SYMMETRIES OF EINSTEIN'S EQUATIONS IN VACUUM AND THEIR GEODESICS

by

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To my parents: for all their unending love, encouragement, and sacrifice

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This thesis explores symmetries of vacuum Einstein equations that are static and at least axially symmetric, i.e., Ricci-flat Lorentzian geometries that admit a timelike Killing vector field and a closed spacelike Killing vector field among their isometries. We study symmetries of the geodesics in these spacetimes as well as symmetries of the system of Einstein equations describing such spacetimes. Geodesics in three dimensions have symmetries and associated conserved quantities absent in four and higher dimensions. We employ the so-called direct method for computing the conserved quantities. For the static axisymmetric system in vacuum, we found all symmetries of the system which enabled us to explain why one cannot obtain algebraic prescriptions for generating new solutions from old ones beyond those already known. Symmetries of the geodesics in spherical symmetry show that there is no general connection between cosmological constant and projective equivalence and that one can find an appropriate coordinate system where the effect of cosmological constant disappears from the bending angle, unlike in the static coordinates.

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CHAPTER 1

INTRODUCTION

> The beauty of mathematics only shows itself to more patient followers.
>
> — *Maryam Mirzakhani*

In this thesis, we study symmetries of Einstein's equations that describe spacetimes that are at least axially symmetric and stationary. In other words, we study systems of Einstein equations that have two Killing vector fields, one timelike and one spacelike with a closed orbit. We will also study symmetries of geodesic equations of such spacetimes and will consider both timelike and null geodesics that describe the free motion of massive and massless particles (i.e., light rays and gravitational waves), respectively. We will look for Lie point symmetries in these systems as well as conserved quantities. In this chapter and the next, we will develop the general differential geometric and symmetry method backgrounds needed for the rest of the thesis. More specific explanations will be provided progressively within each chapter.

1.1 The General Theory of Relativity and Einstein's Equations

Einstein's general theory of relativity is a theory of gravitational fields that supersedes Newton's theory of gravitation. In the latter, gravitation is a universally attractive force which exists between any two massive objects. It is directly proportional to their masses m_1 and m_2:

$$\vec{F} = -\frac{G m_1 m_2}{r^2}\hat{r}, \tag{1.1}$$

where r is the distance between the objects and G is the universal gravitational constant. This "inverse-square" law is a vector equation in three dimensions and has radial symmetry.

Newtonian gravity can equivalently be described by Poisson's equation:

$$\nabla^2\Phi(\vec{x}) = 4\pi G\rho(\vec{x}), \tag{1.2}$$

where $\Phi(\vec{x})$ is the gravitational potential and $\rho(\vec{x})$ the density function. The Newtonian theory of gravitation has been very successful in describing and explaining planetary motion as well as mundane events like the fall of an apple or the motion of a projectile.

Albert Einstein proposed his special theory of relativity in 1905 and showed how it was compatible with Maxwell's equations of electromagnetism. The relevant transformations in special relativity are the Lorentz transformations. However, it became apparent that special relativity and Newtonian gravitation are incompatible, i.e., the latter is not invariant under a Lorentz transformation. In 1915, after ten years of research, Einstein proposed his general theory of relativity, which is not only a theory of gravitational fields but also revolutionized our long-held ideas of space and time. In this theory, gravity is no longer a force but the local curvature of a four-dimensional Lorentzian manifold called spacetime. In the limit of slow motion and small curvature, it reproduces Newtonian theory. The anomalous rate of precession of the perihelion of Mercury's orbit, which could not be explained by Newtonian gravity, was accurately produced by general relativity. The theory also predicted other phenomena that did not occur in the Newtonian framework, such as the bending of light rays around massive objects like stars (which was experimentally verified). Over the years, the general theory of relativity has made further startling predictions, like the expansion of the universe, the existence of black holes, the emission of gravitational waves etc., which are not possible in the Newtonian framework. It has also given rise to many new mathematical concepts in Lorentzian and Riemannian geometries, which have been studied in depth independently of their applications. General relativity is a classical theory, and trying to find a quantum version of gravity has long been an outstanding problem in modern science,. which also required us to understand the classical theory at a deeper level.

1.2 Some Geometric Background

In looking for a theory of gravity compatible with special relativity, Einstein finally found the answer in the geometric program developed by Riemann and realized that spacetime is to be treated as a Lorentzian manifold endowed with a (Lorentzian) metric that is determined by the energy–momentum content given by a set of equations now known as Einstein's equation (see below).

Below we present some basic definitions and concepts of differential geometry that are needed for understanding what Einstein's equations describe. There are more definitions in Appendix A. Further details can be found in the references, especially in [24, 10, 34, 48].

Definition 1 (Curve)**.** *A curve α on a manifold $\mathcal{M}$ is a map from some interval $I \in R$ into $\mathcal{M}$:*

$$\alpha : I \to \mathcal{M}.$$

A curve is closed if I is a closed interval $[a, b]$ and $\alpha(a) = \alpha(b)$. Given a curve α on $\mathcal{M}$, we can naturally assign a tangent vector to each point on the curve. If $\alpha = (\alpha^1, \ldots, \alpha^m)$, then the derivative $\alpha' = (\alpha'^1, \ldots, \alpha'^m)$ defines a tangent vector at each point of the curve. This gives rise to the concept of tangent vector space.

Definition 2 (Tangent Vector Space)**.** *Collection of all tangent vectors at any point $p \in \mathcal{M}$ is called tangent vector space at p, $T_P(\mathcal{M})$. Collection of all tangent vectors is called the tangent space of $\mathcal{M}$, $T\mathcal{M}$.*

Definition 3 (Metric Tensor)**.** *A metric tensor g on the tangent space $T\mathcal{M}$ of a manifold $\mathcal{M}$ defined as an inner product with $\langle , \rangle : \mathscr{U} \subset \mathcal{M} \to R$, where $g(X, Y) = \langle X, Y \rangle \in R$ for tangent vectors.*

Definition 4 (Lorentzian Manifold)**.** *An n-dimensional Lorentzian manifold is a pseudo-Riemannian manifold with a rank-two symmetric tensor g defined on it with $n - 1$ positive and one negative eigenvalues.*

Although a Lorentzian manifold seems to be a simple modification of a Riemannian manifold, there are remarkable differences. For example, compare the two-dimensional Minkowski space (a Lorentzian manifold) with the metric $ds^2 = -dt^2 + dx^2$ in the usual Cartesian coordinates with the Euclidean space (a Riemannian manifold) with the metric $ds^2 = dx^2 + dy^2$. The simple change in the signature of the metric allows the former to have three types of curves and geodesics (see below), which have positive, negative, or zero "distances", i.e., ds^2 is indefinite for the Minkowski metric.

Definition 5 (Integral Curves)**.** *An integral curve of a vector field* $\mathbf{v} \in \mathfrak{X}(\mathcal{M})$ *on a manifold* $\mathcal{M}$ *with a initial condition* p_0*, is a curve* $\alpha : I \to \mathcal{M}$ *such that for every point on the curve,*

$$\alpha'(t) = \mathbf{v}|_{\alpha(\epsilon)} \quad \textit{for all} \quad \epsilon \in I, \quad \alpha(0) = p_0.$$

In other words, the vector field evaluated at any point on the curve is the tangent vector of the curve.

Example 1. *Let* $\mathbf{v}_{(x,y)} = \langle x^2 - y^2, 2xy \rangle$ *be a vector field on* $\mathbb{R}^2$ *(Figure 1.2). To find an integral curve* $\alpha(\epsilon) = (x(\epsilon), y(\epsilon))$ *of* $\mathbf{v}$ *at the point* $(0,1) \in \mathbb{R}^2$ *we need to check the condition* $\alpha'(\epsilon) = \mathbf{v}(\alpha(\epsilon))$

$$\begin{bmatrix} x'(\epsilon) \\ y'(\epsilon) \end{bmatrix} = \begin{bmatrix} x^2(\epsilon) - y^2(\epsilon) \\ 2x(\epsilon)y(\epsilon) \end{bmatrix}.$$

We need to solve the system of first-order ordinary differential equations

$$\begin{cases} x' = x^2 - y^2 \\ y' = 2xy \end{cases}$$

with initial condition $(x(0), y(0)) = (0,1)$.

The integral curve with the initial condition $(0,1)$ *is* $\alpha(\epsilon) = \left(-\frac{\epsilon}{1+\epsilon^2}, \frac{1}{2}(1 + \frac{|\epsilon^2-1|}{\epsilon^2+1})\right)$.

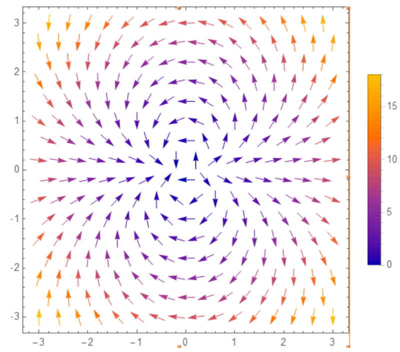


Figure 1.1: The vector field $(x^2 - y^2, 2xy)$ in R^2.

Lie and Covariant Derivatives

On a curved manifold, one can differentiate a function given any local coordinates system. However, there is no natural way to compare the values of vectors and tensors at two points on a manifold, since they belong to two different tangent spaces. However, given two vector fields, one can talk about how one vector changes along (the integral curve of) the other vector. This leads to the definition of the Lie derivative of vectors and tensors, as well as functions. The Lie derivative of a smooth vector field $\mathbf{u}$ with respect to another vector field $\mathbf{v}$ at point p is given by [44]

$$\mathcal{L}_{\mathbf{v}}\mathbf{u}^{\mu} = [\mathbf{u}, \mathbf{v}]^{\mu}. \tag{1.3}$$

For a scalar

$$\mathcal{L}_{\mathbf{v}}f = \mathbf{v}f, \tag{1.4}$$

and for a rank-two tensor

$$\mathcal{L}_{\mathbf{v}}A_{\mu\nu} = Y^{\rho}\partial_{\rho}A_{\mu\nu} + A_{\rho\nu}\partial_{\mu}Y^{\rho} + A_{\mu\rho}\partial_{\nu}Y^{\rho}. \tag{1.5}$$

Covariant derivatives, on the other hand, are defined by

$$\mathbf{v}^{\nu}{}_{;\mu} \equiv \nabla_{\mu}\mathbf{v}^{\nu} = \partial_{\mu}\mathbf{v}^{\nu} + \Gamma^{\nu}{}_{\mu\sigma}\mathbf{v}^{\sigma}, \tag{1.6}$$

where Γ's are the connection coefficients

$$\nabla_\mu \partial_\nu = \sum_\rho \Gamma^\rho_{\mu\nu} \partial_\rho. \tag{1.7}$$

This new extra geometric structure allows comparison between vectors in nearby tangent spaces to be compared. The covariant derivative of a scalar function coincides with partial derivatives: $f_{;\mu} \equiv \nabla_\mu f = \partial_\mu f$. The covariant derivatives of a $(1,1)$ tensor is given by

$$\nabla_\rho A^\mu_\nu = \partial_\rho A^\mu_\nu + \Gamma^\mu_{\rho\sigma} A^\sigma_\nu - \Gamma^\sigma_{\nu\rho} A^\mu_\sigma. \tag{1.8}$$

This is apparent how covariant derivatives apply to covectors and tensors of arbitrary rank. The connection coefficients are often chosen such that the covariant derivative of the metric tensor vanishes

$$\nabla_\rho g_{\mu\nu} = 0. \tag{1.9}$$

These are then referred to as the Christoffel symbols and are uniquely determined by the metric tensor:

$$\Gamma^\rho_{\mu\nu} = \frac{1}{2} g^{\rho\sigma} \left(\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu} \right). \tag{1.10}$$

Riemann, Ricci Tensors

The commutator of covariant derivatives on a vector field defines a rank four tensor

$$[\nabla_\mu, \nabla_\nu] \mathbf{v}_\rho \equiv (\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu) \mathbf{v}_\rho = R_{\mu\nu\rho\sigma} \mathbf{v}^\sigma = R^\sigma{}_{\mu\nu\rho} \mathbf{v}_\sigma. \tag{1.11}$$

This is the famous Riemann curvature tensor which gives all local curvature information of a (pseudo-)Riemannian manifold.

The Riemann curvature tensor can be expressed in terms of Christoffel symbols Stewart-book:

$$R^\sigma{}_{\mu\nu\rho} = \partial_\mu \Gamma^\sigma_{\nu\rho} - \partial_\nu \Gamma^\sigma_{\mu\rho} + \Gamma^\sigma_{\mu\lambda} \Gamma^\lambda_{\nu\rho} - \Gamma^\sigma_{\nu\lambda} \Gamma^\lambda_{\mu\rho}. \tag{1.12}$$

It is evident that it contains second derivatives of the metric tensor since Christoffel symbols contain first derivatives.

The Riemann tensor has certain symmetries which reduce its number of algebraically independent components

$$R^{\sigma}{}_{\mu\nu\rho} = \partial_\mu \Gamma^{\sigma}{}_{\nu\rho} - \partial_\nu \Gamma^{\sigma}{}_{\mu\rho} + \Gamma^{\sigma}{}_{\mu\lambda}\Gamma^{\lambda}{}_{\nu\rho} - \Gamma^{\sigma}{}_{\nu\lambda}\Gamma^{\lambda}{}_{\mu\rho}. \tag{1.13}$$

The following identities of the Riemann tensor components, respective called the first and second Bianchi identities, have important consequences in GR:

$$R_{\mu\nu\rho\sigma} + R_{\mu\sigma\nu\rho} + R_{\mu\sigma\rho\nu} = 0, \tag{1.14}$$

and

$$R^{\mu}{}_{\nu\rho\sigma;\eta} + R^{\mu}{}_{\nu\eta\rho;\sigma} + R^{\mu}{}_{\nu\sigma\eta;\rho} = 0. \tag{1.15}$$

The Ricci curvature tensor is obtained by contraction of the first and third indices of the Riemann tensor:

$$R_{\mu\nu} = R^{\rho}{}_{\mu\rho\nu} = \partial_\rho \Gamma^{\rho}_{\mu\nu} - \partial_\nu \Gamma^{\rho}_{\rho\mu} + \Gamma^{\rho}_{\lambda\sigma}\Gamma^{\sigma}_{\mu\nu} - \Gamma^{\rho}_{\sigma\nu}\Gamma^{\sigma}_{\rho\mu}, \tag{1.16}$$

which by definition is a symmetric, $R_{\mu\nu} = R_{\nu\mu}$. The Ricci scalar, R, is obtained by contracting the Ricci tensor:

$$g^{\mu\sigma} R_{\mu\sigma} = R^{\sigma}{}_{\sigma} = R. \tag{1.17}$$

Like the Riemann curvature tensor the Ricci tensor and the Ricci scalar are both second order in the metric tensor.

1.3 The Einstein Equations

We are now in a position to present Einstein's equations of general relativity: The Lorentzian metric $g_{\mu\nu}$ on a spacetime $\mathcal{M}$ should satisfy the following set of equations

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}, \tag{1.18}$$

where, as defined earlier, $R_{\mu\nu}$ is the Ricci tensor, R is Ricci scalar. These are a set of ten nonlinear coupled second-order differential equations in $g_{\mu\nu}$ and highly nontrivial to solve. The tensor $T_{\mu\nu}$ on the right hand side is the energy-momentum tensor of the matter field and c is the speed of light in vacuum. In other words, the energy momentum tensor of matter fields creates curvature. Even when $T_{\mu\nu} = 0$, one can still have nonvanishing Riemann curvature since then Einstein equation then reduce to Ricci flat metrics upon contraction

$$R_{\mu\nu} = 0 \tag{1.19}$$

or Einstein metrics

$$R_{\mu\nu} = \Lambda g_{\mu\nu} \tag{1.20}$$

when the cosmological constant is present. In either case, one can have multiple solutions even when one severely restricts the spacetime symmetries, as we will see in this thesis.

Einstein's equations can also be obtained from a variational principle — by varying the Einstein-Hilbert action [25, 17] (i.e. $\delta I = 0$)

$$I = \frac{1}{16\pi G}\int \sqrt{-g}\;\;(R + \mathcal{L}_{\mathcal{M}})\, d^4x \tag{1.21}$$

with respect to the metric tensor $g_{\mu\nu}$. Here $\sqrt{-g} \equiv \sqrt{-\det(g_{\mu\nu})}$ and $\mathcal{L}_{\mathcal{M}}$ is the matter Lagrangian that gives rise to[1]

$$T_{\mu\nu} = 2\frac{\delta I_{\text{matter}}}{\delta g^{\mu\nu}}\frac{1}{\sqrt{-g}}. \tag{1.22}$$

One often adds the cosmological constant term to the field equations instead of taking it as a part of $T_{\mu\nu}$:

$$G_{\mu\nu} + \Lambda\, g_{\mu\nu} = 8\pi T_{\mu\nu}, \tag{1.23}$$

where $G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R$ is called the Einstein tensor. One often needs to add appropriate boundary terms to the action (1.22), which plays important role in quantum gravity where the action pays a direct role.

[1] The term $\frac{1}{\sqrt{-g}}$ make $T_{\mu\nu}$ a tensor.

1.4 Solving Einstein's Equations

Because of the high nonlinearity of his equations, Einstein initially believed that there would not be any exact solutions. Soon afterwards, however, Karl Schwarzschild found an exact solution for the spacetime around a static spherically symmetric object, which surprised Einstein:[2]

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 \left(d\theta^2 + \sin^2(\theta) d\phi^2\right). \tag{1.24}$$

Schwarzschild reduced the set of ten Einstein equations to a single ordinary differential equations by using polar coordinates and the spherical symmetry of the problem and staticity (meaning the time independence, since the object is motionless).[3] Following this Einstein's equations have been systematically studied using different matter fields subject to various less restrictive local symmetries, algebraic conditions, and various simplifying assumptions. As a result today we have a good number of exact solutions in four dimensions and many of their properties are well understood [43, 22, 33]. Exact solutions are a concrete way to understand the nonlinearities of the gravitational field, and they shed light on more general non-exact solutions and guide numerical study. They have also played a central role in every program of quantum gravity [5].

1.4.1 Lie Groups in Solving Einstein's Equation

Karl Schwarzschild obtained (1.24) by reducing the ten Einstein equations to a single ordinary differential equation using the explicit spherical symmetry of the problem and staticity

[2]It is perhaps pertinent to recall here Einstein's famous quote: "Do not worry about your difficulties in Mathematics. I can assure you mine are still greater."

[3]Even if one starts with the most general time-dependent solutions with spherical symmetry, (1.24) would be the unique solution to $R_{\mu\nu} = 0$. This uniqueness theorem is known as Birkhoff's theorem.

(meaning time independence, since the object is motionless). He chose spherical coordinates to reflect the former and used a metric ansatz in which all the metric components are independent of time:

$$ds^2 = -f(r)dt^2 + g(r)dr^2 + r^2\left(d\theta^2 + \sin^2(\theta)d\phi^2\right). \tag{1.25}$$

In the language of differential geometry, any spacetime of the form above has $R \times SO(3)$ symmetry. These are Lie groups, although they were not called that at that time. These symmetries leave the metric tensor invariant and thus, are isometries and are described by Killing vector fields. In addition to Killing vectors fields, we would be interested in geodesics. These are a generalization of the shortest distance in Euclidean geometry. Geodesics and isometrics together give rise to conserved quantities. We now explain what this all means.

1.4.2 Killing Vector Fields

We are interested in symmetry transformation that preserve the distance bewteen point which in our context means the metric, i.e., we are interested in isometries. An isometry in (pseudo-)Riemannian manifold (M, g) is a diffeomorphism that preserve the metric [26, 39]:

$$\Phi : \mathcal{M} \longrightarrow \mathcal{M} \quad \text{where } \phi^{\star}(g) = g.$$

Isometries of $(\mathcal{M}^n, g)$ are described by Killing vector fields. A vector field K^μ is Killing if (since $\mathcal{L}_K g_{\mu\nu} = \nabla_{(\mu}K_{\nu)}$ is an identity)

$$\mathcal{L}_K g_{\mu\nu} \equiv \nabla_{(\mu}K_{\nu)} = 0. \tag{1.26}$$

For example, the Killing vector fields of (1.25) are

$$\begin{aligned}
k_1 &= (1,0,0,0) = \frac{\partial}{\partial t}, \\
k_2 &= (0,0,0,1) = \frac{\partial}{\partial \phi}, \\
k_3 &= (0,0,\cos\phi, -\cot\theta\sin\phi) = \cos\phi\frac{\partial}{\partial\theta} - \cot\theta\sin\phi\frac{\partial}{\partial\phi}, \\
k_4 &= (0,0,\sin\phi, +\cot\theta\cos\phi) = \sin\phi\frac{\partial}{\partial\theta} + \cot\theta\cos\phi\frac{\partial}{\partial\phi}.
\end{aligned} \tag{1.27}$$

The Lie algebra spanned by $\{k_2, k_3, k_4\}$ is $\mathfrak{so}(3)$:

$$[k_3, k_2] = k_4, \quad [k_2, k_4] = k_3, \quad [k_4, k_3] = k_2. \tag{1.28}$$

1.5 Geodesics

Parallel Transport: Let $\gamma(\tau)$ be a curve in $(\mathcal{M}^n, g)$ with tangent vector T^μ. If another vector v^μ at each point on the curve satisfies

$$T^\mu \nabla_\mu v^\nu = 0, \tag{1.29}$$

we say that v^μ is parallelly transported along $\gamma(\tau)$.[4] From the definition of covariant derivative of a vector,

$$\nabla_\mu v^\nu = \partial_\mu v^\nu + \Gamma^\nu_{\mu\sigma} v^\sigma, \tag{1.31}$$

it follows that (1.29) gives

$$\frac{dv^\nu}{d\tau} + \Gamma^\nu_{\mu\sigma} v^\sigma T^\mu = 0. \tag{1.32}$$

Note that the parallel transport of v^μ requires v^μ to be defined on $\gamma(\tau)$ and does not require it to be a vector field. The theory of ordinary differential equations tells us that the first-order equation (1.32) will have a unique solution for any given initial value $v^\mu(\tau_0)$.

Geodesics: A curve $\gamma(\lambda)$ with a tangent vector T^μ is called a geodesic if it is parallelly transported along itself:

$$T^\mu \nabla_\mu T^\nu = f(x) T^\nu, \tag{1.33}$$

[4]Likewise, we call a tensor of any rank $A^{\rho_1}_{\sigma_1}{}^{\rho_2 \ldots \rho_n}_{\sigma_2 \ldots \sigma_m}$ parallelly transported along $\gamma(\tau)$ if

$$T^\mu \nabla_\mu A^{\rho_1}_{\sigma_1}{}^{\rho_2 \ldots \rho_n}_{\sigma_2 \ldots \sigma_m} = 0. \tag{1.30}$$

where $f(x)$ is a function on the curve. However, using the freedom of parametrization (since length of a curve between two points is independent of parametrization) one can choose an affine parameter to have

$$T^\mu \nabla_\mu T^\nu = 0, \tag{1.34}$$

which also means that $g_{\mu\nu}T^\mu T^\nu$ will be constant along the curve. Through an linear transformation (that connect different affine parameters) this can further be adjusted to have $g_{\mu\nu}T^\mu T^\nu = -1$ for a timelike geodesic, i.e. this is also the proper time parameter. In a coordinate system (i.e., $T^\mu = \frac{dx^\mu}{d\tau}$) equation (1.34) then gives

$$\frac{d^2 x^\nu}{d\tau^2} + \Gamma^\nu_{\mu\sigma} \frac{dx^\mu}{d\tau} \frac{dx^\sigma}{d\tau} = 0. \tag{1.35}$$

These are a set of n coupled second order ordinary differential equations (ODE). Thus, given any initial data $(x^\mu(\tau_0), \frac{dx^\mu}{d\tau}(\tau_0))$ the theory of ODE implies there will be a unique solution. This means that for any point $p \equiv (x^0, x^1, \ldots, x^{n-1})$ on $(\mathcal{M}^n, g)$ and a tangent vector T^μ, there is a unique geodesics through p. It will have the Taylor expansion:

$$x^\nu(\tau) = \tau T^\nu(\tau_0) - \frac{\tau^2}{2}\Gamma^\nu_{\mu\sigma} T^\mu(\tau_0) T^\sigma(\tau_0) + O(\tau^3) \tag{1.36}$$

1.6 Killing Vectors and Conserved Quantities

If a curve $\gamma(\tau)$ is a geodesic with tangent vector P^μ then the quantity

$$E \equiv P^\mu K_\mu, \tag{1.37}$$

where K_μ is a Killing vector field, is conserved along the geodesic, i.e., its derivative along the geodesic is zero:

$$P^\mu \nabla_\mu E \equiv P^\mu \nabla_\mu (P^\nu K_\nu) = P^\mu(\nabla_\mu P^\nu) K_\nu + P^\mu P^\nu (\nabla_\mu K_\nu) = 0, \tag{1.38}$$

in which the first quantity vanishes due to (1.34) and second one because of contraction between the symmetric $P^\mu P^\nu$ part with the antisymmetric $\nabla_\mu K_\nu$ as in (1.26) part.

In general relativity, the energy momentum tensor is symmetric and divergence free

$$T_{\mu\nu} = T_{\nu\mu}, \tag{1.39}$$

$$\nabla_\mu T^{\mu\nu} = 0. \tag{1.40}$$

The latter follows from the Einstein tensor (i.e., the right hand side of the Einstein equation being divergence free as a consequence of the the contracted Bianchi identities: $\nabla_\mu R^\mu{}_\nu = \frac{1}{2}\nabla_\mu R$.) By a similar argument as above, it follows that the "current"

$$J^\mu = T^{\mu\nu} K_\nu \tag{1.41}$$

is conserved along a geodesic, i.e.,

$$\nabla_\mu J^\mu = 0. \tag{1.42}$$

A generalization of the concept of a Killing vector is a Killing tensor — a symmetric rank-p tensor with vanishing covariant derivative

$$\nabla_{(\mu} K_{\rho_1\rho_2\ldots\rho_p)} = 0. \tag{1.43}$$

They also provide (additional) conserved quantities along geodesics. However, they do not arise from the isometries of $(\mathcal{M}^n, g)$ and their existence is hard to detect. For example, it took some time to realize that the Kerr metric has a rank-two Killing tensor, which, as a consequence showed that geodesics of Kerr metric are integrable. There are similar examples.

In 1915, i.e., the very year in which general relativity was proposed, Emmy Noether showed that symmetries of equations arising from a variational problem admit conserved quantities. The conserved quantities from Killing vector fields of a spacetime can be seen as "Noether charges."

1.6.1 Geodesic from a Lagrangian

Timelike geodesics between any two points in a spacetime can be looked as arsing from extremizing the action

$$I = \int_{\lambda_1}^{\lambda_2} ds = \int_{\lambda_1}^{\lambda_2} \sqrt{-g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}} d\lambda \equiv \int_{\lambda_1}^{\lambda_2} L d\lambda. \tag{1.44}$$

The Euler-Lagrange equation of motion

$$\frac{d}{d\lambda}\left(\frac{\partial L}{\partial\left(dx^\mu/d\lambda\right)}\right) - \frac{\partial L}{\partial x^\mu} = 0, \tag{1.45}$$

for the above Lagrangian gives

$$\frac{d^2x^\mu}{d\lambda^2} + \Gamma^\mu_{\nu\sigma}\frac{dx^\nu}{d\lambda}\frac{dx^\sigma}{d\lambda} = \frac{1}{2}x^\mu\frac{\partial g_{\nu\rho}}{\partial x^\sigma}\left(\frac{dx^\nu}{d\lambda}\frac{dx^\rho}{d\lambda}\frac{dx^\sigma}{d\lambda}\right), \tag{1.46}$$

which is not affinely parametrized. Instead if one considers the Lagrangian

$$L = g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}, \tag{1.47}$$

one obtains

$$\frac{d^2x^\mu}{d\lambda^2} + \Gamma^\mu_{\nu\sigma}\frac{dx^\nu}{d\lambda}\frac{dx^\sigma}{d\lambda} = 0 \tag{1.48}$$

which is now affinely parametrized. It is now clear that isometries of $g_{\mu\nu}$ are symmetries of L. In other words, L will be preserved under the same Lie group action that leaves $g_{\mu\nu}$ invariant.[5] Thus isometries of $g_{\mu\nu}$ would correspond to conserved quantities via Noether's theorem. For example, if the metric does not depend on one of the coordinates, say x^0, then

$$\frac{\partial L}{\partial x^0} = 0 \Rightarrow \frac{d}{d\lambda}\left(\frac{\partial L}{\partial \dot{x}^0}\right) = 0, \tag{1.49}$$

which gives a first integral. We will see how such first integrals can be used to integrate for geodesics.

[5]Since the Christoffel symbols are determined by $g_{\mu\nu}$'s, they are also preserved under the action of the same group.

1.6.2 Lie Symmetry Method of Differential Equations

Sophus Lie (1842–1899) introduced continuous groups to when study differential equations [35]. These groups came to be known as Lie groups and found immediate use in other branches of mathematics, including for obtaining exact solutions of Einstein's equations. Lie's original method has now grown into a vast field with many new techniques that use symmetry in some form or other (see, for example, [42, 35]). In addition, with the availability of symbolic computation, symmetry methods are routinely being used to study nonlinear systems. We will expand on this in Chapter 2 .

Convention

We will follow the $(-,+,+,+)$ signature convention as in [24, 10, 34, 48]. In addition, we will use geometrized unit in which the speed of light $c = 1$ and the gravitational constant $G = 1$.

1.7 Thesis Outline

This thesis is organized as follows. In Chapter 2, we discuss symmetry methods for solving differential equations arising from systems that admit a variational formulation as well as those that do not. In particular, we describe how symmetries can be extracted using Noether's theorem for variational problems and discuss the so-called direct method, which provides symmetries of systems that are not variational.

Chapter 3 analyzes geodesics in three- and higher-dimensional spacetimes that admit codimension-two spherical symmetry. We calculate symmetries of null and timelike geodesics of the BTZ metric and obtain an unexpected class of symmetries for the timelike geodesics that disappear in four or higher dimensions.

In Chapter 4, we study four-dimensional static spherically symmetric spacetimes that share the same null orbital equation. We show that there is no direct connection between the projective tensor of the optical metric and the coincidence of unparametrized null geodesics, irrespective of the presence of a cosmological constant, as was previously thought.

In Chapter 5, we study symmetries of the set of PDEs that describe axially symmetric static metrics and find all the symmetries and their integral curves. We also examine the geodesic equations of axisymmetric static solutions.

In Chapter 6, as the next logical step, we analyze the Kerr metric (which has rotation). We start by looking at the symmetries of null geodesics in the space of all rotating spacetimes, just as we did for static spacetimes. Finding any hidden symmetry, whether precise or approximate, will be instructive and a step toward the complete integration of the Kerr spacetime geodesic equations. This chapter considers the fourth conserved quantity, which is known as the Carter constant, using the Hamilton–Jacobi method. The separability of the Kerr–(anti)-de Sitter space has already been discussed in the literature [4]. We give a necessary and sufficient condition for the separability of the Jacobi action (for a given Hamiltonian) from the Levi-Civita theorem for the generic axisymmetric metric. The presence of the fourth conserved quantity is also linked to the existence of Killing–Yano tensors (square root of the tensor).

We will use the method advanced by Cheviakov and Anco for conserved quantities. We will use Maple and the GEM package developed by Cheviakov in the symbolic computation of symmetries and conservation laws of the system, especially when we have arbitrary functions or constant parameters. We have used Mathematica and Maple to draw the figures and solve some of the equations explicitly. All the work presented herein was done in collaboration with and under the supervision of Dr. Mohammad Akbar. The work on Chapter 6 was done in collaboration with Sai Madhav Modumudi.

CHAPTER 2

SYMMETRY METHODS FOR DIFFERENTIAL EQUATIONS AND CONSERVATION LAWS

> Among all of the mathematical disciplines the theory of differential equations is the most important... It furnishes the explanation of all those elementary manifestations of nature which involve time.
>
> — *Sophus Lie*

Prior to Sophus Lie, the study of differential equations was mostly centered around finding *ad hoc* methods. In 1893, Lie [31] introduced continuous groups to solve differential equations systematically. Such groups came to be known as Lie groups and independently found applications in other branches of mathematics and physics. Lie's original ideas were revived and expanded in the early 60s, starting with the work of Garrett Birkhoff (interestingly, the son of George Birkhoff, the originator of the famous Birkhoff theorem in general relativity) [7, 35, 42].

The main idea in the symmetry methods of differential equations, as we will describe below, is the use of infinitesimal generators. A symmetry group is a local transformation group that transforms a solution into another solution of the system (or maps to a different system, which we do not consider here). These transformations act on the jet space, which is a manifold comprising independent and dependent variables and the derivatives of dependent variables. A jet space does not have a metric, so such transformations do not correspond to Killing vectors. However, other non-metrical concepts that we discuss in the context of general relativity in Chapter 1 do apply, as we will see below. There are some basic definitions in Appendix A.

2.1 Preliminary Definitions and Examples

Below we start with some basic definitions and examples.

Definition 6 (Flows [35, 47])**.** *The flow generated by vector field* $\mathbf{v}$ *through a point* $x \in M$ *is the parameterized maximal integral curve passing through* x*, denoted* $\Psi(\varepsilon, x)$ *where* ε *is the parameter of the curve. Basically each flow line express an integral curve of* α*.*
The flow of a vector field has the following properties

$$\begin{aligned} &(i) \qquad \Psi(\delta, \Psi(\varepsilon, x)) = \Psi(\delta + \varepsilon, x), \\ &(ii) \qquad \Psi(0, x) = x, \\ &(iii) \qquad \frac{d}{d\varepsilon}\Psi(\varepsilon, x) = \mathbf{v}|_{\Psi(\varepsilon,x)}. \end{aligned} \tag{2.1}$$

If $\xi = (\xi^1, \ldots, \xi^m)$ are the coefficients of $\mathbf{v}$, we can use a Taylor expansion to find

$$\Psi(\varepsilon, x) = x + \varepsilon\xi(x) + O(\varepsilon^2).$$

Example 2. *Let* $\Psi : (-\infty, \infty) \times \mathbb{R}^2 \to \mathbb{R}^2$ *be a function,*

$$\Psi\left(t, \begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} e^{-t} & 1 \\ 1 & e^{2t} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix},$$

Then the global flow $\mathbb{R}^2$ *generated by the vector field*

$$\begin{aligned} \mathbf{v}_{(x,y)} &= \frac{\partial \Psi}{\partial t}(t, (x, y))\Big|_{t=0} = \begin{bmatrix} -e^{-t} & 0 \\ 0 & 2e^{2t} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}\Bigg|_{t=0} \\ &= \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -x \\ 2y \end{bmatrix} = -x\frac{\partial}{\partial x} + 2y\frac{\partial}{\partial y}. \end{aligned}$$

Definition 7 (Generators)**.** *Given a parameterized transformation* $\phi(x, \epsilon) : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$*, where* $\phi = (\phi^1, \cdots, \phi^n)$*, the infinitesimal generator* $\mathbf{v} = (\xi^1, \cdots, \xi^n)$ *is*

$$\mathbf{v} = \frac{d\phi}{d\epsilon}|_{\epsilon=0}. \tag{2.2}$$

If we have an infinitesimal generator, we can recover the original transformation by finding the flow of $\mathbf{v}$ through a point x_0. The flow of $\mathbf{v}$ through a point x_0 can be found by solving

$$\frac{d\phi^i}{d\epsilon} = \xi^i(x); \quad \phi(0) = x_0. \tag{2.3}$$

For an ordinary differential equation one can reduce the order of the differential equation by one using a one-parameter symmetry group when it is found.

Definition 8 (Lie Bracket)**.** *If $\mathbf{v}$ and $\mathbf{w}$ are vector fields, then the Lie bracket $[\mathbf{v}, \mathbf{w}]$ is a vector field such that*

$$[\mathbf{v}, \mathbf{w}](f) \equiv \mathbf{v}(\mathbf{w}(f)) - \mathbf{w}(\mathbf{v}(f)) \tag{2.4}$$

for all smooth $f : M \to \mathbb{R}$. The bracket operation $[.,.]$ on $\mathfrak{X}(\mathcal{M})$ (set of all smooth vector fields on M) have the following properties (see, for example, [36]):

i) Bilinearity

$$[a\mathbf{v}_1 + b\mathbf{v}_2, \mathbf{w}] = a[\mathbf{v}_1, \mathbf{w}] + b[\mathbf{v}_2, \mathbf{w}], \tag{2.5}$$

ii) Skew-symmetry

$$[\mathbf{v}, \mathbf{w}] = -[\mathbf{w}, \mathbf{v}], \tag{2.6}$$

iii) Jacobi identity

$$[\mathbf{u}, [\mathbf{v}, \mathbf{w}]] + [\mathbf{w}, [\mathbf{u}, \mathbf{v}]] + [\mathbf{v}, [\mathbf{w}, \mathbf{u}]] = 0, \tag{2.7}$$

for all $\mathbf{u}$, $\mathbf{v}$,$\mathbf{w}$ in $\mathfrak{X}(M)$.

Lie groups and related definitions are given in Appendix B.

Definition 9. *A vector field $\mathbf{v}$ defined on a Lie group G is right-invariant if*

$$dR_g(\mathbf{v}|_h) = \mathbf{v}|_{R_g(h)} = \mathbf{v}|_{hg}, \tag{2.8}$$

for all g and h in G.

That is, if the vector field evaluated at the image point under the right translation is the same as the image of the vector at h under the differential of right translation it is right-invariant. If $\mathbf{v}$ and $\mathbf{w}$ are right-invariant vector fields, then any linear combination is also a right-invariant vector field and the set of right-invariant vector fields forms a vector space.

Definition 10 (Symmetry Group)**.** *A group G is said to be a symmetry group of a system of differential equations $\Delta(x, u) = 0$ if whenever $u = f(x)$ is a solution of the system, then $g \cdot f(x)$ is also a solution.*

In other words, a transformation which maps each solution of the system to another solution of the system is a symmetry of the system.

Example 3. *The transformation $\tilde{x} = cx$, $\tilde{u} = \frac{u}{c}$ is a symmetry group of the equation $xu = 1$ since $\tilde{x}\tilde{u} = 1$ whenever $xu = 1$*

2.1.1 Prolongation or Extension

In the symmetry methods of differential equations the space on which the symmetry should act is not just the set of independent and dependent variables. The concept of the prolongation extends transformation group to act on the derivative of the dependent variables as well and defines a manifold called Jet space [42, 35].

Definition 11 (Jet Space)**.** *The n-th order jet space is the extension of a space of independent and dependent variables $X \times U$ to include the space of all n-th order derivatives of u with respect to combinations of x. This is denoted $X \times U^{(n)}$. In the simple way, the group of dependent, independent and derivative of independent function is jet space.*

Definition 12 (Total Derivative)**.** *The total derivative of an expression is given by the derivative while treating the dependent variables as functions of the independent variables.*

For example, if u is dependent on x, then

$$\begin{aligned} D_x(Q(x,u(x))) &= \frac{\partial}{\partial x}Q(x,u(x)) \\ &= \frac{\partial Q}{\partial x} + \frac{\partial Q}{\partial u}\frac{\partial u}{\partial x} \\ &= \frac{\partial Q}{\partial x} + u_x\frac{\partial Q}{\partial u}. \end{aligned} \tag{2.9}$$

Example 4. *From the example (6) we have*

$$\begin{aligned} \phi^x &= D_x(\phi - \xi u_x) + \xi u_{xx} \\ &= D_x(x - (-u)(u_x)) + (-u)(u_{xx}) \\ &= 1 + {u_x}^2 + u u_{xx} - u u_{xx} \\ &= 1 + {u_x}^2. \end{aligned} \tag{2.10}$$

as previously found.

Definition 13 (Prolongation)**.** *The prolongation* $\mathrm{pr}^{(n)} f$ *of a function* $u = f(x)$ *is the* n*-tuple consisting of* f *and all derivatives up to order* n*.*

Example 5. *Let* $u(x,y) = \cos(x) + e^{2y}$*. Then*

$$\mathrm{pr}^{(2)}(u) = \left(\cos(x) + e^{2y}, -\sin(x), 2e^{2y}, -\cos(x), 0, 4e^{2y}\right).$$

The prolongation of an infinitesimal generator is defined to be the infinitesimal generator of the prolonged transformation. That is if $\mathbf{v}$ is the generator for $\phi(x,u,\epsilon)$, then

$$\mathrm{pr}^{(k)}\mathbf{v} = \frac{d}{d\epsilon}\mathrm{pr}^{(k)}\phi(x,\epsilon)|_{\epsilon=0}$$

Definition 14 (Prolongation Formula)**.** *Let*

$$\mathbf{v} = \sum_{i=1}^{p} \xi^i(x,u)\frac{\partial}{\partial x^i} + \sum_{\alpha=1}^{q} \phi_\alpha(x,u)\frac{\partial}{\partial u^\alpha} \tag{2.11}$$

be a vector field defined on an open subset M of $X \times U$. The n-th prolongation of $\mathbf{v}$ is

$$\mathrm{pr}^{(n)}\mathbf{v} = \mathbf{v} + \sum_{\alpha=1}^{q}\sum_{J}\phi_\alpha^J(x, u^{(n)})\frac{\partial}{\partial u_J^\alpha} \tag{2.12}$$

where the coefficient functions ϕ_α^J are given by

$$\phi_\alpha^J = D_J(\phi_\alpha - \sum_{i=1}^{p}\xi^i u_i^\alpha) + \sum_{i=1}^{p}\xi^i u_{J,i}^\alpha. \tag{2.13}$$

Definition 15. *The prolongation $\mathrm{pr}^{(n)}g$ of a group action is defined such that the prolonged group action acting on a point is the same as the prolongation of the image of the point. That is, if $(\tilde{x}, \tilde{u}) = g \cdot (x, u)$, then*

$$\mathrm{pr}^{(n)}g \cdot (x, u^{(n)}) = (\tilde{x}, \tilde{u}^{(n)}), \tag{2.14}$$

where

$$\tilde{u}^{(n)} = \mathrm{pr}^{(n)}(g \cdot f)(\tilde{x}). \tag{2.15}$$

Example 6. *Consider the group of transformations [35]*

$$(\tilde{x}, \tilde{u}) = (x\cos\varepsilon - u\sin\varepsilon, x\sin\varepsilon + u\cos\varepsilon), \tag{2.16}$$

which has the infinitesimal generator

$$v = -u\frac{\partial}{\partial x} + x\frac{\partial}{\partial u}.$$

We can find the derivative of the transformed function,

$$\tilde{u}_{\tilde{x}} = \frac{d\tilde{u}}{d\tilde{x}} = \frac{\frac{d\tilde{u}}{dx}}{\frac{d\tilde{x}}{dx}} = \frac{\sin\varepsilon + u_x\cos\varepsilon}{\cos\varepsilon - u_x\sin\varepsilon}. \tag{2.17}$$

To find the prolonged infinitesimal generator, we differentiate with respect to ε and set $\varepsilon = 0$,

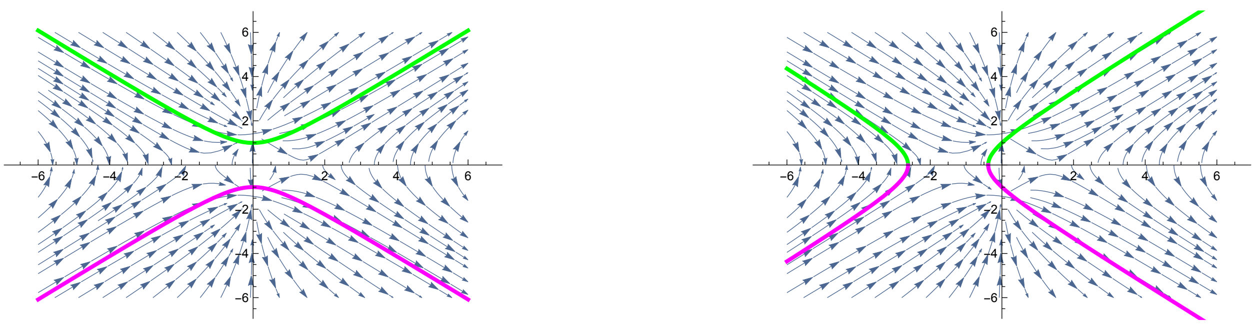
$$\begin{aligned}
&\frac{d}{d\varepsilon}\left(\frac{\sin(\varepsilon) + u_x\cos(\varepsilon)}{\cos(\varepsilon) - u_x\sin(\varepsilon)}\right)|_{\varepsilon=0} \\
&= \frac{(\cos\varepsilon - u_x\sin\varepsilon)(\cos\varepsilon - u_x\sin\varepsilon) - (\sin\varepsilon + u_x\cos\varepsilon)(-\sin\varepsilon - u_x\cos\varepsilon)}{(\cos\varepsilon - u_x\sin\varepsilon)^2}|_{\varepsilon=0} \\
&= 1 + {u_x}^2
\end{aligned} \tag{2.18}$$

Thus,

$$\mathrm{pr}^{(1)}\mathbf{v} = -u\frac{\partial}{\partial x} + x\frac{\partial}{\partial u} + (1 + {u_x}^2)\frac{\partial}{\partial u_x}. \tag{2.19}$$

Definition 16. *Nontrivial symmetry maps a point of the solution curve to another one. (Trivial symmetry maps every solution curve to itself.)*

Example 7. *Consider $y' = \frac{1}{2}\frac{x}{y} + \frac{1}{2}\frac{y}{x} - \frac{1}{xy}$. This admits the general solution $y = \sqrt{x^2 + cx + 1}$ which can be seen as arising from $y = \sqrt{x^2+1}$ via a one-parameter transformation with c as the parameter. Successive transformation T_{c_1} and T_{c_2} gives $T_{c_1+c_2}$. The figures on the left below represents the vector field coresponding to the solution $y = \sqrt{x^2+1}$ and the one on the right represents the transformed solution with $c = 3$.*



(a) Solution curve.

(b) Transformed of solution curve.

Figure 2.1: Nontrivial Symmetry Transformation

2.2 Symmetry analysis of Nonlinear Equation and Examples

Theorem 1. *Let $\Delta(x, u^{(n)}) = 0$ be a differential equation defined on s subset M of $X \times U$ and let G be a local group of transformations acting on M. If*

$$\mathrm{pr}^{(n)}\mathbf{v}[\Delta(x, u^{(n)})] = 0, \quad \textit{whenever} \quad \Delta(x, u^{(n)}) = 0 \tag{2.20}$$

for every infinitesimal generator $\mathbf{v}$ of G, then G is a symmetry group of the equation.

Thus one needs to consider prolongation only up to the order of the differential equation.

Example 8 (Symmetry Analysis of Nonlinear Equation)**.** *Consider the nonlinear equation*

$$uu_x - x = 0. \tag{2.21}$$

We seek to find a transformation $\tilde{x} = \tilde{x}(x,u)$, $\tilde{u} = \tilde{u}(x,u)$ such that $\tilde{u}\tilde{u}_{\tilde{x}} - \tilde{x} = 0$, whenever $uu_x - x = 0$. We begin with the general vector field

$$\mathbf{v} = \xi(x,u)\frac{\partial}{\partial x} + \phi(x,u)\frac{\partial}{\partial u} \tag{2.22}$$

acting on $X \times U$. Since this is a first order equation, we will need the first prolongation of this vector

$$\mathrm{pr}^{(1)}\mathbf{v} = \xi\frac{\partial}{\partial x} + \phi\frac{\partial}{\partial u} + \phi^x\frac{\partial}{\partial u_{,x}}. \tag{2.23}$$

Let $\Delta = uu_x - x$, then

$$\mathrm{pr}^{(1)}\mathbf{v}(\Delta) = -\xi + \phi u_x + \phi^x u \tag{2.24}$$

As we know a vector field $\mathbf{v}$ generates a one-parameter symmetry group if and only if

$$\mathrm{pr}^{(1)}\mathbf{v}(\Delta) = 0, \tag{2.25}$$

that is,

$$-\xi + \phi u_x + \phi^x u = 0 \tag{2.26}$$

We use the prolongation formula (2.22) to find ϕ^x,

$$\begin{aligned}
\phi^x &= D_x(\phi - \xi u_{,x}) + \xi u_{,xx} = D_x(\phi) - D_x(\xi u_x) + u_{xx}\xi \\
&= \phi_x + u_x\phi_u - D_x(\xi)u_x - \xi D_x(u_x) + \xi u_{xx} \\
&= \phi_x + \phi_u u_x - \xi_x u_x - \xi_u u_x^2 - \xi u_{xx} + \xi u_{,xx} \\
&= \phi_x + \phi_u u_x - \xi_x u_x - \xi_u u_x^2,
\end{aligned} \tag{2.27}$$

where D_x is the total derivative with respect to x:

$$D_x = \frac{\partial}{\partial x} + u_{,x}\frac{\partial}{\partial u} + u_{,xx}\frac{\partial}{\partial u_x} + \ldots$$

To find the symmetry group, we now apply this to the differential equation (2.21), which gives

$$\begin{aligned}
&-\xi + \phi u_x + (\phi_x + \phi_u u_x - \xi_x u_x - \xi_u u_x^2)u \\
&-\xi + \phi u_x + \phi_x u + u\phi_u u_x - u\xi_x u_x - u\xi_u u_x^2 = 0.
\end{aligned} \tag{2.28}$$

Setting the coefficients of u, u_x, u_x^2 coefficients to zero:

monomial	coefficient
1	$-\xi + u\phi_x = 0$
u_x	$\phi + u\phi_u - u\xi_x = 0$
u_x^2	$-u\xi_u = 0$

From the last equation we see that $\xi = \xi(x)$. The first equation then gives ϕ. Therefore

$$\xi = C_1 x + C_2, \tag{2.29}$$

$$\phi = \frac{1}{u}\left(\frac{x^2+u^2}{2}C_1 + xC_2 + C_3\right), \tag{2.30}$$

where C_1, C_2, C_3 are arbitrary constants. Thus the Lie algebra of the infinitesimal symmetries of equation (2.21) will be spanned by following three vector fields:

$$v_1 = 2x\frac{\partial}{\partial x} + \frac{x^2+u^2}{u}\frac{\partial}{\partial u} \tag{2.31}$$

$$v_2 = \frac{\partial}{\partial x} + \frac{x}{u}\frac{\partial}{\partial u} \tag{2.32}$$

$$v_3 = \frac{1}{u}\frac{\partial}{\partial u}. \tag{2.33}$$

To find the flow of the infinitesimal generators we need to integrate subject to the initial conditions: $\tilde{x}(0) = x$, and $\tilde{u}(0) = u$. We get

$$G_1 = (\varepsilon + x, \sqrt{2x + 2\varepsilon^2 + u^2}), \tag{2.34a}$$

$$G_2 = (x, \sqrt{2\varepsilon + u}), \tag{2.34b}$$

$$G_3 = (x, e^{\varepsilon}u). \tag{2.34c}$$

For example, G_2 can be computed as

$$\frac{d\tilde{x}}{d\varepsilon} = 1 \longrightarrow \tilde{x} = \varepsilon + c$$

$$\frac{d\tilde{u}}{d\varepsilon} = \frac{\tilde{x}}{\tilde{u}} \longrightarrow \sqrt{2\varepsilon + c}.$$

2.3 Noether's Theorems Vs the Direct Method

As mentioned in Chapter 1, Emmy Noether found the connection between symmetries and local conservation laws for partial differential equations (PDEs) that admit a variational principle, i.e., PDEs that arise from extremizing an action functional (see, for example, [35]). The ODEs and the PDEs that we will study in this thesis are variational. Despite this, we will follow a new method, called the direct method, in computing conserved quantities [6]. This method, unlike the Noether theorems, can compute conserved quantities directly from the differential equations irrespective of whether they arise from a variational principle or not. Another advantage for which this system is preferred over Noether's theorems is that conservation laws based on this method are coordinate-independent, unlike Noether's.

2.4 The Direct Method: Generalizing Noether

Direct method was advanced by Blueman and Anco and is now popular (see, for example, [3]) as it can find possible local conservation laws from any set of differential equations. Consider a system of N PDEs of order k

$$\Delta^\sigma[u] = \Delta^\sigma(x, u, \partial u, ..., \partial^k u) = 0 \quad k = 1, 2, ..., \sigma \tag{2.35}$$

We have n independent and m dependent variables: $x = (x^1, ..., x^n)$, $u(x) = (u^1(x), ..., u^m(x))$.

Definition 17 (Local Conservation Laws)**.** *A divergence expression is zero on solutions of*

$$\nabla(\phi) := D_i \phi^i[u] = 0, \quad i = 1, ..., n \tag{2.36}$$

Here D_i are the **total derivative** operators defined earlier:

$$D_i = \frac{\partial}{\partial x^i} + u_i \frac{\partial}{\partial u} + u_{ij} \frac{\partial}{\partial u^i} + ... \tag{2.37}$$

and $\phi^i[u] = \phi^i(x, u, \partial u, ..., \partial^k u)$ are fluxes which are conservation laws.

Definition 18 (Euler Operator with respect to u^j).

$$E_{u^j} = \frac{\partial}{\partial u^j} - D_i \frac{\partial}{\partial u_i^j} + ... + (-1)^s D_{i_1}...D_{i_s} \frac{\partial}{\partial u_{i_1...i_s}^j} + ..., \quad j = 1, ..., m \tag{2.38}$$

Theorem 2. *The equations*

$$E_{u_j} F(x, u, \partial u, ...) \equiv 0, j = 1, ..., m \tag{2.39}$$

hold for arbitrary $u(x)$ if and only if

$$F(x, u, \partial u, ...) \equiv D_i \Phi^i(x, u, \partial u, ...), \quad i = 1, ..., n. \tag{2.40}$$

Linear determining equation for finding local CL multipliers

Theorem 3. *A set of non-singular local multipliers $\{\mu_\sigma(x, u, \partial u, ..., \partial^l u\}_{\sigma=1}^N$ yields a local conservation law for the PDE system $\Delta^\sigma[u] = 0$ if and only if the set of identities*

$$E_{u_j}(\mu_\sigma(x, u, \partial u, ..., \partial^l u)\Delta^\sigma(x, u, \partial u, ..., \partial^k u) \equiv 0, \quad j = 1, ..., m \tag{2.41}$$

holds for arbitrary functions $u(x)$.

Example 9 (Non-Linear Equation($uu_x = x$)). *For finding conservation laws we need to seek multiplier $\mu(x, u, u_x)$, where the Euler operator is given by*

$$E_u = \frac{\partial}{\partial u} - D_x \frac{\partial}{\partial u_x} + D_x^2 \frac{\partial}{\partial u_{xx}}. \tag{2.42}$$

For finding determining equations we have:

$$\begin{aligned} E_u\Big[\mu(uu_x - x)\Big] &= 0, \\ \left(\frac{\partial}{\partial u} - D_x \frac{\partial}{\partial u_x} + D_x^2 \frac{\partial}{\partial U_{xx}}\right)(\mu(uu_x - x)) &= 0. \end{aligned} \tag{2.43}$$

Simplifying we obtain (the determining equations):

$$\mu_{uu} = \frac{\mu_u}{u}, \quad \mu_x = -\frac{\mu_u x}{u}, \quad \mu_{ux} = 0. \tag{2.44}$$

This gives

$$\mu = C_1 + C_2\left(-u^2 + x^2\right), \tag{2.45}$$

where C_1 *and* C_2 *are arbitrary constants. Now we consider two cases:*

Case 1: $\mu = 1$*:*

$$(u.u_x - x) = D_x(\frac{1}{2}u^2 - \frac{1}{2}x^2) = 0. \tag{2.46}$$

Case 2: $\mu = -u^2 - x^2$

$$(-u^2 - x^2)(u.u_x - x) = D_x(\frac{-\left(u^2 - x^2\right)^2}{4}) = 0. \tag{2.47}$$

In the rest of the dissertation we will be using direct method like the above to find local conservation laws after we have found symmetries by the process described earlier.

CHAPTER 3

GEODESICS IN SPHERICALLY SYMMETRIC STATIC SPACETIMES IN THREE AND HIGHER DIMENSIONS

> Algebras (jabbre and maqabeleh) are geometric facts which are proved by propositions five and six of Book two of Elements.
>
> — *Omar Khayyam*

In this chapter we study symmetries of geodesic equation in three and higher dimensions in the possible presence of a cosmological constant. We will study spherically symmetric static metrics which for $n \geq 3$ dimension would mean that the constant t hypersurfaces of the static metric

$$ds_d^2 = -f(x^i)dt^2 + g_{ij}dx^i dx^j \tag{3.1}$$

admits an $SO(n-2)$ symmetry. Adopting polar coordinates solutions of these metrics are are known: (i) in $n = 3$ Banados–Teitelboim– Zanelli (BTZ) solution with a negative cosmological constant, and (i) in $n \geq 4$ Schwarzschild-(anti-) de Sitter metrics. We are interested in symmetries of their timelike and null geodesics.

3.1 Geodesics in BTZ Metric

Below we first study three dimensions. There is an established tradition of studying $2+1$ dimensional gravity (see, for example, [9]) and many results and insights obtained in three dimensions persist in four and higher dimensions. However, unlike four and higher dimensions, one requires matter fields in three dimensions to produce curvature since vacuum solutions in three dimension necessarily have vanishing Ricci curvature tensor (without any symmetry assumptions), i.e., flat. However, this changes when one adds a negative cosmological constant and there is a unique static axisymmetric static solution — the famous

Banados–Teitelboim– Zanelli (BTZ) solution:

$$ds^2 = -\left(\frac{r^2}{l^2} - M\right) dt^2 + \left(\frac{r^2}{l^2} - M\right)^{-1} dr^2 + r^2 d\phi^2, \tag{3.2}$$

where $l^{-2} = -\Lambda$, where $\Lambda < 0$. This is a black hole solution with a horizon at $r = r_H = \sqrt{M}l$. It can shown that the spacetime is non singular at the horizon by moving to another coordinate system. Note that M, i.e., the mass, is positive on physical consideration, but can be allowed to have negative values in which case there is no horizon. Unlike the Schwarzschild solution described in Chapter 1 (and later in this chapter), $r = 0$ is not a singularity in this spacetime either. This solution came as a surprise when it was first found in 1992, and since then it has been used to study various classical and quantum aspects of gravity. We will show how geodesic equations in BTZ stand in sharp contrast to those of Schwarzschild-(anti) de Sitter metrics in four and higher dimensions in terms of their internal symmetry group.

It is easy to show that, starting with the general static axisymmetric metric in 2 + 1 dimensions

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{g(r)} + r^2 d\phi^2, \tag{3.3}$$

that BTZ is the unique solution of Einstein's equation with a cosmological constant, $R_{\mu\nu} = \Lambda g_{\mu\nu}$. The unique solutions of this system is $f = g = -\Lambda r^2 - M$. It is clear that one needs $\Lambda < 0$ to have a Lorentzian solution with t as the time coordinate.

We will start with Lagrangian derived from the metric (3.3) as described in Chapter 1:

$$L = -f(r)\dot{t}^2 + \frac{1}{g(r)}\dot{r}^2 + r^2\dot{\phi}^2. \tag{3.4}$$

Since the metric (3.3) is independent of the coordinates t and ϕ, we obtain three equations and three first integrals

$$\ddot{t} + \frac{f'(r)}{f(r)}\dot{r}\,\dot{t} = 0, \tag{3.5}$$

$$\ddot{r} + \frac{1}{2}g(r)f'(r)\dot{t}^2 - \frac{1}{2}\frac{g'(r)}{g(r)}\dot{r}^2 - rg(r)\dot{\phi}^2 = 0, \tag{3.6}$$

$$\ddot{\phi} + \frac{2}{r}\dot{r}\dot{\phi} = 0, \tag{3.7}$$

$$E = f(r)\dot{t}, \tag{3.8}$$

$$J = r^2\dot{\phi}, \tag{3.9}$$

$$L = -f(r)\dot{t}^2 + \frac{1}{g(r)}\dot{r}^2 + r^2\dot{\phi}^2. \tag{3.10}$$

ensuring complete integrability. Eliminating $\dot{t}$ and $\dot{\phi}$ using (3.8) and (3.9) one obtains from (3.10):

$$L = -\frac{E^2}{f(u)} + \frac{\dot{u}}{u^4 g(u)} + J^2u^2 \tag{3.11}$$

in which the standard substitution $u(\phi) = 1/r(\phi)$ has been made. For an unparametrized form of the geodesic, we need u as a function of ϕ. Converting the derivative $\dot{u} = \frac{du}{d\phi}\dot{\phi}$ one obtains

$$\left(\frac{du}{d\phi}\right)^2 + g(u)u^2 - \frac{g(u)}{f(u)}\frac{E^2}{J^2} = \frac{Lg(u)}{J^2}, \tag{3.12}$$

Null Geodesics

The unparametrized null geodesics follows upon differentiating (3.12) again with respect to ϕ with $L = 0$:

$$\frac{d^2u}{d\phi^2} = -\left(\frac{1}{2}g(u)u^2\right)' + \frac{1}{2}\frac{E^2}{J^2}\left(\frac{g(u)}{f(u)}\right)'. \tag{3.13}$$

For the BTZ solution $f = g = (\frac{r^2}{l^2} - M)$. The null geodesic is then

$$\frac{d^2u}{d\phi^2} = Mu. \tag{3.14}$$

Therefore, unparametrized null geodesic equation is not dependent on Λ, as in four-dimensional Schwarzschild-(anti) de Sitter spacetimes [28].

Equation (3.14) is the repulsive one dimensional oscillator (for $M < 0$ it a normal harmonic oscillator, which we will discuss shortly). This admits the exact solution:

$$\frac{1}{r} = A\cosh(\phi) + B\sinh\phi \equiv C_1 e^{p\phi} + C_2 e^{-p\phi}. \tag{3.15}$$

Thus clearly no closed null geodesic exists. Of course, the angle measurement of bending of light will depend on the the two-dimensional spatial geometry.

$$ds^2 = \left(\frac{r^2}{l^2} - M\right)^{-1} dr^2 + r^2 d\phi^2, \tag{3.16}$$

and will be affected by Λ. In four dimensions and higher similar absence of the cosmological constant "orbital light bending equation" has been observed and debated and the consensus is that a comoving observer will indeed see the effect of the cosmological constant with which we agree.

Timelike Geodesics

We differentiate (3.12) with respect to ϕ with $L = -1$ and making the substitution $l^{-2} = -\Lambda$ (so that Λ is positive quantity) and obtain for BTZ:

$$\frac{d^2u}{d\phi^2} = \frac{\Lambda}{J^2u^3} + Mu \tag{3.17}$$

The timelike geodesics depends on Λ. This can be integrated exactly to give (for positive M)

$$u(\phi) = \pm\frac{1}{2}\frac{\sqrt{C_1 M\left(4\Lambda C_1^2 J^2 M + 4pe^{-2p\phi}C_2 + 4MC_2^2 + e^{-4p\phi}\right)}}{C_1 J M e^{-p\phi}}. \tag{3.18}$$

3.1.1 Symmetries of Null Geodesics of BTZ with Positive Mass M

The null geodesic equation with positive mass is given by

$$\frac{d^2u}{d\phi^2} = Mu. \tag{3.19}$$

This system is well studied in the literature as repulsive harmonic oscillator and the symmetry group was shown to be $SL(3,R)$ for any real value of M [32, 30]. Below we obtain the generators of this symmetry group in a slightly different form than in [30].

To find the symmetry group of (3.19), as described in Chapter 2, we need the second prolongation $Pr^{(2)}$ (i.e., derivatives of the dependent variable u up to second order) of the infinitesimal generator $\mathbf{v}$:

$$\mathbf{v} = \xi(\phi,u)\frac{\partial}{\partial\phi} + \eta(\phi,u)\frac{\partial}{\partial u}, \tag{3.20}$$

in which $\xi(\phi,u)$ and $\eta(\phi,u)$ are one-parameter groups. We take second order prolongation of the generator as follows

$$Pr^{(2)}\mathbf{v} = \xi\frac{\partial}{\partial\phi} + \eta\frac{\partial}{\partial u} + \eta^{\phi}\frac{\partial}{\partial u_{,\phi}} + \eta^{\phi\phi}\frac{\partial}{\partial u_{,\phi\phi}}. \tag{3.21}$$

Let $\Delta = u_{\phi\phi} - Mu$, then

$$Pr^{(2)}\mathbf{v}(\Delta) = \eta^{\phi\phi} - M\eta, \tag{3.22}$$

where from the Jacobian matrix we have

$$\frac{\partial\Delta}{\partial u_{\phi\phi}} = 1, \quad \frac{\partial\Delta}{\partial u} = -M.$$

By Theorem (2.31) of Olver [35] a vector field $\mathbf{v}$ generates a one-parameter symmetry group if and only if

$$Pr^{(2)}\mathbf{v}(\Delta) = 0, \quad \text{whenever} \quad \Delta = 0. \tag{3.23}$$

That is,

$$\eta^{\phi\phi} - M\eta = 0. \tag{3.24}$$

To find the coefficients of $\frac{\partial}{\partial u_{,\phi}}$ and $\frac{\partial}{\partial u_{,\phi\phi}}$ we apply formula (2.39) from [35]

$$\begin{aligned} \eta^{\phi} &= D_{\phi}(\eta - \xi u_{\phi}) + \xi u_{\phi\phi} \\ &= \eta_{\phi} + \eta_u u_{\phi} - \xi_{\phi} u_{\phi} - \xi_u u_{\phi}^2 - \xi u_{\phi\phi} + \xi u_{\phi\phi} \\ &= \eta_{\phi} + \eta_u u_{\phi} - \xi_{\phi} u_{\phi} - \xi_u u_{\phi}^2, \end{aligned} \tag{3.25}$$

$$\begin{aligned} \eta^{\phi\phi} &= D_{\phi}^2(\eta - \xi u_{\phi}) + \xi u_{\phi\phi\phi} \\ &= \eta_{\phi\phi} + (2\eta_{\phi u} - \xi_{\phi\phi})u_{\phi} + (\eta_{uu} - 2\xi_{\phi u})u_{\phi}^2 - \xi_{uu}u_{\phi}^3 + (\eta_u - 2\xi_{\phi})u_{\phi\phi} - 3\xi_u u_{\phi} u_{\phi\phi}. \end{aligned} \tag{3.26}$$

Where D_{ϕ} is the total derivative with respect to ϕ:

$$D_{\phi} = \frac{\partial}{\partial \phi} + u_{\phi}\frac{\partial}{\partial u} + u_{\phi\phi}\frac{\partial}{\partial u_{\phi}} + ...$$

Now we substitute equation (3.26) into (3.24) :

$$\eta_{\phi\phi} + (2\eta_{\phi u} - \xi_{\phi\phi})u_{\phi} + (\eta_{uu} - 2\xi_{\phi u})u_{\phi}^2 - \xi_{uu}u_{\phi}^3 + (\eta_u - 2\xi_{\phi})u_{\phi\phi} - 3\xi_u u_{\phi} u_{\phi\phi} - M\eta = 0, \tag{3.27}$$

and replace $u_{\phi\phi}$ by Mu in above equation to reduce the order of equation

$$\eta_{\phi\phi} + (2\eta_{\phi u} - \xi_{\phi\phi})u_{\phi} + (\eta_{uu} - 2\xi_{\phi u})u_{\phi}^2 - \xi_{uu}u_{\phi}^3 + (\eta_u - 2\xi_{\phi})Mu - 3\xi_u u_{\phi} Mu - M\eta = 0, \tag{3.28}$$

which upon simplification gives

$$\eta_{\phi\phi} - M\eta - \xi_{uu}u_{\phi}^3 + (\eta_{uu} - 2\xi_{u\phi})u_{\phi}^2 - 3\xi_{uu}u_{\phi}Mu + (\eta_{u\phi} - \xi_{\phi\phi})u_{\phi} + (\eta_u - 2\xi_{\phi})Mu = 0. \tag{3.29}$$

We can write the coefficients of equation of (3.29) in terms of u_{ϕ} which are the determining equations for the Lie point symmetries, following table presents that coefficients:

monomial	coefficient
u_{ϕ}^3	$-\xi_{uu} = 0$
u_{ϕ}^2	$\eta_{uu} - 2\xi_{u\phi} = 0$
u_{ϕ}	$-3\xi_{uu}Mu + (\eta_{u\phi} - \xi_{\phi\phi}) = 0$
1	$\eta_{\phi\phi} - M\eta + (\eta_u - 2\xi_{\phi})Mu = 0$

By solving the last three equations we can obtain ξ and ϕ:

$$\xi = C_1ue^{p\phi} + C_2ue^{-n\phi} + C_4e^{2p\phi} + C_5e^{-2p\phi} + C_3 \tag{3.30}$$

$$\eta = (-u^2pC_2 + C_8)e^{-p\phi} + (u^2pC_1 + C_7)e^{p\phi} - C_5upe^{-2p\phi} + C_4une^{2p\phi} + C_6u, \tag{3.31}$$

where $C_1, C_2, C_3, C_4, C_5, C_6, C_7, C_8$ are arbitrary constants and $p = \sqrt{M}$.

Therefore Lie algebra of the infinitesimal symmetries of equation (3.19) will be spanned by the following eight vector fields:

$$v_1 = \frac{\partial}{\partial \phi} \tag{3.32a}$$

$$v_2 = u\frac{\partial}{\partial u} \tag{3.32b}$$

$$v_3 = e^{p\phi}\frac{\partial}{\partial u} \tag{3.32c}$$

$$v_4 = \left(e^{-p\phi} + u\right)\frac{\partial}{\partial u} \tag{3.32d}$$

$$v_5 = e^{2p\phi}\frac{\partial}{\partial \phi} + e^{2p\phi}pu\frac{\partial}{\partial u} \tag{3.32e}$$

$$v_6 = e^{-2p\phi}\frac{\partial}{\partial \phi} - e^{-2p\phi}pu\frac{\partial}{\partial u} \tag{3.32f}$$

$$v_7 = e^{p\phi}u\frac{\partial}{\partial \phi} + e^{p\phi}u^2p\frac{\partial}{\partial u} \tag{3.32g}$$

$$v_8 = e^{-p\phi}u\frac{\partial}{\partial \phi} - e^{-p\phi}u^2p\frac{\partial}{\partial u}. \tag{3.32h}$$

The vector fields $\{v_1, \cdots, v_8\}$ generate eight dimensional Lie algebra as follows

$$\begin{aligned}
&[v_1, v_3] = pv_3, \quad [v_1, v_4] = p\left(-v_4 + v_2\right), \quad [v_1, v_5] = 2pv_5,\\
&[v_1, v_6] = -2pv_6, \quad [v_1, v_7] = pv_7, \quad [v_1, v_8] = -pv_8, \quad [v_2, v_3] = -v_3,\\
&[v_2, v_4] = v_2 - v_4, \quad [v_2, v_7] = v_7, \quad [v_2, v_8] = v_8, \quad [v_3, v_4] = v_3, \quad [v_3, v_6] = 2p\left(-v_4 + v_2\right)\\
&[v_3, v_7] = v_5, \quad [v_3, v_8] = -3pv_2 + v_1, \quad [v_4, v_5] = 2pv_3, \quad [v_4, v_7] = v_7 + 3pv_2 + v_1,\\
&[v_4, v_8] = v_6 + v_8, \quad [v_5, v_6] = -4pv_1, \quad [v_5, v_8] = -2pv_7, \quad [v_6, v_7] = 2pv_8,
\end{aligned} \tag{3.33}$$

corresponding to $sl(3, R)$ Lie algebra.

3.1.2 Null Geodesic Symmetries of BTZ Solution With negative Mass $M = -m$

Timelike equation for BTZ metric with negative mass $-m$ is obtain by

$$\frac{d^2u}{d\phi^2} = -mu, \tag{3.34}$$

Then infinitesimal generators will be

$$\begin{aligned}
v_1 &= \frac{\partial}{\partial\phi} && (3.35a)\\
v_2 &= u\frac{\partial}{\partial u} && (3.35b)\\
v_3 &= \sin(p\phi)\frac{\partial}{\partial u} && (3.35c)\\
v_4 &= (\cos(p\phi) + u)\frac{\partial}{\partial u} && (3.35d)\\
v_5 &= \sin(2p\phi)\frac{\partial}{\partial\phi} + p\cos(2p\phi)u\frac{\partial}{\partial u} && (3.35e)\\
v_6 &= \cos(2p\phi)\frac{\partial}{\partial\phi} - p\sin(2p\phi)u\frac{\partial}{\partial u} && (3.35f)\\
v_7 &= \sin(p\phi)u\frac{\partial}{\partial\phi} + p\cos(p\phi)u^2\frac{\partial}{\partial u} && (3.35g)\\
v_8 &= \cos(p\phi)u\frac{\partial}{\partial\phi} - p\sin(p\phi)u^2\frac{\partial}{\partial u}. && (3.35h)
\end{aligned}$$

The generators with commutation relations are

$$\begin{aligned}
&[v_1, v_3] = pv_4 - pv_2, \quad [v_1, v_4] = -pv_3, \quad [v_1, v_5] = 2pv_6,\\
&[v_1, v_6] = -2pv_5, \quad [v_1, v_7] = pv_8, \quad [v_1, v_8] = -pv_7, \quad [v_2, v_3] = -v_3,\\
&[v_2, v_4] = v_2 - v_4, \quad [v_2, v_7] = v_7, \quad [v_2, v_8] = v_8, \quad [v_3, v_4] = v_3,\\
&[v_3, v_5] = -pv_3, \quad [v_3, v_6] = -pv_4 + pv_2, \quad [v_3, v_7] = \frac{1}{2}(v_1 - v_6)\\
&[v_3, v_8] = \frac{1}{2}(v_5 - 3pv_2), \quad [v_4, v_5] = p(v_4 - v_2), \quad [v_4, v_6] = -pv_3,\\
&[v_4, v_7] = -v_7 + \frac{1}{2}(v_5 + 3pv_2), \quad [v_4, v_8] = v_8 + \frac{1}{2}(v_1 + v_6), \quad [v_5, v_6] = -2pv_1\\
&[v_5, v_7] = pv_7, \quad [v_5, v_8] = -pv_8, \quad [v_6, v_7] = pv_8, \quad [v_6, v_8] = pv_7, \qquad (3.36)
\end{aligned}$$

corresponding to $sl(3, R)$ Lie algebra.

3.1.3 Symmetries of Timelike Geodesics of BTZ with Positive Mass M

For the timelike geodesic equation with the positive mass M

$$\frac{d^2u}{d\phi^2} = Mu + \frac{\Lambda}{J^2u^3}, \tag{3.37}$$

we follow the same steps as before with the infinitesimal generator $\mathbf{v}$:

$$\mathbf{v} = \xi(\phi, u)\frac{\partial}{\partial\phi} + \eta(\phi, u)\frac{\partial}{\partial u}, \tag{3.38}$$

in which $\xi(\phi, u)$ and $\eta(\phi, u)$ are one-parameter groups and

$$\Delta = u_{\phi\phi} - Mu - \frac{\Lambda}{b^2u^3} \tag{3.39}$$

and the vanishing second order prolongation of vector field $\mathbf{v}$:

$$Pr^{(2)}\mathbf{v}(\Delta) = \eta^{\phi\phi} - M\eta + \frac{3\Lambda}{J^2u^4}\eta = 0, \tag{3.40}$$

to obtain

$$\eta_{\phi\phi} + (2\eta_{\phi u} - \xi_{\phi\phi})u_\phi + (\eta_{uu} - 2\xi_{\phi u})u_\phi^2 - \xi_{uu}u_\phi^3 + (\eta_u - 2\xi_\phi)u_{\phi\phi} - 3\xi_u u_\phi u_{\phi\phi} - M\eta - \frac{3\Lambda}{b^2u^4}\eta = 0. \tag{3.41}$$

Replacing $u_{\phi\phi}$ by $Mu + \frac{\Lambda}{b^2u^3}$ in above equation

$$\begin{aligned} &\eta_{\phi\phi} + (2\eta_{\phi u} - \xi_{\phi\phi})\, u_\phi + (\eta_{uu} - 2\xi_{\phi u})\, u_\phi^2 - \xi_{uu}u_\phi^3 \\ + \quad &(\eta_u - 2\xi_\phi)\left(Mu + \frac{\Lambda}{b^2u^3}\right) - 3\xi_u u_\phi \left(Mu + \frac{\Lambda}{b^2u^3}\right) - M\eta - \frac{3\Lambda}{b^2u^4}\eta = 0, \end{aligned} \tag{3.42}$$

which upon simplification gives

$$\begin{aligned} &\eta_{\phi\phi} + \left(2\eta_{\phi u} - \xi_{\phi\phi} - 3\xi_u(Mu + \frac{\Lambda}{J^2u^3})\right) u_\phi + (\eta_{uu} - 2\xi_{\phi u})\, u_\phi^2 - \xi_{uu}u_\phi^3 + (\eta_u - 2\xi_\phi) \\ &\left(Mu + \frac{\Lambda}{J^2u^3}\right) - M\eta - \frac{3\Lambda}{J^2u^4}\eta = 0. \end{aligned} \tag{3.43}$$

We can write the coefficients of equation of (3.43) in terms of u_ϕ which are the determining equations for the Lie point symmetries. The coefficient are listed below:

monomial	coefficient
u_ϕ^3	$-\xi_{uu} = 0$
u_ϕ^2	$\eta_{uu} - 2\xi_{u\phi} = 0$
u_ϕ	$2\eta_{\phi u} - \xi_{\phi\phi} - 3\xi_u(Mu + \frac{\Lambda}{J^2u^3}) = 0$
1	$\eta_{\phi\phi} + (\eta_u - 2\xi_\phi)\left(Mu + \frac{\Lambda}{J^2u^3}\right) - M\eta - \frac{3\Lambda}{J^2u^4}\eta = 0$

From the last three equations we obtain ξ and ϕ

$$\xi = C_1 + C_2e^{2p\phi} + C_3e^{-2p\phi} \tag{3.44}$$

$$\eta = p\left(C_2e^{2p\phi} - C_3e^{-2p\phi}\right)u, \tag{3.45}$$

where C_1, C_2, C_3 are arbitrary constants and $p = \sqrt{M}$. Therefore Lie algebra of the infinitesimal symmetries of equation (3.37) will be spanned by three vector fields:

$$v_1 = \frac{\partial}{\partial\phi} \tag{3.46}$$

$$v_2 = e^{2p\phi}\frac{\partial}{\partial\phi} + pe^{2p\phi}u\frac{\partial}{\partial u} \tag{3.47}$$

$$v_3 = e^{-2p\phi}\frac{\partial}{\partial\phi} - pe^{-2p\phi}u\frac{\partial}{\partial u} \tag{3.48}$$

Now the following table express the Lie algebra of v_i ,$i = 1, 2, 3$;

	v_1	v_2	v_3
v_1	0	$2pv_2$	$-2pv_3$
v_2	$-2pv_2$	0	$-4pv_1$
v_3	$2pv_3$	$4pv_1$	0

Three generators with commutation relations are

$$[v_1, v_2] = 2pv_2, \quad [v_1, v_3] = -2pv_3, \quad [v_2, v_3] = -4pv_1. \tag{3.49}$$

Then the basis vectors v_1, v_2, v_3 has commutation relations corresponding to $sl(2, R)$ Lie algebra (see the Appendix B).

3.1.4 Symmetries of Timelike Geodesics of BTZ with negative mass ($M = -m$)

Timelike equation for BTZ metric with negative mass $-m$ is obtain by

$$\frac{d^2u}{d\phi^2} = -mu + \frac{\Lambda}{J^2u^3}. \tag{3.50}$$

Then infinitesimal generators will be

$$\begin{aligned} v_1 &= \frac{\partial}{\partial\phi}, & (3.51)\\ v_2 &= \sin(2p\phi)\frac{\partial}{\partial\phi} + p\cos(2p\phi)u\frac{\partial}{\partial u}, & (3.52)\\ v_3 &= \cos(2p\phi)\frac{\partial}{\partial\phi} - p\sin(2p\phi)u\frac{\partial}{\partial u}. & (3.53) \end{aligned}$$

The generators with commutation relations are

$$[v_1, v_2] = 2pv_3, \quad [v_1, v_3] = -2pv_2, \quad [v_2, v_3] = -2pv_1, \tag{3.54}$$

corresponding to $sl(2, R)$ Lie algebra.

3.1.5 Conservation Laws for the Null and Timelike Geodesics

To discover the conservation laws, we need to calculate the multiplier $\mu(\phi, u)$ for the whole calculation of all equations and take the Euler operator as

$$E_U = \frac{\partial}{\partial U} - D_\phi\frac{\partial}{\partial U_\phi} + D_\phi^2\frac{\partial}{\partial U_{\phi\phi}}. \tag{3.55}$$

Null Geodesics

To find the determining equation we have

$$\begin{aligned} E_u\Big[\mu(u_{\phi\phi} - Mu)\Big] &= 0 \\ (\frac{\partial}{\partial u} - D_\phi \frac{\partial}{\partial u_\phi} + D_\phi^2 \frac{\partial}{\partial u_{\phi\phi}})(\mu(u_{\phi\phi} - Mu)) &= 0, \end{aligned} \tag{3.56}$$

where μ is multiplier. Then by simplifying the last equation, we obtain 4 determining equations

$$2\mu_u = 0, \quad 2\mu_{\phi u} = 0, \quad -\mu_u Mu - \mu M + \mu_{\phi\phi} = 0, \quad \mu_{uu} = 0. \tag{3.57}$$

Finally multipliers solution by solving last first order linear partial differential equation given by

$$\mu = C_1 e^{p\phi} + C_2 e^{-p\phi}. \tag{3.58}$$

Now we can consider 2 cases:

Case 1: $\mu = e^{p\phi}$:

$$e^{p\phi}(u_{\phi\phi} - pu) = D_\phi(-e^{p\phi}(pu - u_\phi)) = 0, \tag{3.59}$$

because

$$\begin{aligned} & D_\phi(-e^{p\phi}(pu - u_\phi)) \\ = \ & (\frac{\partial}{\partial \phi} + u_\phi \frac{\partial}{\partial u} + u_{\phi\phi}\frac{\partial}{\partial u_\phi})(-e^{p\phi}(pu - u_\phi)) \\ = \ & -Me^{p\phi}u + pe^{p\phi}u_\phi - pe^{p\phi}u_\phi + u_{\phi\phi}e^{p\phi} \\ = \ & e^{p\phi}(-Mu + u_{\phi\phi}) = 0. \end{aligned} \tag{3.60}$$

Similarly for case 2 we have:

Case 2: $\mu = e^{-p\phi}$

$$e^{-p\phi}(u_{\phi\phi} - pu) = D_\phi(e^{-p\phi}(pu + u_\phi)) = 0. \tag{3.61}$$

Timelike Geodesics

Similarly, from the Euler equation for finding determining equation we have

$$\begin{aligned}
E_u\Big[\mu(u_{\phi\phi} - Mu - \frac{\Lambda}{b^2u^3})\Big] &= 0, \\
(\frac{\partial}{\partial u} - D_\phi\frac{\partial}{\partial u_\phi} + D_\phi^2\frac{\partial}{\partial u_{\phi\phi}})(\mu(u_{\phi\phi} - Mu - \frac{\Lambda}{b^2u^3})) &= 0.
\end{aligned} \tag{3.62}$$

Then simplifying the latter equation one obtains four determining equations

$$\begin{aligned}
\mu_{\phi,\phi} &= \frac{-4\mu_{u_\phi}ku^2u_\phi + 4\mu_{u_\phi}u_\phi^3 + 4\mu(\phi,u,u_\phi)ku^2 - 2\mu_\phi uu_\phi - 4\mu(\phi,u,u_\phi)u_\phi^2}{u^2}, \\
\mu_{u_{\phi\phi}} &= \frac{2\mu_{u_\phi}u_\phi - 2\mu(\phi,u,u_\phi)}{u}, \\
\mu_u &= \frac{-\mu_{u_\phi}ux + \mu(\phi,u,u_\phi)}{u}, \quad \mu_{u_\phi u_\phi} = 0.
\end{aligned} \tag{3.63}$$

From these four determining equations one can calculate $\mu(\phi,u,u_\phi)$ given by

$$\mu(\phi,u,u_\phi) = C_3(up + u_\phi)\mathrm{e}^{-2p\phi} - C_2(up - u_\phi)\mathrm{e}^{2p\phi} + C_1u_\phi. \tag{3.64}$$

Therefore three cases are the following:

Case 1: $\mu = u_\phi$

$$u_\phi\left(u_{\phi\phi} - \frac{\Lambda}{J^2u^3} - Mu\right) = 0, \tag{3.65}$$

Case 2: $\mu = -\left(up - u_\phi\right)e^{2p\phi}$

$$-\left(up - u_\phi\right)e^{2p\phi}\left(u_{\phi\phi} - \frac{\Lambda}{J^2u^3} - Mu\right) = 0, \tag{3.66}$$

Case 3: $\mu = \left(up + u_\phi\right)e^{-2p\phi}$

$$\left(up + u_\phi\right)e^{-2p\phi}\left(u_{\phi\phi} - \frac{\Lambda}{J^2u^3} - Mu\right) = 0. \tag{3.67}$$

3.2 Geodesics in Schwarzschild-(anti-)de Sitter

Geodesics equations for the general static spherically symmetric spacetimes in four dimensions

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{g(r)} + r^2 \left(d\theta^2 + \sin^2\theta d\phi^2\right), \tag{3.68}$$

can be obtained from the Lagrangian

$$L = -f(r)\dot{t}^2 + \frac{1}{g(r)}\dot{r}^2 + r^2\left(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2\right). \tag{3.69}$$

As in the case with Schwarzschild, the equation for θ, and spherical symmetry imply that it is sufficient to consider geodesics in the equatorial plane, $\theta = \pi/2$, without any loss of generality. This leaves one with three equations and three first integrals which are identical to equations (3.5)–(3.10) for BTZ. The same argument can be extended to $n > 4$ to consider only "equatorial plane" of the general metric in dimensions $n \geq 4$

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{g(r)} + r^2 d\Omega^2_{n-2} \tag{3.70}$$

where Ω^2_{n-2} is the unit metric on S^{n-2}. As a consequence the general form of the null and timelike orbital equation will be identical to (3.12):

$$\left(\frac{du}{d\phi}\right)^2 + g(u)u^2 - \frac{g(u)}{f(u)}\frac{E^2}{J^2} = \frac{Lg(u)}{J^2}. \tag{3.71}$$

However, a great deal of difference will come from the form of f and g which varies with n.

If we impose n-dimensional Einstein equation $R_{\mu\nu} = \Lambda g_{\mu\nu}$ on (3.70) it gives uniquely the two-parameter Schwarzschild-(anti)de Sitter metric:

$$ds^2 = -\left(1 - \frac{2M}{r^{n-3}} - \frac{2\Lambda r^2}{(n-1)(n-2)}\right)dt^2 + \left(1 - \frac{2M}{r^{n-3}} - \frac{2\Lambda r^2}{(n-1)(n-2)}\right)^{-1} dr^2 + r^2 d\Omega^2_{n-2}, \tag{3.72}$$

where

$$d\Omega^2_{n-2} = \left(d\alpha^2_{n-2} + \sin^2\alpha_{n-2}(d\alpha^2_{n-3} + \sin^2\alpha_{n-3}(...(d\alpha^2_3 + \sin^2\alpha_3(d\alpha^2_2 + \sin^2\alpha_2))...\right).$$

Since metric coefficients are independent of t and $\alpha_1 \equiv \phi$, energy and angular momentum are conserved as before

$$\begin{aligned} E &= \left(1 - \frac{2M}{r^{n-3}} - \frac{2\Lambda r^2}{(n-1)(n-2)}\right)\dot{t}, \\ J &= r^2\left(\sin^2\alpha_{n-2}\sin^2\alpha_{n-3}...\sin^2\alpha_3\sin^2\alpha_2\right)\dot{\phi}, \end{aligned} \tag{3.73}$$

where $0 < \alpha_i < \pi$, and therefore, $0 < \prod_{i=2}^{n-2}\sin^2(\alpha_i) < 1$. The first-order geodesic equation is then

$$L\left(1 - 2Mu^{n-3} - \frac{2\Lambda}{u^2(n-1)(n-2)}\right) = -E^2 + \frac{1}{u^4}\left(\frac{du}{d\phi}\right)^2\frac{J^2u^4}{\alpha^4} + \frac{J^2u^2}{\alpha^2} - \frac{2MJ^2u^{n-1}}{\alpha^2} \tag{3.74}$$

where we used α^2 to denote $\prod_{i=2}^{n-2}\sin^2(\alpha_i)$. Differentiating (3.74) with respect to ϕ, we obtain

$$-2L(n-3)Mu^{n-4}\frac{du}{d\phi} + \frac{4\Lambda L}{u^3(n-1)(n-2)}\frac{du}{d\phi} = \frac{2J^2}{\alpha^4}\frac{du}{d\phi}\frac{d^2u}{d\phi^2} + \frac{2uJ^2}{\alpha^2}\frac{du}{d\phi} - 2\frac{(n-1)MJ^2u^{n-2}}{\alpha^2}\frac{du}{d\phi}. \tag{3.75}$$

The unparametrized geodesics confined on plane given by different fixed values of α_i's is thus

$$\frac{d^2u}{d\phi^2} = (n-3)Mu^{n-4}\frac{\alpha^4}{J^2}L - \frac{2\Lambda\alpha^4 L}{J^2u^3(n-1)(n-2)} - u\alpha^2 + (n-1)Mu^{n-2}\alpha^2. \tag{3.76}$$

For $L = -1$, we obtain timelike geodesics in n-dimensional

$$\frac{d^2u}{d\phi^2} = (n-3)Mu^{n-4}\frac{\alpha^4}{J^2}L - \frac{2\Lambda\alpha^4\epsilon}{J^2u^3(n-1)(n-2)} - u\alpha^2 + (n-1)Mu^{n-2}\alpha^2. \tag{3.77}$$

For $L = 0$, we obtain null geodesics

$$\frac{d^2u}{d\phi^2} = -u\alpha^2 + (n-1)Mu^{n-2}\alpha^2. \tag{3.78}$$

As in four dimensions, one can choose $\alpha = 1$ without loss of generality. For $\Lambda = 0$, we obtain Schwarzschild in n dimensions.

Geodesics of staic solutions with co-dimension-two spherical symmetry

	de-Sitter($\Lambda > 0$)	anti-de Sitter($\Lambda < 0$)
Timelike geodesic $3-d$ BTZ		$\frac{d^2u}{d\phi^2} = Mu + \frac{\Lambda}{b^2u^3}$
Timelike geodesic $4-d$ Sch	$\frac{d^2u}{d\phi^2} = \frac{M}{L^2} + \frac{\Lambda}{3L^2u^3} - u + 3Mu^2$	$\frac{d^2u}{d\phi^2} = \frac{M}{L^2} - \frac{\Lambda}{3L^2u^3} - u + 3Mu^2$
Timelike geodesic $5-d$ Sch	$\frac{d^2u}{d\phi^2} = \frac{Mu}{2L^2} + \frac{\Lambda}{24L^2u^3} - u + 2Mu^3$	$\frac{d^2u}{d\phi^2} = \frac{Mu}{2L^2} - \frac{\Lambda}{24L^2u^3} - u + 2Mu^3$
Timelike geodesic $n-d$ Sch	$\frac{d^2u}{d\phi^2} = (n-3)Mu^{n-4}\frac{\alpha^4}{L^2} + \frac{2\Lambda\alpha^4}{L^2u^3(n-1)(n-2)}$ $-u\alpha^2 + (n-1)Mu^{n-2}\alpha^2$	$\frac{d^2u}{d\phi^2} = (n-3)Mu^{n-4}\frac{\alpha^4}{L^2} - \frac{2\Lambda\alpha^4}{L^2u^3(n-1)(n-2)}$ $-u\alpha^2 + (n-1)Mu^{n-2}\alpha^2$
Null geodesic $3-d$ BTZ		$\frac{d^2u}{d\phi^2} = Mu$
Null geodesic $4-d$ Sch	$\frac{d^2u}{d\phi^2} = -u + 3Mu^2$	$\frac{d^2u}{d\phi^2} = -u + 3Mu^2$
Null geodesic $5-d$ Sch	$\frac{d^2u}{d\phi^2} = -u + 2Mu^3$	$\frac{d^2u}{d\phi^2} = -u + 2Mu^3$
Null geodesic $n-d$ Sch	$\frac{d^2u}{d\phi^2} = -u\alpha^2 + (n-1)Mu^{n-2}\alpha^2$	$\frac{d^2u}{d\phi^2} = -u\alpha^2 + (n-1)Mu^{n-2}\alpha^2$

3.2.1 Symmetries of Schwarzschild-(A)dS

Following the same steps, we find no symmetries for Schwarzschild-(A)dS geodesics, timelike or null.

3.2.2 Conservation Laws for the Null and Timelike Geodesics Schwarzschild-(anti-) de Sitter in Four dimensions

Timelike Geodesics

To find determining equation we have:

$$\begin{aligned} E_u\Big[\mu(u_{\phi\phi}-\frac{M}{L^2}-\frac{\Lambda}{3L^2u^3}+u-3Mu^2)\Big] &= 0, \\ (\frac{\partial}{\partial u}-D_\phi\frac{\partial}{\partial u_\phi}+D_\phi^2\frac{\partial}{\partial u_{\phi\phi}})(\mu(u_{\phi\phi}-\frac{M}{L^2}-\frac{\Lambda}{3L^2u^3}+u-3Mu^2)) &= 0. \end{aligned} \tag{3.79}$$

$$\begin{aligned} &\mu_u(u_{\phi\phi}-\frac{M}{L^2}-\frac{\Lambda}{3L^2u^3}+u-3Mu^2)+\mu(\frac{\Lambda}{L^2u^4}+1-6Mu)+D_\phi(\mu_\phi+u_\phi\mu_u)=0, \\ &\mu_u(-\frac{M}{L^2}-\frac{\Lambda}{3L^2u^3}+u-3Mu^2)+\mu(\frac{\Lambda}{L^2u^4}+1-6Mu)\mu_u+\mu_{\phi\phi}+2u_\phi\mu_{\phi u} \\ &+u_\phi^2\mu_{uu}+2u_{\phi\phi}=0. \end{aligned} \tag{3.80}$$

Then simplifying recent equation (12.13) and obtain 4 determining equations as follows:

$$2\mu_u=0, \quad 2\mu_{\phi u}=0, \quad \mu_u u+\mu-3\mu_u Mu^2-6\mu Mu-\frac{\mu_u M}{L^2}-\frac{\mu_u\Lambda}{3L^2u^3}+\frac{\mu\Lambda}{L^2u^4}+\mu_{\phi\phi}=0, \quad \mu_{uu}=0. \tag{3.81}$$

From the above equations we have $\mu = 0$, it means there no conservation laws and no integrating factors on $(\phi, u(\phi))$.

Null Geodesics

To find the determining equation we have:

$$\begin{aligned} E_u\Big[\mu(u_{\phi\phi}+u-3Mu^2)\Big] &= 0, \\ \left(\frac{\partial}{\partial u}-D_\phi\frac{\partial}{\partial u_\phi}+D_\phi^2\frac{\partial}{\partial u_{\phi\phi}}\right)(\mu(u_{\phi\phi}+u-3Mu^2)) &= 0. \end{aligned} \tag{3.82}$$

$$\begin{aligned} &\mu_u(u_{\phi\phi}+u-3Mu^2)+\mu(1-6Mu)+D_\phi(\mu_\phi+u_\phi\mu_u)=0, \\ &\mu_u(u-3Mu^2)+\mu(1-6Mu)+\mu_{\phi\phi}+2u_\phi\mu_{\phi u}+u_\phi^2\mu_{uu}+2u_{\phi\phi}\mu_u=0. \end{aligned} \tag{3.83}$$

Then simplifying recent equation (12.17) and obtain 4 determining equations as follows:

$$2\mu_u = 0, \quad 2\mu_{\phi u} = 0, \quad \mu_u u + \mu - 3\mu_u M u^2 - 6\mu M u + \mu_{\phi\phi} = 0, \quad \mu_{uu} = 0. \tag{3.84}$$

From the above equations we have $\mu = 0$, it means there no conservation laws and no integrating factors on $(\phi, u(\phi))$.

3.2.3 Conservation laws of the timelike and null geodesic equation for the (n-Dimension) Schwarzschild black hole

For the timelike geodesics

To find the determining equation we have:

$$E_u\Big[\mu\Big(u_{\phi\phi} - (n-3)Mu^{n-4}\frac{\alpha^4}{L^2} - \frac{2\Lambda\alpha^4}{L^2u^3(n-1)(n-2)} + u\alpha^2 - (n-1)Mu^{n-2}\alpha^2\Big)\Big] = 0. \tag{3.85}$$

We impose equation (3.55) into (3.85), we get

$$\begin{aligned}
&\mu_u\Big(u_{\phi\phi} - (n-3)Mu^{n-4}\frac{\alpha^4}{L^2} - \frac{2\Lambda\alpha^4}{L^2u^3(n-1)(n-2)} + u\alpha^2 - (n-1)Mu^{n-2}\alpha^2\Big)\\
&+\mu\Big(\alpha^2 - (n-1)(n-2)u^{n-3}M\alpha^2 + \frac{6\alpha^2\Lambda}{(n-1)(n-2)L^2u^4} - (n-3)(n-4)Mu^{n-5}\frac{\alpha^2}{L^2}\Big)\\
&+\mu_{\phi\phi} + 2u_\phi\mu_{\phi u} + u_\phi^2\mu_{uu} + 2u_{\phi\phi}\mu_u = 0.
\end{aligned} \tag{3.86}$$

Then simplifying recent equation and obtain 4 determining equations as follows:

$$\begin{aligned}
&2\mu_u = 0, \quad 2\mu_{\phi u} = 0, \quad \mu_{uu} = 0,\\
&-(n-3)\mu_u Mu^{n-4}\frac{\alpha^4}{L^2} - \frac{2\mu_u\Lambda\alpha^4}{L^2u^3(n-1)(n-2)} + \mu_u u\alpha^2 - \mu_u(n-1)Mu^{n-2}\alpha^2\\
&+\mu\alpha^2 - (n-1)(n-2)\mu u^{n-3}M\alpha^2 + \frac{6\alpha^2\mu\Lambda}{(n-1)(n-2)L^2u^4}\\
&-(n-3)(n-4)M\mu u^{n-5}\frac{\alpha^2\mu}{L^2} = 0.
\end{aligned} \tag{3.87}$$

From the above equations we have $\mu = 0$, it means there no conservation laws and no integrating factors on $(\phi, u(\phi))$.

For the null geodesic

The Euler operator is

$$E_U = \frac{\partial}{\partial U} - D_\phi \frac{\partial}{\partial U_\phi} + D_\phi^2 \frac{\partial}{\partial U_{\phi\phi}} \tag{3.88}$$

For finding determining equation we have

$$\begin{aligned} &E_u\Big[\mu(u_{\phi\phi} + u\alpha^2 - (n-1)Mu^{n-2}\alpha^2)\Big] = 0, \\ &(\frac{\partial}{\partial u} - D_\phi \frac{\partial}{\partial u_\phi} + D_\phi^2 \frac{\partial}{\partial u_{\phi\phi}})(\mu(u_{\phi\phi} + u\alpha^2 - (n-1)Mu^{n-2}\alpha^2)) = 0. \end{aligned} \tag{3.89}$$

$$\begin{aligned} &\mu_u(u_{\phi\phi} + u\alpha^2 - (n-1)Mu^{n-2}\alpha^2) + \mu(\alpha^2 - (n-1)(n-2)u^{n-3}M\alpha^2) \\ &+D_\phi(\mu_\phi + u_\phi\mu_u) = 0, \\ &\mu_u(u_{\phi\phi} + u\alpha^2 - (n-1)Mu^{n-2}\alpha^2) + \mu(\alpha^2 - (n-1)(n-2)u^{n-3}M\alpha^2) + \mu_{\phi\phi}, \\ &+2u_\phi\mu_{\phi u} + u_\phi^2\mu_{uu} + u_{\phi\phi}\mu_u = 0 \\ &\mu_u(u\alpha^2 - (n-1)Mu^{n-2}\alpha^2) + \mu(\alpha^2 - (n-1)(n-2)u^{n-3}M\alpha^2) + \mu_{\phi\phi} \\ &+2u_\phi\mu_{\phi u} + u_\phi^2\mu_{uu} + 2u_{\phi\phi}\mu_u = 0. \end{aligned} \tag{3.90}$$

Then simplifying recent equation and obtain 4 determining equations as follows:

$$\begin{aligned} &\alpha^2\mu_u u - \alpha^2(n-1)\mu_u Mu^{n-2} + \mu\alpha^2 - (n-1)(n-2)\mu\alpha^2 u^{n-3}M = 0, \quad \mu_{uu} = 0, \\ &2\mu_u = 0, \quad 2\mu_{\phi u} = 0. \end{aligned} \tag{3.91}$$

From the above equations we have $\mu = 0$, it means there no conservation laws and no integrating factors on $(\phi, u(\phi))$.

3.3 Conclusion

Despite the similarity in how they arise, static solutions in Einstein's gravity with a cosmological constant have a unique character in three dimensions. Here, null geodesics are described by the repulsive harmonic oscillator equation, which has the lie algebra of $sl(3,R)$. Although the null geodesics are different for negative mass, they have the exact same Lie algebra $sl(3,R)$. This perhaps is no surprise since both cases can be mapped into a $Y''=0$ system. It is known from Lie's original work that if a second order ODE with the constant coefficients admits the Lie symmetry algebra of dimension 8, it can be mapped to the canonical linear equation $Y''=0$, which admits a symmetry group isomorphic to the $SL(3,R)$ group [31].

On the other hand, the timelike geodesic equation, which has an additional term of u^{-3}, has $SL(2,R)$ as its symmetry. This is very special and occurs only with the additional term of $1/u^3$. This fact has been noted in the literature, and such equations are called the Ermakov-Pinney equation. What is surprising is that both $sl(3,R)$ and $sl(2,R)$ occur in the same system. This is as if the symmetry has spontaneously broken on moving from null to timelike geodesics. The existence of symmetry, in either case, means that one can obtain any unparametrized geodesic in BTZ spacetime from any other by means of a certain set of transformations. We found local conservation laws for both timelike and null geodesics. In fact, they can be explained from a variational perspective (i.e., via Noerther's theorems) since these equations (equations (3.19) and (3.37) respectively) can be seen as arising from the (reduced) Lagrangians: $L=\frac{1}{2}\left(u'^2+Mu^2\right)$ and $L=\frac{1}{2}\left(u'^2+Mu^2-\frac{1}{2}\frac{\Lambda}{J^2u^2}\right)$.

In four and higher dimensions, on the other hand, there is no symmetry and no conserved quantities for either null or timelike geodesics, despite having only additional polynomial terms, except for the trivial $\phi \to \phi + c$ symmetry.

We close this chapter with the observation that the orbital null geodesic equation in any dimension does not depend on the cosmological constant. This has profound consequences and will be the subject matter of the next chapter.

CHAPTER 4

NULL GEODESICS IN STATIC SPHERICALLY SYMMETRIC SPACETIMES: SYMMETRIES AND EQUIVALENCES

> I consider that I understand an equation when I can predict the properties of its solutions, without actually solving it. God used beautiful mathematics in creating the world.
>
> — *Paul A. M. Dirac*

The remarkable coincidence that Schwarzschild and Kottler (Schwarzschild-de Sitter) spacetimes have identical orbital light-bending equation (equivalently, the absence of the cosmological constant in the orbital light-bending equation in the latter), as we have noticed in chapter 3, have been subject to much discussion and debate in recent times. It has also been shown that these spacetimes has projectively equivalent (three-dimensional Riemannian) optical metrics to explain the coincidence and they have also been shown to have the same projective tensor. To obtain a better understanding of this in terms of Lorentzian geometry, in this chapter, we ask the opposite, and a more general, question: given an arbitrary spherically symmetric static Lorentzian geometry what are the other static spherically symmetric spacetimes that will have the same orbital light-bending equation? We do not assume any of these spacetimes to be *a priori* solutions of Einstein equations, in vacuum or otherwise, and they may represent actual astrophysical situations with unknown matter sources and base our question purely in terms of symmetry. We also test if these metric are projective equivalent in different coordinates and find the answer to be negative. This thus negates the proposed correspondence between projective equivalence via projective tensor.

4.1 Introduction

It has been known that in Schwarzschild-(anti-)de Sitter spacetime

$$ds^2 = -\left(1 - \frac{2M}{r} - \Lambda r^2\right) dt^2 + \frac{dr^2}{1 - \frac{2M}{r} - \Lambda r^2} + r^2 \left(d\theta^2 + \sin^2\theta d\phi^2\right), \tag{4.1}$$

the orbital light bending equation

$$\frac{d^2u}{d\phi^2} + u = 3Mu^2, \tag{4.2}$$

where $u \equiv r^{-1}$), is not affected by the cosmological constant [27]. In recent years, this observation came at the center of attention after it was argued that [40] despite the equation being identical for both metrics, actual observations are affected by the cosmological constant since the geometry of spatial hypersurfaces are different in the presence of the cosmological constant [40]. A great number of works followed analyzing this and related issues from different angles (see citations of [40]).

As for the mysterious origin of this coincidence, it was shown that the optical metric of Schwarzschild and Schwarzschild-(anti-)de Sitter spacetimes are projectively equivalent [20]. More generally, the projective tensor of the spherically symmetric static spacetime

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 \left(d\theta^2 + \sin^2\theta d\phi^2\right), \tag{4.3}$$

remains unchanged by the following symmetry $f(r) \to f(r) + \Lambda r^2$. This, together with the debate of observed angle measurement, mostly driven by exact solutions of the type (4.3) have created a consistent narrative centered around the cosmological constant.

In this chapter we will take the most general static spherically symmetric spacetimes in curvature coordinates with ansatz

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{g(r)} + r^2 \left(d\theta^2 + \sin^2\theta d\phi^2\right), \tag{4.4}$$

where $f(r) \neq g(r)$ in general and test the validity of the above conclusions. We will first ask what spherically symmetric static spacetime share the same orbital null geodesic equation.

We will see that these spacetimes cannot be described in terms of the presence or absence of a cosmological constant unless one pre-imposes $f(r) = g(r)$. In addition, these spacetimes are nor projectively equivalent, unless, again, $f(r) = g(r)$.

Finally we turn our attention the actual light bending measurement and ask whether there exist coordinate systems (i.e., observers) in which one would indeed measure the same bending angle. We find that the answer to this is affirmative in the PG coordinates and their generalization.

4.2 Static Spacetimes with Identical Orbital Null Equation

Geodesics equations of (4.4) can be obtained from the Lagrangian

$$L = -f(r)\dot{t}^2 + \frac{1}{g(r)}\dot{r}^2 + r^2\left(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2\right). \tag{4.5}$$

As in the case with Schwarzschild, the equation for θ, and spherical symmetry imply that it is sufficient to consider geodesics on the equatorial plane, $\theta = \pi/2$, without any loss of generality [26]. This leaves one with tree equations and three first integrals that we described in Chapter 2. As we have shown there, eliminating $\dot{t}$ and $\dot{\phi}$ using (3.8) and (3.9) one obtains from (3.10) for null geodesics ($L = 0$):

$$\left(\frac{du}{d\phi}\right)^2 + g(u)u^2 - \frac{g(u)}{f(u)}\frac{E^2}{J^2} = 0, \tag{4.6}$$

in which the standard substitution $u(\phi) = 1/r(\phi)$ has been made and in which prime represents $d/d\phi$. The orbital equation for null geodesics follows upon differentiating (4.6) again with respect to ϕ:

$$\frac{d^2u}{d\phi^2} = -\left(\frac{1}{2}g(u)u^2\right)' + \frac{1}{2}\frac{E^2}{J^2}\left(\frac{g(u)}{f(u)}\right)'. \tag{4.7}$$

4.2.1 Symmetry of the Orbital Null Equation in Static Coordinates

The first ask what static spherically symmetric spacetimes share the same orbital null equation. When $f(r) = g(r)$, the answer of this question is known, namely all solutions of the form $f(r)+\Lambda r^2$, where $f(r)$ is a specific function. This parameter Λ is easily interpreted as cosmological constant as one can verify by computing the Ricci tensor; the effect of Λr^2 is adding a $\Lambda g_{\mu\nu}$ to the right hand side of: $R_{\mu\nu} = \Lambda g_{\mu\nu} + 8\pi T_{\mu\nu}$. This interpretation will break down as we consider the general case.

Theorem 2.1: The two-parameter family of static spacetimes

$$ds^2 = -\frac{f(r)\left(g(r)+\alpha r^2\right)}{g(r)+\beta f(r)}\,dt^2 + \frac{dr^2}{g(r)+\alpha r^2} + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right). \tag{4.8}$$

share the same orbital null bending equation for any real α and β.

Proof: It follows by direct substitution that the two parameter transformation

$$(f(r), g(r)) \to (F(r), G(r)) \equiv \left(\frac{f(r)\left(g(r)+\alpha r^2\right)}{g(r)+\beta f(r)}, g(r)+\alpha r^2\right) \tag{4.9}$$

leaves (4.7) invariant.

One can check that the transformation $T_{\alpha,\beta}$ forms a two parameter Lie group by checking that two successive transformation $T_{\alpha_1,\beta_1}\circ T_{\alpha_2,\beta_2} = T_{\alpha_1+\alpha_2,\beta_1+\beta_2}$ by calculating

$$\begin{aligned} T_{\alpha_1\beta_1}\circ T_{\alpha_2\beta_2}(f(r),g(r)) &= \left(\frac{\frac{f(r)\left(g(r)+(\alpha_1+\alpha_2)r^2\right)}{g(r)+\beta_1 f(r)}\left(g(r)+\alpha_2 r^2\right)}{g(r)+\beta_2\frac{f(r)(g(r)+\alpha_1 r^2)}{g(r)+\beta_1 f(r)}}, g(r)+(\alpha_1+\alpha_2)r^2\right) \\ &= \left(\frac{f(r)\left(g(r)+(\alpha_1+\alpha_2)r^2\right)}{g(r)+(\beta_1+\beta_2)f(r)}, g(r)+(\alpha_1+\alpha_2)r^2\right) \end{aligned} \tag{4.10}$$

$f(r)=g(r)$: Emergence of the Cosmological Consonant

As we mentioned earlier, most discussion on light bending in static spherically symmetric metric was done using the metric form (4.3) and often using exact solution. For such geometries the general two parameter transformation reduces to

$$(f(r), g(r)\equiv f(r)) \to (F(r), G(r)) \equiv \left(\frac{(f(r)+\alpha r^2)}{1+\beta}, f(r)+\alpha r^2\right). \tag{4.11}$$

The multiplicative factor of $1/(1+\beta)$ can be absorbed into the time coordinate and, as such, the resulting metrics is a one-parameter family of metrics. Computing the components of Riemann curvature tensor, this parameter can be easily be identified with a cosmological constant. This is what is behind the narrative around the cosmological constant. However, such an identification is no longer possible for the general metric (4.4), as we will see below.

The general $f(r) \neq g(r)$: Disappearance of Cosmological Constant

None-zero components of Ricci tensor (4.4) are

$$R_{tt} = \frac{gf'}{r} + \frac{gf''}{2} - \frac{gf'^2}{4F} + \frac{g'f'}{4}, \tag{4.12}$$

$$R_{rr} = -\frac{g'}{rg} - \frac{f''}{2f} + \frac{f'^2}{4f^2} - \frac{g'f'}{4fg}, \tag{4.13}$$

$$R_{\theta\theta} = -g + 1 - \frac{g'r}{2} - \frac{f'gr}{2f} = R_{\phi\phi}/\sin^2\theta. \tag{4.14}$$

It is easy to check that the transformation (4.9) does not return us components of the Ricci tensor with additive terms as was possible for the special case of $f(r) = g(r)$. In other words, it is impossible to identify α as the cosmological constant, for any choice of β including zero, or any combination of α and β, as the cosmological constant. We will, however, see that this general symmetry can be interpreted in terms of anisotropic fluids.

4.2.2 No Projective Equivalence of Optical Metrics

The appearance of the cosmological constant can be understood in terms of geodesics of optical geometry: null geodesics of any static metric,

$$ds^2 = -g_{00}(x)dt^2 + h_{ij}(x)dx^i dx^j, \tag{4.15}$$

projects onto geodesics of its optical metric[1]:

$$d\sigma^2 = \frac{h_{ij}(x)}{g_{00}(x)}dx^i dx^j. \tag{4.16}$$

[1]The simplest example is perhaps light rays in Minkowski spacetime project onto straight lines in Euclidean space.

The optical metrics of Schwarzschild-(anti-)de Sitter for different values of Λ (including $\Lambda = 0$), but the same M, are projectively equivalent. This can be seen by directly computing the projective tensor of (4.3) [20]. Note that having the same projective tensor serves only as necessary condition of projective equivalence [15]. Thus two metric having the same projective tensor may not be projectively equivalent. Below we will see that even this connection (between projective equivalence and having the same optical light bending equation) breaks down as one considers the general metric form.[2]

The non-zero components of projective curvature tensor

$$\begin{aligned} W_1 : &= -W^r_{\theta\theta r} = W^r_{\theta r\theta} = W^\phi_{\theta\theta\phi} = -W^\phi_{\theta\phi\theta} \\ &= -\frac{1}{8f^2}\left(f'^2 g r^2 - f'g'fr^2 - 2f''fgr^2 + 2f'fgr + 2g'f^2 r - 4f^2 g + 4f^2\right) \\ W_2 : &= -W^r_{\phi\phi r} = W^r_{\phi r\phi} = W^\theta_{\phi\phi\theta} = -W^\theta_{\phi\theta\phi} = \sin^2\theta W_1 \end{aligned}$$

clearly is not invariant under the symmetry transformation (4.9) unless $f(r) = g(r)$.

4.3 Projective Equivalence in Static Isotropic Coordinates

Any spherically symmetric static metric can be brought in the isotropic form

$$ds^2 = -h(r)dt^2 + s(r)\left(dr^2 + r^2 d\theta^2 + r^2\sin^2\theta d\phi^2\right) \tag{4.17}$$

Since angular momentum and energy are conserved we can write:

$$\begin{aligned} E &= h(r)\frac{dt}{ds} \\ J &= r^2 s(r)\frac{d\phi}{ds} \end{aligned} \tag{4.18}$$

[2]In general, projective equivalence between spaces (or parts thereof) can be verified easily when they are written using the same coordinate system as Schwarzschild and Kottler spacetimes are and hence their optical metrics. However, the converse is not necessarily true because of the freedom of the conformal factor as we discuss below.

Where $\theta = \frac{\pi}{2}$ then $d\theta = 0$ and $\sin\theta = 1$.

$$\begin{aligned} 0 &= -h(r)\left(\frac{dt}{ds}\right)^2 + s(r)\left(\frac{dr}{ds}\right)^2 + r^2 s(r)\left(\frac{d\phi}{ds}\right)^2 \\ 0 &= -\frac{E^2}{h(r)s(r)} + \left(\frac{dr}{ds}\right)^2 + \frac{J^2}{r^2 s^2(r)} \end{aligned} \tag{4.19}$$

put $(r = \frac{1}{u})$ where $r = r(\phi)$:

$$0 = -\frac{E^2}{h(u)s(u)} + \frac{J^2}{s^2(u)}\left(\frac{du}{d\phi}\right)^2 + \frac{J^2 u^2}{s^2(u)} \tag{4.20}$$

Now we take a derivative of (4.20) with respect to ϕ:

$$\frac{d^2u}{d\phi^2} = \frac{E^2}{2J^2}\left(\frac{s(u)}{h(u)}\right)' - u \tag{4.21}$$

The reason for considering SSS spacetime in isotropic coordinate (4.17) is its optical metric is very intuitive to understand and discuss in terms of Newtonian propagation of light. Its optical metric is

$$ds^2_{opt} = \frac{s(r)}{h(r)}\left(dr^2 + r^2 d\theta^2 + r^2\sin^2(\theta)d\phi^2\right). \tag{4.22}$$

where n plays the role of "refractive index"[3]

$$n^2(r) = \frac{s(r)}{h(r)} \tag{4.23}$$

It is clear from the light-bending equation that two spacetimes will have the same light bending equations is the squares of their refractive indices n_1 and n_2 differs by a constant

$$n_1^2 - n_2^2 = \text{constant}. \tag{4.24}$$

However, unlike in the canonical coordinates, here one notices a difference if one uses the first degree equation of null geodesics that can be obtained by solving (4.20) for $\frac{du}{d\phi}$. For this, $n_2(r) = kn_1(r)$, where k is a constant. Unlike in the canonical coordinates, bending

[3]This is easy to see with since the spatial slices are flat and one can check that the geodesic bending is completely determined by n. See Perlick.

angle comes directly from the orbital equation and there is no ambiguity because the spatial metric is flat. We now compute the projective tensor of the optical metric which has only one algebraically independent component given by

$$W^{\alpha}_{\beta\beta\alpha} = \frac{1}{8}\left(2r^2\frac{(n^2)''}{n^2} - 3\frac{(n^2)'}{n^2}r^2 - 2\frac{(n^2)'}{n^2}r\right) \tag{4.25}$$

which changes if one replaces $n^2(r)$ by $n(r)^2$ + constant or by $k(r)n(r)^2$, unless if k is a constant in which case one has the same spacetime upon the redefinition of the time coordinate.

4.4 More Equivalences

It is easy to see that the orbital null geodesic equations remain unchanged if one moves to a new time coordinate but keeps the other coordinates. For example, in terms of the Painlevè-Gullstrand time coordinate, defined in terms of the original time and radial coordinates [37, 23]

$$\bar{t} = t + 4m\left[\sqrt{\frac{r}{2m}} + \frac{1}{2}\ln\left(\frac{\sqrt{\frac{r}{2m}}-1}{\sqrt{\frac{r}{2m}}+1}\right)\right]. \tag{4.26}$$

turns the Schwarzschild metric (1.24) into the following form

$$ds^2 = -\left(1-\frac{2m}{r}\right)d\bar{t}^2 + 2\sqrt{\frac{2m}{r}}drd\bar{t} + dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2). \tag{4.27}$$

The orbital null geodesic equations are the same. In addition the the $\bar{t} = const$ hypersurface of (4.27) is the three-dimensional flat metric expressed in polar coordinates. These hypersurfaces run across the horizon, $r = 2m$ and has only $r = 0$ as a coordinate singularity (see [1] and references therein for further discussion).

Measurement of light bending on these hypersurfaces would be given directly by the orbital light bending equation. Thus, observers in different static spacetimes in their respective Painlevè-Gullstrand time coordinates would not be able to tell the difference between them, and in particular, when $f = g$ is the spacetime contains a cosmological constant or not.

4.5 Conclusion

As the field of gravitational astronomy is becoming a reality, a valid question is how one can use any result obtained via exact solutions in real astronomical situations where solutions are not exact, and the matter sources that generate them are not completely known. One may also ask how much of these results are relevant in alternative theories of gravity on which there is a great deal of interest these days.

The missing Λ in (4.2) has largely been taken to be a coincidence between Schwarzschild and Schwarzschild-(anti-)de Sitter spacetimes. However, one can directly show from the null geodesics equations for Schwarzschild-(anti-)de Sitter spacetime that different values of Λ can give rise to the same equations (up to parametrization) [20].

In this chapter, we found that there is always a two-parameter family of static spherically symmetric spacetimes which has the same orbital light-bending equation for any value of the parameters, i.e., these spacetimes are projectively equivalent. At the same time, we have found that the Weyl projective curvature tensor is not the same for this two-parameter family unless $f(r) = g(r)$ (as in Kottler). Thus, there is no general correspondence between projective equivalence and the Weyl projective curvature tensor. In addition, even the correspondence between Schwarzschild and Kottler is coordinate-dependent and breaks down if one moves to, say, isotropic coordinates as was recently shown explicitly in [41]. Furthermore, we found that in static isotropic coordinates, the two geometries will have the same second-order light orbital light bending equation if the square of their refractive indices differs by a constant and the same first-order light bending equation if they are, essentially, the same geometry. Finally, we point out that there are coordinates in which the measurement of bending angle in spherically symmetric static spacetimes could directly come from the (same) orbital equation, given that the constant-time hypersurfaces in these coordinates are flat.

CHAPTER 5

STATIC AXISYMMETRIC RICCI-FLAT SOLUTIONS

> Symmetry, as wide or as narrow as you may define its meaning, is one idea by which man through the ages has tried to comprehend and create order, beauty and perfection.
>
> — *Hermann Weyl*

In this chapter, we study symmetries of Einstein's equations describing static axisymmetric metrics in vacuum. In other words, we will study a PDE system of two independent and two dependent variables. This system admits an infinite number of solutions unlike the spherically symmetric system in which the solution is unique. So it is indeed useful to know how solutions map to each other within the space of solutions. An unexpected Lie point symmetry of this system has recently been found which can act on the system to produce a new solution from an existing solution algebraically without solving any equation [2]. We will try to find all symmetries of the system to gain a better understanding. Apart from being a PDE system, another way this system is different from our previous ODE systems is that this system is variational, and one expects conserved quantities from symmetries.

In addition, we will analyze the geodesics of this system. This too differs considerably from geodesics in particular exact solutions that we did earlier in spherical symmetry. We will devote much attention to geodesics on the equatorial plane.

5.1 Symmetry Analysis and Conservation Laws of PDE Systems

The general static axially symmetric vacuum solutions of Einstein's equations ($R_{\mu\nu} = 0$) can be written in Weyl coordinates as (see [2])

$$ds^2 = -e^{2u(\rho,z)}dt^2 + e^{-2u(\rho,z)}\left[e^{2k(\rho,z)}(d\rho^2 + dz^2) + \rho^2 d\phi^2\right], \tag{5.1}$$

where $u(\rho, z)$ and $k(\rho, z)$ satisfy the following field equations [29],

$$\frac{\partial^2 u}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial u}{\partial \rho} + \frac{\partial^2 u}{\partial z^2} = 0, \tag{5.2}$$

$$\frac{\partial k}{\partial \rho} = \rho\left[\left(\frac{\partial u}{\partial \rho}\right)^2 - \left(\frac{\partial u}{\partial z}\right)^2\right], \tag{5.3}$$

$$\frac{\partial k}{\partial z} = 2\rho\frac{\partial u}{\partial \rho}\frac{\partial u}{\partial z}. \tag{5.4}$$

This system involves two independent variables ρ and z and two dependent variables u and k. The first equation is a second-order linear PDE in u, and the remaining two are nonlinear PDEs of the first order.

5.2 Symmetry Analyzing of Axially Symmetric Equation and it's Lie Algebra

It is not difficult to see that the transformation

$$(u_0, k_0) \to (\beta u_0, \beta^2 k_0) \tag{5.5}$$

leaves the system (5.2) invariant; in other words, for any arbitrary solution (u_0, k_0) there is a (non-equivalent) solution $(\beta u_0, \beta^2 k_0)$ for $\beta \in (-\infty, \infty)$. This is an example of point symmetry which maps dependent variables only to (new/transformed) dependent variables and maps the independent variables to themselves. As special case of this, $(u_0, k_0) \to (-u_0, k_0)$, is the very first generation technique given more than 60 years ago – and revisited from time to time for another two decades – by Hans Buchdahl (see [2] for more details). Are there more nontrivial point symmetries? This question was partially addressed in [2] where it was shown that the transformation

$$(u_0, k_0) \to (u_0 + \alpha \ln \rho, k_0 + 2\alpha u_0 + \alpha^2 \ln \rho), \tag{5.6}$$

leaves the system (5.2) invariant. It means one can obtain a solution from another purely algebraically. In this chapter we will understand why it works.

Symmetry of the Laplace Equation

The first equation of the system, equation (5.2), is the two-dimensional axially symmetric Laplace equation in cylindrical coordinates. Here we have two independent variables ρ and z and one dependent variable u. Since equation (5.2) is a second order PDE we need second prolongation:

$$\mathbf{v} = \xi(\rho, z, u)\frac{\partial}{\partial \rho} + \tau(\rho, z, u)\frac{\partial}{\partial z} + \phi(\rho, z, u)\frac{\partial}{\partial u} \tag{5.7}$$

$$Pr^{(2)}\mathbf{v} = \xi\frac{\partial}{\partial \rho} + \tau\frac{\partial}{\partial z} + \phi\frac{\partial}{\partial u} + \phi^{\rho}\frac{\partial}{\partial u_{,\rho}} + \phi^{z}\frac{\partial}{\partial u_{,z}} + \phi^{\rho\rho}\frac{\partial}{\partial u_{,\rho\rho}} + \phi^{\rho z}\frac{\partial}{\partial u_{,\rho z}} + \phi^{zz}\frac{\partial}{\partial u_{,zz}} \tag{5.8}$$

Taking

$$\Delta = u_{,\rho\rho} + \frac{1}{\rho}u_{,\rho} + u_{,zz}, \tag{5.9}$$

$$Pr^{(2)}\mathbf{v}(\Delta) = \phi^{\rho\rho} + \frac{1}{\rho}\phi^{\rho} - \frac{1}{\rho^2}\xi u_{,\rho} + \phi^{zz} \tag{5.10}$$

where

$$\frac{\partial \Delta}{\partial u_{,\rho\rho}} = 1, \quad \frac{\partial \Delta}{\partial u_{,\rho}} = \frac{1}{\rho} \tag{5.11}$$

$$\frac{\partial \Delta}{\partial \rho} = -\frac{1}{\rho^2}\xi u_{,\rho}, \quad \frac{\partial \Delta}{\partial u_{,zz}} = 1 \tag{5.12}$$

As we know, a vector field $\mathbf{v}$ generates a one-parameter symmetry group if and only if

$$Pr^{(2)}\mathbf{v}(\Delta) = 0, \tag{5.13}$$

which in this case is

$$\phi^{\rho\rho} + \frac{1}{\rho}\phi^{\rho} - \frac{1}{\rho^2}\xi u_{,\rho} + \phi^{zz} = 0 \tag{5.14}$$

We apply equation (2.39) of Olver's book[1] [35] to obtain

$$\begin{aligned}
\phi^{\rho} &= D_\rho(\phi - \xi u_{,\rho} - \eta u_{,z}) + \xi u_{,\rho\rho} + \eta u_{,\rho z} \\
&= \phi_{,\rho} + (\phi_{,u} - \xi_{,\rho})u_{,\rho} - \xi_{,u}u_{,\rho}^2 - (\tau_{,\rho} + \tau_{,u}u_{,\rho})u_{,z} \qquad (5.15)\\
\phi^{\rho\rho} &= D_\rho^2(\phi - \xi u_{,\rho} - \eta u_{,z}) + \xi u_{,\rho\rho\rho} + \eta u_{,\rho\rho z} \\
&= \phi_{,\rho\rho} + (2\phi_{,\rho u} - \xi_{,\rho\rho})u_{,\rho} - \tau_{,\rho\rho}u_{,z} + (\phi_{,uu} - 2\xi_{,\rho u})u_{,\rho}^2 - 2\tau_{,\rho u}u_{,\rho}u_{,z} - \xi_{,uu}u_{,\rho}^3 \\
&- \tau_{,uu}u_{,\rho}^2u_{,z} + (\phi_{,u} - 2\xi_{,\rho})u_{,\rho\rho} - 2\tau_{,\rho}u_{,\rho z} - 3\xi_{,u}u_{,\rho}u_{,\rho\rho} - \tau_{,u}u_{,z}u_{,\rho\rho} - 2\tau_{,u}u_{,\rho}u_{,\rho z} \qquad (5.16)\\
\phi^{zz} &= D_z^2(\phi - \xi u_{,\rho} - \eta u_{,z}) + \xi u_{,\rho zz} + \eta u_{,zzz} \\
&= \phi_{,zz} + (2\phi_{,zu} - \tau_{,zz})u_{,z} + (\phi_{,uu} - 2\tau_{,zu})u_{,z}^2 - \xi_{,uu}u_{,\rho}u_{,z}^2 - \tau_{,uu}u_{,z}^3 - 2\xi_{,zu}u_{,\rho}u_{,z} \\
&- \xi_{,u}u_{,\rho}u_{,zz} - 2\xi_{,u}u_{,z}u_{,\rho z} - 3\tau_{,u}u_{,z}u_{,zz} - \xi_{,zz}u_{,\rho} - 2\xi_{,z}u_{,\rho z} + (\phi_{,u} - 2\tau_{,z})u_{,zz}, \qquad (5.17)
\end{aligned}$$

where D_ρ and D_z are total derivative with respect to ρ and z:

$$D_\rho = \frac{\partial}{\partial \rho} + u_{,\rho}\frac{\partial}{\partial u} + u_{,\rho\rho}\frac{\partial}{\partial u_{,\rho}} + u_{,\rho z}\frac{\partial}{\partial u_{,z}} + \cdots$$

$$D_z = \frac{\partial}{\partial z} + u_{,z}\frac{\partial}{\partial z} + u_{,zz}\frac{\partial}{\partial u_{,z}} + u_{,\rho z}\frac{\partial}{\partial u_{,\rho}} + \cdots$$

Now substitute equations (5.15)–(5.17) into (5.14):

$$\begin{aligned}
&- \frac{1}{\rho^2}\xi u_{,\rho} + \phi_{,\rho\rho} + (2\phi_{,\rho u} - \xi_{,\rho\rho})u_{,\rho} - \tau_{,\rho\rho}u_{,z} + (\phi_{,uu} - 2\xi_{,\rho u})u_{,\rho}^2 - 2\tau_{,\rho u}u_{,\rho}u_{,z} - \xi_{,uu}u_{,\rho}^3 \\
&- \tau_{,uu}u_{,\rho}^2u_{,z} + (\phi_{,u} - 2\xi_{,\rho})u_{,\rho\rho} - 2\tau_{,\rho}u_{,\rho z} - 3\xi_{,u}u_{,\rho}u_{,\rho\rho} - \tau_{,u}u_{,z}u_{,\rho\rho} - 2\tau_{,u}u_{,\rho}u_{,\rho z} \\
&+ \frac{1}{\rho}(\phi_{,\rho} + (\phi_{,u} - \xi_{,\rho})u_{,\rho} - \tau u_{,\rho}u_{,z} - \xi_{,u}u_{,\rho}^2 - \tau u_{,u}u_{,\rho}u_{,z}) + \phi_{,zz} + (2\phi_{,zu} - \tau_{,zz})u_{,z} \\
&+ (\phi_{,uu} - 2\tau_{,zu})u_{,z}^2 - \xi_{,uu}u_{,\rho}u_{,z}^2 - \tau_{,uu}u_{,z}^3 - 2\xi_{,zu}u_{,\rho}u_{,z} - \xi_{,u}u_{,\rho}u_{,zz} \\
&- 2\xi_{,u}u_{,z}u_{,\rho z} - 3\tau_{,u}u_{,z}u_{,zz} - \xi_{,zz}u_{,\rho} - 2\xi_{,z}u_{,\rho z} + (\phi_{,u} - 2\tau_{,z})u_{,zz} = 0. \qquad (5.18)
\end{aligned}$$

[1] $\phi_\alpha^j(x, u^{(n)}) = D_j(\phi_\alpha - \sum_{i=1}^p \xi^i u_i^\alpha) + \sum_{i=1}^p \xi^i u_{j,i}^\alpha$, where $j = (j_1, ..., j_k)$, $1 \le j_k \le p$ and $1 \le k \le n$ $u_i^\alpha := \frac{\partial u^\alpha}{\partial x^i}$, $u_{j,i}^\alpha = \frac{\partial u_j^\alpha}{\partial x^i}$

Then replace $u_{,\rho\rho}$ by $-\frac{1}{\rho}u_{,\rho} - u_{,zz}$ and plug in equation (5.18),

$$\begin{aligned}
- &\ \frac{1}{\rho^2}\xi u_{,\rho} + \phi_{,\rho\rho} + (2\phi_{,\rho u} - \xi_{,\rho\rho})u_{,\rho} - \tau_{,\rho\rho}u_{,z} + (\phi_{,uu} - 2\xi_{,\rho u})u_{,\rho}^2 - 2\tau_{,\rho u}u_{,\rho}u_{,z} - \xi_{,uu}u_{,\rho}^3 \\
- &\ \tau_{,uu}u_{,\rho}^2 u_{,z} - \frac{1}{\rho}u_{,\rho}\phi_{,u} - \phi_{,u}u_{,zz} + \frac{2}{\rho}\xi_{,\rho}u_{,\rho} + 2\xi_{,\rho}u_{,zz} - 2\tau_{,\rho}u_{,\rho z} + \frac{3}{\rho}\xi_{,u}u_{,\rho}^2 + 3\xi_{,u}u_{,\rho}u_{,zz} \\
+ &\ u_{,\rho} + \tau_{,u}u_{,z}u_{,zz} - 2\tau_{,u}u_{,\rho}u_{,\rho z} + \frac{1}{\rho}(\phi_{,\rho} + (\phi_{,u} - \xi_{,\rho})u_{,\rho} - \tau u_{,\rho}u_{,z} - \xi_{,u}u_{,\rho}^2 - \tau u_{,u}u_{,\rho}u_{,z}) \\
+ &\ (2\phi_{,zu} - \tau_{,zz})u_{,z} + (\phi_{,uu} - 2\tau_{,zu})u_{,z}^2 - \xi_{,uu}u_{,\rho}u_{,z}^2 - \tau_{,uu}u_{,z}^3 - 2\xi_{,zu}u_{,\rho}u_{,z} - \xi_{,u}u_{,\rho}u_{,zz} \\
- &\ 2\xi_{,u}u_{,z}u_{,\rho z} - 3\tau_{,u}u_{,z}u_{,zz} - \xi_{,zz}u_{,\rho} - 2\xi_{,z}u_{,\rho z} + (\phi_{,u} - 2\tau_{,z})u_{,zz} + \frac{1}{\rho}\tau_{,u}u_{,z} + \phi_{,zz} = 0.
\end{aligned} \tag{5.19}$$

Simplify (5.19) we get:

$$\begin{aligned}
&\ \phi_{,zz} + \phi_{,\rho\rho} + \big(-\xi + \frac{1}{\rho}\xi_{,\rho} - \xi_{,\rho\rho} + 2\phi_{,u\rho} - \xi_{,zz}\big)u_{,\rho} - \big(\frac{1}{\rho}\tau_{,\rho} + \tau_{,zz} + \tau_{,\rho\rho} - 2\phi_{,uz}\big)u_{,z} \\
+ &\ \frac{1}{\rho}\phi_{,\rho} + \big(\frac{2}{\rho}\xi_{,u} + \phi_{,uu} - 2\xi_{,\rho u}\big)u_{,\rho}^2 - 2\tau_{,u}u_{,\rho}u_{,\rho z} - 2\tau_{,u}u_{,z}u_{,zz} \\
- &\ \big(2\tau_{,uz} + \phi_{,uu}\big)u_{,z}^2 - \xi_{,uu}u_{,\rho}^3 - \tau_{,uu}u_{,z}^3 - \big(2\xi_{,uz} + 2\tau_{,\rho u}\big)u_{,\rho}u_{,z} - \xi_{,uu}u_{,\rho}u_{,z}^2 \\
- &\ \tau_{,uu}u_{,\rho}^2 u_{,z} - \big(2\xi_{,z} + 2\tau_{,\rho}\big)u_{,\rho z} - 2\tau_{,z}u_{,zz} + 2\xi_{,u}u_{,\rho}u_{,zz} - 2\xi_{,u}u_{,z}u_{,\rho z} = 0.
\end{aligned} \tag{5.20}$$

This last equation seems complicated but it is generally easy to solve; both ξ and η are independent of ϕ, so (5.20) can be decomposed in to a system of PDE's that are the determining equations for the Lie point symmetries, the procedure is as follows

monomial	coefficient
1	$\frac{1}{\rho}\phi_{,\rho} + \phi_{,zz} + \phi_{,\rho\rho} = 0$
$u_{,\rho}$	$-\frac{1}{\rho^2}\xi + \frac{1}{\rho}\xi_{,\rho} - \xi_{,\rho\rho} + 2\phi_{,u\rho} - \xi_{,zz} = 0$
$u_{,z}$	$-\frac{1}{\rho}\tau_{,\rho} - \tau_{,zz} - \tau_{,\rho\rho} + 2\phi_{,uz} = 0$
$u_{,\rho}^2$	$\frac{2}{\rho}\xi_{,u} + \phi_{,uu} - 2\xi_{,\rho u} = 0$
$u_{,z}^2$	$-2\tau_{,uz} + \phi_{,uu} = 0$
$u_{,\rho}^3$	$-\xi_{,uu} = 0$
$u_{,z}^3$	$-\tau_{,uu} = 0$
$u_{,\rho}u_{,z}$	$2\xi_{,uz} + 2\tau_{,\rho u} = 0$
$u_{,z}^2$	$-\xi_{,uu} = 0$
$u_{,\rho}^2 u_{,z}$	$-\tau_{,uu} = 0$
$u_{,\rho z}$	$-2\xi_{,z} - 2\tau_{,\rho} = 0$
$u_{,zz}$	$-2\tau_{,z} + 2\xi_{,\rho} = 0$
$u_{,\rho}u_{,zz}$	$2\xi_{,u} = 0$
$u_{,z}u_{,\rho z}$	$-2\xi_{,u} = 0$
$u_{,\rho}u_{,\rho z}$	$-2\tau_{,u} = 0$
$u_{,z}u_{,zz}$	$-2\tau_{,u} = 0$

By using 4 last equations of table also by $\xi_{,uu} = 0$ and $\eta_{,uu} = 0$ we will see that neither ξ nor τ depends on u so ξ and τ can be written as $\xi = \xi(\rho, z)$ and $\tau = \tau(\rho, z)$

Now when we solve equations $-2\xi_{,z} - 2\tau_{,\rho} = 0$ and $-2\tau_{,z} + 2\xi_{,\rho} = 0$ by integration we can obtain ξ and τ as follows:

$$\xi = (C_1 z + C_2)\rho \tag{5.21}$$

$$\tau = \frac{1}{2}(z^2 - \rho^2)C_1 + C_2 z + C_3, \tag{5.22}$$

$$\phi = -\frac{1}{2}(C_1 z - 2C_4)u + \alpha(\rho, z) \tag{5.23}$$

For finding ϕ since ξ and τ are not dependant on u so equation 5 of table can be written as $\phi_{,uu} = 0$ so ϕ is linear in u and one can write $\phi = \beta(\rho, z)u + \alpha(\rho, z)$. On the other hand when we use equation (5.21) and (5.22) in equations 2 and 3 from the table one can obtain $\phi_{,u\rho} = 0$ and $\phi_{,uz} = -\frac{1}{2}C_1$, so immediately can see $\beta = -\frac{1}{2}C_1 zu + C_4 u$ by integration. Where $C_1, ..., C_4$ are arbitrary constants and α is arbitrary solution of Laplace equation. Therefore Lie algebra of the infinitesimal symmetries of the axially-symmetric Laplace equation will be spanned by four vector fields and the infinite-dimensional sub algebra v_α where α is an arbitrary solution of the Laplace equation:

$$v_1 = z\rho\frac{\partial}{\partial\rho} + \frac{1}{2}(z^2 - \rho^2)\frac{\partial}{\partial z} - \frac{1}{2}zu\frac{\partial}{\partial u} \tag{5.24a}$$

$$v_2 = \rho\frac{\partial}{\partial\rho} + z\frac{\partial}{\partial z} \tag{5.24b}$$

$$v_3 = \frac{\partial}{\partial z} \tag{5.24c}$$

$$v_4 = u\frac{\partial}{\partial u} \tag{5.24d}$$

$$v_\alpha = \alpha(\rho, z)\frac{\partial}{\partial u} \tag{5.24e}$$

For finding flow of infinitesimal generators it is sufficient just do some integration by impose initial conditions: $\tilde{\rho}(0) = \rho$, $\tilde{z}(0) = z$ and $\tilde{u}(0) = u$:

$$G_1 = \left(\frac{\rho}{1 - z\varepsilon + \left(\frac{\varepsilon}{2}\right)^2(\rho^2 + z^2)}, \frac{z - \frac{\varepsilon}{2}(\rho^2 + z^2)}{1 - z\varepsilon + \left(\frac{\varepsilon}{2}\right)^2(\rho^2 + z^2)}, u\sqrt{1 - z\varepsilon + \left(\frac{\varepsilon}{2}\right)^2(\rho^2 + z^2)}\right),$$

$$G_2 = (\rho e^\varepsilon, z e^\varepsilon, u), \tag{5.25a}$$

$$G_3 = (\rho, z + \varepsilon, u), \tag{5.25b}$$

$$G_4 = (\rho, z, u e^\varepsilon), \tag{5.25c}$$

$$G_\alpha = (\rho, z, u + \alpha(\rho, z)\varepsilon). \tag{5.25d}$$

For example for finding G_1, we needed to solve

$$\begin{cases} \frac{d\tilde{\rho}}{d\varepsilon} = \tilde{\rho}\tilde{z} \\ \frac{d\tilde{z}}{d\varepsilon} = \frac{1}{2}(\tilde{z}^2 - \tilde{\rho}^2) \\ \frac{d\tilde{u}}{d\varepsilon} = -\frac{1}{2}\tilde{z}\tilde{u} \end{cases}$$

When we take first two equations of previous system together as real and imaginary parts of complex number $\mathbb{C} = \tilde{z} + i\tilde{\rho}$ then $\frac{\mathbb{C}^2}{2} = \frac{d\mathbb{C}}{d\varepsilon}$ and solve this to get $\mathbb{C}$ where the real part of $\mathbb{C}$ is $\tilde{z}$ and the imaginary part of $\mathbb{C}$ is $\tilde{\rho}$ as follow

$$\tilde{\rho} = \frac{4b}{(\varepsilon + 2a)^2 + 4b^2} \tag{5.26}$$

$$\tilde{z} = \frac{-2(\varepsilon + 2a)}{(\varepsilon + 2a)^2 + 4b^2}, \tag{5.27}$$

where a and b are the real and imaginary parts of a constant complex number. Using initial conditions $\tilde{\rho}(0) = \rho$, $\tilde{z}(0) = z$ we can get a and b and obtain:

$$\tilde{\rho} = \frac{\rho}{1 - z\varepsilon + \left(\frac{\varepsilon}{2}\right)^2 (\rho^2 + z^2)} \tag{5.28}$$

$$\tilde{z} = \frac{z - \frac{\varepsilon}{2}(\rho^2 + z^2)}{1 - z\varepsilon + \left(\frac{\varepsilon}{2}\right)^2 (\rho^2 + z^2)} \tag{5.29}$$

Then substitute $\tilde{z}$ in third equation $\frac{d\tilde{u}}{d\varepsilon} = -\frac{1}{2}\tilde{z}\tilde{u}$ and integrating we get

$$\tilde{u} = u\sqrt{1 - z\varepsilon + \left(\frac{\varepsilon}{2}\right)^2 (\rho^2 + z^2)}$$

Thus, G_1 is given by

$$G_1 = \left(\frac{\rho}{1 - z\varepsilon + \left(\frac{\varepsilon}{2}\right)^2 (\rho^2 + z^2)}, \frac{z - \frac{\varepsilon}{2}(\rho^2 + z^2)}{1 - z\varepsilon + \left(\frac{\varepsilon}{2}\right)^2 (\rho^2 + z^2)}, u\sqrt{1 - z\varepsilon + \left(\frac{\varepsilon}{2}\right)^2 (\rho^2 + z^2)}\right)$$

For finding G_2, need to solve

$$\frac{d\tilde{\rho}}{d\varepsilon} = \tilde{\rho}$$

$$\ln(\tilde{\rho}) = \varepsilon + c$$

$$\tilde{\rho} = e^{\varepsilon}\rho \tag{5.30}$$

and the same way we get $\tilde{z}$

$$\tilde{z} = e^{\varepsilon} z \tag{5.31}$$

Therefore G_2 is given by

$$G_2 = (e^{\varepsilon}\rho, e^{\varepsilon}z, u) \tag{5.32}$$

Hence we can apply the same procedure and get G_3 , G_4 and G_α.

Lie algebra

	v_1	v_2	v_3	v_4	v_α
v_1	0	$-v_1$	$v_2 - \frac{1}{2}v_4$	0	$v_{\alpha'''}$
v_2	v_1	0	$-v_3$	0	$v_{\alpha''}$
v_3	$-v_2 + \frac{1}{2}v_4$	v_3	0	0	v_{α_z}
v_4	0	0	0	0	$-v_\alpha$
v_α	$-v_{\alpha'''}$	$-v_{\alpha''}$	$-v_{\alpha_z}$	v_α	0

where

$$\alpha''' = z\rho\alpha_{,\rho} + \frac{1}{2}(z^2 - \rho^2)\alpha_{,z} + \frac{1}{2}z\alpha, \quad \alpha'' = \rho\alpha_{,\rho} + z\alpha_{,z}, \quad \alpha_{,z}\frac{\partial}{\partial u} = v_{\alpha,z}$$

$$[v_1, v_\alpha] = v_{\alpha'''}\frac{\partial}{\partial u}, \quad [v_2, v_\alpha] = v_{\alpha''}, \quad [v_3, v_\alpha] = v_{\alpha,z}, \quad [v_4, v_\alpha] = v_\alpha \tag{5.33}$$

Therefore the four generators of Laplace equation with non zero commutation relations are:

$$[v_1, v_2] = -v_1, \tag{5.34}$$

$$[v_1, v_3] = v_2 - \frac{1}{2}v_4, \tag{5.35}$$

$$[v_2, v_3] = -v_3. \tag{5.36}$$

Since v_4 is a center so the vector fields $\{X_1, X_2, X_3\}$ span the symmetry Lie algebra of (5.2) where $X_i = v_i + v_4$

	X_1	X_2	X_3
X_1	0	$-X_1$	X_2
X_2	X_1	0	$-X_3$
X_3	$-X_2$	X_3	0

and

$$[X_1, X_2] = -X_1, \quad [X_1, X_3] = X_2, \quad [X_2, X_3] = -X_3, \tag{5.37}$$

Therefore the Lie algebra of Laplace equation is $\mathfrak{sl}(2)$ (see the Appendix B).

5.2.1 Symmetries of the Entire System

We apply the second prolongation to the whole system (5.2,5.3,5.4). Setting the coefficients of u_ρ,u_z,$u_{\rho z}$,u_{zz} obtained from equations to be vanish, in which the determining equations for the Lie point symmetries as follows:

$$\begin{gathered}\Psi_{uu} = 0, \quad \Psi_{ku} = 0, \quad \Psi_{kk} = 0, \quad \xi_\rho - \frac{\xi}{\rho} = 0, \quad \eta_\rho = 0, \quad \Psi_\rho = 0, \quad \phi_\rho - \frac{\Psi_u}{2\rho} = 0, \quad \xi_z = 0 \\ \eta_z - \frac{\xi}{\rho} = 0, \quad \Psi_z = 0, \quad \phi_z = 0, \quad \xi_u = 0, \quad \eta_u = 0, \quad \phi_u - \frac{1}{2}\Psi_k\xi_k = 0, \quad \eta_k = 0, \quad \phi_k = 0\end{gathered} \tag{5.38}$$

By solving above 16 equations one can obtain expression for each ξ, η, ϕ and Ψ as follows:

$$\xi = C_5\rho, \tag{5.39}$$

$$\eta = C_5 z + C_6, \tag{5.40}$$

$$\phi = C_1\rho + C_3, \ln(\rho) + C_2 \tag{5.41}$$

$$\Psi = 2(C_3 u + C_1 k) + C_4, \tag{5.42}$$

where $C_1, ..., C_6$ are arbitrary constants.

Hence Lie algebra of the infinitesimal symmetries of system of equations (5.39),...,(5.42) will be spanned by six vector fields:

$$v_1 = \frac{\partial}{\partial z}, \quad (5.43)$$
$$v_2 = \rho\frac{\partial}{\partial \rho} + z\frac{\partial}{\partial z}, \quad (5.44)$$
$$v_3 = \frac{\partial}{\partial u}, \quad (5.45)$$
$$v_4 = \frac{\partial}{\partial k}, \quad (5.46)$$
$$v_5 = u\frac{\partial}{\partial u} + 2k\frac{\partial}{\partial k}, \quad (5.47)$$
$$v_6 = \ln(\rho)\frac{\partial}{\partial u} + 2u\frac{\partial}{\partial k}. \quad (5.48)$$

To find the flows of infinitesimal generators, it is sufficient to do some integration by imposing initial conditions: $\tilde{\rho}(0) = \rho$,$\tilde{z}(0) = z$, $\tilde{u}(0) = u$ and $\tilde{k}(0) = k$:

$$G_1 = (\rho, z + \varepsilon, u, k), \quad (5.49)$$
$$G_2 = (\rho e^{\varepsilon}, z e^{\varepsilon}, u, k), \quad (5.50)$$
$$G_3 = (\rho, z, u + \varepsilon, k), \quad (5.51)$$
$$G_4 = (\rho, z, u, k + \varepsilon), \quad (5.52)$$
$$G_5 = (\rho, z, u e^{\varepsilon}, k e^{2\varepsilon}), \quad (5.53)$$
$$G_6 = (\rho, z, \ln(\rho)\varepsilon + u, \ln(\rho)\varepsilon^2 + 2u\varepsilon + k). \quad (5.54)$$

Where ε is a real number.

Lie algebra of v_i, $i = 1, 2, 3, 4, 5, 6$

	v_1	v_2	v_3	v_4	v_5	v_6
v_1	0	v_1	0	0	0	0
v_2	$-v_1$	0	0	0	0	v_3
v_3	0	0	0	0	v_3	$2v_4$
v_4	0	0	0	0	$2v_4$	0
v_5	0	0	$-v_3$	$-2v_4$	0	$-v_6$
v_6	0	$-v_3$	$-2v_4$	0	v_6	0

$$[v_1, v_2] = v_1, \quad [v_2, v_6] = v_3, \quad [v_3, v_5] = v_3, \quad [v_3, v_6] = 2v_4, \quad [v_4, v_5] = 2v_4, \quad [v_5, v_6] = -v_6. \tag{5.55}$$

5.3 Conserved Quantities of the System of Static Axial Symmetry Vacuum Solution

To find conservation laws of the system equations (5.2) we need to seek three multipliers first

$$\xi = \Lambda^1(\rho, z, u, k, u_\rho, u_z), \quad \eta = \Lambda^2(\rho, z, u, k, u_\rho, u_z), \quad \zeta = \Lambda^3(\rho, z, u, k, u_\rho, u_z) \tag{5.56}$$

Euler Operators are given by

$$E_u = \frac{\partial}{\partial u} - D_\rho \frac{\partial}{\partial u_\rho} - D_z \frac{\partial}{\partial u_z} + D_\rho^2 \frac{\partial}{\partial u_{\rho\rho}} + D_\rho D_z \frac{\partial}{\partial u_{\rho z}} + D_z^2 \frac{\partial}{\partial u_{zz}} \tag{5.57}$$

$$E_k = \frac{\partial}{\partial k} - D_\rho \frac{\partial}{\partial k_\rho} - D_z \frac{\partial}{\partial k_z} \tag{5.58}$$

Hence determining equations are as follows

$$E_u\Big[\xi(u_{\rho\rho} + \frac{1}{\rho}u_\rho + u_{zz}) + \eta(k_\rho - \rho u_\rho^2 - \rho u_z^2) + \zeta(k_z - 2\rho u_\rho u_z)\Big] = 0$$

$$E_k\Big[\xi(u_{\rho\rho} + \frac{1}{\rho}u_\rho + u_{zz}) + \eta(k_\rho - \rho u_\rho^2 - \rho u_z^2) + \zeta(k_z - 2\rho u_\rho u_z)\Big] = 0$$

Then multipliers solution by simplifying determining equation gives turns to

$$\begin{aligned}
\xi &= (C_5\sin(\sqrt{-L}z)+C_6\cos(\sqrt{L}z))C_3\rho J_0(\sqrt{-L}\rho)+(C_5\sin(\sqrt{L}z),\\
&+F_1(\rho,z,u,k)u_z+\rho(\rho u_\rho+\frac{1}{2}u)(C_1z+C_2)+C_6\cos(\sqrt{L}z))C_4\rho Y_0(\sqrt{-L}\rho), && (5.59)\\
\eta &= \frac{2F_{1k}u_{z\rho}u_\rho+F_{1u}u_z+F_{1z}-(C_1z-C_2)\rho}{2\rho} && (5.60)\\
\zeta &= \frac{1}{2}\left(-u_\rho^2+u_z^2)F_{1k}-C_1\rho-F_{1u}u_\rho-F_{1\rho}\rho+F_1(\rho,z,u,k)\right). && (5.61)
\end{aligned}$$

Thus one can find the fluxes for seven linearly independence conservation laws from multipliers

$$\begin{aligned}
&\xi^1=\frac{1}{2}\rho z(2\rho u_\rho+u),\quad \xi^2=\frac{1}{2}(2u_\rho+u),\quad \xi^3=u_z,\quad \eta^1=-\frac{1}{2}z,\\
&\eta^2=-\frac{1}{2},\quad \zeta^1=-\frac{1}{2}\rho,\quad \zeta^3=-\frac{1}{2\rho^2}. && (5.62)
\end{aligned}$$

Case 1: $\xi^1=\frac{1}{2}\rho z(2\rho u_\rho+u)$, $\eta^1=-\frac{1}{2}z$, $\zeta^1=-\frac{1}{2}\rho$:

$$\begin{aligned}
&(\frac{1}{2}\rho z(2\rho u_\rho+u))(u_{\rho\rho}+\frac{1}{\rho}u_\rho+u_{zz})+(-\frac{1}{2}z)(k_\rho-\rho u_\rho^2-\rho u_z^2)-(\frac{1}{2}\rho)(k_z-2\rho u_\rho u_z)=0\\
&D_\rho(-\frac{1}{4}k_z\rho^2+\frac{1}{2}\rho^2 z(u_\rho^2-u_z^2)+\frac{1}{2}\rho u_\rho u_z-\frac{1}{2}kz)\\
&+D_z(z\rho^2u_\rho u_z+\frac{1}{2}\rho z u u_z-\frac{1}{4}\rho(u^2+\rho k_\rho))=0. && (5.63)
\end{aligned}$$

Case 2: $\xi^2=\frac{1}{2}\rho(2u_\rho+u)$, $\eta=-\frac{1}{2}$, $\zeta=0$:

$$\begin{aligned}
&\frac{1}{2}\rho(2u_\rho+u)(u_{\rho\rho}+\frac{1}{\rho}u_\rho+u_{zz})-\frac{1}{2}(k_\rho-\rho u_\rho^2-\rho u_z^2)=0\\
&D_\rho(\frac{1}{2}\rho^2(u_\rho^2-u_z^2)+\frac{1}{2}\rho u u_\rho-\frac{1}{2}k)+D_z(\rho^2u_\rho u_z+\frac{1}{2}\rho u u_z)=0 && (5.64)
\end{aligned}$$

Case 3: $\xi^3=u_z$, $\eta=0$, $\zeta=-\frac{1}{2\rho^2}$:

$$\begin{aligned}
&u_z(u_{\rho\rho}+\frac{1}{\rho}u_\rho+u_{zz})+-\frac{1}{2\rho^2}(k_z-2\rho u_\rho u_z)=0\\
&D_\rho(u_\rho u_z)+\frac{1}{2}D_z(u_z^2+\frac{k}{\rho^2}-u_\rho^2)=0 && (5.65)
\end{aligned}$$

5.4 Symmetries of Geodesics

We now turn our attention to geodeics of the general system with unknown u and k. From the Lagrangian $L = g_{\mu\nu}\dot{x}^{\mu}\dot{x}^{\nu}$ we obtain

$$-2e^{2u}\dot{t} = 0, \tag{5.66}$$

$$2e^{-2u+2k}\dot{\rho} - 2u_{\rho}e^{2u}\dot{t}^2 + (-2u_{\rho} + 2k_{\rho})\, e^{-2u+2k}(\dot{\rho}^2 + \dot{z}^2) + 2\rho\,(1 - \rho u_{\rho})\, e^{-2u}\dot{\phi}^2 = 0, \tag{5.67}$$

$$2e^{-2u+2k}\dot{z} - 2u_{z}e^{2u}\dot{t}^2 + (-2u_{z} + 2k_{z})\, e^{-2u+2k}(\dot{\rho}^2 + \dot{z}^2) - 2\rho^2 u_z e^{-2u}\dot{\phi}^2 = 0, \tag{5.68}$$

$$\rho^2 e^{-2u}\dot{\phi} = 0 \tag{5.69}$$

where $\frac{\partial u}{\partial \rho} \equiv u_{\rho}, \frac{\partial k}{\partial \rho} \equiv k_{\rho}$ etc. Two conserved quantities immediately follow:

$$E = e^{2u}\dot{t}, \quad J = \rho^2 e^{-2u}\dot{\phi}. \tag{5.70}$$

in terms of which the geodesic equations are:

$$\frac{\mathrm{d}^2 t}{\mathrm{d}s^2} + 2u_{\rho}\frac{\mathrm{d}t}{\mathrm{d}s}\frac{\mathrm{d}\rho}{\mathrm{d}s} + 2u_{z}\frac{\mathrm{d}t}{\mathrm{d}s}\frac{\mathrm{d}z}{\mathrm{d}s} = 0 \tag{5.71a}$$

$$\begin{aligned}\frac{\mathrm{d}^2 \rho}{\mathrm{d}s^2} + (e^{4u-2k}u_{\rho})\left(\frac{\mathrm{d}t}{\mathrm{d}s}\right)^2 + (-u_{\rho} + k_{\rho})\left(\frac{\mathrm{d}\rho}{\mathrm{d}s}\right)^2 + 2(-u_z + k_z)\frac{\mathrm{d}z}{\mathrm{d}s}\frac{\mathrm{d}\rho}{\mathrm{d}s} \\ +(u_{\rho} - k_{\rho})\left(\frac{\mathrm{d}z}{\mathrm{d}s}\right)^2 + e^{-2k}(\rho^2 u_{\rho} - \rho)\left(\frac{\mathrm{d}\phi}{\mathrm{d}s}\right)^2 = 0\end{aligned} \tag{5.71b}$$

$$\begin{aligned}\frac{\mathrm{d}^2 z}{\mathrm{d}s^2} + (e^{4u-2k}u_{z})\left(\frac{\mathrm{d}t}{\mathrm{d}s}\right)^2 + (u_{z} - k_{z})\left(\frac{\mathrm{d}\rho}{\mathrm{d}s}\right)^2 + 2(-u_{\rho} + k_{\rho})\frac{\mathrm{d}z}{\mathrm{d}s}\frac{\mathrm{d}\rho}{\mathrm{d}s} \\ +(-u_{z} + k_{z})\left(\frac{\mathrm{d}z}{\mathrm{d}s}\right)^2 + e^{-2k}\rho^2 u_{z}\left(\frac{\mathrm{d}\phi}{\mathrm{d}s}\right)^2 = 0\end{aligned} \tag{5.71c}$$

$$\frac{\mathrm{d}^2 \phi}{\mathrm{d}s^2} + 2(-u_{\rho} + \frac{1}{\rho})\frac{\mathrm{d}\rho}{\mathrm{d}s}\frac{\mathrm{d}\phi}{\mathrm{d}s} - 2u_{z}\frac{\mathrm{d}z}{\mathrm{d}s}\frac{\mathrm{d}\phi}{\mathrm{d}s} = 0. \tag{5.71d}$$

5.4.1 Orbital Motion

Now we can calculate orbital motion $\frac{d^2\rho}{d\phi^2}$ directly from geodesics equations using the chain rule

$$\begin{aligned}\frac{d^2\rho}{d\phi^2} &= \frac{d}{d\phi}(\frac{d\rho}{d\phi}) = \frac{\frac{d}{ds}}{\frac{d\phi}{ds}}(\frac{\frac{d\rho}{ds}}{\frac{d\phi}{ds}}) \\ &= \frac{\ddot{\rho}\dot{\phi} - \ddot{\phi}\dot{\rho}}{\dot{\phi}^3},\end{aligned} \tag{5.72}$$

where $\dot{\rho} = \frac{d\rho}{ds}$ and $\ddot{\rho} = \frac{d^2\rho}{ds^2}$. Now if we assume $\frac{dz}{ds} = 0$ and multiply equation (5.71b) by $\frac{1}{\dot{\phi}^2}$ and (5.71d) by $\frac{\dot{\rho}}{\dot{\phi}^3}$ then subtract equations, which yields

$$\frac{d^2\rho}{d\phi^2} = -\left(e^{4u-2k}u_\rho\right)\left(\frac{\dot{t}}{\dot{\phi}}\right)^2 + (u_\rho - k_\rho)\left(\frac{\dot{\rho}}{\dot{\phi}}\right)^2 + e^{-2k}\rho^2\left(\frac{1}{\rho} - u_\rho\right) + 2\left(\frac{1}{\rho} - u_\rho\right)\left(\frac{\dot{\rho}}{\dot{\phi}}\right)^2$$

$$\frac{d^2\rho}{d\phi^2} = -\left(e^{4u-2k}u_\rho\right)\left(\frac{dt}{d\phi}\right)^2 + \left(-u_\rho - k_\rho + \frac{2}{\rho}\right)\left(\frac{d\rho}{d\phi}\right)^2 + e^{-2k}\rho^2\left(\frac{1}{\rho} - u_\rho\right) \tag{5.73}$$

Since L is another constant of motion, setting it equal to zero would lead to null geodesics, as before:

$$\frac{d^2\rho}{d\phi^2} + \left(2\left(\frac{d\rho}{d\phi}\right)^2 + 2e^{-2k}\rho^2\right)u_\rho + \left(k_\rho - \frac{2}{\rho}\right)\left(\frac{d\rho}{d\phi}\right)^2 - \rho e^{-2k} = 0 \tag{5.74}$$

Therefore we have

$$\frac{\mathrm{d}^2\rho}{\mathrm{d}\phi^2} + \left(k_\rho + 2u_\rho - \frac{2}{\rho}\right)\left(\frac{\mathrm{d}\rho}{\mathrm{d}\phi}\right)^2 + \left(2e^{-2k}\rho^2 u_\rho - e^{-2k}\rho\right) = 0. \tag{5.75}$$

This equation is the same as (5.89) because when we take $\frac{dz}{ds} = 0$, u_z and k_z will automatically drop out from the equations.

5.4.2 Analyzing Special Case on the Equatorial Plane

If we initially take $z = z_0$ and $\frac{dz}{ds} = 0$, then from the geodesic equation (5.71c) we have

$$\frac{\mathrm{d}^2 z}{\mathrm{d}s^2} + (e^{4u-2k}u_z)\left(\frac{\mathrm{d}t}{\mathrm{d}s}\right)^2 + (u_z - k_z)\left(\frac{\mathrm{d}\rho}{\mathrm{d}s}\right)^2 + e^{-2k}(\rho^2 u_z)\left(\frac{\mathrm{d}\phi}{\mathrm{d}s}\right)^2 = 0. \tag{5.76}$$

Now to have $\frac{d^2z}{ds^2}=0$ (means equatorial motion), we need to satisfy the following condition

$$u_z\left(e^{4u-2k}\left(\frac{\mathrm{d}t}{\mathrm{d}s}\right)^2+\left(\frac{\mathrm{d}\rho}{\mathrm{d}s}\right)^2+e^{-2k}\rho^2\left(\frac{\mathrm{d}\phi}{ds}\right)^2\right)-k_z\left(\frac{d\rho}{ds}\right)^2=0. \tag{5.77}$$

Using $L=0$ leads to further simplification:

$$-e^{4u-2k}\left(\frac{\mathrm{d}t}{\mathrm{d}s}\right)^2+\left(\frac{\mathrm{d}\rho}{\mathrm{d}s}\right)^2+\rho^2e^{-2k}\left(\frac{\mathrm{d}\phi}{ds}\right)^2=0. \tag{5.78}$$

Now when we plugging (5.78) into (5.77), we will have

$$u_z\left(2\left(\frac{\mathrm{d}\rho}{\mathrm{d}s}\right)^2+2\rho^2e^{-2k}\left(\frac{\mathrm{d}\phi}{\mathrm{d}s}\right)^2\right)=k_z\left(\frac{\mathrm{d}\rho}{\mathrm{d}s}\right)^2. \tag{5.79}$$

Then we divide both sides of equation (5.79) by $\left(\frac{\mathrm{d}\phi}{\mathrm{d}s}\right)^2$:

$$(2u_z-k_z)\left(\frac{\mathrm{d}\rho}{\mathrm{d}\phi}\right)^2+2\rho^2e^{-2k}u_z \;=\; 0, \tag{5.80}$$

which yields

$$\left(\frac{\mathrm{d}\rho}{\mathrm{d}\phi}\right)^2=\frac{-2e^{-2k}\rho^2u_z}{2u_z-k_z} \tag{5.81}$$

This means the the right hand side of (5.81) has to be non negative. We now consider some particular cases:

Case (i) $2u_z-k_z>0$ and $u_z<0$, then we have

$$\frac{k_z}{2}>u_z>0. \tag{5.82}$$

Case (ii) $2u_z-k_z<0$ and $u_z>0$, then we have

$$\frac{k_z}{2}<u_z<0, \tag{5.83}$$

so it is not possible to have u_z and k_z with different sign when they are nonzero.

Case (iii) If $u_z=0$, and $2u_z-k_z\neq 0$ meaning $k_z\neq 0$, then we have

$$\frac{d\rho}{d\phi}=0, \tag{5.84}$$

therefore we have circular orbit. Note that if

$$u_z = 0, \quad k_z = 0 \quad (i.e. \quad u = u(\rho), \quad k = k(\rho)) \tag{5.85}$$

we have the cylindrical case, and for

$$u_z = 0, \quad k_z \neq 0 \quad (i.e. \quad u = u(\rho), \quad k = k(\rho, z)) \tag{5.86}$$

we have circular orbit.

We are looking for general solution of all geodesic equations when we have null path and $z = z_0$, $\frac{dz}{ds} = 0$, $u_z = 0$ and $k_z = 0$, then with all this information we can write geodesic equations as follows:

$$\frac{\mathrm{d}^2 t}{\mathrm{d}s^2} + 2u_\rho \frac{\mathrm{d}t}{\mathrm{d}s}\frac{\mathrm{d}\rho}{\mathrm{d}s} = 0 \tag{5.87a}$$

$$\frac{\mathrm{d}^2 \rho}{\mathrm{d}s^2} + (e^{4u-2k}u_\rho)\left(\frac{\mathrm{d}t}{\mathrm{d}s}\right)^2 + (-u_\rho + k_\rho)\left(\frac{\mathrm{d}\rho}{\mathrm{d}s}\right)^2 + e^{-2k}(\rho^2 u_\rho - \rho)\left(\frac{\mathrm{d}\phi}{\mathrm{d}s}\right)^2 = 0 \tag{5.87b}$$

$$\frac{\mathrm{d}^2 \phi}{\mathrm{d}s^2} + 2(-u_\rho + \frac{1}{\rho})\frac{\mathrm{d}\rho}{\mathrm{d}s}\frac{\mathrm{d}\phi}{\mathrm{d}s} = 0. \tag{5.87c}$$

We apply equation (1.30) in the previous equations and we will have

$$\frac{\mathrm{d}^2 t}{\mathrm{d}s^2} + 2u_\rho \frac{\mathrm{d}t}{\mathrm{d}s}\frac{\mathrm{d}\rho}{\mathrm{d}s} = 0 \tag{5.88a}$$

$$\frac{\mathrm{d}^2 \rho}{\mathrm{d}s^2} + (2e^{-2k}\rho^2 u_\rho)\left(\frac{\mathrm{d}\phi}{\mathrm{d}s}\right)^2 + k_\rho\left(\frac{\mathrm{d}\rho}{\mathrm{d}s}\right)^2 - e^{-2k}\rho\left(\frac{\mathrm{d}\phi}{\mathrm{d}s}\right)^2 = 0 \tag{5.88b}$$

$$\frac{\mathrm{d}^2 \phi}{\mathrm{d}s^2} + 2(-u_\rho + \frac{1}{\rho})\frac{\mathrm{d}\rho}{\mathrm{d}s}\frac{\mathrm{d}\phi}{\mathrm{d}s} = 0. \tag{5.88c}$$

Then by some manipulations and using the idea (5.72) we have

$$\frac{\mathrm{d}^2 \rho}{\mathrm{d}\phi^2} + \left(2e^{-2k}\rho^2 u_\rho - e^{-2k}\rho\right) + \left(k_\rho + 2u_\rho - \frac{2}{\rho}\right)\left(\frac{\mathrm{d}\rho}{\mathrm{d}\phi}\right)^2 = 0. \tag{5.89}$$

Now if we apply $\frac{dz}{ds} \neq 0$ and $\frac{d\rho}{ds} = 0$ simultaneously, we have geodesic equations as follows

$$\frac{\mathrm{d}^2 t}{\mathrm{d}s^2} + 2u_z \frac{\mathrm{d}t}{\mathrm{d}s}\frac{\mathrm{d}z}{\mathrm{d}s} = 0 \tag{5.90a}$$

$$(e^{4u-2k}u_\rho)\left(\frac{\mathrm{d}t}{\mathrm{d}s}\right)^2 + (u_\rho - k_\rho)\left(\frac{\mathrm{d}z}{\mathrm{d}s}\right)^2 + e^{-2k}(\rho^2 u_\rho - \rho)\left(\frac{\mathrm{d}\phi}{\mathrm{d}s}\right)^2 = 0 \tag{5.90b}$$

$$\frac{\mathrm{d}^2 z}{\mathrm{d}s^2} + (e^{4u-2k}u_z)\left(\frac{\mathrm{d}t}{\mathrm{d}s}\right)^2 + (-u_z + k_z)\left(\frac{\mathrm{d}z}{\mathrm{d}s}\right)^2 + e^{-2k}(\rho^2 u_z)\left(\frac{\mathrm{d}\phi}{\mathrm{d}s}\right)^2 = 0 \tag{5.90c}$$

$$\frac{\mathrm{d}^2 \phi}{\mathrm{d}s^2} - 2u_z \frac{\mathrm{d}z}{\mathrm{d}s}\frac{\mathrm{d}\phi}{\mathrm{d}s} = 0. \tag{5.90d}$$

Then by taking equations (5.90c) and (5.90d), and do some manipulation, we have

$$\frac{\mathrm{d}^2 z}{\mathrm{d}\phi^2} + \left(e^{4u-2k}u_z\right)\left(\frac{\mathrm{d}t}{\mathrm{d}\phi}\right)^2 + (u_z + k_z)\left(\frac{\mathrm{d}z}{\mathrm{d}\phi}\right)^2 + e^{-2k}\rho^2 u_z = 0. \tag{5.91}$$

Now if we take Lagrangian zero, means

$$0 = -e^{4u-2k}\left(\frac{\mathrm{d}t}{\mathrm{d}s}\right)^2 + \left(\frac{\mathrm{d}z}{\mathrm{d}s}\right)^2 + \rho^2 e^{-2k}\left(\frac{\mathrm{d}\phi}{ds}\right)^2 \tag{5.92}$$

So, (5.91) turns to

$$\frac{\mathrm{d}^2 z}{\mathrm{d}\phi^2} + 2e^{-2k}\rho^2 u_z + (2u_z + k_z)\left(\frac{\mathrm{d}z}{\mathrm{d}\phi}\right)^2 = 0. \tag{5.93}$$

Let $\frac{dz}{ds} \neq 0$ and $\frac{d\phi}{ds} \neq 0$, taking $\rho = \rho_0$, $\frac{d\rho}{ds} = 0$ and $\frac{d^2\rho}{ds^2} = 0$

We start with the equation (5.71b) and take $\frac{d\rho}{ds} = 0$.

$$\frac{\mathrm{d}^2 \rho}{\mathrm{d}s^2} + (e^{4u-2k}u_\rho)\left(\frac{\mathrm{d}t}{\mathrm{d}s}\right)^2 + (u_\rho - k_\rho)\left(\frac{\mathrm{d}z}{\mathrm{d}s}\right)^2 + e^{-2k}(\rho^2 u_\rho - \rho)\left(\frac{\mathrm{d}\phi}{\mathrm{d}s}\right)^2 = 0. \tag{5.94}$$

Then to have $\frac{\mathrm{d}^2\rho}{\mathrm{d}s^2} = 0$, we need to have the following condition

$$u_\rho\left(e^{4u-2k}\left(\frac{\mathrm{d}t}{\mathrm{d}s}\right)^2 + \left(\frac{\mathrm{d}z}{\mathrm{d}s}\right)^2 + e^{-2k}\rho^2\left(\frac{\mathrm{d}\phi}{ds}\right)^2\right) - k_\rho\left(\frac{dz}{ds}\right)^2 - \rho^2 e^{-2k}\left(\frac{\mathrm{d}\phi}{\mathrm{d}s}\right)^2 = 0. \tag{5.95}$$

On the other hand when we put Lagrangian zero we will have

$$0 = -e^{2u}\left(\frac{\mathrm{d}t}{\mathrm{d}s}\right)^2 + e^{-2u+2k}\left(\frac{\mathrm{d}z}{\mathrm{d}s}\right)^2 + \rho^2 e^{-2u}\left(\frac{\mathrm{d}\phi}{ds}\right)^2 \tag{5.96}$$

So

$$0 = -e^{4u-2k}\left(\frac{\mathrm{d}t}{\mathrm{d}s}\right)^2 + \left(\frac{\mathrm{d}z}{\mathrm{d}s}\right)^2 + \rho^2 e^{-2k}\left(\frac{\mathrm{d}\phi}{ds}\right)^2. \tag{5.97}$$

Now by applying (5.97) in (5.95), we will have

$$u_\rho\left(2\left(\frac{\mathrm{d}z}{\mathrm{d}s}\right)^2 + 2\rho^2 e^{-2k}\left(\frac{\mathrm{d}\phi}{\mathrm{d}s}\right)^2\right) = k_\rho\left(\frac{\mathrm{d}z}{\mathrm{d}s}\right)^2 + \rho^2 e^{-2k}\left(\frac{\mathrm{d}\phi}{\mathrm{d}s}\right)^2. \tag{5.98}$$

Then we divide both sides of equation (5.98) by $\left(\frac{\mathrm{d}\phi}{\mathrm{d}s}\right)^2$, we obtain

$$\begin{aligned} u_\rho\left(2\left(\frac{\mathrm{d}z}{\mathrm{d}\phi}\right)^2 + 2\rho^2 e^{-2k}\right) &= k_\rho\left(\frac{\mathrm{d}z}{\mathrm{d}\phi}\right)^2 + \rho e^{-2k} \\ (2u_\rho - k_\rho)\left(\frac{\mathrm{d}z}{\mathrm{d}\phi}\right)^2 &= \rho e^{-2k} - 2\rho^2 e^{-2k} u_\rho, \end{aligned} \tag{5.99}$$

therefore we have

$$\left(\frac{\mathrm{d}z}{\mathrm{d}\phi}\right)^2 = \frac{e^{-2k}\rho^2\left(\frac{1}{\rho} - u_\rho\right)}{2u_\rho - k_\rho}. \tag{5.100}$$

We can consider some cases of the equation (5.100)

Case (i) If $k = 0$, then $u_\rho < \frac{1}{2\rho}$ and $u_\rho > 0$, then we have

$$0 < u_\rho < \frac{1}{2\rho}. \tag{5.101}$$

or $u_\rho > \frac{1}{2\rho}$ and $u_\rho < 0$, then we have

$$0 > u_\rho > \frac{1}{2\rho}. \tag{5.102}$$

Case (ii) If $u_\rho = 0$,

$$k_\rho > 0 \text{ and } \rho > 0 \quad \text{or} \quad k_\rho > 0 \text{ and } \rho < 0 \tag{5.103}$$

Case (iii) If $k_\rho = 0$,

$$0 < u_\rho < \frac{1}{2\rho} \quad \text{or} \quad 0 > u_\rho > \frac{1}{2\rho} \tag{5.104}$$

Case (iv) If $\frac{1}{\rho} - 2u_\rho = 0$ and $k_\rho \neq \frac{1}{\rho}$, then

$$\frac{dz}{d\phi} = 0 \rightarrow z(\phi) = C$$

If $\frac{dz}{ds} = 0$ and $L = 0$

$$\frac{\mathrm{d}^2\rho}{\mathrm{d}\phi^2} + \left(k_\rho + 2u_\rho - \frac{2}{\rho}\right)\left(\frac{\mathrm{d}\rho}{\mathrm{d}\phi}\right)^2 + \left(2e^{-2k}\rho^2 u_\rho - e^{-2k}\rho\right) = 0. \tag{5.105}$$

If $\frac{d\rho}{ds} = 0$ and $L = 0$

$$\frac{\mathrm{d}^2 z}{\mathrm{d}\phi^2} + (2u_z + k_z)\left(\frac{\mathrm{d}z}{\mathrm{d}\phi}\right)^2 + 2e^{-2k}\rho^2 u_z = 0. \tag{5.106}$$

Now to check the consistency of equations (5.105) and (5.106), one can put $\rho = \rho_0$ and $z = z_0$, then from the equation (5.105) we have

$$u_\rho = \frac{1}{2\rho} \longrightarrow u = \frac{1}{2}\ln(\rho) + f(z) \tag{5.107}$$

and from equation (5.106) we have

$$u_z = 0 \longrightarrow u = u(\rho) \tag{5.108}$$

then we have

$$u = \ln\rho^2 + C_1$$

Now if we take (5.106)

$$u_z = 0 \longrightarrow u = f(\rho) \tag{5.109}$$

From equation (5.105) we have

$$u_\rho = \frac{2}{\rho} \tag{5.110}$$

So we have

$$u = \ln\rho^2 + C_1$$

5.5 Conclusions

We find that things become more interesting as one relaxes symmetry. The Einstein equations of the static vacuum system have an infinite number of solutions, and there are a total of six symmetries that leave it invariant. Using these, one can obtain one solution from another. Looking at G_6, it is now clear why the unexpected symmetry found in [2] was to be expected due to the additive $\ln \rho$ term that appears in G_6. This also shows that one cannot expect another algebraic prescription.

The geodesic equations are complicated and general. However, an orbital light bending equation can be found relating ρ with ϕ. This is not unexpected given that there are no cross-terms in the metric, and one could thus eliminate the parameter. We analyzed the orbital equation for various special cases and hope that they and the general equations derived herein will find future applications.

CHAPTER 6

STATIONARY AXIALLY SYMMETRIC METRICS

The highest reach of human science is the scientific recognition of human ignorance...

— *William R. Hamilton*

This chapter is devoted to analyzing geodesics of the most general stationary axisymmetric metric and Lie point symmetries arising from them. Stationary axially symmetric Kerr metric is the first rotating solution obtained in 1963. The static solution of axially symmetric metric in cylindrical coordinates was discovered in 1917 by Weyl as was dicussed in the previous chapter. Later in 1924, a stationary, axially symmetric solution of the Einstein field equation was known. After the Kerr solution, another set of attractive rotating solutions, called the Tomimatsu-Sato solutions, were found in 1972.

Although both Kerr and Tomimatsu solutions rotate and describe spherically symmetric metrics' exterior fields, they have an essential difference. If we omit the angular momentum in the Kerr metric, the solution reduces to the Schwarzschild solution. Static implies non-rotating, in the Tomimatsu-Sato, by eliminating the angular momentum, so the limit would not tend to Schwarzschild, but instead tends to the Weyl solution [28].

Kerr solution intended to some different kinds of coordinates. Usually, we consider that in Boyer-Lindquist coordinate, with two Killing vectors such as $k^1 = (1, 0, 0, 0)$, and $k^2 = (0, 0, 0, 1)$, which are taking to account for stationary and axial symmetry.

In addition we study the condition for separability of geodesics of general static axially symmetric metric specially the Weyl metric.

For additional detail, we start with the most general static case (i.e. $k = 0$), then one can consider projective geometry, in the sense that the geodesics are the same as null geodesics; (see Chap3 for the proof). Therefore, we will have three-dimensional metric.

Then we considered necessary and sufficient conditions for the separability and went through the Hamilton Jacobi, which is corresponding to the fourth conserved quantity (i.e., the Carter constant).

6.1 Stationary Axially Symmetric Vacuum Solutions of Einstein's Equations

The most general axially symmetric rotating metric is given by ([11, 12, 38])

$$\mathrm{d}s^2 = f\mathrm{d}t^2 - 2k\mathrm{d}t\mathrm{d}\phi + l\mathrm{d}\phi^2 + \mathrm{e}^u\left(\mathrm{d}\rho^2 + \mathrm{d}z^2\right), \tag{6.1}$$

where all functions f , k , l and u are function of ρ and z.

First integrals are differential equations of first order that defined on the tangent bundle of a manifold (i.e. TM). There are 4 first integrals or constant of motion exists L, E, ds^2, K for stationary axisymmetric solution in vacuum , constant of motions are playing the significant roles because without solving the equations we can find some properties of the motion. Then by comparing to metric (8.2) we will have canonical form

$$\begin{aligned}
\mathrm{d}s^2 &= f\left(\mathrm{d}t^2 - 2\frac{k}{f}\mathrm{d}t\mathrm{d}\phi + \frac{k^2}{f^2}\mathrm{d}\phi^2\right) + \left(-\frac{k^2}{f} + l\right)\mathrm{d}\phi^2 + \mathrm{e}^{\lambda}\left(\mathrm{d}\rho^2 + \mathrm{d}z^2\right) \\
&= \mathrm{e}^{2u}\left(dt - \Omega(\rho, z)\mathrm{d}\phi\right)^2 + \left(\rho^2\mathrm{e}^{-2u}\right)\mathrm{d}\phi^2 + \mathrm{e}^{-2u+2k}\left(\mathrm{d}\rho^2 + \mathrm{d}z^2\right),
\end{aligned} \tag{6.2}$$

$$f = -\mathrm{e}^{2u}, \quad \frac{k}{f} = \Omega(\rho, z), \quad \lambda = -2u + 2k, \quad -\frac{k^2}{f} + l = \mathrm{e}^{-2u} \tag{6.3}$$

Then (8.64) form as follows

$$\mathrm{d}s^2 = -\mathrm{e}^{2u}\mathrm{d}t^2 + 2\Omega\mathrm{e}^{2u}\mathrm{d}t\mathrm{d}\phi + \rho^2\mathrm{e}^{-2u}\mathrm{d}\phi^2 + \mathrm{e}^{-2u+2k}\left(\mathrm{d}\rho^2 + \mathrm{d}z^2\right), \tag{6.4}$$

Then two immediate first integrals are as follows

$$E = e^{2u}\left(-\frac{\mathrm{d}t}{ds} + \Omega\frac{\mathrm{d}\phi}{\mathrm{d}s}\right), \quad L = \Omega\mathrm{e}^{2u}\frac{dt}{\mathrm{d}s} + \rho^2\mathrm{e}^{-2u}\frac{\mathrm{d}\phi}{\mathrm{d}s} \tag{6.5}$$

where the metric components are:

$$g_{tt} = -\mathrm{e}^{2u}, \quad g_{t\phi} = g_{\phi t} = \Omega\mathrm{e}^{2u}, \quad g_{\phi\phi} = \rho^2\mathrm{e}^{-2u}, \quad g_{\rho\rho} = g_{zz} = \mathrm{e}^{-2u+2k}. \tag{6.6}$$

When $\phi = 0$ cross term is remove and we go back to static axisymmetric, when we have time reversal, t goes to $-t$, then ϕ goes to $-\phi$ simultaneously means we have isometry, (i.e., the metric remains invariant, because we have $g_{t\phi}$ term which is not zero), so we can not have isometry when only have time reversal or ϕ goes to $-\phi$ alone. The geodesics equations are

$$\begin{aligned}
&\frac{\mathrm{d}^2 t}{\mathrm{d}s^2} + 2\frac{2\mathrm{e}^{4u}u_\rho\Omega^2 + 2\rho^2 u_\rho + \mathrm{e}^{4u}\Omega_\rho\Omega}{2\rho^2 + 2\Omega^2\mathrm{e}^{4u}}\frac{\mathrm{d}t}{\mathrm{d}s}\frac{\mathrm{d}\rho}{\mathrm{d}s} + 2\frac{2\mathrm{e}^{4u}u_z\Omega^2 + 2\rho^2 u_z + \mathrm{e}^{4u}\Omega_z\Omega}{2\rho^2 + 2\Omega^2\mathrm{e}^{4u}}\frac{\mathrm{d}t}{\mathrm{d}s}\frac{\mathrm{d}z}{\mathrm{d}s} \\
&+2\frac{-\rho\left(4u_\rho\Omega\rho + \Omega_\rho\rho - 2\Omega\right)}{2\rho^2 + 2\Omega^2\mathrm{e}^{4u}}\frac{\mathrm{d}\phi}{\mathrm{d}s}\frac{\mathrm{d}\rho}{\mathrm{d}s} + 2\frac{-\rho^2\left(4u_z\Omega + \Omega_z\right)}{2\rho^2 + 2\Omega^2\mathrm{e}^{4u}}\frac{\mathrm{d}\phi}{\mathrm{d}s}\frac{\mathrm{d}z}{\mathrm{d}s} = 0 \qquad (6.7a)
\end{aligned}$$

$$\begin{aligned}
&\frac{\mathrm{d}^2 \rho}{\mathrm{d}s^2} + e^{4u-2k}u_\rho\left(\frac{\mathrm{d}t}{\mathrm{d}s}\right)^2 + (-u_\rho + k_\rho)\left(\frac{\mathrm{d}\rho}{\mathrm{d}s}\right)^2 + 2(-u_z + k_z)\frac{dz}{\mathrm{d}s}\frac{\mathrm{d}\rho}{\mathrm{d}s} \\
&-2\frac{\mathrm{e}^{4u-2k}\left(2u_\rho\Omega + \Omega_\rho\right)}{2}\frac{dt}{\mathrm{d}s}\frac{\mathrm{d}\phi}{\mathrm{d}s} + (u_\rho - k_\rho)\left(\frac{\mathrm{d}z}{\mathrm{d}s}\right)^2 + e^{-2k}(\rho^2 u_\rho - \rho)\left(\frac{\mathrm{d}\phi}{\mathrm{d}s}\right)^2 = 0 \qquad (6.7b)
\end{aligned}$$

$$\begin{aligned}
&\frac{\mathrm{d}^2 z}{\mathrm{d}s^2} + (e^{4u-2k}u_z)\left(\frac{\mathrm{d}t}{\mathrm{d}s}\right)^2 + (u_z - k_z)\left(\frac{\mathrm{d}\rho}{\mathrm{d}s}\right)^2 + 2(-u_\rho + k_\rho)\frac{\mathrm{d}z}{\mathrm{d}s}\frac{\mathrm{d}\rho}{\mathrm{d}s} \\
&+2(-\frac{\mathrm{e}^{4u-2k}\left(2u_z\Omega + \Omega_z\right)}{2})\frac{\mathrm{d}t}{\mathrm{d}s}\frac{\mathrm{d}\phi}{\mathrm{d}s} + (-u_z + k_z)\left(\frac{\mathrm{d}z}{\mathrm{d}s}\right)^2 + e^{-2k}(\rho^2 u_z)\left(\frac{\mathrm{d}\phi}{\mathrm{d}s}\right)^2 = 0 \qquad (6.7c)
\end{aligned}$$

$$\begin{aligned}
&\frac{\mathrm{d}^2 \phi}{\mathrm{d}s^2} + 2(\frac{\mathrm{e}^{2u}\Omega_\rho}{2\rho^2\mathrm{e}^{-2u} + 2\Omega^2\mathrm{e}^{2u}})\frac{\mathrm{d}\rho}{\mathrm{d}s}\frac{\mathrm{d}t}{\mathrm{d}s} + 2(\frac{\mathrm{e}^{2u}\Omega_z}{2\rho^2\mathrm{e}^{-2u} + 2\Omega^2\mathrm{e}^{2u}})\frac{\mathrm{d}z}{\mathrm{d}s}\frac{\mathrm{d}t}{\mathrm{d}s} \\
&+2(\frac{2\mathrm{e}^{4u}u_\rho\Omega^2 - 2\rho^2 u_\rho + \mathrm{e}^{4u}\Omega_\rho\Omega + 2\rho}{2\rho^2 + 2\Omega^2\mathrm{e}^{4u}})\frac{\mathrm{d}\rho}{\mathrm{d}s}\frac{\mathrm{d}\phi}{\mathrm{d}s} + 2(\frac{2\mathrm{e}^{4u}u_z\Omega^2 - 2\rho^2 u_z + \mathrm{e}^{4u}\Omega_z\Omega}{2\rho^2 + 2\Omega^2\mathrm{e}^{4u}})\frac{\mathrm{d}z}{\mathrm{d}s}\frac{\mathrm{d}\phi}{\mathrm{d}s} = 0, \\
&\qquad (6.7d)
\end{aligned}$$

where $\frac{\partial u}{\partial \rho} \equiv u_\rho, \frac{\partial k}{\partial \rho} \equiv k_\rho$ etc.

6.2 Separability of Hamilton-Jacobi on Geodesics and Finding the Fourth Conserved Quantity (Carter's Constant)

Let M be a Riemannian manifold, then Lagrangian $L = g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu$ define on the tangent bundle TM of the Riemann manifold and Hamiltonian, $H = \frac{1}{2}g^{\mu\nu}p_\mu p_\nu$, is defined on the cotangent bundle T^*M. The *fourth conserved quantity* will be calculated from *Hamilton-Jacobi equation*. Note that

$$p_\mu = \frac{\partial L}{\partial \dot{x}^\mu} \quad \Longrightarrow \quad p_\mu = 2g_{\mu\nu}\dot{x}^\nu \qquad (6.8)$$

for the given Lagrangian. Hence,

$$\frac{1}{2}g^{\mu\nu}p_\mu = \dot{x}^\nu \ . \tag{6.9}$$

Hence, the Hamiltonian

$$\mathcal{H} \ = \ p_\mu \dot{q}^\mu - L \ = \ g^{\mu\nu}p_\mu p_\nu \tag{6.10}$$

The Hamilton-Jacobi equation is given by

$$\mathcal{H} + \frac{\partial S}{\partial \sigma} = 0 \tag{6.11}$$

where S is the Jacobi action. In the Hamilton-Jacobi formalism, we have $p_\mu = \frac{\partial S}{\partial x^\mu}$. Hence, we finally have

$$\frac{\partial S}{\partial \sigma} = -g^{\mu\nu}\frac{\partial S}{\partial x^\mu}\frac{\partial S}{\partial x^\nu} \ . \tag{6.12}$$

To get to the 4$^{\text{th}}$ conserved quantity, we assume the Jacobi action is separable, i.e. we will have

$$S = m_0^2\sigma - Et + L\phi + S_r(r) + S_\theta(\theta) \ . \tag{6.13}$$

6.2.1 Fourth Conserved Quantity of the General Stationary Axially Symmetric Metric

As we mentioned in previous section metric is the form

$$ds^2 = -f dt^2 + 2k\, dt\, d\phi + l d\phi^2 + \mathrm{e}^u \left(d\rho^2 + dz^2\right), \tag{6.14}$$

where $f,\ k,\ l$ and u are functions of ρ and z.

Levi- Civita theorem proved that Hamilton-Jacobi equation admits a completely separable solution if and only if satisfy for $\mu \neq \nu$

$$\partial^\mu H \partial^\nu H \partial_{\mu\nu} H + \partial_\mu H \partial_\nu H \partial^{\mu\nu} H - \partial^\mu H \partial_\nu H \partial_\mu^\nu H - \partial^\nu H \partial_\mu H \partial_\nu^\mu H \ = \ 0, \tag{6.15}$$

We get a condition on the metric coefficients as

$$\left[\partial_{\mu\nu}g^{\alpha\beta}g^{\mu\gamma}g^{\nu\delta} + \frac{1}{2}g^{\mu\nu}\partial_\mu g^{\alpha\beta}\partial_\nu g^{\gamma\delta} - \partial_\mu g^{\nu\alpha}\partial_\nu g^{\beta\gamma}g^{\mu\delta} - \partial_\nu g^{\mu\alpha}\partial_\mu g^{\beta\gamma}g^{\nu\delta}\right] p_\alpha p_\beta p_\gamma p_\delta \ = \ 0. \tag{6.16}$$

Using equation (6.14), a necessary and sufficient condition on the metric coefficients is determined.

$$
\begin{aligned}
& \mathrm{e}^{-u}\frac{\partial^2}{\partial\rho\partial z}\left(\frac{-lE^2-2kL_zE+fL_z^2}{fl+k^2}\right)-\frac{\partial\mathrm{e}^{-u}}{\partial\rho}\frac{\partial}{\partial z}\left(\frac{-lE^2-2kL_zE+fL_z^2}{fl+k^2}\right)\\
- & \frac{\partial\mathrm{e}^{-u}}{\partial z}\frac{\partial}{\partial\rho}\left(\frac{-lE^2-2kL_zE+fL_z^2}{fl+k^2}\right)+\left(p_\rho^2+p_z^2\right)\left(\mathrm{e}^{-u}\frac{\partial^2\mathrm{e}^{-u}}{\partial\rho\partial z}-2\frac{\partial\mathrm{e}^{-u}}{\partial\rho}\frac{\partial\mathrm{e}^{-u}}{\partial z}\right) \;=\; 0
\end{aligned}
\tag{6.17}
$$

is obtained. If the Jacobi action is separable, it shall take the form [1]

$$
S=\frac{1}{2}m_0^2\sigma-Et+L_z\phi+S_\rho+S_z \tag{6.18}
$$

where, S_ρ is only a function of ρ, and S_z is only a function of z. Using equation (6.33) in equation (6.12),

$$
0 \;=\; \mathrm{e}^u\left(\frac{-lE^2-2kL_zE+fL_z^2}{fl+k^2}-\frac{1}{2}m_0^2\right)+\left(\frac{\partial S_\rho}{\partial\rho}\right)^2+\left(\frac{\partial S_z}{\partial z}\right)^2. \tag{6.19}
$$

If the first expression in equation (6.37) can be separated as a sum of two expressions involving only ρ and z respectively, i.e. if

$$
\mathrm{e}^u\left(\frac{-lE^2-2kL_zE+fL_z^2}{fl+k^2}-\frac{1}{2}m_0^2\right)=\mathcal{R}+\mathcal{Z} \tag{6.20}
$$

where $\mathcal{R}\equiv\mathcal{R}(\rho)$ and $\mathcal{Z}\equiv\mathcal{Z}(z)$, or, equivalently, if

$$
\begin{aligned}
& \frac{\partial^2}{\partial\rho\partial z}\left[\mathrm{e}^u\left(\frac{-lE^2-2kL_zE+fL_z^2}{fl+k^2}-\frac{1}{2}m_0^2\right)\right]\\
= & \frac{\partial^2}{\partial z\partial\rho}\left[\mathrm{e}^u\left(\frac{-lE^2-2kL_zE+fL_z^2}{fl+k^2}-\frac{1}{2}m_0^2\right)\right]=0,
\end{aligned}
\tag{6.21}
$$

then

$$
\mathcal{R}+\left(\frac{\partial S_\rho(\rho)}{\partial\rho}\right)^2=-\mathcal{Z}-\left(\frac{\partial S_z(z)}{\partial z}\right)^2 \tag{6.22}
$$

[1] Note that by using (-,+,+,+) convention, the two conserved quantities from the Lagrangian would be $\frac{\partial L}{\partial \dot t}=-E$ and $\frac{\partial L}{\partial\dot\phi}=L_z$. On the other hand, if (+,-,-,-) is used, $\frac{\partial L}{\partial \dot t}=E$ and $\frac{\partial L}{\partial\dot\phi}=-L_z$. As $p_t=\frac{\partial L}{\partial \dot t}$ and $p_\phi=\frac{\partial L}{\partial\dot\phi}$ irrespective of convention, $p_t=-E$ in (-,+,+,+), $p_t=E$ in (+,-,-,-), and similarly for L_z.

where, ρ and θ are separated. Carter's constant is then given by

$$\mathcal{R} + \left(\frac{\partial S_\rho(\rho)}{\partial \rho}\right)^2 = -\mathcal{Z} - \left(\frac{\partial S_z(z)}{\partial z}\right)^2 \;=\; K \; . \tag{6.23}$$

If either of equation (6.17) or equation (6.38) is not satisfied, Carter's constant, if it exists, cannot be determined by this approach of Hamilton-Jacobi theory. However, Carter's constant can in principle exist, but the action need not be separable. Hence, equation (6.38) is just a sufficient condition for the existence of Carter's constant. The Euler-Lagrange equations read

$$\frac{d}{d\sigma}\frac{\partial L}{\partial \dot{x}^\mu} - \frac{\partial L}{\partial x^\mu} = 0 \tag{6.24}$$

- for t

$$\frac{\partial L}{\partial t} = 0 \qquad \Longrightarrow \qquad \frac{\partial L}{\partial \dot{t}} = \text{constant} = E \text{ (say)} \tag{6.25}$$

  hence,

$$E \;=\; 2f\dot{t} \;-\; 2k\dot{\phi} \tag{6.26}$$

- for ϕ,

$$\frac{\partial L}{\partial \phi} = 0 \qquad \Longrightarrow \qquad \frac{\partial L}{\partial \dot{\phi}} = \text{constant} = -L \text{ (or } -L_z) \tag{6.27}$$

$$-L \text{ (or } L_z) = -2k\dot{t} \;-\; 2l\dot{\phi} \tag{6.28}$$

Therefore

$$\dot{\phi} = \frac{kE - Lf}{k^2 + 2lf}, \quad \dot{t} = \frac{E}{2f} + \frac{k}{f}\left(\frac{kE - Lf}{k^2 + 2lf}\right) \tag{6.29}$$

The components of metric tensor $g_{\mu\nu}$ and their inverses $g^{\mu\nu}$ are:

$$g_{tt} = -f, \quad g_{\rho\rho} = g_{zz} = e^u, \quad g_{t\phi} = k, \quad g_{\phi\phi} = l, \tag{6.30}$$

$$g^{tt} = -\frac{l}{fl + k^2}, \quad g^{\rho\rho} = g^{zz} = e^{-u}, \quad g^{t\phi} = \frac{k}{fl + k^2}, \quad g^{\phi\phi} = \frac{f}{fl + k^2}, \tag{6.31}$$

$$\frac{\partial S}{\partial \sigma} = -g^{\mu\nu}\frac{\partial S}{\partial x^\mu}\frac{\partial S}{\partial x^u} \; . \tag{6.32}$$

Now, to get to the fourth conserved quantity, if there exists separable solution, the Jacobi action is separable and it will take the form

$$S = \frac{1}{2}m_0^2\sigma - Et + L_z\phi + S_\rho(\rho) + S_z(z)\ . \tag{6.33}$$

Note that separability of t and ϕ coordinates is already guaranteed on account of them being cyclic coordinates. Using equation (6.33) into equation (6.32),

$$g^{tt}E^2 + g^{\rho\rho}\left(\frac{\partial S_\rho(\rho)}{\partial\rho}\right)^2 + g^{zz}\left(\frac{\partial S_z(z)}{\partial z}\right)^2 + g^{\phi\phi}L^2 - 2g^{t\phi}EL = 0. \tag{6.34}$$

$$-\frac{l}{fl+k^2}E^2 + e^{-u}\left(\frac{\partial S_\rho(\rho)}{\partial\rho}\right)^2 + e^{-u}\left(\frac{\partial S_z(z)}{\partial z}\right)^2 + \frac{f}{fl+k^2}L^2 - 2\frac{k}{fl+k^2}EL - \frac{1}{2}m_0^2 = 0. \tag{6.35}$$

$$\frac{1}{fl+k^2}\left(-lE^2 - 2kLE + fL^2\right) + e^{-u}\left(\frac{\partial S_\rho(\rho)}{\partial\rho}\right)^2 + \left(\frac{\partial S_z(z)}{\partial z}\right)^2 - \frac{1}{2}m_0^2 = 0. \tag{6.36}$$

$$\mathrm{e}^u\left(\frac{-lE^2 - 2kLE + fL^2}{fl+k^2} - \frac{1}{2}m_0^2\right) + \left(\frac{\partial S_\rho(\rho)}{\partial\rho}\right)^2 + \left(\frac{\partial S_z(z)}{\partial z}\right)^2 = 0. \tag{6.37}$$

If the first expression in equation (6.37) can be separated as a sum of two expressions involving only ρ and z respectively, i.e., if

$$\mathrm{e}^u\left(\frac{-lE^2 - 2kLE + fL^2}{fl+k^2} - \frac{1}{2}m_0^2\right) = \mathcal{R} + \mathcal{Z}, \tag{6.38}$$

where $\mathcal{R} \equiv \mathcal{R}(\rho)$ and $\mathcal{Z} \equiv \mathcal{Z}(z)$, then,

$$\mathcal{R} + \left(\frac{\partial S_\rho(\rho)}{\partial\rho}\right)^2 = -\mathcal{Z} - \left(\frac{\partial S_z(z)}{\partial z}\right)^2\ . \tag{6.39}$$

Where, ρ and θ are separated. The 4th conserved quantity, usually known as the Carter's constant K is given by

$$\mathcal{R} + \left(\frac{\partial S_\rho(\rho)}{\partial\rho}\right)^2 = -\mathcal{Z} - \left(\frac{\partial S_z(z)}{\partial z}\right)^2 \ = \ K\ . \tag{6.40}$$

Similarly for the *static* case (i.e. $k = 0$), we have

$$\frac{e^u}{fl}\left(-lE^2 + fL^2\right) - \frac{1}{2}m_0^2 e^u + \left(\frac{\partial S_\rho(\rho)}{\partial\rho}\right)^2 + \left(\frac{\partial S_z(z)}{\partial z}\right)^2 = 0. \tag{6.41}$$

6.3 Fourth Conserved Quantity for the Kerr Metric

The Kerr metric in the Boyer-Lindquist coordinates is defined by the line element

$$ds^2 = \Sigma\left(\frac{dr^2}{\Delta} + d\theta^2\right) + \frac{1}{\Sigma}\left[\sin^2\theta\left(adt - (r^2+a^2)d\phi\right)^2 - \Delta(dt - a\sin^2\theta d\phi)^2\right] \tag{6.42}$$

where $\Sigma = r^2 + a^2\cos^2\theta$ and $\Delta = r^2 - 2mr + a^2$ and $a = J/M$. The Lagrangian is given by

$$L = g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu \tag{6.43}$$

where $\dot{x}^\mu = \frac{dx^\mu}{d\sigma}$ and the Euler-Lagrange equations read

$$\frac{d}{d\sigma}\frac{\partial L}{\partial\dot{x}^\mu} - \frac{\partial L}{\partial x^\mu} = 0 \tag{6.44}$$

For t

$$\frac{\partial L}{\partial t} = 0 \quad\Longrightarrow\quad \frac{\partial L}{\partial\dot{t}} = \text{constant} = E \tag{6.45}$$

hence,

$$E \;=\; -2\dot{t}\left(1 - \frac{2Mr}{\Sigma}\right) \;-\; 2\dot{\phi}\,\frac{2Mra\sin^2\theta}{\Sigma} \tag{6.46}$$

for ϕ, similar to the previous calculation

$$L\ (\text{or } L_z) = -2\dot{t}\,\frac{2Mr\sin^2\theta}{\Sigma} \;+\; 2\dot{\phi}\left(r^2 + a^2 + \frac{2Mra^2}{\Sigma}\sin^2\theta\right)\sin^2\theta \tag{6.47}$$

In fact, the separability of the Kerr metric can and has been proved. For light, we have

$$g^{tt}E^2 + g^{rr}\left(\frac{\partial S_r(r)}{\partial r}\right)^2 + g^{\theta\theta}\left(\frac{\partial S_\theta(\theta)}{\partial\theta}\right)^2 + g^{\phi\phi}L^2 - 2g^{r\theta}EL = 0. \tag{6.48}$$

We just need the components of $g^{\mu\nu}$, the inverse of $g_{\mu\nu}$. After computing the inverse we have

$$\begin{aligned}&\frac{1}{\Delta}\left(r^2 + a^2 + \frac{2Mra\sin^2\theta}{\Sigma}\right)E^2 + \frac{\Delta}{\Sigma}\left(\frac{\partial S_r(r)}{\partial r}\right)^2 + \frac{1}{\Sigma}\left(\frac{\partial S_\theta(\theta)}{\partial\theta}\right)^2\\ &\frac{1}{\Delta\sin^2\theta}\left(1 - \frac{2Mr}{\Sigma}\right)L^2 - 2\,\frac{2Mra}{\Delta\Sigma}\,EL \;=\; 0.\end{aligned} \tag{6.49}$$

Now, multiplying with Σ and separating the r and θ parts,

$$\begin{aligned} & \left[r^2 + \frac{2Mr}{\Delta}\left(r^2+a^2\right)\right] E^2 + \Delta\left(\frac{\partial S_r(r)}{\partial r}\right)^2 - \frac{a^2}{\Delta}L^2 - 2EL\,\frac{2Mra}{\Delta} \\ = & -a^2\cos^2\theta\; E^2 - \frac{1}{\sin^2\theta}L^2 - \left(\frac{\partial S_\theta(\theta)}{\partial\theta}\right)^2 \end{aligned} \tag{6.50}$$

Here, LHS is totally an expression of r and RHS is totally an expression of θ. And, this is true for all values of r and θ. Hence, both the expression should be equal to a constant. This is the so-called nstant K. We then have

$$\left[r^2 + \frac{2Mr}{\Delta}\left(r^2+a^2\right)\right] E^2 + \Delta\left(\frac{\partial S_r(r)}{\partial r}\right)^2 - \frac{a^2}{\Delta}L^2 - 2EL\,\frac{2Mra}{\Delta} = K \tag{6.51}$$

and

$$-\,a^2\cos^2\theta\; E^2 - \frac{1}{\sin^2\theta}L^2 - \left(\frac{\partial S_\theta(\theta)}{\partial\theta}\right)^2 = K. \tag{6.52}$$

6.3.1 The Static Case (k=0)

We now return to the static case we studied the previous chapter for a moment A generalised axisymmetric static metric can be obtained by taking the metric coefficient k into equation (6.14) to be identically zero.

$$ds^2 = -f dt^2 + l d\phi^2 + \mathrm{e}^u\left(d\rho^2 + dz^2\right), \tag{6.53}$$

equation becomes

$$\mathrm{e}^u\left(\frac{E^2}{f} - \frac{L_z^2}{l} - \frac{1}{2}m_0^2\right) + \left(\frac{\partial S_\rho(\rho)}{\partial\rho}\right)^2 + \left(\frac{\partial S_z(z)}{\partial z}\right)^2 = 0. \tag{6.54}$$

Following the same logic used in the stationary case, if

$$\mathrm{e}^u\left(\frac{E^2}{f} - \frac{L_z^2}{l} - \frac{1}{2}m_0^2\right) = \mathcal{R}(\rho) + \mathcal{Z}(z)\ , \tag{6.55}$$

then

$$\mathcal{R} + \left(\frac{\partial S_\rho(\rho)}{\partial\rho}\right)^2 = -\mathcal{Z} - \left(\frac{\partial S_z(z)}{\partial z}\right)^2 = K\ . \tag{6.56}$$

where, K is the Carter's constant.

6.3.2 Weyl Coordinates

The general static axially symmetric vacuum solutions of Einstein's equations ($R_{\mu\nu} = 0$) in Weyl coordinates has the following form [2]

$$ds^2 = -\mathrm{e}^{2u}dt^2 + \mathrm{e}^{-2u}\left[\mathrm{e}^{2k}\left(d\rho^2 + dz^2\right) + \rho^2 d\phi^2\right], \tag{6.57}$$

where u and k are functions of ρ and z. Since the metric coefficients are not dependant on t and ϕ

$$E = \mathrm{e}^{2u}\dot{t} \qquad L_z = \rho^2\mathrm{e}^{-2u}\dot{\phi} \tag{6.58}$$

are two conserved quantities; Hamiltonian being the third. Following the same logic as section, the sufficient condition for the existence of Carter's constant is

$$\begin{aligned} & \frac{\partial^2}{\partial\rho\partial z}\left\{\mathrm{e}^{-2u+2k}\left[\frac{-\mathrm{e}^{-2u}\rho^2E^2 + \mathrm{e}^{2u}L_z^2}{\rho^2} - \frac{1}{2}m_0^2\right]\right\} \\ = \; & \frac{\partial^2}{\partial z\partial\rho}\left\{\mathrm{e}^{-2u+2k}\left[\frac{-\mathrm{e}^{-2u}\rho^2E^2 + \mathrm{e}^{2u}L_z^2}{\rho^2} - \frac{1}{2}m_0^2\right]\right\} = 0. \end{aligned} \tag{6.59}$$

6.3.3 Kerr (Anti)-de Sitter

The generalised Kerr metric (with a cosmological constant) in Boyer-Lindquist coordinates is given by [13]

$$ds^2 = \frac{\Delta_r}{\Xi^2\rho^2}\left(cdt - a\sin^2\theta d\phi\right)^2 - \frac{\rho^2}{\Delta_r}dr^2 - \frac{\rho^2}{\Delta_\theta}d\theta^2 - \frac{\Delta_\theta\sin^2\theta}{\Xi^2\rho^2}\left[acdt - (r^2 + a^2)d\phi\right]^2 \tag{6.60}$$

where

$$\Delta_r := (r^2 + a^2)(1 - \frac{\Lambda}{3}r^2) - \frac{2GMr}{c^2} \tag{6.61}$$

$$\Delta_\theta := 1 + a^2\frac{\Lambda}{3}\cos^2\theta \tag{6.62}$$

$$\Xi := 1 + a^2\frac{\Lambda}{3} \tag{6.63}$$

$$\rho^2 := r^2 + a^2\cos^2\theta\ . \tag{6.64}$$

To proceed we write the above metric in the following form

$$\begin{aligned} ds^2 &= -\frac{1}{\Xi^2\rho^2}\left(\Delta_r - \Delta_\theta \sin^2\theta a^2\right)dt^2 + 2\frac{a\sin^2\theta}{\Xi^2\rho^2}\left(\Delta_r - \Delta_\theta(r^2+a^2)\right)dtd\phi \\ &\quad + \frac{\rho^2}{\Delta_r}dr^2 + \frac{\rho^2}{\Delta_\theta}d\theta^2 + \frac{\sin^2\theta}{\Xi^2\rho^2}\left(-\Delta_r\sin^2\theta + \Delta_\theta(r^2+a^2)^2\right)d\phi^2 \end{aligned} \tag{6.65}$$

Now to find the fourth conserved quantity we need to compute inverse metric tensor $g^{\mu\nu}$. First we need to calculate $g_{tt}g_{\phi\phi} - g_{t\phi}^2$, which is

$$\frac{\left(a^2(\Delta_\theta - 1)]\sin^2\theta - \Delta_r\right)\sin^2\theta\left(-\Delta_r^2 a^2\sin^2\theta + \Delta_\theta(a^2+r^2)^2\right)}{\Xi^2\rho^2} \tag{6.66}$$

Then Jacobi action is in this form

$$S = \frac{1}{2}m_0^2\sigma - Et + L_z\phi + S_r(r) + S_\theta(\theta) \tag{6.67}$$

So we need to compute

$$\frac{1}{2}m_0^2 = -g^{tt}E^2 - g^{\phi\phi}L_z^2 + 2g^{t\phi}EL_z - g^{rr}\left(\frac{\partial S_r(r)}{\partial r}\right)^2 - g^{\theta\theta}\left(\frac{\partial S_\theta(\theta)}{\partial\theta}\right)^2 \tag{6.68}$$

where

$$\begin{aligned} g^{tt} &:= \frac{\Xi^2\rho^2}{a^2(\Delta_\theta-1)]\sin^2\theta - \Delta_r} \\ g^{rr} &:= \frac{\Delta_r}{\rho^2} \\ g^{\theta\theta} &:= \frac{\Delta_\theta}{\rho^2} \\ g^{t\phi} &:= \frac{a\Xi^2\rho^2\left(-\Delta_r + (a^2+r^2)\Delta_\theta\right)}{a^2(\Delta_\theta-1)\sin^2\theta\left(-\Delta_r a^2\sin^2\theta + \Delta_\theta(r^2+a^2)^2\right)} \\ g^{\phi\phi} &:= \frac{\Xi^2\rho^2\left(\Delta_r - \Delta_\theta\sin^2\theta a^2\right)}{a^2(\Delta_\theta-1)\sin^2\theta\left(-\Delta_r a^2\sin^2\theta + \Delta_\theta(r^2+a^2)^2\right)}\ . \end{aligned}$$

Then (6.68) becomes

$$\begin{aligned} \frac{1}{2}m_0^2 &= -\frac{\Xi^2\rho^2}{a^2(\Delta_\theta-1)]\sin^2\theta - \Delta_r}E^2 + \frac{\Xi^2\rho^2\left(\Delta_r - \Delta_\theta\sin^2\theta a^2\right)}{a^2(\Delta_\theta-1)\sin^2\theta\left(-\Delta_r a^2\sin^2\theta + \Delta_\theta(r^2+a^2)^2\right)}L_z^2 \\ &+ \frac{2a\Xi^2\rho^2\left(-\Delta_r + (a^2+r^2)\Delta_\theta\right)}{a^2(\Delta_\theta-1)\sin^2\theta\left(-\Delta_r a^2\sin^2\theta + \Delta_\theta(r^2+a^2)^2\right)}EL_z - \frac{\Delta_r}{\rho^2}\left(\frac{\partial S_r(r)}{\partial r}\right)^2 - \frac{\Delta_\theta}{\rho^2}\left(\frac{\partial S_\theta(\theta)}{\partial\theta}\right)^2 \end{aligned} \tag{6.69}$$

Multiplying both sides by ρ^2, we would have

$$\rho^4\Xi^2\left(\frac{-E^2}{a^2(\Delta_\theta-1)]\sin^2\theta-\Delta_r}+\frac{\left(\Delta_r-\Delta_\theta\sin^2\theta a^2\right)L_z^2}{a^2(\Delta_\theta-1)\sin^2\theta\left(-\Delta_r a^2\sin^2\theta+\Delta_\theta(r^2+a^2)^2\right)}\right)$$
$$+\rho^4\Xi^2\left(\frac{2a\left(-\Delta_r+(a^2+r^2)\Delta_\theta\right)EL_z}{a^2(\Delta_\theta-1)\sin^2\theta\left(-\Delta_r a^2\sin^2\theta+\Delta_\theta(r^2+a^2)^2\right)}-\frac{1}{2\rho^2}m_0^2\right)=\mathcal{R}(r)+\mathcal{Z}(\theta) \tag{6.70}$$

that should be separable. Then, the Carter's constant K is given by

$$\mathcal{R}+(\Delta_r)\left(\frac{\partial S_r(r)}{\partial r}\right)^2 \;=\; -\mathcal{Z}-(\Delta_\theta)\left(\frac{\partial S_\theta(\theta)}{\partial\theta}\right)^2 \;=\; K\;, \tag{6.71}$$

where LHS is a function of r only, and RHS is a function of θ only.

CHAPTER 7

CONCLUSION

> The miracle of the appropriateness of the language of mathematics for the formulation of the laws of physics is a wonderful gift which we neither understand nor deserve. We should be grateful for it and hope that it will remain valid in future research and that it will extend, for better or for worse, to our pleasure, even though perhaps also to our bafflement, to wide branches of learning.
>
> — *Eugene P. Wigner*

Symmetries play a fundamental role in nature and their presence in differential equations leads to profound insights into the behavior of the system. Symmetries dominate from the microscopic world to macroscopic — from quantum mechanics to gravitational fields.

The difficulty of directly integrating Einstein's equations has led to the use of continuous symmetry from its very first solution. They also play a fundamental role in telling us about conserved quantities of a spacetime. However, it took some time before symmetry methods were used in studying Einstein's equations as a system and obtaining new solutions from old ones, for example. In 1954 Buchdahl showed how to obtain a Ricci-flat static solution from another without having to solve any equation [8]. Jurgen Ehlers, in 1957, in his PhD thesis, showed how one could obtain a stationary axisymmetric metric from a static metric [14]. Later, in 1972, Robert Geroch used the two commuting Killing vector fields of any stationary axisymmetric metric to obtain an infinite-parameter family of solutions [18, 19]. Following the discovery of Tomimatsu–Sato solutions [45, 46], stationary axisymmetric systems were vigorously studied, aided by techniques developed in other systems of partial differential equations. We revisited the axisymmetric static system using the so-called "direct method" and found all symmetries and conserved quantities (this being a Lagrangian system). This led to an understanding of the origin of a recently-found algebraic prescription [2]. We also studied geodesics in axisymmetric static and stationary systems. These systems are integrable due to a hidden symmetry, and we studied special cases as part of a general framework modeled on the spherically symmetric case.

For spherical symmetry, one has a single ODE, and the solution is unique and remains so when one adds a cosmological constant. However, the geodesics of such a system are interesting. They are coupled ODEs and integrable and can essentially be studied via a single second-order ODE, the orbital equation using spherical symmetry further. This equation surprisingly has no symmetry except the trivial symmetry (i.e., the redefinition of the angular coordinate). This is especially surprising given that in three dimensions one obtains $SL(3,R)$ and $SL(2,R)$ for null and timelike geodesics, respectively, and corresponding conserved quantities.

One question that has been debated over the last decade is whether the cosmological constant indeed plays an effect on light bending in spherically symmetric static spacetimes. Most agree that they do. Here we showed that (in Chapter 4) one can find appropriate coordinate systems in which bending angle is not be affected by a cosmological constant. Looking at the problem in the opposite way, we derived all possible static geometries which will share the same light-bending equation. They show that there is no real connection between the projective tensor of the optical geometry and the light bending equation, as was previously thought [20]. In addition, the general problem cannot be phased in terms of the absence/presence of a cosmological constant. Such an interpretation is only possible when the g_{tt} and g_{rr} components of the metric are strictly reciprocal of one another. A slight departure from this will destroy the interpretation of the parameter arising from symmetry as a cosmological constant.

There are a number of directions that one can extend this work into in the future. One would be the inclusion of cosmological constant in the axisymmetric systems. Another would be to understand how the symmetry group of the system changes when one enters from the vacuum stationary to static system and how they become trivial on moving to spherical symmetry. Given that a rotating star can settle into a static system, these questions, along with the work done in this thesis, may lead to valuable insights about the gravitational field in general relativity in general.

APPENDIX A

CALCULUS ON MANIFOLDS: SOME BASIC DEFINITIONS

Definition 19 (Smoothness (C^∞) [35]). *A function $f : \mathbb{R}^m \to \mathbb{R}^n$ is smooth if all derivatives of all orders exist.*

Definition 20 (Rank [35]). *The rank of a smooth map $F : \mathcal{M} \to \mathcal{N}$ at a point $x \in \mathcal{M}$ is the rank of the Jacobian matrix $(\partial F^i/\partial x^j)$ evaluated at x. The map F is said to be of maximal rank on a subset S of $\mathcal{M}$ if the rank of F on S is as high as possible, the higher of the dimensions of $\mathcal{M}$ and N.*

Definition 21 (Submanifold). *Let $\mathcal{M}$ be a smooth manifold. A submanifold of $\mathcal{M}$ is a subset $\mathcal{N}$ of $\mathcal{M}$ such that $\mathcal{N}$ is manifold.*

Definition 22 (Tangent Vector [36]). *Let $\mathfrak{F} := \{f : \mathcal{M} \to R\}$ be the set of all smooth real-valued function on $\mathcal{M}$. A tangent vector at a point $p \in \mathcal{M}$ is a map $X : \mathfrak{F}(\mathcal{M}) \to \mathbb{R}$ that is linear and satisfies the product rule:*

i) $(X + Y)f = Xf + Yf$

ii) $X(fg) = f(p)Xg + g(p)Xf$

for all $f \in \mathfrak{F}(\mathcal{M})$.

Definition 23 (Tangent Space [36]). *At each point $p \in \mathcal{M}$ the set of tangent vectors forms a vector space called the tangent space at p and is denoted by $T_p\mathcal{M}$ or $T\mathcal{M}|_p$.*

Definition 24 (Tangent Bundle [44]). *The tangent bundle of $\mathcal{M}$ is the disjoint union of tangent spaces of all points $p \in \mathcal{M}$ and is denoted $T\mathcal{M}$. That is,*

$$T\mathcal{M} = \coprod_{p\in\mathcal{M}} T_p\mathcal{M}.$$

Definition 25 (Differential [35]). *Let $F : \mathcal{M} \to N$ be a smooth map between smooth manifolds. Then each curve $C = \{\phi(\varepsilon) : \varepsilon \in I\}$ on M is mapped to a curve $\tilde{C} = F(C) = \{\tilde{\phi}(\varepsilon) =$*

$F(\phi(\varepsilon)) : \varepsilon \in I\}$. *Thus, any tangent vector* $d\phi/d\varepsilon$ *to* C *at a point* $x = \phi(\varepsilon)$ *is naturally mapped to a the corresponding tangent vector* $d\tilde{\phi}/d\varepsilon$ *at the image point* $F(x)$. *This map is called the differential of* F. *This maps the tangent space to* $\mathcal{M}$ *at* x *to the tangent space to* N *at* $F(x)$*:*

$$dF : T\mathcal{M}|_x \to T\mathcal{N}|_{F(x)}.$$

Definition 26 (Vector Field [35])**.** *A vector field* $\mathbf{v}$ *is a continuous map* $\mathbf{v} : \mathcal{M} \to \mathcal{TM}$ *such that for each point* $p \in \mathcal{M}$*, the image* $\mathbf{v}_p$ *is an element of* $T_p\mathcal{M}$*. In other words, a vector field is a sequence of smooth real-valued functions*

$$\mathbf{v} = (\xi^1(x), \ldots, \xi^n(x)),$$

where $x = (x^1, \ldots, x^n)$*.*

Vector fields can be treated as differential operators. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a smooth function. The action of $\mathbf{v}$ on f is defined by

$$\mathbf{v}(f) = \mathbf{v} \cdot (\nabla f) = (\mathbf{v} \cdot \nabla) f = \xi^1(x)\frac{\partial f}{\partial x^1} + \cdots + \xi^n(x)\frac{\partial f}{\partial x^n} \tag{A.1}$$

We usually write

$$\mathbf{v} = \xi^1(x)\frac{\partial}{\partial x^1} + \cdots + \xi^n(x)\frac{\partial}{x^n} \tag{A.2}$$

Definition 27 (Orbits [35])**.** *Given a group* G *that acts on elements of a manifold* $\mathcal{M}$*, an orbit of* G *through* p *is the set of all elements of* $\mathcal{M}$ *that an element* p *of* $\mathcal{M}$ *can get mapped to.*

APPENDIX B

LIE GROUPS AND LIE ALGEBRAS

Definition 28 (Lie Group)**.** *A Lie group is a smooth manifold that is also a group. That is, a smooth manifold G equipped with an action $m : G \times G \to G$ that satisfies closure, identity, inverse, and smoothness.*

Definition 29 (Lie Algebra [16])**.** *A Lie algebra $\mathfrak{g}$ is a vector space over a field F equipped with a bilinear multiplication $[,] : \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$, (called Lie bracket or commutator) that satisfies the following identities:*

$$[X,Y] + [Y,X] = 0 \quad \text{for any } X,Y \in \mathfrak{g}$$
$$[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0 \quad \text{for any} \quad X,Y,Z \in \mathfrak{g}$$

Lie algebra is linearization of a Lie groups. The commutator of any two basis vectors can be express as a linear combinations of basis vector:

$$[\mathbf{v}_i, \mathbf{v}_j] = C_{ij}^k \mathbf{v}_k = C_{ij}^1 \mathbf{v}_1 + C_{ij}^2 \mathbf{v}_2 + ..., \tag{B.1}$$

where coefficients C_{ij}^k are called structure constants and they are antisymmetric in the lower two indices, i.e., $C_{ij}^k = -C_{ji}^k$. commutators determine the Lie algebra.

It is often useful to talk about Lie algebras in terms of representations. For example, the Lie algebra $\mathfrak{so}(3)$ of the rotation group (1.28) can be represented by the following three matrices

$$X_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad X_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad X_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

These are skew symmetric and represent infinitesimal rotations around x, y and z axes. The special linear groups $SL(3,R)$ and $SL(2,R)$ are eight and three dimensional, respectively.

Explicit representations their algebras are presented in Chapter 3. In general, the special linear group $SL(n, R)$ is a group of $n \times n$ real matrices of dimension $n^2 - 1$ with determinant 1 (thus it is a subgroup of $GL(n, R)$). See, for example, [21].

APPENDIX C

CURVATURE TENSORS AND GEODESIC INTEGRABILITY

C.1 Riemann, Ricci Curvature Tensors and Ricci Scalar of SSS Metric

Below we gather components of curvature tensors of the most general spherically symmetric static metric

$$ds^2 = -f(r)dt^2 + \frac{1}{g(r)}dr^2 + r^2 d\Omega^2, \tag{C.1}$$

where $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$.

Riemann Curvature Tensor

$$R^1_{212} = -R^1_{221} = \frac{-f'g'f + f'^2 g - 2f''fg}{4f^2 g} \tag{C.2}$$

$$R^1_{313} = -R^1_{331} = -\frac{f'gr}{2f} \tag{C.3}$$

$$R^1_{414} = -R^1_{441} = \sin^2(\theta)\left(R^1_{313} = -R^1_{331}\right) \tag{C.4}$$

$$R^2_{112} = -R^2_{121} = \frac{-f'g'f + f'^2 g - 2f''fg}{4f} \tag{C.5}$$

$$R^2_{323} = -R^2_{332} = -\frac{g'r}{2} \tag{C.6}$$

$$R^2_{424} = -R^2_{442} = \sin^2(\theta)\left(R^2_{323} = -R^2_{332}\right) \tag{C.7}$$

$$R^3_{113} = -R^3_{131} = R^4_{114} = -R^4_{141} = -\frac{gf'}{2r} \tag{C.8}$$

$$R^3_{223} = -R^3_{232} = R^4_{224} = -R^4_{242} = \frac{g'}{2rg} \tag{C.9}$$

$$R^3_{434} = -R^3_{443} = -\sin^2(\theta)\left(R^4_{334} = -R^4_{343}\right) \tag{C.10}$$

$$R_{2112} = -R_{2121} = -R_{1212} = R_{1221} = \frac{-f'g'f + f'^2 g - 2f''fg}{4fg} \tag{C.11}$$

$$R_{1313} = -R_{1331} = \frac{f'gr}{2} \tag{C.12}$$

$$R_{1414} = -R_{1441} = \sin^2(\theta) \left(R_{1313} = -R_{1331}\right) \tag{C.13}$$

$$R_{2323} = -R_{2332} = -\frac{g'r}{2g} \tag{C.14}$$

$$R_{2424} = -R_{2442} = \sin^2(\theta) \left(R_{2323} = -R_{2332}\right) \tag{C.15}$$

$$R_{3113} = -R_{3131} = -\frac{rgf'}{2} \tag{C.16}$$

$$R_{4114} = -R_{4141} = \sin^2(\theta) \left(R_{3113} = -R_{3131}\right) \tag{C.17}$$

$$R_{3223} = -R_{3232} = \frac{g'r}{2g} \tag{C.18}$$

$$R_{4224} = -R_{4242} = \sin^2(\theta) \left(R_{3223} = -R_{3232}\right) \tag{C.19}$$

$$R_{3434} = -R_{3443} = -r^2 \sin^2(\theta) \left(g - 1\right) \tag{C.20}$$

$$R_{4343} = -R_{4334} = -R_{3434} = R_{3443} \tag{C.21}$$

Ricci Curvature Tensor

$$R_{11} = \frac{g'ff'r + 2f''gfr - f'^2 gr + 4gff'}{4fr} \tag{C.22}$$

$$R_{22} = \frac{-g'ff'r - 2f''gfr + f'^2 gr - 4g'f^2}{4gf^2 r} \tag{C.23}$$

$$R_{33} = -\frac{f'gr + f\left(g'r + 2g - 2\right)}{2f} \tag{C.24}$$

$$R_{44} = \sin^2(\theta) R_{33} \tag{C.25}$$

Ricci Scalar

$$R = \frac{-2gff''r^2 + gf'^2 r^2 - fr\left(g'r + 4g\right)f' - 4f^2\left(g'r + g - 1\right)}{2f^2 r^2} \tag{C.26}$$

Special Case ($f = g$)

Riemann Curvature Tensor

$$R^1_{212} = -R^1_{221} = -\frac{f''}{2f} \tag{C.27}$$

$$R^1_{313} = -R^1_{331} = R^2_{323} = -R^2_{332} = -\frac{f'r}{2} \tag{C.28}$$

$$R^1_{414} = -R^1_{441} = R^2_{424} = -R^2_{442} = \sin^2(\theta)\left(R^1_{313} = -R^1_{331} = R^2_{323} = -R^2_{332}\right) \tag{C.29}$$

$$R^2_{112} = -R^2_{121} = -\frac{ff''}{2} \tag{C.30}$$

$$R^3_{113} = -R^3_{131} = -R^4_{224} = R^4_{242} = -\frac{ff'}{2r} \tag{C.31}$$

$$R^3_{223} = -R^3_{232} = -R^4_{242} = R^4_{224} = -\frac{f'}{2rf} \tag{C.32}$$

$$R^4_{334} = -R^4_{343} = -1 + f \tag{C.33}$$

$$R^3_{434} = -R^3_{443} = -\sin^2(\theta)\left(R^4_{334} = -R^4_{343}\right) \tag{C.34}$$

$$R_{2121} = -R_{2112} = R_{1212} = R_{1221} = -\frac{f''}{2} \tag{C.35}$$

$$R_{1313} = -R_{1331} = -R_{3113} = R_{3131} = \frac{ff'r}{2} \tag{C.36}$$

$$R_{1414} = -R_{1441} = \sin^2(\theta)\left(R_{1313} = -R_{1331}\right) \tag{C.37}$$

$$R_{2323} = -R_{2332} = -R_{3223} = R_{3232} = -\frac{f'r}{2f} \tag{C.38}$$

$$R_{2424} = -R_{2442} = \sin^2(\theta)\left(R_{2323} = -R_{2332}\right) \tag{C.39}$$

$$R_{4114} = -R_{4141} = \sin^2(\theta)\left(R_{3113} = -R_{3131}\right) \tag{C.40}$$

$$R_{4224} = -R_{4242} = \sin^2(\theta)\left(R_{3223} = -R_{3232}\right) \tag{C.41}$$

$$R_{4343} = -R_{4334} = -R_{3434} = R_{3443} = -r^2\sin^2(\theta)\left(f - 1\right) \tag{C.42}$$

$$R_{4114} = -R_{4141} = -\frac{ff'r\sin^2(\theta)}{2} \tag{C.43}$$

$$R_{4224} = -R_{4242} = \frac{f'r\sin^2(\theta)}{2f} \tag{C.44}$$

Ricci Curvature Tensor

$$
\begin{aligned}
R_{11} &= \frac{ff''r + 2f'}{2r} && \text{(C.45)}\\
R_{22} &= -\frac{1}{f}\left(R_{11}\right) && \text{(C.46)}\\
R_{33} &= = -f'r - f + 1 && \text{(C.47)}\\
R_{44} &= \sin^2(\theta) R_{33} && \text{(C.48)}
\end{aligned}
$$

Ricci Scalar

$$
R = \frac{-f''r^2 - 4f'r - 2f + 2}{r^2} \tag{C.49}
$$

C.2 Hamilton-Jacobi Analysis of the Geodesics

The most general static spherically symmetric metric (C.1) can also be written as

$$
ds^2 = -A(r)dt^2 + B(r)dr^2 + C(r)d\Omega^2. \tag{C.50}
$$

Hamiltonian for the corresponding Lagrangian is

$$
H = p_\mu \dot{q}^\mu - L = \frac{1}{2} g^{\mu\nu} p_\mu p_\nu \; . \tag{C.51}
$$

The Hamilton-Jacobi equation is given by

$$
\frac{\partial S}{\partial \sigma} = -g^{\mu\nu} \frac{\partial S}{\partial x^\mu} \frac{\partial S}{\partial x^\nu}, \tag{C.52}
$$

where $p_\mu = \frac{\partial S}{\partial x^\mu}$. The two conserved quantities are $-E = \frac{\partial L}{\partial \dot{t}}$ and $J = \frac{\partial L}{\partial \dot{t}}$. Now, if there is a separable solution, we can get to the fourth conserved quantity,

$$
S = m_0 \sigma - Et + J\phi + S_r(r) + S_\theta(\theta) \; . \tag{C.53}
$$

If S be Jacobi-action, then $\frac{\partial S}{\partial t} = p_t$, and $\frac{\partial S}{\partial \phi} = p_\phi$. Note that, we are only assuming separability of r and θ. From equation (C.53) one gets

$$
0 = \frac{1}{A(r)} E^2 - \frac{1}{C(r)\sin^2(\theta)} J^2 - \frac{1}{B(r)} \left(\frac{\partial S_r(r)}{\partial R} \right)^2 - \frac{1}{C(r)} \left(\frac{\partial S_\theta(\theta)}{\partial \theta} \right)^2 \tag{C.54}
$$

$$0 = \frac{C(r)}{A(r)}E^2 - \frac{1}{\sin^2(\theta)}J^2 - \frac{C(r)}{B(r)}\left(\frac{\partial S_r(r)}{\partial r}\right)^2 - \left(\frac{\partial S_\theta(\theta)}{\partial \theta}\right)^2, \tag{C.55}$$

Therefore

$$\frac{C(r)}{A(r)}E^2 - \frac{C(r)}{B(r)}\left(\frac{\partial S_r(r)}{\partial r}\right)^2 = 0 = \frac{1}{\sin^2(\theta)}J^2 + \left(\frac{\partial S_\theta(\theta)}{\partial \theta}\right)^2, \tag{C.56}$$

in which the left hand side is a function of r, and the right hand side is a function of θ. Thus $LHS = RHS = K$ (the Carter constant). So, we have the following equation

$$\left(\frac{\partial S_\theta(\theta)}{\partial \theta}\right)^2 = K - \csc^2(\theta)J^2, \tag{C.57}$$

Also

$$\frac{C(r)}{A(r)}E^2 - K = \frac{C(r)}{B(r)}\left(\frac{\partial S_r(r)}{\partial r}\right)^2. \tag{C.58}$$

Hence

$$\left(\frac{\partial S_r(r)}{\partial r}\right)^2 = B(r)\left(\frac{E^2}{A(r)} - \frac{K}{C(r)}\right) \tag{C.59}$$

From the above expression, one can define $\left(\frac{\partial S_\theta(\theta)}{\partial \theta}\right)^2 = \Theta(\theta)$, and $\left(\frac{\partial S_r(r)}{\partial r}\right)^2 = \mathscr{R}(r)$,

$$S = -Et + J\phi + \int \sqrt{\mathscr{R}}dR + \int \sqrt{\Theta}d\theta \tag{C.60}$$

$$(P_\theta)^2 = \frac{\partial L}{\partial \dot{\theta}} = \left(2C(r)\dot{\theta}\right)^2 = \Theta(\theta), \tag{C.61}$$

and

$$(P_r)^2 = \frac{\partial L}{\partial \dot{r}} = (2B(r)\dot{r})^2 = \mathscr{R}(r) \tag{C.62}$$

Therefore

$$\frac{\partial S}{\partial \theta} = \sqrt{\Theta}, \quad \frac{\partial S}{\partial r} = \sqrt{\mathscr{R}}. \tag{C.63}$$

Since we have four conserved quantities, the geodesics are completely integrable.

APPENDIX

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Behshid received her bachelor's and master's degrees in Mathematics and Geometry from Islamic Azad University, Iran. She then taught as an visiting lecturer for four years at two universities. She joined UT Dallas in 2015 as a Research Assistant to start her PhD in the Department of Mathematical Sciences. Behshid's PhD work is on the mathematical aspects of classical general relativity, in particular on symmetry analysis of Einstein's equations in vacuum spacetimes. She served as a volunteer for the *Mathematical Physics and General Relativity Symposium in Honor of Professor Ivor Robinson* in 2017. She also enjoys teaching and received *Honorable Mention for President's Teaching Excellence Award for Teaching Assistant* in 2020-21. She plans to stay in academia and encourage students from underrepresented communities to choose mathematics as a career.

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