration before the negotiation process to reduce the buyer side uncertainty in the valuation. Our results show that the presence of a moderately high outside option can trigger a demonstration. Interestingly, the seller is better off providing a noisy demonstration even when they do not have an outside option but the chances of a high value data is relatively high.

There are several instances where proprietary data can be quite rich and complex and extracting meaningful insights could require considerable effort in such situations. Sometimes the buyer may not always have the ability to effectively analyze the data and is willing to hire a consultant for this task. The consultant acts as a gatekeeper and works with the buyer. Once a decision is made to purchase the data, the buyer pays the seller for the data and the consultant for their services. The prices are arrived at through a negotiation process involving all three parties. The third essay investigates the outcome of the negotiation in the presence of a consultant and observes that proposing a demonstration is not beneficial for the seller in a three-party negotiation. However, it is interesting to find that if the consultant or the seller is aware of the true value while the buyer remains unaware as before, this does not help the consultant or the seller.

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CHAPTER 1

INTRODUCTION

In this chapter, we describe the three research problems studied in the dissertation and highlight our contribution with respect to the related literature.

1.1 The Design of Feature-limited Demonstration Software: Choosing the Right Features to Include

Software products are typically classified as experience goods (Chellappa and Shivendu, 2005), as their value is difficult to estimate before actual use (Nelson, 1970). Vendors often circumvent this difficulty by letting consumers "use the software for free under certain limitations such as time and functionality" (Ilan, 2001). These free evaluation (or demonstration) versions, typically referred to as *demos*, allow the customer to experience the product directly, and to estimate the values of the features included in the demo more precisely.

This study was motivated by a demo design problem drawn from the telecommunications industry. For telecommunications providers, provisioning — the setup and configuration of equipment to activate telecommunications services with customer information — plays a crucial role in improving efficiency and reducing costs. The vendors of provisioning software — telecommunications equipment manufacturers — usually signal the value of their product by providing free, feature-limited versions of their software to potential customers. Similarly, companies like Oracle, Salesforce.com and SAP use feature-limited versions of their enterprise software to entice customers into buying the full product.

Feature-limited demos eliminate the possibility of a comprehensive evaluation. However, if a carefully designed demo is made available, it helps the consumer estimate the value of the full software product more accurately (Li and Gery, 2000). Conceptually, a demo provides a potential customer with a signal of the product's utility. This signal, which is the expected value of the full version over the demo, should influence a customer's willingness to purchase the software product, and therefore, it is crucial that the signal be chosen such that it maximizes the customer's willingness to pay. The key criterion that should determine the signal, and consequently drive the design of feature-limited demos, is elucidated in a posting on the discussion board *The Business of Software*,[1] which states that, by limiting the features, "you are selling the difference between the trial version and the full version, and you make that difference bigger by disabling some features." We recognize that the vendor has to incorporate the customer's evaluation criterion into their optimization model, and explicitly incorporate this idea into the demo-design process.

Buying a full-feature version based on a free demo is similar to purchasing an upgraded version based on a basic version. Several factors, such as the obsolescence of an existing product (Mehra et al., 2014) or the cost of upgrading (Bala and Carr, 2009), affect a customer's decision to upgrade. While there is consensus in the literature on the positive value of demos, there is little research to assist with the configuration and design of demo software. The research that does exist focuses primarily on various aspects of quality uncertainty, and the role of demos in mitigating this uncertainty. Feature-limited software can be viewed through the lens of versioning, where a product is vertically differentiated into free and premium versions. Versioning of products using free trials and samples to increase the customer's willingness to pay for the full version has been shown to be an effective strategy for promotions (Lewis and Sappington, 1994; Wang and Zhang, 2009). Wei and Nault (2014) consider the monopolist strategy of selling feature-limited information goods to customers who differ in individual tastes for quality, and find that any horizontal differentiation favors versioning. In the context of learning by word-of-mouth, Niculescu and Wu (2014) find that feature-limited software is optimal when the prior on features not in the demo is either relatively low or high, but not in between. Li and Gery (2000) suggest that demos allow customers to

[1] discuss.joelonsoftware.com/default.asp?biz.5.499021.12

make a more informed estimation of a product's true value, thereby reducing the uncertainty faced by the customer. Roberts and Urban (1988) argue that by reducing uncertainty, demos help software vendors increase the consumer's perceived value of the product. Seufert (2013) shows how a data-driven approach is best implemented in the development of feature-limited software. Using case studies of successful feature limited demos, Semenzin et al. (2012) suggest that it might be better to make the core features of the software available for free. Lee et al. (2013) consider consumer usage, upgrades and referrals to find the value of the demo relative to the premium product. Similarly, freemium.org has a model that uses conversion rate, number of free users, marginal cost per free user and contribution per premium product to decide the optimum mix of free and charged features. However, these approaches are less appropriate in the context of designing customized industrial software, where the customers are few but large, and where vendors usually has a deeper understanding of the tasks being performed by end users.

This study makes various contributions, the most important of which are a formal representation of the design of feature-limited demo as an NP-complete optimization problem, and fast, effective procedures to solve it. While prior research has investigated strategies for providing demos (e.g., Niculescu and Wu, 2014; Cheng et al., 2015), very little research exists that provides a scientific basis for the *design* of feature-limited demos.

In chapter 2, we present the customer's problem of estimating the incremental benefit of purchasing the complete software product, given a feature-limited demo. The benefits could be through *productivity* gains resulting from the increased efficiency with which a customer can carry out the tasks they need to perform, or from a *re-engineering* effect which enables the customer to execute tasks that were not being performed before. We formulate the customer's problem as an integer program, and show that it is easy to solve. Next, we present the vendor's problem of designing the best feature-limited demo to provide to a specific customer, as a non-linear integer program. We show that the vendor's problem is NP-complete in the strong sense, and that state-of-the-art solvers like BARON and CPLEX have difficulty solving realistic versions of the problem. We solve the problem effectively using a fast converging "scenario aggregation" (SA) approach based on Lagrangian decomposition (Guignard and Kim, 1987). As solving the problem in situations where the parameter variability is low is observed to be time-consuming, we develop a continuous version of the problem that exploits the fact that the parameters have low variability. We incorporate the results from the continuous model into an "augmented heuristic" that solves the problem very quickly.

1.2 Bargaining Over Data: When does Making the Buyer More Informed Help?

The explosive growth of eBusiness has allowed many companies to accumulate a repertoire of unique data sets that can provide substantial value to other firms. These data sets are a growing source of revenue for their owners — one that can generate millions of dollars each year. Hence, the natural question arises: How much is a specific data set worth? This turns out to be a difficult question to answer, as neither data owners nor potential buyers are usually aware of the full potential of the data in question. According to the *Wall Street Journal* "data isn't a physical asset like a factory or cash, and there aren't any official guidelines for assessing its value" (Monga, 2014).

Perhaps the best known example of data owners profiting from the sale of proprietary data is the retail industry, where retailers are aware that consumer goods manufacturers also have a lot to gain from their point-of-sale data. Monga (2014) reports that Kroger, a well-known supermarket operator, collects information on customer purchases at more than 2,600 stores, while tracking approximately 55 million loyalty-card members. Kroger's revenues from the sale of their data is estimated to be in the order of $100 million a year (Monga, 2014). This situation is not unique to the supermarket context. For example, Global Distribution Systems (GDSs) — information systems created for travel agencies to manage reservations and purchases of airline tickets, hotel rooms, rental cars, and other tourist services — collect substantial volumes of data on customer shopping activity. This data can provide airlines and hotel chains with considerable insights — an understanding of how well they are converting travel demand into bookings, for instance — and enable them to make better decisions on new promotions, routes, schedules and room capacity. There is a growing list of companies who sell their proprietary data — for example, credit card companies sell consumer purchase data to advertisers who can target consumers with their advertisements (Edwards, 2013), Facebook sells data to advertisers (Moran, 2015), banks sell consumer data (Jones, 2013), while Yodlee, a provider of online personal-finance tools to many large banks, sells data from credit and debit card transactions to investors and research firms, who mine it for clues on trends that can affect stock prices (Hope, 2015).

Our focus in this study is the sale of proprietary data through a negotiation (or bargaining) process in a B2B context. Unfortunately, while data owners know that their clients will be able to leverage their unique data to gain insights, they are often unclear on the price they can charge for it. Their knowledge of the client's business needs is limited, and this limits the extent to which they can estimate the value of the data to the firm buying it. This raises a fundamental question — how should the data owners price their data? There is uncertainty on the buyer's side as well — while they know their business objectives, they have only limited knowledge about the data (for example, through a data dictionary provided by the seller). Consequently, the buyer does not know exactly how useful the data will be to them, which limits the price they are willing to pay for it. In this context, the price of the data is often arrived at through a negotiation process between the two parties. This two-sided uncertainty in the valuation of data is, to some extent, similar to that of intangible goods like intellectual property (e.g., patents), and negotiation has been studied in that domain (Kishimoto and Muto, 2012; Gans et al., 2008; Walden, 2005; Lai and Qiu, 2003; Gans and Stern, 2000). Problems involving incomplete information between a buyer and a seller has been extensively studied from the perspective of a bargaining process where each player knows their own valuation of the good, but not that of the other (Ausubel et al., 2002; Fudenberg and Tirole, 1983; Myerson and Satterthwaite, 1983). In our context, neither player initially knows their own valuation of the data product.

There is evidence that providing additional information in the form of demonstrations and samples in advance can reduce valuation uncertainty and increase a buyer's willingness to pay, even when it puts the seller at an information disadvantage (Shapiro, 1983; Lewis and Sappington, 1994; Chellappa and Shivendu, 2005; Wang and Zhang, 2009; Niculescu and Wu, 2014; Cheng et al., 2015; Bhargava and Chen, 2012). In our context, these demonstrations typically involve or presentations that showcase the value of the data through dashboards that illustrate its benefits (for example, in the form of analytical tools, summary statistics, relevant plots and decision criteria). Seller can also provide a sample data as a free trial which can be used by the buyer in their own model to learn the value of the data. Davie (2015) notes that "the most common approach for companies who have embarked on data monetization is to develop a dashboard or application for the data...". It is to be noted that the sole purpose of such samples and dashboard is to let the buyer update their belief about the value of the data. Buyer can use this sample data in their own model and learn some of the nuances about the data but not all which will result in partial learning on the part of the buyer. As an example, a sample data can provide a noisy information about the distribution of the variables (fields), the actual distribution can only be learned after using the entire data. Our model captures this important aspect of data where the seller can control the amount of information revealed to the buyer through a data demonstration. The level of granularity in the data provides enough flexibility to control the accuracy of the signal which is difficult to achieve in other information goods. Although showing a free data demonstration in advance can increase a buyer's willingness to pay, conversely it puts the seller at an information disadvantage (Shapiro, 1983; Niculescu and Wu, 2014). Therefore, the decision of whether or not a demonstration should be provided is an important one, and this paper provides insights into situations where a seller should (and should not) propose a demonstration to the buyer before the negotiation process.

It is known that buyers often underestimate value in situations when the true value is unknown (Shapiro, 1983; Heiman and Muller, 1996; Chellappa and Shivendu, 2005; Cheng and Liu, 2012; Wei and Nault, 2013; Cheng et al., 2015; Niculescu and Wu, 2014). As a result, it might become necessary for the seller to signal to the buyer that the data is worth more than their (under)estimated value, and a well designed demonstration is an effective tool to accomplish this objective. This demonstration clarifies to the buyer that they have underestimated its value, by letting them realize the true value of the data. This creates an interesting situation that could benefit both parties — while more information could benefit the buyer in the negotiation process, correcting underestimation could benefit the seller. Our result corroborate the previous findings by Shapiro (1983); Cheng and Liu (2012), and Niculescu and Wu (2014) which states that the extent of underestimation affects the outcome — the extent of underestimation has to be enough to make the seller better off by making a demonstration that reveals the true value to the buyer. Buyer can also underestimate the probability distribution of the value in which case we find that it is equivalent to the situation when buyer underestimate the value. What sets our paper apart is that providing demonstration is optimal even in the absence of underestimation when seller has a backup option.

We consider the situation where the seller also has a fixed outside option, and bargains with the buyer in the hope of getting a higher expected price. We assume that both the players are aware of this outside option. Seller has an option to offer a free data demonstration to the buyer to help them learn the value of the data. We represent the negotiation between the buyer and the seller as a generalized Nash bargaining process initially proposed by Harsanyi and Selten (1972) to incorporate the information asymmetry. We adopt an incentive compatible decision mechanism from Myerson (1979, 1984) which maps the players' information about the value of the data to bargaining outcomes. Thus our model is structurally different from the previous models on trials which consider a monopolist selling information products in a market with heterogenous preferences. In prior studies the price of the product is fixed by the seller however, we consider that the price is mutually decided by the seller and the buyer through a negotiation. Our result shows that the seller will strictly benefit by making the buyer fully aware of the data (thereby increasing information asymmetry and putting the seller at an information disadvantage) when the value of the outside option to the seller is moderately high. Interestingly, it is optimum for the seller to let the buyer *partially* learn the value of the data by providing a noisy signal through a demonstration even in the absence of an outside option and at an information disadvantage position. This is in contrast to the previous literature on learning which states that a monopolist seller always finds it most profitable either to provide the best possible information to the buyer through free pre-sale promotions, or not to provide any new information at all (Lewis and Sappington, 1994). Equally interesting is the fact that this noisy signal is *detrimental* for the buyer even when the buyer is at an information advantageous position. In addition, when the value of the outside option is high, a demonstration has the potential to make otherwise unsuccessful negotiations successful.

This paper makes significant contributions to the existing literature on software trails. It is the first to systematically analyze the negotiation process involved in the monetization of proprietary data. We find that there are circumstances under which the presence of an outside option can trigger a demonstration. A demonstration can also be critical when the buyer has underestimated the value of the data, and can make otherwise unsuccessful negotiations successful. One interesting insight is that it does not matter whether seller update their belief about the value as only the belief of the buyer governs the negotiation outcome.

1.3 Bargaining Over Data with the Consultant as a Gatekeeper

Often, proprietary datasets are rich and complex, with the potential to provide meaningful insights. For example, the search history of travellers before they made their final booking can provide airlines with a wealth of information on customer preferences which can be used to offer customized recommendations (for hotel rooms, car rentals, etc.). Extracting reliable and actionable insights in such contexts are usually nontrivial and requires experienced, sophisticated analysis. There are instances where buyer had hired a consultant to provide data analytic services. In such situations, buyer also involve the consultant in the negotiation process where all three firms, seller, buyer, and consultant collectively decide the price of the data and the service charge for the consultant respectively. In this essay we develop a three-party bargaining framework to analyze this data monetization process where the consultant plays the role of a gatekeeper and incur a processing cost to provide data analytic services.

Chapter 4 addresses the data monetization process in a three-player bargaining setup. We adopt the Nash bargaining process (Nash, 1950) to calculate the expected payoffs to the players. We do not consider any outside option for the seller as in the previous essay with two-player negotiation. As before, seller has an option to offer a data demonstration to the buyer. We assume that the demonstration is not noisy and will provide a complete information to the buyer about the value of the data. If the seller proposes a demonstration, the buyer has the option to invite the consultant to attend. Two related questions in this context are whether the buyer should attend, and if so, whether they should invite the consultant — i.e., will the buyer gain more if the consultant is also aware of the value before the negotiation? If invited by the buyer, the consultant can choose to accept or decline the invitation, depending on what is best for them. The buyer also has the option of sharing the true value of the report with the seller after the demonstration if sharing is to the buyer's benefit. In order to better understand the options, we analyze each of four possible scenarios — (i) when the true value of the report is unknown to all three players (corresponding to the scenario where no demonstration is offered to the buyer, or where one is offered but declined by the buyer), (ii) when only the buyer is aware of the true value with the others knowing its distribution (corresponding to the scenario where the buyer does not invite the consultant, or where the consultant declines the buyer's invitation), (iii) when both the buyer and the consultant are aware of the true value, with the seller knowing only the distribution (corresponding to the situation where a demonstration is offered, and accepted by both the buyer and the consultant), and (iv) when both the buyer and the seller are aware of the true value, with the consultant knowing only the distribution (corresponding to the situation where the buyer reveals the true value to the seller, and the consultant does not attend the demonstration). Note that the scenario when all the players become aware of the true value is effectively equivalent to the scenario when the true value is unknown to all the players. The only difference is that the players now divide the true value rather than the expected value equally among them.

Our results suggest that the the seller does not gain by offering a demonstration to the buyer. Similar to the two-player negotiation, if the buyer underestimate the true value then seller has the opportunity to offer a demonstration. However, this opportunity of demonstration is less when the consultant is also involved in the negotiation process. It is interesting to find that if the consultant or the seller is aware of the true value while the buyer remains unaware as before, does not help the consultant or the seller.

The rest of the dissertation proceeds as follows: Chapter 2 analyses the design of the feature-limited demonstration. Chapter 3 and chapter 4 investigate the data demonstration under two-player and three-player bargaining setups.

CHAPTER 2

THE DESIGN OF FEATURE-LIMITED DEMONSTRATION SOFTWARE: CHOOSING THE RIGHT FEATURES TO INCLUDE[1]

2.1 The Customer's Problem

Given a feature-limited demo, the customer has to estimate the incremental benefit the complete software product will provide over the demo. This incremental benefit includes the benefit from new tasks that the customer can perform (re-engineering effect) and from the ability to perform existing tasks more easily (productivity effect).

2.1.1 Terminology and Related Notation

We consider customers who routinely perform a set of *tasks*. The relevant set of tasks $\mathcal{T}$ consists of all the tasks that can be performed profitably with the full version of the software. The profit from performing task j without the software is π_j. We assume that it is not difficult for the customers to estimate π_j, as the revenue and cost associated with each task j can be estimated quite precisely. If the customer can profitably perform task j without the software, π_j is positive. On the other hand, π_j will be negative for those tasks for which the costs exceed the benefits (without the software). The software product increases the efficiency of tasks currently being done (i.e., tasks with $\pi_j > 0$), and potentially enables tasks that are currently too expensive (i.e., tasks with $\pi_j < 0$) to be done profitably. The product is composed of a set of *features* $\mathcal{F}$, that contribute to improving the net benefit of performing the tasks.

[1]Reprinted with permission from Ray, J., Samuel, J., Menon, S. and Mookerjee, V., "The Design of Feature-Limited Demonstration Software: Choosing the Right Features to Include." *Production and Operations Management*, 26(1): 9-30. Jan. 2017. Published by John Wiley and Sons. © 2016 Production and Operations Management Society.

The set of features $\mathcal{D} \subseteq \mathcal{F}$ included in the evaluation software by the vendor comprises the *demo*. Given a demo $\mathcal{D}$, the customer experiences the features included in it, and determines the benefit β_{ij} from feature $i \in \mathcal{D}$ to task $j \in \mathcal{T}$ for each feature included in the demo. β_{ij} is the gain in productivity and efficiency over the cost of effort and time to learn the feature. The software provides the customer the ability to perform tasks more efficiently than before, by reducing the cost associated with doing them. This reduction in cost is captured by the β_{ij}. Given that the customers can experience these features for as long as needed, it is reasonable to assume that they will arrive at relatively precise values for the β_{ij}. However, the β_{ij} values cannot be estimated very precisely for features not included in the demo. Consequently, the customer estimates the β_{ij} values as b_{ij} for all features $i \notin \mathcal{D}$, based on the help files and user guides that accompany the software. Existing research suggests that there is reason to expect that b_{ij} will be consistently lower than the corresponding β_{ij} (Shapiro, 1983; Heiman and Muller, 1996).

The problem facing the customer is that of identifying the tasks to perform had the complete software product been available, and of estimating the total benefit from performing these tasks. The original total profit, when no software or demo is available, is $\Pi_{\varnothing} = \sum_{\{j \in \mathcal{T} | \pi_j > 0\}} \pi_j$, while the total profit with the complete software product is denoted $\Pi_{\mathcal{F}}$. A demo allows the customer to better estimate $\Pi_{\mathcal{F}}$ using the features included in it. The value of $\Pi_{\mathcal{F}}$ estimated by the customer when given a demo $\mathcal{D}$ is denoted $\tilde{\Pi}_{\mathcal{F}}^{\mathcal{D}}$, while the value of $\Pi_{\mathcal{F}}$ estimated in the absence of a demo is denoted $\tilde{\Pi}_{\mathcal{F}}^{\varnothing}$. Note that $\tilde{\Pi}_{\mathcal{F}}^{\mathcal{D}}$ is estimated using the β_{ij} values for all features $i \in \mathcal{D}$ and the b_{ij} values for all features $i \notin \mathcal{D}$, while $\tilde{\Pi}_{\mathcal{F}}^{\varnothing}$ is obtained using the estimates b_{ij} for all features $i \in \mathcal{F}$. As customers tend to underestimate their evaluations in the face of uncertainty, one would expect $\tilde{\Pi}_{\mathcal{F}}^{\mathcal{D}} \geq \tilde{\Pi}_{\mathcal{F}}^{\varnothing}$ in general. The demo itself might have intrinsic value, and the profit with just the demo $\mathcal{D}$ is denoted as $\Pi_{\mathcal{D}}$. If the demo has intrinsic value, $\Pi_{\mathcal{D}}$ will be greater than $\Pi_{\varnothing}$.

The *value* of the software product to the customer, $\Upsilon_{\mathcal{F}}$, is the incremental profit resulting from the use of the software product, i.e., $\Upsilon_{\mathcal{F}} = (\Pi_{\mathcal{F}} - \Pi_{\varnothing})$. The estimated value of the product given a demo $\mathcal{D}$ is $\tilde{\Upsilon}_{\mathcal{F}}^{\mathcal{D}} = (\tilde{\Pi}_{\mathcal{F}}^{\mathcal{D}} - \Pi_{\varnothing})$. In the absence of a demo, the estimated value of the product is $\tilde{\Upsilon}_{\mathcal{F}}^{\varnothing} = (\tilde{\Pi}_{\mathcal{F}}^{\varnothing} - \Pi_{\varnothing})$. The intrinsic value of the demo $\Upsilon_{\mathcal{D}} = (\Pi_{\mathcal{D}} - \Pi_{\varnothing})$ is the increase in profit as a result of using only the features in $\mathcal{D}$; this comprises the re-engineering benefit from any new tasks enabled by the demo, and the productivity benefit from the increased efficiency in performing currently executed tasks.

The *signal* Ψ is defined as the gap between the estimated product value and the value of the demo, i.e., $\Psi = (\tilde{\Upsilon}_{\mathcal{F}}^{\mathcal{D}} - \Upsilon_{\mathcal{D}})$ when a demo $\mathcal{D}$ is provided, and $\Psi = \tilde{\Upsilon}_{\mathcal{F}}^{\varnothing}$ when there is no demo. Note that given a demo $\mathcal{D}$, Ψ can also be expressed as $(\tilde{\Pi}_{\mathcal{F}}^{\mathcal{D}} - \Pi_{\mathcal{D}})$. Table 2.1 summarizes the notation used in this paper for easy reference.

Table 2.1. Relevant Notation

$\mathcal{T}$	Set of relevant tasks	$\tilde{\Pi}_{\mathcal{F}}^{\mathcal{D}}$	Customer's estimate of $\Pi_{\mathcal{F}}$, given demo $\mathcal{D}$
$\mathcal{F}$	Set of software features	$\tilde{\Pi}_{\mathcal{F}}^{\varnothing}$	Customer's estimate of $\Pi_{\mathcal{F}}$, without demo
π_j	Pre-software profit from task j	$\Pi_{\mathcal{D}}$	Profit with demo $\mathcal{D}$
$\mathcal{D}$	Demo ($\subseteq \mathcal{F}$)	$\Upsilon_{\mathcal{F}}$	Incremental profit with software = $(\Pi_{\mathcal{F}} - \Pi_{\varnothing})$
β_{ij}	Benefit from feature i to task j	$\tilde{\Upsilon}_{\mathcal{F}}^{\mathcal{D}}$	Estimate of $\Upsilon_{\mathcal{F}}$, given demo $\mathcal{D} = (\tilde{\Pi}_{\mathcal{F}}^{\mathcal{D}} - \Pi_{\varnothing})$
b_{ij}	Customer's estimate of β_{ij}	$\tilde{\Upsilon}_{\mathcal{F}}^{\varnothing}$	Estimate of $\Upsilon_{\mathcal{F}}$, without demo = $(\tilde{\Pi}_{\mathcal{F}}^{\varnothing} - \Pi_{\varnothing})$
$\Pi_{\varnothing}$	Total pre-software profit	$\Upsilon_{\mathcal{D}}$	Intrinsic value of demo $\mathcal{D} = (\Pi_{\mathcal{D}} - \Pi_{\varnothing})$
$\Pi_{\mathcal{F}}$	Profit with software	Ψ	The signal $\begin{cases} = (\tilde{\Upsilon}_{\mathcal{F}}^{\mathcal{D}} - \Upsilon_{\mathcal{D}}) \text{ with demo } \mathcal{D} \\ = \tilde{\Upsilon}_{\mathcal{F}}^{\varnothing} \text{ without demo} \end{cases}$

2.1.2 An Illustrative Example

Consider the simplified network monitoring software in Figure 2.1. It has six features, represented by the nodes in the lower part of the figure; i.e., $\mathcal{F} = \{f_1,\ f_2,\ f_3,\ f_4,\ f_5,\ f_6\}$. Features are viewed from a logical (rather than an implementation) perspective, and we assume that the contribution of any given feature to a task is not dependent on the contributions of the other features. This software allows the customer to potentially perform the two tasks represented at the top of the figure; i.e., the task set is $\mathcal{T} = \{t_1,\ t_2\}$. Each task has a profit

π_j, as noted in the figure. When the profit from a task is positive (as is the case for task t_1), that task will be performed even without the software. Any contributions of the software to tasks with positive π_j will be through productivity impacts. If π_j is negative (as is the case for task t_2), it is not cost-effective for the customer to perform the task without the software. The contribution of the software to tasks with negative π_j will potentially allow for new tasks to be performed, which is a re-engineering impact.

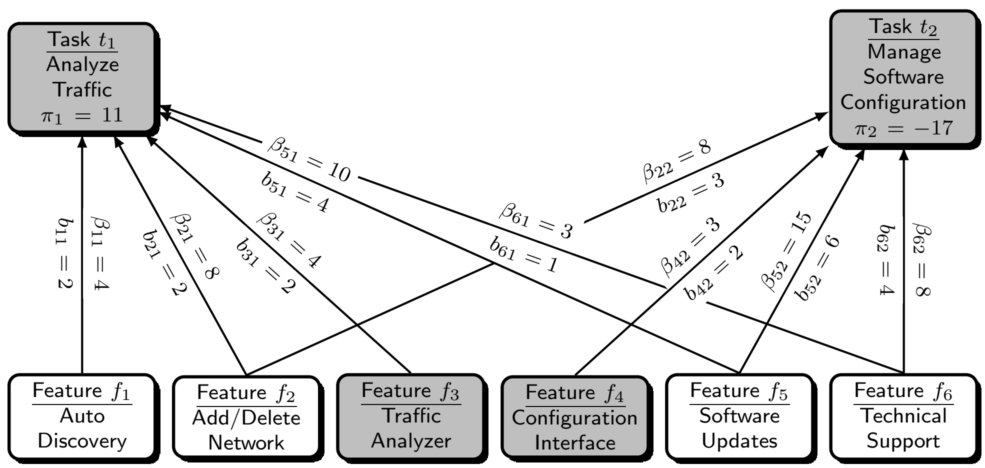


Figure 2.1. Simplified Example

Feature i and task j are connected by an edge if the availability of feature i will improve the profit from task i (by β_{ij}). The customer can experience all features included in the demo, and arrive at precise estimates of the β_{ij} associated with each. If a feature i is not included in the demo, the customer will have to use the product description to estimate (as b_{ij}) its contribution to each task. The values of β_{ij} and b_{ij} for relevant feature-task pairs are indicated on the links in the figure. Note that a task could benefit even when a subset of the features relevant to that task is included in the demo.

Prior to the software being considered for evaluation, π_1 was 11 (positive) and π_2 was -17 (negative). Therefore, task t_1 was being performed, while task t_2 was not; the total profit $\Pi_\varnothing$ was simply that from task t_1, i.e., 11. Let features f_3 and f_4 (i.e., Traffic Analyzer and Configuration Interface) comprise the demo provided to the customer, i.e., $\mathcal{D} = \{f_3,\ f_4\}$. f_3

contributes only to t_1, while f_4 contributes only towards t_2. Consequently, the customer will experience the demo and identify the correct values of β_{31} and β_{42}. For all other feature-task pairs, the customer estimates the β_{ij} as b_{ij}.

2.1.3 Incorporating Uncertainty: The Scenario Approach

Recall that the customer approximates the β_{ij} values associated with the features not in the demo as b_{ij}. Based on existing literature (e.g., Shapiro, 1983; Heiman and Muller, 1996), it is reasonable to assume that b_{ij} will be lower than β_{ij} (though the models developed in this paper do not require this to be the case). The experience-good nature of software products implies that there is uncertainty in estimating the β_{ij} values. To obtain these estimates, customers can refer to user manuals, obtain expert opinions, or contact the vendor for more information. As far as the customer is concerned, we assume that b_{ij} is a random draw from $[0, \beta_{ij}]$, to account for underestimation.

The customer's problem can be viewed as a multi-period planning problem under uncertainty. The decision to purchase the software has to be made in the first period in the presence of uncertainty. If the product is purchased, the estimated incremental profit is $\Psi = (\tilde{\Upsilon}^{\mathcal{D}}_{\mathcal{F}} - \Upsilon_{\mathcal{D}})$ as explained earlier. On the other hand, if the product is not purchased, the associated profit is the original profit plus the intrinsic value of the demo.

Given the inherent non-linearity associated with integer programming, we cannot replace a random quantity (β_{ij}) by its expected value. For instance, if expected values we used for β_{ij} in the example in Section 2.1.2, the customer incorrectly infer that no new tasks will be performed with the new software. This erroneous conclusion hurts both the customer and the vendor. To estimate the value of the full product, customers must first arrive at an estimate of the set of tasks that will be performed with the full set of features. This set of tasks is identified for the sole purpose of arriving at an estimate of product value — if the product is purchased, all β_{ij} values will be precisely estimated, and the customer will be able to identify the actual set of tasks that can be performed effectively with the software. The customer's decision problem therefore is a stochastic program with *complete* recourse.

One approach to solving stochastic programs is to approximate the uncertainty in the parameters through a set of scenarios with associated probabilities. A scenario can be interpreted as a state of nature, or equivalently, as one possible outcome of the uncertain parameters. In most cases, incorporating all possible scenarios into the decision making process is not practical as the number of scenarios is too large. Consequently, the *deterministic equivalent*[2] is usually approximated by a model involving fewer scenarios. If the number of scenarios used is large enough, the approximation obtained will be of good quality.

In the context of the customer's problem, each scenario s is represented by a random draw b_{ij}^s from $[0, \beta_{ij}]$ for each β_{ij} that needs to be estimated. The customer needs to estimate the likely value of the complete product, given that the β_{ij} values are known for each feature i included in the demo $\mathcal{D}$, and have been estimated as b_{ij}^s for each feature not included in the demo. The estimated benefit from doing a specific task j comprises (i) the original profit π_j, (ii) the improvements β_{ij} guaranteed by all features $i \in \mathcal{D}$ included in the demo, and (iii) the estimated improvements b_{ij}^s from the features $i \in \mathcal{F} \setminus \mathcal{D}$. Given a set $\mathcal{S}$ of scenarios and probabilities p^s associated with each scenario $s \in \mathcal{S}$, formulation (CS) will provide the incremental value of the product, $\tilde{\Upsilon}_{\mathcal{F}}^{\mathcal{D}} = (\tilde{\Pi}_{\mathcal{F}}^{\mathcal{D}} - \Pi_{\varnothing})$, after defining the task selection variable y_j^s, which is 1 if task j is selected for execution under scenario s, and 0 otherwise.

$$(CS) \quad \text{Max} \quad \tilde{\Upsilon}_{\mathcal{F}}^{\mathcal{D}} = \sum_{s \in \mathcal{S}} p^s \left(\sum_{j \in \mathcal{T}} \left(\pi_j + \sum_{i \in \mathcal{D}} \beta_{ij} + \sum_{i \in \mathcal{F} \setminus \mathcal{D}} b_{ij}^s \right) y_j^s - \sum_{\{j \in \mathcal{T} | \pi_j > 0\}} \pi_j \right) \tag{2.1}$$

$$\text{s.t.} \quad y_j^s \in \{0, 1\} \qquad \forall j \in \mathcal{T}; \forall s \in \mathcal{S} \tag{2.2}$$

The objective function coefficient of variable y_j^s comprises (i) the value of performing task j without the software (π_j), (ii) the contribution to task j of all features $i \in \mathcal{D}$ included in the

[2] A deterministic equivalent of a stochastic program is a deterministic model that incorporates in it every possible scenario.

demo ($\sum_{i\in\mathcal{D}} \beta_{ij}$), and (iii) the estimated contribution to task j under scenario s of all features $i \in \mathcal{F} \setminus \mathcal{D}$ that have been left out of the demo ($\sum_{i\in\mathcal{F}\setminus\mathcal{D}} b^s_{ij}$). The constant term $\sum_{\{j\in\mathcal{T}|\pi_j>0\}} \pi_j$ that is subtracted from the objective function is $\Pi_\varnothing$, the benefit that was being derived from the tasks before the demo became available. It is easy to see that the optimal solution to (CS) will comprise those tasks that will have a positive net benefit after considering all three components of the objective function coefficient, i.e., the optimal solution is $y^s_j = 1 \ \forall j \ni (\pi_j + \sum_{i\in\mathcal{D}} \beta_{ij} + \sum_{i\in\mathcal{F}\setminus\mathcal{D}} b^s_{ij}) > 0$, and $y^s_j = 0$ for all other tasks. Effectively, the estimated net contribution from a newly enabled task j is $(\pi_j + \sum_{i\in\mathcal{D}} \beta_{ij} + \sum_{i\in\mathcal{F}\setminus\mathcal{D}} b^s_{ij})$, while that from a task that is currently being done is $(\sum_{i\in\mathcal{D}} \beta_{ij} + \sum_{i\in\mathcal{F}\setminus\mathcal{D}} b^s_{ij})$ — for such tasks, the benefit of π_j was already being generated, and cannot be attributed to the software. The tasks selected by (CS) are those that the customer would expect to perform once the complete version of the software is available. Note that the tasks identified to be performed do not have to be identical across the scenarios, as the intent is to obtain a good estimate of the signal, and not to precisely pinpoint the tasks to be performed with the software. This implies that the decision variables can be defined distinctly for each scenario, and consequently, formulation (CS) decomposes into disconnected subproblems (CS^s) for each scenario s.

$$(CS^s) \quad \text{Max} \quad \tilde{\Upsilon}^{\mathcal{D}s}_{\mathcal{F}} = \sum_{j\in\mathcal{T}} \left(\pi_j + \sum_{i\in\mathcal{D}} \beta_{ij} + \sum_{i\in\mathcal{F}\setminus\mathcal{D}} b^s_{ij} \right) y^s_j - \sum_{\{j\in\mathcal{T}|\pi_j>0\}} \pi_j \tag{2.3}$$

$$\text{s.t.} \quad y^s_j \in \{0,\ 1\} \qquad \forall j \in \mathcal{T}; \forall s \in \mathcal{S} \tag{2.4}$$

Based on our initial motivation of the telecommunication provisioning problem, we have considered that the customer can perform each task independent of the others. If a task j_1 is dependent on another task j_2, that is, if task j_1 can only be performed when task j_2 is complete, we need to add a constraint $y^s_{j_2} \geq y^s_{j_1}$ to (CS^s). Given the parameters for a scenario, the customer's problem remains easy — all the customer needs to do is solve (CS^s) to obtain the optimal objective function value $\tilde{\Upsilon}^{\mathcal{D}s}_{\mathcal{F}}$ associated with scenario s. Therefore, given the set of enumerated scenarios $\mathcal{S}$, and the probabilities p^s associated with each scenario $s \in \mathcal{S}$, the customer can estimate the value of the product as $\tilde{\Upsilon}^{\mathcal{D}}_{\mathcal{F}} = \sum_{s \in \mathcal{S}} p^s \tilde{\Upsilon}^{\mathcal{D}s}_{\mathcal{F}}$. The probabilities p^s can be obtained from any distribution that is appropriate in a given context; if the scenarios is equally likely, p^s is simply $\frac{1}{|\mathcal{S}|}$.

As the demo being considered is a feature-limited one, the customer has to identify the intrinsic value of the demo and subtract this value from the optimal objective function value of (CS) in order to identify the real benefit from buying the software. The intrinsic value of the demo $\Upsilon_{\mathcal{D}}$ can be obtained by solving a variant of formulation (CS), where the third term is removed from the objective function coefficient of the y^s_j variables. Now there is no need to consider the scenarios as there is no uncertainty involved.

2.2 The Vendor's Problem

The vendor has to design the best feature-limited demo possible. The best feature-limited demo will comprise those features that will maximize the value of the signal detected by the customer. The process of identifying the features to include in the best feature-limited demo has to incorporate in it the customer's evaluation approach. In the previous section, we established that the best way in which the customer can identify the signal is through the scenario approach. The vendor needs to include this process into the design process and therefore the customer's problem and the vendor's problem are intricately linked together. Identifying the best signal is in the interest of the customer, as is providing a demo that will result in the best signal in the interest of the vendor. To this extent, the objectives of both parties are aligned, and there is no reason for the customer to intentionally use an inferior approach to estimate the signal.

2.2.1 The Discrete Model With Uncertainty: The Scenario Approach

The discrete version of the vendor's problem can be formulated as (VS), after defining the following additional variables — (i) x_i, which is 1 if feature i is included in the demo, and

0 otherwise, and (ii) ρ_j, which is the actual benefit to the customer from task j, when only the features included in the demo are considered. For tasks with $\left(\pi_j + \sum_{i \in \mathcal{F}} \beta_{ij} x_i\right) < 0$, the features included in the demo are not enough to make the task profitable, and therefore, they result in no value being given away. However, when $\left(\pi_j + \sum_{i \in \mathcal{F}} \beta_{ij} x_i\right) > 0$ for any task, the vendor is giving some value of the associated features away for free. To the extent that there may be tasks with $\left(\pi_j + \sum_{i \in \mathcal{F}} \beta_{ij} x_i\right) < 0$ involving some of the same features however, not all of the value associated with the features are being given away. It is also possible for a demo to contain many features, but have no intrinsic value to the customer. The variables ρ_j capture the value given away towards task j through features included in the demo. When added up over all tasks $j \in \mathcal{T}$, the ρ_j variables give the total profit to the customer from using the demo.

$$
\begin{aligned}
(VS) \quad \text{Max} \quad & \Psi^{VS} = \sum_{s \in \mathcal{S}} p^s \left(\sum_{j \in \mathcal{T}} \left(\pi_j + \sum_{i \in \mathcal{F}} \beta_{ij} x_i + \sum_{i \in \mathcal{F}} b^s_{ij} (1 - x_i) \right) y^s_j \right) - \sum_{j \in T} \rho_j && (2.5) \\
\text{s.t.} \quad & \rho_j \geq \left(\pi_j + \sum_{i \in \mathcal{F}} \beta_{ij} x_i \right) \qquad \forall j \in \mathcal{T} && (2.6) \\
& \rho_j \geq 0 \qquad \forall j \in \mathcal{T} && (2.7) \\
& x_i \in \{0,\ 1\} \qquad \forall i \in \mathcal{F} && (2.8) \\
& y^s_j \in \{0,\ 1\} \qquad \forall j \in \mathcal{T}; \forall s \in \mathcal{S} && (2.9)
\end{aligned}
$$

The objective function maximizes the estimated strength of the signal by taking the expectation over the signals from each of the scenarios. The estimate of the signal obtained by the vendor need not be identical to that obtained by the customer as the scenarios generated by the two parties are likely to be different, and the vendor is constrained in the number of scenarios that can be used when solving formulation (VS). As with the customer's problem, the objective function coeffiecient of y^s_j has three components — π_j, the value of performing task j without the software, $\sum_{i \in \mathcal{F}} \beta_{ij} x_i$, the contribution to task j of all features $i \in \mathcal{D}$ included in the demo, and $\sum_{i \in \mathcal{F}} b^s_{ij} (1 - x_i)$, the estimated contribution to task j of all features $i \in \mathcal{F} \setminus \mathcal{D}$ that have been left out of the demo. Constraints (2.6) and (2.7) together ensure that all tasks that provide a net benefit with the demo count towards the profit from the demo, and that no task with a negative net benefit is performed. As the customer has the option not to perform task j, the intrinsic value of the demo cannot be negative. This is captured by constraint (2.7), the nonnegativity constraint on ρ_j. If the demo has intrinsic (positive) value — i.e., if $\left(\pi_j + \sum_{i \in \mathcal{D}} \beta_i\right) > 0$ — ρ_j will be set to that value, and subtracted from the objective function. The value of the features included has to be greater than $(-\pi_j)$ before the customer can start benefiting from performing task j. As with the customer's problem, the probability p^s is $\frac{1}{|\mathcal{S}|}$ if the scenarios are equally likely. Depending on the number of scenarios used and the specific values of the b^s_{ij} that are realized in each scenario, the value of the signal obtained can vary. As the number of scenarios increases however, this variation can be expected to decrease.

As we are considering customized software where the vendor has a good understanding of the customer's cost structure, we have assumed that it is possible for the vendor to obtain reasonable estimates of a customer's π_j, and β_{ij} parameters. While this may not be possible precisely, it will be possible for the vendor to get a good idea by consulting domain experts, sales managers and others who know the customer well to get good estimates of these parameters. We are only assuming that the vendor has *distributional* information concerning the value customers have for tasks and for the amount a feature contributes to a particular task. The exact values of these parameters are not known to the vendor. Even if there is some uncertainty in these estimates, the models developed in this paper can accommodate it with very minimal modification. Once the distribution of β_{ij} is known the vendor can use the customer's scenario selection criteria to estimate the b^s_{ij}. Assuming that the customer consistently underestimates, the vendor will estimate the b^s_{ij} from the range $[0,\ \beta_{ij}]$. We have assumed that the vendor knows this information with certainty for ease of illustration.

Problem (VS) is a non-linear integer program, with the product of the x_i and y^s_j variables introducing the non-linearity. It can be linearized by introducing variables z^s_{ij}, which are defined to be 1 when both x_i and y_j^s are 1, and 0 otherwise — that is, z_{ij}^s is 1 if (i) feature i is included in the demo, and (ii) task j is selected for execution under scenario s, and 0 otherwise. The linearized version of (VS) is (VS_L) below.

$$
\begin{aligned}
(VS_L) \quad \text{Max} \quad & \sum_{s\in\mathcal{S}} p^s \left[\sum_{j\in\mathcal{T}} \left(\left(\pi_j + \sum_{i\in\mathcal{F}} b_{ij}^s \right) y_j^s + \sum_{i\in\mathcal{F}} (\beta_{ij} - b_{ij}^s) z_{ij}^s \right) \right] - \sum_{j\in\mathcal{T}} \rho_j && (2.10)\\
\text{s.t.} \quad & z_{ij}^s \leq x_i && \forall s \in \mathcal{S}; \forall ij \in \mathcal{E} \quad (2.11)\\
& z_{ij}^s \leq y_j^s && \forall s \in \mathcal{S}; \forall ij \in \mathcal{E} \quad (2.12)\\
& z_{ij}^s \geq (x_i + y_j^s - 1) && \forall s \in \mathcal{S}; \forall ij \in \mathcal{E} \quad (2.13)\\
& \rho_j \geq \left(\pi_j + \sum_{i\in\mathcal{F}} \beta_{ij} x_i \right) && \forall j \in \mathcal{T} \quad (2.14)\\
& \rho_j \geq 0 && \forall j \in \mathcal{T} \quad (2.15)\\
& x_i \in \{0,\ 1\} && \forall i \in \mathcal{F} \quad (2.16)\\
& y_j^s \in \{0,\ 1\} && \forall j \in \mathcal{T}; \forall s \in \mathcal{S} \quad (2.17)
\end{aligned}
$$

The objective function can be interpreted as follows. For any task j that is selected to be performed in scenario s, the associated benefit is *at least* equal to $(\pi_j + \sum_{i\in\mathcal{F}} b_{ij}^s)$, which is the total of the original profit and the estimated b_{ij}^s values. If any feature i that contributes positively to task j is included in the demo, this will result in an extra benefit of $(\beta_{ij} - b_{ij}^s)$ to task j in scenario s; the total benefit to task j from all features included in the demo is accounted for by the term $\sum_{i\in\mathcal{F}} (\beta_{ij} - b_{ij}^s) z_{ij}^s$. Note that constraint (2.13) can be eliminated whenever $\beta_{ij} \geq b_{ij}^s$, and constraints (2.11) and (2.12) can be eliminated whenever $\beta_{ij} < b_{ij}^s$. As the customer will underestimate the parameter values in general, the former situation is more likely to occur in reality than the latter.

The feature inclusion decisions x_i connect the scenarios together, and consequently, (VS_L) cannot be decomposed into easier subproblems. The vendor's problem is NP-complete in the strong sense as shown in Proposition 2.1 (proof in Appendix A.1), and the nonlinearity introduced by the product of the x_i and y_j^s variables only adds to the difficulty.

Proposition 2.1 *The vendor's problem is NP-complete in the strong sense.*

As the vendor's problem is NP-complete, the number of scenarios the vendor chooses to use will play a key role in its solvability from a practical perspective, with the vendor's problem taking longer to solve as the number of scenarios increases. While it is feasible (and realistic) for the customer to generate thousands of scenarios (given that the customer's problem is not NP-complete), this is not an option for the vendor. Ideally, the number of scenarios used by the vendor has to be small enough to ensure solvability and large enough to ensure that the optimal demo stays relatively unchanged if (VS_L) is solved multiple times (with different scenarios in each run).

One characteristic of the optimal demo is that the effective re-engineering benefit signaled by any feature included in the demo will exceed the productivity benefits given away as a result of this feature being part of the demo. This is because the signal is obtained as the difference between the estimated value of the software given the demo, and the value that is given away through the demo. Consequently, if a feature gives away more than it helps improve the value of the signal, a better signal can obtained by removing this feature from the demo.

2.3 Performance of Commercial Solvers on Large Problems

The larger version of WiNetPro has over 100 features. In this section, we conduct additional experiments on problems of realistic size to investigate how effective commercial solvers are in solving the vendor's problem. The Branch-And-Reduce Optimization Navigator (BARON) (Sahinidis, 1996) is a state-of-the-art computational system for solving non-convex optimization problems, while IBM (2011) is recognized as the leading software for solving integer programs. CPLEX is run on Intel Core i5 CPU running at 3.20GHz while the BARON solver is run on NEOS servers. The server used for our experiments had 2 Intel Xeon X5660 CPUs running at 2.8GHz (12 cores total). We also conducted experiments to evaluate the effectiveness of the scenario aggregation algorithm.

There are various aspects of the problem that could potentially affect the effectiveness and quality of the solutions obtained. Two critical ones are the variability of the parameters and network structure (the density of the network linking features and tasks). The average connections per feature (r) and the average connections per task (q) are set to 10% in the sparse network, 50% in the medium network and 90% in the dense network.

The results presented in this section are for a software product with 75 features that can (profitably) perform 100 tasks for the customer. We recognize that in general, enterprise software programs contain more than 75 features and 100 tasks. For example, Oracle's PeopleTools 8.53 has 425 features (PeopleTools, 2013). The results of these experiments make clear why we chose not to consider more than 75 features and 100 tasks — neither of the commercial solvers were able to provide any result even after 2 days. The parameters π_j and β_{ij} for these experiments are drawn from discrete uniform distributions $U(-160, 40)$ and $U(10, 50)$ respectively. These parameter values are chosen such that about 20% of the tasks would be performed without the software. The means μ_π and μ_β are -60 and 30 respectively, while the standard deviations σ_π and σ_β are 58 and 11.8 respectively. Four values are considered for the number of scenarios (25, 50, 75 and 100) and the b_{ij}^s values are drawn from $U(0, \beta_{ij})$ (i.e., the vendor is assuming that the customer is always underestimating, to be consistent with existing literature). The linearized formulation (VS_L) is solved using CPLEX, while the non-linear formulation (VS) is solved using BARON. Table 2.2 summarizes the experimental results, including the solution time in seconds.

While BARON is able to solve the smaller problems relatively quickly, it encounters difficulty as the problem sizes get larger (in terms of network density and the number of scenarios used). Moreover, we find that the solutions found by BARON are often suboptimal, and hence not reliable. CPLEX, on the other hand, is able to solve all the problems but one. However, it takes a considerable amount of time to solve them. The largest problem in the sparse network was not solved even after 50 hours. As more scenarios give more precise

Table 2.2. Performance of BARON and CPLEX on Large Problems

Network Structure ($r\%, q\%$)	**Scenarios**	**BARON** **Demo Size**	**BARON** **Objective Value**	**BARON** **Time (sec)**	**CPLEX** **Demo Size**	**CPLEX** **Objective Value**	**CPLEX** **Time (sec)**
(10,10)	25	13	7431	98	14	7463	3041
	50	14	7421	421	14	7497.2	78234
	75	NA	NA	NA	14	7484.7	141020
	100	NA	NA	NA	NA	NA	NA
(50,50)	25	4	51810	362	3	51840	1058
	50	NA	NA	NA	4	51722	4810
	75	NA	NA	NA	3	51825	10194
	100	NA	NA	NA	3	51852	16435
(90,90)	25	2	96767	498	2	96784	419
	50	NA	NA	NA	2	96712	2093
	75	NA	NA	NA	2	96723	5773
	100	NA	NA	NA	2	96722	8973

estimates of the signal, these results make clear that neither solver will be able to solve problems of realistic size, when more features and tasks are involved.

2.4 The Scenario Aggregation Approach

As the number of scenarios increases, so does the difficulty of solving formulations (VS) and (VS_L). As the results in Table 2.2 show, solving the models directly in a commercial solver is not practical for problems of realistic size. We pointed out when solving the customer's problem that this problem decomposes into easily solved subproblems once the demo is available. As the variables x_i and ρ_j link the scenarios together, this is equivalent to saying that (VS) and (VS_L) decompose into $|\mathcal{S}|$ easy problems (one per scenario), once the x_i and ρ_j variables are fixed. (VS_L) falls under the general category of scenario aggregation formulations. It is well-known that scenario aggregation formulations for real-world problems can be so large as to make it impossible to solve optimally.

Lagrangian decomposition (Guignard and Kim, 1987) based approaches have been successfully used to solve scenario aggregation problems (e.g., Glockner et al., 2001; Jönsson et al., 1993), and the approach proposed by Jönsson et al. (1993) can be easily adapted to our context. This involves defining scenario-specific "clones" x_i^s and ρ_j^s of the x_i and ρ_j variables, and adding constraints that force the clones in each scenario to be equal, as follows:

$$x_i^1 = x_i^2,\ x_i^2 = x_i^3, \ldots,\ x_i^k = x_i^{k+1}, \ldots,\ x_i^{S-1} = x_i^S \quad \forall i \in \mathcal{F} \quad (2.18)$$

$$\text{and, } \rho_j^1 = \rho_j^2,\ \rho_j^2 = \rho_j^3, \ldots,\ \rho_j^k = \rho_j^{k+1}, \ldots,\ \rho_j^{S-1} = \rho_j^S \quad \forall j \in \mathcal{T} \quad (2.19)$$

These constraints are then relaxed, with each violation being penalized. Specifically, Lagrangian multipliers λ_i^k and μ_j^k are associated with the k^{th} equality constraints in (2.18) and (2.19) respectively ($k = 1, 2, \ldots, S-1$), and the following terms are added to the objective function:

$$\sum_{i\in\mathcal{F}}\sum_{k=1}^{S-1} \lambda_i^k \left(x_i^k - x_i^{k+1}\right) = \sum_{i\in\mathcal{F}} \left(\lambda_i^1 x_i^1 + \left(-\lambda_i^1 + \lambda_i^2\right) x_i^2 + \ldots - \lambda_i^{S-1} x_i^S\right) = \sum_{k\in\mathcal{S}}\sum_{i\in\mathcal{F}} w_i^k x_i^k \quad (2.20)$$

$$\sum_{j\in\mathcal{T}}\sum_{k=1}^{S-1} \mu_j^k \left(\rho_j^k - \rho_j^{k+1}\right) = \sum_{j\in\mathcal{T}} \left(\mu_j^1 \rho_j^1 + \left(-\mu_j^1 + \mu_j^2\right) \rho_j^2 + \ldots - \mu_j^{S-1} \rho_j^S\right) = \sum_{k\in\mathcal{S}}\sum_{j\in\mathcal{T}} u_j^k \rho_j^k \quad (2.21)$$

The multipliers w_i^s and u_j^s in the above equations are obtained by collecting the Lagrangian multipliers for each scenario s as follows

$$w_i^1 = \lambda_i^1,\ w_i^2 = -\lambda_i^1 + \lambda_i^2, \ldots,\ w_i^k = -\lambda_i^{k-1} + \lambda_i^k, \ldots,\ w_i^S = -\lambda_i^{S-1} \quad (2.22)$$

$$\text{and, } u_j^1 = \mu_j^1,\ u_j^2 = -\mu_j^1 + \mu_j^2, \ldots,\ u_j^k = -\mu_j^{k-1} + \mu_j^k, \ldots,\ u_j^S = -\mu_j^{S-1} \quad (2.23)$$

The Lagrangian dual problem (VS_D) of (VS_L) can now be written as

$$(VS_D) \qquad \text{Min } LD(w,u), \text{where}$$

$$LD(w,u) = \text{Max} \sum_{s\in\mathcal{S}} p^s \left[\sum_{j\in\mathcal{T}} \left(\left(\pi_j + \sum_{i\in\mathcal{F}} b_{ij}^s\right) y_j^s + \sum_{i\in\mathcal{F}} (\beta_{ij} - b_{ij}^s) z_{ij}^s\right) - \sum_{i\in\mathcal{F}} w_i^s x_i^s - \sum_{j\in\mathcal{T}} u_j^s \rho_j^s - \sum_{j\in\mathcal{T}} \rho_j^s\right] \quad (2.24)$$

$$\text{s.t.} \quad z_{ij}^s \leq x_i^s \quad \forall s \in \mathcal{S}; \forall ij \in \mathcal{E} \quad (2.25)$$

$$z_{ij}^s \leq y_j^s \quad \forall s \in \mathcal{S}; \forall ij \in \mathcal{E} \quad (2.26)$$

$$z_{ij}^s \geq (x_i^s + y_j^s - 1) \quad \forall s \in \mathcal{S}; \forall ij \in \mathcal{E} \quad (2.27)$$

$$\rho_j^s \geq \left(\pi_j + \sum_{i\in\mathcal{F}} \beta_{ij} x_i^s\right) \quad \forall s \in \mathcal{S}; \forall j \in \mathcal{T} \quad (2.28)$$

$$\rho_j^s \geq 0 \quad \forall s \in \mathcal{S}; \forall j \in \mathcal{T} \quad (2.29)$$

$$x_i^s, y_j^s \in \{0,\ 1\} \quad \forall s \in \mathcal{S}; \forall i \in \mathcal{F}; \forall j \in \mathcal{T} \quad (2.30)$$

This dual problem not only decomposes into separate smaller problems for each scenario, it also provides an upper bound to the objective function value of (VS_L). While solutions to each scenario are easy to obtain, feasible solutions necessitate that the x_i^s be set to the same values in each scenario. The *scenario aggregation algorithm* described in the Appendix A.1 combines the scenario solutions systematically to find feasible solutions.

2.4.1 Performance of Scenario Aggregation

As we observed in our earlier experiments, neither BARON nor CPLEX can solve problems of realistic size efficiently. We apply the scenario aggregation algorithm to the same problems, and the results are provided in Table 2.3. We also repeat the results presented in Table 2.2 for easy comparison. The maximum iteration counter is set to 30, and the procedure terminates when the gap between the upper and lower bounds falls below 1%. The upper and lower bounds are obtained through the scenario aggregation procedure described in the Appendix A.1. Note that the lower bound is always a feasible solution to the original integer program (VS_L). As we can see, scenario aggregation is able to solve all of these large problems within two hours.

Table 2.3. Performance of Scenario Aggregation

Network Structure $(r\%, q\%)$	**Scen.**	**Scenario Aggregation** Demo Size	Upper Bound	Lower Bound	Time (sec)	**BARON Solver** Demo Size	Objective Value	Time (sec)	**CPLEX Solver** Demo Size	Objective Value	Time (sec)
(10,10)	25	14	7529.6	7461.1	1197	13	7431	98	14	7463	3041
	50	14	7575.2	7497.2	2726	14	7421	421	14	7497.2	78234
	75	14	7565.3	7484.7	5171	NA	NA	NA	14	7484.7	141020
	100	14	7576.7	7498.3	7142	NA	NA	NA	NA	NA	NA
(50,50)	25	4	52045	51826	65	4	51810	362	3	51840	1058
	50	4	51921	51722	130	NA	NA	NA	4	51722	4810
	75	3	52029	51825	193	NA	NA	NA	3	51825	10194
	100	4	52050	51852	282	NA	NA	NA	3	51852	16435
(90,90)	25	2	96944	96784	64	2	96767	498	2	96784	419
	50	2	96888	96712	119	NA	NA	NA	2	96712	2093
	75	2	96873	96723	192	NA	NA	NA	2	96723	5773
	100	2	96877	96722	321	NA	NA	NA	2	96722	8973

Scenario aggregation identifies optimal or near-optimal solutions extremely quickly, taking less than 25 minutes on average, across all the 12 problems solved. The sparse network problems took around 5 to 10 iterations (mostly to reduce the upper bound to reach the tolerance of 1%) while the medium and dense network problems took only a single iteration to find the solution. The best feasible solution identified is the same as the optimal one obtained from CPLEX in 9 out of 11 problems — for the remaining 2 problems, the feasible solution is less than 0.03% away from the optimum. The relative effectiveness of scenario aggregation gets more pronounced as the number of scenarios approaches 100. In the largest problem involving the sparse network, both BARON and CPLEX failed to provide any solution within a reasonable time while scenario aggregation found a solution in less than 2 hours. These results show that scenario aggregation can find near-optimal solutions quickly, with the benefits being more apparent when the number of scenarios is large. A large number of scenarios is preferred, as it gives more precise estimates of the signal. However, that is precisely when both the optimal approaches are likely to fail.

Table 2.4. Performance of BARON and Scenario Aggregation (Low Parameter Variability)

Network Structure		**Scenario Aggregation**				**BARON Solver**		
($r\%, q\%$)	**Scenarios**	**Demo Size**	**Upper Bound**	**Lower Bound**	**Time (sec)**	**Demo Size**	**Objective Value**	**Time (sec)**
(10,10)	25	19	7939.2	7724.0[a]	43406	17	7462	107
	50	19	7934.2	7703.2[a]	184250	18	7542	466
	75	19	7924.9	7698.5[a]	116460	NA	NA	NA
	100	19	7941.2	7710.9[a]	176960	NA	NA	NA
(50,50)	25	4	52973	52782	556	0	50437	367
	50	4	52812	52613	943	NA	NA	NA
	75	4	52838	52636	1596	NA	NA	NA
	100	4	52791	52591	2017	NA	NA	NA
(90,90)	25	2	98141	97849	81	NA	NA	NA
	50	2	98411	98092	150	NA	NA	NA
	75	2	98170	97837	251	NA	NA	NA
	100	2	98235	97887	345	NA	NA	NA

[a] A tolerance of 3% is used.

Next, we conduct a similar set of experiments to study the impact on the performance of scenario aggregation as the variability of parameters decreases. In these experiments, we use $\sigma_\pi = 1$ and $\sigma_\beta = 1$; the corresponding results are presented in Table 2.4. The π_j and β_{ij} are drawn from $U(-61.3, -58.7)$ and $U(28.7, 31.3)$ respectively, to maintain the same values of $\mu_\pi = -60$ and $\mu_\beta = 30$ as in the higher variability case. CPLEX was not able to solve even the smallest problem after 50 hours of computation. This is expected, as the near-homogeneous nature of the parameters induces symmetry in the integer program (VS_L). Integer programs with symmetry are well known to be difficult to solve via traditional branch and bound (Margot, 2009). Therefore, results from solving the problems in CPLEX are not presented in Table 2.4. Baron also performed poorly, solving only 4 out of the 12 problems (providing suboptimal solutions in each case). These experiments confirm our earlier results — neither BARON nor CPLEX will be effective on problems of realistic size.

Scenario aggregation solves all the problems involving medium and dense networks relatively quickly, taking 1278 and 207 seconds respectively on average. BARON had difficulty with these problems and could only solve the smallest problem in the medium network suboptimally. The performance of scenario aggregation deteriorates considerably when the network is sparse; it takes 5 to 8 iterations and over 36 hours on average for the heuristic to converge within a gap of 3%. While the best solution found was always better than that provided by BARON, the slow convergence is a concern.

2.5 A Continuous Approximation and The Augmented Heuristic

We see that BARON, CPLEX and scenario aggregation are all unable to solve problems efficiently when the variabilities in π_j and β_{ij} are very small. As mentioned earlier, this is likely to be a result of symmetry in the integer program (VS_L). To overcome this difficulty, we first solve a continuous approximation to estimate the number of features in the demo, and use the result of this analytical model in Section 2.5.2 to develop an *augmented heuristic* (AH) to find fast and near-optimal solutions.

2.5.1 The Continuous Model

We start with a simplified model where the feature-to-task mapping is assumed to be homogenous. This implies that we have $N = |\mathcal{F}|$ homogenous features catering to $T = |\mathcal{T}|$ homogenous tasks. The vendor needs to decide on the number of features n to include in the demo such that the value of the signal detected by the customer is maximized.

We define q (≥ 1) to be the average number of features connected to a task and r (≥ 1) to be the average number of tasks connected to a feature. Therefore, the total number of connections between features and tasks is $qT = rN$. On average, each task will map to $\frac{nr}{T}$ features in the demo and $\frac{(N-n)r}{T}$ features not in the demo. The customer will perform all tasks for which the expected net benefit is positive i.e., $\mathbb{E}_{\pi,b}\left[\pi + \frac{\beta nr}{T} + \frac{(N-n)br}{T}\right] > 0$. Therefore, $\tilde{\Pi}_{\mathcal{F}}^{\mathcal{D}}$ which is the revised estimate of the profit with the complete software given a demo is $\mathbb{E}_{\pi,b}\left[T\left(\pi + \frac{\beta nr}{T} + \frac{(N-n)br}{T}\right)\right]^{+}$. Here, $\mathbb{E}_{\pi,b}$ is the expectation over the variables π and b while $\mathbb{E}^{+}_{\pi,b}$ is the expectation over π and b when the argument is positive. To obtain the signal, the customer needs to subtract the value $\Pi_{\mathcal{D}}$ of the demo $\mathcal{D}$ from $\tilde{\Pi}_{\mathcal{F}}^{\mathcal{D}}$. The profit $\Pi_{\mathcal{D}}$ of using only the features in $\mathcal{D}$ can be obtained by removing the term $\frac{(N-n)br}{T}$ from the formulation of $\tilde{\Pi}_{\mathcal{F}}^{\mathcal{D}}$. Consequently, the vendor will formulate problem (VC) to obtain the optimum number of features n in the demo that maximizes the signal Ψ^{VC}. We denote the vendor's estimate of the signal as Ψ^{VC}, to distinguish it from the more precise estimate of Ψ calculated by the customer.

$$(VC) \quad \underset{0\leq n\leq N}{\text{Max}} \quad \Psi^{VC} = \mathbb{E}_{\pi,b}\left[T\left(\pi + \frac{\beta nr}{T} + \frac{(N-n)br}{T}\right)\right]^{+} - \mathbb{E}_{\pi}\left[T\left(\pi + \frac{\beta nr}{T}\right)\right]^{+} \qquad (2.31)$$

(VC) can be solved by treating $\hat{\pi}$ and $\hat{b}$ as point estimates. The result is captured in Proposition 2.2 (proof in Appendix A.2).

Proposition 2.2 *For a homogenous feature-to-task mapping with $\hat{\pi}$ and $\hat{b} \leq \beta$ as point estimates, the optimal number of features $n^{\star}$ to be included in the demo and the corresponding* *signal* $\Psi^{VC}(n^\star)$ *are*

$$\left(n^\star, \Psi^{VC}(n^\star)\right) = \begin{cases} (0,0), & \textit{if } \hat{\pi} \in \left(-\infty, -\frac{\beta r N}{T}\right) \quad (2.1) \\ \left(-\frac{\hat{\pi}T}{\beta r}, \hat{b}r\left(N + \frac{\hat{\pi}T}{\beta r}\right)\right), & \textit{if } \hat{\pi} \in \left[-\frac{\beta r N}{T}, 0\right] \quad (2.2) \\ \left(0, \hat{b}rN\right) & \textit{if } \hat{\pi} \in [0, \infty) \quad (2.3) \end{cases}$$

Propositions 2.2.1 and 2.2.3 identify situations where it would not be in the best economic interest of the vendor to provide a feature-limited demo. Proposition 2.2.1 says that the vendor will not provide a demo if the pre-software profit $\hat{\pi}$ from a task is less than $-\frac{\beta r N}{T}$, the total profit from a task when the entire software is available. Therefore, the customer will not be able to perform any task even with the entire software, since $\left(\hat{\pi} + \frac{\beta r N}{T}\right) < 0$. Hence, providing a free demo would not generate productivity or re-engineering gains. Similarly, the vendor is better off by not providing any demo when $\hat{\pi}$ is non-negative. In this case, the customer will be able to profitably perform all the tasks without the software and the software provides no re-engineering benefits. On the other hand, when $\hat{\pi} \in \left[-\frac{\beta r N}{T}, 0\right]$, there will be both re-engineering and productivity benefits. In this case a more negative π will lead to a large demo as more features are needed to signal the re-engineering benefits for a task. The optimal demo size $n^\star$ decreases with β — as the value per feature increases, fewer features are needed for the same level of profit. Along the same lines, $n^\star$ is smaller while the strength of the signal is higher for denser networks. As r increases, more tasks get connected to a feature. Hence, fewer features in a dense network will have the same effect on profit as more features in a sparse network. It is interesting to note that the optimal number of features $n^\star$ is independent of the estimate $\hat{b}$.

2.5.2 Incorporating Insights from the Continuous Model

We saw earlier that both the optimal and the scenario aggregation approaches have difficulty when the parameter variability is low, and the network is sparse. In this section, we develop an *Augmented Heuristic* (AH) that explicitly incorporates the results from the continuous model into formulation (VS_L), to solve it effectively.

First, we obtain an estimate of the optimal demo size $n^\star$ by replacing the point estimates $\hat{\pi}$ and $\hat{\beta}$ with the corresponding mean values $\overline{\pi}$ and $\overline{\beta}$, and solving the continuous version using Proposition 2.2. We then find the smallest number of features required to make the expected benefit of performing a task positive. If q is the average number of features contributing to a task, then on average, $\frac{n^* q}{N}$ features in the demo and $\frac{(N-n^*)q}{N}$ features not in the demo will be contributing to a task. Therefore, the total expected benefit of performing a task is $\left(\overline{\pi} + \overline{\beta}\frac{n^* q}{N} + \overline{b}\frac{(N-n^*)q}{N}\right)$. The smallest value of q that makes this expected benefit positive (say, q_l) is obtained by finding the smallest value of q that makes $\left(\overline{\pi} + \overline{\beta}\frac{n^* q}{N} + \overline{b}\frac{(N-n^*)q}{N}\right) > 0$. Next, we identify all tasks that are connected to at least q_l features — performing these tasks is expected to result in positive benefits. We fix the variable y_j^s to 1 for all such tasks j in every scenario s, and to 0 for all other tasks in every scenario. Once the y_j^s variables are fixed, formulation (VS_L) is substantially reduced; this reduced version of (VS_L) is solved in CPLEX to obtain the solution to the original problem.

Table 2.5. Performance of Augmented Heuristic (Low Parameter Variability)

Network Structure ($r\%, q\%$)	**Scen.**	**Cont. Demo Size**	**Augmented Heuristic**			**Scenario Aggregation**			**BARON Solver**		
			Demo Size	**Objective Value**	**Time (sec)**	**Demo Size**	**Lower Bound**	**Time (sec)**	**Demo Size**	**Objective Value**	**Time (sec)**
(10,10)	25	20	19	7727.6	15	19	7724.0[a]	43406	17	7462	107
	50	20	20	7691.8	32	19	7703.2[a]	184250	18	7542	466
	75	20	20	7661.7	32	19	7698.5[a]	116460	NA	NA	NA
	100	20	19	7676.5	49	19	7710.9[a]	176960	NA	NA	NA
(50,50)	25	4	4	52782	14	4	52782	556	0	50437	367
	50	4	4	52613	29	4	52613	943	NA	NA	NA
	75	4	4	52636	36	4	52636	1596	NA	NA	NA
	100	4	4	52591	37	4	52591	2017	NA	NA	NA
(90,90)	25	2	2	97849	13	2	97849	81	NA	NA	NA
	50	2	2	98098	15	2	98092	150	NA	NA	NA
	75	2	2	97837	29	2	97837	251	NA	NA	NA
	100	2	2	97902	32	2	97887	345	NA	NA	NA

[a] A tolerance of 3% is used.

Table 2.5 compares the results of the Augmented Heuristic and scenario aggregation for problems involving low parameter variability ($\sigma_\pi = 1$ and $\sigma_\beta = 1$). The mean values of π_j and

β_{ij} are set to -60 and 30 respectively. As before, three density levels are considered — sparse, medium and dense. The same datasets from in Table 2.4 are used in these experiments.

The quality of the signal degrades slightly as the network becomes sparser. However, the signal from the Augmented Heuristic is almost the same as the best one obtained from scenario aggregation. In fact, in several cases the Augmented Heuristic is able to improve upon the signal from scenario aggregation. This is probably because the scenario aggregation algorithm is terminated at the convergence tolerance of 3% for the sparse network and 1% for the medium and dense networks. The fact that these near-optimal solutions are identified in a matter of seconds — the average solution time for the 12 problems we tried was only 27.8 seconds — makes the Augmented Heuristic particularly appealing for low parameter variability contexts. We see that the optimal demo size decreases as the network density increases, as predicted by the continuous model.

Table 2.6 provides a quick reference for the best method to adopt for each combination of parameter variability, network structure and number of scenarios. The Augmented Heuristic is preferred for problems with low parameter variability. It is able to find a near-optimal demo within 28 seconds on average with an accuracy within 3% of the optimal. For problems with high parameter variability, the scenario aggregation approach is able to find the optimal signal in less than 25 minutes on average. Moreover, scenario aggregation is the only method that is able to solve all 12 trials. In the sparse network with high variability, scenario aggregation occasionally gives suboptimal signals when the number of scenarios is fewer than 25. CPLEX is practical here, and we recommend using it when the number of scenarios is fewer than 25.

2.6 Extending to a Multi-Segment Model

So far, we have considered a context where the demo is being constructed either for a single customer, or for a single segment of homogeneous customers. This is because our work is

Table 2.6. Performance Comparison

Parameter Variability	Network Structure	Comparative Performance			Best Method
		Method	Optimality	Average Time (seconds)	
Low	Sparse	BARON	Unreliable	—	AH
		SA	within 3%	36 hours	
		AH	within 3%	32	
	Medium	BARON	Unreliable	—	AH
		SA	within 1%	1278	
		AH	within 1%	29	
	Dense	BARON	Unreliable	—	AH
		SA	within 1%	207	
		AH	within 1%	22	
High	Sparse	CPLEX	optimal	1 to 50 hours	SA/CPLEX[a]
		BARON	Unreliable	—	
		SA	optimal	4059	
	Medium	CPLEX	optimal	8124	SA
		BARON	Unreliable	—	
		SA	optimal	168	
	Dense	CPLEX	optimal	4315	SA
		BARON	Unreliable	—	
		SA	optimal	174	

[a] CPLEX is the preferred method for scenarios ≤ 25.

motivated by a problem from the telecommunications industry where, even when the demo is not being built for a single customer, the customer segments were clearly identified. Many software vendors do not have that luxury, and would need to either design a single demo to serve across multiple customer segments, or spend time and effort to identify the customers in each segment (which would enable them to provide customized demos to each segment). All the formulations and solution approaches we have presented so far easily extend to the context of building a single demo in a multi-segment environment.

Usually, vendors can segment the market according to customer needs. Suppose $\mathcal{K}$ represents the set of customer segments, and p_k the proportion of customers in segment k. As before, benefits are a function of how important a task is to the customer, and there will be segment specific values π_j^k and β_{ij}^k for each segment $k \in \mathcal{K}$. If a single demo needs to be assembled to serve all three customer segments, the first change needed is in how the scenarios are generated — we need to generate scenarios in the proportion customers exist in the population. So we generate $\mathcal{S}_k$ scenarios from each segment $k \in \mathcal{K}$, such that $\frac{\mathcal{S}_k}{\bigcup_{k \in \mathcal{K}} \mathcal{S}_k} = p_k$, where $\bigcup_{k \in \mathcal{K}} \mathcal{S}_k$ is the set of all scenarios, $\mathcal{S}$. Once the scenarios are drawn appropriately, the objective function and constraints need to be adjusted to reflect the segments. The modified formulation is (VS_M) below. Other than the x_i variables, all the parameters and variables in (VS_M) have a new index k to identify the customer segment. The x_i variables are independent of the customer segments as there has to be one demo across all the segments.

$$(VS_M) \quad \text{Max} \quad \sum_{k\in\mathcal{K}} p_k \left(\sum_{s\in\mathcal{S}_k} p^{ks} \left(\sum_{j\in\mathcal{T}} \left(\pi_j^k + \sum_{i\in\mathcal{F}} \beta_{ij}^k x_i + \sum_{i\in\mathcal{F}} b_{ij}^{ks}(1-x_i) \right) y_j^{ks} \right) - \sum_{j\in T} \rho_j^k \right) \tag{2.32}$$

$$\text{s.t.} \quad \rho_j^k \geq \left(\pi_j^k + \sum_{i\in\mathcal{F}} \beta_{ij}^k x_i \right) \quad \forall j \in \mathcal{T}; \forall k \in \mathcal{K} \tag{2.33}$$

$$\rho_j^k \geq 0 \quad \forall j \in \mathcal{T}; \forall k \in \mathcal{K} \tag{2.34}$$

$$x_i \in \{0,\ 1\} \quad \forall i \in \mathcal{F} \tag{2.35}$$

$$y_j^{ks} \in \{0,\ 1\} \quad \forall j \in \mathcal{T}; \forall s \in \mathcal{S}_k; \forall k \in \mathcal{K} \tag{2.36}$$

If the vendor had perfect information on the segments each customer belonged to, they could provide customized demos to each customer segment and do better. The difference between the expected signal from three customized demos and from the signal from the single demo is the expected value of knowing the customer segments perfectly. This expected value of perfect information (EVPI) provides the loss in the signal resulting from the need to build a single demo. Conversely, it provides the vendor an estimate of the amount they should be willing to spend in order to identify the segments their customers belong to.

Table 2.7 shows results from computational experiments involving three customer segments — a “low” segment comprising customers who generate limited benefits from the software, a “high” segment comprising customers who generate substantial benefits from the software, and a “medium” segment comprising customers who generate benefits that are somewhere in between. The “Low”, “Medium” and “High” columns in Table 2.7 present results based on providing a customized demo to each segment. Results for providing a single demo that hedges across all three customer segments is provided under “Combined”. The “EVPI” column provides the expected value of perfect information — this is the expected benefit to be gained by identifying the members of each customer segment perfectly.

Table 2.7. Multiple Segments and EVPI

Network Structure ($r\%, q\%$)	Low			Medium			High			Combined			
	Scen.	**Demo Size**	**Signal**	**Scen.**	**Demo Size**	**Signal**	**Scen.**	**Demo Size**	**Signal**	**Scen.**	**Demo Size**	**Signal**	**EVPI**
(10,10)	50	4	952	25	17	1512	25	9	3930	100	7	1704	132.5
	25	5	952	50	18	1487	25	9	3930	100	9	1794	170.0
	25	5	952	25	17	1512	50	10	3918	100	10	2473	102.0
(50,50)	50	8	6235	25	10	17822	25	4	32535	100	5	15345	361.8
	25	9	6204	50	10	17750	25	4	32535	100	4	17985	574.8
	25	9	6204	25	10	17822	50	4	32650	100	5	22005	326.5
(90,90)	50	8	14345	25	5	37346	25	2	62100	100	3	31651	383.0
	25	8	14423	50	5	37299	25	2	62100	100	3	37322	458.3
	25	8	14423	25	5	37346	50	2	62246	100	3	43736	329.3

For these experiments, the segments are created by keeping the coefficients of variation (CV) for π_j and β_{ij} the same across all the segments. The π_j values for the low, medium and high types are drawn from uniform distributions with means -100, -80 and -60 respectively. The standard deviations for π_j are taken as 96.7, 77.3 and 58 for low, medium and high types respectively. Likewise, the β_{ij} values are also drawn from uniform distributions with means 10, 20 and 30 for low, medium and high respectively. The standard deviations for β_{ij} are 3.9, 7.8 and 11.8 for low, medium and high respectively. This maintains the CV for π_j and β_{ij} at -0.97 and 0.39 respectively across the three segments.

Experiments are conducted on three customer segment type distributions for each network structure. A total number of 100 scenarios are generated in each experiment; the first category has $p_{low} = 0.5$, $p_{medium} = p_{high} = 0.25$, the second has $p_{medium} = 0.5$, $p_{low} = p_{high} = 0.25$, and the third has $p_{high} = 0.5$ $p_{low} = p_{medium} = 0.25$. The scenarios from each segment are drawn from $U[0, \beta_{ij}]$ in the same proportion as p_{low}, p_{medium} and p_{high}.

We see that the EVPI is smallest when the probability of a customer belonging to a high type segment is high. This is partly a result of the fact that the high segment drives the combined demo to a great extent, and the loss in signal is primarily a result of some features not being in the demo to help the low and medium types. Overall however, Table 2.7 suggests that the loss in the signal by combining all the segments is not very high. It is possible however, that the magnitude of the loss will increase as the number of segments increase.

From a business perspective, the decision of whether to provide a single demo or not must be made carefully. As we now provide a single demo that, in a crude sense, *averages* across different customer segments, the single demo is clearly not optimal for any *single* segment. Ideally, the best option is to customize a demo for each segment. However, this requires that the firm distinguish between customers in different segments. Table 2.7 provides the benefits of identifying customer segments perfectly. However, there is benefit to be gained even if the customer segments are not identified perfectly, i.e., even if the firm can make noisy observations about the segment to which a customer belongs. A very similar analysis can provide the vendor with the expected value of imperfect information as well. That is, the models presented in this study offer a sound basis for the firm to choose between providing a single demo and providing several customized demos based on a noisy signal about the segment to which a customer belongs.

2.7 Concluding Remarks

An important problem for many companies is that of designing a feature limited version of their product targeted to specific customers in order to signal the value of the full product. Such feature-limited products have been used successfully in various industries. In spite of the popularity and advantages of this model, there has been very little research into the optimal design of feature-limited demonstration software. This study is the first attempt to develop a theoretical basis for designing feature-limited demos. The formulations and solution approaches presented in the paper can easily be extended into a context where multiple segments of customers exist — the vendor can either create a single demo across all customer segments, or segment-specific demos for each segment. The vendor can use the ideas presented in this paper to estimate the expected value of identifying the segment to which each customer belongs. This provides the vendor with an estimate of how much they should be willing to spend to identify the customers precisely.

CHAPTER 3

BARGAINING OVER DATA: WHEN DOES MAKING THE BUYER MORE INFORMED HELP?

3.1 Role of a Data Demonstration

Consider a seller who has a unique, proprietary data set that is of value to the buyer. The price of the data set is jointly decided by the seller and the buyer through a negotiation process. We consider that seller has an outside option for their data and expects to gain more from the negotiation with the buyer. While a thorough analysis of the data to evaluate its full potential is not possible before the negotiation process, the buyer can reduce uncertainty by asking the seller to provide a free demonstration/presentation to evaluate the value of the data. Bain (1956) showed that with free trials buyers can effectively gain information regarding how well a product fits their needs. This informational advantage will help them to make purchasing decision. The seller will provide a free demonstration only if it increases their expected payoff. Thus, a demonstration can help the buyer to learn the value of the data and has the potential to partially or completely mitigate buyer-side uncertainty about the value. For example, seller can provide a sample data as a free trial. Heiman and Muller (1996) states that buyers change their priors on the product value after sampling it. This sample data will act as a signal based on which buyer will update their belief about the value of the data. As an example, a sample data can provide a noisy signal about the distribution of the variables (fields), the actual distribution can only be learned after using the entire data. The level of granularity in the data provides enough flexibility to control the accuracy of the signal which is difficult to achieve in other information goods. Seller can manage the amount of information revealed by the data sample by appropriately selecting the fields, instances, time period, and amount of data. This makes the learning endogenous which is inline with prior literature on learning (Lewis and Sappington, 1994; Dey et al., 2013; Lahiri and Dey, 2016). While the buyer updates their belief about the value after a demonstration and *before* the negotiation, the seller stays uncertain as before as the buyer has no incentive to let the seller know how much they have learned from the demonstration. The seller will however, be aware that the buyer has updated their belief about the value of the data set. Given the asymmetry in information between the seller and the buyer, the resulting negotiation calls for a bargaining framework under asymmetric information.

Harsanyi and Selten (1972) established the theory of cooperative games with incomplete information by generalizing Nash's (Nash, 1950) two player complete information bargaining game. They modeled a two player incomplete information bargaining problem as a pair (X, p) where X, the feasible bargaining set, is the convex hull of the expected payoffs of the players generated by *strict* equilibria, and p is the joint probability matrix of the states of the players. They defined a strict equilibrium as the equilibrium point where each player plays their best response strategy against the others and any deviation by a player to an alternative best response strategy does not affect the payoffs of the other players. They extended the axioms developed by Nash and showed that a unique solution of the two person incomplete bargaining game exists in the set X which satisfy all the axioms. Later, Myerson (1979) proposed an incentive compatible direct revelation mechanism and showed that it coincides with the approach developed by Harsanyi and Selten (1972). We incorporate the approach of Myerson (1979) to solve our asymmetric information bargaining game.

3.1.1 The Model

We adopt the binary distribution model of the value of the data as proposed in Lewis and Sappington (1994). The value is either D_L with probability p_L, or D_H $(0 < D_L < D_H)$ with probability $p_H = (1 - p_L)$, with an expected value of D and is a common knowledge. Let the signal received by the buyer after the demonstration be S_t with probability $P(S_t)$ where t represent the states L and H of the value. The signal provides information about the value or desirability of the data. Let $P(S_L|L) = P(S_H|H) = \alpha$ and $P(S_H|L) = P(S_L|H) = 1 - \alpha$. Then $P(S_H) = P(S_H|H)P(H) + P(S_H|L)P(L) = \alpha p_H + (1 - \alpha)p_L$ and $P(S_L) = (1 - \alpha)p_H + \alpha p_L$. Buyer will update their belief about the value after the demonstration as $P(H|S_H) = \frac{P(S_H|H)P(H)}{P(S_H)} = \frac{\alpha p_H}{\alpha p_H + (1-\alpha)p_L}$ and $P(L|S_L) = \frac{\alpha p_L}{\alpha p_L + (1-\alpha)p_H}$. In the absence of a demo, the signal will be uninformative (useless) i.e. $P(H|S_H) = P(H|S_L) = p_H$ and $P(L|S_L) = P(L|S_H) = p_L$. At the other extreme, the signal will be fully informative (accurate) i.e. $P(H|S_H) = 1 = P(L|S_L)$ and $P(H|S_L) = 0 = P(L|S_H)$. Note that an accurate signal is obtained when $\alpha = 1$ and a useless signal is obtained when $\alpha = 0.5$. Signal will be noisy for $0.5 < \alpha < 1$. The seller is interested in providing a signal through a demo that maximizes their expected payoff.

Let Φ be the set of possible prices that the players can potentially agree upon. Each element $d \in \Phi$ is a possible mutually agreeable price $q(d)$ in the range $[0, D_H]$. The payoff to the buyer in each alternative d is $u_B(d, t) = D_t - q(d)$, where t is either H or L depending upon the state of the data. The payoff to the seller, $u_S(d)$ is $q(d)$ irrespective of the state. As an agreement is not guaranteed, a *disagreement alternative* d^* is included in Φ; essentially, it represents the status quo that would prevail if the players fail to arrive at a mutually acceptable price. The disagreement alternative d^* is characterized by an outside option available to the seller before agreeing to a negotiation with the buyer, involving a fixed value, r. The buyer will not receive anything in the case of a disagreement, and consequently, the payoffs in alternative d^* is $u_S(d^*) = r$ and $u_B(d^*, t) = 0$. Rather than agreeing upon a single alternative, Myerson (1979) proposed that the players collectively agree on a decision rule, or mechanism, $\mu(d|S_t)$ defined by

$$\sum_{d \in \Phi} \mu(d|S_t) = 1; \quad t \in \{L, H\} \tag{3.1}$$

$$\mu(d|S_t) \geq 0; \quad \forall d \in \Phi, \quad t \in \{L, H\} \tag{3.2}$$

The randomized strategy $\mu(d|S_t)$ is the probability of selecting the alternative d when the signal reported by the buyer is S_t. Once the buyer receives the signal after attending the demonstration, the bargaining process begins. In the bargaining game, both players agree on a mechanism $\mu(d|S_t)$ to be implemented when the signal S_t is revealed by the buyer to the seller *after* the bargaining process. That is, in the bargaining process the players mutually agree upon two sets of probability distributions $\mu(d|S_L)$ and $\mu(d|S_H)$ over the set of alternatives Φ. This allows the buyer to agree on mechanism μ without actually revealing the signal during the bargaining process. When the bargaining is complete and mechanism μ is mutually decided, the buyer reveals the signal S_t. The players then execute the agreed probability distribution $\mu(d|S_t)$ to select one of the alternative $d \in \Phi$ and receive corresponding payoffs $\big(u_S(d), u_B(d,t)\big)$. Rather than selecting a single price as a solution, the decision variable μ is adopted as a distribution over the set Φ to make it a truth-revealing mechanism.

Let $U_B(\mu, S_{t'}|S_t)$ be the conditional expected payoff to the buyer when the received signal is S_t while reported signal is $S_{t'}$ (where $t, t' \in \{L, H\}$), when mechanism μ is implemented. Let $U_S(\mu)$ be the expected payoff to the seller in mechanism μ. The buyer will update the low and the high value after receiving the signal as $V_L = D_L P(L|S_L) + D_H P(H|S_L) = \frac{(1-\alpha)p_H D_H + \alpha p_L D_L}{P_L}$, and $V_H = D_L P(L|S_H) + D_H P(H|S_H) = \frac{\alpha p_H D_H + (1-\alpha)p_L D_L}{P_H}$, where $P_H = \alpha p_H + (1-\alpha)p_L$, and $P_L = (1-\alpha)p_H + \alpha p_L$. The expected payoffs can be calculated as follows:

$$\begin{aligned}
U_B(\mu, S_H|S_H) &= \sum_{d\in\Phi\setminus\{d^*\}} \mu(d|S_H)\big(u_B(d,H)P(H|S_H) + u_B(d,L)P(L|S_H)\big) \\
&= \sum_{d\in\Phi\setminus\{d^*\}} \mu(d|S_H)\Big((D_H - u_S(d))\tfrac{\alpha p_H}{P(S_H)} + (D_L - u_S(d))\tfrac{(1-\alpha)p_L}{P(S_H)}\Big) \\
&= \sum_{d\in\Phi\setminus\{d^*\}} \mu(d|S_H)\Big(\tfrac{\alpha p_H D_H + (1-\alpha)p_L D_L}{\alpha p_H + (1-\alpha)p_L} - u_S(d)\Big) \\
&= \sum_{d\in\Phi\setminus\{d^*\}} \mu(d|S_H)\big(V_H - u_S(d)\big) \qquad (3.3)
\end{aligned}$$

Similarly, we get,

$$
\begin{aligned}
U_B(\mu, S_L|S_H) &= \sum_{d\in\Phi\setminus\{d^*\}} \mu(d|S_L)\big(V_H - u_S(d)\big) && (3.4)\\
U_B(\mu, S_H|S_L) &= \sum_{d\in\Phi\setminus\{d^*\}} \mu(d|S_H)\big(V_L - u_S(d)\big) && (3.5)\\
U_B(\mu, S_L|S_L) &= \sum_{d\in\Phi\setminus\{d^*\}} \mu(d|S_L)\big(V_L - u_S(d)\big) && (3.6)\\
U_S(\mu) &= U_S(\mu, S_L|S_L)P(S_L) + U_S(\mu, S_H|S_H)P(S_H)\\
&= \sum_{d\in\Phi} \mu(d|S_L)\Big(u_S(d)P(L|S_L) + u_S(d)P(H|S_L)\Big)P(S_L)\\
&\quad + \sum_{d\in\Phi} \mu(d|S_H)\Big(u_S(d)P(L|S_H) + u_S(d)P(H|S_H)\Big)P(S_H)\\
&= \sum_{d\in\Phi} \Big(\mu(d|S_L)P(S_L) + \mu(d|S_H)P(S_H)\Big)u_S(d) && (3.7)
\end{aligned}
$$

A mechanism is said to be *feasible* if it satisfies conditions (3.1) and (3.2). It also needs to be *Bayesian Incentive Compatible* (BIC) and *Individually Rational* (IR) i.e., μ must also satisfy:

$$
\begin{aligned}
U_B(\mu, S_L|S_L) &\geq U_B(\mu, S_H|S_L); \quad U_B(\mu, S_H|S_H) \geq U_B(\mu, S_L|S_H) && (3.8)\\
U_B(\mu, S_t|S_t) &\geq 0; \quad U_S(\mu) \geq r; \quad t, t' \in \{L, H\} && (3.9)
\end{aligned}
$$

Constraints (3.8) ensure that the buyer will have no incentive to lie about the realized state as there is no positive gain from lying. The IR constraint (3.9) implies that buyer expects to get a non-negative payoff. The seller would expect to make at least r from the negotiation. Therefore, the presence of an outside option places the restriction $U_S(\mu) \geq r$ (Binmore, 1985). The set of possible expected payoff pairs $\mathcal{F} = \Big\{\big(U_S(\mu), U_B(\mu, S_L|S_L), U_B(\mu, S_H|S_H)\big) : \mu \text{ is feasible, and satisfies IR and BIC}\Big\}$ is a finite, compact and convex set. Let Ψ be the set of feasible mechanisms for the bargaining game. Myerson (1979) showed that this truth-telling mechanism can be determined through the solution approach of Harsanyi and Selten (1972). Following their method, the solution to this Bayesian bargaining game ($S_N B_Y$) is the mechanism $\mu \in \Psi$ that maximizes the generalized Nash product, subject to the constraints discussed earlier.

$$
\begin{array}{llllllll}
(S_N B_Y) & \max\limits_{\mu\in\Psi} & \multicolumn{6}{l}{U_S(\mu)\cdot\big(U_B(\mu,S_L|S_L)\big)^{(1-p_H)}\cdot\big(U_B(\mu,S_H|S_H)\big)^{p_H}} \\
 & \text{s.t.} & U_B(\mu,S_L|S_L) & \geq & U_B(\mu,S_H|S_L); & U_B(\mu,S_H|S_H) & \geq & U_B(\mu,S_L|S_H) \\
 & & U_S(\mu) & \geq & r & & & \\
 & & U_B(\mu,S_L|S_L) & \geq & 0; & U_B(\mu,S_H|S_H) & \geq & 0 \\
 & & \sum\limits_{d\in\Phi}\mu(d|S_L) & = & 1; & \sum\limits_{d\in\Phi}\mu(d|S_H) & = & 1 \\
 & & \mu(d|S_L) & \geq & 0; & \mu(d|S_H) & \geq & 0 \;\; \forall\, d\in\Phi \\
 & & u_B(d,S_L)+u_S(d) & = & V_L; & \forall\, d\in\Phi\backslash\{d^*\} & & \\
 & & u_B(d,S_H)+u_S(d) & = & V_H; & \forall\, d\in\Phi\backslash\{d^*\} & & \\
 & & \multicolumn{3}{l}{u_B(d^*,S_t)=0;\quad u_S(d^*)=r;} & t\in\{L,H\} & &
\end{array}
$$

Maximizing the objective function identifies a unique solution from $\mathcal{F}$ on the Pareto-efficient frontier constructed by the conditional expected payoffs $U_S(\mu)$, $U_B(\mu, S_L|S_L)$, and $U_B(\mu, S_H|S_H)$. The exponents $(1-p_H)$ and p_H loosely signify the influences of the likely states in the bargaining process. The solution to $(S_N B_Y)$ and its proof are provided in Appendix B.1. Following Lemma 3.1 states the equilibrium outcome when no demonstration is provided i.e. $\alpha = 0.5$ (proof is in Appendix B.1).

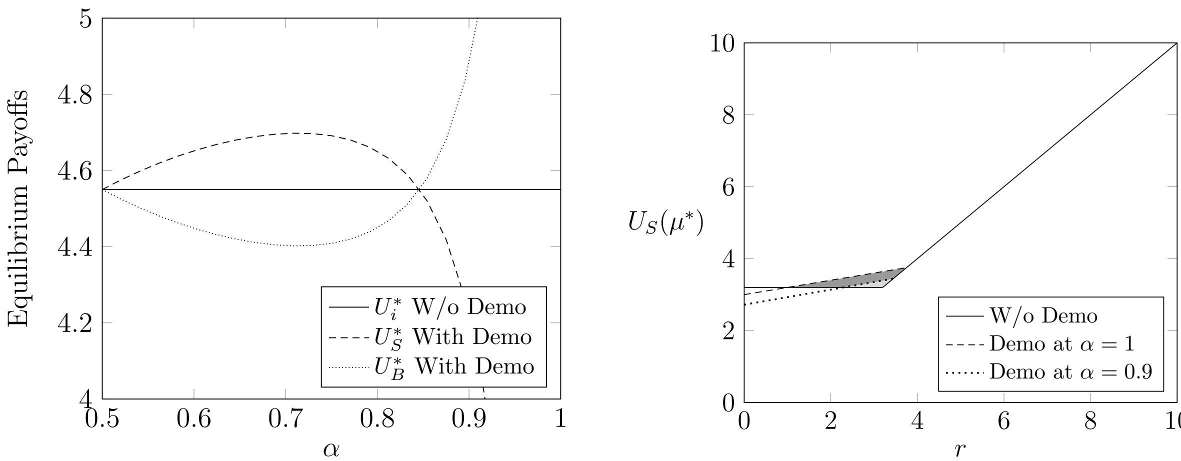
Lemma 3.1 *When the distribution of the value is common knowledge and the outside option for the seller is r, the negotiation without a demonstration will result in the following expected payoffs U_S^* and U_B^* to the seller and the buyer respectively, where D is the expected value of the data product.*

$$
(U_S^*, U_B^*) = \begin{cases} \left(\frac{D}{2}, \frac{D}{2}\right), & \text{if } r \leq \frac{D}{2} \\ (r, D-r), & \text{if } \frac{D}{2} \leq r \leq D \\ (r, 0), & \text{if } D \leq r \end{cases}
$$

Lemma 3.1 implies that when the value of the outside option is low ($r < \frac{D}{2}$), the seller expects to gain from the negotiation and will be willing to negotiate. If $r \geq \frac{D}{2}$, the seller will be indifferent to a negotiation. The buyer's expected payoffs decrease beyond $r > \frac{D}{2}$, as they need to pay at least r to make the negotiation a success; a failure to negotiate leaves them empty handed. Consequently, the negotiation will break down only when $r \geq D$. Seller will offer a demonstration if their expected payoff is more than that stated in Lemma 3.1. It is to be noted that the comparison between the demonstration scenario and the no-demonstration scenario is not only a comparison of two information structures where under one structure the buyer is informed whereas under the other she is uninformed, but more importantly a comparison between two follow-up mechanisms that are used in conjunction with their corresponding information structures. Following proposition B.1 states the conditions under which the seller should/(should not) propose a free demo and buyer should/(should not) accept the offer (proof in Appendix B.1).

Proposition 3.1 *Seller's decision on offering a demonstration is based on following criteria:*
B.1.1. Seller will not offer a demo when $0 \leq r \leq \min\left\{D_L, \frac{1}{2}D\right\}$ *and* $p_H \leq 0.5$.
B.1.2. Seller will offer a noisy demo of accuracy $\alpha^* = \frac{1}{2k_1}\left(k_2 + \sqrt{k_2^2 + 4k_1k_3}\right)$ *when* $0 \leq r \leq \frac{2r^-(\alpha^*) - p_H D_H}{p_L}$ *and* $p_H \geq 0.5$. *Buyer will reject this offer as it is detrimental for them.*
B.1.3. Seller will offer a demo of accuracy $\alpha = 1$ *when — (i)* $D_L \leq r \leq \frac{p_H D_H}{1+p_H}$ *and* $p_H \leq 0.5$, *(ii)* $\frac{2r^-(\alpha^*) - p_H D_H}{p_L} \leq r \leq \frac{p_H D_H}{1+p_H}$ *and* $p_H \geq 0.5$. *Buyer is indifferent in accepting or rejecting this offer.*
B.1.4. Seller will be indifferent in offering a demo otherwise.
where $D = p_H D_H + p_L D_L$,
$r^- = \frac{1}{4}\left(2(V_L + V_H) - (p_L V_L + p_H V_H) - \sqrt{(2(V_L + V_H) - (p_L V_L + p_H V_H))^2 - 8V_L V_H}\right)$,
$k_1 = (p_H D_H - p_L D_L)(p_H - p_L)^2$, $k_2 = (p_H^2 D_H + p_L^2 D_L)(p_H - p_L) - 2k_3$, *and*
$k_3 = (p_H^2 D_H - p_L^2 D_L)\sqrt{p_L p_H} + p_L p_H((2 - p_L)D_H - (2 - p_H)D_L)$

Criteria B.1.1 states that seller gets hurt after demonstration when the outside option is small ($r < \min\left(D_L, \frac{D}{2}\right)$) and chances of getting a high value is also small. By providing a demonstration ($\alpha > 0.5$), the seller is signalling the buyer's that there is a high chance of a low value D_L. Scott and Yalch (1980) alert that if buyer interpret sampling as a signal for a poorly performing product, it might affect them negatively. Thus the fear of getting a negative payoff in state L will compel the buyer to transfer the positive probabilities $\mu^*(d|S_L) > 0$ from prices higher than D_L, which is higher than r and $\frac{D}{2}$, to the disagreement alternative where their payoff is zero. This in turn hurts the seller as the expected price falls below $\frac{D}{2}$ (obtained when no demonstration is provided). Therefore, in this situation providing more information to the buyer becomes detrimental for the seller. Consequently, seller will not let the buyer update their belief by offering a demonstration. However, criteria B.1.2 states that when chances of a high value is high ($p_H > 0.5$) then seller can actually gain by offering a demonstration even when they are at an information disadvantage with a small or no outside option as a backup. It is interesting to observe that the seller will gain more by deliberately providing a noisy signal ($0.5 < \alpha^* < 1$) rather than an accurate signal ($\alpha = 1$). To understand this better we need to consider the forces that start working as α increases from 0.5 to 1. After receiving the signal the buyer updates the value to $V_L = D_L P(L|S_L) + D_H P(H|S_L)$ and $V_H = D_L P(L|S_H) + D_H P(H|S_H)$. With increase in α, additional high prices become available for negotiation in state H as $V_H > D$ which helps the seller's payoff. On the other hand, buyer will not consider the prices between V_L and D ($> V_L$) which they would have considered if the demo is not provided. This hurts the seller's payoff. In addition, for $p_H > 0.5$, we have $D - V_L > V_H - D$ which implies that the number of additional prices considered by the buyer in high state is less than the prices forgone in the low state. This again hurts the seller as the the consideration set of prices shrinks with increase in α. The interplay of these counter forces give an internal optimum in favour of the seller when outside option is small. Figure 3.1(a) shows this internal optimum which is achieved at $\alpha^* = 0.714$. However, this noisy signal *hurts* buyer since it is a zero-sum game under criteria B.1.2. Therefore, buyer will not agree to attend the demonstration. As α increases beyond $\alpha_T = \frac{p_H D_H - p_L D_L}{2(p_H D_H - p_L D_L) - (p_H - p_L) D} = 0.846$, the seller's expected payoff reduces below the no-demonstration payoff as shown in Figure 3.1(a) consequently seller will not offer any demo. This is in contrast with the prior literature on trail and learning which states that seller will either choose not to provide any private information or provide perfect information to all potential buyers (Lewis and Sappington, 1994; Johnson and Myatt, 2006).



(a) $D_L = 1$, $D_H = 10$, $p_H = 0.9$, $r = 0$ (b) $D_L = 1$, $D_H = 10$, $p_H = 0.6$

Figure 3.1. Effect of Learning from the Demo on the Seller's Equilibrium Payoff

Criteria B.1.3 illustrates the importance of the outside option. Consider the situation where $D_L < \min\left(\frac{D}{2}, r\right)$, $\alpha = 1$, and the state turns out to be L. In the no-demonstration negotiation, the buyer will be hurt as they would have paid a price $\frac{D}{2}$ (when $r \leq \frac{D}{2}$) or r (when $r \geq \frac{D}{2}$), both of which are greater than D_L. Similar to the criteria B.1.1, the fear of getting a negative payoff for prices higher than D_L in state L will convince the buyer to transfer positive probabilities from prices higher than D_L to the disagreement alternative. However, in this case seller gains from this action of the buyer as each price choice $q(d)$ in the range $D_L < q(d) < r$ is strictly dominated by the disagreement alternative d^* under mechanism $\mu^*(d|S_L)$. This is because when the state is L then the buyer's payoffs from alternatives $\{d : D_L < q(d)\}$ are negative. In addition, the seller's payoffs under state L from alternatives $\{d : q(d) < r\}$ are less than the disagreement payoff r. As a result, both players will agree to select the disagreement alternative d^* under mechanism $\mu^*(d|S_L)$ over any price in the range $D_L < q(d) < r$. By providing a demonstration, the seller is able to convey this information to the buyer and thereby persuade them to give positive probability (i.e. $\mu^*(d^*|S_L) > 0$) to the disagreement alternative rather than to a price between D_L and r if the data turns out to be of low value. The presence of a relatively high valued outside option makes it possible for the seller to take the risk and provide full information ($\alpha = 1$) about the state of the data as it helps them increase their equilibrium payoff beyond the no-demonstration situation. When $p_H > 0.5$ then seller has the option of providing a noisy demo α^* when r is small however as discussed earlier, the buyer will reject such offers. When outside option is relatively high $\frac{2r^-(\alpha^*)-p_H D_H}{p_L} \leq r$ then seller gains more by offering a demonstration with full information. As shown in Appendix B.1, buyer is indifferent in accepting or rejecting this offer. To improve the expected payoff beyond the no-demonstration negotiation payoff, the probability p_H needs to be high ($\geq \frac{r}{D_H - r}$) giving the condition $r \leq \frac{p_H D_H}{1+p_H}$. The seller will not achieve this advantage beyond $r = \frac{D_H}{2}$ corresponding to $p_H = 1$. Figure 3.1(b) illustrates that an outside option for the seller increases their equilibrium payoff. The presence of an outside option makes the seller less vulnerable even when the buyer is aware of the true value. The shaded area in Figure 3.1(b) depicts the seller's gain with the demonstration. It also shows that the seller gains more by providing complete information ($\alpha = 1$) via demo.

Observe that the buyer will never be worse off after a demo with complete information as it removes their uncertainty in the valuation of the data product. Note however, that the buyer will loose this advantage if they make the seller aware of the true value after the demonstration (*before* the negotiation). If both players become aware of the true value, the negotiation will happen under full information, where their ex-ante equilibrium expected payoffs are $\frac{D}{2}$ for $r \leq \frac{D}{2}$. The solution to problem $(S_N B_Y)$ (provided in Appendix B.1) indicates that without a demonstration ($\alpha = 0.5$) the negotiation is unsuccessful for $D < r$ however, a demonstration can make it a successful negotiation even when $D < r < D_H$.

An interesting aside is that any updated belief by the seller regarding the distribution of the value will not affect the outcome of the negotiation. This is in contrast with the buyer's belief where any learning via a demonstration is affecting the equilibrium expected payoffs. To better comprehend this, we need to understand how the updated belief is changing players' payoff structure in each alternative. The buyer's payoff in any alternative $d \neq d^*$ depends on the realization of the true state of the value i.e. $u_B(d,t) = D_t - q(d)$, where $t \in \{L, H\}$. Therefore, an incentive compatible mechanism is needed to ensure that the buyer does not lie about the true value and get undue advantage. This changes the negotiation problem to a generalized Nash bargaining problem. Any update about the belief by the buyer will change the expected payoffs of the negotiation according to equations (3.3) to (3.7). However, the seller's payoff $u_S(d) = q(d)$ remains unaffected by the true state of the value. Therefore, the seller do not have any incentive to lie about the true value and no incentive compatible mechanism is required for the seller. Therefore, as stated in Lemma 3.2 and shown in Appendix B.2, the expected payoff remains same as before even when seller update their belief about the distribution of the value. This is precisely the reason that *only buyer's belief governs the price negotiation* and signaling through a demonstration becomes even more important.

Lemma 3.2 *The outcome of the negotiation does not change when the seller update their belief about the probability distribution of the value.*

We illustrate the results of Proposition B.1 by considering two numerical examples, involving a demonstration with partial and complete information.

3.1.2 Numerical Example 1

Suppose the value of the data is \$5 in state H and \$0.5 in state L. We consider the case when there is no outside option for the seller i.e. $r = 0$. Suppose the probabilities p_H and

p_L are 0.9 and 0.1 respectively. The expected value $D = (0.1 \times 0.5 + 0.9 \times 5) = \$4.55$. For ease of illustration, we assume that only integer prices between \$0 to \$5 are considered by the players. Since $r < \frac{1}{2}D$, both the players will receive an expected payoff of $\frac{D}{2} = \$2.275$ if no demo is offered. Suppose the seller designs a demo through which the buyer receives the noisy signals S_L and S_H such that $P(S_L|L) = P(S_H|H) = \alpha = 0.7$. Then $P(S_H) = \alpha p_H + (1-\alpha)p_L = 0.66$, $P(S_L) = 1 - P(S_H) = 0.34$, $V_H = \frac{\alpha p_H D_H + (1-\alpha)p_L D_L}{P(S_H)} = \$4.795$, and $V_L = \frac{(1-\alpha)p_H D_H + \alpha p_L D_L}{P(S_L)} = \$4.074$. The set of possible alternatives will now consists of the 7 elements given in Table 3.1.

Table 3.1. Payoff Table with Noisy Demonstration

Φ	d_0	d_1	d_2	d_3	d_4	d_5	d^*
$u_S(d)$	0	1	2	3	4	5	0
$u_B(S_L)$	4.074	3.074	2.074	1.074	0.074	-0.926	0
$u_B(S_H)$	4.795	3.795	2.795	1.795	0.795	-0.205	0
$\mu^*(d|S_L)$	0.410	0.057	0.113	0.010	0.014	0.396	0
$\mu^*(d|S_H)$	0.265	0.331	0	0	0	0.403	0

The equilibrium mechanism is provided in the last two rows. The equilibrium expected payoffs are calculated using equations (3.3), (3.6), and (3.7); and are found to be $U_B(\mu, S_H|S_H) = \$2.447$, $U_B(\mu, S_L|S_L) = \$1.725$, and $U_S(\mu) = \$2.348$. The seller's equilibrium payoff has increased from \$2.275 to \$2.348 even when the demo is a noisy one. In fact, if the demo had revealed the full information about the data i.e. $\alpha = 1$, then the equilibrium payoff for the seller would have reduced to \$2.25 which is lower than their no-demo expected payoff. The equilibrium expected payoff of the buyer is $P(S_H)U_B(\mu, S_H|S_H) + P(S_L)U_B(\mu, S_L|S_L) = \$2.201$ which is less than \$2.275 hence, buyer will reject any offer by the seller to attend a demonstration. For $\alpha = 1$, $P(S_H) = p_H$, $P(S_L) = p_L$, $V_H = D_H$, and $V_L = D_L$. Table 3.2 shows the equilibrium mechanism for a demo with complete information ($\alpha = 1$). The equilibrium payoffs are $U_B(\mu, H|H) = \$2.5$, $U_B(\mu, L|L) = \$0.25$, $U_S(\mu) = \$2.25$, and $U_B(\mu) = \$2.275$. This illustrates that providing full information through a demo actually hurt the seller while maintaining buyer's payoff same as before.

Table 3.2. Payoff Table with a Complete Information Demonstration

Φ	d_0	d_1	d_2	d_3	d_4	d_5	d^*
$u_S(d)$	0	1	2	3	4	5	0
$u_B(L)$	0.5	-0.5	-1.5	-2.5	-3.5	-4.5	0
$u_B(H)$	5	4	3	2	1	0	0
$\mu^*(d\|L)$	0.5	0	0	0	0	0	0.5
$\mu^*(d\|H)$	0.296	0.219	0.018	0.027	0.037	0.403	0

3.1.3 Numerical Example 2

In this example we consider the outside option to be \$2.3 while keeping rest of parameters same as in Example 3.1.2. Table 3.3 shows the equilibrium mechanism for a demo with complete information ($\alpha = 1$). The equilibrium payoffs are $U_B(\mu, H|H) = \$2.5$, $U_B(\mu, L|L) = \$0.25$, $U_S(\mu) = \$2.365$, and $U_B(\mu) = \$2.275$. Therefore offering complete information demonstration is beneficial for the seller as it increases their equilibrium payoff from \$2.275 to \$2.365 while buyer remains indifferent as before.

Table 3.3. Payoff Table with a Complete Information Demonstration

Φ	d_0	d_1	d_2	d_3	d_4	d_5	d^*
$u_S(d)$	0	1	2	3	4	5	2.3
$u_B(L)$	0.5	-0.5	-1.5	-2.5	-3.5	-4.5	0
$u_B(H)$	5	4	3	2	1	0	0
$\mu^*(d\|L)$	0.5	0	0	0	0	0	0.5
$\mu^*(d\|H)$	0.5	0	0	0	0	0.5	0

3.2 Underestimation

As mentioned earlier, buyers often underestimate either the value of experience goods when the true value is not known (Shapiro, 1983; Heiman and Muller, 1996) or the probability distribution of the true value. As underestimation on the part of the buyer reduces the price they would be willing to pay, the seller might potentially wish to propose a demonstration before the negotiation to make the buyer realize that they are underestimating. Shapiro

(1983) have shown that when buyer' beliefs about the value are initially lower than their true values, the seller engages in providing product samples to promote buyer learning. Doing so allows the seller to mitigate the impact of any underestimation on the part of buyer. In the following two section we consider two types of underestimation to verify whether the findings of previous literature are also consistent in the bargaining context.

3.2.1 Buyer Underestimating the Value

In this section we analyze the effect of demonstration in the bargaining context when the buyer underestimate the true value. As before, buyer assume the distribution of the underestimated value to be D_L and D_H with probabilities p_L and p_H respectively however, unknown to the buyer, the true value can be as high as $D_H^+ > D_H$. Without a demonstration buyer do not know that they are actually underestimating. As previously explained in Lemma 3.2 and shown in Appendix B.2, the negotiation will proceed as if both the players assume the same underestimated distribution even though the seller know the true distribution of the data to be D_L and D_H^+ with probabilities p_L and p_H respectively. The equilibrium prices decided by the negotiation will be based on the buyer's belief of the value and will remain unaffected by any belief perceived by the seller. This results in an equilibrium expected price of $\frac{D}{2}$ when $r \leq \frac{D}{2}$ or r when $\frac{D}{2} \leq r \leq D$ as shown in Lemma 3.1 where $D = p_L D_L + p_H D_H$. However, after a successful negotiation without a demonstration and subsequent analysis of the data, buyer would realize that the true value is either D_L or D_H^+. Therefore a priori, buyer's equilibrium expected payoff is given by D^+ minus the equilibrium price where $D^+ = p_L D_L + p_H D_H^+$, as stated in Lemma 3.3.

Lemma 3.3 *When the buyer underestimates the value of the data and the seller does not provide a demonstration, a negotiation will result in following expected payoffs for the seller*

(U_S^*) *and the buyer* (U_B^*) *respectively.*

$$(U_S^*, U_B^*) = \begin{cases} \left(\frac{D}{2}, D^+ - \frac{D}{2}\right), & \textit{if } r \leq \frac{D}{2} \\ \left(r, D^+ - r\right), & \textit{if } \frac{D}{2} \leq r \leq D \\ \left(r, 0\right), & \textit{if } D < r \end{cases}$$

The buyer's expected payoff increases even without a demonstration when the value of the data is underestimated. On the other hand, seller would have received a higher expected payoff of $\frac{D^+}{2}$ instead of $\frac{D}{2}$ for a wider range of r $\left(r \leq \frac{D^+}{2}\right)$ if the buyer had not underestimated. One possible approach the seller could take is to use a demonstration to let the buyer know that the value of the data could be higher than previously assumed. This will help on two fronts — first, it increases buyer's willingness to pay more for the data which could result in higher equilibrium price, and second, it increases the chances of a successful negotiation, as a wider range of prices become feasible. For example, Lemma 3.3 states that the negotiation breaks down when $r > D$ even though the buyer could have received a positive payoff from a successful negotiation for $D < r < D^+$. This lost opportunity results from a lack of information sharing. If a demonstration can signal that the value of the data could potentially be greater than previously thought, the buyer would be more willing to negotiate even when $r > D$. After attending the demonstration the buyer realize that they were underestimating and the true value of the data could be D_L or $D_H^+ > D_H$. We assume that the demonstration is accurate and provides full information to the buyer.

3.2.2 Role of Demonstration when Value is Underestimated

Consider the situation where the seller do not have an outside option for their proprietary data set. According to proposition B.1, the seller should not propose a demonstration to the buyer if buyer are not underestimating. This raises the question whether there is any situation when proposing a demonstration is beneficial to the seller. If the seller knows that the value of the data is being underestimated, they might be interested in providing a demonstration to the buyer. Suppose the seller proposes a demonstration which helps the buyer realize that the value of the data product could be as high as D_H^+ after attending the demonstration. That is, the demonstration not only helps the buyer realize the true state of the data, but also reduces the extent by which they underestimate the value. Players will now consider an extended set of possible prices Φ ranging from 0 to D_H^+. The payoffs for the seller and the buyer are $u_S(d) = q(d)$ and $u_B(d,t) = D_t - q(d)$ respectively with $t = L, H^+$. The negotiation problem $(S_N B_Y)_V$ given below is obtained form $(S_N B_Y)$ by substituting $\alpha = 1$, $D_H = D_H^+$, and $r = 0$.

$$
\begin{array}{llllllll}
(S_N B_Y)_V & \max\limits_{\mu\in\Psi} & \multicolumn{6}{l}{U_S(\mu)\cdot\big(U_B(\mu,L|L)\big)^{(1-p_H)}\cdot\big(U_B(\mu,H^+|H^+)\big)^{p_H}} \\
 & \text{s.t.} & U_B(\mu,L|L) & \geq & U_B(\mu,H^+|L); & U_B(\mu,H^+|H^+) & \geq & U_B(\mu,L|H^+) \\
 & & U_S(\mu) & \geq & 0 & & & \\
 & & U_B(\mu,L|L) & \geq & 0; & U_B(\mu,H^+|H^+) & \geq & 0 \\
 & & \sum\limits_{d\in\Phi}\mu(d|L) & = & 1; & \sum\limits_{d\in\Phi}\mu(d|H^+) & = & 1 \\
 & & \mu(d|L) & \geq & 0; & \mu(d|H^+) & \geq & 0 \ \ \forall\, d\in\Phi \\
 & & u_B(d,L)+u_S(d) & = & D_L; & \forall\, d\in\Phi\backslash\{d^*\} & & \\
 & & u_B(d,H^+)+u_S(d) & = & D_H^+; & \forall\, d\in\Phi\backslash\{d^*\} & & \\
 & & \multicolumn{3}{l}{u_B(d^*,t)=0;\quad u_S(d^*)=0;} & t\in\{L,H^+\} & &
\end{array}
$$

The equilibrium outcome is stated in Lemma 3.4 and is proved in Appendix B.3.

Lemma 3.4 *When the buyer underestimate the distribution of the value to be D_L and D_H with probabilities p_L and $p_H = (1 - p_L)$ respectively while the seller knows the true distribution to be D_L and $D_H^+ > D_H$ with probabilities p_L and p_H respectively, the outcome of the negotiation will give the following expected payoffs to the players in the absence of an outside option and after a demonstration with accurate signal,*

$$
(U_S^*, U_B^*) = \begin{cases} (A, D^+ - A) & \text{if } 0 \leq p_H \leq \frac{2D_L}{2D_H^+ - D_L} \\ \left(\frac{p_H D_H^+}{2}, \frac{(D^+ - D_L)(p_H D_H^+ - p_L D_L)}{2(p_H D_H^+ - D_L)}\right), & \text{if } \frac{2D_L}{2D_H^+ - D_L} < p_H < \frac{D_L}{D_H^+ - D_L} \\ \left(\frac{p_H D_H^+}{2}, \frac{D^+}{2}\right), & \text{if } \frac{D_L}{D_H^+ - D_L} \leq p_H \leq 1 \end{cases}
$$

where, $A = \frac{1}{4}\left(2(D_L + D_H^+) - D^+ - \sqrt{(2(D_L + D_H^+) - D^+)^2 - 8D_L D_H^+}\right)$, $D^+ = p_L D_L + p_H D_H^+$.

The condition $\left(\frac{2D_L}{2D_H^+ - D_L} \leq p_H \leq 1\right)$ implies that $\frac{D_L}{D_H^+} \leq \frac{2}{3}$. In other words, while the probability of receiving D_L is low, D_L is relatively lower than D_H^+. So, the expected loss under L will be high for the buyer if they agree for prices greater than D_L under mechanism $\mu(d|L)$. As a result, the buyer is better off by choosing the disagreement alternative d^* rather than giving positive probabilities to prices above D_L. The seller agrees with this decision, as $U_B(\mu, L|H^+) = \sum_{d \in \Phi \setminus \{d^*\}} \mu(d|L)\big(D_H^+ - u_S(d)\big)$ will become high otherwise, and the buyer will have a greater incentive to lie. Therefore, both players will agree to shift positive probabilities from prices above D_L to the disagreement alternative d^* under mechanism $\mu^*(d|L)$. This agreement will reduce the seller's expect payoff in state L as all positive prices above D_L are given zero weights. However, the seller compensates for this loss in state L by a gain in state H^+ if the extent of underestimation is large enough. When $\frac{D_L}{D_H^+} \geq \frac{2}{3}$, then $\frac{2D_L}{2D_H^+ - D_L} \geq 1$ and the first condition will hold for the entire range of p_H. Loosely speaking, this means that D_L is close to D_H^+, and therefore, the players are reasonably certain that the true value of the data is high. In this situation, both players benefit from a successful negotiation. Consequently, they will refrain from attaching a positive probability to the disagreement alternative and, as shown in the proof in Appendix B.2, $\mu^*(d^*|L) = \mu^*(d^*|H^+) = 0$. Therefore, the entire expected payoff D^+ is shared between the players. It is easy to show using Lemma 3.4 that U_S^* increases from $\frac{D_L}{2}$ to $\frac{D_H^+}{2}$ as p_H increases from 0 to 1.

Before proposing a demonstration the seller would like to know what D_H^+ needs to be before a demonstration is warranted, that is, what improvement in the upper bound will ensure that they will get an expected payoff greater than that implied by Lemma 3.3; such a demonstration is an effective one for the seller. For this we compare the equilibrium expected payoff of the seller with a demo provided in Lemma 3.4 with $\frac{D}{2}$ which is their payoff without a demo when $r = 0$. Proposition 3.2 establishes the threshold value for D_H^+; the proof is in Appendix B.2.

Proposition 3.2 *If the demonstration helps the buyer realize that the upper bound of the value could be as high as $D_H^+ > D_H$ while all other parameters stay unchanged, the seller will propose a demonstration to the buyer before the negotiation when*

$$D_H^+ > \begin{cases} \frac{D}{p_H}, & \text{if } 0 < \frac{D_L}{D_H} \leq \frac{2}{3} \\ D\left(\frac{2D_L - D_H}{D_L - (2-p_H)(D_H - D_L)}\right), & \text{if } \frac{2}{3} < \frac{D_L}{D_H} \leq 1 \end{cases}$$

where, $D = p_L D_L + p_H D_H$.

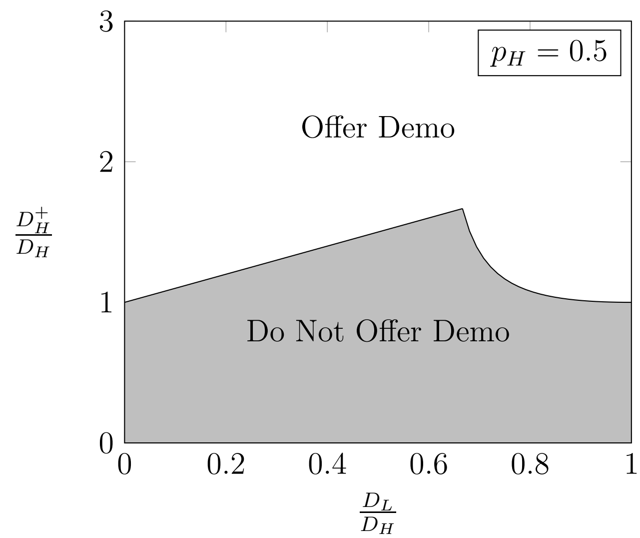


Figure 3.2. Criteria for the Seller to Propose a Demonstration With No Outside Option

Proposition 3.2 states that the seller can gain from proposing a demonstration when buyer underestimates the value even though the seller does not have any outside option as a backup. This corroborates previous findings by Shapiro (1983); Cheng and Liu (2012), and Niculescu and Wu (2014) that a trail is optimum when underestimation is sufficiently large. However, their findings are based on a monopoly market where the seller sets the price to gain benefit from the customer heterogeneity. Our context is different where price of the data is a joint decision of both seller and buyer which is arrived through a bargaining process. The seller is better off only when the demonstration is able to indicate that the value of the data is higher than a threshold. If the extent of underestimation is less than the threshold, it is better for the seller not to offer a demonstration — this is illustrated by the shaded region in Figure 3.2. Figure 3.2 plots the threshold of $\frac{D_H^+}{D_H}$ for $p_H = 0.5$, above which a demonstration is beneficial. As discussed before, when D_L is small and p_H is moderately high, both players will agree to assign zero probability to prices greater than D_L under mechanism $\mu(d|L)$. This in turn, reduces the seller's expected payoff from $\frac{D}{2}$ in the no-demonstration case to $\frac{p_H D_H^+}{2}$ after the demonstration. The seller will propose a demonstration only if the increase from D_H to D_H^+ is large enough to overcome this loss. This loss steadily increases with D_L, and consequently, we see a steady increase in the threshold $\frac{D_H^+}{D_H}$ as $\frac{D_L}{D_H}$ increases in Figure 3.2. When D_L is close to D_H (i.e. $\frac{D_L}{D_H} > \frac{2}{3}$), both players are recognize that the value is high, and will ensure that the negotiation will succeed. In this case, the purpose of the demonstration — to help the buyer realize that the data has high value — is no longer vital, and the effect of underestimation becomes less important. Therefore, we see a decline in the threshold when $\frac{D_L}{D_H}$ is high ($> \frac{2}{3}$), with $\frac{D_H^+}{D_H}$ approaching 1 as $\frac{D_L}{D_H}$ approaches 1.

3.2.3 Numerical Example 3

We now reconsider Example 1, in light of Proposition 3.2, with $r = 0$. Without a demonstration, both the players will receive an expected payoff of $\frac{D}{2} = \$2.275$. Since $\frac{D_L}{D_H} = \frac{1}{10} < \frac{2}{3}$, Proposition 3.2 implies that a demonstration can be a viable option for the seller if it indicates a new upper bound that is greater than $\frac{D}{p_H} = \$5.055$ to the buyer. Suppose the seller designs a demonstration which improves D_H from \$5 to \$6 while keeping other parameters unchanged. After the demonstration the buyer will realize the state H^+ if the value is \$6 or the state L if the value of \$1. The set of possible alternatives will now consists of the 8 elements given in Table 3.4.

Since $\frac{D_L}{D_H^+ - D_L}$ $(\frac{0.5}{6-0.5} = 0.09) < p_H$ $(= 0.9)$, the expected payoffs (calculated using Lemma 3.4) are $U_S^{S_N B_Y} = \frac{p_H D_H^+}{2} = \$2.7$ and $U_B^{S_N B_Y} = \frac{D^+}{2} = \frac{1}{2}(p_L D_L + p_H D_H^+) = \$2.725$.

Table 3.4. Payoff Table: Demonstration when the value is underestimated

Φ	d_0	d_1	d_2	d_3	d_4	d_5	d_6	d^*
$u_S(d)$	0	1	2	3	4	5	6	0
$u_B(L)$	0.5	-0.5	-1.5	-2.5	-3.5	-4.5	-5.5	0
$u_B(H^+)$	6	5	4	3	2	1	0	0
$\mu^*(d\|L)$	0.5	0	0	0	0	0	0	0.5
$\mu^*(d\|H^+)$	0.268	0.173	0.007	0.011	0.140	0.180	0.220	0

Providing the demonstration has increased the seller's expected payoff from \$2.275 to \$2.7. The buyer is always better off after attending a demonstration; in this example, the buyer's expected payoff has increased from \$2.275 to \$2.725. This illustrates that the seller can gain from demonstration even when they do not have an outside option.

3.2.4 Buyer Underestimating the Probability Distribution

Let the true distribution of the value be D_H with probability p_H and D_L with probability $p_L = 1 - p_H$ and is known to the seller. Suppose the buyer underestimate the probability distribution and assume it to be D_H with probability $p_H^B < p_H$ and D_L with probability $p_L^B = 1 - p_H^B$. As explained in Lemma 3.2, the equilibrium prices will depend upon buyer's believe about the distribution. Consequently, the negotiation without a demonstration will result in an equilibrium expected price of $\frac{D^B}{2}$ when $r \leq \frac{D^B}{2}$ or r when $\frac{D^B}{2} \leq r \leq D^B$ which is obtained from Lemma 3.1 by replacing D with $D^B = p_H^B D_H + p_L^B D_L$. However, after a successful negotiation and subsequent analysis of the data, buyer would know that the value is either D_L or D_H which is realized from the true probability distribution p_L and p_H respectively. Therefore a priori, the equilibrium expected payoff for the buyer is given by D minus the equilibrium price where $D = p_H D_H + p_L D_L$ as stated in Lemma 3.5.

Lemma 3.5 *When the buyer underestimates the distribution of the data to be D_H with probability $p_H^B < p_H$ and D_L with probability $p_L^B = 1 - p_H^B$ and the seller does not provide a demonstration, a negotiation will result in following expected payoffs for the seller (U_S^*) and* *the buyer* (U_B^*) *respectively where* $D^B = p_H^B D_H + p_L^B D_L$ *and* $D = p_H D_H + p_L D_L$.

$$(U_S^*, U_B^*) = \begin{cases} \left(\frac{D^B}{2}, D - \frac{D^B}{2}\right), & \textit{if } r \leq \frac{D^B}{2} \\ (r, D - r), & \textit{if } \frac{D^B}{2} \leq r \leq D^B \\ (r, 0), & \textit{if } D^B < r \end{cases}$$

The buyer's expected payoff increases even without a demonstration as $D > D^B$. On the other hand, seller would have received a higher expected payoff of $\frac{D}{2}$ instead of $\frac{D^B}{2}$ for a wider range of r $\left(r \leq \frac{D}{2}\right)$ if the buyer had not underestimated. Furthermore, underestimation decreases the chances of a successful negotiation from $r < D$ to $r < D^B$.

3.2.5 Role of Demonstration when Probability Distribution is Underestimated

Consider the situation where the buyer underestimate the probability distribution to be p_H^B and p_L^B for values D_H and D_L respectively. Suppose that seller provides a demonstration whose sole purpose is to mitigate buyer's uncertainty about the state of the data and does not attempt to reduce the extent by which they underestimate the probability distribution. That is, after the demonstration buyer will know that the value is either D_L or D_H realized from their underestimated probability distribution p_L^B and p_H^B respectively. As stated in Lemma 3.2, the price negotiation will not be affected by seller's knowledge of the distribution and will proceed as below,

$$\begin{array}{llllllll} (S_N B_Y)_P & \max\limits_{\mu \in \Psi} & \multicolumn{6}{l}{U_S(\mu) \cdot \left(U_B(\mu, L|L)\right)^{(1-p_H^B)} \cdot \left(U_B(\mu, H|H)\right)^{p_H^B}} \\ & \text{s.t.} & U_B(\mu, L|L) & \geq & U_B(\mu, H|L); & U_B(\mu, H|H) & \geq & U_B(\mu, L|H) \\ & & U_S(\mu) & \geq & 0 & & & \\ & & U_B(\mu, L|L) & \geq & 0; & U_B(\mu, H|H) & \geq & 0 \\ & & \sum\limits_{d \in \Phi} \mu(d|L) & = & 1; & \sum\limits_{d \in \Phi} \mu(d|H) & = & 1 \\ & & \mu(d|L) & \geq & 0; & \mu(d|H) & \geq & 0 \ \ \forall\, d \in \Phi \\ & & u_B(d, t) + u_S(d) & = & D_t; & \multicolumn{3}{l}{\forall\, d \in \Phi \backslash \{d^*\}; \quad t \in \{L, H\}} \\ & & \multicolumn{3}{l}{u_B(d^*, t) = 0; \quad u_S(d^*) = 0;} & t \in \{L, H\} & & \end{array}$$

In the above problem we assume that there is no outside option for the seller. The equilibrium outcome is provided in Lemma 3.6 (proof in Appendix B.2).

Lemma 3.6 *When the buyer underestimate the probability distribution to be p_H^B and $p_L^B = 1 - p_H^B$ for values D_H and D_L respectively while seller knows the true distribution to be $p_H > p_H^B$ and $p_L = 1 - p_H$ for values D_H and D_L respectively, the outcome of the negotiation will give the following expected payoffs to the players in the absence of an outside option and after a demonstration with accurate signal.*

$$(U_S^*, U_B^*) = \begin{cases} (A, D - A) & \text{if } 0 \leq p_H \leq \frac{2A}{D_H} \\ \left(\frac{p_H D_H}{2}, \frac{(D-D_L)(p_H D_H - p_L^B D_L)}{2(p_H D_H - D_L)}\right), & \text{if } \frac{2A}{D_H} < p_H < \frac{D_L}{D_L + D_H - D^B} \\ \left(\frac{p_H D_H}{2}, \frac{D}{2}\right), & \text{if } \frac{D_L}{D_L + D_H - D^B} \leq p_H \leq 1 \end{cases}$$

where, $A = \frac{1}{4}\left(2(D_L + D_H) - D^B - \sqrt{(2(D_L + D_H) - D^B)^2 - 8 D_L D_H}\right)$ *and* $D^B = p_L^B D_L + p_H^B D_H$

Now, suppose that seller shows a demonstration that not only mitigates the uncertainty about the state of the data but also corrects the underestimation in the probability distribution of the states assumed by the buyer. That is, after a demonstration, buyer knows the value to be either D_L or D_H realized from the true probability distribution p_L and p_H respectively. This can be achieved by a data dashboard which can be used multiple times by the buyer to know the true distribution of the states of the data as well its value. After such a demonstration, the price negotiation will be same as problem $(S_N B_Y)_V$ of section 3.2.2 by replacing D_H^+ by D_H and solution in Lemma 3.4 will follow.

3.2.6 Equivalence of Underestimations

In the previous section we stated that if the demonstration is able to correct the underestimation and also help the buyer realize the true state of the data then players will solve the same negotiation problem irrespective of whether the underestimation is in the value or in the probability distribution. This conveys an important insight — these two types of underestimations are equivalent. To understand this better let us suppose that the true states of the data is distributed as D_L and D_H with probabilities p_L and $p_H = 1 - p_L$. Consider two cases — (i) buyer underestimates the value i.e. buyer assumes the distribution to be D_L and D_H^- ($D_L < D_H^- < D_H$) with probabilities p_L and p_H, (ii) buyer underestimates the probability distribution i.e. buyer assumes the distribution to be D_L and D_H with probabilities p_L^B and $p_H^B < p_H$. Using lemmas 3.3 and 3.5, the negotiation outcome without a demonstration for cases (i) and (ii) are given as $(U_S^*, U_B^*)_{(i)} = \left(\frac{D^-}{2}, D - \frac{D^-}{2}\right)$ and $(U_S^*, U_B^*)_{(ii)} = \left(\frac{D^B}{2}, D - \frac{D^B}{2}\right)$ respectively where $D^- = p_L D_L + p_H D_H^-$ and $D^B = p_L^B D_L + p_H^B D_H$. Suppose in case (i), a demonstration is able to reveal the true state D_L or D_H to the buyer thereby, correcting the underestimation. Similarly, in case (ii), a demonstration is able to reveal the true state D_L or D_H to the buyer and they know that the true state is realized from the true probability distribution p_L and p_H respectively thereby correcting the underestimation. After the demonstration, the price negotiation in both these cases will proceed as problem $(S_N B_Y)_V$ where D_H^+ is replaced by D_H. It is easy to observe that the two cases will be equivalent when $D^- = D^B$. This implies that underestimating the value to be D_H^- has the same effect as underestimating the probability of high state to be $p_H \left(\frac{D_H^- - D_L}{D_H - D_L}\right)$. Note that this equivalence will also hold when seller has an outside option.

3.3 Concluding Remarks

Many companies possess data that is of value to other firms. However, monetizing such proprietary data can be difficult since its value is often uncertain. When large firms are involved, and both sides are uncertain about the value of the data, the exchange usually occurs through a process of bargaining. This is the first paper to analyze this bargaining process for the selling and buying of data. At a higher level, our paper is about monetizing data so sellers can extract proper value in a bargaining process. The monetization of data is an important and increasingly relevant consideration because it facilitates the selling and buying of data, and fuels the growth of data-driven decision-making in business. Future research on this subject could examine the role of a third party — one that provides data-analytic services — in the bargaining process. We expect that examining such a three-party bargaining problem could provide new insights about the role of demonstrations in the bargaining process. Also, while our paper considered a monetization problem involving a seller and a single buyer, it will be interesting to examine optimal data monetization strategies when multiple buyers (but a single seller) are involved.

CHAPTER 4

BARGAINING OVER DATA WITH CONSULTANT AS A GATEKEEPER

This research is motivated by the problem faced by a major Global Distribution System (GDS) provider. GDS is an online travel reservation system traditionally used by the travel agents to make airline, hotel and rental car bookings. The company has collected a large volume of proprietary and unique data by the virtue of their business. Collected data is of considerable importance to the airlines because of its rich content as it not only includes the booking details of individual travellers but also their search details. While bookings provide future travel demand, search data on the other hand can provide traveller's search behavior previously unknown to the airlines. The client airline of GDS is interested in purchasing this data from them and using it in their business model to make informed decisions about flight schedule, routing, aircraft capacity and other such business aspects. While both GDS and the airline recognize the importance of this unique dataset, its actual worth is unclear to both the companies. GDS is unsure about the price they should ask for the data as they have limited knowledge about the business model for which this data will be used by the airline. Moreover, data being an experience goods requires a comprehensive analysis to ascertain its full value; consequently, airlines are also uncertain about the price they should be willing to pay for the data. To overcome this two sided uncertainty, both the parties agree on price negotiation. Furthermore, there are instances where the airline had hired a consultant to provide data analytic services to the airline. In such situations, the airline also involves the consultant in the negotiation process where all the three companies collectively decide the price of the data and the service charge to be paid by the airline to the GDS and the consultant respectively. In this chapter we develop a three-party bargaining framework to analyze this data monetization process where the consultant plays the role of a gatekeeper and incurs a processing cost to provide data analytic services. Note that the model studied here can be applied to other product selling context with uncertain valuation where the price of the product and consultant's process cost are decided through a three-party negotiation process.

4.1 True Value Unknown to All Three Players

We assume that all three players seller, buyer, and consultant, have a common belief about the distribution of the true value of the business report generated by the consultant; the value is either R_L with probability p_L, or R_H with probability $p_H = (1-p_L)$ $(0 < R_L < R_H)$, with an expected value of R. Let Φ be the set of possible agreeable alternatives of price-pair (q_S, q_C) available to the players where q_S is the price paid to the seller and q_C is the price paid to the consultant. As the buyer will not agree to pay more than R_H — the highest value they can obtain from the report — and both the seller and the consultant will not agree for a negative price, we have $\Phi = [0, R_H] \times [0, R_H]$. If the players agree on an alternative (q_S, q_C), the payoffs to the seller, buyer and the consultant are $u_S(q_S) = q_S$, $u_B(q_S, q_C) = R - q_S - q_C$, and $u_C(q_C) = q_C$ respectively. The set of possible payoff pairs $\mathcal{F} = \big\{\big(u_S(q_S), u_B(q_S, q_C), u_C(q_C)\big) : 0 \leq q_S, q_C \leq R_H\big\}$ is a finite, compact and convex set known to all the players. As an agreement is not guaranteed, a *disagreement outcome* v is included in $\mathcal{F}$; essentially, it represents the status quo that would prevail if the players fail to arrive at a mutually acceptable price, and consequently, $v = (0, 0, 0)$. We assume that the consultant has a reservation price r which includes the cost of analyzing the data and preparing the business report. The consultant is expected to receive a price more than r. The players enter a negotiation to agree upon a mutually acceptable price-pair (q_S^*, q_C^*). Using a set of axioms, Nash (1950) was able to show that rational players will eventually select a unique, Pareto-efficient payoffs $\big(u_S(q_S^*), u_B(q_S^*, q_C^*), u_C(q_C^*)\big)$ from the set $\mathcal{F}$ corresponding to the prices q_S^* and q_C^* that maximizes the product $u_S(q_S) \cdot u_B(q_S, q_C) \cdot u_C(q_C)$ in the space $\mathcal{F}$.

This Nash bargaining problem ($S_N B_N C_N$) is given below.

$$
\begin{aligned}
(S_N B_N C_N) \quad & \max_{(u_S, u_B, u_C) \in \mathcal{F}} && u_S \cdot u_B \cdot u_C \\
& \text{s.t.} && u_S + u_B + u_C = R; \quad u_S \geq 0; \quad u_B \geq 0; \quad u_C \geq r; \quad (u_S, u_B, u_C) \in \mathcal{F} \backslash v \\
& && u_S = 0; \quad u_B = 0; \quad u_C = 0; \qquad (u_S, u_B, u_C) \in v
\end{aligned}
$$

Following Lemma 4.1 (see Appendix C.1 for the proof) states that all three players will divide the expected value equally among themselves when the processing cost is small. As r increases beyond $\frac{R}{3}$, the payoffs for the seller and buyer decreases and the negotiation fails when the processing cost is too high ($r > R$). The consultant only benefits when r is small ($< \frac{R}{3}$) beyond which there is no gain.

Lemma 4.1 *When the distribution of the value is a common knowledge and the processing cost for the consultant is r, the negotiation will result in the following expected payoffs U_S^*, U_B^*, and U_C^* to the seller, buyer, and the consultant respectively, where R is the expected value of the report.*

$$
(U_S^*, U_B^*, U_C^*) = \begin{cases} \left(\frac{R}{3}, \frac{R}{3}, \frac{R}{3}\right), & \text{if } r \leq \frac{R}{3} \\ \left(\frac{R-r}{2}, r, \frac{R-r}{2}\right), & \text{if } \frac{R}{3} \leq r \leq R \\ (0, 0, 0), & \text{if } R \leq r \end{cases}
$$

4.2 Buyer Learns the True Value Through a Demonstration

One relevant question pertaining to this negotiation is whether either player can improve on the result obtained in Lemma 4.1. One possible strategy by the seller could be to offer a demonstration to the buyer which mitigates buyer-side uncertainty by revealing the true value of the report to the buyer. The seller will propose a demonstration to the buyer only if their expected payoff from the three-player negotiation exceeds the expected payoff from the no-demonstration scenario. Furthermore, the buyer will accept the proposal only if they benefit from it. In this section, we consider the situation where the seller proposes a demonstration and buyer accepts it. Additionally, the buyer either does not invite the consultant, or the consultant declines the invitation — that is, only the buyer knows the true value of the report after the demonstration.

As before, the value can be either R_H or R_L with probabilities p_H and $p_L = (1 - p_H)$ respectively, resulting in an expected value of $R = p_L R_L + p_H R_H$. After the demonstration, the state of the data is revealed to the buyer and the buyer realizes that its value to them is either R_H (under state H) or R_L (under state L). The seller and the consultant have the same distributions as earlier, except that they now know that the true state has been revealed to the buyer after the demonstration. So, both the seller and the consultant will assign probability p_t to the event that the revealed state is $t = L, H$. As part of the three-party negotiation (under asymmetric information) the players collectively agree on a decision mechanism $\mu(d|t)$ to be implemented if the true value announced after the negotiation is t.

Let Φ denote a set of possible alternatives. Each element $d \in \Phi$ is corresponds to prices $q_S(d)$ and $q_C(d)$ in the range $[0, R_H]$. For alternative d, the buyer will receive a payoff of $u_B(d,t) = R_t - q_S(d) - q_C(d)$ while payoffs of seller and consultants are $u_S(d) = q_S(d)$ and $u_C(d) = q_C(d)$ where the state $t = L, H$. The expected payoff for player $i \in \{S, C\}$, where S is the seller and C is the consultant, in mechanism $\mu(d|t)$ is given by $U_i(\mu) = \sum\limits_{d \in \Phi} \big(p_L \mu(d|L) + p_H \mu(d|H)\big) u_i(d)$ where $p_L = 1 - p_H$. The expected payoff for the buyer revealing the true state $t = L, H$ is $U_B(\mu, t|t) = \sum\limits_{d \in \Phi} \mu(d|t) u_B(d,t)$. The set of possible expected payoffs $\mathcal{F} = \big\{\big(U_S(\mu), U_B(\mu, L|L), U_B(\mu, H|H), U_C(\mu)\big) : \mu \text{ is feasible, and satisfies IR and BIC}\big\}$ is a finite, compact and convex set. Note that the set $\mathcal{F}$ also contains the disagreement outcome $v = (0, 0, 0, 0)$. Let Ψ be the set of feasible mechanisms for the bargaining game. The generalized Nash bargaining problem with asymmetric information can be written as follows:

$$
\begin{aligned}
(S_N B_Y C_N) \quad & \max_{\mu \in \Psi} \; U_S(\mu) \cdot U_C(\mu) \cdot \big(U_B(\mu, L|L)\big)^{(1-p_H)} \cdot \big(U_B(\mu, H|H)\big)^{p_H} \\
\text{s.t.} \quad & U_B(\mu, L|L) \geq U_B(\mu, H|L); \quad U_B(\mu, H|H) \geq U_B(\mu, L|H) \\
& U_S(\mu) \geq 0 \\
& U_C(\mu) \geq r \\
& U_B(\mu, L|L) \geq 0; \quad U_B(\mu, H|H) \geq 0 \\
& \sum_{d \in \Phi} \mu(d|L) = 1; \quad \sum_{d \in \Phi} \mu(d|H) = 1 \\
& \mu(d|L) \geq 0; \quad \mu(d|H) \geq 0 \;\; \forall\, d \in \Phi \\
& u_B(d,t) + u_S(d) + u_C(d) = R_t; \quad \forall\, d \in \Phi \backslash \{d^*\}; \;\; t \in \{L, H\} \\
& u_B(d^*, t) = 0; \;\; u_S(d^*) = 0; \;\; u_C(d^*) = 0 \quad t \in \{L, H\} \\
& u_S(d), u_C(d) \in [0, R_H]
\end{aligned}
$$

The solution to $(S_N B_Y C_N)$ and its proof is provided in Appendix C.1. After comparing the equilibrium expected payoff of the seller before and after the demonstration we observe that seller's equilibrium payoff reduces after the demonstration (see Appendix C.1). Without any outside option, the seller will not offer any demonstration that reveals full information to the buyer. This is in accordance with the findings in a two-player negotiation with no outside option and complete information through a demonstration. Since there is no fall-back option for the seller hence buyer gains an upper hand in the negotiation after knowing the true value. This helps the buyer to gain more while both seller and consultant get hurt.

Proposition 4.1 *Seller will not offer a demonstration to the buyer which reveals true value of the report.*

One might ask whether it is beneficial for the seller to offer a demonstration when both buyer and consultant attend it. Appendix C.3 states that the players will receive the same payoff as if only buyer is aware of the true value. Therefore, it does not matter whether consultant is aware of the true value or not. This is because consultant's knowledge of the true value does not change the payoffs of any player in any alternative and the negotiation problem remains same as before (see Appendix C.3). In addition, only consultant knowing the true value (seller offers demo only to the consultant who accept the offer), will result in a negotiation outcome given by Lemma 4.1 as if none of the players no the true value. Since, buyer's uncertainty about the value remains same hence their payoff in each alternative is based on the expected value i.e. $u_B(q_S, q_C) = R - q_S - q_C$ which eliminates the need for an incentive compatible mechanism leading to a negotiation outcome where none of the players know the true value. Similarly, seller knowing the true value of the report will not affect the payoffs of any player and hence, will not affect the negotiation outcome. In nutshell, only situation that matters is whether buyer is aware of the true value. If the buyer knows the true value then the negotiation outcome is given by the solution of $(S_N B_Y C_N)$ problem irrespective of other players knowing or not knowing the true value. Conversely, if the buyer assumes a distribution of of the true value then the negotiation outcome is given by the Lemma 4.1 irrespective of other players knowing or not knowing the true value. Following Lemma illustrates this interesting fact.

Lemma 4.2 *The following relationships hold for the expected payoffs of the seller, the buyer and the consultant in a three-party negotiation:*

$(U_i^*)^{S_N B_Y C_N} = (U_i^*)^{S_N B_Y C_Y} = (U_i^*)^{S_Y B_Y C_N}$

$(U_i^*)^{S_N B_N C_N} = (U_i^*)^{S_N B_N C_Y} = (U_i^*)^{S_Y B_N C_N}$

where $i = S, B, C$ *and* S_Y (S_N)*,* B_Y (B_N) *and* C_Y (C_N) *represent the seller, buyer and consultant knowing (not knowing) the true value.*

4.3 Concluding Remarks

The buyers of data often hire a consultant to do the analysis for them, either because they do not have, or do not wish to tie up, internal resources on the project. We examine the impact of the consultant on the negotiation process. As with the two-player case, we find that the seller will be worse off if they propose a demonstration and either the buyer, or the buyer and the consultant attend (if the sole purpose of the demonstration is to inform the attendees of the value of the report). Therefore, the seller will not propose a demonstration in this case either.

Interestingly, if either the consultant or the seller knows the true value of the report apart from the buyer then both the consultant and the seller gets hurt. Thus, the knowledge of the true value does not help the consultant and the seller unless all the players are aware of the true value. What remains to be studied is whether it is possible for the seller to propose a demonstration and increase their payoff if there is significant underestimation of report value.

APPENDIX A

THE DESIGN OF FEATURE-LIMITED DEMONSTRATION SOFTWARE: CHOOSING THE RIGHT FEATURES TO INCLUDE

Proposition A.1 *The vendor's problem is NP-complete in the strong sense.*

Proof: The formulation for the vendor's problem is

$$
\begin{aligned}
(VS) \quad \text{Max} \quad & \Psi^{VS} = \sum_{s \in \mathcal{S}} p^s \left(\sum_{j \in \mathcal{T}} \left(\pi_j + \sum_{i \in \mathcal{F}} \beta_{ij} x_i + \sum_{i \in \mathcal{F}} b^s_{ij}(1 - x_i) \right) y^s_j \right) - \sum_{j \in T} \rho_j && \text{(A.1)} \\
\text{s.t.} \quad & \rho_j \geq \left(\pi_j + \sum_{i \in \mathcal{F}} \beta_{ij} x_i \right) \quad \forall j \in \mathcal{T} && \text{(A.2)} \\
& \rho_j \geq 0 \quad \forall j \in \mathcal{T} && \text{(A.3)} \\
& x_i \in \{0,\ 1\} \quad \forall i \in \mathcal{F} && \text{(A.4)} \\
& y^s_j \in \{0,\ 1\} \quad \forall j \in \mathcal{T}; \forall s \in \mathcal{S} && \text{(A.5)}
\end{aligned}
$$

Given an instance of (VS) verification involves (i) calculating the objective function value and comparing it to Ψ^{VS}, and (ii) checking whether all the constraints are satisfied. Each of these checks involve additions, and can be done in polynomial time. Therefore, (VS) is in class NP.

Now consider a restricted version of the vendor's problem involving only one scenario, where some of the variables are fixed. Specifically, suppose the ρ_j are fixed at 0 for all tasks $j \in \mathcal{T}$, while the y_j are fixed at 1 for some subset of tasks $j \in \mathcal{T}_1$, and at 0 for the rest. Incorporating the values of ρ_j and y_j into (VS) results in the simplified formulation (VS_R) below.

$$
\begin{aligned}
(VS_R) \quad \text{Max} \quad & \Psi^{VS_R} = \sum_{j \in \mathcal{T}_1} \pi_j + \sum_{j \in \mathcal{T}_1} \sum_{i \in \mathcal{F}} b_{ij} + \sum_{i \in \mathcal{F}} \left[\sum_{j \in \mathcal{T}_1} (\beta_{ij} - b_{ij}) \right] x_i && \text{(A.6)} \\
\text{s.t.} \quad & \sum_{i \in \mathcal{F}} \beta_{ij} x_i \leq -\pi_j \quad \forall j \in \mathcal{T} && \text{(A.7)} \\
& x_i \in \{0,\ 1\} \quad \forall i \in \mathcal{F} && \text{(A.8)}
\end{aligned}
$$

Note that $\sum_{j \in \mathcal{T}_1} \pi_j + \sum_{j \in \mathcal{T}_1} \sum_{i \in \mathcal{F}} b_{ij}$ is a constant, and does not play a role in identifying the optimal solution to (VS_R). Consequently, an equivalent objective is to maximize $\sum_{i \in \mathcal{F}} \left[\sum_{j \in \mathcal{T}_1} (\beta_{ij} - b_{ij}) \right] x_i$.

We will show that (VS_R) (and therefore, (VS)) is NP-complete in the strong sense by showing that any multidimensional knapsack problem can be reduced to (VS_R). The multidimensional knapsack problem is a well-studied combinatorial optimization problem that NP-complete in the strong sense (Puchinger et al., 2010). Given a set of items $\mathcal{N}$, a set of resources $\mathcal{M}$, profits $p_i \geq 0$ associated with each item $i \in \mathcal{N}$, capacities $k_j \geq 0$ for each resource $j \in \mathcal{M}$, quantities $a_{ij} \geq 0$, of resource $j \in \mathcal{M}$ consumed by item $i \in \mathcal{N}$, and binary decision variables w_i indicating which items are selected, the multidimensional knapsack problem can be formulated as (MKP) below.

$$
\begin{aligned}
(MKP) \quad \text{Max} \quad & \sum_{i \in \mathcal{N}} p_i w_i \\
\text{s.t.} \quad & \sum_{i \in \mathcal{N}} a_{ij} w_i \leq k_j \quad \forall j \in \mathcal{M} \\
& w_i \in \{0,1\} \qquad \forall i \in \mathcal{N}
\end{aligned}
$$

We can assume without loss of generality that $p_i \leq a_{ij}\ \forall i \in \mathcal{N},\ j \in \mathcal{M}$ (otherwise, the p_i can be scaled down such that this is the case, without affecting the problem). The following mapping implies that any instance of (MKP) is equivalent to the restricted version of the vendor's problem (VS_R): $\mathcal{M} \to \mathcal{T}$, $\mathcal{N} \to \mathcal{F}$, $k_j \to (-\pi_j)$, $a_{ij} \to \beta_{ij}$, $p_i \to \sum_{j \in \mathcal{T}_1}(\beta_{ij} - b_{ij})$, and $w_i \to x_i$. This implies that if we have a solution to (VS_R), we have a solution to (MKP). Conversely, if (VS_R) does not have a solution, neither does (MKP). Therefore, (VS_R) is NP-complete in the strong sense. As (VS_R) is a restricted version of (VS), (VS) is also NP-complete in the strong sense. ■

A.1 The Scenario Aggregation Algorithm

The scenario aggregation algorithm was first formally proposed by Rockafellar and Wets (1991). It converges to the solution of the stochastic problem by aggregating successive solutions of scenario subproblems. The key components of the scenario aggregation algorithm are described below.

Initialization: All the Lagrangian multipliers are initialized to 0, which automatically sets all the w_i^s and u_j^s values to 0. The iteration counter v ranges from 1 to 30. The upper and lower bounds (UBD^v and LBD^v) of the signal are set to ∞ and $-\infty$ respectively.

Updating the Bounds: To calculate the lower bound, each scenario subproblem of (VS_D) (see Section 2.4) is solved first, resulting in a solution vector $(\hat{x}^{s,v}, \hat{\rho}^{s,v})$ for each scenario s and iteration v. Each of the $\hat{x}^{s,v}$ values represents a feasible demo. Therefore, feasible solutions to the original problem (VS) can be obtained by substituting these $\hat{x}^{s,v}$ and $\hat{\rho}^{s,v}$ values for the x_i and ρ_j respectively. This is done for each distinct scenario pair $(\hat{x}^{s,v}, \hat{\rho}^{s,v})$, and the best, feasible solution $(\bar{\bar{x}}^v, \bar{\bar{\rho}}^v)$ is identified from among the current scenario solutions.

$$
\overline{VS}^{\ v} = \max_{\{\bar{\bar{x}}^v, \bar{\bar{\rho}}^v\} \in \{\hat{x}^{s,v}, \hat{\rho}^{s,v}\}_{s \in S}} \left[\max_{y_j^{s,v}} \sum_{s \in \mathcal{S}} p^s \sum_{j \in \mathcal{T}} \left(\pi_j + \sum_{i \in \mathcal{F}} \beta_{ij} \bar{\bar{x}}^v + \sum_{i \in \mathcal{F}} b_{ij}^s (1 - \bar{\bar{x}}^v) \right) y_j^{s,v} - \sum_{j \in T} \bar{\bar{\rho}}^v \right]
$$

If the objective function value $\overline{VS}^{\,v}$ improves the existing lower bound, we have a new feasible solution that gives a better lower bound; the lower bound can then be updated as $LBD^v = \max\left(LBD^{v-1}, \overline{VS}^{\,v}\right)$. The upper bound is obtained by solving the Lagrangian dual problem (VS_D). This provides an initial solution vector $(\bar{x}^{s,v}, \bar{\rho}^{s,v})$ for each scenario $s \in \mathcal{S}$ and iteration v which is then used to update the multiplier values. The upper bound can be updated as the smaller of the previous upper bound and the value of the Lagrangian dual LD^v (equation (2.24)) in the current iteration, i.e., $UBD^v = \min\left(UBD^{v-1}, LD^v\right)$.

Updating the multipliers: In each iteration, the Lagrangian multipliers are updated using subgradient optimization,

$$\lambda_i^{s,v+1} = \lambda_i^{s,v} + \theta^v\left(\bar{x}_i^{s,v} - \bar{x}_i^{s+1,v}\right) \text{ for } s = 1, \ldots, S-1; \quad \forall i \in \mathcal{F} \quad \text{(A.9)}$$

$$\mu_j^{s,v+1} = \mu_j^{s,v} + \theta^v\left(\bar{\rho}_j^{s,v} - \bar{\rho}_j^{s+1,v}\right) \text{ for } s = 1, \ldots, S-1; \quad \forall j \in \mathcal{T} \quad \text{(A.10)}$$

where the $\bar{x}^{s,v}$ and $\bar{\rho}^{s,v}$ are obtained by solving (VS_D) in iteration v. We adapt the commonly used step size formula (A.11) from Jönsson et al. (1993) .

$$\theta^v = \frac{\gamma(LD^v(w,u) - LBD^v)}{\sum_{s=1}^{S-1}\left(\sum_{i\in\mathcal{F}} \|\bar{x}_i^{s,v} - \bar{x}_i^{s+1,v}\|^2 + \sum_{j\in\mathcal{T}} \|\bar{\rho}_j^{s,v} - \bar{\rho}_j^{s+1,v}\|^2\right)} \quad \text{(A.11)}$$

Where, $\gamma \in (0, 2]$, LBD^v is the lower bound, and LD^v is the objective value of problem (VS_D). In the first iteration, LD^1 is computed for $w = u = 0$. For each iteration the steplength θ^v is halved each time LD^v fails to improve. Trial computations indicate that the scalar γ should be small for smaller convergence rate therefore the γ is adjusted accordingly.

Termination: The algorithm stops when any of the following criteria are met — (i) maximum number of iterations is reached, (ii) upper and lower bounds are within a specified limit i.e. $\frac{UBD^v - LBD^v}{|LBD^v|} < \varepsilon$, or (iii) the $\bar{x}^s$ obtained from solving the Lagrangian dual problem (VS_D), are identical for all the scenarios. We terminate with a near optimal solution vector $\bar{\bar{x}}$ if criterion (ii) is met, while we stop with an optimal solution vector $\bar{x}$ if we stop based on criterion (iii).

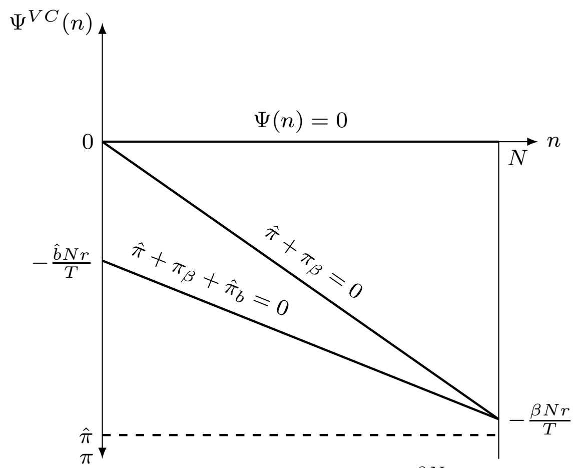
Proposition A.2 *For a homogenous feature-to-task mapping with $\hat{\pi}$ and $\hat{b} \leq \beta$ as point estimates, the optimal number of features $n^\star$ to be included in the demo and the corresponding signal $\Psi^{VC}(n^\star)$ are,*

$$\left(n^\star, \Psi^{VC}(n^\star)\right) = \begin{cases} (0,0), & \text{if } \hat{\pi} \in \left(-\infty, -\frac{\beta r N}{T}\right) \\ \left(-\frac{\hat{\pi}T}{\beta r}, \hat{b}r\left(N + \frac{\hat{\pi}T}{\beta r}\right)\right), & \text{if } \hat{\pi} \in \left[-\frac{\beta r N}{T}, 0\right] \\ \left(0, \hat{b}rN\right) & \text{if } \hat{\pi} \in [0, \infty) \end{cases}$$

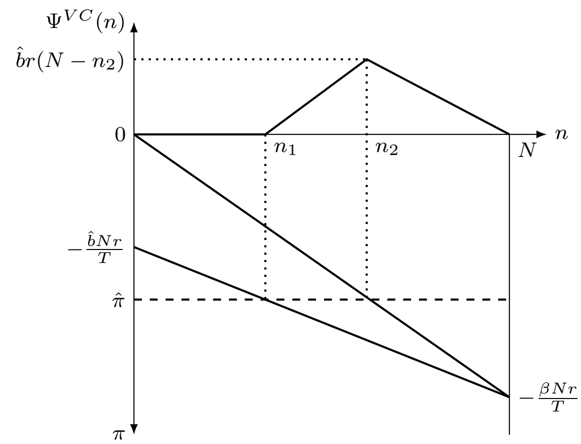
Proof: Given $\hat{\pi}$ and $\hat{b} \leq \beta$ as point estimates the vendor will obtain the following signal $\Psi^{VC} = \tilde{\Pi}_{\mathcal{F}}^{\mathcal{D}} - \Pi_{\mathcal{D}}$:

$$(VC) \qquad \underset{0 \leq n \leq N}{\text{Max}} \quad \Psi^{VC} = T\left(\hat{\pi} + \frac{\beta n r}{T} + \frac{(N-n)\hat{b}r}{T}\right)^{+} - T\left(\hat{\pi} + \frac{\beta n r}{T}\right)^{+} \qquad \text{(A.12)}$$

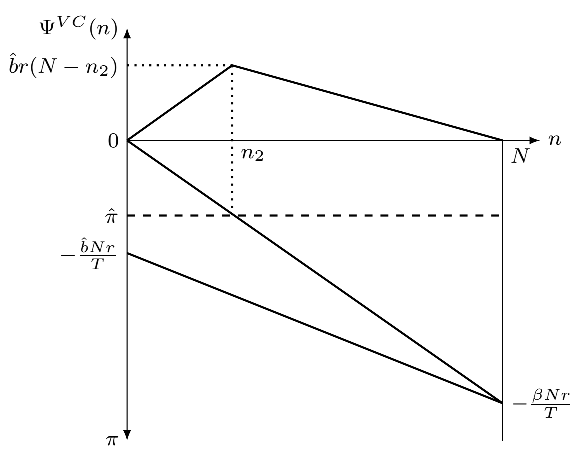
The term $\tilde{\Pi}_{\mathcal{F}}^{\mathcal{D}}$ is the revised estimate of the benefit with the complete software given a demo while $\Pi_{\mathcal{D}}$ is the actual benefit to the customer from the demo. Let $\pi_\beta = \frac{\beta n r}{T}$ and $\hat{\pi}_b = \frac{\hat{b}(N-n)r}{T}$ where r is the number of tasks connected to a feature. Figure A.1(a) plots $\hat{\pi} + \pi_\beta + \hat{\pi}_b = 0$ and $\hat{\pi} + \pi_\beta = 0$ with π on the negative vertical axis and $\Psi^{VC}(n)$ on the positive vertical axis. Decision variable n is on the horizontal axis. Following four cases can be formed depending on the position of $\hat{\pi}$.



(a) Case 1. $-\infty \leq \hat{\pi} \leq -\frac{\beta N r}{T}$.



(b) Case 2. $-\frac{\beta N r}{T} \leq \hat{\pi} \leq -\frac{\hat{b}Nr}{T}$.



(c) Case 3. $-\frac{\hat{b}Nr}{T} \leq \hat{\pi} \leq 0$.

$\hat{\pi}$	$n^\star$	$\Psi^{VC}(n^\star)$
$\left(-\infty, -\frac{\beta r N}{T}\right)$	0	0
$\left[-\frac{\beta r N}{T}, 0\right]$	$-\frac{\hat{\pi}T}{\beta r}$	$\hat{b}r\left(N + \frac{\hat{\pi}T}{\beta r}\right)$
$[0, \infty)$	0	$\hat{b}rN$

(d) Summary table

Figure A.1. Proposition 1 when $\hat{b} \leq \beta$

Case 1 $-\infty \leq \hat{\pi} \leq -\frac{\beta N r}{T}$ (Figure A.1(a)). This is an extreme case where even after including all the features in the demo none of the tasks make any positive contribution. Hence the customer would not get any benefit even if the software is provided free of cost. Here we have $\hat{\pi} + \pi_\beta + \hat{\pi}_b \leq 0$ and $\hat{\pi} + \pi_\beta \leq 0$ which implies that $\tilde{\Pi}^{\mathcal{D}}_{\mathcal{F}} = 0$ and $\Pi_{\mathcal{D}} = 0$. Therefore, $\Psi^{VC}(n^\star) = \tilde{\Pi}^{\mathcal{D}}_{\mathcal{F}} - \Pi_{\mathcal{D}} = 0$ and $n^\star = 0$.

Case 2 $-\frac{\beta N r}{T} \leq \hat{\pi} \leq -\frac{\hat{b} N r}{T}$ (Figure A.1(b)). This is divided into following three subcases – (2.i) $0 \leq n \leq n_1$ where $n_1 = -\frac{\hat{\pi}T/r + \hat{b}N}{\beta - \hat{b}}$. This is same as Case 1 where the vendor's signal is 0. (2.ii) $n_1 \leq n \leq n_2$ where $n_2 = -\frac{\hat{\pi}T}{\beta r}$. In this subcase $\hat{\pi} + \pi_\beta + \hat{\pi}_b \geq 0$ and $\hat{\pi} + \pi_\beta \leq 0$. Therefore, $\Psi^{VC} = T\left(\hat{\pi} + \frac{\beta n r}{T} + \frac{\hat{b}(N-n)r}{T}\right) = \hat{\pi}T + (\beta - \hat{b})nr + \hat{b}Nr$. Signal is maximized at $n^\star = n_2 = -\frac{\hat{\pi}T}{\beta r}$. Therefore, $\Psi^{VC}(n_2) = \hat{b}r(N - n_2) = \frac{\hat{\pi}\hat{b}T}{\beta} + \hat{b}Nr$. (2.iii) $n_2 \leq n \leq N$. For this subcase $\hat{\pi} + \pi_\beta + \hat{\pi}_b \geq 0$ and $\hat{\pi} + \pi_\beta \geq 0$. Hence,$\Psi^{VC} = T\left(\hat{\pi} + \frac{\beta n r}{T} + \frac{\hat{b}(N-n)r}{T}\right) - T\left(\hat{\pi} + \frac{\beta n r}{T}\right) = \hat{b}r(N-n)$. The maximum signal is obtained at $n = n_2 = -\frac{\hat{\pi}T}{\beta r}$ and $\Psi^{VC}(n_2) = \hat{b}r(N - n_2)$. Since for Case 2, $-\frac{\beta N r}{T} \leq \hat{\pi}$ therefore, $\hat{b}Nr + \frac{\hat{\pi}\hat{b}T}{\beta} \geq 0$. Hence, comparing the solutions of (2.i), (2.ii), and (2.iii) we get $n^\star = -\frac{\hat{\pi}T}{\beta r}$ and $\Psi^{VC}(n^\star) = \hat{b}r\left(\frac{\hat{\pi}T}{\beta r} + N\right)$.

Case 3 $-\frac{\hat{b}Nr}{T} \leq \hat{\pi} \leq 0$ (Figure A.1(c)). This can be further divided into two subcases – (3.i) $0 \leq n \leq n_2$. This is same as Subcase (2.ii). (3.ii) $n_2 \leq n \leq N$. This is same as Subcase (2.iii). Therefore, the optimum signal for this entire range is $\Psi^{VC}(n_2) = \hat{b}r\left(\frac{\hat{\pi}T}{\beta r} + N\right)$ and $n^\star = n_2 = -\frac{\hat{\pi}T}{\beta r}$.

Case 4 $0 \leq \hat{\pi} \leq \infty$. This is the case where the customer is able to perform all the tasks without needing any software. The only benefit the software can provide is the productivity gain. In this case $\hat{\pi}+\hat{\pi}_\beta+\hat{\pi}_b \geq 0$ and $\hat{\pi} + \hat{\pi}_\beta \geq 0$. Hence, $\Psi^{VC} = T\left(\hat{\pi} + \frac{\beta n r}{T} + \frac{\hat{b}(N-n)r}{T}\right) - T\left(\hat{\pi} + \frac{\beta n r}{T}\right) = \hat{b}r(N-n)$. Since the signal is decreasing in n hence the vendor will be better off by not providing any demo. Therefore, $n^\star = 0$ and $\Psi^{VC}(0) = \hat{b}Nr$. It is interesting to note that the vendor will provide a demo only when there is a re-engineering effect. All the above cases of Proposition 2.2 are summarized in Table A.1(d). ∎

APPENDIX B

BARGAINING OVER DATA: WHEN DOES MAKING THE BUYER MORE INFORMED HELP?

Property B.1 *At the equilibrium of the negotiation problem* $(S_N B_Y)$, $\mu^*(d^*|S_H)$ *is zero when* $r < V_H$ *while both* $\mu^*(d^*|S_H)$ *and* $\mu^*(d^*|S_L)$ *are one when* $r \geq V_H$ *where* d^* *is the disagreement alternative.*

PROOF: Problem $(S_N B_Y)$ is

$$
\begin{array}{llllll}
(S_N B_Y) & \max\limits_{\mu \in \Psi} & U_S(\mu) \cdot \big(U_B(\mu, S_L|S_L)\big)^{(1-p_H)} \cdot \big(U_B(\mu, S_H|S_H)\big)^{p_H} & & & \\
& \text{s.t.} & U_B(\mu, S_L|S_L) \geq U_B(\mu, S_H|S_L); & U_B(\mu, S_H|S_H) & \geq & U_B(\mu, S_L|S_H) \\
& & U_S(\mu) \geq r & & & \\
& & U_B(\mu, S_L|S_L) \geq 0; & U_B(\mu, S_H|S_H) & \geq & 0 \\
& & \sum\limits_{d \in \Phi} \mu(d|S_L) = 1; & \sum\limits_{d \in \Phi} \mu(d|S_H) & = & 1 \\
& & \mu(d|S_L) \geq 0; & \mu(d|S_H) & \geq & 0 \ \forall\, d \in \Phi \\
& & u_B(d, S_L) + u_S(d) = V_L; & \forall\, d \in \Phi \backslash \{d^*\} & & \\
& & u_B(d, S_H) + u_S(d) = V_H; & \forall\, d \in \Phi \backslash \{d^*\} & & \\
& & u_B(d^*, S_t) = 0; \quad u_S(d^*) = r; & t \in \{L, H\} & &
\end{array}
$$

where,

$$U_B(\mu, S_H|S_H) = \sum_{d \in \Phi \backslash \{d^*\}} \mu(d|S_H)\Big(V_H - u_S(d)\Big) \tag{B.1}$$

$$U_B(\mu, S_L|S_H) = \sum_{d \in \Phi \backslash \{d^*\}} \mu(d|S_L)\Big(V_H - u_S(d)\Big) \tag{B.2}$$

$$U_B(\mu, S_H|S_L) = \sum_{d \in \Phi \backslash \{d^*\}} \mu(d|S_H)\Big(V_L - u_S(d)\Big) \tag{B.3}$$

$$U_B(\mu, S_L|S_L) = \sum_{d \in \Phi \backslash \{d^*\}} \mu(d|S_L)\Big(V_L - u_S(d)\Big) \tag{B.4}$$

$$U_S(\mu) = \sum_{d \in \Phi} \Big(P_L \mu(d|S_L) + P_H \mu(d|S_H)\Big) u_S(d) \tag{B.5}$$

Since $\frac{\partial V_H}{\partial \alpha} = \frac{p_L p_H (D_H - D_L)}{P_H^2} > 0$ hence, V_H is an increasing function in α. Also, $\frac{\partial V_L}{\partial \alpha} = -\frac{p_L p_H (D_H - D_L)}{P_L^2} < 0$ hence, V_L is a decreasing function in α. At $\alpha = 0.5$, we get $V_H = V_L = D$ where $D = p_H D_H + p_L D_L$. Furthermore, $V_H = D_H$ and $V_L = D_L$ at $\alpha = 1$. Therefore, $0 \leq D_L \leq V_L \leq D \leq V_H \leq D_H$.

(i) When $0 \leq r < V_H$. Define Φ^- to be $\{d \in \Phi : V_H \geq u_S(d) \geq \max\{V_L, r\}; d \neq d^*\}$. Suppose Property B.1 was not true, and that $\mu^*(d^*|S_H) = \epsilon > 0$ in the optimal equilibrium solution. This implies that $\sum_{d \in \Phi \backslash \{d^*\}} \mu^*(d|S_H) = (1 - \epsilon)$, which implies that every $\mu^*(d|S_H)_{d \in \Phi \backslash \{d^*\}} \leq (1 - \epsilon)$. This means that we can always increase the value of one of the variables $\mu^*(d'|S_H)$, where $d' \in \Phi^-$, by ϵ, while reducing $\mu^*(d^*|S_H)$ to 0 and leaving all other variables $\mu^*(d|S_H)$ $(d \in \Phi; d \neq d', d^*)$ and $\mu^*(d|S_L)_{d\in\Phi}$ unchanged.

Increasing $\mu^*(d'|S_H)$ by ϵ and decreasing $\mu^*(d^*|S_H)$ by ϵ to 0 (i) increases $U_B(\mu^*, S_H|S_H)$ by $\epsilon\big(V_H - u_S(d')\big)$ since $u_S(d') \leq V_H$, (ii) decreases $U_B(\mu^*, S_H|S_L)$ by $\epsilon\big(V_L - u_S(d')\big)$ since $u_S(d') \geq V_L$, and (iii) increases $U_S(\mu^*)$ by $\epsilon P_H(u_S(d') - r)$ since $u_S(d') \geq r$, while maintaining the probability balance equation $\sum_{d\in\Phi} \mu^*(d|S_H) = 1$. $U_B(\mu^*, S_L|S_H)$, $U_B(\mu^*, S_L|S_L)$, and $\sum_{d\in\Phi} \mu^*(d|S_L) = 1$ are unchanged since the values of the variables $\mu^*(d|S_L)$ are unaffected by the change.

All IR constraints remain satisfied since $U_B(\mu^*, S_H|S_H)$ and $U_S(\mu^*)$ are increasing as a result of the change. $U_B(\mu^*, S_H|S_H)$ increases by $\epsilon\big(V_H - u_S(d')\big)$ while $U_B(\mu^*, S_L|S_H)$ is unchanged, ensuring that the BIC constraint $U_B(\mu^*, S_H|S_H) \geq U_B(\mu^*, S_L|S_H)$ is not violated. The BIC constraint $U_B(\mu, S_L|S_L) \geq U_B(\mu, S_H|S_L)$ continues to hold since $U_B(\mu, S_H|S_L)$ decreases by $\epsilon\big(V_L - u_S(d')\big)$ while $U_B(\mu, S_L|S_L)$ remains unaffected. Therefore, the new solution resulting from this change remains feasible.

The net effect therefore, is an increase in the objective function value, as both components of the objective function — $U_S(\mu^*)$ and $U_B(\mu^*, S_H|S_H)$ — have increased, and the objective function is an increasing function of each of these components. That is, we have identified a feasible solution to $(S_N B_Y)$ that has a better objective function value. Therefore the original solution with $\mu^*(d^*|S_H) = \epsilon > 0$ could not have been optimal.

(ii) When $r \geq V_H$. Suppose Property B.1 was not true, that is $\mu^*(d^*|S_H) = 1 - \epsilon' < 1$ and $\mu^*(d^*|S_L) = 1 - \epsilon'' < 1$ in the optimal equilibrium solution. Suppose $\mu^*(d'|S_H) = \epsilon'$ and $\mu^*(d''|S_L) = \epsilon''$ for $\{d', d''\} \in \Phi\backslash\{d^*\}$ while all other alternatives have zero probability in the equilibrium. This means $U_B(\mu^*, S_H|S_H) = \epsilon'(V_H - d')$ and $U_B(\mu^*, S_L|S_L) = \epsilon''(V_L - d'')$. Both $U_B(\mu^*, S_H|S_H)$ and $U_B(\mu^*, S_L|S_L)$ will be non-negative if $V_H \geq d'$ and $V_L \geq d''$. Also, $U_S(\mu^*) = P_H\Big(\epsilon' d' + (1-\epsilon')r\Big) + P_L\Big(\epsilon'' d'' + (1-\epsilon'')r\Big) = r - P_H\epsilon'(r - d') - P_L\epsilon''(r - d'')$. Since $r > V_H \geq \{d', d''\}$ hence $U_S(\mu^*) < r$ which violates the IR constraint for the seller. Therefore the original solution with $\mu^*(d^*|S_H) = 1 - \epsilon' < 1$ and $\mu^*(d^*|S_L) = 1 - \epsilon'' < 1$ could not have been optimal. ■

Property B.2 *At equilibrium of the negotiation problem $(S_N B_Y)$ the BIC constraint $U_B(\mu, S_H|S_H) \geq U_B(\mu, S_L|S_H)$ is binding.*

PROOF: Property B.2 essentially states that $U_B(\mu, S_H|S_H) = U_B(\mu, S_L|S_H)$, or equivalently,
$\sum_{d\in\Phi\backslash\{d^*\}} \mu^*(d|S_H)\big(V_H - u_S(d)\big) = \sum_{d\in\Phi\backslash\{d^*\}} \mu^*(d|S_L)\big(V_H - u_S(d)\big)$.

Suppose Property B.2 was not true, and that $U_B(\mu^*, S_H|S_H) = U_B(\mu^*, S_L|S_H) + \delta$ for some $\delta > 0$ in the optimal equilibrium solution. From Property B.1, we know that $\sum_{d\in\Phi\backslash\{d^*\}} \mu^*(d|S_H) = 1$. Therefore, we can write the BIC constraint $\sum_{d\in\Phi\setminus\{d^*\}} \mu^*(d|S_H)\big(V_H - u_S(d)\big) = \sum_{d\in\Phi\setminus\{d^*\}} \mu^*(d|S_L)\big(V_H - u_S(d)\big) + \delta$ as,

$$\begin{aligned} V_H - \sum_{d\in\Phi\setminus\{d^*\}} \mu^*(d|S_H)u_S(d) &= V_H \sum_{d\in\Phi\setminus\{d^*\}} \mu^*(d|S_L) - \sum_{d\in\Phi\setminus\{d^*\}} \mu^*(d|S_L)u_S(d) + \delta \\ \Rightarrow\ V_H\big(1 - \sum_{d\in\Phi\setminus\{d^*\}} \mu^*(d|S_L)\big) &= \sum_{d\in\Phi\setminus\{d^*\}} \big(\mu^*(d|S_H) - \mu^*(d|S_L)\big)u_S(d) + \delta \qquad \text{(B.6)} \end{aligned}$$

The other BIC constraint $U_B(\mu^*, S_L|S_L) \geq U_B(\mu^*, S_H|S_L)$ can be written as

$$\begin{aligned} \sum_{d\in\Phi\setminus\{d^*\}} \mu^*(d|S_L)\big(V_L - u_S(d)\big) &\geq \sum_{d\in\Phi\setminus\{d^*\}} \mu^*(d|S_H)\big(V_L - u_S(d)\big) \\ \Rightarrow\ \sum_{d\in\Phi\setminus\{d^*\}} \big(\mu^*(d|S_H) - \mu^*(d|S_L)\big)u_S(d) &\geq V_L\big(1 - \sum_{d\in\Phi\setminus\{d^*\}} \mu^*(d|S_L)\big) \qquad \text{(B.7)} \end{aligned}$$

Define Φ^+ to be $\{d : V_L - u_S(d) \geq 0; d \neq d^*\}$. From equations (C.9) and (C.10) we get $(V_H - V_L)\big(1 - \sum_{d\in\Phi\setminus\{d^*\}} \mu^*(d|S_L)\big) \geq \delta$, which implies $\sum_{d\in\Phi\setminus\{d^*\}} \mu^*(d|S_L) \leq 1 - \frac{\delta}{V_H - V_L}$. Therefore, none of the variables $\mu^*(d|S_L)_{d\in\Phi\setminus\{d^*\}}$ are greater than $1 - \frac{\delta}{V_H-V_L}$. This implies that it is feasible to increase the value of one of the variables $\mu^*(d'|S_L)_{d'\in\Phi^+}$ by $\epsilon = \frac{\delta}{V_H - u_S(d')} > 0$, while keeping all other variables fixed, since

$$\mu^*(d'|S_L) + \frac{\delta}{V_H - u_S(d')} \leq 1 - \left(\frac{\delta}{V_H - V_L} - \frac{\delta}{V_H - u_S(d')}\right) < 1.$$

which is true since $V_H - u_S(d') > V_H - V_L$ for $d' \in \Phi^+$.

Therefore, we choose a variable $\mu^*(d'|S_L)$ for some $d' \in \Phi^+$, and increase it by ϵ to be $\frac{\delta}{V_H - u_S(d')}$, while simultaneously decreasing $\mu^*(d^*|S_L)$, the probability corresponding to the disagreement alternative d^*, by the same amount, ϵ. We know that $\sum_{d\in\Phi\setminus\{d^*\}} \mu^*(d|S_L) \leq 1 - \frac{\delta}{V_H-V_L}$ which implies $\mu^*(d^*|S_L) \geq \frac{\delta}{V_H-V_L} > 0$. Subtracting $\epsilon = \frac{\delta}{V_H - u_S(d')}$ from $\mu^*(d^*|S_L)$, we get

$$\mu^*(d^*|S_L) - \frac{\delta}{V_H - u_S(d')} \geq \frac{\delta}{V_H - V_L} - \frac{\delta}{V_H - u_S(d')} > 0$$

and therefore, it is feasible to decrease $\mu^*(d^*|S_L)$ by ϵ. Increasing $\mu^*(d'|S_L)$ and decreasing $\mu^*(d^*|S_L)$ by ϵ

1. increases $U_B(\mu^*, S_L|S_L)$ by $\delta\left(\frac{V_L - u_S(d')}{V_H - u_S(d')}\right) > 0$,

2. increases $U_B(\mu^*, S_L|S_H)$ by δ, because

$$\begin{aligned} U_B(\mu^*, S_L|S_H) &= \sum_{d\in\Phi\setminus\{d',d^*\}} \mu^*(d|S_L)\big(V_H - u_S(d)\big) + \left(\mu^*(d'|S_L) + \frac{\delta}{V_H - u_S(d')}\right)\big(V_H - u_S(d')\big) \\ &= \sum_{d\in\Phi\setminus\{d',d^*\}} \mu^*(d|S_L)\big(V_H - u_S(d)\big) + \mu^*(d'|S_L)\big(V_H - u_S(d')\big) + \delta \\ &= \sum_{d\in\Phi\setminus\{d^*\}} \mu^*(d|S_L)\big(V_H - u_S(d)\big) + \delta \end{aligned}$$

3. increases $U_S(\mu^*)$ by $\frac{\delta P_L u_S(d')}{V_H - u_S(d')} > 0$.

$\sum_{d\in\Phi} \mu^*(d|S_H) = 1$ since $\mu^*(d'|S_L)_{d'\in\Phi^+}$ increases by ϵ and $\mu^*(d^*|S_L)$ decreases by ϵ. $U_B(\mu^*, S_H|S_H)$, $U_B(\mu^*, S_H|S_L)$, and $\sum_{d\in\Phi} \mu^*(d|S_H) = 1$ are not affected since none of the variables $\mu^*(d|S_H)$ have been changed. Since $U_B(\mu^*, S_L|S_L)$ and $U_S(\mu^*)$ increase as a result of the change, the IR constraints remain satisfied. As $U_B(\mu^*, S_L|S_L)$ increases by $\delta\frac{V_L - u_S(d')}{V_H - u_S(d')}$ while $U_B(\mu^*, S_H|S_L)$ stays unchanged, the BIC constraint $U_B(\mu^*, S_L|S_L) \geq U_B(\mu^*, S_H|S_L)$ is not violated. The BIC constraint $U_B(\mu^*, S_H|S_H) \geq U_B(\mu^*, S_L|S_H)$ is now a binding constraint since $U_B(\mu^*, S_L|S_H)$ increases by δ, the exact value of the gap between $U_B(\mu^*, S_H|S_H)$ and $U_B(\mu^*, S_L|S_H)$.

The net effect therefore, is an increase in the objective function value, as the objective function is an increasing function of both $U_S(\mu^*)$ and $U_B(\mu^*, S_L|S_L)$, while still satisfying all the constraints. That is, we have identified a feasible solution to $(S_N B_Y)$ that has a better objective function value, and the original solution with $U_B(\mu^*, S_H|S_H) > U_B(\mu^*, S_L|S_H)$ could not have been optimal. ■

B.1 Solving Problem $(S_N B_Y)$

Property B.1 states that for $r > V_H$, $\mu^*(d^*|S_H) = 1$ and $\mu^*(d^*|S_L) = 1$. This implies that the negotiation breaks down when $r > V_H$ and the equilibrium payoffs are $U_S(\mu^*) = r$, and $U_B(\mu^*, S_L|S_L) = U_B(\mu^*, S_H|S_H) = 0$. Therefore, we solve the $(S_N B_Y)$ problem for $r < V_H$. We exploit Properties B.1 and B.2 and convert $(S_N B_Y)$ without losing any information to a simpler problem $(S_N B_Y)'$, by aggregating the μ variables. Property B.1 implies $\sum_{d\in\Phi\backslash\{d^*\}} \mu^*(d|S_H) = 1$, while Property B.2 implies $U_B(\mu^*, S_H|S_H) = U_B(\mu^*, S_L|S_H)$. Define three new variables $A_L(\mu) = \sum_{d\in\Phi\backslash\{d^*\}} \mu(d|S_L)u_S(d)$, $A_H(\mu) = \sum_{d\in\Phi\backslash\{d^*\}} \mu(d|S_H)u_S(d)$, and $\delta(\mu) = \mu(d^*|S_L)$. Note that A_L and A_H are both non-negative, as $u_S(d) \geq 0$ for each $d \in \Phi$. Using equations $u_B(d, S_H) + u_S(d) = V_H$ for $d \in \Phi\backslash\{d^*\}$, and $u_B(d^*, S_H) = 0$, the binding equation $U_B(\mu^*, S_H|S_H) = U_B(\mu^*, S_L|S_H)$ can be written as

$$\begin{aligned}
\sum_{d\in\Phi\backslash\{d^*\}} \mu^*(d|S_H)\big(V_H - u_S(d)\big) &= \sum_{d\in\Phi\backslash\{d^*\}} \mu^*(d|S_L)\big(V_H - u_S(d)\big) \\
\Rightarrow \qquad V_H - A_H &= (1-\delta)V_H - A_L \\
\Rightarrow \qquad \delta &= \frac{1}{V_H}(A_H - A_L) \qquad \text{(B.8)}
\end{aligned}$$

because $\sum_{d\in\Phi\backslash\{d^*\}} \mu(d|S_L) + \delta = 1$.

Equations $u_B(d, S_L) + u_S(d) = A_L$ for $d \in \Phi \backslash \{d^*\}$, and $u_B(d^*, S_L) = 0$ along with (B.8) can be used to reduce the inequality $U_B(\mu, S_L|S_L) \geq U_B(\mu, S_H|S_L)$ to $A_H \geq A_L$ as shown below.

$$\begin{array}{lrcl} & \sum_{d\in\Phi\backslash\{d^*\}} \mu^*(d|S_L)\big(V_L - u_S(d)\big) & \geq & \sum_{d\in\Phi\backslash\{d^*\}} \mu^*(d|S_H)\big(V_L - u_S(d)\big) \\ \Rightarrow & (1-\delta)V_L - A_L & \geq & V_L - A_H \\ \Rightarrow & A_H - A_L & \geq & \frac{V_L}{V_H}(A_H - A_L) \\ \Rightarrow & (V_H - V_L)(A_H - A_L) & \geq & 0 \\ \Rightarrow & A_H & \geq & A_L \end{array}$$

The expected payoffs $U_S(\mu)$, $U_B(\mu, S_L|S_L)$ and $U_B(\mu, S_H|S_H)$ can now be written in terms of $A_L(\mu)$, $A_H(\mu)$, and δ as

$$\begin{array}{lcll} U_S(\mu) & = & P_L A_L + P_H A_H + \delta P_L r & \text{(B.9)} \\ U_B(\mu, S_L|S_L) & = & \sum_{d\in\Phi\backslash\{d^*\}} \mu(d|S_L)\big(V_L - u_S(d)\big) & \\ & = & (1 - \frac{1}{V_H}(A_H - A_L))V_L - A_L & \\ & = & \frac{1}{V_H}\Big(V_L(V_H - A_H) - A_L(V_H - V_L)\Big) & \text{(B.10)} \\ U_B(\mu, S_H|S_H) & = & \sum_{d\in\Phi\backslash\{d^*\}} \mu(d|S_H)\big(V_H - u_S(d)\big) & \\ & = & V_H - A_H & \text{(B.11)} \end{array}$$

where $P_L = 1 - P_H$. The IR constraint $U_S(\mu) \geq r$ can now be represented by $P_L A_L + P_H A_H + \delta P_L r \geq r$. Also, the IR constraint $U_B(\mu, S_H|S_H) \geq 0$ is trivially satisfied by $U_B(\mu, S_L|S_L) \geq 0$ (see equation (B.10)). The constraint $\mu(d|S_L) \geq 0$ along with $u_S(d) \geq 0$ for each $d \in \Phi$ implies $A_L \geq 0$ and $\delta \geq 0$. We can now formulate the simplified three-variable problem $(S_N B_Y)'$ that is equivalent to $(S_N B_Y)$ as

$$\begin{array}{lllcll} (S_N B_Y)' & \max\limits_{A_L, A_H, \delta} & \multicolumn{4}{l}{(P_L A_L + P_H A_H + \delta P_L r) \cdot \big(V_L(V_H - A_H) - A_L(V_H - V_L)\big)^{(1-p_H)} \cdot (V_H - A_H)^{p_H}} \\ & \text{s.t.} & P_L A_L + P_H A_H + \delta P_L r & \geq & r & \text{(B.12)} \\ & & V_L(V_H - A_H) - A_L(V_H - V_L) & \geq & 0 & \text{(B.13)} \\ & & \delta & = & \frac{1}{V_H}(A_H - A_L) & \text{(B.14)} \\ & & A_H & \geq & A_L & \text{(B.15)} \\ & & A_L \geq 0; \quad \delta \geq 0 & & & \end{array}$$

We have ignored the factor $\left(\frac{1}{V_H}\right)$ of equation (B.10) in this transformation, as it is a constant. The solutions to $(S_N B_Y)'$, once obtained, gives $U_S(\mu^*)$, $U_B(\mu^*, S_L|S_L)$ and $U_B(\mu^*, S_H|S_H)$ from equations (B.9), (B.10) and (B.11) respectively. To get back the mechanism μ^*, we need to use the equations $A_L(\mu^*)$, $A_H(\mu^*)$, and $\delta(\mu^*)$, along with the feasibility constraints $\sum_{d\in\Phi} \mu^*(d|S_t) = 1$ for $t \in \{L, H\}$, $\mu^*(d^*|S_H) = 0$, and

$\mu^*(d|S_t) \geq 0$ for $d \in \Phi$. μ^* will not be unique if the number of μ^* variables is greater than the number of equations. It is easy to observe that a feasible mechanism always exists. For example, the following μ^*-vector is a feasible solution: $\mu^*(0|S_L) = \mu^*(0|S_H) = 1 - \frac{A_H(\mu^*)}{V_H}$, $\mu^*(V_H|S_L) = \frac{A_L(\mu^*)}{V_H}$, $\mu^*(V_H|S_H) = \frac{A_H(\mu^*)}{V_H}$, $\mu^*(d^*|S_L) = \delta(\mu^*) = \frac{1}{V_H}(A_H(\mu^*) - A_L(\mu^*))$, with the rest of the $\mu^*(d)$'s equal to zero. This solution satisfies $A_L(\mu^*) = \sum_{d \in \Phi \setminus \{d^*\}} \mu^*(d|S_L) u_S(d) = 0 \cdot \mu^*(0|S_L) + V_H \cdot \mu^*(V_H|S_L)$, and $A_H(\mu^*) = \sum_{d \in \Phi \setminus \{d^*\}} \mu^*(d|S_H) u_S(d) = 0 \cdot \mu^*(0|S_H) + V_H \cdot \mu^*(V_H|S_H)$. This is a valid solution because $0 \leq A_L(\mu^*) \leq A_H(\mu^*) \leq V_H$ which makes each $\mu^*(d) \in [0,1]$. Additionally, $\sum_{d \in \Phi} \mu^*(d|S_L) = \mu^*(0|S_L) + \mu^*(V_H|S_L) + \mu^*(d^*|S_L) = 1$ and $\sum_{d \in \Phi} \mu^*(d|S_H) = \mu^*(0|S_H) + \mu^*(V_H|S_H) = 1$.

None of the expected payoffs $U_S(\mu)$, $U_B(\mu, S_L|S_L)$ and $U_B(\mu, S_H|S_H)$ are zero, as this will make the objective value zero which is not a local maximum. Hence, let $P_L A_L + P_H A_H + \delta P_L r > 0$, and $V_L(V_H - A_H) - A_L(V_H - V_L) > 0$ (which makes $V_H - A_H > 0$); these are later shown to be satisfied by the optimal solution. After taking a log-transformation of the objective, the Lagrangian of the problem is

$$\begin{aligned}\mathcal{L}(A_L, A_H, \delta, \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5) \quad = \quad & \ln(P_L A_L + P_H A_H + \delta P_L r) + p_L \ln(V_L(V_H - A_H) - A_L(V_H - V_L)) \\ & + p_H \ln(V_H - A_H) - \lambda_1(P_L A_L + P_H A_H + \delta P_L r - r) - \lambda_2(A_H - A_L) \\ & + \lambda_3\left(\delta - \tfrac{1}{V_H}(A_H - A_L)\right) - \lambda_4 A_L - \lambda_5 \delta\end{aligned}$$

The KKT conditions generate the following equations along with the constraints of the problem $(S_N B_Y)'$, $\lambda_1 \geq 0$, $\lambda_2 \geq 0$, $\lambda_4 \geq 0$, and $\lambda_5 \geq 0$.

$$\frac{\partial \mathcal{L}}{\partial A_L}: \quad \frac{P_L}{P_L A_L + P_H A_H + \delta P_L r} - \frac{p_L(V_H - V_L)}{V_L(V_H - A_H) - A_L(V_H - V_L)} - \lambda_1 P_L + \lambda_2 + \frac{\lambda_3}{V_H} - \lambda_4 = 0 \quad \text{(B.16)}$$

$$\frac{\partial \mathcal{L}}{\partial A_H}: \quad \frac{P_H}{P_L A_L + P_H A_H + \delta P_L r} - \frac{p_L V_L}{V_L(V_H - A_H) - A_L(V_H - V_L)} - \frac{p_H}{V_H - A_H} - \lambda_1 P_H - \lambda_2 - \frac{\lambda_3}{V_H} = 0 \quad \text{(B.17)}$$

$$\frac{\partial \mathcal{L}}{\partial \delta}: \quad \frac{P_L r}{P_L A_L + P_H A_H + \delta P_L r} - \lambda_1 P_L r + \lambda_3 - \lambda_5 = 0 \quad \text{(B.18)}$$

$$\lambda_1(P_L A_L + P_H A_H + \delta P_L r - r) = 0 \quad \text{(B.19)}$$

$$\lambda_2(A_H - A_L) = 0 \quad \text{(B.20)}$$

$$\lambda_4 A_L = 0 \quad \text{(B.21)}$$

$$\lambda_5 \delta = 0 \quad \text{(B.22)}$$

Case I: $A_L > 0$ and $A_H \neq A_L$

In this case both λ_4 and λ_2 will be 0. Furthermore, since $A_H > A_L$, equation (B.14) implies that $\delta > 0$. Using equation (B.22), we get $\lambda_5 = 0$. Substituting δ from equation (B.14) into equation (B.9), gives

$$U_S = \frac{1}{V_H}\big((V_H - r)A_L P_L + A_H(V_H P_H + P_L r)\big) \quad \text{(B.23)}$$

Case I(a): $\lambda_1 = 0$. Using equation (B.18) to eliminate λ_3 from equations (B.16) and (B.17) gives

$$\frac{P_L(V_H - r)}{V_H(P_L A_L + P_H A_H + \delta P_L r)} = \frac{p_L(V_H - V_L)}{V_L(V_H - A_H) - A_L(V_H - V_L)} \quad \text{(B.24)}$$

and $$\frac{P_H V_H + P_L r}{V_H(P_L A_L + P_H A_H + \delta P_L r)} = \frac{p_L V_L}{V_L(V_H - A_H) - A_L(V_H - V_L)} + \frac{p_H}{V_H - A_H} \quad \text{(B.25)}$$

Eliminating δ from equations (B.24) and (B.25) yields

$$p_L(V_H - A_H)(P_H V_H + P_L r)(V_H - V_L) = P_L(V_H - r)\Big(p_L V_L(V_H - A_H) + p_H(V_L(V_H - A_H) - A_L(V_H - V_L))\Big)$$

$$\Rightarrow\ -A_L(V_H - r)(V_H - V_L)p_H P_L = (V_H - A_H)\Big(p_L(P_H V_H + P_L r)(V_H - V_L) - (V_H - r)P_L V_L\Big) \quad \text{(B.26)}$$

Substituting δ from equation (B.14) into equation (B.24) gives

$$\frac{P_L(V_H-r)}{(V_H-r)P_L A_L + A_H(P_H V_H + P_L r)} = \frac{p_L(V_H-V_L)}{V_L(V_H-A_H) - A_L(V_H-V_L)}$$

which after simplification yields

$$P_L(V_H - r)V_L V_H - A_L P_L(V_H - r)(V_H - V_L)(1 + p_L) = A_H\Big(p_L(V_H - V_L)(P_H V_H + P_L r) + P_L V_L(V_H - r)\Big)$$

Substituting A_L from equation (B.26) in the above equation gives

$$\begin{aligned}
& P_L(V_H - r)V_L V_H + \Big(\frac{1+p_L}{p_H}\Big)(V_H - A_H)\Big(p_L(P_H V_H + P_L r)(V_H - V_L) - (V_H - r)P_L V_L\Big) \\
& = A_H\Big(p_L(V_H - V_L)(P_H V_H + P_L r) + P_L V_L(V_H - r)\Big) \\
\Rightarrow\ & P_L V_L V_H(V_H - r)\big(1 - \tfrac{1+p_L}{p_H}\big) + p_L V_H(P_H V_H + P_L r)(V_H - V_L)\big(\tfrac{1+p_L}{p_H}\big) \\
& = A_H\Big(p_L(P_H V_H + P_L r)(V_H - V_L)\big(\tfrac{1+p_L}{p_H} + 1\big) + P_L V_L(V_H - r)\big(1 - \tfrac{1+p_L}{p_H}\big)\Big) \\
\Rightarrow\ & (1 + p_L)(V_H - V_L)p_L P_H V_H^2 + p_L P_L r(1 + p_L)(V_H - V_L)V_H - 2p_L P_L V_L V_H^2 + 2p_L P_L r V_L V_H \\
& = 2A_H\big(p_L P_H V_H(V_H - V_L) + p_L P_L r(V_H - V_L) - p_L P_L V_L V_H + p_L P_L V_L r\big) \\
\Rightarrow\ & A_H = \frac{V_H\Big((1+p_L)(V_H-V_L)P_H - 2P_L V_L\Big) + P_L r(V_H + V_L + p_L(V_H - V_L))}{2(P_H V_H - V_L + P_L r)} \\
\Rightarrow\ & = \frac{V_H}{2} - \frac{(V_H-r)P_L V_L - p_L(P_H V_H + P_L r)(V_H - V_L)}{2(P_H V_H - V_L + P_L r)}
\end{aligned}$$

For a finite A_H we need $P_H V_H - V_L + P_L r \neq 0$. From equation (B.11), we get $U_B(\mu^*, S_H|S_H)$ as

$$\begin{aligned}
U_B(\mu^*, S_H|S_H) &= V_H - \frac{V_H\Big((1+p_L)(V_H-V_L)P_H - 2P_L V_L\Big) + P_L r(V_H + V_L + p_L(V_H - V_L))}{2(P_H V_H - V_L + P_L r)} \\
&= \frac{V_H(P_H p_H(V_H - V_L) + 2P_L r) - P_L r(V_H + V_L + p_L(V_H - V_L))}{2(P_H V_H - V_L + P_L r)} \\
&= \frac{p_H(V_H - V_L)(P_H V_H + P_L r)}{2(P_H V_H - V_L + P_L r)} \qquad \text{(B.27)}
\end{aligned}$$

For $U_B(\mu^*, S_H|S_H) > 0$ we need $P_H V_H - V_L + P_L r > 0$, i.e., $P_H > \frac{V_L - r}{V_H - r}$ (B.28)

From equations (B.26) and (B.27), we get

$$A_L = \frac{(P_H V_H + P_L r)\Big((V_H - r)P_L V_L - p_L(P_H V_H + P_L r)(V_H - V_L)\Big)}{2(P_H V_H - V_L + P_L r)(V_H - r)P_L} \quad \text{(B.29)}$$

Substituting A_H and A_L in equation (B.10), we get

$$\begin{aligned}
U_B(\mu^*, S_L|S_L) &= \frac{1}{V_H}\Big(V_L(V_H - A_H) - A_L(V_H - V_L)\Big) \\
&= \frac{V_L p_H(V_H - V_L)(P_H V_H + P_L r)}{2(P_H V_H - V_L + P_L r)V_H} - \frac{(V_H - V_L)(P_H V_H + P_L r)\Big((V_H - r)P_L V_L - p_L(P_H V_H + P_L r)(V_H - V_L)\Big)}{2(P_H V_H - V_L + P_L r)(V_H - r)P_L V_H} \\
&= \frac{(V_H - V_L)(P_H V_H + P_L r)\Big(V_L P_L p_H(V_H - r) - (V_H - r)P_L V_L - p_L(P_H V_H + P_L r)(V_H - V_L)\Big)}{2(P_H V_H - V_L + P_L r)(V_H - r)P_L V_H} \\
&= \frac{(V_H - V_L)(P_H V_H + P_L r)p_L\Big((P_H V_H + P_L r)(V_H - V_L) - P_L V_L(V_H - r)\Big)}{2(P_H V_H - V_L + P_L r)(V_H - r)P_L V_H} \\
&= \frac{p_L(P_H V_H + P_L r)(V_H - V_L)}{2P_L(V_H - r)} \qquad \text{(B.30)}
\end{aligned}$$

We observe that $U_B(\mu^*, S_L|S_L) > 0$. We can now calculate $U_B(\mu^*) = P_L U_B(\mu^*, S_L|S_L) + P_H U_B(\mu^*, S_H|S_H)$ as

$$\begin{aligned} U_B(\mu^*) &= \frac{p_L(P_HV_H+P_Lr)(V_H-V_L)}{2(V_H-r)} + \frac{p_HP_H(V_H-V_L)(P_HV_H+P_Lr)}{2(P_HV_H-V_L+P_Lr)} \\ &= \frac{(P_HV_H+P_Lr)(V_H-V_L)(p_L(P_HV_H-V_L+P_Lr)+P_Hp_H(V_H-r))}{2(P_HV_H-V_L+P_Lr)(V_H-r)} \\ &= \frac{(P_HV_H+P_Lr)(V_H-V_L)(P_HV_H+P_Lr-p_LV_L-p_Hr)}{2(P_HV_H-V_L+P_Lr)(V_H-r)} \end{aligned} \quad (B.31)$$

From equation (C.31), we get $U_S = \frac{1}{V_H}\big((V_H-r)A_LP_L + A_H(V_HP_H+P_Lr)\big)$. Substituting A_L and A_H gives

$$\begin{aligned} U_S(\mu^*) &= \frac{(P_HV_H+P_Lr)\Big((V_H-r)P_LV_L-p_L(P_HV_H+P_Lr)(V_H-V_L)\Big)}{2(P_HV_H-V_L+P_Lr)V_H} \\ &\quad + \frac{\Big(V_H((1+p_L)(V_H-V_L)P_H-2P_LV_L)+P_Lr(V_H+V_L+p_L(V_H-V_L))\Big)(P_HV_H+P_Lr)}{2(P_HV_H-V_L+P_Lr)V_H} \\ &= \frac{(P_HV_H+P_Lr)\Big(-P_LV_L-P_Hp_L(V_H-V_L)+P_H(V_H-V_L)+P_Hp_L(V_H-V_L)+P_Lr\Big)}{2(P_HV_H-V_L+P_Lr)} \\ &= \frac{P_HV_H+P_Lr}{2} \end{aligned} \quad (B.32)$$

Clearly, $U_S(\mu^*) \geq r$ if $P_H \geq \frac{r}{V_H-r}$. (B.33)

We also need $A_L > 0$. From equation (B.29), we get

$$\begin{aligned} \frac{(P_HV_H+P_Lr)\Big((V_H-r)P_LV_L-p_L(P_HV_H+P_Lr)(V_H-V_L)\Big)}{2(P_HV_H-V_L+P_Lr)(V_H-r)P_L} &> 0 \\ \Rightarrow \quad P_L\big((V_H-r)V_L - p_Lr(V_H-V_L)\big) - p_LP_HV_H(V_H-V_L) &> 0 \\ \Rightarrow \quad (V_H-r)V_L - p_Lr(V_H-V_L) - P_H\big((V_H-r)V_L - p_Lr(V_H-V_L) + p_LV_H(V_H-V_L)\big) &> 0 \\ \Rightarrow \quad \frac{V_LV_H-r(p_HV_L+p_LV_H)}{(V_H-r)(p_HV_L+p_LV_H))} &> P_H \end{aligned} \quad (B.34)$$

The condition $A_H > A_L$, after some algebra, yields,

$$P_H^2(V_H-r)^2 - P_H(V_H-r)\big(V_H-2r+2V_L+p_L(V_H-V_L)\big) + (2V_L-r)(V_H-r) - p_Lr(V_H-V_L) < 0 \quad (B.35)$$

The discriminant of the above quadratic equation is positive as shown below,

$$\begin{aligned} &\big(V_H-2r+2V_L+p_L(V_H-V_L)\big)^2 - 4(2V_L-r)(V_H-r) + 4p_Lr(V_H-V_L) \\ = \; &(V_H+2V_L-2r)^2 + p_L^2(V_H-V_L)^2 + 2p_L(V_H-V_L)(V_H+2V_L-2r) - 4(2V_L-r)(V_H-r) + 4p_Lr(V_H-V_L) \\ = \; &(V_H-2V_L)^2 + p_L^2(V_H-V_L)^2 + 2p_L(V_H-V_L)(V_H+2V_L) \\ = \; &(V_H-2V_L+p_L(V_H-V_L))^2 + 8p_LV_L(V_H-V_L) \;\geq\; 0 \end{aligned}$$

Therefore, $A_H > A_L$ will be satisfied if $P_H^- < P_H < P_H^+$ where P_H^- and P_H^+ are the smaller and larger roots of (B.35) respectively. From equations (B.28), (B.33), and (B.34), Case I(a) will have a solution when $P_H > \frac{V_L-r}{V_H-r}$, $P_H \geq \frac{r}{V_H-r}$, $\frac{V_LV_H-r(p_HV_L+p_LV_H)}{(V_H-r)(p_HV_L+p_LV_H))} > P_H$, and $P_H^- < P_H < P_H^+$. Now, $P_H^- \geq \frac{V_L-r}{V_H-r}$ is always true as shown below,

$$\begin{aligned}
\frac{V_H+2V_L-2r+p_L(V_H-V_L)-\sqrt{(V_H-2V_L+p_L(V_H-V_L))^2+8p_LV_L(V_H-V_L)}}{2(V_H-r)} &\geq \frac{V_L-r}{V_H-r} \\
\Rightarrow \quad (V_H-2V_L+p_L(V_H-V_L))^2+8p_LV_L(V_H-V_L) &\leq (V_H+p_L(V_H-V_L))^2 \\
\Rightarrow \quad 8p_LV_L(V_H-V_L) &\leq 4V_L(V_H-V_L+p_L(V_H-V_L)) \\
\Rightarrow \quad p_L &\leq 1
\end{aligned}$$

Furthermore, $P_H^- \geq \frac{r}{V_H-r}$, after some algebra yields, $2r^2-r(V_L+V_H+p_LV_H+p_HV_L)+V_LV_H \geq 0$ (B.36) Following shows that the discriminant of the above quadratic is non-negative, i.e.,

$$\begin{aligned}
(V_L+V_H+p_LV_H+p_HV_L)^2-8V_LV_H &= (2V_L+V_H+(V_H-V_L)p_L)^2-8V_LV_H \\
&\geq (2V_L+V_H)^2-8V_LV_H \\
&= (2V_L-V_H)^2 \quad \geq \quad 0
\end{aligned}$$

Hence, $P_H^- \geq \frac{r}{V_H-r}$ is possible when $r \leq r^-$ or $r \geq r^+$ where, r^- and r^+ are the smaller and larger roots of (B.36) respectively. Also, $\frac{r}{V_H-r} \geq P_H^-$ is possible if $r^- \leq r \leq r^+$. For $r^- \leq r \leq r^+$, we will have a valid solution if,

$$\begin{aligned}
\frac{V_LV_H-r(p_HV_L+p_LV_H)}{(V_H-r)(p_HV_L+p_LV_H))} &\geq \frac{r}{V_H-r} \\
\Rightarrow \quad \frac{V_LV_H}{2(p_HV_L+p_LV_H)} &\geq r
\end{aligned}$$

Define $V'=p_HV_H+p_LV_L$ then, $p_HV_L+p_LV_H=V_L+V_H-V'$ and $p_L(V_H-V_L)=V_H-V'$. Following shows that $r^- \leq \frac{V_LV_H}{2(p_HV_L+p_LV_H)}$.

$$\begin{aligned}
2V_H+2V_L-V'-\sqrt{(2V_H+2V_L-V')^2-8V_LV_H} &\leq \frac{2V_LV_H}{V_L+V_H-V'} \\
\Rightarrow \quad \big((2V_L+2V_H-V')(V_L+V_H-V')-2V_LV_H\big)^2 &\leq (V_L+V_H-V')^2\big((2V_L+V_H-V')^2-8V_LV_H\big) \\
\Rightarrow \quad V_LV_H-(V_L+V_H-V')(2V_L+2V_H-V')+2(V_L+V_H-V)^2 &\leq 0 \\
\Rightarrow \quad V_LV_H-V'(V_L+V_H-V') &\leq 0 \\
\Rightarrow \quad (V'-V_H)(V'-V_L) &\leq 0
\end{aligned}$$

Above is always true since $V_L \leq p_HV_H+p_LV_L \leq V_H$. Similar algebra shows that $r^+ \geq \frac{V_LV_H}{2(p_HV_L+p_LV_H)}$. Furthermore, $P_H^- \leq \frac{V_LV_H-r(p_HV_L+p_LV_H)}{(V_H-r)(p_HV_L+p_LV_H))}$ as shown below.

$$\begin{aligned}
\frac{V_H+2V_L-2r+p_L(V_H-V_L)-\sqrt{(V_H-2V_L+p_L(V_H-V_L))^2+8p_LV_L(V_H-V_L)}}{2(V_H-r)} &\leq \frac{V_LV_H-r(p_HV_L+p_LV_H)}{(V_H-r)(p_HV_L+p_LV_H))} \\
\Rightarrow \quad \tfrac{1}{2}\big(2V_H+2V_L-2r-V'-\sqrt{(2V_H-2V_L-V')^2+8V_L(V_H-V')}\big) &\leq \frac{V_LV_H}{p_HV_L+p_LV_H}-r \\
\Rightarrow \quad \tfrac{1}{2}\big(2V_H+2V_L-V'-\sqrt{(2V_H-V')^2+4V_L^2-4V_L(2V_H-V')+8V_L(V_H-V')}\big) &\leq \frac{V_LV_H}{p_HV_L+p_LV_H} \\
\Rightarrow \quad \tfrac{1}{4}\big(2V_H+2V_L-V'-\sqrt{(2V_H-V')^2+4V_L^2+4V_L(2V_H-V')-8V_LV_H}\big) &\leq \frac{V_LV_H}{2(p_HV_L+p_LV_H)} \\
\Rightarrow \quad \tfrac{1}{4}\big(2V_H+2V_L-V'-\sqrt{(2V_H+2V_L-V')^2-8V_LV_H}\big) &\leq \frac{V_LV_H}{2(p_HV_L+p_LV_H)}
\end{aligned}$$

Above inequality is $r^- \leq \frac{V_L V_H}{2(p_H V_L + p_L V_H)}$ which is previously shown to be true. Hence, Case I(a) is valid for

$$\begin{array}{lllll} \frac{V_L V_H - r(p_H V_L + p_L V_H)}{(V_H - r)(p_H V_L + p_L V_H))} > P_H > P_H^- & \text{if} & r \leq r^- \text{ or } r \geq r^+ \\ \frac{V_L V_H - r(p_H V_L + p_L V_H)}{(V_H - r)(p_H V_L + p_L V_H))} > P_H > \frac{r}{V_H - r} & \text{if} & \frac{V_L V_H}{2(p_H V_L + p_L V_H)} \geq r \geq r^- \end{array}$$

Case I(b): $\lambda_1 \neq 0$ From equation (B.19), we get $U_S(\mu^*) = r$. Equation (C.31) yields $(V_H - r)A_L P_L + A_H(V_H P_H + P_L r) = V_H r$ (B.37). Adding equations (B.16) and (B.17), and using equation (B.37), we get $\frac{1}{r} - \frac{p_L V_H}{V_L(V_H - A_H) - A_L(V_H - V_L)} - \frac{p_H}{V_H - A_H} = \lambda_1$. Substituting λ_1 in equation (B.18), we get $-\lambda_3 = P_L r \left(\frac{p_L V_H}{V_L(V_H - A_H) - A_L(V_H - V_L)} + \frac{p_H}{V_H - A_H} \right)$. Substituting λ_1 and λ_3 in equations (B.16) gives

$$\begin{aligned} \frac{p_L(P_L V_H - V_H + V_L)}{V_L(V_H - A_H) - A_L(V_H - V_L)} + \frac{p_H P_L}{V_H - A_H} - \frac{P_L r}{V_H}\left(\frac{p_L V_H}{V_L(V_H - A_H) - A_L(V_H - V_L)} + \frac{p_H}{V_H - A_H} \right) &= 0 \\ \Rightarrow \quad \frac{p_H P_L(V_H - r)}{V_H(V_H - A_H)} - \frac{p_L(P_H V_H - D_L + P_L r)}{V_L(V_H - A_H) - A_L(V_H - V_L)} &= 0 \\ \Rightarrow \quad p_H P_L(V_H - r)\left(V_L(V_H - A_H) - A_L(V_H - V_L)\right) - p_L V_H(V_H - A_H)(P_H(V_H - r) + r - V_L) &= 0 \\ \Rightarrow \quad (V_H - A_H)\left(p_L V_H(V_L - r) - (V_H - r)(p_L P_H V_H - p_H P_L V_L)\right) = A_L p_H P_L(V_H - r)(V_H - V_L) & \quad \text{(B.38)} \end{aligned}$$

Substituting A_L from equation (B.37),

$$\begin{aligned} &(V_H - A_H)\left(p_L V_H(V_L - r) - (V_H - r)(p_L P_H V_H - p_H P_L V_L)\right) = p_H \left(V_H r - A_H(P_H V_H + P_L r)\right)(V_H - V_L) \\ \Rightarrow \quad & A_H\Big(p_H(V_H - V_L)(P_H V_H + P_L r) - p_L V_H(V_L - r) + (V_H - r)(p_L P_H V_H - p_H P_L V_L)\Big) = \\ & \qquad V_H\Big(p_H r(V_H - V_L) - p_L V_H(V_L - r) + (V_H - r)(p_L P_H V_H - p_H P_L V_L)\Big) \\ \Rightarrow \quad & A_H = \frac{p_H r(V_H - V_L) - p_L V_H(V_L - r) + (V_H - r)(p_L P_H V_H - p_H P_L V_L)}{P_H V_H - V_L + P_L r} \end{aligned}$$

From equation (B.11), we get $U_B(\mu^*, S_H|S_H)$ as

$$\begin{aligned} U_B(\mu^*, S_H|S_H) &= V_H - \frac{p_H r(V_H - V_L) - p_L V_H(V_L - r) + (V_H - r)(p_L P_H V_H - p_H P_L V_L)}{P_H V_H - V_L + P_L r} \\ &= \frac{P_H V_H^2 - V_L V_H + P_L V_H r - p_H V_H r + p_H V_L r + p_L V_L V_H - p_L V_H r - (V_H - r)(p_L P_H V_H - p_H P_L V_L)}{P_H V_H - V_L + P_L r} \\ &= \frac{P_H V_H^2 - p_H V_L V_H + P_L V_H r - V_H r + p_H V_L r - (V_H - r)(p_L P_H V_H - p_H P_L V_L)}{P_H V_H - V_L + P_L r} \\ &= \frac{(V_H - r)(P_H V_H - p_H V_L - p_L P_H V_H + p_H P_L V_L)}{P_H V_H - V_L + P_L r} \\ &= \frac{p_H P_H(V_H - V_L)(V_H - r)}{P_H V_H - V_L + P_L r} \quad \text{(B.39)} \end{aligned}$$

$U_B(\mu^*, S_H|S_H) > 0$ is true for $P_H V_H - V_L + P_L r > 0$, i.e., $P_H > \frac{V_L - r}{V_H - r}$. From equation (B.38), we get $A_L = \frac{P_H\left(p_L V_H(V_L - r) - (V_H - r)(p_L P_H V_H - p_H P_L V_L)\right)}{P_L(P_H V_H - V_L + P_L r)}$. Substituting A_H and A_L in equation (B.10) we get

$$\begin{aligned} U_B(\mu^*, S_L|S_L) &= \frac{p_H P_H P_L V_L(V_H - V_L)(V_H - r) - P_H(V_H - V_L)\left(p_L V_H(V_L - r) - (V_H - r)(p_L P_H V_H - p_H P_L V_L)\right)}{P_L V_H(P_H V_H - V_L + P_L r)} \\ &= \frac{P_H(V_H - V_L)\left((V_H - r)p_L P_H V_H - p_L V_H(V_L - r)\right)}{P_L V_H(P_H V_H - V_L + P_L r)} \\ &= \frac{p_L P_H(V_H - V_L)}{P_L} \quad \text{(B.40)} \end{aligned}$$

Observe that $U_B(\mu^*, S_L|S_L) > 0$. $U_B(\mu^*) = P_L U_B(\mu^*, S_L|S_L) + P_H U_B(\mu^*, S_H|S_H)$ is given by

$$\begin{aligned} U_B(\mu^*) &= \frac{p_H P_H^2 (V_H - V_L)(V_H - r)}{P_H V_H - V_L + P_L r} + p_L P_H (V_H - V_L) \\ &= \frac{P_H (V_H - V_L)\left(p_H P_H (V_H - r) + p_L P_H V_H - p_L V_L + p_L P_L r\right)}{P_H V_H - V_L + P_L r} \\ &= \frac{P_H (V_H - V_L)\left(P_H V_H - p_H P_H r - p_L V_L + p_L P_L r\right)}{P_H V_H - V_L + P_L r} \end{aligned} \quad \text{(B.41)}$$

For $P_H V_H - V_L + P_L r > 0$, the condition $A_H > A_L$ will be satisfied if,

$$\begin{aligned} & p_H r (V_H - V_L) - p_L V_H (V_L - r) + (V_H - r)(p_L P_H V_H - p_H P_L V_L) > \frac{P_H \left(p_L V_H (V_L - r) - (V_H - r)(p_L P_H V_H - p_H P_L V_L)\right)}{P_L} \\ \Rightarrow \quad & P_L r p_H - p_L V_L + p_L r + p_L P_H V_H - p_H P_L V_L - p_L P_H r > 0 \\ \Rightarrow \quad & P_H > \frac{V_L - r}{p_H V_L + p_L V_H - r} \quad \text{(B.42)} \end{aligned}$$

Also for $P_H V_H - V_L + P_L r > 0$ the condition $A_L > 0$ will be satisfied if $p_L V_H (V_L - r) - (V_H - r)(p_L P_H V_H - p_H P_L V_L) > 0$, i.e., $P_H < \frac{V_L V_H - r(p_H V_L + p_L V_H)}{(V_H - r)(p_H V_L + p_L V_H)}$. (B.43)

Therefore, from equations (B.42), and (B.43), Case I(b) will have a solution when $P_H > \frac{V_L - r}{V_H - r}$, $P_H > \frac{V_L - r}{p_H V_L + p_L V_H - r}$, and $P_H < \frac{V_L V_H - r(p_H V_L + p_L V_H)}{(V_H - r)(p_H V_L + p_L V_H)}$. There is no solution in Case I(b) when $r \geq \frac{V_L V_H}{p_H V_L + p_L V_H}$. For $V_L \geq r$, $\frac{V_L - r}{V_H - r} \leq \frac{V_L - r}{p_H V_L + p_L V_H - r}$ implies $p_L V_L + p_H V_H \geq V_L \geq r$ which is always true. Also, $\frac{V_L - r}{p_H V_L + p_L V_H - r} < \frac{V_L V_H - r(p_H V_L + p_L V_H)}{(V_H - r)(p_H V_L + p_L V_H)}$ implies,

$$\begin{aligned} (V_L - r)(V_H - r)(p_H V_L + p_L V_H) &< (V_L V_H - r(p_H V_L + p_L V_H))(p_H V_L + p_L V_H - r) \\ \Rightarrow \quad (p_H V_L + p_L V_H - V_H)(p_H V_L + p_L V_H - V_L) &< 0 \\ \Rightarrow \quad p_H (V_L - V_H) p_L (V_H - V_L) &< 0 \end{aligned}$$

Above inequality is always true. Furthermore, since $\frac{V_L V_H}{p_H V_L + p_L V_H} > V_L$ hence, for $r \geq V_L$ we have, $0 \leq P_H < \frac{V_L V_H - r(p_H V_L + p_L V_H)}{(V_H - r)(p_H V_L + p_L V_H)}$. Therefore, Case I(b) is valid for

$$\begin{aligned} \frac{V_L V_H - r(p_H V_L + p_L V_H)}{(V_H - r)(p_H V_L + p_L V_H)} > P_H &> \frac{V_L - r}{p_H V_L + p_L V_H - r} \quad \text{if} \quad r \leq V_L \\ \frac{V_L V_H - r(p_H V_L + p_L V_H)}{(V_H - r)(p_H V_L + p_L V_H)} > P_H &\geq 0 \quad \text{if} \quad V_L \leq r < \frac{V_L V_H}{p_H V_L + p_L V_H} \end{aligned}$$

Case II: $A_L = 0$ **and** $A_H \neq A_L$

In this case, both λ_2 and λ_5 will be 0. From equations (C.31) and (B.14) we have, $U_S = \frac{A_H}{V_H}(P_H V_H + P_L r)$ and $\delta = \frac{A_H}{V_H}$.

Case II(a): $\lambda_1 = 0$**.** Substituting δ in (B.18), we get $-\lambda_3 = \frac{P_L r V_H}{A_H (P_H V_H + P_L r)}$. Substituting λ_3 in equation (B.17) gives

$$\frac{P_H V_H}{A_H (P_H V_H + P_L r)} - \frac{p_L}{V_H - A_H} - \frac{p_H}{V_H - A_H} + \frac{P_L r}{A_H (P_H V_H + P_L r)} = 0$$

This equation yields $A_H = \frac{1}{2}V_H$. Therefore, $\delta = 0.5$ and $U_S(\mu^*) = \frac{1}{2}(P_H V_H + P_L r)$. Clearly, $U_S(\mu^*) \geq r$ for $P_H \geq \frac{r}{V_H - r}$. Substituting A_H and A_L in equations (B.10) and (B.11), we get $U_B(\mu^*, S_L|S_L) = \frac{V_L}{2}$ and $U_B(\mu^*, S_H|S_H) = \frac{V_H}{2}$. This gives, $U_B(\mu^*) = \frac{1}{2}(P_H V_H + P_L V_L)$. We observe that all the constraints of problem $(S_N B_Y)'$ are satisfied. Since $1 \geq P_H \geq \frac{r}{V_H - r}$, Case II(a) is valid for $P_H \geq \frac{r}{V_H - r}$ and $r \leq \frac{V_H}{2}$.

Case II(b): $\lambda_1 \neq 0$**.** Here we have $U_S = \frac{A_H}{V_H}(P_H V_H + P_L r) = r$, i.e., $A_H = \frac{V_H r}{P_H V_H + P_L r}$. Also, $\delta = \frac{r}{P_H V_H + P_L r}$. $U_B(\mu^*, S_H|S_H) = V_H - A_H = V_H(1-\delta) = \frac{P_H V_H (V_H - r)}{P_H V_H + P_L r}$. $U_B(\mu^*, S_L|S_L) = \frac{V_L}{V_H}(V_H - A_H) = (1-\delta)V_L$. Therefore, $U_B(\mu^*) = (1-\delta)(P_H V_H + P_L V_L) = (P_H V_H + P_L V_L)\left(1 - \frac{r}{P_H V_H + P_L r}\right)$. $U_B(\mu^*, S_L|S_L) > 0$ implies $P_H V_H + P_L r > r$ which is always true for $P_H > 0$ and $V_H > r$.

Case III: $A_L \neq 0$ **and** $A_H = A_L = A(say)$

In this case, $\lambda_4 = 0$. From equations (C.31) and (B.14), we have $U_S = A_H = A_L = A$ and $\delta = 0$. Property B.1 along with $\delta = 0$ implies that no positive probability is given to the disagreement alternatives under states L or H.

Case III(a): $\lambda_1 = 0$. Adding equations (B.16) and (B.17), we get

$$\begin{aligned} \frac{1}{A} - \frac{p_L}{V_L - A} - \frac{p_H}{V_H - A} &= 0 \\ \Rightarrow \quad 2A^2 - A(V_L + V_H + p_L V_H + p_H V_L) + V_L V_H &= 0 \quad \text{(B.44)} \end{aligned}$$

The quadratic equation in (B.44) is same as that in (B.36). The smaller and larger roots are r^- and r^+ respectively as found in Case I(a). From equation (B.16), we get $U_B(\mu^*, S_L|S_L) = V_L - A$. Hence, for a feasible solution, we need $V_L > A$. We will show using contradiction that the smaller root r^- satisfies $V_L > r^-$ while the larger root r^+ does not. Let $r^- \geq V_L$

$$\begin{aligned} \frac{1}{4}\left(2(V_L + V_H) - (p_L V_L + p_H V_H) - \sqrt{(2(V_L + V_H) - (p_L V_L + p_H V_H))^2 - 8V_L V_H}\right) &\geq V_L \\ \Rightarrow \quad (2(V_H - V_L) - (p_L V_L + p_H V_H))^2 &\geq (2(V_L + V_H) - (p_L V_L + p_H V_H))^2 - 8V_L V_H \\ \Rightarrow \quad 8V_L V_H &\geq 4V_L(4V_H - 2(p_L V_L + p_H V_H)) \\ \Rightarrow \quad p_L V_L + p_H V_H &\geq V_H \end{aligned}$$

This is not true for $p_H < 1$. Hence, the smaller root r^- satisfies $U_B(\mu^*, S_L|S_L) > 0$ when $p_H \neq 1$. A similar algebra shows that $V_L < r^+$. Hence only the smaller root r^- of the quadratic (B.44) is a solution for this case. We have $U_B(\mu^*, S_H|S_H) = V_H - r^- > 0$. The expected payoffs for this case is calculated as $U_S = r^-$

and $U_B = P_L V_L + P_H V_H - r^-$. We need $U_S = r^- \geq r$ to be satisfied, i.e.,

$$
\begin{aligned}
& 2(V_L+V_H) - (p_L V_L + p_H V_H) - \sqrt{(2(V_L+V_H)-(p_L V_L + p_H V_H))^2 - 8V_L V_H} \geq 4r \\
\Rightarrow \quad & (2(V_H+V_L) - (p_L V_L + p_H V_H) - 4r)^2 \geq (2(V_L+V_H)-(p_L V_L + p_H V_H))^2 - 8V_L V_H \\
\Rightarrow \quad & 8V_L V_H \geq 4r\,(4(V_H+V_L) - 2(p_L V_L + p_H V_H) - 4r) \\
\Rightarrow \quad & V_L V_H \geq 2r(V_H+V_L) - r p_H (V_H - V_L) - r V_L - 2r^2 \\
\Rightarrow \quad & p_H \geq \tfrac{(2r-V_L)(V_H-r)}{r(V_H-V_L)} \qquad \text{(B.45)}
\end{aligned}
$$

For a valid solution, we need $1 \geq p_H \geq \frac{(2r-V_L)(V_H-r)}{r(V_H-V_L)}$, i.e.,

$$
\begin{aligned}
(V_H - V_L)r &\geq (2r - V_L)(V_H - r) \\
\Rightarrow \quad 2r^2 - rV_H - 2rV_L + V_L V_H &\geq 0 \\
\Rightarrow \quad (2r - V_H)(r - V_L) &\geq 0
\end{aligned}
$$

This is true if either $r \leq \min(V_L, \frac{1}{2}V_H)$, or $r \geq \max(V_L, \frac{1}{2}V_H)$.

Case III(b): $\lambda_1 \neq 0$. For this case we have $U_S = A_H = A_L = A = r$. Therefore, $U_B(\mu^*, S_L|S_L) = V_L - r$, $U_B(\mu^*, S_H|S_H) = V_H - r$, and $U_B(\mu^*) = P_L V_L + P_H V_H - r$. For a valid solution, we need $U_B(\mu^*, S_L|S_L) > 0$, i.e., $V_L > r$.

From the above six cases we observe that there are five critical points for r: r^-, $\frac{V_L V_H}{2(p_H V_L + p_L V_H)}$, $\frac{V_L V_H}{p_H V_L + p_L V_H}$, V_L, and $\frac{1}{2}V_H$ where r^- is the smaller root of the quadratic equation (B.36) of Case I(a). We know that $r^- \leq V_L$ as shown in Case III(a). $r^- \leq \frac{V_L V_H}{2(p_H V_L + p_L V_H)}$ as shown in Case I(a). Also, $V_L \leq \frac{V_L V_H}{p_H V_L + p_L V_H} \leq V_H$ is easily shown to be true. Furthermore, $\frac{V_L V_H}{p_H V_L + p_L V_H} \leq \frac{V_H}{2}$ if $p_H V_H + p_L V_L \leq V_H - V_L$ and $V_L \geq \frac{V_L V_H}{2(p_H V_L + p_L V_H)}$ if $V_L + \frac{1}{2}V_H \geq p_H V_H + p_L V_L$. Moreover, $V_L + \frac{1}{2}V_H \leq V_H - V_L$ if $V_L \leq \frac{1}{4}V_H$.

We consider the following five ranges — Range 1: $V_L \leq \frac{V_H}{2}$ and $p_H V_H + p_L V_L \leq \min\{V_H - V_L, V_L + \frac{1}{2}V_H\}$, Range 2: $V_L \leq \frac{1}{4}V_H$ and $V_L + \frac{1}{2}V_H \leq p_H V_H + p_L V_L \leq V_H - V_L$, Range 3: $\frac{1}{4}V_H \leq V_L \leq \frac{1}{2}V_H$ and $V_H - V_L \leq p_H V_H + p_L V_L \leq V_L + \frac{1}{2}V_H$, Range 4: $V_L \leq \frac{1}{2}V_H$ and $p_H V_H + p_L V_L \geq \max\{V_H - V_L, V_L + \frac{1}{2}V_H\}$, and Range 5: $\frac{1}{2}V_H \leq V_L \leq V_H$.

Range 1: $V_L \leq \frac{V_H}{2}$ and $p_H V_H + p_L V_L \leq \min\{V_H - V_L, V_L + \frac{1}{2}V_H\}$.

In this range we have $r^- \leq \frac{V_L V_H}{2(p_H V_L + p_L V_H)} \leq V_L \leq \frac{1}{2}V_H < V_H$. We will obtain the global solution for the following five ranges of r.

(i) $0 \leq r \leq r^-$: From Case I(a) we know that $P_H^- \leq \frac{V_L V_H - r(p_H V_L + p_L V_H)}{(V_H - r)(p_H V_L + p_L V_H)}$ where $P_H^- = \frac{2V_H + 2V_L - 2r - (p_H V_H + p_L V_L) - \sqrt{(2V_H + 2V_L - (p_H V_H + p_L V_L))^2 - 8V_L V_H}}{2(V_H - r)}$.

The global equilibrium solution is obtained by comparing the objective values of the valid cases in the following three ranges of P_H. Note that for $p_H \leq 0.5$, $p_H \leq P_H \leq 0.5$ and for $p_H \geq 0.5$, $0.5 \leq P_H \leq p_H$.

$$\begin{array}{llrcccl} \text{Case III(a)} & \text{if} & 0 & \leq & P_H & \leq & P_H^- \\ \text{Case I(a)} & \text{if} & P_H^- & \leq & P_H & \leq & \frac{V_L V_H - r(p_H V_L + p_L V_H)}{(V_H - r)(p_H V_L + p_L V_H)} \\ \text{Case II(a)} & \text{if} & \frac{V_L V_H - r(p_H V_L + p_L V_H)}{(V_H - r)(p_H V_L + p_L V_H)} & \leq & P_H & \leq & 1 \end{array}$$

(ii) $r^- \leq r \leq \frac{V_L V_H}{2(p_H V_L + p_L V_H)}$: We consider four ranges of P_H with breakpoints at $\frac{V_L - r}{p_H V_L + p_L V_H - r}$, $\frac{r}{V_H - r}$, and $\frac{V_L V_H - r(p_H V_L + p_L V_H)}{(V_H - r)(p_H V_L + p_L V_H)}$. $\frac{r}{V_H - r} \geq \frac{V_L - r}{p_H V_L + p_L V_H - r}$ yields $2r^2 - r(V_H + V_L + p_L V_H + p_H V_L) + V_L V_H \leq 0$. This quadratic inequality is true for $r^- \leq r \leq \frac{V_L V_H}{2(p_H V_L + p_L V_H)} \leq r^+$, as shown in Case I(a) hence, $\frac{r}{V_H - r} \geq \frac{V_L - r}{p_H V_L + p_L V_H - r}$. Also, $\frac{r}{V_H - r} \leq \frac{V_L V_H - r(p_H V_L + p_L V_H)}{(V_H - r)(p_H V_L + p_L V_H)}$ yields $r \leq \frac{V_L V_H}{2(p_H V_L + p_L V_H)}$ which is true. The global solution for these ranges of P_H is

$$\begin{array}{llrcccl} \text{Case III(b)} & \text{if} & 0 & \leq & P_H & \leq & \frac{V_L - r}{p_H V_L + p_L V_H - r} \\ \text{Case I(b)} & \text{if} & \frac{V_L - r}{p_H V_L + p_L V_H - r} & \leq & P_H & \leq & \frac{r}{V_H - r} \\ \text{Case I(a)} & \text{if} & \frac{r}{V_H - r} & \leq & P_H & \leq & \frac{V_L V_H - r(p_H V_L + p_L V_H)}{(V_H - r)(p_H V_L + p_L V_H)} \\ \text{Case II(a)} & \text{if} & \frac{V_L V_H - r(p_H V_L + p_L V_H)}{(V_H - r)(p_H V_L + p_L V_H)} & \leq & P_H & \leq & 1 \end{array}$$

(iii) $\frac{V_L V_H}{2(p_H V_L + p_L V_H)} \leq r \leq V_L$: We consider four ranges of P_H with breakpoints at $\frac{V_L - r}{p_H V_L + p_L V_H - r}$, $\frac{V_L V_H - r(p_H V_L + p_L V_H)}{(V_H - r)(p_H V_L + p_L V_H)}$, and $\frac{r}{V_H - r}$. Range 1(ii) shows that $\frac{V_L V_H - r(p_H V_L + p_L V_H)}{(V_H - r)(p_H V_L + p_L V_H)} \leq \frac{r}{V_H - r}$ for $\frac{V_L V_H}{2(p_H V_L + p_L V_H)} \leq r$. Also, Case I(b) shows $\frac{V_L - r}{p_H V_L + p_L V_H - r} \leq \frac{V_L V_H - r(p_H V_L + p_L V_H)}{(V_H - r)(p_H V_L + p_L V_H)}$. The global solution for these ranges of P_H is

$$\begin{array}{llrcccl} \text{Case III(b)} & \text{if} & 0 & \leq & P_H & \leq & \frac{V_L - r}{p_H V_L + p_L V_H - r} \\ \text{Case I(b)} & \text{if} & \frac{V_L - r}{p_H V_L + p_L V_H - r} & \leq & P_H & \leq & \frac{V_L V_H - r(p_H V_L + p_L V_H)}{(V_H - r)(p_H V_L + p_L V_H)} \\ \text{Case II(b)} & \text{if} & \frac{V_L V_H - r(p_H V_L + p_L V_H)}{(V_H - r)(p_H V_L + p_L V_H)} & \leq & P_H & \leq & \frac{r}{V_H - r} \\ \text{Case II(a)} & \text{if} & \frac{r}{V_H - r} & \leq & P_H & \leq & 1 \end{array}$$

(iv) $V_L \leq r \leq \frac{V_H}{2}$: For this range of r, Case II(b) is the global solution when $0 \leq P_H \leq \frac{r}{V_H - r}$ and Case II(a) is the global solution when $\frac{r}{V_H - r} \leq P_H \leq 1$.

(v) $\frac{V_H}{2} \leq r < V_H$: Case II(b) is the global solution for the entire range of P_H.

Range 2: $V_L \leq \frac{1}{4} V_H$ **and** $V_L + \frac{1}{2} V_H \leq p_H V_H + p_L V_L \leq V_H - V_L$**.**

As shown earlier, $\frac{V_L V_H}{2(p_H V_L + p_L V_H)} \leq \frac{1}{2} V_H$. We will consider the following three ranges of r.

(i) $0 \leq r \leq \frac{V_L V_H}{2(p_H V_L + p_L V_H)}$: Case II(a) is the global solution for the entire range of P_H.

(ii) $\frac{V_L V_H}{2(p_H V_L + p_L V_H)} \leq r \leq \frac{1}{2} V_H$: Case II(b) is the global solution when $0 \leq P_H \leq \frac{r}{V_H - r}$ and Case II(a) is the global solution when $\frac{r}{V_H - r} \leq P_H \leq 1$.

(iii) $\frac{1}{2} V_H \leq r \leq V_H$: Case II(b) is the global solution for the entire range of P_H.

Range 3: $\frac{1}{4}V_H \leq V_L \leq \frac{1}{2}V_H$ **and** $V_H - V_L \leq p_H V_H + p_L V_L \leq V_L + \frac{1}{2}V_H$**.**

In this range we have $r^- \leq \frac{V_L V_H}{2(p_H V_L + p_L V_H)} \leq V_L \leq \frac{1}{2}V_H < V_H$. Therefore, we will consider five ranges of r.

(i) $0 \leq r \leq r^-$: The optimal solution remains the same as that of Range 1(i).

(ii) $r^- \leq r \leq \frac{V_L V_H}{2(p_H V_L + p_L V_H)}$: Case I(a) is the global solution when $0 \leq P_H \leq \frac{V_L V_H - r(p_H V_L + p_L V_H)}{(V_H - r)(p_H V_L + p_L V_H)}$ and Case II(a) is the global solution when $\frac{V_L V_H - r(p_H V_L + p_L V_H)}{(V_H - r)(p_H V_L + p_L V_H)} \leq P_H \leq 1$.

(iii) $\frac{V_L V_H}{2(p_H V_L + p_L V_H)} \leq r \leq V_L$: From Range 1(iii) we know that $\frac{V_L V_H - r(p_H V_L + p_L V_H)}{(V_H - r)(p_H V_L + p_L V_H)} \leq \frac{r}{V_H - r}$ for $\frac{V_L V_H}{2(p_H V_L + p_L V_H)} \leq r$. We obtain the global solution for the following three ranges of P_H

Case I(a)	if	0	$\leq$	P_H	$\leq$	$\frac{V_L V_H - r(p_H V_L + p_L V_H)}{(V_H - r)(p_H V_L + p_L V_H)}$
Case II(b)	if	$\frac{V_L V_H - r(p_H V_L + p_L V_H)}{(V_H - r)(p_H V_L + p_L V_H)}$	$\leq$	P_H	$\leq$	$\frac{r}{V_H - r}$
Case II(a)	if	$\frac{r}{V_H - r}$	$\leq$	P_H	$\leq$	1

For $V_L \leq r \leq \frac{1}{2}V_H$ and $\frac{1}{2}V_H \leq r < V_H$ the global solution remains the same as that of Range 1(iv) and Range 1(v) respectively.

Range 4: $V_L \leq \frac{1}{2}V_H$ **and** $p_H V_H + p_L V_L \geq \max\{V_H - V_L, V_L + \frac{1}{2}V_H\}$**.**

In this range, we have $r^- \leq \frac{V_L V_H}{2(p_H V_L + p_L V_H)} \leq \frac{1}{2}V_H < V_H$. We will consider four ranges of r.

(i) $0 \leq r \leq r^-$: There is a single breakpoint $\frac{V_L V_H - r(p_H V_L + p_L V_H)}{(V_H - r)(p_H V_L + p_L V_H)}$ for P_H between 0 to 1. Case I(a) is the global solution when $0 \leq P_H \leq \frac{V_L V_H - r(p_H V_L + p_L V_H)}{(V_H - r)(p_H V_L + p_L V_H)}$ and Case II(a) is the global solution when $\frac{V_L V_H - r(p_H V_L + p_L V_H)}{(V_H - r)(p_H V_L + p_L V_H)} \leq P_H \leq 1$.

(ii) $r^- \leq r \leq \frac{V_L V_H}{2(p_H V_L + p_L V_H)}$: Case II(a) is the global solution for the entire range of P_H.

(iii) $\frac{V_L V_H}{2(p_H V_L + p_L V_H)} \leq r < \frac{1}{2}V_H$: The global solution is the same as that of Range 2(ii).

(iv) $\frac{1}{2}V_H \leq r < V_H$: Case II(b) is the global solution for the entire range of P_H.

Range 5: $\frac{1}{2}V_H \leq V_L \leq V_H$**.**

In this range we have $r^- \leq \frac{V_L V_H}{2(p_H V_L + p_L V_H)} \leq \frac{1}{2}V_H \leq V_L < V_H$. We will obtain the global solution for the following five ranges of r.

(i) $0 \leq r \leq r^-$: The solution is same as that of Range 1(i).

(ii) $r^- \leq r \leq \frac{V_L V_H}{2(p_H V_L + p_L V_H)}$: The solution is same as that of Range 1(ii).

(iii) $\frac{V_L V_H}{2(p_H V_L + p_L V_H)} \leq r \leq \frac{1}{2}V_H$: The global solution remains same as that of Range 1(iii).

(iv) $\frac{V_H}{2} \leq r \leq V_L$: We divide the range of P_H in three parts with breakpoints at $\frac{V_L - r}{p_H V_L + p_L V_H - r}$ and $\frac{V_L V_H - r(p_H V_L + p_L V_H)}{(V_H - r)(p_H V_L + p_L V_H)}$. From Case I(b) we have $\frac{V_L - r}{p_H V_L + p_L V_H - r} \leq \frac{V_L V_H - r(p_H V_L + p_L V_H)}{(V_H - r)(p_H V_L + p_L V_H)}$. The global solution is given as follows.

Case III(b)	if	0	$\leq$	P_H	$\leq$	$\frac{V_L - r}{p_H V_L + p_L V_H - r}$
Case I(b)	if	$\frac{V_L - r}{p_H V_L + p_L V_H - r}$	$\leq$	P_H	$\leq$	$\frac{V_L V_H - r(p_H V_L + p_L V_H)}{(V_H - r)(p_H V_L + p_L V_H)}$
Case II(b)	if	$\frac{V_L V_H - r(p_H V_L + p_L V_H)}{(V_H - r)(p_H V_L + p_L V_H)}$	$\leq$	P_H	$\leq$	1

(v) $V_L \leq r < V_H$: Case II(b) is the global solution for the entire range of P_H.

Tables B.3 to B.7 summarize the optimal solutions of problem $(S_N B_Y)$ in the five cases where, $r^- = \frac{1}{4}\Big(2(V_L + V_H) - (p_L V_L + p_H V_H) - \sqrt{(2(V_L + V_H) - (p_L V_L + p_H V_H))^2 - 8 V_L V_H}\Big)$, and $V = P_H V_H + P_L V_L$. Table B.1 provides the notations A to F denoting the equilibrium payoffs (U_S^*, U_B^*) for the seller and the buyer respectively. The notations a to d provided in Table B.2 denote the probabilities between which the given solutions are valid.

Table B.1. Notation for Optimal Solutions

Notation	(U_S^*, U_B^*)
A	$(r, V - r)$
B	$(r^-, V - r^-)$
C	$\left(\frac{1}{2}(P_H V_H + P_L r), \frac{(P_H V_H + P_L r)(V_H - V_L)(P_H V_H + P_L r - p_L V_L - p_H r)}{2(P_H V_H - V_L + P_L r)(V_H - r)}\right)$
D	$\left(\frac{1}{2}(P_H V_H + P_L r), \frac{V}{2}\right)$
E	$\left(r, \frac{(V - V_L)\left(P_H V_H - p_H P_H r - p_L V_L + p_L P_L r\right)}{P_H V_H - V_L + P_L r}\right)$
F	$\left(r, V\left(1 - \frac{r}{P_H V_H + P_L r}\right)\right)$

Table B.2. Notation for Probability Breakpoints

Notation	**Probability breakpoints**
a	$\frac{r}{V_H - r}$
b	$\frac{2(V_H + V_L - r) - (p_H V_H + p_L V_L) - \sqrt{(2(V_H + V_L) - (p_H V_H + p_L V_L))^2 - 8 V_L V_H}}{2(V_H - r)}$
c	$\frac{V_L - r}{p_H V_L + p_L V_H - r}$
d	$\frac{V_L V_H - r(p_H V_L + p_L V_H)}{(V_H - r)(p_H V_L + p_L V_H))}$

Table B.3. Case 1: When $V_L \leq \frac{1}{2} V_H$ and $p_H V_H + p_L V_L \leq \min\{V_H - V_L, V_L + \frac{1}{2} V_H\}$

$0 \leq r \leq r^-$	$r^- \leq r \leq \frac{V_L V_H}{2(p_H V_L + p_L V_H)}$	$\frac{V_L V_H}{2(p_H V_L + p_L V_H)} \leq r \leq V_L$	$V_L \leq r \leq \frac{V_H}{2}$	$\frac{V_H}{2} \leq r < V_H$
B if $0 \leq P_H \leq b$ C if $b \leq P_H \leq d$ D if $d \leq P_H \leq 1$	A if $0 \leq P_H \leq c$ E if $c \leq P_H \leq a$ C if $a \leq P_H \leq d$ D if $d \leq P_H \leq 1$	A if $0 \leq P_H \leq c$ E if $c \leq P_H \leq d$ F if $d \leq P_H \leq a$ D if $a \leq P_H \leq 1$	F if $0 \leq P_H \leq a$ D if $a \leq P_H \leq 1$	F if $0 \leq P_H \leq 1$

Table B.4. Case 2: When $V_L \le \frac{1}{4}V_H$ and $V_L + \frac{1}{2}V_H \le p_H V_H + p_L V_L \le V_H - V_L$

$0 \le r \le \frac{V_L V_H}{2(p_H V_L + p_L V_H)}$	$\frac{V_L V_H}{2(p_H V_L + p_L V_H)} \le r \le \frac{V_H}{2}$	$\frac{V_H}{2} \le r < V_H$
D if $0 \le P_H \le 1$	F if $0 \le P_H \le a$ D if $a \le P_H \le 1$	F if $0 \le P_H \le 1$

Table B.5. Case 3: When $\frac{1}{4}V_H \le V_L \le \frac{1}{2}V_H$ and $V_H - V_L \le p_H V_H + p_L V_L \le V_L + \frac{1}{2}V_H$

$0 \le r \le r^-$	$r^- \le r \le \frac{V_L V_H}{2(p_H V_L + p_L V_H)}$	$\frac{V_L V_H}{2(p_H V_L + p_L V_H)} \le r \le V_L$	$V_L \le r \le \frac{V_H}{2}$	$\frac{V_H}{2} \le r < V_H$
B if $0 \le P_H \le b$ C if $b \le P_H \le d$ D if $d \le P_H \le 1$	C if $0 \le P_H \le d$ D if $d \le P_H \le 1$	C if $0 \le P_H \le d$ F if $d \le P_H \le a$ D if $a \le P_H \le 1$	F if $0 \le P_H \le a$ D if $a \le P_H \le 1$	F if $0 \le P_H \le 1$

Table B.6. Case 4: When $V_L \le \frac{1}{2}V_H$ and $p_H V_H + p_L V_L \ge \max\{V_H - V_L, V_L + \frac{1}{2}V_H\}$

$0 \le r \le r^-$	$r^- \le r \le \frac{V_L V_H}{2(p_H V_L + p_L V_H)}$	$\frac{V_L V_H}{2(p_H V_L + p_L V_H)} \le r \le \frac{V_H}{2}$	$\frac{V_H}{2} \le r < V_H$
C if $0 \le P_H \le d$ D if $d \le P_H \le 1$	D if $0 \le P_H \le 1$	F if $0 \le P_H \le a$ D if $a \le P_H \le 1$	F if $0 \le P_H \le 1$

Table B.7. Case 5: When $\frac{1}{2}V_H \le V_L \le V_H$

$0 \le r \le r^-$	$r^- \le r \le \frac{V_L V_H}{2(p_H V_L + p_L V_H)}$	$\frac{V_L V_H}{2(p_H V_L + p_L V_H)} \le r \le \frac{V_H}{2}$	$\frac{V_H}{2} \le r \le V_L$	$V_L \le r < V_H$
B if $0 \le P_H \le b$ C if $b \le P_H \le d$ D if $d \le P_H \le 1$	A if $0 \le P_H \le c$ E if $c \le P_H \le a$ C if $a \le P_H \le d$ D if $d \le P_H \le 1$	A if $0 \le P_H \le c$ E if $c \le P_H \le d$ F if $d \le P_H \le a$ D if $a \le P_H \le 1$	A if $0 \le P_H \le c$ E if $c \le P_H \le d$ F if $d \le P_H \le 1$	F if $0 \le P_H \le 1$

Lemma B.1 *When the distribution of the value is common knowledge and the outside option for the seller is r, the negotiation without a demonstration will result in the following expected payoffs U_S^* and U_B^* to the seller and the buyer respectively, where D is the expected value of the data product.*

$$(U_S^*, U_B^*) = \begin{cases} \left(\frac{D}{2}, \frac{D}{2}\right), & \text{if } r \le \frac{D}{2} \\ (r, D-r), & \text{if } \frac{D}{2} \le r \le D \\ (r, 0), & \text{if } D \le r \end{cases}$$

PROOF: Since no demonstration is offered we have $\alpha = 0.5$ and both the players assume the distribution of the value to be D_L with probability p_L and D_H with probability $p_H = 1 - p_L$. Then $P_H = P_L = 0.5$ and $V_L = V_H = D$. From equations (B.1), (B.2), (B.3), (B.4), and (B.5) we get, $U_B(\mu, S_H|S_H) = \sum_{d\in\Phi\setminus\{d^*\}} \mu(d|S_H)\big(D - u_S(d)\big)$, $U_B(\mu, S_L|S_H) = \sum_{d\in\Phi\setminus\{d^*\}} \mu(d|S_L)\big(D - u_S(d)\big)$, $U_B(\mu, S_H|S_L) = \sum_{d\in\Phi\setminus\{d^*\}} \mu(d|S_H)\big(D - u_S(d)\big)$, $U_B(\mu, S_L|S_L) = \sum_{d\in\Phi\setminus\{d^*\}} \mu(d|S_L)\big(D - u_S(d)\big)$, and $U_S(\mu) = \frac{1}{2}\sum_{d\in\Phi}\big(\mu(d|S_H) + \mu(d|S_L)\big)u_S(d)$. Property B.2 states that $U_B(\mu, S_H|S_H) = U_B(\mu, S_L|S_H)$ i.e. $\sum_{d\in\Phi\setminus\{d^*\}}\big(\mu(d|S_H) - \mu(d|S_L)\big)\big(D - u_S(d)\big) = 0$. For $D - u_S(d) \ne 0$

we have $\mu(d|S_H) = \mu(d|S_L)$ for each $d \in \Phi\backslash\{d^*\}$. Also, $\mu(d^*|S_H) = 1 - \sum\limits_{d\in\Phi\backslash\{d^*\}} \mu(d|S_H)$ and $\mu(d^*|S_L) = 1 - \sum\limits_{d\in\Phi\backslash\{d^*\}} \mu(d|S_L)$. This implies $\mu(d^*|S_H) = \mu(d^*|S_L)$. Consequently, we need a single decision vector $\mu(d) = \mu(d|S_H) = \mu(d|S_L)$ for $d \in \Phi$. Therefore, the problem $(S_N B_Y)$ can be written as the following problem $(S_N B_N)$:

$$
\begin{array}{llllllll}
(S_N B_N) & \max\limits_{\mu\in\Psi} & \multicolumn{6}{l}{\left(\sum\limits_{d\in\Phi} \mu(d) u_S(d)\right) \cdot \left(\sum\limits_{d\in\Phi} \mu(d) u_B(d)\right)} \\
& \text{s.t.} & \sum\limits_{d\in\Phi} \mu(d) u_S(d) & \geq & r; & \sum\limits_{d\in\Phi} \mu(d) u_B(d) & \geq & 0 \\
& & \sum\limits_{d\in\Phi} \mu(d) & = & 1; & \mu(d) & \geq & 0 \ \forall\, d \in \Phi \\
& & u_B(d) + u_S(d) & = & D & \forall\, d \in \Phi\backslash\{d^*\} & & \\
& & \multicolumn{6}{l}{u_B(d^*) = 0; \quad u_S(d^*) = r;}
\end{array}
$$

Here $\sum\limits_{d\in\Phi} \mu(d) u_S(d)$ and $\sum\limits_{d\in\Phi} \mu(d) u_B(d)$ represents the expected payoffs U_S and U_B of seller and buyer respectively. Note that any disagreement ($\mu(d^*) = 1$) will breakdown the negotiation giving $U_S = r$ and $U_B = 0$ and the Nash product will become zero. The players can do better by agreeing on a price. For any price q, the expected payoffs to the seller and the buyer are $U_S(q) = q$ and $U_B(q) = D - q$ respectively. The Nash product is given by, $\max\limits_{r\leq q\leq D} q\cdot(D-q)$. The Lagrangian of the problem is $\mathcal{L}(q, \lambda_1, \lambda_2) = q(D-q) - \lambda_1(q-r) - \lambda_2(D-q)$. The KKT conditions are represented by the following equations, along with the non-negativity constraints $\lambda_1 \geq 0$, and $\lambda_2 \geq 0$:

$$
\begin{array}{rrcll}
\frac{\partial\mathcal{L}}{\partial q}: & D - 2q - \lambda_1 + \lambda_2 & = & 0 & \text{(B.46)} \\
& \lambda_1(q-r) & = & 0 & \text{(B.47)} \\
& \lambda_2(D-q) & = & 0 & \text{(B.48)} \\
& q - r & \geq & 0 & \text{(B.49)} \\
& D - q & \geq & 0 & \text{(B.50)}
\end{array}
$$

Case I: $q \neq r$. This implies that $\lambda_1 = 0$ and $\lambda_2 = 2q - D$. From (B.48) we get $(2q-D)(q-D) = 0$ which gives $q = D, \frac{D}{2}$. The objective is 0 when $q = D$, and $\left(\frac{D}{2}\right)^2$ when $q = \frac{D}{2}$, and therefore, the optimal solution when $q \neq r$ is $q = \frac{D}{2}$ Inequality (B.49) implies that $(U_S, U_B) = \left(\frac{D}{2}, \frac{D}{2}\right)$ is feasible only when $r \leq \frac{D}{2}$.

Case II: $q = r$. This implies that $U_B(q) = D - r$. For a feasible solution, we need $D \geq r$. Since both $q = r$ and $q = \frac{D}{2}$ are feasible solutions in the range $r < \frac{D}{2}$, we need to compare their objective values $r(D-r)$ and $\left(\frac{D}{2}\right)^2$ to find the global optimum. Since $\frac{1}{2}\left(r + (D-r)\right) \geq \sqrt{r(D-r)}$, $\left(\frac{D}{2}\right)^2 \geq r(D-r)$. Therefore, $\left(\frac{D}{2}, \frac{D}{2}\right)$ is the global solution when $r < \frac{D}{2}$ and $(r, D-r)$ is the global solution in the range $\frac{D}{2} \leq r \leq D$. For $r \geq D$, none of the prices satisfy all the conditions, and therefore, there will be no agreement — the disagreement outcome $(r, 0)$ will result. ■

Proposition B.1 *Seller's decision on offering a demonstration is based on following criteria:*

B.1.1. Seller will not offer a demo when $0 \le r \le \min\left\{D_L, \frac{1}{2}D\right\}$ *and* $p_H \le 0.5$.

B.1.2. Seller will offer a noisy demo of accuracy $\alpha^* = \frac{1}{2k_1}\left(k_2 + \sqrt{k_2^2 + 4k_1k_3}\right)$ *when* $0 \le r \le \frac{2r^-(\alpha^*) - p_H D_H}{p_L}$ *and* $p_H \ge 0.5$. *Buyer will reject this offer as it is detrimental for them.*

B.1.3. Seller will offer a demo of accuracy $\alpha = 1$ *when — (i)* $D_L \le r \le \frac{p_H D_H}{1+p_H}$ *and* $p_H \le 0.5$, *(ii)* $\frac{2r^-(\alpha^*) - p_H D_H}{p_L} \le r \le \frac{p_H D_H}{1+p_H}$ *and* $p_H \ge 0.5$. *Buyer is indifferent in accepting or rejecting this offer.*

B.1.4. Seller will be indifferent in offering a demo otherwise.

where $D = p_H D_H + p_L D_L$,

$r^- = \frac{1}{4}\left(2(V_L + V_H) - (p_L V_L + p_H V_H) - \sqrt{(2(V_L + V_H) - (p_L V_L + p_H V_H))^2 - 8V_L V_H}\right)$,

$k_1 = (p_H D_H - p_L D_L)(p_H - p_L)^2$, $k_2 = (p_H^2 D_H + p_L^2 D_L)(p_H - p_L) - 2k_3$, *and*

$k_3 = (p_H^2 D_H - p_L^2 D_L)\sqrt{p_L p_H} + p_L p_H((2 - p_L)D_H - (2 - p_H)D_L)$

PROOF: From Table B.1 we observe that seller's equilibrium payoff can take following three values (i) r^-, (ii) $\frac{1}{2}(P_H V_H + P_L r)$, and (iii) r. In the following we discuss when the seller will refuse to offer a demo, offer a noisy demo, offer a demo with complete information, and remain indifferent in offering a demo.

When $U_S^{(S_N B_Y)} = r^-$

From Tables B.3, B.5, and B.7, we observe that the equilibrium solution for seller is r^- when $r \le r^-$ and $P_H \le P_H^-$ where P_H^- and r^- are obtained in section B.1. We also need $V_L \le \frac{1}{2}V_H$ with $p_H V_H + p_L V_L \le V_H - V_L$ or $V_L \ge \frac{1}{2}V_H$ to hold. From case B.1 we know that $P_H^- = \frac{2r^- - r}{V_H - r}$. Hence, $P_H^- = \frac{2r^- - r}{V_H - r} \ge P_H$ implies $\frac{2r^- - P_H V_H}{P_L} \ge r$. Therefore, $U_S^{(S_N B_Y)} = r^-$ holds for $r \le \min\left\{r^-, \frac{2r^- - P_H V_H}{P_L}\right\}$. At $\alpha = 0.5$, we have, $V_L = V_H = D$ hence, $U_S^{(S_N B_Y)}(\alpha = 0.5) = \frac{1}{4}\left(2(D + D) - (p_L D + p_H D) - \sqrt{(2(D + D) - (p_L D + p_H D))^2 - 8D^2}\right) = \frac{D}{2}$. Observe that $\alpha = 0.5$ implies to the situation when the demo is useless or equivalently, no demo is provided. In the following we will find a range of $\alpha > 0.5$ for which a demo is viable for the seller i.e. $U_S^{(S_N B_Y)}(\alpha) > \frac{D}{2}$.

$$
\begin{aligned}
& \tfrac{1}{4}\left(2(V_L + V_H) - (p_L V_L + p_H V_H) - \sqrt{(2(V_L + V_H) - (p_L V_L + p_H V_H))^2 - 8V_L V_H}\right) &>& \tfrac{1}{2}D \\
\Rightarrow\; & (2(V_L + V_H) - (p_L V_L + p_H V_H))^2 - (2(V_L + V_H) - (p_L V_L + p_H V_H) - 2D)^2 &<& 8V_L V_H \\
\Rightarrow\; & (2(V_L + V_H) - (p_L V_L + p_H V_H) - P_H V_H - P_L V_L)D &<& 2V_L V_H \\
\Rightarrow\; & (2V_H - (V_H - V_L)(P_H + p_H))(P_H(V_H - V_L) + V_L) &<& 2V_L V_H \\
\Rightarrow\; & (V_H - V_L)(P_H^2(V_H - V_L) - 2P_H V_H + P_H p_H(V_H - V_L) + (P_H + p_H)V_L) &>& 0 \\
\Rightarrow\; & D(P_H + p_H) - 2P_H V_H &>& 0 \quad \text{(B.51)} \\
\Rightarrow\; & \big(p_H(D_H - D_L) + D_L\big)(2\alpha p_H + 1 - \alpha) - 2\big(\alpha p_H D_H + (1 - \alpha)(1 - p_H)D_L\big) &>& 0 \\
\Rightarrow\; & 2\alpha(D_H - D_L)p_H^2 - \big((3D_H - D_L)\alpha - D_L - D_H\big)p_H - 4(1 - \alpha)D_L &>& 0
\end{aligned}
$$

Since the product of the roots of the above quadratic equation is negative hence, the smaller root p_H^- is negative. Following shows that the larger root p_H^+ is more than 0.5.

$$\begin{aligned}
\frac{(3D_H-D_L)\alpha-D_L-D_H-\sqrt{((3D_H-D_L)\alpha-D_L-D_H)^2+8\alpha(1-\alpha)(D_H-D_L)D_L}}{4\alpha(D_H-D_L)} &> \frac{1}{2} \\
\Rightarrow \quad \big(2\alpha(D_H-D_L)-(3D_H-D_L)\alpha-D_L-D_H\big)^2-((3D_H-D_L)\alpha-D_L-D_H)^2 &< 8\alpha(1-\alpha)(D_H-D_L)D_L \\
\Rightarrow \quad (D_H-D_L)\alpha-(3D_H-D_L)\alpha+D_L+D_H &< 2(1-\alpha)D_L \\
\Rightarrow \quad (2\alpha-1)(D_H-D_L) &> 0
\end{aligned}$$

Above is always true. Similar algebra shows that $p_H^+ < 1$. Equation (B.51) can be further extended as follows,

$$\begin{aligned}
D(P_H+p_H)-2P_HV_H &> 0 \\
\Rightarrow \quad D(\alpha(p_H-p_L)+1)-2\big(\alpha(p_HD_H-p_LD_L)+p_LD_L\big) &> 0 \\
\Rightarrow \quad \alpha\big((p_H-p_L)D-2(p_HD_H-p_LD_L)\big)-2p_LD_L+D &> 0 \\
\Rightarrow \quad \frac{p_HD_H-p_LD_L}{2(p_HD_H-p_LD_L)-(p_H-p_L)D} &> \alpha
\end{aligned}$$

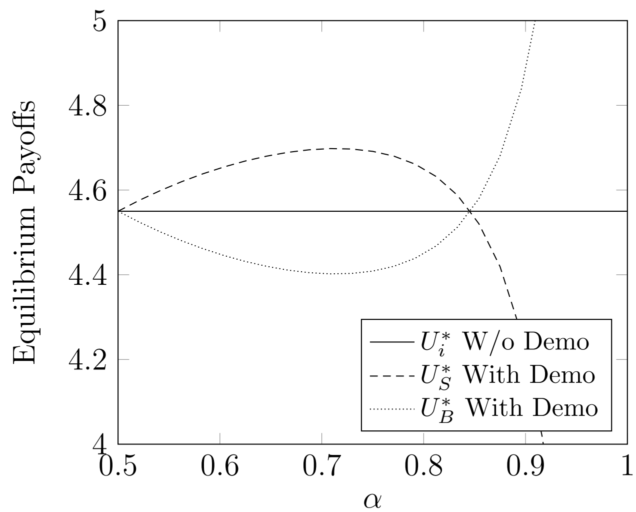


Figure B.1. $D_L = 1$, $D_H = 10$, $p_H = 0.9$, $r = 0$

Define $\alpha_T = \frac{p_HD_H-p_LD_L}{2(p_HD_H-p_LD_L)-(p_H-p_L)D}$. For $r^- > \frac{1}{2}D$ we need $\alpha_T > \alpha \geq 0.5$ or equivalently, $0.5 < p_H^+ < p_H$. Moreover, for $\alpha_T > \alpha$ we have $\frac{2r^- - P_HV_H}{P_L} > r^-$. Therefore $U_S^{(S_NB_Y)}(\alpha) \geq U_S^{(S_NB_Y)}(\alpha = 0.5)$ if $r \leq r^-$ and $\alpha \leq \alpha_T$. Observe that, $p_H > 0.5$ implies $\alpha_T > 0.5$. It is easy to show that $\alpha_T < 1$. $\frac{p_HD_H-p_LD_L}{2(p_HD_H-p_LD_L)-(p_H-p_L)D} < 1$ implies $D_L < D$ which is always true for $p_H > 0.5$. Furthermore, since $U_S^{(S_NB_Y)}(\alpha = 0.5) = U_S^{(S_NB_Y)}(\alpha_T) = \frac{D}{2}$ and $U_S^{(S_NB_Y)}(\alpha) \geq U_S^{(S_NB_Y)}(\alpha = 0.5)$ for $0.5 \leq \alpha \leq \alpha_T$ hence, $U_S^{(S_NB_Y)}(\alpha)$ must attain its maximum in the interval $(0.5, \alpha_T)$ as shown in Figure B.1. Let $x = 2(V_L +$

$V_H) - (p_LV_L + p_HV_H) = (2-p_L)V_L + (2-p_H)V_H$ then $r^- = \frac{1}{4}\left(x - \sqrt{x^2 - 8V_LV_H}\right)$. We have $\frac{\partial V_H}{\partial \alpha} = \frac{p_Lp_H(D_H-D_L)}{P_H^2} > 0$ and $\frac{\partial V_L}{\partial \alpha} = -\frac{p_Lp_H(D_H-D_L)}{P_L^2} < 0$. Therefore, $\frac{\partial x}{\partial \alpha} = p_Lp_H(D_H-D_L)\left(\frac{2-p_H}{P_H^2} - \frac{2-p_L}{P_L^2}\right)$. First order condition $\frac{\partial U_S^{(S_NB_Y)}}{\partial \alpha} = 0$ gives,

$$\frac{1}{4}\left(\frac{\partial x}{\partial \alpha} - \frac{2x\frac{\partial x}{\partial \alpha} - 8V_L\frac{\partial V_H}{\partial \alpha} - 8V_H\frac{\partial V_L}{\partial \alpha}}{2\sqrt{x^2-8V_LV_H}}\right) = 0$$

$$\Rightarrow \left(\frac{\partial x}{\partial \alpha}\right)^2(x^2 - 8V_LV_H) = \left(x\frac{\partial x}{\partial \alpha} - 4V_L\frac{\partial V_H}{\partial \alpha} - 4V_H\frac{\partial V_L}{\partial \alpha}\right)^2$$

$$\Rightarrow \left(x\frac{\partial x}{\partial \alpha} - 2V_L\frac{\partial V_H}{\partial \alpha} - 2V_H\frac{\partial V_L}{\partial \alpha}\right)\left(V_L\frac{\partial V_H}{\partial \alpha} + V_H\frac{\partial V_L}{\partial \alpha}\right) = V_LV_H\left(\frac{\partial x}{\partial \alpha}\right)^2$$

$$\Rightarrow \left(((2-p_L)V_L + (2-p_H)V_H)\left(\frac{2-p_H}{P_H^2} - \frac{2-p_L}{P_L^2}\right) - 2\left(\frac{V_L}{P_H^2} - \frac{V_H}{P_L^2}\right)\right)\left(\frac{V_L}{P_H^2} - \frac{V_H}{P_L^2}\right) = V_LV_H\left(\frac{2-p_H}{P_H^2} - \frac{2-p_L}{P_L^2}\right)^2$$

$$\Rightarrow \left(((2-p_L)(2-p_H) - 2)\left(\frac{V_L}{P_H^2} - \frac{V_H}{P_L^2}\right) - \frac{V_L(2-p_L)^2}{P_L^2} + \frac{V_H(2-p_H)^2}{P_H^2}\right)\left(\frac{V_L}{P_H^2} - \frac{V_H}{P_L^2}\right) = V_LV_H\left(\frac{2-p_H}{P_H^2} - \frac{2-p_L}{P_L^2}\right)^2$$

$$\Rightarrow p_Lp_H\left(\frac{V_L}{P_H^2} - \frac{V_H}{P_L^2}\right)^2 - \frac{V_L^2(2-p_L)^2}{P_L^2P_H^2} - \frac{V_H^2(2-p_H)^2}{P_L^2P_H^2} = -2V_LV_H\frac{(2-p_L)(2-p_H)}{P_L^2P_H^2}$$

$$\Rightarrow \left(V_LP_L^2 - V_HP_H^2\right)\sqrt{p_Lp_H} = P_LP_H\left(V_H(2-p_H) - V_L(2-p_L)\right)$$

$$\Rightarrow \Big(\big(\alpha p_LD_L + (1-\alpha)p_HD_H\big)\big(\alpha p_L + (1-\alpha)p_H\big) - \big(\alpha p_HD_H + (1-\alpha)p_LD_L\big)\big(\alpha p_H + (1-\alpha)p_L\big)\Big)\sqrt{p_Lp_H}$$

$$= \big(\alpha p_HD_H + (1-\alpha)p_LD_L\big)\big(\alpha p_L + (1-\alpha)p_H\big)(2-p_H) - \big(\alpha p_LD_L + (1-\alpha)p_HD_H\big)\big(\alpha p_H + (1-\alpha)p_L\big)(2-p_L)$$

$$\Rightarrow (\alpha^2 - (1-\alpha)^2)(p_L^2D_L - p_H^2D_H)\sqrt{p_Lp_H} = 2(\alpha^2 - (1-\alpha)^2)p_Lp_H(D_H - D_L)$$
$$-(\alpha^2 - \alpha(1-\alpha))p_Lp_H\big(p_HD_H - p_LD_L\big) - \alpha(1-\alpha)\big(p_H^3D_H - p_L^3D_L\big) - (1-\alpha)^2p_Lp_H(p_HD_L - p_LD_H)$$

$$\Rightarrow (2\alpha-1)\big((p_L^2D_L - p_H^2D_H)\sqrt{p_Lp_H} - 2p_Lp_H(D_H-D_L)\big) + (2\alpha^2-\alpha)p_Lp_H(p_HD_H - p_LD_L)$$
$$+(\alpha-\alpha^2)\big(p_H^3D_H - p_L^3D_L\big) + (1-\alpha)^2p_Lp_H(p_HD_L - p_LD_H) = 0$$

$$\Rightarrow (p_HD_H - p_LD_L)(p_H-p_L)^2\alpha^2 - (p_H^2D_H - p_L^2D_L)\sqrt{p_Lp_H} - p_Lp_H((2-p_L)D_H - (2-p_H)D_L)$$
$$-\Big((p_H^2D_H + p_L^2D_L)(p_H-p_L) - 2(p_H^2D_H - p_L^2D_L)\sqrt{p_Lp_H} - 2p_Lp_H((2-p_L)D_H - (2-p_H)D_L)\Big)\alpha = 0$$

Define $k_1 = (p_HD_H - p_LD_L)(p_H-p_L)^2$, $k_2 = (p_H^2D_H + p_L^2D_L)(p_H-p_L) - 2k_3$, and $k_3 = (p_H^2D_H - p_L^2D_L)\sqrt{p_Lp_H} + p_Lp_H((2-p_L)D_H - (2-p_H)D_L)$. Since $p_H > 0.5$, we get, $k_1 > 0$ and $k_3 > 0$. Therefore, the smaller root $\alpha_s = \frac{1}{2k_1}\left(k_2 - \sqrt{k_2^2 + 4k_1k_3}\right)$ is negative. We now show that the larger root $\alpha_l = \frac{1}{2k_1}\left(k_2 + \sqrt{k_2^2 + 4k_1k_3}\right)$ lies between 0.5 and α_T. We know $U_S^{(S_NB_Y)}(\alpha_T) = U_S^{(S_NB_Y)}(0.5) = \frac{D}{2}$. In addition, $U_S^{(S_NB_Y)} = r^-$ is continuous and differentiable in the interval $(0.5, \alpha_T)$. Therefore from Rolle's theorem, there exists at least one α in $(0.5, \alpha_T)$ such that $\frac{\partial U_S^{(S_NB_Y)}}{\partial \alpha} = 0$. Since, $\alpha_s < 0$ hence $0.5 \le \alpha_l \le \alpha_T$. Therefore, $U_S^{(S_NB_Y)}$ achieves it maximum at $\alpha^* = \alpha_l$. As stated earlier, $p_H > 0.5$ implies $\alpha_T > 0.5$. Therefore, there always exists α^* in $(0.5, \alpha_T)$ when $p_H > 0.5$. Seller will then offer a noisy demo with accuracy of α^* when $p_H > 0.5$ and $r \le r^-(\alpha^*)$. Conversely, for $p_H \le 0.5$ we get $\alpha_T \le 0.5$. Therefore $U_S^{(S_NB_Y)}(\alpha) \le U_S^{(S_NB_Y)}(0.5) = \frac{1}{2}D$ consequently, seller will not offer any demo when $p_H \le 0.5$ and $r \le r^-$.

When $U_S^{(S_NB_Y)} = \frac{1}{2}(P_HV_H + P_Lr)$

$\frac{1}{2}(P_HV_H + P_Lr)$ can be written as $\frac{1}{2}\big(\alpha(p_HD_H - p_LD_L - (p_H-p_L)r) + p_LD_L + p_Hr\big)$. The equilibrium payoff of the seller will be increasing in α when $r > \frac{p_H D_H - p_L D_L}{p_H - p_L}$ with $p_H \le 0.5$ or when $r < \frac{p_H D_H - p_L D_L}{p_H - p_L}$ with $p_H \ge 0.5$. Also, Tables B.3 to B.7 indicate that $U_S^{(S_N B_Y)} = \frac{1}{2}(P_H V_H + P_L r)$ when $0 \le r \le r^-$ with $P_H^- \le P_H$ or $r^- \le r < V_H$ with $\frac{r}{V_H - r} \le P_H$ holds.

When $0 \le r \le r^-$.

From case B.1 we know that $P_H^- = \frac{2r^- - r}{V_H - r}$. Therefore, $U_S^{(S_N B_Y)} = \frac{1}{2}(P_H V_H + P_L r)$ will hold if $0 \le r \le r^-$ and $P_H^- = \frac{2r^- - r}{V_H - r} \le P_H$ i.e. $\frac{2r^- - P_H V_H}{P_L} \le r$. Furthermore, $\frac{2r^- - r^-}{V_H - r^-} \le P_H$ implies $\frac{2r^- - P_H V_H}{P_L} \le r^-$. Hence, $U_S^{(S_N B_Y)} = \frac{1}{2}(P_H V_H + P_L r)$ if $\frac{2r^- - P_H V_H}{P_L} \le r \le r^-$. In the following we show that for $p_H < 0.5$, $\frac{2r^- - P_H V_H}{P_L} \ge \frac{p_H D_H - p_L D_L}{p_H - p_L}$ i.e.,

$$\begin{aligned} r^- &\ge \tfrac{P_L(p_H D_H - p_L D_L) + (p_H - p_L) P_H V_H}{2(p_H - p_L)} \\ &= \tfrac{(\alpha p_L + (1-\alpha) p_H)(p_H D_H - p_L D_L) + (p_H - p_L)(\alpha p_H D_H + (1-\alpha) p_L D_L)}{2(p_H - p_L)} \\ &= \tfrac{p_H^2 D_H - p_L^2 D_L}{2(p_H - p_L)} \end{aligned} \tag{B.52}$$

To prove inequality (B.52), we first show that r^- is increasing in p_H while $\frac{p_H^2 D_H - p_L^2 D_L}{2(p_H - p_L)}$ is decreasing in p_H. Let $x = 2(V_L + V_H) - (p_L V_L + p_H V_H) = (2 - p_L) V_L + (2 - p_H) V_H$ then $r^- = \frac{1}{4}\left(x - \sqrt{x^2 - 8 V_L V_H}\right)$. We have $\frac{\partial V_H}{\partial p_H} = \frac{\alpha(1-\alpha)(D_H - D_L)}{P_H^2} \ge 0$ and $\frac{\partial V_L}{\partial p_H} = \frac{\alpha(1-\alpha)(D_H - D_L)}{P_L^2} \ge 0$. Therefore, $\frac{\partial x}{\partial p_H} = \alpha(1-\alpha)(D_H - D_L)\left(\frac{2 - p_H}{P_H^2} + \frac{2 - p_L}{P_L^2}\right)$. We show that $\frac{\partial U_S^{(S_N B_Y)}}{\partial p_H} \ge 0$ i.e.,

$$\begin{aligned} \tfrac{1}{4}\left(\tfrac{\partial x}{\partial p_H} - \tfrac{2x\frac{\partial x}{\partial p_H} - 8V_L\frac{\partial V_H}{\partial p_H} - 8V_H\frac{\partial V_L}{\partial p_H}}{2\sqrt{x^2 - 8V_L V_H}}\right) &\ge 0 \\ \Rightarrow \left(\tfrac{\partial x}{\partial p_H}\right)^2 (x^2 - 8V_L V_H) \ge \left(x\tfrac{\partial x}{\partial p_H} - 4V_L\tfrac{\partial V_H}{\partial p_H} - 4V_H\tfrac{\partial V_L}{\partial p_H}\right)^2 & \\ \Rightarrow \left(x\tfrac{\partial x}{\partial p_H} - 2V_L\tfrac{\partial V_H}{\partial p_H} - 2V_H\tfrac{\partial V_L}{\partial p_H}\right)\left(V_L\tfrac{\partial V_H}{\partial p_H} + V_H\tfrac{\partial V_L}{\partial p_H}\right) &\ge V_L V_H\left(\tfrac{\partial x}{\partial p_H}\right)^2 \\ \Rightarrow \left(((2-p_L)V_L + (2-p_H)V_H)\left(\tfrac{2-p_H}{P_H^2} + \tfrac{2-p_L}{P_L^2}\right) - 2\left(\tfrac{V_L}{P_H^2} + \tfrac{V_H}{P_L^2}\right)\right)\left(\tfrac{V_L}{P_H^2} + \tfrac{V_H}{P_L^2}\right) &\ge V_L V_H\left(\tfrac{2-p_H}{P_H^2} + \tfrac{2-p_L}{P_L^2}\right)^2 \\ \Rightarrow \left(((2-p_L)(2-p_H) - 2)\left(\tfrac{V_L}{P_H^2} + \tfrac{V_H}{P_L^2}\right) + \tfrac{V_L(2-p_L)^2}{P_L^2} + \tfrac{V_H(2-p_H)^2}{P_H^2}\right)\left(\tfrac{V_L}{P_H^2} + \tfrac{V_H}{P_L^2}\right) &\ge V_L V_H\left(\tfrac{2-p_H}{P_H^2} + \tfrac{2-p_L}{P_L^2}\right)^2 \\ \Rightarrow p_L p_H\left(\tfrac{V_L}{P_H^2} + \tfrac{V_H}{P_L^2}\right)^2 + \tfrac{V_L^2(2-p_L)^2}{P_L^2 P_H^2} + \tfrac{V_H^2(2-p_H)^2}{P_L^2 P_H^2} &\ge 2V_L V_H\tfrac{(2-p_L)(2-p_H)}{P_L^2 P_H^2} \\ \Rightarrow p_L p_H\left(\tfrac{V_L}{P_L^2} + \tfrac{V_H}{P_H^2}\right)^2 + \left(\tfrac{V_H(2-p_H) - V_L(2-p_L)}{P_L P_H}\right)^2 &\ge 0 \end{aligned}$$

Above is always true hence, r^- is increasing in p_H. Furthermore, $\frac{p_H^2 D_H - p_L^2 D_L}{2(p_H - p_L)}$ is decreasing in p_H if,

$$\begin{aligned} \tfrac{\partial}{\partial p_H}\left(\tfrac{p_H^2(D_H - D_L) + 2p_H D_L - D_L}{2(2p_H - 1)}\right) &< 0 \\ \Rightarrow \tfrac{(2p_H - 1)(2p_H(D_H - D_L) + 2D_L) - 2(p_H^2(D_H - D_L) + 2p_H D_L - D_L)}{2(2p_H - 1)^2} &< 0 \\ \Rightarrow -\tfrac{p_H p_L (D_H - D_L)}{(p_H - p_L)^2} &< 0 \end{aligned}$$

Above is true for $p_H \neq 0.5$. At $p_H = 0$, $V_L = V_H = D_L$ which gives $r^- = \frac{D_L}{2}$ and $\frac{p_H^2 D_H - p_L^2 D_L}{2(p_H - p_L)} = \frac{D_L}{2}$. Furthermore, at $p_H = 1$, $r^- = \frac{p_H^2 D_H - p_L^2 D_L}{2(p_H - p_L)} = \frac{D_H}{2}$. In addition, when $p_H \to 0.5^-$, $\frac{p_H^2 D_H - p_L^2 D_L}{2(p_H - p_L)} \to -\infty$

and when $p_H \to 0.5^+$, $\frac{p_H^2 D_H - p_L^2 D_L}{2(p_H - p_L)} \to \infty$. Since r^- is increasing while $\frac{p_H^2 D_H - p_L^2 D_L}{2(p_H - p_L)}$ is decreasing in p_H therefore, $r^- \geq \frac{p_H^2 D_H - p_L^2 D_L}{2(p_H - p_L)}$ for $0 \leq p_H < 0.5$ and $r^- \leq \frac{p_H^2 D_H - p_L^2 D_L}{2(p_H - p_L)}$ for $0.5 < p_H \leq 1$. This implies $\frac{2r^- - P_H V_H}{P_L} \geq \frac{p_H D_H - p_L D_L}{p_H - p_L}$ for $0 \leq p_H \leq 0.5$ and $\frac{2r^- - P_H V_H}{P_L} \leq \frac{p_H D_H - p_L D_L}{p_H - p_L}$ for $0.5 \leq p_H \leq 1$. Since $\frac{2r^- - P_H V_H}{P_L} \leq r \leq r^-$ hence, $\frac{p_H D_H - p_L D_L}{p_H - p_L} \leq r$ for $0 \leq p_H \leq 0.5$. This implies that the equilibrium payoff of the seller is increasing in α for $0 \leq p_H \leq 0.5$. Since, $\frac{\partial}{\partial p_H}\left(\frac{p_H D_H - p_L D_L}{p_H - p_L}\right) = -\frac{D_H - D_L}{(p_H - p_L)^2} < 0$ hence, $\frac{p_H D_H - p_L D_L}{p_H - p_L}$ is decreasing in p_H. When $p_H \to 0.5^+$, $\frac{p_H D_H - p_L D_L}{p_H - p_L} \to \infty$. At $p_H = 1$, $\frac{p_H D_H - p_L D_L}{p_H - p_L} = D_H$ and $r^- = \frac{D_H}{2}$. This implies, $r^- < \frac{p_H D_H - p_L D_L}{p_H - p_L}$ for $0.5 \leq p_H \leq 1$. Therefore, $\frac{2r^- - P_H V_H}{P_L} \leq r \leq r^- < \frac{p_H D_H - p_L D_L}{p_H - p_L}$ for $0.5 < p_H \leq 1$. Hence, the equilibrium payoff of the seller is increasing in α for $0.5 \leq p_H \leq 1$.

When $r^- \leq r < V_H$.

As stated in B.43, $U_S^{(S_N B_Y)} = \frac{1}{2}(P_H V_H + P_L r)$ if $\frac{r}{V_H - r} \leq P_H$ i.e. $r \leq \frac{P_H V_H}{1+P_H}$. Since $\frac{P_H V_H}{1+P_H} \leq V_H$ hence, $U_S^{(S_N B_Y)} = \frac{1}{2}(P_H V_H + P_L r)$ if $r^- \leq r \leq \frac{P_H V_H}{1+P_H}$. Using the inequality $\frac{2r^- - P_H V_H}{P_L} \geq \frac{p_H D_H - p_L D_L}{p_H - p_L}$ for $0 \leq p_H < 0.5$ as proved in the previous section B.43, we show that $r^- \leq \frac{P_H V_H}{1+P_H}$ implies $r \geq \frac{p_H D_H - p_L D_L}{p_H - p_L}$ when $p_H < 0.5$.

$$
\begin{aligned}
\frac{P_H V_H}{1+P_H} - r^- &\geq 0 \\
\Rightarrow \quad P_H V_H - (2 - P_L) r^- &\geq 0 \\
\Rightarrow \quad P_L r^- - (2r^- - P_H V_H) &\geq 0 \\
\Rightarrow \quad P_L r^- - P_L \frac{p_H D_H - p_L D_L}{p_H - p_L} &\geq 0 \text{ when } p_H < 0.5
\end{aligned}
$$

Therefore as stated in B.43, $r \geq \frac{p_H D_H - p_L D_L}{p_H - p_L}$ implies that $U_S^{(S_N B_Y)}$ is increasing in α when $p_H \leq 0.5$. We now show that $\frac{P_H V_H}{1+P_H} < \frac{p_H D_H - p_L D_L}{p_H - p_L}$ when $p_H \geq 0.5$. Derivative of $\frac{P_H V_H}{1+P_H}$ with respect to p_H gives $\frac{\partial}{\partial p_H}\left(\frac{P_H V_H}{1+P_H}\right) = \frac{\alpha(2-\alpha) D_H - (1-\alpha^2) D_L}{(1+P_H)^2}$. Since $\alpha(2-\alpha) \geq 1 - \alpha^2$ for $\alpha \geq 0.5$ hence, $\frac{P_H V_H}{1+P_H}$ is increasing in p_H. As shown previously, $\frac{p_H D_H - p_L D_L}{p_H - p_L}$ is decreasing in p_H. In addition, at $p_H = 1$, $\frac{P_H V_H}{1+P_H} = \frac{\alpha}{1+\alpha} D_H < D_H$ and $\frac{p_H D_H - p_L D_L}{p_H - p_L} = D_H$. Therefore, $\frac{P_H V_H}{1+P_H} < \frac{p_H D_H - p_L D_L}{p_H - p_L}$. This implies, $r < \frac{p_H D_H - p_L D_L}{p_H - p_L}$. Therefore as stated in B.43, seller's equilibrium payoff is increasing in α.

From B.43 and B.43, we observe that $U_S^{(S_N B_Y)} = \frac{1}{2}(P_H V_H + P_L r)$ is increasing in α for $\frac{2r^- - P_H V_H}{P_L} \leq r \leq \frac{P_H V_H}{1+P_H}$. Therefore, seller will offer a demo that will reveal full information about the data i.e. $\alpha = 1$.

When $U_S^{(S_N B_Y)} = r$

Since the equilibrium payoff of the seller does not depend on α hence seller is indifferent to offering a demo.

From the above cases we find that the seller has three choices when $p_H \leq 0.5$ — (i) do not offer a demo i.e. $\alpha = 0.5$, (ii) offer a demo without noise i.e. $\alpha = 1$, and (iii) indifferent in offering a demo. Seller will not offer a demo if $U_S^{(S_N B_Y)}(\alpha = 0.5) \geq U_S^{(S_N B_Y)}(\alpha = 1)$. From B.43 we get, $U_S^{(S_N B_Y)}(\alpha = 0.5) = r^-(\alpha = 0.5) = \frac{1}{2} D$. From B.43 we get, $U_S^{(S_N B_Y)}(\alpha = 1) = \frac{1}{2}(P_H V_H + P_L r)_{\alpha=1} = \frac{1}{2}(p_H D_H + p_L r)$.

$U_S^{(S_N B_Y)}(\alpha = 0.5) \geq U_S^{(S_N B_Y)}(\alpha = 1)$ implies $D_L \geq r$. We also need $r \leq r^-(\alpha = 0.5) = \frac{1}{2}D$. Therefore, seller will not offer a demo if $r \leq \min\left\{D_L, \frac{1}{2}D\right\}$ and $p_H \leq 0.5$. Seller will offer a demo with full information when $U_S^{(S_N B_Y)}(\alpha = 1) > U_S^{(S_N B_Y)}(\alpha = 0.5)$ i.e. $D_L < r$. Also, we need $\frac{2r^- - P_H V_H}{P_L} \leq r \leq \frac{P_H V_H}{1+P_H}$ to hold at $\alpha = 1$ i.e. $\frac{2r^-(\alpha=1) - p_H D_H}{p_L} \leq r \leq \frac{p_H D_H}{1+p_H}$. Following shows that $D_L \geq \frac{2r^-(\alpha=1)-p_H D_H}{p_L}$.

$$\begin{aligned}
p_L D_L &\geq 2r^-(\alpha = 1) - p_H D_H \\
\Rightarrow \quad 2(D_L + D_H) - D - \sqrt{(2(D_L + D_H) - D)^2 - 8D_L D_H} &\leq 2D \\
\Rightarrow \quad (2(D_L + D_H) - 3D)^2 &\leq (2(D_L + D_H) - D)^2 - 8D_L D_H \\
\Rightarrow \quad D_L D_H &\leq D(D_L + D_H - D) \\
\Rightarrow \quad (D - D_L)(D - D_H) &\leq 0
\end{aligned}$$

Above is true hence, seller will offer a complete information demo if $D_L < r \leq \frac{p_H D_H}{1+p_H}$ and $p_H \leq 0.5$. For $r \geq \frac{p_H D_H}{1+p_H}$, seller will become indifferent in offering a demo as $U_S^{(S_N B_Y)} = r$.

When $p_H \geq 0.5$ then seller has following three choices — (i) offer a noisy demo with $\alpha = \alpha^* = \frac{1}{2k_1}\left(k_2 + \sqrt{k_2^2 + 4k_1 k_3}\right)$ where the constants k_1, k_2, and k_3 are given in B.43, (ii) offer a complete information demo i.e. $\alpha = 1$, and (iii) indifferent in offering a demo. Seller will offer a noisy demo when $U_S^{(S_N B_Y)}(\alpha = \alpha^*) \geq U_S^{(S_N B_Y)}(\alpha = 1)$. This implies $r^-(\alpha^*) \geq \frac{1}{2}(p_H D_H + p_L r)$ i.e. $r \leq \frac{2r^-(\alpha^*)-p_H D_H}{p_L}$. We also need $r \leq \min\left\{r^-, \frac{2r^- - P_H V_H}{P_L}\right\}_{\alpha=\alpha^*}$. For $p_H \geq 0.5$ we have, $\frac{2r^- - P_H V_H}{P_L} \geq \frac{2r^-(\alpha^*)-p_H D_H}{p_L}$. Therefore, seller will offer a noisy demo if $r \leq \frac{2r^-(\alpha^*)-p_H D_H}{p_L}$ and $p_H \geq 0.5$. Buyer's equilibrium payoff in this range is $V - r^-(\alpha^*)$. $r^-(\alpha^*) > \frac{D}{2}$ implies that $V - r^-(\alpha^*) < \frac{D}{2}$ since $V = D$. This shows that the buyer's equilibrium payoff decreases below the no-demo payoff hence, buyer will reject such an offer by the seller. Seller will offer a demo with full information when $U_S^{(S_N B_Y)}(\alpha = 1) \geq U_S^{(S_N B_Y)}(\alpha^*)$ i.e. $\frac{2r^-(\alpha^*)-p_H D_H}{p_L} \leq r$. Also, we need $\frac{2r^- - P_H V_H}{P_L} \leq r \leq \frac{P_H V_H}{1+P_H}$ to hold at $\alpha = 1$ i.e. $\frac{2r^-(\alpha=1)-p_H D_H}{p_L} \leq r \leq \frac{p_H D_H}{1+p_H}$. Since r^- attains its maximum at α^* hence $r^-(\alpha^*) > r^-(\alpha = 1)$. This implies $\frac{2r^-(\alpha=1)-p_H D_H}{p_L} \leq \frac{2r^-(\alpha^*)-p_H D_H}{p_L}$. Therefore, a complete information demo will be offered if $\frac{2r^-(\alpha^*)-p_H D_H}{p_L} < r \leq \frac{p_H D_H}{1+p_H}$ and $p_H \geq 0.5$. Buyer's equilibrium payoff in this range is $\frac{V}{2} = \frac{D}{2}$ hence, buyer will be indifferent in either accepting or rejecting this offer. Seller will become indifferent in offering a demo when $r \geq \frac{p_H D_H}{1+p_H}$. ■

Lemma B.2 *The outcome of the negotiation does not change when the seller update their belief about the probability distribution of the value.*

Proof: We will consider two situations — (i) when $\alpha = 0.5$ i.e. no demonstration is offered and the buyer assumes the distribution of the value to be D_L with probability p_L and D_H with probability $p_H = 1 - p_L$, and (ii) when $\alpha > 0.5$ i.e. the demonstration is offered and buyer reports the signal S_t after the negotiation where $t = L, H$.

When $\boldsymbol{\alpha = 0.5}$. Since no demonstration is provided, the decision mechanism needed is a single probability vector $\mu(d)$ over the alternatives $d \in \Phi$. Suppose the seller believes that the value of their data is R_s where $s = L, H$ with probability $P(R_s)$. Let $P(t|R_s)$ be the conditional probability of the true value to be $t = L, H$ given that the seller believes it to be R_s. Then $U_S(\mu)$ is written as,

$$\begin{aligned}
U_S(\mu) &= U_S(\mu|R_L)P(R_L) + U_S(\mu|R_H)P(R_H) \\
&= \Big(\sum_{d\in\Phi} \mu(d)u_S(d)P(L|R_L) + \sum_{d\in\Phi} \mu(d)u_S(d)P(H|R_L)\Big)P(R_L) \\
&\quad +\Big(\sum_{d\in\Phi} \mu(d)u_S(d)P(L|R_H) + \sum_{d\in\Phi} \mu(d)u_S(d)P(H|R_H)\Big)P(R_H) \\
&= \sum_{d\in\Phi} \Big(P(L, R_L) + P(H, R_L)\Big)\mu(d)u_S(d) + \sum_{d\in\Phi} \Big(P(L, R_H) + P(H, R_H)\Big)\mu(d)u_S(d) \\
&= \sum_{d\in\Phi} \Big(P(R_L) + P(R_H)\Big)\mu(d)u_S(d) \\
&= \sum_{d\in\Phi} \mu(d)u_S(d) \qquad \text{(B.53)}
\end{aligned}$$

Equation (B.53) of $U_S(\mu)$ is independent of any belief by the seller and is same as that in problem $(S_N B_N)$ (see section B.1) where both the players assume a common distribution with no updated believes. This is as though both the players are equally unaware of the value and the negotiation prices will be calculated by equating $\alpha = 0.5$ in the general solution.

When $\boldsymbol{\alpha > 0.5}$. Suppose after the negotiation buyer report the signal to be S_t where $t = L, H$. Let $P(S_t, R_s)$ be the joint probability that the reported signal is S_t while seller believe it to be R_s where $s = L, H$. Then $U_S(\mu)$ is written as,

$$\begin{aligned}
U_S(\mu) &= \textstyle\sum_{S_t} U_S(\mu|S_t, R_L)P(S_t, R_L) + \sum_{S_t} U_S(\mu|S_t, R_H)P(S_t, R_H) \\
&= \textstyle\sum_{S_t} \Big(\displaystyle\sum_{d\in\Phi} \mu(d|S_t)u_S(d)P(S_t, R_L) + \sum_{d\in\Phi} \mu(d|S_t)u_S(d)P(S_t, R_H)\Big) \\
&= \textstyle\sum_{S_t} \displaystyle\sum_{d\in\Phi} \mu(d|S_t)\Big(P(S_t, R_L) + P(S_t, R_H)\Big)u_S(d) \\
&= \sum_{d\in\Phi} \textstyle\sum_{S_t} \mu(d|S_t)P(S_t)u_S(d) \\
&= \sum_{d\in\Phi} \Big(P(S_L)\mu(d|S_L) + P(S_H)\mu(d|S_H)\Big)u_S(d) \qquad \text{(B.54)}
\end{aligned}$$

Equation (B.54) of $U_S(\mu)$ is same as equation (B.5) of problem $(S_N B_Y)$ in section 3.1.1 and is independent of seller's belief. This implies that beliefs of the seller do not change the expected payoffs of the players. The equilibrium prices decided by the negotiation will be based on the buyer's belief of the value and will remain unaffected by any belief perceived by the seller. ■

Lemma B.3 *When the buyer knows the true value of the data to be either D_L or D_H^+ through a demonstration, and the seller assumes it to be distributed as D_L with probability p_L and D_H^+ with probability $p_H = (1 - p_H)$, the outcome of the negotiation will give the following expected payoffs to the players in the absence of an outside option.*

$$(U_S^*, U_B^*) = \begin{cases} (S_s, D^+ - S_s) & \text{if } 0 \le p_H \le \frac{2D_L}{2D_H^+ - D_L} \\ \left(\frac{p_H D_H^+}{2}, \frac{(D^+ - D_L)(p_H D_H^+ - p_L D_L)}{2(p_H D_H^+ - D_L)}\right), & \text{if } \frac{2D_L}{2D_H^+ - D_L} < p_H < \frac{D_L}{D_H^+ - D_L} \\ \left(\frac{p_H D_H^+}{2}, \frac{D^+}{2}\right), & \text{if } \frac{D_L}{D_H^+ - D_L} \le p_H \le 1 \end{cases}$$

where, $S_s = \frac{1}{4}\left(2(D_L + D_H^+) - D^+ - \sqrt{(2(D_L + D_H^+) - D^+)^2 - 8D_L D_H^+}\right)$, $D^+ = p_L D_L + p_H D_H^+$.

Proof: To obtain the solution, we need to consider the solution tables of Appendix B.1, by replacing $\alpha = 1$. This gives $P_H = p_H$, $P_L = p_L$, $V_L = D_L$, and $V_H = D_H^+$ (which is the true upper value). Since $r = 0 < r^-$, we will consider the following solution given under column $0 \le r \le r^-$ in Table B.3 and Table B.7.

$$\begin{cases} (r^-, D^+ - r^-) & \text{if } 0 \le p_H \le \frac{(2D_H^+ - r)(D_L - r)}{(D_H^+ - r)(2D_H^+ - D_L - r)} \\ \left(\frac{1}{2}(p_H D_H^+ + p_L r), \frac{(D_H^+ - D_L)(p_H(D_H^+ - r) - p_L(D_L - r))(p_H D_H^+ + p_L r)}{2(D_H^+ - r)(p_H D_H^+ + p_L r - D_L)}\right) & \text{if } \frac{(2D_H^+ - r)(D_L - r)}{(D_H^+ - r)(2D_H^+ - D_L - r)} \le p_H \le \frac{D_H^+(D_L - r)}{(D_H^+ - D_L)(D_H^+ - r)} \\ \left(\frac{1}{2}(p_H D_H^+ + p_L r), \frac{D^+}{2}\right) & \text{if } \frac{D_H^+(D_L - r)}{(D_H^+ - D_L)(D_H^+ - r)} \le p_H \le 1 \end{cases}$$

We will get the desired result of Lemma 3.4 by replacing r with zero in each of the above cases. ■

Proposition B.2 *If the demonstration helps the buyer realize that the upper bound of the value could be as high as $D_H^+ > D_H$ while all other parameters stay unchanged, the seller will propose a demonstration to the buyer before the negotiation when*

$$D_H^+ > \begin{cases} \frac{D}{p_H}, & \text{if } 0 < \frac{D_L}{D_H} \le \frac{2}{3} \\ D\left(\frac{2D_L - D_H}{D_L - (2 - p_H)(D_H - D_L)}\right), & \text{if } \frac{2}{3} < \frac{D_L}{D_H} \le 1 \end{cases}$$

where, $D = p_L D_L + p_H D_H$.

Proof: The proof consists of the following two parts.

(1) When $\frac{2D_L}{2D_H^+ - D_L} < p_H \le 1$. The demonstration will give a higher expected payoff to the seller if $\frac{p_H D_H^+}{2} > \frac{D}{2}$ i.e. $D_H^+ > \frac{D}{p_H} = \frac{D_L}{p_H} + D_H - D_L$ along with $p_H > \left(\frac{2D_L}{2D_H^+ - D_L}\right)$ i.e. $D_H^+ > \frac{D_L}{p_H} + \frac{D_L}{2}$. It is easy to observe that

$$D_H^+ > \begin{cases} \frac{D}{p_H}, & \text{if } 0 < \frac{D_L}{D_H} \le \frac{2}{3} \\ \frac{D_L}{p_H} + \frac{D_L}{2}, & \text{if } \frac{2}{3} \le \frac{D_L}{D_H} \le 1 \end{cases}$$

(2) When $0 < p_H \leq \frac{2D_L}{2D_H^+ - D_L}$. In this case, $D_H^+ \leq \frac{D_L}{p_H} + \frac{D_L}{2}$. Let $D^+ = p_L D_L + p_H D_H^+$. The demonstration will give a higher expected payoff to the seller if $\frac{1}{4}\left(2(D_L + D_H^+) - D^+ - \sqrt{(2(D_L + D_H^+) - D^+)^2 - 8D_L D_H^+}\right) > \frac{D}{2}$.

$$\begin{aligned}
2(D_L + D_H^+) - D^+ - \sqrt{(2(D_L + D_H^+) - D^+)^2 - 8D_L D_H^+} &> 2D \\
\Rightarrow \quad (2(D_L + D_H^+) - D^+ - 2D)^2 &> (2(D_L + D_H^+) - D^+)^2 - 8D_L D_H^+ \\
\Rightarrow \quad 2D_L D_H^+ &> D(2(D_L + D_H^+) - D^+ - D) \\
\Rightarrow \quad D_H^+(2D_L - 2D + p_H D) &> D(2D_L - p_L D_L - D) \\
\Rightarrow \quad D_H^+(2D_L - (p_H D_H + D_L(1 - p_H))(2 - p_H)) &> D(2D_L - p_L D_L - p_H D_H - p_L D_L) \\
\Rightarrow \quad D_H^+(2D_L - (p_H(D_H - D_L) + D_L)(2 - p_H)) &> D(2D_L p_H - p_H D_H) \\
\Rightarrow \quad D_H^+(D_L - (2 - p_H)(D_H - D_L)) &> D(2D_L - D_H) \quad \text{(B.55)}
\end{aligned}$$

(i) When $D_L - (2 - p_H)(D_H - D_L) > 0$: This implies $\frac{D_L}{D_H} > \frac{2-p_H}{3-p_H} > \frac{1}{2}$, since $p_H < 1$. Let $\frac{D_L}{D_H} = \beta$. Then equation (B.55) can be written as $\frac{D_H^+}{D_H} > \frac{\left(p_H(1-\beta)+\beta\right)(2\beta-1)}{\beta-(2-p_H)(1-\beta)}$, and $D_H^+ \leq \frac{D_L}{p_H} + \frac{D_L}{2}$ can be written as $\frac{D_H^+}{D_H} \leq \beta\left(\frac{1}{p_H} + \frac{1}{2}\right)$. For a valid D_H^+ we need

$$\begin{aligned}
\frac{\left(p_H(1-\beta)+\beta\right)(2\beta-1)}{\beta-(2-p_H)(1-\beta)} &< \beta\left(\frac{1}{p_H} + \frac{1}{2}\right) \\
\Rightarrow \quad 2p_H^2(1-\beta)(2\beta-1) + 2p_H\beta(2\beta-1) &< 2\beta^2 + p_H\beta^2 - 4\beta(1-\beta) + \beta(1-\beta)p_H^2 \\
\Rightarrow \quad p_H^2(1-\beta)(4\beta - 2 - \beta) + p_H\beta(4\beta - 2 - \beta) - 2\beta(\beta - 2 + 2\beta) &< 0 \\
\Rightarrow \quad \left(p_H^2(1-\beta) + p_H\beta - 2\beta\right)(3\beta - 2) &< 0 \\
\Rightarrow \quad \left(p_H + \frac{\beta+\sqrt{\beta(8-7\beta)}}{2(1-\beta)}\right)\left(p_H - \frac{-\beta+\sqrt{\beta(8-7\beta)}}{2(1-\beta)}\right)(3\beta - 2) &< 0 \quad \text{(B.56)}
\end{aligned}$$

Note that $p_H + \frac{\beta+\sqrt{\beta(8-7\beta)}}{2(1-\beta)}$ is always positive, and $\frac{-\beta+\sqrt{\beta(8-7\beta)}}{2(1-\beta)} > 1$ for $\beta > \frac{1}{2}$ as shown below.

$$\begin{aligned}
\frac{-\beta+\sqrt{\beta(8-7\beta)}}{2(1-\beta)} &> 1 \\
\Rightarrow \quad \beta(8 - 7\beta) &> (2 - \beta)^2 \\
\Rightarrow \quad 0 &> 2\beta^2 - 3\beta + 1 \\
\Rightarrow \quad 0 &> (2\beta - 1)(\beta - 1) \\
\Rightarrow \quad \beta &> \frac{1}{2}
\end{aligned}$$

This implies that $p_H - \frac{-\beta+\sqrt{\beta(8-7\beta)}}{2(1-\beta)} < 0$. Hence, inequality (B.56) is true for $\beta > \frac{2}{3}$, which also makes $D_L - (2 - p_H)(D_H - D_L) > 0$. Therefore, the seller can provide a demonstration such that D_H^+ is above the threshold $D^B = D\left(\frac{2D_L - D_H}{D_L-(2-p_H)(D_H-D_L)}\right)$ when $\frac{D_L}{D_H} > \frac{2}{3}$. If $D_H^+ \geq \frac{D_L}{p_H} + \frac{D_L}{2}$ while $\frac{D_L}{D_H} > \frac{2}{3}$ then we are in the previous region, where $\frac{2D_L}{2D_H^+ - D_L} < p_H \leq 1$. Therefore, $D_H^+ > D^B$ will always make the seller better off than in the no-demonstration scenario when $\frac{D_L}{D_H} > \frac{2}{3}$. We show below that $D\left(\frac{2D_L - D_H}{2D_L - D_H - p_L(D_H - D_L)}\right) \geq D_H$ when $D_L - (2 - p_H)(D_H - D_L) > 0$.

$$\begin{array}{rrcl} & D\left(\frac{2D_L - D_H}{D_L - (2-p_H)(D_H - D_L)}\right) & \geq & D_H \\ \Rightarrow & ((D_H - D_L)p_H + D_L)(2D_L - D_H) & \geq & D_H D_L - D_H(2 - p_H)(D_H - D_L) \\ \Rightarrow & (D_H - D_L)(2D_L p_H - D_H p_H + 2D_H - p_H D_H) + 2D_L^2 - 2D_L D_H & \geq & 0 \\ \Rightarrow & (D_H - D_L)(2D_L p_H + 2D_H p_L) - 2D_L(D_H - D_L) & \geq & 0 \\ \Rightarrow & p_L(D_H - D_L)^2 & \geq & 0 \end{array}$$

(ii) When $D_L - (2 - p_H)(D_H - D_L) < 0$: Similar algebra as above shows that $D\left(\frac{2D_L - D_H}{2D_L - D_H - p_L(D_H - D_L)}\right) < D_H$ when $D_L - (2 - p_H)(D_H - D_L) < 0$. Therefore this situation is not feasible as it makes $D_H^+ < D_H$. ∎

B.2 Proof of Lemma 3.6.

Lemma B.4 *When the buyer underestimate the probability distribution to be p_H^B and $p_L^B = 1 - p_H^B$ for values D_H and D_L respectively while seller knows the true distribution to be $p_H > p_H^B$ and $p_L = 1 - p_H$ for values D_H and D_L respectively, the outcome of the negotiation will give the following expected payoffs to the players in the absence of an outside option and after a demonstration with accurate signal.*

$$(U_S^*, U_B^*) = \begin{cases} (A, D - A) & \text{if } 0 \leq p_H \leq \frac{2A}{D_H} \\ \left(\frac{p_H D_H}{2}, \frac{(D - D_L)(p_H D_H - p_L^B D_L)}{2(p_H D_H - D_L)}\right), & \text{if } \frac{2A}{D_H} < p_H < \frac{D_L}{D_L + D_H - D^B} \\ \left(\frac{p_H D_H}{2}, \frac{D}{2}\right), & \text{if } \frac{D_L}{D_L + D_H - D^B} \leq p_H \leq 1 \end{cases}$$

where, $A = \frac{1}{4}\left(2(D_L + D_H) - D^B - \sqrt{(2(D_L + D_H) - D^B)^2 - 8D_L D_H}\right)$, $D^B = p_L^B D_L + p_H^B D_H$, *and* $D = p_L D_L + p_H D_H$

PROOF: Problem $(S_N B_Y)_P$ is given as,

$$\begin{array}{llllll} (S_N B_Y)_P & \max\limits_{\mu \in \Psi} & U_S(\mu) \cdot \left(U_B(\mu, L|L)\right)^{(1-p_H^B)} \cdot \left(U_B(\mu, H|H)\right)^{p_H^B} & & & \\ & \text{s.t.} & U_B(\mu, L|L) \geq U_B(\mu, H|L); & U_B(\mu, H|H) & \geq & U_B(\mu, L|H) \\ & & U_S(\mu) \geq 0 & & & \\ & & U_B(\mu, L|L) \geq 0; & U_B(\mu, H|H) & \geq & 0 \\ & & \sum_{d \in \Phi} \mu(d|L) = 1; & \sum_{d \in \Phi} \mu(d|H) & = & 1 \\ & & \mu(d|L) \geq 0; & \mu(d|H) & \geq & 0 \ \forall\, d \in \Phi \\ & & u_B(d, t) + u_S(d) = D_t; & \multicolumn{3}{l}{\forall\, d \in \Phi \backslash \{d^*\}; \quad t \in \{L, H\}} \\ & & u_B(d^*, t) = 0; \quad u_S(d^*) = 0; & \multicolumn{3}{l}{t \in \{L, H\}} \end{array}$$

where,

$$U_B(\mu, H|H) = \sum_{d\in\Phi\backslash\{d^*\}} \mu(d|H)\big(D_H - u_S(d)\big) \quad \text{(B.57)}$$

$$U_B(\mu, L|H) = \sum_{d\in\Phi\backslash\{d^*\}} \mu(d|L)\big(D_H - u_S(d)\big) \quad \text{(B.58)}$$

$$U_B(\mu, H|L) = \sum_{d\in\Phi\backslash\{d^*\}} \mu(d|H)\big(D_L - u_S(d)\big) \quad \text{(B.59)}$$

$$U_B(\mu, L|L) = \sum_{d\in\Phi\backslash\{d^*\}} \mu(d|L)\big(D_L - u_S(d)\big) \quad \text{(B.60)}$$

$$U_S(\mu) = \sum_{d\in\Phi} \big(p_L\mu(d|L) + p_H\mu(d|H)\big)u_S(d) \quad \text{(B.61)}$$

The problem $(S_NB_Y)_P$ is structurally same as the problem (S_NB_Y). As can be observed from the proofs that both the properties B.1 and B.2 also hold for $(S_NB_Y)_P$. From Property B.1 we get $\mu^*(d^*|H) = 0$ where d^* is the disagreement alternative. Property B.2 implies $U_B(\mu^*, H|H) = U_B(\mu^*, L|H)$. Similar to the solution of (S_NB_Y), we exploit Properties B.1 and B.2 and convert $(S_NB_Y)_P$ without losing any information to a simpler problem $(S_NB_Y)'_P$, by aggregating the μ variables. Define three new variables $A_L(\mu) = \sum_{d\in\Phi\backslash\{d^*\}} \mu(d|L)u_S(d)$, $A_H(\mu) = \sum_{d\in\Phi\backslash\{d^*\}} \mu(d|H)u_S(d)$, and $\delta(\mu) = \mu(d^*|L)$. Note that A_L and A_H are both non-negative, as $u_S(d) \geq 0$ for each $d \in \Phi$. Using equations $u_B(d,H)+u_S(d) = D_H$ for $d \in \Phi\backslash\{d^*\}$, and $u_B(d^*, H) = 0$, the binding equation $U_B(\mu^*, H|H) = U_B(\mu^*, L|H)$ can be written as

$$\begin{aligned}
\sum_{d\in\Phi\backslash\{d^*\}} \mu^*(d|H)\big(D_H - u_S(d)\big) &= \sum_{d\in\Phi\backslash\{d^*\}} \mu^*(d|L)\big(D_H - u_S(d)\big) \\
\Rightarrow \quad D_H - A_H &= (1-\delta)D_H - A_L \\
\Rightarrow \quad \delta &= \tfrac{1}{D_H}(A_H - A_L) \qquad \text{(B.62)}
\end{aligned}$$

because $\sum_{d\in\Phi\backslash\{d^*\}} \mu(d|L) + \delta = 1$.

Equations $u_B(d, L) + u_S(d) = D_L$ for $d \in \Phi\backslash\{d^*\}$, and $u_B(d^*, L) = 0$ along with (B.62) can be used to reduce the inequality $U_B(\mu, L|L) \geq U_B(\mu, H|L)$ to $A_H \geq A_L$ as shown below.

$$\begin{aligned}
\sum_{d\in\Phi\backslash\{d^*\}} \mu^*(d|L)\big(D_L - u_S(d)\big) &\geq \sum_{d\in\Phi\backslash\{d^*\}} \mu^*(d|H)\big(D_L - u_S(d)\big) \\
\Rightarrow \quad (1-\delta)D_L - A_L &\geq D_L - A_H \\
\Rightarrow \quad A_H - A_L &\geq \tfrac{D_L}{D_H}(A_H - A_L) \\
\Rightarrow \quad (D_H - D_L)(A_H - A_L) &\geq 0 \\
\Rightarrow \quad A_H &\geq A_L
\end{aligned}$$

The expected payoffs $U_S(\mu)$, $U_B(\mu, L|L)$ and $U_B(\mu, H|H)$ can now be written in terms of $A_L(\mu)$, $A_H(\mu)$, and δ as

$$\begin{aligned} U_S(\mu) &= p_L A_L + p_H A_H && \text{(B.63)} \\ U_B(\mu, L|L) &= \sum_{d\in\Phi\setminus\{d^*\}} \mu(d|L)\big(D_L - u_S(d)\big) \\ &= (1 - \tfrac{1}{D_H}(A_H - A_L))D_L - A_L \\ &= \tfrac{1}{D_H}\Big(D_L(D_H - A_H) - A_L(D_H - D_L)\Big) && \text{(B.64)} \\ U_B(\mu, H|H) &= \sum_{d\in\Phi\setminus\{d^*\}} \mu(d|H)\big(D_H - u_S(d)\big) \\ &= D_H - A_H && \text{(B.65)} \end{aligned}$$

where $p_L = 1 - p_H$. The IR constraint $U_S(\mu) \geq 0$ can now be represented by $p_L A_L + p_H A_H \geq 0$. Also, the IR constraint $U_B(\mu, H|H) \geq 0$ is trivially satisfied by $U_B(\mu, L|L) \geq 0$ (see equation (B.64)). The constraint $\mu(d|L) \geq 0$ along with $u_S(d) \geq 0$ for each $d \in \Phi$ implies $A_L \geq 0$ and $\delta \geq 0$. We can now formulate the simplified three-variable problem $(S_N B_Y)'_P$ that is equivalent to $(S_N B_Y)_P$ as

$$(S_N B_Y)'_P \quad \max_{A_L, A_H, \delta} \quad (p_L A_L + p_H A_H) \cdot \big(D_L(D_H - A_H) - A_L(D_H - D_L)\big)^{(1-p_H^B)} \cdot (D_H - A_H)^{p_H^B}$$

$$\begin{aligned} \text{s.t.} \quad A_H &\geq A_L && \text{(B.66)} \\ D_L(D_H - A_H) - A_L(D_H - D_L) &\geq 0 && \text{(B.67)} \\ \delta &= \tfrac{1}{D_H}(A_H - A_L) && \text{(B.68)} \\ A_L \geq 0; \quad \delta \geq 0 \end{aligned}$$

We have ignored the factor $\left(\frac{1}{D_H}\right)$ of equation (B.64) in this transformation, as it is a constant. The solutions to $(S_N B_Y)'_P$, once obtained, gives $U_S(\mu^*)$, $U_B(\mu^*, L|L)$ and $U_B(\mu^*, H|H)$ from equations (B.63), (B.64) and (B.65) respectively. To get back the mechanism μ^*, we need to use the equations $A_L(\mu^*)$, $A_H(\mu^*)$, and $\delta(\mu^*)$, along with the feasibility constraints $\sum_{d\in\Phi} \mu^*(d|t) = 1$ for $t \in \{L, H\}$, $\mu^*(d^*|H) = 0$, and $\mu^*(d|t) \geq 0$ for $d \in \Phi$. μ^* will not be unique if the number of μ^* variables is greater than the number of equations. It is easy to observe that a feasible mechanism always exists. For example, the following μ^*-vector is a feasible solution: $\mu^*(0|L) = \mu^*(0|H) = 1 - \frac{A_H(\mu^*)}{D_H}$, $\mu^*(H|L) = \frac{A_L(\mu^*)}{D_H}$, $\mu^*(H|H) = \frac{A_H(\mu^*)}{D_H}$, $\mu^*(d^*|L) = \delta(\mu^*) = \frac{1}{D_H}(A_H(\mu^*) - A_L(\mu^*))$, with the rest of the $\mu^*(d)$'s equal to zero. This solution satisfies $A_L(\mu^*) = \sum_{d\in\Phi\setminus\{d^*\}} \mu^*(d|L)u_S(d) = 0 \cdot \mu^*(0|L) + D_H \cdot \mu^*(H|L)$, and $A_H(\mu^*) = \sum_{d\in\Phi\setminus\{d^*\}} \mu^*(d|H)u_S(d) = 0 \cdot \mu^*(0|H) + D_H \cdot \mu^*(H|H)$. This is a valid solution because $0 \leq A_L(\mu^*) \leq A_H(\mu^*) \leq D_H$ which makes each $\mu^*(d) \in [0, 1]$. Additionally, $\sum_{d\in\Phi} \mu^*(d|L) = \mu^*(0|L) + \mu^*(H|L) + \mu^*(d^*|L) = 1$ and $\sum_{d\in\Phi} \mu^*(d|H) = \mu^*(0|H) + \mu^*(H|H) = 1$.

None of the expected payoffs $U_S(\mu)$, $U_B(\mu, L|L)$ and $U_B(\mu, H|H)$ are zero, as this will make the objective value zero which is not a local maximum. Hence, let $p_L A_L + p_H A_H > 0$, and $D_L(D_H - A_H) - A_L(D_H - D_L) >$

0 (which makes $D_H - A_H > 0$); these are later shown to be satisfied by the optimal solution. After taking a log-transformation of the objective, the Lagrangian of the problem is

$$\begin{aligned}\mathcal{L}(A_L, A_H, \delta, \lambda_1, \lambda_2, \lambda_3, \lambda_4) &= \ln(p_L A_L + p_H A_H) + p_L^B \ln(D_L(D_H - A_H) - A_L(D_H - D_L)) \\ &\quad + p_H^B \ln(D_H - A_H) - \lambda_1 A_L - \lambda_2(A_H - A_L) + \lambda_3\Big(\delta - \tfrac{1}{D_H}(A_H - A_L)\Big) - \lambda_4\delta\end{aligned}$$

where $p_L^B = 1 - p_H^B$. The KKT conditions will generate the following equations along with the constraints of the problem $(S_N B_Y)'_P$, $\lambda_1 \geq 0$, $\lambda_2 \geq 0$, and $\lambda_4 \geq 0$,

$$\begin{aligned}
\tfrac{\partial \mathcal{L}}{\partial A_L}: \quad & \tfrac{p_L}{p_L A_L + p_H A_H} - \tfrac{p_L^B (D_H - D_L)}{D_L(D_H - A_H) - A_L(D_H - D_L)} - \lambda_1 + \lambda_2 + \tfrac{\lambda_3}{D_H} = 0 && \text{(B.69)}\\
\tfrac{\partial \mathcal{L}}{\partial A_H}: \quad & \tfrac{p_H}{p_L A_L + p_H A_H} - \tfrac{p_L^B D_L}{D_L(D_H - A_H) - A_L(D_H - D_L)} - \tfrac{p_H^B}{D_H - A_H} - \lambda_2 - \tfrac{\lambda_3}{D_H} = 0 && \text{(B.70)}\\
\tfrac{\partial \mathcal{L}}{\partial \delta}: \quad & \lambda_3 - \lambda_4 = 0 && \text{(B.71)}\\
& \lambda_1 A_L = 0 && \text{(B.72)}\\
& \lambda_2 (A_H - A_L) = 0 && \text{(B.73)}\\
& \lambda_4 \delta = 0 && \text{(B.74)}
\end{aligned}$$

Case I: $A_L > 0$ and $A_H \neq A_L$

In this case both λ_1 and λ_2 will be 0. Furthermore, from equation (B.68) $A_H > A_L$ implies $\delta > 0$. Which implies $\lambda_4 = \lambda_3 = 0$ (see equations (B.74), and (B.71)respectively). After some lengthy algebra, equation (B.69) and (B.70) yields A_L and A_H as

$$A_L = \frac{p_H D_H (p_L D_L - p_L^B p_H (D_H - D_L))}{2 p_L (p_H D_H - D_L)}; \quad A_H = \frac{D_H}{2}\left(1 - \frac{p_L D_L - p_L^B p_H (D_H - D_L)}{p_H D_H - D_L}\right)$$

A_H can be written as $A_H = \frac{D_H}{2} - \frac{p_L A_L}{p_H}$. The expected payoff of the seller is given by $U_S(\mu^*) = p_L A_L + p_H A_H = \frac{p_H D_H}{2}$. Also,

$$\begin{aligned}
U_B(\mu^*, H|H) &= D_H - A_H &&= \tfrac{p_H^B p_H D_H (D_H - D_L)}{2(p_H D_H - D_L)} && \text{(B.75)}\\
U_B(\mu^*, L|L) &= \tfrac{1}{D_H}\left(D_L(D_H - A_H) - A_L(D_H - D_L)\right) &&= \tfrac{p_L^B p_H (D_H - D_L)}{2 p_L} && \text{(B.76)}\\
U_B(\mu^*) &= p_L U_B(\mu^*, L|L) + p_H U_B(\mu^*, H|H) &&= \tfrac{p_H (D_H - D_L)(p_H D_H - p_L^B D_L)}{2(p_H D_H - D_L)} && \text{(B.77)}\\
\delta &= \tfrac{1}{D_H}(A_H - A_L) &&= \tfrac{D_H}{2} - \tfrac{A_L}{p_H} && \text{(B.78)}
\end{aligned}$$

Equations (B.75) implies that $U_B(\mu^*, H|H) > 0$ is true for $p_H > \frac{D_L}{D_H}$ and $p_H^B \neq 0$. $U_B(\mu^*, L|L) > 0$ if $p_L > 0$, $p_H > 0$, and $p_L^B \neq 0$. Furthermore, $A_L > 0$ is satisfied by $\frac{D_L}{D_H} < p_H$ and $p_L D_L - p_L^B p_H (D_H - D_L) > 0$ i.e. $\frac{D_L}{D_H} < p_H < \frac{D_L}{p_L^B D_H + p_H^B D_L}$. Also $A_H > A_L$, after some algebra yields,

$$p_H^2 D_H - p_H\big(D_L + D_H + p_L^B D_H + p_H^B D_L\big) + 2D_L < 0 \quad \text{(B.79)}$$

The discriminant of the above quadratic equation is non-negative as shown below,

$$\begin{aligned}(D_L + D_H + p_L^B D_H + p_H^B D_L)^2 - 8D_L D_H &= (2D_L + D_H + (D_H - D_L)p_L^B)^2 - 8D_L D_H \\ &\geq (2D_L + D_H)^2 - 8D_L D_H \\ &= (2D_L - D_H)^2 \geq 0\end{aligned}$$

Therefore, $A_H > A_L$ will be satisfied if $p_H^- < p_H < p_H^+$ where p_H^- and p_H^+ are the smaller and larger roots of (B.79) respectively. Case I will have a solution when $\frac{D_L}{D_H} < p_H < \frac{D_L}{p_L^B D_H + p_H^B D_L}$ and $p_H^- < p_H < p_H^+$. Let $D^B = p_H^B D_H + p_L^B D_L$ then, $p_H^- \geq \frac{D_L}{D_H}$ is always true as shown below,

$$\begin{aligned}\frac{2D_H + 2D_L - D^B - \sqrt{(2D_H + 2D_L - D^B)^2 - 8D_L D_H}}{2D_H} &\geq \frac{D_L}{D_H} \\ \Rightarrow \quad (2D_H + 2D_L - D^B - 2D_L)^2 &\geq (2D_H + 2D_L - D^B)^2 - 8D_L D_H \\ \Rightarrow \quad 8D_L D_H &\geq 4D_L(2D_H + D_L - D^B) \\ \Rightarrow \quad D^B &\geq D_L\end{aligned}$$

Following shows that $p_H^- \leq \frac{D_L}{p_L^B D_H + p_H^B D_L} = \frac{D_L}{D_L + D_H - D^B}$.

$$\begin{aligned}2D_H + 2D_L - D^B - \sqrt{(2D_H + 2D_L - D^B)^2 - 8D_L D_H} &\leq \frac{2D_L D_H}{D_L + D_H - D^B} \\ \Rightarrow \quad \big((2D_L + 2D_H - D^B)(D_L + D_H - D^B) - 2D_L D_H\big)^2 &\leq (D_L + D_H - D^B)^2\big((2D_L + 2D_H - D^B)^2 - 8D_L D_H\big) \\ \Rightarrow \quad D_L D_H - (D_L + D_H - D^B)(2D_L + 2D_H - D^B) + 2(D_L + D_H - D^B)^2 &\leq 0 \\ \Rightarrow \quad D_L D_H - D^B(D_L + D_H - D^B) &\leq 0 \\ \Rightarrow \quad (D^B - D_H)(D^B - D_L) &\leq 0\end{aligned}$$

Above inequality is always true. Similar analysis shows that $p_H^+ > \frac{D_L}{D_L + D_H - D^B}$ hence, Case I is valid for $\frac{D_L}{D_L + D_H - D^B} > p_H > p_H^-$

Case II: $A_L = 0$ **and** $A_H \neq A_L$

For this case we have $\lambda_2 = 0$. Similar to Case I, we get $\delta > 0$, and $\lambda_4 = \lambda_3 = 0$. From (B.70) we get $A_H = \frac{D_H}{2} > A_L$. Also, $U_B(\mu^*, H|H) = D_H - A_H = \frac{D_H}{2} > 0$ and $U_B(\mu^*, L|L) = \frac{1}{D_H}\left(D_L(D_H - A_H) - A_L(D_H - D_L)\right) = \frac{D_L}{2} > 0$. Furthermore, $\delta = \frac{1}{D_H}(A_H - A_L) = 0.5$. Hence all conditions are satisfied for Case II. The expected payoffs for the seller and the buyer can be easily calculated as $U_S = \frac{p_H D_H}{2}$ and $U_B = \frac{D}{2}$.

Case III: $A_L > 0$ **and** $A_H = A_L = A$

For this case we have $\lambda_1 = 0$ and $\delta = 0$. Note that Property B.1 along with $\delta = 0$ implies that no positive probability is given to any of the disagreement alternative d^* in states L and H. Adding equations (B.69) and (B.70) we get

$$\begin{aligned}\frac{1}{A} - \frac{p_L^B}{D_L - A} - \frac{p_H^B}{D_H - A} &= 0 \\ \Longrightarrow \quad 2A^2 - A(D_L + D_H + p_L^B D_H + p_H^B D_L) + D_L D_H &= 0 \quad \text{(B.80)}\end{aligned}$$

The discriminant of the quadratic (B.80) is same as that of (B.79) hence, is non-negative. It can be easily verified that the roots are also positive. For this case $U_B(\mu^*, L|L) = D_L - A$. Hence the solution to be feasible, we need $D_L > A > 0$. Using contradiction we will show that the smaller root A_s satisfies $D_L > A_s$ while the larger root A_l does not. Let $D^B = p_H^B D_H + p_L^B D_L$ then $A_s < D_L$ gives,

$$\begin{aligned} \tfrac{1}{4}\left(2(D_L + D_H) - D^B - \sqrt{(2(D_L + D_H) - D^B)^2 - 8D_L D_H}\right) &< D_L \\ \implies \quad (2(D_H - D_L) - D^B)^2 &< (2(D_L + D_H) - D^B)^2 - 8D_L D_H \\ \implies \quad 8D_L D_H &< 4D_L(4D_H - 2D^B) \\ \implies \quad D^B &< D_H \end{aligned}$$

Above is true for $p_H^B < 1$. Hence the smaller root A_s satisfies the condition $U_B(\mu^*, L|L) > 0$ when $p_H^B \neq 1$. A similar algebra will show that $D_L < A_l$. Hence only the smaller root A_s of the quadratic (B.80) is the solution of Case III. We have $U_B(\mu^*, H|H) = D_H - A_s > 0$. The expected payoffs for this case is calculated as $U_S = A_s$ and $U_B = D - A_s$. Comparing p_H^- and A_s we observe that $p_H^- = \frac{2A_s}{D_H}$.

To find which of the three cases is the global solution of the problem $(S_N B_Y)_P$ we need to compare the objective values of these cases. We find that Case I is the global solution under condition $\frac{D_L}{D_L + D_H - D^B} > p_H > \frac{2A_s}{D_H}$. When $0 \leq p_H \leq \frac{2A_s}{D_H}$ Case III is the global solution while when $\frac{D_L}{D_L + D_H - D^B} \leq p_H \leq 1$ Case II is the global solution. ■

APPENDIX C

BARGAINING OVER DATA WITH THE CONSULTANT AS A GATEKEEPER

Lemma C.1 *When the distribution of the value is a common knowledge and the reservation price for the consultant is r, the negotiation will result in the following expected payoffs U_S^*, U_B^*, and U_C^* to the seller, buyer, and the consultant respectively, where R is the expected value of the report.*

$$(U_S^*, U_B^*, U_C^*) = \begin{cases} \left(\frac{R}{3}, \frac{R}{3}, \frac{R}{3}\right), & \text{if } r \leq \frac{R}{3} \\ \left(\frac{R-r}{2}, r, \frac{R-r}{2}\right), & \text{if } \frac{R}{3} \leq r \leq R \\ (0,0,0), & \text{if } R \leq r \end{cases}$$

Proof: Note that the players can do better by agreeing on a price, otherwise, a disagreement will breakdown the negotiation giving zero payoff to each player. For prices q_S and q_C, the Nash product is given by,

$$\begin{array}{lll} (S_N B_N C_N) & \max\limits_{(q_S, q_C)} & q_S \cdot q_C \cdot (R - q_S - q_C) \\ & \text{s.t.} & q_S \geq 0; \quad q_C \geq r; \quad R - q_S - q_C \geq 0 \end{array}$$

The Lagrangian of the problem is given as,

$$\mathcal{L}(q_S, q_C, \lambda) = q_S \cdot q_C \cdot (R - q_S - q_C) - \lambda_1 q_S - \lambda_2 (q_C - r) - \lambda_3 (R - q_S - q_C)$$

The KKT conditions are represented by the following equations, the constraints of problem $(S_N B_N C_N)$, and the non-negativity constraints $\lambda_1 \geq 0$, $\lambda_2 \geq 0$, and $\lambda_3 \geq 0$:

$$\frac{\partial \mathcal{L}}{\partial q_S}: \quad q_C(R - 2q_S - q_C) - \lambda_1 + \lambda_3 \quad \text{(C.1)}; \quad \frac{\partial f}{\partial q_C}: \quad q_S(R - q_S - 2q_C) - \lambda_2 + \lambda_3 \quad \text{(C.2)}$$
$$\lambda_1 q_S = 0; \quad \lambda_2(q_C - r) = 0; \quad \lambda_3(R - q_S - q_C) = 0$$

Case I: $q_C \neq r$**,** $q_S \neq 0$**,** $R - q_S - q_C \neq 0$

This implies $\lambda_1 = \lambda_2 = \lambda_3 = 0$. From equations (C.1) and (C.2) we get four critical points, $(q_S, q_C) = \{(0,0), (R,0), (0,R), \left(\frac{R}{3}, \frac{R}{3}\right)\}$. The critical points $(0,R)$ and $\left(\frac{R}{3}, \frac{R}{3}\right)$ satisfy all the constraints for $r \leq R$ and $r \leq \frac{R}{3}$ respectively. Since the objective is zero for $(0,R)$ and $\left(\frac{R}{3}\right)^3$ for $\left(\frac{R}{3}, \frac{R}{3}\right)$, therefore, the local optimum for $0 \leq r \leq \frac{R}{3}$ is $\left(\frac{R}{3}, \frac{R}{3}\right)$ and for $\frac{R}{3} \leq r \leq R$ is $(0,R)$.

Case II: $q_C = r$**,** $q_S \neq 0$**,** $R - q_S - q_C \neq 0$

This gives $\lambda_1 = \lambda_3 = 0$. From equations (C.1) we get one critical point $\left(\frac{R-r}{2}, r\right)$ which satisfies all the conditions for $r \leq R$.

We will not consider the cases where $q_S = 0$ or $R - q_S - q_C = 0$ as the objective value becomes zero. To find the global optimum for $0 \leq r \leq \frac{R}{3}$, we need to compare the objective values $\left(\frac{R}{3}\right)^3$ and $r\left(\frac{R-r}{2}\right)^2$ for critical points $\left(\frac{R}{3}, \frac{R}{3}\right)$ and $\left(\frac{R-r}{2}, r\right)$ respectively. Since $\frac{1}{3}\Big(\frac{R-r}{2} + \frac{R-r}{2} + r\Big) \geq \Big(\left(\frac{R-r}{2}\right) \cdot \left(\frac{R-r}{2}\right) \cdot r\Big)^{\frac{1}{3}}$ implies $\left(\frac{R}{3}\right)^3 \geq r\left(\frac{R-r}{2}\right)^2$, therefore, $\left(\frac{R}{3}, \frac{R}{3}\right)$ is the global optimum when $0 \leq r \leq \frac{R}{3}$. In addition, $r\left(\frac{R-r}{2}\right)^2 \geq 0$ when $\frac{R}{3} \leq r \leq R$, therefore, $\left(\frac{R-r}{2}, r\right)$ is the global optimum in this range. There is no price pair for $r > R$ which can satisfy all the constraints of the problem $(S_N B_N C_N)$. ■

Property C.1 *In the negotiation problem $(S_N B_Y C_N)$, the probability of selecting the disagreement alternative $d^* \in \Phi$ by the mechanism $\mu^*(d|H)$ is zero.*

Proof: Problem $(S_N B_Y C_N)$ is

$$
\begin{array}{lll}
(S_N B_Y C_N) & \max\limits_{\mu \in \Psi} & U_S(\mu) \cdot U_C(\mu) \cdot \big(U_B(\mu, L|L)\big)^{(1-p_H)} \cdot \big(U_B(\mu, H|H)\big)^{p_H} \\
& \text{s.t.} & U_B(\mu, L|L) \geq U_B(\mu, H|L); \quad U_B(\mu, H|H) \geq U_B(\mu, L|H) \\
& & U_S(\mu) \geq 0 \\
& & U_C(\mu) \geq r \\
& & U_B(\mu, L|L) \geq 0; \quad U_B(\mu, H|H) \geq 0 \\
& & \sum_{d\in\Phi} \mu(d|L) = 1; \quad \sum_{d\in\Phi} \mu(d|H) = 1 \\
& & \mu(d|L) \geq 0; \quad \mu(d|H) \geq 0 \ \forall\, d \in \Phi \\
& & u_B(d,t) + q_S(d) + q_C(d) = R_t; \quad \forall\, d \in \Phi\backslash\{d^*\}; \ \ t \in \{L, H\} \\
& & u_B(d^*, t) = 0; \ \ q_S(d^*) = 0; \ \ q_C(d^*) = 0 \quad t \in \{L, H\} \\
& & q_S(d), q_C(d) \in [0, R_H]
\end{array}
$$

where,

$$U_B(\mu, H|H) = \sum_{d\in\Phi\backslash\{d^*\}} \mu(d|H)\big(R_H - q_S(d) - q_C(d)\big) \quad \text{(C.3)}$$

$$U_B(\mu, L|H) = \sum_{d\in\Phi\backslash\{d^*\}} \mu(d|L)\big(R_H - q_S(d) - q_C(d)\big) \quad \text{(C.4)}$$

$$U_B(\mu, H|L) = \sum_{d\in\Phi\backslash\{d^*\}} \mu(d|H)\big(R_L - q_S(d) - q_C(d)\big) \quad \text{(C.5)}$$

$$U_B(\mu, L|L) = \sum_{d\in\Phi\backslash\{d^*\}} \mu(d|L)\big(R_L - q_S(d) - q_C(d)\big) \quad \text{(C.6)}$$

$$U_S(\mu) = \sum_{d\in\Phi} \big(p_L\mu(d|L) + p_H\mu(d|H)\big)q_S(d) \quad \text{(C.7)}$$

$$U_C(\mu) = \sum_{d\in\Phi} \big(p_L\mu(d|L) + p_H\mu(d|H)\big)q_C(d) \quad \text{(C.8)}$$

Define Φ^m to be $\{d \in \Phi : R_H \geq q_S(d) + q_C(d) \geq R_L; d \neq d^*\}$. Evidently, Φ^m is a non-empty set. Suppose Property C.1 was not true, and that $\mu^*(d^*|H) = \epsilon > 0$ in the optimal equilibrium solution. This implies that $\sum_{d\in\Phi\setminus\{d^*\}} \mu^*(d|H) = (1-\epsilon)$, which implies that every $\mu^*(d|H)_{d\in\Phi\setminus\{d^*\}} \leq (1-\epsilon)$. This means that we can always increase the value of one of the variables $\mu^*(d'|H)$, where $d' \in \Phi^m$, by ϵ, while reducing $\mu^*(d^*|H)$ to 0 and leaving all other variables $\mu^*(d|H)$ $(d \in \Phi; d \neq d', d^*)$ and $\mu^*(d|L)_{d\in\Phi}$ unchanged.

Increasing $\mu^*(d'|H)$ by ϵ and decreasing $\mu^*(d^*|H)$ by ϵ to 0 (i) increases $U_B(\mu^*, H|H)$ by $\epsilon\big(R_H - q_S(d') - q_C(d')\big)$ since $q_S(d') + q_C(d') \leq R_H$, (ii) decreases $U_B(\mu^*, H|L)$ by $\epsilon\big(R_L - q_S(d') - q_C(d')\big)$ since $q_S(d') + q_C(d') \geq R_L$, (iii) increases $U_S(\mu^*)$ by $\epsilon p_H q_S(d')$, and (iv) increases $U_C(\mu^*)$ by $\epsilon p_H q_C(d')$, while maintaining the probability balance equation $\sum_{d\in\Phi} \mu^*(d|H) = 1$. $U_B(\mu^*, L|H)$, $U_B(\mu^*, L|L)$, and $\sum_{d\in\Phi} \mu^*(d|L) = 1$ are unchanged since the values of the variables $\mu^*(d|L)$ are unaffected by the change.

All IR constraints remain satisfied since $U_B(\mu^*, H|H)$, $U_S(\mu^*)$, and $U_C(\mu^*)$ are increasing as a result of the change. $U_B(\mu^*, H|H)$ increases by $\epsilon\big(R_H - q_S(d') - q_C(d')\big)$ while $U_B(\mu^*, L|H)$ is unchanged, ensuring that the BIC constraint $U_B(\mu^*, H|H) \geq U_B(\mu^*, L|H)$ is not violated. The BIC constraint $U_B(\mu, L|L) \geq U_B(\mu, H|L)$ continues to hold since $U_B(\mu, H|L)$ decreases by $\epsilon\big(R_L - q_S(d') - q_C(d')\big)$ while $U_B(\mu, L|L)$ remains unaffected. Therefore, the new solution resulting from this change remains feasible.

The net effect therefore, is an increase in the objective function value, as the three components of the objective function — $U_S(\mu^*)$, $U_C(\mu^*)$, and $U_B(\mu^*, H|H)$ — have increased, and the objective function is an increasing function of each of these components. That is, we have identified a feasible solution to $(S_N B_Y)$ that has a better objective function value. Therefore the original solution with $\mu^*(d^*|H) = \epsilon > 0$ could not have been optimal. ■

Property C.2 *In the negotiation problem $(S_N B_Y C_N)$ the expected payoff to the buyer in state H is not affected even if they pretend to be in state L.*

Proof: Property C.2 essentially states that the BIC constraint $U_B(\mu, H|H) \geq U_B(\mu, L|H)$ will be binding in the optimal equilibrium mechanism. That is, we need to show that $U_B(\mu, H|H) = U_B(\mu, L|H)$, or equivalently, $\sum_{d\in\Phi\setminus\{d^*\}} \mu^*(d|H)\big(R_H - q_S(d) - q_C(d)\big) = \sum_{d\in\Phi\setminus\{d^*\}} \mu^*(d|L)\big(R_H - q_S(d) - q_C(d)\big)$.

Define Φ^+ to be $\{d : R_L - q_S(d) - q_C(d) > 0; d \neq d^*\}$. Suppose Property C.2 was not true, and that $U_B(\mu^*, H|H) = U_B(\mu^*, L|H) + \delta$ for some $\delta > 0$ in the optimal equilibrium solution. From Property C.1, we know that $\sum_{d\in\Phi\setminus\{d^*\}} \mu^*(d|H) = 1$. Therefore, we can write the BIC constraint $\sum_{d\in\Phi\setminus\{d^*\}} \mu^*(d|H)\big(R_H - q_S(d) - q_C(d)\big) = \sum_{d\in\Phi\setminus\{d^*\}} \mu^*(d|L)\big(R_H - q_S(d) - q_C(d)\big) + \delta$ as,

$$\begin{aligned} R_H - \sum_{d\in\Phi\setminus\{d^*\}} \mu^*(d|H)\big(q_S(d) + q_C(d)\big) &= R_H \sum_{d\in\Phi\setminus\{d^*\}} \mu^*(d|L) - \sum_{d\in\Phi\setminus\{d^*\}} \mu^*(d|L)\big(q_S(d) + q_C(d)\big) + \delta \\ \Rightarrow \quad R_H\big(1 - \sum_{d\in\Phi\setminus\{d^*\}} \mu^*(d|L)\big) &= \sum_{d\in\Phi\setminus\{d^*\}} \big(\mu^*(d|H) - \mu^*(d|L)\big)\big(q_S(d) + q_C(d)\big) + \delta \qquad \text{(C.9)} \end{aligned}$$

The other BIC constraint $U_B(\mu^*, L|L) \geq U_B(\mu^*, H|L)$ can be written as

$$\begin{aligned} \sum_{d\in\Phi\setminus\{d^*\}} \mu^*(d|L)\big(R_L - q_S(d) - q_C(d)\big) &\geq \sum_{d\in\Phi\setminus\{d^*\}} \mu^*(d|H)\big(R_L - q_S(d) - q_C(d)\big) \\ \Rightarrow \sum_{d\in\Phi\setminus\{d^*\}} \big(\mu^*(d|H) - \mu^*(d|L)\big)\big(q_S(d) + q_C(d)\big) &\geq R_L\big(1 - \sum_{d\in\Phi\setminus\{d^*\}} \mu^*(d|L)\big) \qquad \text{(C.10)} \end{aligned}$$

From equations (C.9) and (C.10) we get $(R_H - R_L)\big(1 - \sum_{d\in\Phi\setminus\{d^*\}} \mu^*(d|L)\big) \geq \delta$, which implies $\sum_{d\in\Phi\setminus\{d^*\}} \mu^*(d|L) \leq 1 - \frac{\delta}{R_H - R_L}$. Therefore, none of the variables $\mu^*(d|L)_{d\in\Phi\setminus\{d^*\}}$ are greater than $1 - \frac{\delta}{R_H - R_L}$. This implies that it is feasible to increase the value of one of the variables $\mu^*(d'|L)_{d'\in\Phi^+}$ by $\epsilon = \frac{\delta}{R_H - q_S(d') - q_C(d')} > 0$, while keeping all other variables fixed, since

$$\mu^*(d'|L) + \frac{\delta}{R_H - q_S(d') - q_C(d')} \leq 1 - \left(\frac{\delta}{R_H - R_L} - \frac{\delta}{R_H - q_S(d') - q_C(d')}\right) < 1.$$

which is true since $R_H - q_S(d') - q_C(d') > R_H - R_L$ for $d' \in \Phi^+$.

Therefore, we choose a variable $\mu^*(d'|L)$ for some $d' \in \Phi^+$, and increase it by ϵ to be $\frac{\delta}{R_H - q_S(d') - q_C(d')}$, while simultaneously decreasing $\mu^*(d^*|L)$, the probability corresponding to the disagreement alternative d^*, by the same amount, ϵ. We know that $\sum_{d\in\Phi\setminus\{d^*\}} \mu^*(d|L) \leq 1 - \frac{\delta}{R_H - R_L}$ which implies $\mu^*(d^*|L) \geq \frac{\delta}{R_H - R_L} > 0$. Subtracting $\epsilon = \frac{\delta}{R_H - q_S(d') - q_C(d')}$ from $\mu^*(d^*|L)$, we get

$$\mu^*(d^*|L) - \frac{\delta}{R_H - q_S(d') - q_C(d')} \geq \frac{\delta}{R_H - R_L} - \frac{\delta}{R_H - q_S(d') - q_C(d')} > 0$$

and therefore, it is feasible to decrease $\mu^*(d^*|L)$ by ϵ. Increasing $\mu^*(d'|L)$ and decreasing $\mu^*(d^*|L)$ by ϵ

1. increases $U_B(\mu^*, L|L)$ by $\delta\left(\frac{R_L - q_S(d') - q_C(d')}{R_H - q_S(d') - q_C(d')}\right) > 0$,

2. increases $U_B(\mu^*, L|H)$ by δ, because

$$\begin{aligned} U_B(\mu^*, L|H) &= \sum_{d\in\Phi\setminus\{d',d^*\}} \mu^*(d|L)\big(R_H - q_S(d) - q_C(d)\big) + \left(\mu^*(d'|L) + \frac{\delta}{R_H - q_S(d') - q_C(d')}\right)\big(R_H - q_S(d') - q_C(d')\big) \\ &= \sum_{d\in\Phi\setminus\{d',d^*\}} \mu^*(d|L)\big(R_H - q_S(d) - q_C(d)\big) + \mu^*(d'|L)\big(R_H - q_S(d') - q_C(d')\big) + \delta \\ &= \sum_{d\in\Phi\setminus\{d^*\}} \mu^*(d|L)\big(R_H - q_S(d) - q_C(d)\big) + \delta \end{aligned}$$

3. increases $U_S(\mu^*)$ by $\frac{\delta p_L q_S(d')}{R_H - q_S(d') - q_C(d')} > 0$.

$\sum_{d\in\Phi} \mu^*(d|H) = 1$ since $\mu^*(d'|L)_{d'\in\Phi^+}$ increases by ϵ and $\mu^*(d^*|L)$ decreases by ϵ. $U_B(\mu^*, H|H)$, $U_B(\mu^*, H|L)$, and $\sum_{d\in\Phi} \mu^*(d|H) = 1$ are not affected since none of the variables $\mu^*(d|H)$ have been changed. Since $U_B(\mu^*, L|L)$ and $U_S(\mu^*)$ increase as a result of the change, the IR constraints remain satisfied. As $U_B(\mu^*, L|L)$ increases by $\delta\frac{R_L - q_S(d') - q_C(d')}{R_H - q_S(d') - q_C(d')}$ while $U_B(\mu^*, H|L)$ stays unchanged, the BIC constraint $U_B(\mu^*, L|L) \geq$

$U_B(\mu^*, H|L)$ is not violated. The BIC constraint $U_B(\mu^*, H|H) \geq U_B(\mu^*, L|H)$ is now a binding constraint since $U_B(\mu^*, L|H)$ increases by δ, the exact value of the gap between $U_B(\mu^*, H|H)$ and $U_B(\mu^*, L|H)$.

The net effect therefore, is an increase in the objective function value, as the objective function is an increasing function of both $U_S(\mu^*)$ and $U_B(\mu^*, L|L)$, while still satisfying all the constraints. That is, we have identified a feasible solution to $(S_N B_Y)$ that has a better objective function value, and the original solution with $U_B(\mu^*, H|H) > U_B(\mu^*, L|H)$ could not have been optimal. ■

C.1 Solving Problem $(S_N B_Y C_N)$

We exploit Properties C.1 and C.2 and convert $(S_N B_Y C_N)$ without losing any information to a simpler problem $(S_N B_Y C_N)'$, by aggregating the μ variables. Property C.1 implies $\sum_{d\in\Phi\setminus\{d^*\}} \mu^*(d|H) = 1$, while Property C.2 implies $U_B(\mu^*, H|H) = U_B(\mu^*, L|H)$. Define five new variables $S_t(\mu) = \sum_{d\in\Phi\setminus\{d^*\}} \mu(d|t) q_S(d)$, $C_t(\mu) = \sum_{d\in\Phi\setminus\{d^*\}} \mu(d|t) q_C(d)$, and $\delta(\mu) = \mu(d^*|L)$ where $t = L, H$. Note that S_t and C_t are both non-negative, as $q_S(d) \geq 0$ and $q_C(d) \geq 0$ for each $d \in \Phi$. Using equations $u_B(d, H) + q_S(d) + q_C(d) = R_H$ for $d \in \Phi\setminus\{d^*\}$, and $u_B(d^*, H) = 0$, the binding equation $U_B(\mu^*, H|H) = U_B(\mu^*, L|H)$ can be written as

$$\begin{aligned}
\sum_{d\in\Phi\setminus\{d^*\}} \mu^*(d|H)\big(R_H - q_S(d) - q_C(d)\big) &= \sum_{d\in\Phi\setminus\{d^*\}} \mu^*(d|L)\big(R_H - q_S(d) - q_C(d)\big) \\
\Rightarrow \quad R_H - S_H - C_H &= (1-\delta)R_H - S_L - C_L \\
\Rightarrow \quad \delta &= \tfrac{1}{R_H}(S_H - S_L + C_H - C_L) \qquad \text{(C.11)}
\end{aligned}$$

because $\sum_{d\in\Phi\setminus\{d^*\}} \mu(d|L) + \delta = 1$.

Equations $u_B(d, L) + q_S(d) + q_C(d) = R_L$ for $d \in \Phi\setminus\{d^*\}$, and $u_B(d^*, L) = 0$ along with (C.24) can be used to reduce the inequality $U_B(\mu, L|L) \geq U_B(\mu, H|L)$ to $S_H + C_H \geq S_L + C_L$ as shown below.

$$\begin{aligned}
\sum_{d\in\Phi\setminus\{d^*\}} \mu^*(d|L)\big(R_L - q_S(d) - q_C(d)\big) &\geq \sum_{d\in\Phi\setminus\{d^*\}} \mu^*(d|H)\big(R_L - q_S(d) - q_C(d)\big) \\
\Rightarrow \quad (1-\delta)R_L - S_L - C_L &\geq R_L - S_H - C_H \\
\Rightarrow \quad S_H - S_L + C_H - C_L &\geq \tfrac{R_L}{R_H}(S_H - S_L + C_H - C_L) \\
\Rightarrow \quad (R_H - R_L)(S_H - S_L + C_H - C_L) &\geq 0 \\
\Rightarrow \quad S_H + C_H &\geq S_L + C_L
\end{aligned}$$

The expected payoffs $U_S(\mu)$, $U_B(\mu, L|L)$ and $U_B(\mu, H|H)$ can now be written in terms of $S_t(\mu)$, $C_t(\mu)$, and δ as

$$\begin{aligned}
U_S(\mu) &= p_L S_L + p_H S_H && \text{(C.12)}\\
U_C(\mu) &= p_L C_L + p_H C_H && \text{(C.13)}\\
U_B(\mu, L|L) &= \sum_{d\in\Phi\setminus\{d^*\}} \mu(d|L)\big(R_L - q_S(d) - q_C(d)\big)\\
&= \Big(1 - \tfrac{1}{R_H}(S_H - S_L + C_H - C_L)\Big)R_L - S_L - C_L\\
&= \tfrac{1}{R_H}\Big(R_L(R_H - S_H - C_H) - (S_L + C_L)(R_H - R_L)\Big) && \text{(C.14)}\\
U_B(\mu, H|H) &= \sum_{d\in\Phi\setminus\{d^*\}} \mu(d|H)\big(R_H - q_S(d) - q_C(d)\big)\\
&= R_H - S_H - C_H && \text{(C.15)}
\end{aligned}$$

where $p_L = 1 - p_H$. The IR constraint $U_C(\mu) \geq r$ can now be represented by $p_L C_L + p_H C_H \geq r$. Also, the IR constraint $U_B(\mu, H|H) \geq 0$ is trivially satisfied by $U_B(\mu, L|L) \geq 0$ (see equation (C.14)). We can now formulate the simplified five-variable problem $(S_N B_Y C_N)'$ that is equivalent to $(S_N B_Y C_N)$ as

$$\begin{aligned}
(S_N B_Y C_N)' \quad & \max_{S_L, S_H, C_L, C_H, \delta} \Big[(p_L S_L + p_H S_H)\cdot(p_L C_L + p_H C_H)\cdot(R_H - S_H - C_H)^{p_H}\\
& \qquad \cdot\big(R_L R_H - S_H R_L - S_L(R_H - R_L) - C_H R_L - C_L(R_H - R_L)\big)^{(1-p_H)}\Big]\\
\text{s.t.} \quad & S_H + C_H \geq S_L + C_L && \text{(C.16)}\\
& \delta = \tfrac{1}{R_H}(S_H + C_H - S_L - C_L) && \text{(C.17)}\\
& R_L R_H - S_H R_L - S_L(R_H - R_L) - C_H R_L - C_L(R_H - R_L) \geq 0 && \text{(C.18)}\\
& p_L C_L + p_H C_H \geq r && \text{(C.19)}\\
& S_L \geq 0;\quad S_H \geq 0;\quad C_L \geq 0;\quad C_H \geq 0;\quad \delta \geq 0
\end{aligned}$$

We have ignored the factor $\left(\frac{1}{R_H}\right)$ of equation (C.14) in this transformation, as it does not affect the optimum solutions $S_t(\mu^*)$, $C_t(\mu^*)$, and $\delta(\mu^*)$. The solutions to $(S_N B_Y C_N)'$, once obtained, gives $U_S(\mu^*)$, $U_C(\mu^*)$, $U_B(\mu^*, L|L)$, and $U_B(\mu^*, H|H)$ from equations (C.12), (C.13), (C.14) and (C.15) respectively. To get back the mechanism μ^*, we need to use the equations $S_t(\mu^*)$, $C_t(\mu^*)$, and $\delta(\mu^*)$, along with the feasibility constraints $\sum_{d\in\Phi} \mu^*(d|t) = 1$ for $t \in \{L, H\}$, $\mu^*(d^*|H) = 0$, and $\mu^*(d|t) \geq 0$ for $d \in \Phi$. μ^* will not be unique if the number of μ^* variables is greater than the number of equations. It is easy to observe that a feasible mechanism always exists. For example, let us consider three price alternatives $(q_S(d_1), q_C(d_1)) = (0, 0)$, $(q_S(d_2), q_C(d_2)) = (R_H, 0)$, and $(q_S(d_3), q_C(d_3)) = (0, R_H)$. Let the optimal mechanism be $\mu^*(d_1|t) = 1 - \frac{S_H(\mu^*)+C_H(\mu^*)}{R_H}$, $\mu^*(d_2|t) = \frac{S_t(\mu^*)}{R_H}$, $\mu^*(d_3|t) = \frac{C_t(\mu^*)}{R_H}$, $\mu^*(d^*|L) = \delta(\mu^*) = \frac{1}{R_H}(S_H(\mu^*) + C_H(\mu^*) - S_L(\mu^*) - C_L(\mu^*))$, with the rest of the $\mu^*(d|t)$'s equal to zero. This solution satisfies $S_t(\mu^*) = \sum_{d\in\Phi\setminus\{d^*\}} \mu^*(d|t) q_S(d) = 0 \cdot \mu^*(d_1|t) + R_H \cdot \mu^*(d_2|t) + 0 \cdot \mu^*(d_3|t)$, and $C_t(\mu^*) = \sum_{d\in\Phi\setminus\{d^*\}} \mu^*(d|t) q_C(d) = 0 \cdot \mu^*(d_1|t) + 0 \cdot \mu^*(d_2|t) +$

$R_H \cdot \mu^*(d_3|t)$. This is a valid solution because $0 \leq S_L(\mu^*) + C_L(\mu^*) \leq S_H(\mu^*) + C_H(\mu^*) \leq R_H$ which makes each $\mu^*(d) \in [0,1]$. Additionally, $\sum_{d\in\Phi} \mu^*(d|L) = \mu^*(d_1|L) + \mu^*(d_2|L) + \mu^*(d_3|L) + \mu^*(d^*|L) = 1$ and $\sum_{d\in\Phi} \mu^*(d|H) = \mu^*(d_1|H) + \mu^*(d_2|H) + \mu^*(d_3|H) = 1$.

None of the expected payoffs $U_S(\mu)$, $U_C(\mu)$, $U_B(\mu, L|L)$ and $U_B(\mu, H|H)$ are zero, as this will make the objective value zero which is not a local maximum. Hence, let $p_L S_L + p_H S_H > 0$, and $R_L(R_H - S_H - C_H) - (S_L + C_L)(R_H - R_L) > 0$ (which makes $R_H - S_H - C_H > 0$); these are later shown to be satisfied by the optimal solution. After taking a log-transformation of the objective, the Lagrangian of the problem is

$$\begin{aligned}\mathcal{L}(S_L, S_H, C_L, C_H, \delta, \lambda) \quad = \quad & \ln(p_L S_L + p_H S_H) + \ln(p_L C_L + p_H C_H) + p_H \ln(R_H - S_H - C_H) \\ & + p_L \ln\Big(R_L(R_H - S_H - C_H) - (S_L + C_L)(R_H - R_L)\Big) \\ & - \lambda_1 S_L - \lambda_2 S_H - \lambda_3 C_L - \lambda_4 C_H - \lambda_5(S_H + C_H - S_L - C_L) \\ & + \lambda_6\Big(\delta - \tfrac{1}{R_H}(S_H + C_H - S_L - C_L)\Big) + \lambda_7(p_L C_L + p_H C_H - r) - \lambda_8 \delta\end{aligned}$$

where $p_L = 1 - p_H$. The KKT conditions will generate the following equations along with the constraints of the problem $(S_N B_Y C_N)'$, $\lambda_1 \geq 0$, $\lambda_2 \geq 0$, $\lambda_3 \geq 0$, $\lambda_4 \geq 0$, $\lambda_5 \geq 0$, and $\lambda_8 \geq 0$,

$$\frac{\partial \mathcal{L}}{\partial S_L}: \quad \frac{p_L}{p_L S_L + p_H S_H} - \frac{p_L(R_H - R_L)}{R_L(R_H - S_H - C_H) - (S_L + C_L)(R_H - R_L)} - \lambda_1 + \lambda_5 + \frac{\lambda_6}{R_H} = 0 \quad \text{(C.20)}$$

$$\frac{\partial \mathcal{L}}{\partial S_H}: \quad \frac{p_H}{p_L S_L + p_H S_H} - \frac{p_H}{R_H - S_H - C_H} - \frac{p_L R_L}{R_L(R_H - S_H - C_H) - (S_L + C_L)(R_H - R_L)} - \lambda_2 - \lambda_5 - \frac{\lambda_6}{R_H} = 0 \quad \text{(C.21)}$$

$$\frac{\partial \mathcal{L}}{\partial C_L}: \quad \frac{p_L}{p_L C_L + p_H C_H} - \frac{p_L(R_H - R_L)}{R_L(R_H - S_H - C_H) - (S_L + C_L)(R_H - R_L)} - \lambda_3 + \lambda_5 + \frac{\lambda_6}{R_H} + p_L \lambda_7 = 0 \quad \text{(C.22)}$$

$$\frac{\partial \mathcal{L}}{\partial C_H}: \quad \frac{p_H}{p_L C_L + p_H C_H} - \frac{p_H}{R_H - S_H - C_H} - \frac{p_L R_L}{R_L(R_H - S_H - C_H) - (S_L + C_L)(R_H - R_L)} - \lambda_4 - \lambda_5 - \frac{\lambda_6}{R_H} + p_H \lambda_7 = 0 \quad \text{(C.23)}$$

$$\frac{\partial \mathcal{L}}{\partial \delta}: \quad \lambda_6 - \lambda_8 = 0 \quad \text{(C.24)}$$

$$\lambda_1 S_L = 0; \quad \lambda_2 S_H = 0; \quad \lambda_3 C_L = 0; \quad \lambda_4 C_H = 0; \quad \lambda_8 \delta = 0 \quad \text{(C.25)}$$

$$\lambda_5(S_H + C_H - S_L - C_L) = 0 \quad \text{(C.26)}$$

$$\lambda_7(p_L C_L + p_H C_H - r) = 0 \quad \text{(C.27)}$$

Eliminating $\lambda_5 + \frac{\lambda_6}{R_H}$ from (C.20) and (C.22) and also from (C.21) and (C.23) gives following equations,

$$p_L\left(\frac{1}{p_L S_L + p_H S_H} - \frac{1}{p_L C_L + p_H C_H}\right) - \lambda_1 + \lambda_3 - p_L \lambda_7 = 0 \quad \text{(C.28)}$$

$$p_H\left(\frac{1}{p_L S_L + p_H S_H} - \frac{1}{p_L C_L + p_H C_H}\right) - \lambda_2 + \lambda_4 - p_H \lambda_7 = 0 \quad \text{(C.29)}$$

We consider following possible cases to solve the KKT equations.

Case 1: $S_L > 0$**,** $S_H > 0$**,** $C_L > 0$**,** $C_H > 0$ **and** $p_L C_L + p_H C_H \neq r$

For this case $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = \lambda_7 = 0$. From (C.28) and (C.29) we get $p_L S_L + p_H S_H = p_L C_L + p_H C_H$ i.e. $U_S = U_C$.

Case 1.a: $S_H + C_H = S_L + C_L$

Equations (C.17) gives $\delta = 0$ which implies $\mu^*(d^*|L) = 0$. Property C.1 along with $\mu^*(d^*|L) = 0$ implies that no positive probability is given to the disagreement alternatives under states L or H. Eliminating C_L from $p_L S_L + p_H S_H = p_L C_L + p_H C_H$ and $S_H + C_H = S_L + C_L$ gives $S_L = \frac{1}{2p_L}\big((p_L - p_H)S_H + C_H\big)$. Also, $R_L(R_H - S_H - C_H) - (S_L + C_L)(R_H - R_L) = R_H(R_L - S_H - C_H)$. Adding (C.20) and (C.21) then replacing S_L gives $\frac{2}{S_H+C_H} - \frac{p_L}{R_L - S_H - C_H} - \frac{p_H}{R_H - S_H - C_H} = 0$. Assigning $S_H + C_H = x$ forms the quadratic equation $3x^2 - \big(3(R_L + R_H) - R\big)x + 2R_L R_H = 0$ (C.30)

The discriminant of the quadratic (C.30) is non-negative since,

$$\begin{aligned}(3(R_L + R_H) - p_L R_L - p_H R_H)^2 - 24R_L R_H &= (3R_L + 2R_H + p_L(R_H - R_L))^2 - 24R_L R_H \\ &\geq (3R_L + 2R_H)^2 - 24R_L R_H \\ &= (3R_L - 2R_H)^2 \;\geq\; 0\end{aligned}$$

Above is always true hence both roots of (C.30) are real. It can be easily verified that the roots are also positive. Using equation (C.14) and the case condition, we get $U_B(\mu, L|L) = R_L - S_L - C_L$. Hence for the solution to be feasible, we need $R_L > x > 0$. Using contradiction, we will show that the smaller root x_s always satisfies the condition $R_L > x_s$ while the larger root x_l does not. Let $R_L \leq x_s$.

$$\begin{aligned}R_L &\leq \tfrac{1}{6}\left(3(R_L + R_H) - R - \sqrt{(3(R_L + R_H) - R)^2 - 24R_L R_H}\right) \\ \Rightarrow\quad (3(R_L + R_H) - R)^2 - 24R_L R_H &\leq (3(R_H - R_L) - R)^2 \\ \Rightarrow\quad 6R_L(6R_H - 2R) &\leq 24R_L R_H \\ \Rightarrow\quad R_H &\leq R\end{aligned}$$

Above is not true for $p_H < 1$. Hence the smaller root x_s satisfies the condition $U_B(\mu, L|L) > 0$ when $p_H \neq 1$. Similar algebra shows that $R_L < x_l$. Hence only the smaller root x_s of the quadratic (C.30) is the solution of this case. From equation(C.15), we have $U_B(\mu, H|H) = R_H - x_s > 0$. The expected payoffs for this case is calculated as $U_S = p_L S_L + p_H S_H = \frac{1}{2}(S_H + C_H) = \frac{1}{2}x_s$, $U_B = p_L U_B(\mu, L|L) + p_H U_B(\mu, H|H) = R - x_s$ and $U_C = \frac{1}{2}x_s$, where $x_s = \frac{1}{6}\left(3(R_L + R_H) - R - \sqrt{(3(R_L + R_H) - R)^2 - 24R_L R_H}\right)$. We need $U_C > r$, i.e.,

$$\begin{aligned}3(R_L + R_H) - R - \sqrt{(3(R_L + R_H) - R)^2 - 24R_L R_H} &> 12r \\ \Rightarrow\quad (3(R_L + R_H) - R)^2 - (3(R_L + R_H) - R - 12r)^2 &< 24R_L R_H \\ \Rightarrow\quad (3(R_L + R_H) - R - 6r)r &< R_L R_H \\ \Rightarrow\quad 3r(R_L + R_H) - p_H(R_H - R_L)r - R_L r - 6r^2 &< R_L R_H \\ \Rightarrow\quad \frac{(R_H - 2r)(3r - R_L)}{r(R_H - R_L)} &< p_H \qquad \text{(C.31)}\end{aligned}$$

For a valid solution, we need $1 \geq p_H > \frac{(R_H - 2r)(3r - R_L)}{r(R_H - R_L)}$, i.e.,

$$\begin{aligned}(R_H - R_L)r &\geq (R_H - 2r)(3r - R_L) \\ \Rightarrow\quad 6r^2 - 2rR_H - 3rR_L + R_L R_H &\geq 0 \\ \Rightarrow\quad (3r - R_H)(2r - R_L) &\geq 0\end{aligned}$$

This is true if either $r \leq \min(\frac{1}{2}R_L, \frac{1}{3}R_H)$, or $r \geq \max(\frac{1}{2}R_L, \frac{1}{3}R_H)$.

Case 1.b: $S_H + C_H > S_L + C_L$

This condition makes $\lambda_5 = 0$. Equation (C.17) gives $\delta > 0$. Therefore from equation (C.25) and (C.24) we get $\lambda_8 = \lambda_6 = 0$ i.e. all the λ's are zero. From (C.20) we get, $R_L(R_H - S_H - C_H) - (S_L + C_L)(R_H - R_L) = (p_L S_L + p_H S_H)(R_H - R_L)$. Replacing this in (C.21) and after some algebra we get,

$$(R_H - S_H - C_H)(p_H R_H - R_L) = p_H(R_H - R_L)(p_L S_L + p_H S_H) \tag{C.32}$$

After eliminating C_L from $p_L S_L + p_H S_H = p_L C_L + p_H C_H$ and (C.20) we get,

$$R_L R_H - (S_H + C_H)\big(R_L + \tfrac{p_H}{p_L}(R_H - R_L)\big) - 2S_L(R_H - R_L) = (R_H - R_L)(p_L S_L + p_H S_H) \tag{C.33}$$

After eliminating C_H from (C.32) and (C.33) we get $p_L S_L + p_H S_H = \frac{1}{3}p_H R_H$. Since $U_S = U_C$ hence, $U_C = \frac{1}{3}p_H R_H$. The IR constraint of the consultant will be satisfied if $p_H > \frac{3r}{R_H}$ and $r < \frac{R_H}{3}$. We can now calculate $U_B(\mu, L|L) = \frac{1}{R_H}(p_L S_L + p_H S_H)(R_H - R_L) = \frac{1}{3}p_H(R_H - R_L) > 0$. Also, from equation (C.32) $U_B(\mu, H|H) = R_H - S_H - C_H = \frac{p_H^2 R_H(R_H - R_L)}{3(p_H R_H - R_L)}$. For $U_B(\mu, H|H) > 0$ we need $p_H > \frac{R_L}{R_H}$. U_B is given by $p_H U_B(\mu, H|H) + p_L U_B(\mu, L|L) = \frac{p_H(R_H - R_L)(p_H R_H - p_L R_L)}{3(p_H R_H - R_L)}$. We need to check if Case I.b condition $S_H + C_H > S_L + C_L$ is satisfied. Writing $U_B(\mu, L|L)R_H$ and $U_B(\mu, H|H)R_H$ as,

$$\begin{aligned} R_L R_H - R_L(S_H + C_H - S_L - C_L) - R_H(S_L + C_L) &= \tfrac{1}{3}p_H R_H(R_H - R_L) &\text{(C.34)} \\ R_H(R_H - S_H - C_H) &= \tfrac{p_H^2 R_H^2(R_H - R_L)}{3(p_H R_H - R_L)} &\text{(C.35)} \end{aligned}$$

Subtracting equation (C.34) from equation (C.35) and after some algebra, we get, $S_H + C_H - S_L - C_L = R_H\left(1 - \frac{p_H R_L}{3(p_H R_H - R_L)}\right)$. Therefore, using equation (C.17), δ is calculated as $\delta = 1 - \frac{p_H R_L}{3(p_H R_H - R_L)}$. For $S_H + C_H > S_L + C_L$ to satisfy, we need $p_H > \frac{3R_L}{3R_H - R_L}$ and $\frac{3}{4}R_H > R_L$. This also satisfies $p_H > \frac{R_L}{R_H}$ and $\delta > 0$. Now, $\frac{3R_L}{3R_H - R_L} \geq \frac{3r}{R_H}$ if $\frac{R_L R_H}{3R_H - R_L} \geq r$. Since $\frac{R_H}{3} > \frac{R_L R_H}{3R_H - R_L}$ for $\frac{3}{4}R_H > R_L$, therefore, one set of feasible conditions is $p_H > \frac{3R_L}{3R_H - R_L}$, $\frac{R_L R_H}{3R_H - R_L} \geq r$, and $\frac{3}{4}R_H > R_L$. Another set of feasible conditions is $p_H > \frac{3r}{R_H}$, $\frac{R_L R_H}{3R_H - R_L} \leq r < \frac{R_H}{3}$, and $\frac{3}{4}R_H > R_L$. There is no feasible solution for this case when $\frac{3}{4}R_H \leq R_L$.

Case 2: $S_L > 0$**,** $S_H > 0$**,** $C_L > 0$**,** $C_H > 0$ **and** $p_L C_L + p_H C_H = r$

For this case $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = 0$ and $U_C = r$.

Case 2.a: $S_H + C_H = S_L + C_L$

Equations (C.17) implies $\mu^*(d^*|L) = 0$. Eliminating C_L from $p_L C_L + p_H C_H = r$ and $S_H + C_H = S_L + C_L$ gives $p_L S_L = p_L S_H + C_H - r$. Also, $R_L(R_H - S_H - C_H) - (S_L + C_L)(R_H - R_L) = R_H(R_L - S_H - C_H)$. Adding (C.20) and (C.21) then replacing S_L gives $\frac{1}{S_H + C_H - r} - \frac{p_L}{R_L - S_H - C_H} - \frac{p_H}{R_H - S_H - C_H} = 0$. Assigning $S_H + C_H = x$ forms the quadratic equation $2x^2 - \big(2(R_L + R_H) - R + r\big)x + R_L R_H + r(R_L + R_H - R) = 0$ (C.36)

Let the smaller and larger roots be x_s and x_l respectively. Using equation (C.14) and the case condition, we get $U_B(\mu, L|L) = R_L - x$. For a feasible solution we need $R_L > x > 0$. Following shows that the smaller root x_s satisfies the condition $R_L > x_s$ when $R_L > r$.

$$\begin{array}{lrcl} & \frac{1}{4}\left(2(R_L+R_H)-R+r-\sqrt{(2(R_L+R_H)-R+r)^2-8(R_LR_H+r(R_L+R_H-R))}\right) & < & R_L \\ \Rightarrow & (2(R_L+R_H)-R+r)^2-(2(R_H-R_L)-R+r)^2-8(R_LR_H+r(R_L+R_H-R)) & > & 0 \\ \Rightarrow & R_L(2R_H-R+r)-R_LR_H-r(R_L+R_H-R) & > & 0 \\ \Rightarrow & (R_H-R_L)(R_L-r) & > & 0 \end{array}$$

Above is true for $R_L > r$. We also have, $U_S = x - r$. Following shows that $R_L > r$ also satisfies $U_S > 0$.

$$\begin{array}{lrcl} & 2(R_L+R_H)-R+r-\sqrt{(2(R_L+R_H)-R+r)^2-8(R_LR_H+r(R_L+R_H-R))} & > & 4r \\ \Rightarrow & (2(R_L+R_H)-R+r)^2-(2(R_L+R_H)-R-3r)^2-8(R_LR_H+r(R_L+R_H-R)) & < & 0 \\ \Rightarrow & r(2(R_L+R_H)-R-r)-R_LR_H-r(R_L+R_H-R) & < & 0 \\ \Rightarrow & (r-R_H)(R_L-r) & < & 0 \end{array}$$

Above is true for $R_L > r$. Therefore, x_s satisfies all the conditions and is the solution when $R_L > r$. Similar analysis shows that the larger root x_l is the solution when $R_L < r < R_H$. From (C.15) we get $U_B(\mu, H|H) = R_H - x$. Therefore, $U_B = R - x$.

Case 2.b: $S_H + C_H > S_L + C_L$

This condition makes $\lambda_5 = 0$ and $\delta > 0$. Equation (C.32) is obtained as described in section C.30. After eliminating C_L from $p_LC_L + p_HC_H = r$ and (C.20) we get,

$$p_LR_L(R_H - S_H) + C_H(p_HR_H - R_L) - (p_LS_L + r)(R_H - R_L) = p_L(R_H - R_L)(p_LS_L + p_HS_H) \quad \text{(C.37)}$$

Adding equations (C.32) and (C.37) we get $p_LS_L + p_HS_H = \frac{1}{2}(p_HR_H - r)$. The IR constraint for the seller will be satisfied if $p_H > \frac{r}{R_H}$ and $r \leq R_H$. From (C.20) we get $U_B(\mu, L|L) = \frac{1}{R_H}(p_LS_L + p_HS_H)(R_H - R_L) = \frac{1}{2R_H}(p_HR_H - r)(R_H - R_L)$. $U_B(\mu, L|L) > 0$ for $p_H > \frac{r}{R_H}$. Also, from equation (C.32) $U_B(\mu, H|H) = R_H - S_H - C_H = \frac{p_H(p_HR_H - r)(R_H - R_L)}{2(p_HR_H - R_L)}$. Since $p_H > \frac{r}{R_H}$, for $U_B(\mu, H|H) > 0$ we need $p_H > \frac{R_L}{R_H}$. U_B is given by $p_HU_B(\mu, H|H) + p_LU_B(\mu, L|L) = \frac{(p_HR_H - r)(R_H - R_L)(p_HR_H - p_LR_L)}{2R_H(p_HR_H - R_L)}$. We need to check if Case 2.b condition $S_H + C_H > S_L + C_L$ is satisfied. Writing $U_B(\mu, L|L)R_H$ and $U_B(\mu, H|H)R_H$ as,

$$R_LR_H - R_L(S_H + C_H - S_L - C_L) - R_H(S_L + C_L) = \frac{1}{2}(p_HR_H - r)(R_H - R_L) \quad \text{(C.38)}$$

$$R_H(R_H - S_H - C_H) = \frac{p_HR_H(p_HR_H - r)(R_H - R_L)}{2(p_HR_H - R_L)} \quad \text{(C.39)}$$

Subtracting equation (C.38) from equation (C.39) we get,

$$\begin{array}{lrcl} & (R_H-R_L)(R_H-S_H-C_H+S_L+C_L) & = & \frac{1}{2}(p_HR_H-r)(R_H-R_L)\left(\frac{p_HR_H}{p_HR_H-R_L}-1\right) \\ \Rightarrow & R_H-(S_H+C_H-S_L-C_L) & = & \frac{R_L(p_HR_H-r)}{2(p_HR_H-R_L)} \\ \Rightarrow & S_H+C_H-S_L-C_L & = & \frac{p_HR_H(2R_H-R_L)-R_L(2R_H-r)}{2(p_HR_H-R_L)} \end{array}$$

For $S_H+C_H > S_L+C_L$ to satisfy, we need $p_H > \frac{R_L(2R_H-r)}{R_H(2R_H-R_L)}$. For $R_L > r$ we have $\frac{R_L(2R_H-r)}{R_H(2R_H-R_L)} > \frac{R_L}{R_H} > \frac{r}{R_H}$. Therefore, this case is feasible if either $p_H > \frac{R_L(2R_H-r)}{R_H(2R_H-R_L)}$ with $R_L > r$ or $p_H > \frac{r}{R_H}$ with $R_H \geq r > R_L$.

Case 3: $S_L = 0$**,** $S_H > 0$**,** $C_L > 0$**,** $C_H > 0$**, and** $p_LC_L + p_HC_H \neq r$

We have $\lambda_2 = \lambda_3 = \lambda_4 = \lambda_7 = 0$ and $p_HS_H = p_LC_L + p_HC_H$ i.e. $U_S = U_C$. From equation (C.28) $\lambda_1 = 0$.

Case 3.a: $S_H + C_H = C_L$

From $p_HS_H = p_LC_L + p_HC_H$ and $S_H + C_H = C_L$ we get, $2p_HS_H = C_L$. Adding (C.20) and (C.21) and using $2p_HS_H = C_L$ gives $\frac{2}{C_L} - \frac{p_L}{R_L-C_L} - \frac{p_H}{R_H-C_L} = 0$. Rest of the analysis is same as Case 1.a of section C.1 where $C_L = x_s$. This gives, $S_H = \frac{x_s}{2p_H}$ and $C_H = S_H(2p_H - 1)$. $C_H > 0$ will be satisfied for $p_H > 0.5$. U_S, U_C, $U_B(\mu, H|H)$ and $U_B(\mu, L|L)$ remains same as that in Case 1.a. Therefore, the solution is feasible for $p_H > 0.5$ and the range of r discussed in Case 1.a.

Case 3.b: $S_H + C_H > C_L$

Similar to Case 1.b we get $\delta > 0$. Undergoing the same process as that of Case 1.b gives $U_S = \frac{1}{3}p_HR_H$ which implies $S_H = \frac{R_H}{3}$. Also, $U_B(\mu, H|H) = R_H - S_H - C_H = \frac{p_H^2R_H(R_H-R_L)}{3(p_HR_H-R_L)}$. This will result in $C_H = \frac{R_H}{3}\left(2 - \frac{p_H^2(R_H-R_L)}{p_HR_H-R_L}\right)$. Furthermore, $U_B(\mu, H|H) > 0$ implies $p_H > \frac{R_L}{R_H} = \beta$ (say). Using $p_LC_L + p_HC_H = \frac{p_HR_H}{3}$ we get $C_L = \frac{p_HR_H}{3p_L}\left(\frac{p_H^2(R_H-R_L)}{p_HR_H-R_L} - 1\right)$. Now, $C_H > 0$ is satisfied if $2 > \frac{p_H^2(R_H-R_L)}{p_HR_H-R_L}$ i.e. $p_H^2(1-\beta) - 2p_H + 2\beta < 0$ (since $p_H > \beta$). The larger root $\frac{1+\sqrt{(1-\beta)^2+\beta^2}}{1-\beta}$ of this quadratic is greater than 1 hence, $p_H > \frac{1-\sqrt{(1-\beta)^2+\beta^2}}{1-\beta}$. Also, $C_L > 0$ if $\frac{p_H^2(R_H-R_L)}{p_HR_H-R_L} > 1$ i.e. $p_H < \frac{\beta}{1-\beta}$. From Case 1.b, we observe that $S_H + C_H > C_L$ is satisfied for $p_H > \frac{3\beta}{3-\beta}$ and $\frac{3}{4} > \beta$. It can be easily shown that $\frac{\beta}{1-\beta} > \frac{3\beta}{3-\beta} \geq \beta$ and $\frac{\beta}{1-\beta} > \frac{1-\sqrt{(1-\beta)^2+\beta^2}}{1-\beta}$ are always true for $0 < \beta < 1$. In addition, $U_C > r$ is satisfied for $p_H > \frac{3r}{R_H}$ and $r < \frac{R_H}{3}$. Furthermore, $\frac{R_L}{R_H-R_L} > \frac{3r}{R_H}$ when $r \leq \frac{R_LR_H}{3(R_H-R_L)}$. Therefore for $\frac{R_L}{R_H-R_L} > p_H > \frac{3r}{R_H}$ to satisfy we need $\frac{R_L}{R_H-R_L} > \frac{3r}{R_H} > \frac{3R_L}{3R_H-R_L}$ i.e. $\frac{R_LR_H}{3(R_H-R_L)} > r > \frac{R_LR_H}{3R_H-R_L}$. Since $1 \leq p_H$, therefore we also need $r \leq \frac{R_H}{3}$. Moreover, $\frac{R_L}{R_H-R_L} > p_H > \frac{3R_L}{3R_H-R_L}$ is satisfied when $\frac{R_L}{R_H-R_L} > \frac{3R_L}{3R_H-R_L} > \frac{3r}{R_H}$ i.e. $\frac{R_LR_H}{3R_H-R_L} > r$. For $p_H < 1$ we need $\frac{3}{4}R_H > R_L$.

Case 4: $S_L = 0$**,** $S_H > 0$**,** $C_L > 0$**,** $C_H > 0$**, and** $p_LC_L + p_HC_H = r$

We have $\lambda_2 = \lambda_3 = \lambda_4 = 0$ and $U_C = r$.

Case 4.a: $S_H + C_H = C_L$

Eliminating C_H from $p_LC_L + p_HC_H = r$ and $S_H + C_H = C_L$ gives, $p_HS_H = C_L - r$. Multiplying equation (C.22) by p_H and equation (C.23) by p_L and eliminating λ_7 yields

$$\frac{p_Lp_H}{R_H-S_H-C_H} + \frac{p_L^2R_L-p_Lp_H(R_H-R_L)}{R_H(R_L-S_H-C_H)} + \lambda_5 + \frac{\lambda_6}{R_H} = 0 \tag{C.40}$$

Adding equation (C.21) to (C.40) gives, $\frac{1}{p_HS_H} - \frac{p_H}{R_H-S_H-C_H} - \frac{p_L}{R_L-S_H-C_H} = 0$ which, after using $p_HS_H = C_L - r$ and $S_H + C_H = C_L$, becomes $\frac{1}{C_L-r} - \frac{p_H}{R_H-C_L} - \frac{p_L}{R_L-C_L} = 0$. Rest of the analysis is same as that in Case 2.a where,

$S_H + C_H = C_L = x = \frac{1}{4}\left(2(R_L + R_H) - R + r \pm \sqrt{(2(R_L + R_H) - R + r)^2 - 8(R_L R_H + r(R_L + R_H - R))}\right)$

U_S, U_C, $U_B(\mu, H|H)$ and $U_B(\mu, L|L)$ remains same as that in Case 2.a. In addition, $p_L x + p_H C_H = r$ gives, $C_H = \frac{1}{p_H}(r - p_L x)$. From $S_H + C_H = C_L = x$ we get, $S_H = \frac{1}{p_H}(x - r)$. The smaller root x_s is the solution of Case 2.a when $R_L > r$ while the larger root x_l is the solution when $R_L < r$. However, we need to add the constraint $r > p_L x$ corresponding to $C_H > 0$. Constraint $S_H > 0$ is satisfied by $U_S > 0$ which is the solution of Case 2.a.

Case 4.b: $S_H + C_H > C_L$

We have $\lambda_5 = \lambda_8 = \lambda_6 = 0$. Multiplying equation (C.22) by p_H and equation (C.23) by p_L and eliminating λ_7 will give equation (C.40) which can be written as,

$$R_L(R_H - S_H - C_H) - C_L(R_H - R_L) = \frac{1}{p_H}(R_H - S_H - C_H)(p_H R_H - R_L) \tag{C.41}$$

Replacing R.H.S. of (C.41) in (C.21) gives,

$$(R_H - S_H - C_H)(p_H R_H - R_L) = p_H^2 S_H (R_H - R_L) \tag{C.42}$$

Eliminating C_L from (C.41) and $p_L C_L + p_H C_H = r$ gives following after some algebra,

$$(R_L - p_H(R_H - R_L))(R_H - S_H)p_L + C_H(p_H R_H - R_L) = r p_H (R_H - R_L) \tag{C.43}$$

Adding (C.42) and (C.43) gives $U_S = p_H S_H = \frac{1}{2}(p_H R_H - r)$ after some algebra. Rest of the analysis is same as of Case 2.b. where same values of $U_B(\mu, L|L)$, $U_B(\mu, H|H)$, and U_B are obtained under same conditions.

Case 5: $S_L > 0$**,** $S_H = 0$**,** $C_L > 0$**,** $C_H > 0$**, and** $p_L C_L + p_H C_H \neq r$

We have $\lambda_1 = \lambda_3 = \lambda_4 = \lambda_7 = 0$. For $p_L \neq 0$ we have from equation (C.28) $p_L S_L = p_L C_L + p_H C_H$ i.e. $U_S = U_C$. Hence from equation (C.29) $\lambda_2 = 0$.

Case 5.a: $C_H = S_L + C_L$

The analysis is same as that in Case 1.a. where,

$C_H = S_L + C_L = x_s = \frac{1}{6}\left(3(R_L + R_H) - R - \sqrt{(3(R_L + R_H) - R)^2 - 24 R_L R_H}\right)$

From $p_L S_L = p_L C_L + p_H x_s$ and $S_L + C_L = x_s$ we get, $S_L = \frac{x_s}{2p_L}$ and $C_L = S_L(2p_L - 1)$. Along with the conditions of Case 1.a we also need $p_H < 0.5$ which satisfies $C_L > 0$.

Case 5.b: $C_H > S_L + C_L$

The analysis is same as that in Case 1.b. Therefore, $S_L = \frac{p_H R_H}{3p_L}$ and $R_H - C_H = \frac{p_H^2 R_H (R_H - R_L)}{3(p_H R_H - R_L)}$. This will result in $C_H = \frac{R_H}{3}\left(3 - \frac{p_H^2(R_H - R_L)}{p_H R_H - R_L}\right)$. Using $p_L C_L + p_H C_H = \frac{p_H R_H}{3}$ we get $C_L = \frac{p_H R_H}{3p_L}\left(\frac{p_H^2(R_H - R_L)}{p_H R_H - R_L} - 2\right)$. The feasibility condition of Case 1.b states $p_H > \max\left(\frac{3r}{R_H}, \frac{3R_L}{3R_H - R_L}\right) > \frac{R_L}{R_H} = \beta$ (say). $C_H > 0$ will be satisfied if $(1-\beta)p_H^2 - 3p_H + 3\beta < 0$. In addition, $C_L > 0$ will be satisfied if $(1-\beta)p_H^2 - 2p_H + 2\beta > 0$. It is observed that to satisfy both $C_H > 0$ and $C_L > 0$ we need $\frac{3-\sqrt{9-12\beta(1-\beta)}}{2(1-\beta)} < p_H < \frac{1-\sqrt{(1-\beta)^2+\beta^2}}{1-\beta}$. Since $\frac{3\beta}{3-\beta} > \frac{1-\sqrt{(1-\beta)^2+\beta^2}}{1-\beta}$ is always true for $0 < \beta < 1$, hence there is no solution for $p_H > \beta$.

Case 6: $S_L > 0$**,** $S_H = 0$**,** $C_L > 0$**,** $C_H > 0$**, and** $p_L C_L + p_H C_H = r$

We have $\lambda_1 = \lambda_3 = \lambda_4 = 0$ and $U_C = r$.

Case 6.a: $C_H = S_L + C_L$

The analysis remains same as that in Case 2.a where $C_H = S_L + C_L = x$ and x is the solution of the quadratic equation (C.36). In addition, $p_L C_L + p_H x = r$ gives, $C_L = \frac{1}{p_L}(r - p_H x)$. From $C_H = S_L + C_L = x$ we get, $S_L = \frac{1}{p_H}(x - r)$. Along with the feasibility condition of Case 2.a we also need $r > p_H x$ corresponding to $C_L > 0$.

Case 6.b: $C_H > S_L + C_L$

The analysis is same as in Case 2.b i.e. $U_S = p_L S_L = \frac{1}{2}(p_H R_H - r)$. From $R_H - C_H = \frac{p_H(p_H R_H - r)(R_H - R_L)}{2(p_H R_H - R_L)}$ we get, $C_H = R_H - \frac{p_H(p_H R_H - r)(R_H - R_L)}{2(p_H R_H - R_L)}$. For $C_H > 0$ we need $\frac{p_H(2R_H^2 + r(R_H - R_L)) - 2R_L R_H - (R_H - R_L)R_H p_H^2}{2(p_H R_H - R_L)} > 0$. Replacing C_H in $p_L C_L + p_H C_H = r$ gives, $C_L = \frac{(p_H R_H - r)((R_H - R_L)p_H^2 - 2R_H p_H + 2R_L)}{2p_L(p_H R_H - R_L)}$. In Case 2.b it is shown that for a feasible solution we need $p_H > \max\left(\frac{r}{R_H}, \frac{R_L}{R_H}\right)$. Therefore, for $C_H > 0$ and $C_L > 0$ we need,

$$(1-\beta)p_H^2 - \left(2 + (1-\beta)\frac{r}{R_H}\right)p_H + 2\beta < 0 \quad \text{(C.44)}$$

$$(1-\beta)p_H^2 - 2p_H + 2\beta > 0 \quad \text{(C.45)}$$

where, $\beta = \frac{R_L}{R_H}$. As discussed in Case 3.b, the larger root of quadratic (C.45) is greater than 1. Also, the larger root of quadratic (C.44) is greater than that of quadratic (C.45). The feasible value of p_H is found between the smaller roots of (C.44) and (C.45) along with the feasibility condition obtained in Case 2.b.

Case 7: $S_L > 0$**,** $S_H > 0$**,** $C_L = 0$**,** $C_H > 0$**, and** $p_H C_H \neq r$

We have $\lambda_1 = \lambda_2 = \lambda_4 = \lambda_7 = 0$, and $U_C = U_S$ i.e. $p_L S_L + p_H S_H = p_H C_H$. Therefore, $\lambda_2 = 0$.

Case 7.a: $S_H + C_H = S_L$

The analysis is same as that in Case 1.a. where,

$S_H + C_H = S_L = x_s = \frac{1}{6}\left(3(R_L + R_H) - R - \sqrt{(3(R_L + R_H) - R)^2 - 24R_L R_H}\right)$

From $p_L x_s + p_H S_H = p_H C_H$ and $S_H + C_H = x_s$ we get, $C_H = \frac{x_s}{2p_H}$ and $S_H = C_H(2p_H - 1)$. Along with the conditions of Case 1.a we also need $p_H > 0.5$ which satisfies $S_H > 0$.

Case 7.b: $S_H + C_H > S_L$

Similar to Case 1.b we get $\delta > 0$. Undergoing the same process as that of Case 1.b gives $U_S = U_C = \frac{1}{3}p_H R_H$ which implies $C_H = \frac{R_H}{3}$. Also, $U_B(\mu, H|H) = R_H - S_H - C_H = \frac{p_H^2 R_H(R_H - R_L)}{3(p_H R_H - R_L)}$ will result in $S_H = \frac{R_H}{3}\left(2 - \frac{p_H^2(R_H - R_L)}{p_H R_H - R_L}\right)$. Rest of the analysis is similar to Case 3.b where, $S_L = \frac{p_H R_H}{3p_L}\left(\frac{p_H^2(R_H - R_L)}{p_H R_H - R_L} - 1\right)$. The feasible range is same as that of Case 3.b.

Case 8: $S_L > 0$**,** $S_H > 0$**,** $C_L = 0$**,** $C_H > 0$**, and** $p_H C_H = r$

We have $\lambda_1 = \lambda_2 = \lambda_4 = 0$, $U_C = r$ and $C_H = \frac{r}{p_H}$.

Case 8.a: $C_H + S_H = S_L$

The analysis remains same as that in Case 2.a where $C_H + S_H = S_L = x$. In addition, $S_H = x - \frac{r}{p_H}$. Along with the feasibility condition of Case 2.a we also need $p_H x > r$ corresponding to $S_H > 0$.

Case 8.b: $C_H + S_H > S_L$

The analysis is same as in Case 2.b i.e. $U_S = \frac{1}{2}(p_H R_H - r)$. From $R_H - S_H - C_H = \frac{p_H(p_H R_H - r)(R_H - R_L)}{2(p_H R_H - R_L)}$ we get, $S_H = \frac{(p_H R_H - r)(2p_H R_H - (R_H - R_L)p_H^2 - 2R_L)}{2p_H(p_H R_H - R_L)}$. Replacing S_H in $p_L S_L + p_H S_H = \frac{1}{2}(p_H R_H - r)$ gives, $S_L = \frac{(p_H R_H - r)((R_H - R_L)p_H^2 - p_H R_H + R_L)}{2p_L(p_H R_H - R_L)}$. In Case 2.b it is shown that for a feasible solution we need $p_H > \max\left(\frac{r}{R_H}, \frac{R_L}{R_H}\right)$. Therefore, for $S_H > 0$ and $S_L > 0$ we need,

$$\begin{aligned}(1-\beta)p_H^2 - 2p_H + 2\beta &< 0 \quad \text{(C.46)}\\ (1-\beta)p_H^2 - p_H + \beta &> 0 \quad \text{(C.47)}\end{aligned}$$

where, $\beta = \frac{R_L}{R_H}$. Quadratic (C.46) is solved in Case 3.b which gives $p_H > \frac{1-\sqrt{(1-\beta)^2+\beta^2}}{1-\beta}$. Also, quadratic (C.47) gives $p_H < \frac{\beta}{1-\beta}$. Since $\frac{\beta}{1-\beta} > \frac{1-\sqrt{(1-\beta)^2+\beta^2}}{1-\beta}$ for $0 < \beta < 1$ hence, this case will be feasible for $\frac{\beta}{1-\beta} > p_H > \frac{1-\sqrt{(1-\beta)^2+\beta^2}}{1-\beta}$ along with the feasibility condition obtained in Case 2.b.

Case 9: $S_L > 0$**,** $S_H > 0$**,** $C_L > 0$**,** $C_H = 0$**, and** $p_L C_L \neq r$

We have $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_7 = 0$, and $U_C = U_S$ i.e. $p_L S_L + p_H S_H = p_L C_L$.

Case 9.a: $S_H = S_L + C_L$

The analysis is same as that in Case 1.a. where $S_H = S_L + C_L = x_s$. From $p_L S_L + p_H x = p_L C_L$ we get, $C_L = \frac{x_s}{2p_L}$ and $S_L = C_L(2p_L - 1)$. Along with the conditions of Case 1.a we also need $p_H < 0.5$ which satisfies $S_L > 0$.

Case 9.b: $S_H > S_L + C_L$

The analysis is similar to Case 1.b. Therefore, $C_L = \frac{p_H R_H}{3p_L}$ and $R_H - S_H = \frac{p_H^2 R_H(R_H - R_L)}{3(p_H R_H - R_L)}$. This will result in $S_H = \frac{R_H}{3}\left(3 - \frac{p_H^2(R_H - R_L)}{p_H R_H - R_L}\right)$. Using $p_L S_L + p_H S_H = \frac{p_H R_H}{3}$ we get $S_L = \frac{p_H R_H}{3p_L}\left(\frac{p_H^2(R_H - R_L)}{p_H R_H - R_L} - 2\right)$. The feasibility condition of Case 1.b states $p_H > \max\left(\frac{3r}{R_H}, \frac{3R_L}{3R_H - R_L}\right) > \frac{R_L}{R_H} = \beta$ (say). $S_H > 0$ will be satisfied if $(1-\beta)p_H^2 - 3p_H + 3\beta < 0$. In addition, $S_L > 0$ will be satisfied if $(1-\beta)p_H^2 - 2p_H + 2\beta > 0$. It is observed that to satisfy both $S_H > 0$ and $S_L > 0$ we need $\frac{3-\sqrt{9-12\beta(1-\beta)}}{2(1-\beta)} < p_H < \frac{1-\sqrt{(1-\beta)^2+\beta^2}}{1-\beta}$. Since $\frac{3\beta}{3-\beta} > \frac{1-\sqrt{(1-\beta)^2+\beta^2}}{1-\beta}$ is always true for $0 < \beta < 1$, hence there is no solution for $p_H > \beta$.

Case 10: $S_L > 0$**,** $S_H > 0$**,** $C_L > 0$**,** $C_H = 0$**, and** $p_L C_L = r$

We have $\lambda_1 = \lambda_2 = \lambda_3 = 0, U_C = r$ and $C_L = \frac{r}{p_L}$

Case 10.a: $S_H = S_L + C_L$

The analysis remains same as that in Case 2.a where $S_H = S_L + C_L = x$ and x is the solution of the quadratic equation (C.36). This gives, $S_L = x - \frac{r}{p_L}$. Along with the feasibility condition of Case 2.a we also need $p_L x > r$ corresponding to $S_L > 0$.

Case 10.b: $C_H > S_L + C_L$

The analysis is same as in Case 2.b i.e. $U_S = \frac{1}{2}(p_H R_H - r)$. From $R_H - S_H = \frac{p_H(p_H R_H - r)(R_H - R_L)}{2(p_H R_H - R_L)}$ we get, $S_H = R_H - \frac{p_H(p_H R_H - r)(R_H - R_L)}{2(p_H R_H - R_L)} = \frac{p_H(2R_H^2 + r(R_H - R_L)) - 2R_L R_H - (R_H - R_L) R_H p_H^2}{2(p_H R_H - R_L)}$. Replacing S_H in $p_L S_L + p_H S_H = \frac{1}{2}(p_H R_H - r)$ gives, $S_L = \frac{(p_H R_H - r)(R_H - R_L) p_H^2 - (p_H R_H - R_L)(p_H R_H + r)}{2 p_L (p_H R_H - R_L)}$. The rest of the analysis is similar to Case 8.b where a feasible value of p_H is obtained to satisfy $S_H > 0$ and $S_L > 0$ along with the conditions stated in Case 2.b.

Case 11: $S_L = 0$, $S_H > 0$, $C_L = 0$, $C_H > 0$, **and** $p_H C_H \neq r$

This gives $\lambda_2 = \lambda_4 = \lambda_7 = 0$, and $U_C = U_S$ i.e. $S_H = C_H$. This automatically satisfies $S_H + C_H > S_L + C_L$. Therefore, $\lambda_5 = \lambda_8 = \lambda_6 = 0$. From equation (C.21) we get $S_H = C_H = \frac{1}{3} R_H$. Therefore, $U_S = U_C = p_H S_H = \frac{1}{3} p_H R_H$, $U_B(\mu, H|H) = \frac{1}{3} R_H$, $U_B(\mu, L|L) = \frac{1}{3} R_L$, and $U_B = \frac{1}{3} R$. For feasibility we need $U_C > r$ i.e. $p_H > \frac{3r}{R_H}$.

Case 12: $S_L = 0$, $S_H > 0$, $C_L = 0$, $C_H > 0$, **and** $p_H C_H = r$

For this case we have $\lambda_2 = \lambda_4 = 0$ and $C_H = \frac{r}{p_H}$. Since $S_H + C_H > S_L + C_L$, we get $\lambda_5 = \lambda_8 = \lambda_6 = 0$. From equation (C.21) we get $S_H = \frac{1}{2}\left(R_H - \frac{r}{p_H}\right)$. Therefore, $U_S = p_H S_H = \frac{1}{2}(p_H R_H - r)$, $U_B(\mu, H|H) = R_H - S_H - C_H = S_H = \frac{1}{2}\left(R_H - \frac{r}{p_H}\right)$, $U_B(\mu, L|L) = \frac{R_L}{R_H}(R_H - S_H - C_H) = \frac{R_L}{2R_H}(R_H - \frac{r}{p_H})$, and $U_B = \frac{R}{2}\left(1 - \frac{r}{p_H R_H}\right)$. This case is feasible if $p_H > \frac{r}{R_H}$.

Case 13: $S_L = 0$, $S_H > 0$, $C_L > 0$, $C_H = 0$, **and** $p_L C_L \neq r$

This gives $\lambda_2 = \lambda_3 = \lambda_7 = 0$. When $S_H = C_L$, we add equations (C.21) and (C.22) to get the quadratic equation (C.30) where $S_H = C_L = x_s$. The analysis is same as Case 1.a. where

$x_s = \frac{1}{6}\left(3(R_L + R_H) - R - \sqrt{(3(R_L + R_H) - R)^2 - 24 R_L R_H}\right)$

Therefore, $U_S = p_H x_s$, $U_C = p_L x_s$, $U_B(\mu, H|H) = R_H - x_s$, $U_B(\mu, L|L) = R_L - x_s$, and $U_B = R - x_s$. For a feasible solution we need $R_L > x_s$ which is shown to be true in Case 1.a and we also need $U_C > r$ i.e. $p_L x_s > r$.

When $S_H > C_L$, we get $\lambda_5 = 0$. Since $\delta > 0$ hence, $\lambda_8 = \lambda_6 = 0$. Therefore, from equations (C.21) and (C.22) we get,

$$\begin{aligned} \frac{1}{S_H} - \frac{p_H}{R_H - S_H} - \frac{p_L R_L}{R_L(R_H - S_H) - C_L(R_H - R_L)} &= 0 \\ \frac{1}{C_L} - \frac{p_L(R_H - R_L)}{R_L(R_H - S_H) - C_L(R_H - R_L)} &= 0 \end{aligned}$$

Solving above equations gives $S_H = \frac{R_H}{3}$ and $C_L = \frac{2R_LR_H}{3(1+p_L)(R_H-R_L)}$. Therefore, $U_S = p_HS_H = \frac{1}{3}p_HR_H$, $U_C = p_LC_L = \frac{2p_LR_LR_H}{3(1+p_L)(R_H-R_L)}$. Also, $U_B(\mu, H|H) = R_H - S_H = \frac{2}{3}R_H$ and $U_B(\mu, L|L) = \frac{2}{3}\frac{p_LR_L}{1+p_L}$. This gives, $U_B = \frac{2}{3}\frac{p_LR+p_HR_H}{1+p_L}$ where $R = p_LR_L + p_HR_H$. To satisfy the condition $S_H > C_L$ we need $p_H < \frac{2R_H-4R_L}{R_H-R_L}$. For $p_H \geq 0$ we need $\frac{R_H}{2} \geq R_L$.

Case 14: $S_L = 0$**,** $S_H > 0$**,** $C_L > 0$**,** $C_H = 0$**, and** $p_LC_L = r$

$U_C = p_LC_L = r$ gives $C_L = \frac{r}{p_L}$. When $S_H = C_L$, we get $U_S = p_HS_H = \frac{rp_H}{p_L}$, $U_B(\mu, H|H) = R_H - S_H = R_H - \frac{r}{p_L}$ and $U_B(\mu, L|L) = R_L - \frac{r}{p_L}$. Therefore, $U_B = R - \frac{r}{p_L}$. For $U_B(\mu, L|L) > 0$ we need, $p_H < 1 - \frac{r}{R_L}$. For feasible p_H we need $r < R_L$.

When $S_H > C_L > 0$, we have $\lambda_5 = \lambda_8 = \lambda_6 = 0$ along with $\lambda_2 = \lambda_3 = 0$. Denoting $R_H - S_H = y$, equation (C.21) becomes,

$$\begin{aligned} \frac{1}{R_H-y} - \frac{p_H}{y} - \frac{p_LR_L}{R_Ly-C_L(R_H-R_L)} &= 0 \\ \Rightarrow \quad y(R_Ly - C_L(R_H-R_L)) - p_H(R_H-y)(R_Ly - C_L(R_H-R_L)) - p_LR_Ly(R_H-y) &= 0 \\ \Rightarrow \quad 2R_Ly^2 - (R_LR_H + r(R_H-R_L))y - p_HC_LR_H(R_H-R_L) &= 0 \end{aligned}$$

It is easy to check that the determinant of the above quadratic is always non-negative. Since, the smaller root will be negative hence, we get,

$y = \frac{1}{4R_L}\left(R_LR_H + r(R_H-R_L) + \sqrt{(R_LR_H + r(R_H-R_L))^2 + 8R_LR_HC_L(R_H-R_L)}\right)$

Therefore, $U_S = p_HS_H = p_H(R_H - y)$, $U_B(\mu, H|H) = y$, $U_B(\mu, L|L) = \frac{1}{R_H}(R_Ly - C_L(R_H-R_L))$, and $U_B = \frac{1}{R_H}(Ry - r(R_H-R_L))$. Also, $S_H > C_L$ gives $R_H - \frac{r}{p_L} > y$ and $U_B(\mu, L|L) > 0$ gives $y > \frac{r}{R_Lp_L}(R_H-R_L)$. For a feasible solution we need, $R_H - \frac{r}{p_L} > \frac{r}{R_Lp_L}(R_H-R_L)$ i.e. $p_H < 1 - \frac{r}{R_L}$. Since $p_H \geq 0$ therefore, we also need $r < R_L$.

Case 15: $S_L > 0$**,** $S_H = 0$**,** $C_L = 0$**,** $C_H > 0$**, and** $p_HC_H \neq r$

This gives $\lambda_1 = \lambda_4 = \lambda_7 = 0$. When $S_L = C_H$, we add equations (C.20) and (C.23) to get the quadratic equation (C.30) where $S_L = C_H = x_s$. The x_s is given in Case 1.a. Therefore, $U_S = p_Lx_s$, $U_C = p_Hx_s$, $U_B(\mu, H|H) = R_H - x_s$, $U_B(\mu, L|L) = R_L - x_s$, and $U_B = R - x_s$. For a feasible solution we need $R_L > x_s$ which is shown to be true in Case 1.a and we also need $U_C > r$ i.e. $p_Hx_s > r$.

When $C_H > S_L$, we get $\lambda_5 = 0$. Since $\delta > 0$ hence, $\lambda_8 = \lambda_6 = 0$. Therefore, from equations (C.20) and (C.23) we get,

$$\begin{aligned} \frac{1}{S_L} - \frac{p_L(R_H-R_L)}{R_L(R_H-C_H)-S_L(R_H-R_L)} &= 0 \\ \frac{1}{C_H} - \frac{p_H}{R_H-C_H} - \frac{p_LR_L}{R_L(R_H-C_H)-S_L(R_H-R_L)} &= 0 \end{aligned}$$

Solving above equations gives $C_H = \frac{R_H}{3}$ and $S_L = \frac{2R_LR_H}{3(1+p_L)(R_H-R_L)}$. Therefore, $U_C = p_HC_H = \frac{1}{3}p_HR_H$, $U_S = p_LS_L = \frac{2p_LR_LR_H}{3(1+p_L)(R_H-R_L)}$. Also, $U_B(\mu, H|H) = R_H - C_H = \frac{2}{3}R_H$ and $U_B(\mu, L|L) = \frac{2}{3}\frac{p_LR_L}{1+p_L}$. This gives,

$U_B = \frac{2}{3}\frac{p_L R + p_H R_H}{1+p_L}$ where $R = p_L R_L + p_H R_H$. To satisfy the condition $C_H > S_L$ we need $p_H < \frac{2R_H - 4R_L}{R_H - R_L}$. For $p_H \geq 0$ we need $\frac{R_H}{2} \geq R_L$.

Case 16: $S_L > 0$, $S_H = 0$, $C_L = 0$, $C_H > 0$, and $p_H C_H = r$

$U_C = p_H C_H = r$ gives $C_H = \frac{r}{p_H}$. When $S_L = C_H$, we get $U_S = p_L S_L = \frac{r p_L}{p_H}$, $U_B(\mu, H|H) = R_H - C_H = R_H - \frac{r}{p_H}$ and $U_B(\mu, L|L) = R_L - \frac{r}{p_H}$. Therefore, $U_B = R - \frac{r}{p_H}$. For $U_B(\mu, L|L) > 0$ we need, $p_H > \frac{r}{R_L}$. For feasible p_H we need $r < R_L$.

When $C_H > S_L > 0$, we have $\lambda_5 = \lambda_8 = \lambda_6 = 0$ along with $\lambda_1 = \lambda_4 = 0$. From equation (C.20) we get, $S_L = \frac{R_L(p_H R_H - r)}{(1-p_L^2)(R_H - R_L)}$. This gives $U_B(\mu, L|L) = \frac{p_L R_L(p_H R_H - r)}{p_H(1+p_L)}$, $U_B(\mu, H|H) = \frac{1}{p_H}(p_H R_H - r)$, and $U_B = \frac{(p_H R_H - r)(p_L^2 R_L - p_H(1+p_L))}{p_H(1+p_L)}$. For $S_L > 0$ we need $p_H > \frac{r}{R_H}$. In addition, we need $C_H > S_L$ i.e. $\frac{r}{p_H} > \frac{R_L(p_H R_H - r)}{(1-p_L^2)(R_H - R_L)}$ which after some algebra gives $p_H < \frac{r(2R_H - R_L)}{R_L R_H + r(R_H - R_L)}$. For feasible p_H we require $\frac{r}{R_H} < \frac{r(2R_H - R_L)}{R_L R_H + r(R_H - R_L)}$. This yields $r < 2R_H$ which is always true.

Tables C.3 to C.7 summarize the optimal solutions of problem $(S_N B_Y C_N)$ in five cases where,

$S_1 = \frac{1}{6}\left(3(R_L + R_H) - R - \sqrt{(3(R_L + R_H) - R)^2 - 24 R_L R_H}\right)$

$S_2 = \frac{1}{4}\left(2(R_L + R_H) - R + r - \sqrt{(2(R_L + R_H) - R + r)^2 - 8\left(R_L R_H + r(R_L + R_H - R)\right)}\right)$

and $R = p_H R_H + p_L R_L$. Table C.1 provides the notations A to F denoting the equilibrium payoffs (U_S^*, U_B^*, U_C^*) for seller, buyer, and consultant respectively. The notations a to f provided in Table C.2 denote the probabilities between which the given solution is valid. NA denotes that there is no feasible solution in this range and negotiation breaks down.

Table C.1. Notation for Optimal Solutions

Notation	(U_S^*, U_B^*, U_C^*)
A	$\left(\frac{1}{2}S_1, R - S_1, \frac{1}{2}S_1\right)$
B	$\left(\frac{1}{3}p_H R_H, \frac{(R-R_L)(p_H R_H - p_L R_L)}{3(p_H R_H - R_L)}, \frac{1}{3}p_H R_H\right)$
C	$(S_2 - r, R - S_2, r)$
D	$\left(\frac{1}{2}(p_H R_H - r), \frac{(p_H R_H - r)(R_H - R_L)(p_H R_H - p_L R_L)}{2R_H(p_H R_H - R_L)}, r\right)$
E	$\left(\frac{1}{3}p_H R_H, \frac{1}{3}R, \frac{1}{3}p_H R_H\right)$
F	$\left(\frac{1}{2}(p_H R_H - r), \frac{R}{2}(1 - \frac{r}{p_H R_H}), r\right)$

Table C.2. Notation for Probability

Notation	Probability
a	$\frac{(R_H - 2r)(3r - R_L)}{r(R_H - R_L)}$
b	$\frac{3R_L}{3R_H - R_L}$
c	$\frac{R_L}{R_H - R_L}$
d	$\frac{R_L(2R_H - r)}{R_H(2R_H - R_L)}$
e	$\frac{r}{R_H}$
f	$\frac{3r}{R_H}$

Table C.3. Case 1: When $R_L \leq \frac{1}{4}R_H$

$0 \leq r \leq \frac{R_L R_H}{3R_H - R_L}$	$\frac{R_L R_H}{3R_H - R_L} \leq r \leq \frac{R_L R_H}{3(R_H - R_L)}$	$\frac{R_L R_H}{3(R_H - R_L)} \leq r \leq R_L$	$R_L \leq r \leq \frac{R_L R_H}{R_H - R_L}$	$\frac{R_L R_H}{R_H - R_L} \leq r \leq \frac{1}{3}R_H$	$\frac{1}{3}R_H \leq r < R_H$
C if $0 \leq p_H \leq a$	C if $0 \leq p_H \leq d$	C if $0 \leq p_H \leq d$	NA if $0 \leq p_H \leq e$	NA if $0 \leq p_H \leq e$	NA if $0 \leq p_H \leq e$
A if $a \leq p_H \leq b$	D if $d \leq p_H \leq f$	D if $d \leq p_H \leq c$	D if $e \leq p_H \leq c$	F if $e \leq p_H \leq f$	F if $e \leq p_H \leq 1$
B if $b \leq p_H \leq c$	B if $f \leq p_H \leq c$	F if $c \leq p_H \leq f$	F if $c \leq p_H \leq f$	E if $f \leq p_H \leq 1$	
E if $c \leq p_H \leq 1$	E if $c \leq p_H \leq 1$	E if $f \leq p_H \leq 1$	E if $f \leq p_H \leq 1$		

Table C.4. Case 2: When $\frac{1}{4}R_H \leq R_L \leq \frac{1}{3}R_H$

$0 \leq r \leq \frac{R_L R_H}{3R_H - R_L}$	$\frac{R_L R_H}{3R_H - R_L} \leq r \leq \frac{R_L R_H}{3(R_H - R_L)}$	$\frac{R_L R_H}{3(R_H - R_L)} \leq r \leq R_L$	$R_L \leq r \leq \frac{1}{3}R_H$	$\frac{1}{3}R_H \leq r \leq \frac{R_L R_H}{R_H - R_L}$	$\frac{R_L R_H}{R_H - R_L} \leq r < R_H$
C if $0 \leq p_H \leq a$	C if $0 \leq p_H \leq d$	C if $0 \leq p_H \leq d$	NA if $0 \leq p_H \leq e$	NA if $0 \leq p_H \leq e$	NA if $0 \leq p_H \leq e$
A if $a \leq p_H \leq b$	D if $d \leq p_H \leq f$	D if $d \leq p_H \leq c$	D if $e \leq p_H \leq c$	D if $e \leq p_H \leq c$	F if $e \leq p_H \leq 1$
B if $b \leq p_H \leq c$	B if $f \leq p_H \leq c$	F if $c \leq p_H \leq f$	F if $c \leq p_H \leq f$	F if $c \leq p_H \leq 1$	
E if $c \leq p_H \leq 1$	E if $c \leq p_H \leq 1$	E if $f \leq p_H \leq 1$	E if $f \leq p_H \leq 1$		

Table C.5. Case 3: When $\frac{1}{3}R_H \leq R_L \leq \frac{1}{2}R_H$

$0 \leq r \leq \frac{R_L R_H}{3R_H - R_L}$	$\frac{R_L R_H}{3R_H - R_L} \leq r \leq \frac{R_L R_H}{3(R_H - R_L)}$	$\frac{R_L R_H}{3(R_H - R_L)} \leq r \leq \frac{1}{3}R_H$	$\frac{1}{3}R_H \leq r \leq R_L$	$R_L \leq r \leq \frac{R_L R_H}{R_H - R_L}$	$\frac{R_L R_H}{R_H - R_L} \leq r < R_H$
C if $0 \leq p_H \leq a$	C if $0 \leq p_H \leq d$	C if $0 \leq p_H \leq d$	C if $0 \leq p_H \leq d$	NA if $0 \leq p_H \leq e$	NA if $0 \leq p_H \leq e$
A if $a \leq p_H \leq b$	D if $d \leq p_H \leq f$	D if $d \leq p_H \leq c$	D if $d \leq p_H \leq c$	D if $e \leq p_H \leq c$	F if $e \leq p_H \leq 1$
B if $b \leq p_H \leq c$	B if $f \leq p_H \leq c$	F if $c \leq p_H \leq f$	F if $c \leq p_H \leq 1$	F if $c \leq p_H \leq 1$	
E if $c \leq p_H \leq 1$	E if $c \leq p_H \leq 1$	E if $f \leq p_H \leq 1$			

Table C.6. Case 4: When $\frac{1}{2}R_H \leq R_L \leq \frac{3}{4}R_H$

$0 \leq r \leq \frac{R_L R_H}{3R_H - R_L}$	$\frac{R_L R_H}{3R_H - R_L} \leq r \leq \frac{1}{3}R_H$	$\frac{1}{3}R_H \leq r \leq R_L$	$R_L \leq r \leq R_H$
C if $0 \leq p_H \leq a$ A if $a \leq p_H \leq b$ B if $b \leq p_H \leq 1$	C if $0 \leq p_H \leq d$ D if $d \leq p_H \leq f$ B if $f \leq p_H \leq 1$	C if $0 \leq p_H \leq d$ D if $d \leq p_H \leq 1$	NA if $0 \leq p_H \leq e$ D if $e \leq p_H \leq 1$

Table C.7. Case 5: When $\frac{3}{4}R_H \leq R_L < R_H$

$0 \leq r \leq \frac{1}{3}R_H$	$\frac{1}{3}R_H \leq r \leq \frac{R_L R_H}{3R_H - R_L}$	$\frac{R_L R_H}{3R_H - R_L} \leq r < R_L$	$R_L \leq r < R_H$
C if $0 \leq p_H \leq a$ A if $a \leq p_H \leq 1$	C if $0 \leq p_H \leq 1$	C if $0 \leq p_H \leq d$ D if $d \leq p_H \leq 1$	NA if $0 \leq p_H \leq e$ D if $e \leq p_H \leq 1$

Proposition C.1 *Seller will not offer a demonstration to the buyer which reveals true value of the report.*

PROOF: From table C.1 we observe that the equilibrium payoff of the seller in $(S_N B_Y C_N)$ problem is one of the followings: $\frac{1}{2}S_1$, $\frac{1}{3}p_H R_H$, $S_2 - r$, and $\frac{1}{2}(p_H R_H - r)$ where

$S_1 = \frac{1}{6}\left(3(R_L + R_H) - R - \sqrt{(3(R_L + R_H) - R)^2 - 24R_L R_H}\right)$

$S_2 = \frac{1}{4}\left(2(R_L + R_H) - R + r - \sqrt{(2(R_L + R_H) - R + r)^2 - 8\left(R_L R_H + r(R_L + R_H - R)\right)}\right)$

We will consider following three cases — (i) $0 \leq r \leq \frac{R}{3}$, (ii) $\frac{R}{3} \leq r \leq R$, and (iii) $R \leq r \leq R_H$.

(i) When $\boldsymbol{0 \leq r \leq \frac{R}{3}}$. In this case Lemma 4.1 reveals $U_S^{S_N B_N C_N} = \frac{R}{3}$. We will show that all four equilibrium solutions of the seller are less than $\frac{R}{3}$. Following shows that $\frac{1}{2}S_1 \leq \frac{R}{3}$.

$$\begin{aligned} 3(R_L + R_H) - R - \sqrt{(3(R_L + R_H) - R)^2 - 24R_L R_H} &\leq 4R \\ \Rightarrow \quad (3(R_L + R_H) - R)^2 - (3(R_L + R_H) - R - 4R)^2 &\geq 24R_L R_H \\ \Rightarrow \quad (R_L + R_H - R)R &\geq R_L R_H \\ \Rightarrow \quad 0 &\geq (R - R_L)(R - R_H) \end{aligned}$$

Above is true since $R_L \leq R \leq R_H$. Also, $\frac{1}{3}p_H R_H \leq \frac{1}{3}R$ is always true. Tables C.3 to C.6 indicate that the solution $\frac{1}{2}(p_H R_H - r)$ is feasible for $r \leq \frac{R}{3} \leq \frac{R_H}{3}$ when $p_H \leq \frac{3r}{R_H}$. At $p_H = \frac{3r}{R_H}$ we get, $\frac{1}{2}(p_H R_H - r) = r$ and $\frac{R}{3} = r\left(1 - \frac{R_L}{R_H}\right) + \frac{R_L}{3}$. Now, $\frac{1}{2}(p_H R_H - r)$ is linearly increasing in p_H with slope $\frac{1}{2}R_H$ and $\frac{R}{3}$ is also linearly increasing in p_H with slope $\frac{1}{3}(R_H - R_L)$. Therefore, $U_S^{S_N B_Y C_N}$ is linearly increasing at a higher rate than $U_S^{S_N B_N C_N}$ in p_H. At $p_H = \frac{3r}{R_H}$, $U_S^{S_N B_Y C_N} \leq U_S^{S_N B_N C_N}$ if $r \leq r\left(1 - \frac{R_L}{R_H}\right) + \frac{R_L}{3}$ i.e. $r \leq \frac{R_H}{3}$ which is true. Therefore, $U_S^{S_N B_Y C_N} = \frac{1}{2}(p_H R_H - r) \leq \frac{R}{3}$. Solution of $(S_N B_Y C_N)$ indicate that $S_2 - r$ is feasible when $r \leq \frac{R_L R_H}{3R_H - R_L}$ with $p_H \leq \frac{(R_H - 2r)(3r - R_L)}{r(R_H - R_L)}$ or when $\frac{R_L R_H}{3R_H - R_L} \leq r \leq R_L$ with $p_H \leq \frac{R_L(2R_H - r)}{R_H(2R_H - R_L)}$. To prove that $S_2 - r \leq \frac{R}{3}$, we first show that $S_2 - r$ is an increasing convex function in p_H.

Let $x = \sqrt{(2(R_L+R_H)-R+r)^2 - 8\left(R_LR_H + r(R_L+R_H-R)\right)}$. Following shows that $\frac{\partial}{\partial p_H}(S_2 - r) \geq 0$ for $r \leq R_L$,

$$\begin{aligned}
-(R_H - R_L) - \tfrac{2(2(R_L+R_H)-R+r)(-(R_L-R_H))-8r(-(R_L+R_H))}{2x} &\geq 0 \\
\Rightarrow \quad \sqrt{(2(R_L+R_H)-R+r)^2 - 8(R_LR_H + r(R_L+R_H-R))} &\leq 2(R_L+R_H)-R+r-4r \\
\Rightarrow \quad (2(R_L+R_H)-R+r)^2 - (2(R_L+R_H)-R+r-4r)^2 &\leq 8(R_LR_H + r(R_L+R_H-R)) \\
\Rightarrow \quad (2(R_L+R_H)-R-r)r &\leq R_LR_H + r(R_L+R_H-R) \\
\Rightarrow \quad 0 &\leq (r-R_L)(r-R_H)
\end{aligned}$$

Therefore, $S_2 - r$ is an increasing function in p_H. Following shows that $\frac{\partial^2}{\partial p_H^2}(S_2 - r) \geq 0$,

$$\begin{aligned}
-(R_H-R_L)x - \tfrac{(2(2(R_L+R_H)-R+r)(-(R_L-R_H))-8r(-(R_L+R_H)))(2(R_L+R_H)-R+r-4r)}{2x} &\geq 0 \\
\Rightarrow \quad -(R_H-R_L)\left(x - \tfrac{(2(R_L+R_H)-R-3r)^2}{x}\right) &\geq 0 \\
\Rightarrow \quad -(R_H-R_L)\left((2(R_L+R_H)-R+r)^2 - 8(R_LR_H + r(R_L+R_H-R)) - (2(R_L+R_H)-R-3r)^2\right) &\geq 0 \\
\Rightarrow \quad -8(R_H-R_L)\left((2(R_L+R_H)-R-r)r - (R_LR_H + r(R_L+R_H-R))\right) &\geq 0 \\
\Rightarrow \quad 8(R_H-R_L)(r-R_L)(r-R_H) &\geq 0
\end{aligned}$$

At $p_H = 0$ we have $\frac{R}{3} = \frac{R_L}{3}$ and $S_2 - r = \frac{1}{4}\left(R_L + 2R_H + r - \sqrt{(R_L+2R_H+r)^2 - 8R_H(R_L+r)}\right) - r$. Following shows that $U_S^{S_NB_NC_N} = U_S^{S_NB_YC_N}$ at $p_H = 0$,

$$\begin{aligned}
R_L + 2R_H + r - \sqrt{(R_L+2R_H+r)^2 - 8R_H(R_L+r)} &= 4\left(\tfrac{R_L}{3} + r\right) \\
\Rightarrow \quad (R_L+2R_H+r)^2 - \left(R_L+2R_H+r-4(\tfrac{R_L}{3}+r)\right)^2 &= 8R_H(R_L+r) \\
\Rightarrow \quad \left(\tfrac{R_L}{3} + 2R_H - r\right)\left(\tfrac{R_L}{3} + r\right) &= R_H(R_L+r) \\
\Rightarrow \quad r^2 - rR_H + \tfrac{1}{3}R_LR_H - (\tfrac{R_L}{3})^2 &= 0 \\
\Rightarrow \quad \left(r - \tfrac{R_L}{3}\right)\left(r + \tfrac{R_L}{3} - R_H\right) &= 0
\end{aligned}$$

For this case $r \leq \frac{R}{3} = \frac{R_L}{3} \leq \frac{R_LR_H}{3R_H - R_L}$ at $p_H = 0$. Therefore $S_2 - r$ is feasible if $p_H \leq \frac{(R_H-2r)(3r-R_L)}{r(R_H-R_L)}$ i.e. $\frac{R_L}{3} \leq r \leq \frac{R_H}{2}$. This implies $r = \frac{R_L}{3}$ and $U_S^{S_NB_NC_N} = U_S^{S_NB_YC_N}$ at $p_H = 0$. Similarly, it can be shown that $U_S^{S_NB_NC_N} > U_S^{S_NB_YC_N}$ at $p_H = \frac{(R_H-2r)(3r-R_L)}{r(R_H-R_L)}$ and at $p_H = \frac{R_L(2R_H-r)}{R_H(2R_H-R_L)}$. Now $U_S^{S_NB_NC_N} = \frac{R}{3}$ is linearly increasing in p_H and $U_S^{S_NB_YC_N} = S_2 - r$ is an increasing convex function in p_H. Also, $U_S^{S_NB_NC_N} = U_S^{S_NB_YC_N}$ at $p_H = 0$ while $U_S^{S_NB_NC_N} > U_S^{S_NB_YC_N}$ at $p_H = \frac{(R_H-2r)(3r-R_L)}{r(R_H-R_L)}$ and at $p_H = \frac{R_L(2R_H-r)}{R_H(2R_H-R_L)}$. This implies $S_2 - r \leq \frac{R}{3}$. Therefore, seller would not gain by providing a demo when $r \leq \frac{R}{3}$.

(ii) When $\boldsymbol{\frac{R}{3} \leq r \leq R}$. Lemma 4.1 reveals $U_S^{S_NB_NC_N} = \frac{1}{2}(R - r)$. Also, $\frac{R}{3} \leq r$ implies $p_H \leq \frac{3r-R_L}{R_H-R_L}$. Therefore, $U_S^{S_NB_YC_N} = \frac{S_1}{2}$ is a feasible solution if $\frac{(R_H-2r)(3r-R_L)}{r(R_H-R_L)} \leq p_H \leq \frac{3r-R_L}{R_H-R_L}$ which implies $r \geq \frac{R_H}{3}$. However, Tables C.3 to C.7 indicate that $\frac{S_1}{2}$ is feasible when $r < \frac{R_H}{3}$. Therefore, $U_S^{S_NB_YC_N} = \frac{S_1}{2}$ is not an equilibrium solution when $\frac{R}{3} \leq r$. $U_S^{S_N B_Y C_N} = \frac{1}{3} p_H R_H$ is a feasible solution when $r \leq \frac{R_L R_H}{3R_H - R_L}$ with $p_H \geq \frac{3R_L}{3R_H - R_L}$ or when $\frac{R_L R_H}{3R_H - R_L} \leq r$ with $p_H \geq \frac{3r}{R_H}$. Consider $r \leq \frac{R_L R_H}{3R_H - R_L}$ then for feasibility we need $\frac{3r - R_L}{R_H - R_L} \geq p_H \geq \frac{3R_L}{3R_H - R_L}$ i.e. $r \geq \frac{2R_L(3R_H - 2R_L)}{3(3R_H - R_L)}$. Therefore we need $\frac{2R_L(3R_H - 2R_L)}{3(3R_H - R_L)} \leq r \leq \frac{R_L R_H}{3R_H - R_L}$ which implies $\frac{3}{4} R_H \leq R_L$. However, Table C.7 indicates that $U_S^{S_N B_Y C_N} = \frac{1}{3} p_H R_H$ is not an equilibrium solution when $\frac{3}{4} R_H \leq R_L$. Next, consider $\frac{R_L R_H}{3R_H - R_L} \leq r$ with $p_H \geq \frac{3r}{R_H}$ i.e. $r \leq \frac{1}{3} p_H R_H$. However, $\frac{1}{3} p_H R_H \leq \frac{R}{3} \leq r$. Therefore, $U_S^{S_N B_Y C_N} = \frac{1}{3} p_H R_H$ is not feasible when $\frac{R}{3} \leq r$. Furthermore, $U_S^{S_N B_Y C_N} = \frac{1}{2}(p_H R_H - r) \leq \frac{1}{2}(R - r)$ since $p_H R_H \leq R$. Following shows that $U_S^{S_N B_Y C_N} = S_2 - r \leq \frac{1}{2}(R - r)$.

$$
\begin{aligned}
& 2R_L + 2R_H - R + r - \sqrt{(2R_L + 2R_H - R + r)^2 - 8(R_L R_H + r(R_L + R_H - R))} - 4r \leq 2R - 2r \\
\Rightarrow \quad & (2R_L + 2R_H - R + r)^2 - (2R_L + 2R_H - R + r - 2R - 2r)^2 \quad \geq \quad 8(R_L R_H + r(R_L + R_H - R)) \\
\Rightarrow \quad & (R_L + R_H - R)(R + r) \quad \geq \quad R_L R_H + r(R_L + R_H - R) \\
\Rightarrow \quad & 0 \quad \geq \quad (R - R_L)(R - R_H)
\end{aligned}
$$

Above is always true hence seller would not gain by providing a demo when $\frac{R}{3} \leq r \leq R$.

(iii) When $\boldsymbol{R \leq r}$. Lemma 4.1 reveals $U_S^{S_N B_N C_N} = 0$. $R \leq r$ implies $p_H R_H \leq r$ i.e. $p_H \leq \frac{r}{R_H}$. However, Tables C.3 to C.7 state that the negotiation breaks down when $p_H \leq \frac{r}{R_H}$ implying $U_S^{S_N B_Y C_N} = 0$. ■

Property C.3 *The outcome of the negotiation does not change when the buyer knows the true value of the report and consultant update their belief about the value while the seller assumes it to be distributed as R_L and R_H with probability p_L and $p_H = 1 - p_L$ respectively.*

Proof: Suppose the consultant update their believes about the value of the report to be $s = L, H$ with probability $P(s)$. Let $P(t|s)$ be the conditional probability of the true value to be $t = L, H$ given that the consultant believes it to be s. If $U_C(\mu)$ be the expected payoff to the consultant in mechanism μ then,

$$
\begin{aligned}
U_C(\mu) &= U_C(\mu|s = L)P(s = L) + U_C(\mu|s = H)P(s = H) \\
&= \Big(\sum_{d \in \Phi} \mu(d|L) u_C(d) P(t = L|s = L) + \sum_{d \in \Phi} \mu(d|H) u_C(d) P(t = H|s = L) \Big) P(s = L) \\
&\quad + \Big(\sum_{d \in \Phi} \mu(d|L) u_C(d) P(t = L|s = H) + \sum_{d \in \Phi} \mu(d|H) u_C(d) P(t = H|s = H) \Big) P(s = H) \\
&= \sum_{d \in \Phi} \mu(d|L) u_C(d) P(t = L, s = L) + \sum_{d \in \Phi} \mu(d|H) u_C(d) P(t = H, s = L) \\
&\quad + \sum_{d \in \Phi} \mu(d|L) u_C(d) P(t = L, s = H) + \sum_{d \in \Phi} \mu(d|H) u_C(d) P(t = H, s = H) \\
&= \sum_{d \in \Phi} \mu(d|L) u_C(d) \Big(P(t = L, s = L) + P(t = L, s = H) \Big) \\
&\quad + \sum_{d \in \Phi} \mu(d|H) u_C(d) \Big(P(t = H, s = L) + P(t = H, s = H) \Big) \\
&= \sum_{d \in \Phi} \Big(p_L \mu(d|L) + p_H \mu(d|H) \Big) u_C(d) \qquad \text{(C.48)}
\end{aligned}
$$

Equation (C.48) is same as that of the consultant's expected payoff when only buyer knows the true value (see section 4.2). This implies that believes of the consultant does not change their expected payoff which remains equivalent to the situation when the consultant knows the distribution of the true value to be R_L and R_H with probabilities p_L and $p_H = 1 - p_L$ respectively. This is because the payoffs depend on the actual value of the report realized by the buyer and it will remain unaffected by any belief perceived by the consultant or seller. Therefore, we can use the solution of negotiation problem $(S_N B_Y C_N)$ to get an expected payoff from this negotiation. ■

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BIOGRAPHICAL SKETCH

Jyotishka Ray was born in Patna, India, to Dr. Manoranjan Ray and Professor Ratri Ray. He is trained as a Mechanical Engineer (B.Tech from Indian Institute of Technology, Banaras Hindu University, 2002), and was working as a Manager (Customer Case – CVBU) in Tata Motors Ltd. of India, Mumbai when he left work and came to USA (2008) for studies. He completed his Masters from The University of North Carolina at Chapel Hill (2011) in Operations Research and Statistics and joined PhD program at UT Dallas. He also obtained Post Graduate Diploma in General Management from XLRI – Xavier School of Management (2006). He is married to Ratna, and is blessed with their two sons Arit and Ankit. Jyotishka loves playing Tabla, an Indian percussion musical instrument, and likes to sketch. He has taught the undergraduate courses in Management Information Systems, Database fundamentals, and Java Programming at UTD since Fall 2014. He has a published paper in Production and Operations Management (2017). Jyotishka hopes to bring his managerial experiences into the classrooms at California State University – East Bay where he will be joining as an Assistant Professor of Management Information Systems in August 2017.

CURRICULUM VITAE

JYOTISHKA RAY

Areas of Interests

Research: Software Demonstration, Commercialization of Software Products, Negotiation.
Teaching: Data-driven Decision Making, e-Negotiation in IT, JAVA Programming, Database Fundamentals, Information Technology for Business, Introduction to Management Information Systems.

Education

The University of Texas at Dallas, TX, U.S.A.
Ph.D., Information Systems Management — August 2017
Dissertation Title: *Essays on Commercialization of Information Technology Products.*
Advisors: Vijay Mookerjee and Syam Menon

The University of North Carolina at Chapel Hill, NC, U.S.A.
M.S., Operations Research and Statistics — Aug. 08, 2011
Thesis Title: *A Review of Component Commonality in an Assemble-to-Order System.*
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Xavier Labour Relations Institute at Jamshedpur, India.
PGDM (GM), Information Systems and Operations Management — May 06, 2006

Indian Institute of Technology at BHU, India.
B. Tech., Mechanical Engineering — March 23, 2003
Thesis Title: *Identification of the System Parameters of a Mechanical System by using Neural Network Analysis.*
Advisor: Mihir K. Ghosh

Research Papers

1. "The Design of Feature-Limited Demonstration Software: Choosing the Right Features to Include." Ray, J., Samuel, J., Menon, S. and Mookerjee, V. (2017), **Production and Operations Management**, 26(1): 9-30.
2. "Bargaining over Data: When does Making the Buyer More Informed Help?" Ray, J., Menon, S. and Mookerjee, V. (*Major revision at* **Information Systems Research**, *2017*)
3. "Bargaining over Data with Consultant as a Gatekeeper" Ray, J., Menon, S. and Mookerjee, V. (*Working paper. Target Journal:* **Management Information Systems Quarterly**, *2017*)

Teaching Experience

Teaching Instructor
Nine undergraduate courses taught in The University of Texas at Dallas with full responsibility

ITSS 3300.5U1.17Su	Information Technology for Business	ongoing	Summer 2017
ITSS 3300.005.17S	Information Technology for Business	4.3/5.0	Spring 2017
ITSS 3300.008.16F	Information Technology for Business	4.4/5.0	Fall 2016
ITSS 3300.501.16F	Information Technology for Business	4.3/5.0	Fall 2016
ITSS 3211.5U1.16U	Introduction to JAVA Programming	4.7/5.0	Summer 2016
ITSS 3300.501.15F	Information Technology for Business	4.3/5.0	Spring 2016
ITSS 3300.006.14F	Information Technology for Business	4.5/5.0	Fall 2015
MIS 3300.502.14S	Introduction to Management Information System	4.5/5.0	Fall 2014
MIS/ACCT 4300.502.14S	Database Fundamentals	4.4/5.0	Spring 2014

Teaching Assistant Fall 2011 - 2017
Information Systems Management, Naveen Jindal School of Management
The University of Texas at Dallas, Richardson, TX

Other Experiences

Research Assistant at Kenan-Flagler Business School, UNC Fall 2008 - Fall 2010
Worked with faculties to derive steady state policy for optimal inventory replenishment of partially substitutable old and new items.

Industry Interface Fall 2011 - 2017

- Delivered a talk at a leading wireless telecommunication company and proposed solution procedure on designing a feature-limited version of their wireless network provisioning software. A simplified version of the software is adapted as a **case study** in our research paper.
- Delivered a talk at a major Global Distribution Systems provider about monetizing data product. The discussion resulted in two research papers on *"Bargaining over Data"* and started several new research initiatives.

Manager, Customer Care July 2002 - May 2008
Commercial Vehicle Business Unit
Tata Motors Ltd., Mumbai, India

Professional Memberships

- Certified SAS Base Programmer
- Institute for Operations Research and the Management Sciences (INFORMS)
- Association for Information Systems (AIS)

Conference Presentations

Designing Feature-limited Demonstration Software: Choosing the Right Features to Include.

- 2016 May, Production and Operations Management Society (POMS), Orlando, FL

The Design of Feature-limited Evaluation Software

- 2013 Oct., INFORMS Annual Meeting, Minneapolis, MN
- 2013 Apr., Big XII+ MIS Research Symposium, Stillwater, OK

Bargaining over Data: An Analysis of B2B Negotiations in the Context of Data Products

- 2016 Oct., INFORMS Annual Meeting, Nashville, TN

Monetizing Data Through B2B Negotiation: When is a Demonstration Appropriate?

- 2015 Oct., INFORMS Conference on Information Systems and Technology (CIST), Philadelphia, PA
- 2017 Jan., Conference on the Digital Economy (CODE), Indian School of Business

Professional Services

Session Chair

2016 May Health Information Technology at POMS, Orlando, FL

Invited Reviewer

- Decision Support Systems, 2016
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- Interfaces, 2015
- Information Technology and Management, 2014

Computer Skills

Statistical Packages: SAS, R.
Optimization Packages: CPLEX, BARON, Mathematica, Matlab.
Languages: Python, JAVA, AMPL, PHP, GAMS.
Simulation: Arena, @Risk.
Applications: LaTeX, MS-Office, Spreadsheet, and presentation software.