PROOF OF THE STRONG AJ CONJECTURE FOR THE FIGURE 8 KNOT

by

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The AJ conjecture, formulated by Garoufalidis [7], relates the A-polynomial of a knot and the colored Jones polynomial of a knot. The strong AJ conjecture first proposed in [6] and then modified by Sikora [14], relates the orthogonal ideal to the classical peripheral ideal. The orthogonal ideal is an ideal of the skein module of the torus and the classical peripheral ideal is an ideal of the coordinate ring of the $SL(2, \mathbb{C})$ character variety. This conjecture could be seen as the topological and algebraic structure that underlies the AJ conjecture. The strong AJ conjecture has been confirmed for all torus knots and cables over torus knots. As such, the conjecture has only been confirmed for cases of non-hyperbolic knots. It should be noted that most knots fall into the class of being hyperbolic. In this thesis we confirm the strong AJ conjecture for the figure 8 knot which is the simplest hyperbolic knot.

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CHAPTER 1

INTRODUCTION

The Jones polynomial introduced by Jones [10] in 1985, is a knot invariant which had surprised much of the mathematics community. Furthermore, the Jones polynomial was then related to quantum field theory by Witten [17]. This could be seen as the beginning of quantum topology as a subject, and has since gave birth to many interesting connections to other fields of mathematics. One such example is the AJ conjecture, where the J denotes the Jones polynomial and the A denotes the A-polynomial.

During the same time period in which the Jones polynomial was introduced, the A-polynomial was introduced in [4] but used a more classical framework. Namely, it was a description of the fundamental group of a knot complement viewed from the boundary.

For some time, their was no apparent connection between the two invariants, as they were constructed from disparate ideas. But this would soon change, as it was noticed in [6] that the A-polynomial seemed to be determined by a non-commutative ideal of the quantum torus known as the quantum peripheral ideal. This relationship, after slight modification at a later time period, would become the strong AJ conjecture. It was also shown that this ideal gave recurrence relations to the colored Jones polynomial. Unfortunately their was an obstruction of being unable to show that this ideal was non trivial.

Motivated by this work, the AJ conjecture was introduced by Garoufalidis in [7]. This conjecture states that the polynomial describing the recurrence relation of the colored Jones polynomial in the classical limit becomes the A-polynomial. From this view point, Garoufalidis was able to circumvent the obstruction in [6], but as we will see, the structure and ideas in [6] are fundamental to the problem. The AJ conjecture has been proven for many knots. A proof of the conjecture in terms of a large family of knots was first done by Thang Le in [11], where he showed that a large class of two-bridge knots hold for the conjecture. The use of skein modules were impertinent in the proof and the structures in [6] are used.

The modification to the relationship noticed in [6] was done in [14], and is often called the strong AJ conjecture. For the explicit statement of the conjecture, we follow [15]. This conjecture relates the orthogonal ideal and the classical peripheral ideal. Where the orthogonal ideal is an ideal of the skein module of the torus, and was first introduced in [6]. The classical peripheral ideal is an ideal of the coordinate ring of the $SL(2, \mathbb{C})$ character variety, and was also introduced in [6]. It should be noted that it was also conjectured in [6] that the orthogonal ideal and the quantum peripheral ideal are equivalent, and was shown in [11] that this conjecture implies that the Jones polynomial distinguishes the unknot.

In this thesis we review the material needed to state the strong AJ conjecture. Then as the main result of the thesis, we confirm the strong AJ conjecture for the figure 8 knot.

1.1 Overview

This thesis is organized as follows, in the first chapter we go over prerequisites needed to state the AJ conjecture and the strong AJ conjecture. More specifically, we first go over the Kauffman bracket and use this to define the Jones Polynomial. We then work towards defining the colored Jones polynomial using Chebychev polynomials of the second kind. With this, we then look at recurrence relations and the colored Jones polynomial.

Having gone over one side of conjecture, we move towards defining the A-polynomial. To do this, we introduce representations of a group into $SL(2, \mathbb{C})$ and the $SL(2, \mathbb{C})$ character variety. This is then followed by the specific case of when our manifold is a knot complement, from there we can then define the A-polynomial. Which is then followed by the statement of the AJ conjecture.

Having now introduced the AJ conjecture, we work towards the strong AJ conjecture. To do this, we revisit the character variety, study regular functions on the variety, and introduce the classical peripheral ideal or also known as the A-ideal.

We then study the quantum analog, namely the quantum peripheral ideal. Before this, we first go over the Kauffman bracket skein module and the case of when our manifold is a knot complement. After this, we then introduce the quantum peripheral ideal and the orthogonal ideal. This is then followed by the statement of the strong AJ conjecture.

Having gone over the basic background of the strong AJ conjecture, we arrive at the main result of the thesis. Namely we confirm that the strong AJ conjecture holds true for the figure 8 knot. As the proof relies on the existence of a polynomial in which we explicitly provide, chapter 2 simply shows that the polynomial indeed satisfies the conditions needed to prove the theorem, and chapter 3 provides the construction of this polynomial.

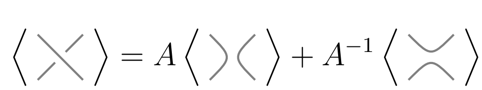
1.2 The Colored Jones Polynomial

The colored Jones polynomial is family of invariants indexed by integers, when $n = 2$ one recovers the original Jones polynomial. Their are many ways to define the colored Jones polynomial, but in this thesis we will take the approach using the Kauffman bracket and Chebychev polynomials. Unfortunately, going this route does not uncover its relations to quantum groups and TQFT's. For a more in depth discussion on the subject, we recommend the excellent expository paper [13].

In this section we will be following [11] and [12], and refer to them for more information on the subjects we discuss.

1.2.1 Kauffman Bracket and the Jones Polynomial

Let $\mathfrak{D}$ be the set of knot diagrams of knots in S^3, then the **Kauffman Bracket** is a function $\langle\cdot\rangle : \mathfrak{D} \to \mathcal{R} := \mathbb{Z}[A^{\pm 1}]$. We denote by $\langle D\rangle$, the polynomial assigned to a diagram D via the Kauffman bracket. The Kauffman bracket is computed using the following relations,



$$\left\langle \bigcirc \cup D \right\rangle = (-A^2 - A^{-2})\langle D \rangle$$
$$\left\langle \bigcirc \right\rangle = (-A^2 - A^{-2}).$$

An example of using the bracket can be seen in Figure 1.1, there the bracket of the trefoil would be the sum of the polynomials at the end of the table.

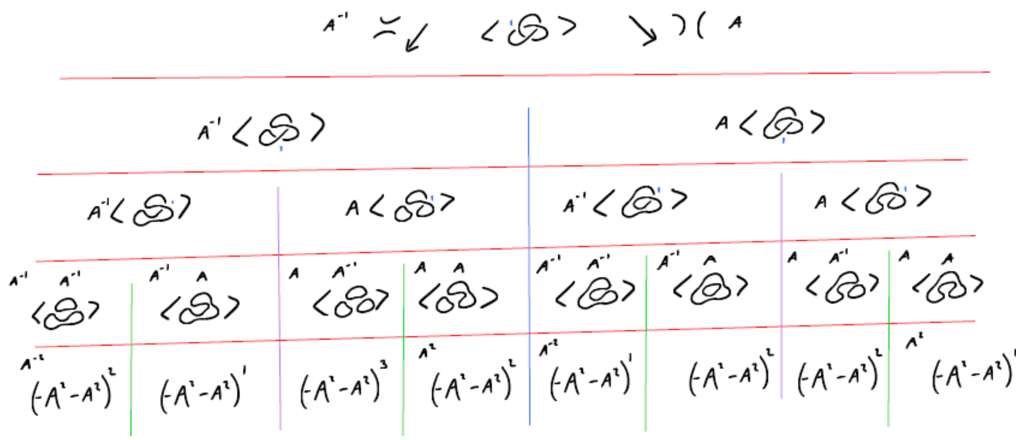


Figure 1.1. A table evaluating the Kauffman bracket of the trefoil knot.

The **Jones polynomial** is then defined as

$$J(K) = \frac{(-A^3)^{w(D)}\langle D \rangle}{-A^2 - A^{-2}}\Big|_{q:=A^4}.$$

Where D is the diagram for the knot K and $w(D)$ is the writhe of the knot with diagram D. One can show that $J(K)$ is indeed a invariant of knots and links. We will see that this is the 2-dimensional colored Jones polynomial $J_K(2)$.

1.2.2 The Colored Jones Polynomial

Let $S_n(z)$ be the Chebychev polynomials the second kind. Recall that these polynomials are defined by the following relation.

$$\begin{aligned} S_0(z) &= 1 \\ S_1(z) &= z \\ S_{n+1}(z) &= zS_n(z) - S_{n-1}(z) \end{aligned}$$

for all $n \in \mathbb{Z}$.

Now for a framed knot K in S^3 and an integer $n \geq 0$, we define the n-th power K^n as the link consisting of n parallel copies of K. An example of K^3 can be seen in Figure 1.2, note that the number of crossings increases. Moreover, we see that the Kauffman bracket can be extended linearly to be defined on $\mathcal{RK}$. Where $\mathcal{RK}$ is the free module over $\mathcal{R}$ with basis $\mathcal{K}$, and where $\mathcal{K}$ the set of ambient isotopic knots in S^3. With this we can define the

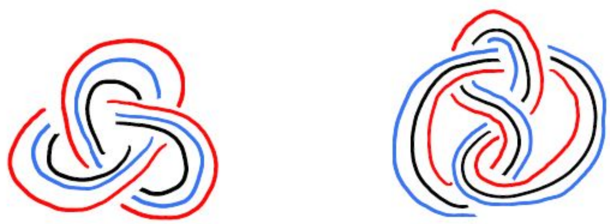


Figure 1.2. An example of K^3 for the trefoil and figure 8 knot.

colored Jones polynomial as

$$J_K(n+1) := (-1)^n \langle\, S_n(K) \,\rangle\,.$$

Note here, since it can be shown that the colored Jones polynomial does not depend on the diagram of the knot, we refer to the bracket as a function on knots instead of knot diagrams.

1.3 The Recurrence Ideal and Recurrence Polynomial

Consider a function with domain being the set of integers, $f : \mathbb{Z} \to \mathcal{R} := \mathbb{C}[q^{\pm 1}]$, and define the operators L and M, with an action on the functions by:

$$(Mf)(n) = q^n f(n), \quad (Lf)(n) = f(n+1).$$

We see that $LM = qML$ and that $M^{\pm 1}, L^{\pm}1$ generate the **quantum torus** $\mathcal{T}$, a noncommutative ring with presentation

$$\mathcal{T} = \mathcal{R}\langle M^{\pm 1}, L^{\pm 1}\rangle/(LM = qML).$$

The **reccurence ideal** of the discrete function f is the left ideal $\mathcal{A}$ in $\mathcal{T}$ that annihilates f, namely, we have

$$\mathcal{A} = \{P \in T \mid Pf = 0\}.$$

We denote by $\mathcal{A}_K$, the recurrence ideal of J_K. Where $J_K(n)$ is the n-colored Jones polynomial.

Note that the quantum torus $\mathcal{T}$ is not a principal ideal domian, and hence $\mathcal{A}_K$ might not be generated by a single element. It was noticed in [7] that by adding to $\mathcal{T}$ all the inverses of polynomials in M one gets a principal ideal domain $\tilde{\mathcal{T}}$.

Formally one constructs $\tilde{\mathcal{T}}$ as follows. Let $\mathcal{R}(M)$ be the fractional field of the polynomial ring $\mathcal{R}[M]$. Explicitly, we have that

$$\mathcal{R}(M) = \left\{ \frac{P(M)}{Q(M)} \mid P(M), Q(M) \in \mathcal{R}[M] \right\}.$$

Let $\tilde{\mathcal{T}}$ be the set of all Laurent polynomials in the variable L with coefficients in $\mathcal{R}(M)$:

$$\tilde{\mathcal{T}} = \left\{ \sum_{k\in\mathbb{Z}} a_k(M)L^k \mid a_k(M) \in \mathcal{R}(M), \;\; a_k = 0 \text{ almost everywhere} \right\},$$

and define the product in $\tilde{\mathcal{T}}$ by $a(M)L^k b(M)L^l = a(M)b(q^k M)L^{k+l}$.

It is known that every left ideal in $\tilde{\mathcal{T}}$ is principal, and that $\mathcal{T}$ embeds as a subring of $\tilde{\mathcal{T}}$. The extension $\tilde{\mathcal{A}_K} := \tilde{\mathcal{T}}\mathcal{A}_K$ of $\mathcal{A}_K$ in $\tilde{\mathcal{T}}$ is then generated by a single polynomial

$$\alpha_K(q; M, L) = \sum_{i=0}^{n} \alpha_{K,i}(q; M)L^i,$$

where the degree in L is assumed to be minimal and all the coefficients $\alpha_{K,i}(q; M) \in \mathbb{Z}[q^{\pm 1}, M]$ are assumed to be co-prime. We also see that α_K is defined up to a polynomial in $\mathbb{Z}[q^{\pm 1}, M]$. We call α_K the **recurrence polynomial** of K. It also often called the non-commutative A-polynomial, and denoted by $\hat{A}(M, L; q)$. It should be noted that the notation of $q = t^2$ is often used.

1.4 The A-polynomial

A classical invariant in topology is the fundamental group and it is known to be a very strong invariant. But, given two groups and their presentations, it is hard to tell if the two groups are isomorphic. For example, the following group with presentation,

$$\langle\ x, y, z \mid xyx^{-1}y^{-2},\ yzy^{-1}z^{-2},\ zxz^{-1}x^{-2}\ \rangle$$

is equivalent to the trivial group.

One method to study these groups is to study homorphisms of groups into matrices in hopes to study the group through more familiar objects. Such homorphisms are called representations of the group into matrices, and these representations are described by their characters.

The A-polynomial describes the $SL(2,\mathbb{C})$ character variety of the knot complement when viewed from the boundary, and was first introduced in [4]. We will see that the formulation of this polynomial is quite different from the Jones polynomial, but none the less the two polynomials are intimately related.

In this section, we will be following [11], [12], and [9].

1.4.1 Representations into $SL(2,\mathbb{C})$

Let G be a group, a representation of G into $GL(V)$ is a group homorphism

$$\rho : G \to GL(V).$$

We denote the set of all representations of G into $GL(V)$ as $Hom(G, GL(V))$.

Let $G = \langle\ a_1, \ldots, a_n \mid r_1, \ldots, r_k\ \rangle$, we want to study $Hom(G, SL(2,\mathbb{C}))$. Note that this set can be described algebraically. Namely every representation ρ assigns $a_i \in G$ to $A_i \in SL(2,\mathbb{C})$. Note for ρ to be a representation the assigned matrices must also satisfy the relations of the group. With this, we get the following polynomial equations,

$$\begin{aligned} det(A_i) - 1 &= 0 \\ R_j - I &= 0. \end{aligned}$$

where $i = 1, \ldots n$ and $j = 1, \ldots k$. We see that we have $n + 4k$ polynomial equations. Thus one can show that any representation of the group satisfies these polynomial equations and conversely any solution to these equations would give a representation. Meaning one could view $Hom(G, SL(2, \mathbb{C}))$ as an affine variety, namely the zeros to polynomial equations.

1.4.2 Conjugation

Given a representation ρ, one can construct another representation via conjugation.

Example 1.4.1. *Let us look at the knot group of the trefoil,*

$$G = \langle\ a, b \mid aba = bab\ \rangle.$$

Let us now assume we have a representation ρ, where $\rho(a) = A$ and $\rho(b) = B$ are in $SL(2, \mathbb{C})$. This means that we have $ABA = BAB$.

With this we claim that ρ_C is also a representation. Where ρ_C is the representation defined by

$$\rho_C(a) = C\rho(a)C^{-1},$$

where $C \in SL(2, \mathbb{C})$. This means that we have to check if

$$\rho_C(a)\rho_C(b)\rho_C(a) = \rho_C(b)\rho_C(a)\rho_C(b).$$

We see that this indeed holds,

$$\begin{aligned}\rho_C(a)\rho_C(b)\rho_C(a) &= CAC^{-1}CBC^{-1}CAC^{-1}\\ &= CABAC^{-1}\\ &= CBABC^{-1}\\ &= CBC^{-1}CAC^{-1}CBC^{-1}\\ &= \rho_C(b)\rho_C(a)\rho_C(b).\end{aligned}$$

Hence we see that given a representation, we can construct a family of representations from the original one. One also notes that conjugate matrices have the same trace. This motivates us to study representations up to conjugation and trace.

1.4.3 Character Variety

Let $\rho : G \to SL(2,\mathbb{C})$ be a representation, we define the **trace** of a representation to be the function

$$\chi_\rho : G \to \mathbb{C}$$

$$g \mapsto Tr(\rho(g))$$

where Tr is the trace of the matrix in $SL(2,\mathbb{C})$. The function χ_ρ is often called the **character** of a representation ρ.

Given two representations, ρ_1, ρ_2 we say that they are **trace equivalent** if

$$Tr(\rho_1(g)) = Tr(\rho_2(g)) \quad \forall g \in G.$$

We see two trace equivalent representations have the same character, namely $\chi_{\rho_1} = \chi_{\rho_2}$.

Let M be a manifold, and $\pi_1(M)$ its fundamental group. We define the $SL(2,\mathbb{C})$ **character variety** of M, denoted by $\chi(M)$, as

$$\chi(M) = Hom(\pi_1(M), SL(2,\mathbb{C}))//\sim,$$

where two representations ρ_1, ρ_2 are equivalent if and only if they are trace equivalent. One could think of $\chi(M)$ as the set of characters, and hence the name character variety.

Recall that conjugation by matrices does not affect the trace, and hence conjugate representations are equivalent under this trace equivalence. Moreover, when looking at irreducible representations, the equivalence of trace and conjugation are the same. Hence often times one will look at the set,

$$Hom(\pi_1(M), SL(2, \mathbb{C}))/conjugation.$$

1.4.4 Knot Complement and Torus Boundary

Note that for a knot complement, the manifold has torus boundary, and we have that $\pi_1(\partial M = T^2) = \mathbb{Z} \oplus \mathbb{Z}$. So let us now look at representations up to conjugation of $\pi_1(T^2)$ into $SL(2, \mathbb{C})$.

Recall that if two matrices $A, B \in SL(2, \mathbb{C})$ commute they can both be conjugated to be simultaneously in Jordan normal form.

Hence up to conjugation we have that $\rho \in \chi(\partial M)$ has the following form,

$$\rho(\gamma_m) = \begin{pmatrix} m & * \\ 0 & m^{-1} \end{pmatrix}, \quad \rho(\gamma_l) = \begin{pmatrix} l & ** \\ 0 & l^{-1} \end{pmatrix}.$$

Where γ_l and γ_m are the longitude and meridian generators of $\pi_1(\partial M)$. Note that the entries with the $*'s$ can be changed by conjugation while keeping Jordan normal form. This means that each representation corresponds to a choice of $l, m \in \mathbb{C}^*$. We see that the choice of l or l^{-1} coorespond to the same representation, hence one mods out by $\mathbb{Z}_2$. Thus we see that

$$\chi(\partial M) \cong \mathbb{C}^* \oplus \mathbb{C}^*/\mathbb{Z}_2$$

1.4.5 The A-polynomial

Note that a representation ρ on $\pi_1(M)$ induces a representation on $\pi_1(\partial M)$. The induced representation can be seen as follows. Recall that there is a natural inclusion map $i : \partial M \to$

M. This inclusion map induces the map $\theta : \pi_1(\partial M) \to \pi_1(M)$, namely it sends paths on ∂M to paths on M through the inclusion map. We then finally have the induced map on representations,

$$\theta^* : \chi(M) \to \chi(\partial M) \cong C^* \otimes C^*/\mathbb{Z}_2$$

where

$$\theta^*(\rho) := \rho \circ \theta : \pi_1(\partial M) \to SL(2, \mathbb{C}).$$

Using this induced map, we have that every representation on M corresponds to a representation ∂M. With this, one asks which representation on ∂M can be lifted to representations on M. To find such a set one looks at $im(\theta^*)$.

Miraculously, it turns out that

$$im(\theta^*) = \{\ (m,l) \in C^* \otimes C^*/\mathbb{Z}_2 \ \mid \ A(m,l) = 0\ \}.$$

Namely, the representations on ∂M that can be lifted to M are described by a single polynomial, we call this polynomial the A-polynomial.

1.5 The AJ Conjecture

We are now ready to introduce the AJ conjecture.

Conjecture 1.5.1. *For every knot K in S^3, $\alpha_K|_{q=1}$ is equal to the A-polynomial, up to a polynomial depending on M only.*

The conjecture has been comfirmed for the trefoil, figure 8, all torus knots, and some classes of two bridge knots.

As to emphasisize an over arching structure to the conjecture, it should be noted that in the proof[11] for a large class of knots the use of structures seen in [6] were used. We now work towards these structures, and see how it motivates the strong AJ conjecture.

It should also be noted that the AJ conjecture was also independently conjectured by Gukov in [8], but was done from a more physical view point. Namely the A-polynomial could be seen as a classical constraint on the phase space for Chern-Simons theory. Hence when quantizing the theory, the classical constraint becomes an operator that acts on the partition function which is the colored Jones polynomial. This approach uses more a differential geometric viewpoint and is related to the physical interpretation of the Jones polynomial first introduced by Witten [17]. This viewpoint is very interesting and has also led to new more relations between knot theory and physics. Unfortunately this matter is currently out of scope of the author and current thesis, but for more information, we refer to [8] and [9].

1.6 The Classical Peripheral Ideal

One of the structures studied in the strong AJ conjecture is the classical peripheral ideal or also known as the A-ideal. As in the case of the A-polynomial, the definition of the ideal comes from using the boundary torus. Here we follow [11] and [12].

1.6.1 Coordinate Ring on the $SL(2,\mathbb{C})$ Character Variety

Given an affine variety V in the n-dimensional affine space K^n, where K is an algebraically closed field, the coordinate ring of V is the quotient ring

$$K[V] = K[x_1, \ldots, x_n]/I(V)$$

where $I(V)$ is the ideal formed by all polynomials $f(x_1, \ldots, x_n)$ with coefficients in K such that $f(x_1, \ldots, x_n) = 0$ on V.

Recall they are two ways to view $K[V]$, one as cosets and the other as an equivalence relation. Namely in terms of an equivalence relation, we have $f(x) \sim g(x)$ if and only if $f(x) - g(x) \in I(V)$. We see that if this is true, we have $f(x) \in g(x) + I(V)$ which means they are in the same coset.

Example 1.6.1. *Let*

$$V = \{ \ (x, y) \in \mathbb{C}^2 \ | \ y - x^2 = 0 \ \}$$

and $f(x, y) = y^2$ *and* $g(x, y) = x^4$. *We see that in* $\mathbb{C}[V]$, $f(x, y) \sim g(x, y)$ *since* $f(x, y) - g(x, y) = 0$ *which means they are in the same coset.*

One can think of the coordinate ring as the set of functions on our variety. It is used in algebraic geometry and often carries more information then the original variety.

Now, recall that the set of representations up to conjugation could be viewed as an affine variety. With this we can study the coordinate ring $\mathbb{C}[\chi(M)]$ of the variety $\chi(M)$.

One such example of a function on representations is the the trace function

$$Tr_a : \chi(M) \to \mathbb{C}$$

where $Tr_a(\rho) = Tr(\rho(a))$ with $a \in \pi_1(M)$. When M is a surface, it is known that $\mathbb{C}[\chi(M)]$ is generated by the trace functions subject to relations. Furthermore, it can be shown that in the case of $SL(2, \mathbb{C})$ we have

$$Tr(A)Tr(B) = Tr(AB) + Tr(AB^{-1}).$$

We will see that this identity is related to the Kauffman bracket relation. We will touch on the subject more in the next section.

1.6.2 The Classical Peripheral Ideal

Recall than when our manifold is a knot complement, we have the natural inclusion map $i : \partial M \to M$. This map induces the map

$$\theta : \pi_1(\partial M) \to \pi_1(M),$$

which then leads to the map

$$\theta^* : \chi(M) \to \chi(\partial M),$$

where $\theta^*(\rho) = \rho \circ \theta$.

Now let $f : \chi(\partial M) \to \mathbb{R}$, where $f \in \mathbb{C}[\chi(\partial M)]$. We then can define

$$\theta_* : \mathbb{C}[\chi(\partial M)] \to \mathbb{C}[\chi(M)]$$

where

$$\theta_*(f) = f \circ \theta^* : \chi(M) \to \mathbb{R},$$

We then define the **classical peripheral ideal** as

$$\mathfrak{p} = ker\theta_*.$$

It also often called the A-ideal, as it determines the A-polynomial. In fact, with the action $\sigma(l^a m^b) = l^{-a}m^{-b}$, we have that $\mathfrak{p} = (A(m,l)\mathfrak{t})^\sigma$. Namely we have that $\mathfrak{p}$ is the σ invariant part of the ideal $A(m,l)\mathfrak{t} \subset \mathfrak{t} := \mathbb{C}[m^{\pm 1}, l^{\pm 1}]$ generated by $A(m,l)$.

1.7 The Quantum Peripheral and Orthogonal Ideals

The quantum peripheral and orthogonal ideals were first introduced in [6]. These ideals capture the relation between the skein module and recurrence relations for the the colored Jones polynomial.

Before studying these ideals we first introduce the Kuaffman bracket skein module and then study the case when our manifold is a knot complement. Here we follow [11] and [5].

1.7.1 The Kauffman Bracket Skein Module

Recall that a framed("black board") link is a link in which one has the choice of a normal vector always pointing out of the "black board", one could also replace the link with an annulus to better visulize the normal vector. The choice of such a normal vector allows one to avoid things such as the mobius band.

Now let $\mathcal{R} := \mathbb{C}[A^{\pm 1}]$ and $\mathcal{L}$ be the set of isotopy classes of framed links in the manifold Y, including the empty link. We then let $\mathcal{RL}$ be the $\mathcal{R}$-module with basis $\mathcal{L}$, and $\mathcal{K}(Y)$ be the smallest submodule containing all the expressions of the form

We define the **Kauffman bracket skein module** as the quotient

$$\mathcal{RL}/\mathcal{K}(Y).$$

This qoutient is denoted by $\mathcal{S}(Y)$ and is sometimes called the **skein module** of Y.

When $Y = \Sigma \times I$, where Σ is a 2-manifold, $\mathcal{S}(Y)$ has the structure of an algebra where the multiplication is the operation of gluing the manifolds on top of each other. An example, of the gluing operation can be seen in Figure 1.3. One notes that this a non-commutative algebra.

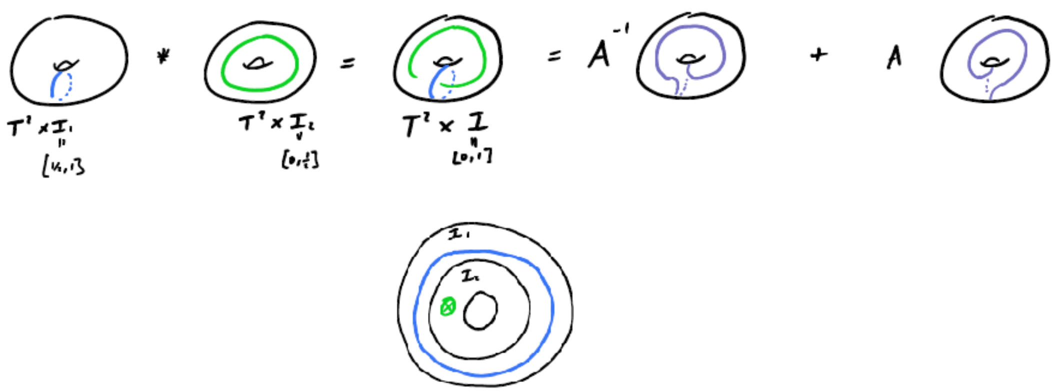


Figure 1.3. The multiplication of two elements in the skein module of the torus.

It is known that the Kuaffman bracket skein module is a quantization of the coordinate ring on the character variety. Note that on $\mathbb{C}[\chi(Y)]$, we have the following relation,

$$Tr(A)Tr(B) = Tr(AB) + Tr(AB^{-1}).$$

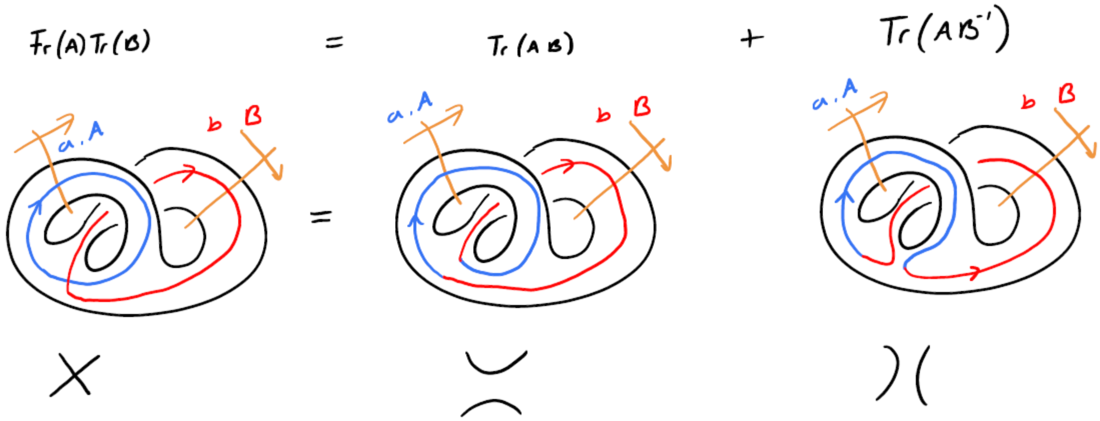


Figure 1.4. A pictorial representation of the trace relation.

This relation can be seen pictorially in Figure 1.4. One sees that this relation is the Kauffman bracket relation at $A = -1$. With this we see that skein modules and character varieties are intimately related. For more details on quantization of character varieties and its relation to skein modules, we refer to [2].

1.7.2 Skein Module of the Torus and the Quantum Torus

Let T^2 be the torus, it known that $S(T^2)$ has basis consisting of the set of torus links, their parallels, and the trivial link. Recall that torus links are simply link on the torus that can be drawn on the torus with out crossing. Torus knots can be seen as curves on the torus and can be denoted by $\lambda_{p,r}$, which is the (p, r)-torus knot. Here, p denotes the number of times the curve goes around in the meridian direction, and r the number of times the curve goes around in the longitue direction. We also denote the $(1, 0)$-torus knot as μ and $(0, 1)$-torus knot as λ, which correspond to the meridian and longitude. Some examples of (p, r)-torus knots are shown in Figure 1.5.

Now we shall see how $S(T^2)$ relates to the quantum torus $\mathcal{T}$. Recall that the quantum torus is defined as (with $q = t^2$ and $\mathcal{R} := \mathbb{C}[t^{\pm 1}]$)

$$\mathcal{T} = \mathcal{R}\langle\, M^{\pm 1}, L^{\pm 1}\,\rangle/(LM - t^2 ML).$$

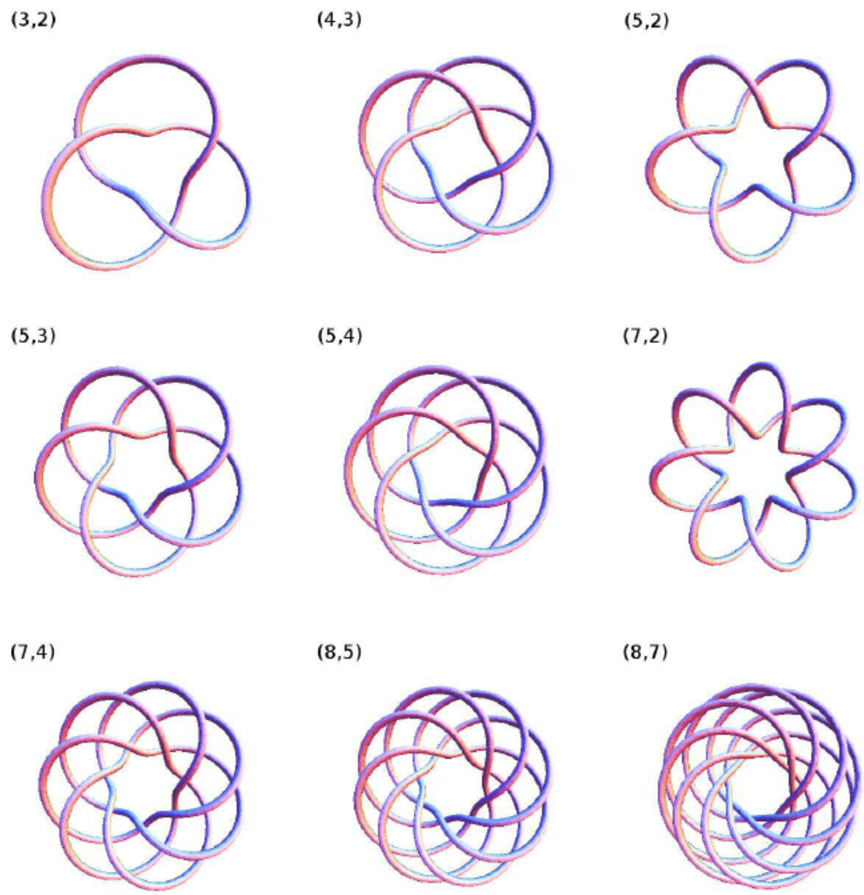


Figure 1.5. Some examples of Torus knots. This figure is from [1].

Frohman and Gelca showed that there is a unique algebra homomorphism $\mathcal{Y} : S(T^2) \to \mathcal{T}$ such that

$$\mathcal{Y}(\mu) = -(M + M^{-1}), \quad \mathcal{Y}(\lambda) = -(L + L^{-1}),$$

or more explicitly,

$$\mathcal{Y}(\lambda_{p,r}) = (-1)^{p+r} t^{pr} (M^p L^r + M^{-p} L^{-r}).$$

If we let $\sigma : \mathcal{T} \to \mathcal{T}$ be the involution defined by $\sigma(M^k L^l) := M^{-k} L^{-l}$, we have that $\mathcal{Y}$ maps $S(T^2)$ isomorphicaly onto the symmetric part of $\mathcal{T}$, which we will denote by $\mathcal{T}^\sigma$.

Recall that $\mathcal{A}_K$ is an ideal of the quantum torus, and as such we see the beginning of the relation between skein modules and reccurence relations. We delve deeper into these relations in the following subsections, when discussing the quantum peripheral ideal and the orthogonal ideal.

1.7.3 Quantum Peripheral Ideal

As we have seen the classical peripheral ideal, for the coordinate ring on the character variety, we study a similar notion for the skein module. Once again, letting $M = S^3 - N(K)$, the inclusion map $\partial M \hookrightarrow M$ induces the map

$$\Theta : S(\partial M) \to S(M), \quad \Theta(l) = l \cdot \emptyset.$$

The kernel $\mathcal{P} := ker\Theta$ is called the **quantum peripheral ideal**, which was first introduced in [6]. It was shown in [6] that every element of $\mathcal{P}$ gives rise to a reccurence relation for the colored Jones polynomial.

The problem with this was that it could not be proved that $\mathcal{P}$ is non-trivial. To circumvent this we introduce the orthogonal ideal.

1.7.4 Othogonal Ideal

Note that T^2 is the boundary of both $M = S^3 - N(K)$ and $N(K)$. With this one can view $S(M)$ as a left $S(\partial M)$-module and $S(N(K))$ as a right $S(\partial M)$-module.

The module structure can be seen as pushing the link in T^2 into M or $N(K)$, which would then define an action via the union of the links. For example in the case of $S(M)$ as a left $S(\partial M)$-module, let $L_M \in S(M)$ and $L_{\partial M} \in S(\partial M)$ be the evaluation of links $\mathfrak{L}_M \in M$ and $\mathfrak{L}_{\partial M} \in \partial M$. Then we let $L_{\partial M} \cdot L_M$ be defined as the evaulation of $i(\mathfrak{L}_{\partial M}) \cup \mathfrak{L}_M \in M$. The case of $S(N(K))$ as a right $S(\partial M)$-module can be seen in a similar fashion.

One can also see that there is a $\mathcal{R}$-bilinear form

$$\langle \cdot, \cdot \rangle : S(N(K)) \otimes_{S(T^2)} S(M) \to S(S^3) = \mathcal{R}$$

given by

$$\langle l', l'' \rangle = \langle l' \cup l'' \rangle,$$

where l' and l'' are links in $N(K)$ and M respectively. With this one can define $\mathcal{O}$, the **orthogonal ideal**, as

$$\mathcal{O} := \{ \ l \in S(\partial M) \ | \ \langle l', \Theta(l) \rangle = 0 \text{ for every } l' \in S(N(K)) \ \} .$$

It is known that $\mathcal{O}$ is nonempty, and is related to the peripheral and reccurence ideal in the following way.

Proposition 1.7.1. *For every knot one has*

$$\mathcal{P} \subset \mathcal{O} \subset \mathcal{A}_K .$$

Moreover,

$$\mathcal{O} = \mathcal{A}_K \cap \mathcal{T}^\sigma .$$

We will sometimes denote $\mathcal{A} \cap \mathcal{T}^\sigma$ as $\mathcal{A}_K^\sigma$. One should also note that it is conjectured that $\mathcal{P} = \mathcal{O}$.

1.8 The Strong AJ Conjecture

Here we let ϵ be the map that sends t to -1, namely it sets $t = -1$.

Conjecture 1.8.1. *Suppose K is a knot in S^3, then*

$$\sqrt{\epsilon(\mathcal{A}_K^\sigma)} = \mathfrak{p}.$$

Where $\sqrt{\epsilon(\mathcal{A}_K^\sigma)}$ denotes the radical of the ideal $\epsilon(\mathcal{A}_K^\sigma)$.

The strong AJ conjecture has been confirmed for the trefoil knot[14], all torus knots [15], and cables of torus knots[16].

1.9 Summary

A summary of the structure of the problem we have discussed so far can be see in Figure 1.6. Namely, it shows that the AJ conjecute could be seen from the point of view of the Skein module and character variety.

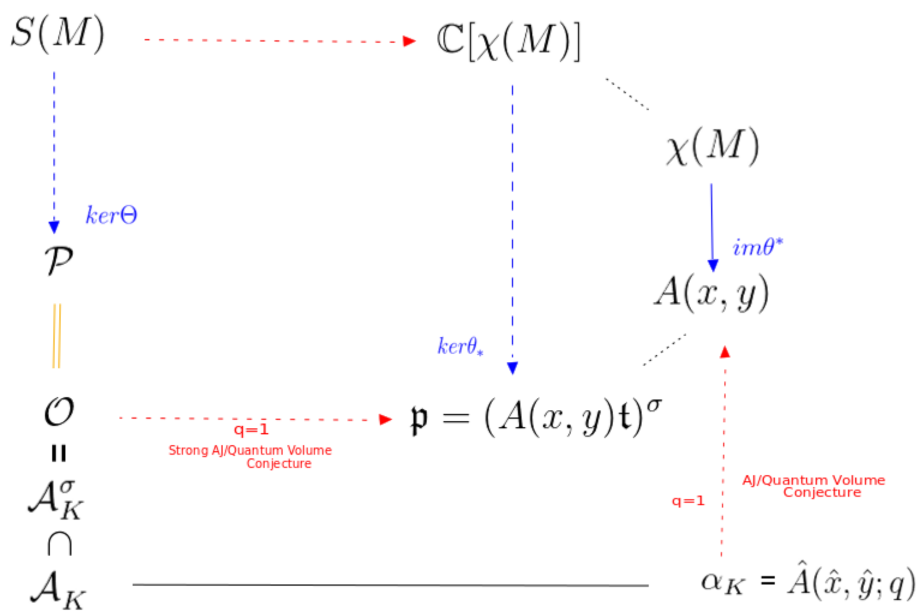


Figure 1.6. A diagram of the relations of objects discussed so far.

The items in Figure 1.6 are listed as,

- M : The knot complement $S^3 - K$,
- $\chi(M)$ and $\chi(\partial M)$: The $SL(2, \mathbb{C})$ Character variety of $\pi_1(M)$ and $\pi_1(\partial M)$,
- $\mathbb{C}[\chi(M)]$ and $\mathbb{C}[\chi(\partial M)]$: The coordinate ring of $\chi(M)$ and $\chi(\partial M)$,
- $S(M)$ and $S(\partial M)$: Kauffman bracket skein module of M and ∂M.
- From the natural embedding of $\partial M \to M$ we have the following maps,

  $\theta : \pi_1(\partial M) \to \pi_1(M)$,

$\theta^* : \chi(M) \to \chi(\partial M)$,

$\theta_* : \mathbb{C}[\chi(\partial M)] \to \mathbb{C}[\chi(M)]$,

$\Theta : S(\partial M) \to S(M)$.

- With these maps, we have the following,

  $A(x, y)$: the A-polynomial, whose zero set is $im\theta^*$. Also note that x, y can be used to denote l, m, and associated to this one can use $\hat{x}, \hat{y}$ to denote the operators L, M.

  $\mathfrak{p} = ker\theta_*$ known as the classical peripheral ideal or A-ideal,

  $\mathcal{P} = ker\Theta$ known as the quantum peripheral ideal,

  $\mathcal{O} = \{\ l' \in S(\partial M)\ |\ \langle\ l,\ \Theta(l')\ \rangle = 0\ \}$ known as the orthogonal ideal.

  Note that the quantum and orthogonal ideals are in the $S(T^2) \cong \mathcal{T}^\sigma$.

- We also have $\mathcal{A}_K$ as the reccurence ideal, and $\mathcal{A}_K^\sigma$ as the sigma invariant part of the reccurence ideal. Where sigma is the map $\sigma(ML) = M^{-1}L^{-1}$.

  It is known that $\mathcal{P} \subset \mathcal{O} \subset \mathcal{A}_K$.

  - It is currently conjectured that $\mathcal{P} = \mathcal{O}$.

CHAPTER 2

THE STRONG AJ CONJECTURE FOR THE FIGURE 8 KNOT

In this chapter we go over the proof of theorem 2.1.1, which relies on the existence of a polynomial that satisfies the conditions of lemma 2.3.4. We will often denote this polynomial as $Q(t, M, L)$, and its existence is proven by explicitly providing such a polynomial. In the next chapter we go over the construction of the explicit polynomial.

2.1 Main Result

The main result of this thesis is the following.

Theorem 2.1.1. *The Strong AJ conjecture holds true for the figure 8 knot.*

The theorem will be proven by the string of lemmas and propositions stated in the following sections. We state the proof in section 2.5.

2.2 Notation

For conveniece we use the following notation. We have $J_K(n) \in \mathbb{Z}[t^{\pm 1}]$ and use the following normalization

$$J_U(n) = \frac{t^{2n} - t^{-2n}}{t^2 - t^{-2}},$$

where U is the unknot. Also note that we have the change of variables $q = t^2$, and have $\mathcal{R} := \mathbb{C}[t^{\pm 1}]$.

We will also often use two letters denote a variable, for example ae_0 is a quantity. In the case that it does represent a single quantity will state that it does not.

2.3 Main Lemmas

The following lemmas are instrumental in the proof of theorem 2.1.1, and is the general method(so far) that has been used in proofs for the strong AJ conjecture in other cases of knots. These lemmas can be found in [15].

Lemma 2.3.1. *Suppose the AJ conjecture holds true for a knot K and the A-polynomial A_K does not have any non-trivial M-factors. Then $\sqrt{\epsilon(\mathcal{A}^\sigma_K)} \subset \mathfrak{p}$.*

Proof. Proof can be found in [15]. □

The above lemma shows that we only need to show one side of the inclusion if the AJ conjecture holds for the particular knot we are interested in.

Lemma 2.3.2. *Suppose there exist $P \in \mathcal{A}^\sigma_K$ such that $\epsilon(P) = M^k L^l (A_K)^{2m}$ for some integers k, l, m, then $\mathfrak{p} \subset \sqrt{\epsilon(\mathcal{A}^\sigma_K)}$*

Proof. Proof can be found in [15]. □

The above lemma shows that if we can find a polynomial P with the conditions stated in the lemma, then we have the other side of the inclusion.

Lemma 2.3.3. *Suppose $h_1(n), \dots, h_k(n) \in \mathcal{R}[M^{\pm 1}]$. There exist $R(t, L) \in \mathcal{R}[L^{\pm 1}]$ such that $\sigma(R(t, L)) = R(t, L)$, $\epsilon(R(t, L)) = (L + L^{-1} - 2)^m$ for some $m \geq 1$, and $R(t, L)h_i = 0$ for all $i = 1, \dots, k$.*

Proof. Proof can be found in [15]. □

Note that $(L + L^{-1} - 2)^m = L^{-m}(L - 1)^{2m}$, namely it is the homongeonus part of the A-polynomial raised to an even power up to some power of L. With this, the above lemma will allow us to weaken the requirements of lemma 2.3.2. We use the following notation, $A_K = (L - 1)A'_K$. Namely, the A-polynomial has a factor of $L - 1$, we will be looking at the part of the A-polynomial without the $L - 1$ factor.

Lemma 2.3.4. *If there exist* $Q(t,M,L) \in \mathcal{R}[M^{\pm 1}, L^{\pm 1}]$ *such that* $Q(t,M,L)J_K \in \mathcal{R}[M]$, $\sigma(Q(t,M,L)) = Q(t,M,L)$, *and* $\epsilon(Q(t,M,L)) = M^n L^b (A'_K)^{2r}$ *for some integers* n, b, r, *then* $\mathfrak{p} \subset \sqrt{\epsilon(\mathcal{A}_K^\sigma)}$.

Proof. Note with $Q(t,M,L)J \in \mathcal{R}[M^\pm]$, we have by lemma 2.3.3 that there exist R such that $\sigma(R(t,L)) = R(t,L)$, $\epsilon(R(t,L)) = (L + L^{-1} - 2)^m$, and $RQJ_K = 0$. Now note that $(R(t,L)Q(t,M,L))^{2rm} J_K = 0$, $\sigma((R(t,L)Q(t,M,L))^{rm}) = (R(t,L)Q(t,M,L))^{rm}$, and $\epsilon((R(t,L)Q(t,M,L))^{mr}) = M^a L^d (A_K)^{2mr}$ where $a = nm \in \mathbb{Z}$ and $d = bm - rm \in \mathbb{Z}$. Thus $(R(t,L)Q(t,M,L))^{rm}$ satisfies the requirements of lemma 2.3.2, and hence $\mathfrak{p} \subset \sqrt{\epsilon(\mathcal{A}_K^\sigma)}$. □

The proof of lemma 2.3.4 was done implicitly in [15]. We will be using this lemma to show one side of the inclusion. Namely, we will construct a polynomial Q that satisfies the condition stated in the lemma.

2.4 Propositions and Lemmas

We now go over some propositions and lemmas that are particular to this case of the conjecture.

For $P \in \mathbb{C}[t,M,x]$, define $\langle P \rangle : \mathbb{Z} \to \mathcal{R} := \mathbb{C}[t^{\pm 1}]$ by

$$\langle P \rangle (n) := [n] \sum_{k=0}^{n-1} P(t, t^{2n}, t^{4k}) \prod_{l=1}^{k} (t^{4n} + t^{-4n} - t^{4l} - t^{-4l})$$

for $n \geq 1$, and $\langle P \rangle (-n) = - \langle P \rangle (n)$ for $n \leq 0$. It is known that $\langle 1 \rangle$ is equal to the colored Jones polynomial of the figure 8 knot. With this we can study how the operator L acts on the colored Jones polynomial of the figure 8 knot.

For $P, Q \in \mathbb{C}[t,M,x]$, we write $P \equiv Q$ if $P - Q \in \mathbb{C}[t,M]$. With this notation, we have the following lemma.

Lemma 2.4.1.

$$
\begin{aligned}
L\langle 1\rangle &\equiv \left(t^{-2}M^{-4}-t^{-2}M^{-2}-t^{2}\right)\langle 1\rangle+\left(t^{6}M^{2}-t^{2}M^{-2}\right)\langle x\rangle\\
L^{2}\langle 1\rangle &\equiv \Big\{t^{-12}M^{-8}-\left(t^{-12}+t^{-8}\right)M^{-6}-t^{12}M^{4}-(t^{-4}+1)M^{-4}\\
&+\left(t^{-4}+1\right)M^{-2}+t^{8}+t^{4}+1\Big\}\langle 1\rangle\\
&+\Big\{t^{16}M^{6}-t^{-8}M^{-6}-t^{12}M^{4}+t^{-4}M^{-4}-\left(t^{12}+t^{8}+t^{4}\right)M^{2}\\
&+\left(t^{4}+t^{-4}+1\right)M^{-2}\Big\}\langle x\rangle\\
L^{-1}\langle 1\rangle &\equiv \left(t^{-2}M^{4}-t^{-2}M^{2}-t^{2}\right)\langle 1\rangle+\left(t^{6}M^{-2}-t^{2}M^{2}\right)\langle x\rangle\\
L^{-2}\langle 1\rangle &\equiv \Big\{t^{-12}M^{8}-\left(t^{-12}+t^{-8}\right)M^{6}-t^{12}M^{-4}-\left(t^{-4}+1\right)M^{4}\\
&+(t^{-4}+1)M^{2}+t^{8}+t^{4}+1\Big\}\langle 1\rangle\\
&+\Big\{t^{16}M^{-6}-t^{-8}M^{6}-t^{12}M^{-4}+t^{-4}M^{4}-\left(t^{12}+t^{8}+t^{4}\right)M^{-2}\\
&+\left(t^{4}+t^{-4}+1\right)M^{2}\Big\}\langle x\rangle
\end{aligned}
$$

Proof. From ([3], Proposition 4.5), we have the expression for $L\langle 1\rangle$ and

$$L\langle x\rangle \equiv (t^{-2}M^{-2}-t^{2}M^{2})\langle 1\rangle+(t^{6}M^{4}-t^{2}M^{2}-t^{2})\langle x\rangle.$$

With this and using the fact that $L^2\langle 1\rangle = L(L\langle 1\rangle)$, we obtain the rest of the formulas by direct calculation. □

The following two lemmas explicitly show that there is a polynomial $Q(t,M,L)$ that satisfies the condition stated in lemma 2.3.4. This shown by explicitly providing $Q(t,M,L)$, in the next chapter we will give the construction and method to obtaining this polynomial.

Lemma 2.4.2. *One has*

$$(L^2+L^{-2}+e_1(t,M)L+e_{-1}(t,M)L^{-1}+e_0(t,M))J_K \in \mathcal{R}[M^{\pm 1}]$$

where

$$\begin{aligned}
e_1(t,M) &= \frac{-1-t^{16}+M^4t^8(1+t^4)^2+M^2(t^4+t^{12})+M^6(t^8+t^{16})-M^8(t^{12}+t^{20})}{M^4t^{10}}\\
e_0(t,M) &= \frac{1}{M^8t^4}\Big(t^8+M^{16}t^8-M^4t^8(2+t^4)-M^{12}t^8(2+t^4)-M^2(t^4+t^8)\\
&+ M^6(t^4+t^8)+M^{10}(t^4+t^8)-M^{14}(t^4+t^8)+2M^8(1+t^4+2t^8+t^{12})\Big)\\
e_{-1}(t,M) &= \frac{M^4t^8(1+t^4)^2-t^{12}(1+t^8)+M^6(t^4+t^{12})-M^8(1+t^{16})+M^2(t^8+t^{16})}{M^4t^{10}}
\end{aligned}$$

Proof. Using lemma 2.4.1 and direct calculation we see that

$$\begin{aligned}
&(L^2+L^{-2}+e_1(t,M)L+e_{-1}(t,M)L^{-1}+e_0(t,M))J_K\\
&=\langle x\rangle\frac{(-1+M)(1+M)(1+M^2)(1+M^4+M^2t^8+M^2t^{12}+M^2t^{16})}{M^4t^4}.
\end{aligned}$$

□

Lemma 2.4.3. *Let K be the figure 8 knot, then there exist $Q(t,M,L)$ such that $Q(t,M,L)J_K \in \mathcal{R}[M]$, $\sigma(Q(t,M,L)) = Q(t,M,L)$, and $\epsilon(Q(t,M,L)) = M^kL^l(A'_K)^{2m}$ for some integers k,l,m.*

Proof. Let

$$Q(t,M,L) = L^2+L^{-2}+e_1L+e_{-1}L^{-1}+e_0.$$

We see that the symmetry conditions

$$\begin{aligned}
e_1(t,M) &= e_{-1}(t,M^{-1})\\
e_0(t,M) &= e_0(t,M^{-1})
\end{aligned}$$

are satisfied, and hence we have $\sigma(Q)=Q$. From lemma 2.4.2 we have that $QJ_K \in \mathcal{R}[M^{\pm1}]$.

Finally, with

$$A'_K(L,M) = L+L^{-1}+\frac{-1+M^2+2M^4+M^6-M^8}{M^4}$$

we see that

$$L^2+L^{-2}+e_1(1,M)L+e_{-1}(1,M)L^{-1}+e_0(1,M) = \frac{A'_K(L,M)^2}{L^2M^8}.$$

□

2.5 Proof of Theorem 2.1.1

Proof. It was shown in [7], that the AJ conjecture holds true for the figure 8 knot. Hence by lemma 2.3.1, we have that

$$\sqrt{\epsilon(\mathcal{A}_K^\sigma)} \subset \mathfrak{p}.$$

We now show the other side of the inclusion. Note by lemma 2.4.2 we have

$$(L^2 + L^{-2} + e_1(t, M)L + e_{-1}(t, M)L^{-1} + e_0(t, M))J_K \in \mathcal{R}[M^{\pm 1}].$$

Letting $Q(t, M, L) = L^2 + L^{-2} + e_1(t, M)L + e_{-1}(t, M)L^{-1} + e_0(t, M)$, we have by lemma 2.4.3 that $Q(t, M, L)$ satisfies the requirements of lemma 2.3.4, and hence

$$\mathfrak{p} \subset \sqrt{\epsilon(\mathcal{A}_K^\sigma)}.$$

□

CHAPTER 3

CONSTRUCTION OF PARTICULAR POLYNOMIAL IN PROOF OF THEOREM 2.1.1

3.1 Overview

In this chapter we go over the construction of the polynomial $Q(t, M, L)$ that was used in the proof of theorem 2.1.1. In particular, in section 3.2 we give the setup to obtaining a polynomial $Q(t, M, L)$ that satisfies the conditions of lemma 2.3.4, and show that the conditions lead to a system of equations. We then give a solution to the system of equations and show that it leads to the polynomial $Q(t, M, L)$ used in the proof of theorem 2.1.1. In section 3.3, we give a detailed step-by-step construction to the solution of the system of equations.

3.2 Main Idea of Construction

We now give the setup that leads to a system of equations whose solution leads to a polynomial that satisfies the conditions of lemma 2.3.4, and provide a solution to the system of equations and show that it leads to the polynomial $Q(t, M, L)$ used in the proof theorem 2.1.1.

3.2.1 The A-polynomial and Recurrence Relation for the figure 8 Knot

Firstly, we recall the A-polynomial and a recurrence relation for the Figure 8 knot, as we will build upon this information.

For the figure 8 knot, it is known that the A'_K the A-polynomial with out the $L-1$ factor, up to a power of M and L, is

$$A'_K = L + L^{-1} + \frac{-1 + M^2 + 2M^4 + M^6 - M^8}{M^4}.$$

As it will be needed for lemma 2.3.2, we see

$$(A'_K)^2 = ae_2(M)L^2 + ae_{-2}(M)L^{-2} + ae_1(M)L + ae_{-1}(M)L^{-1} + ae_0(M)$$

where

$$\begin{aligned} ae_2(M) &= 1 \\ ae_1(M) &= \frac{-2M^4 + 2M^6 + 4M^8 + 2M^{10} - 2M^{12}}{M^8} \\ ae_0(M) &= \frac{1 - 2M^2 - 3M^4 + 2M^6 + 10M^8 + 2M^{10} - 3M^{12} - 2M^{14} + M^{16}}{M^8} \\ ae_{-1}(M) &= \frac{-2M^4 + 2M^6 + 4M^8 + 2M^{10} - 2M^12}{M^8} \\ ae_{-2}(M) &= 1 \end{aligned}$$

We have the following Recurrence relation from [**?**], which can also be derived directly from proposition 2.4.1,

$$\alpha(t, M, L) = a_1(t, M)L + a_0(t, M) + a_{-1}(t, M)L^{-1} \tag{3.1}$$

where

$$\begin{aligned} a_1(t, M) &= t^{-2}M^2 - t^2M^{-2} \\ a_{-1}(t, M) &= t^2M^2 - t^{-2}M^{-2} \\ a_0(t, M) &= (M^2 - M^{-2})(-M^4 - M^{-4} + M^2 + M^{-2} + t^4 + t^{-4}). \end{aligned}$$

We see that $\alpha(t, M, L)J_K(N) \in R[M^{\pm 1}]$.

3.2.2 Set up and Obtaining Conditions

We wish to find $Q(t,M,L)$ such that $QJ_K(n) \in \mathcal{R}[M^{\pm 1}]$ and $\sigma(Q(t,M,L)) = Q(t,M,L)$, or more explicity $Q(t,M^{-1},L^{-1}) = Q(t,M,L)$.

To do this we will work off the recurrence relation 3.1, $\alpha(t,M,L)$. Namely we will multiply by a rational function $B(t,M,L)$ on the left and solve for $B(t,M,L)$ such that $B(t,M,L)\alpha(t,M,L)$ becomes $Q(t,M,L)$ in lemma 2.3.4.

We let $B(t,M,L)$ be the following,

$$B(t,M,L) = b_1(t,M)L + b_0(t,M) + b_{-1}(t,M)L^{-1}, \tag{3.2}$$

where $b_i(t,M)$ are rational functions in t and M. Multiplying the recurrence relation $\alpha(t,M,L)$ by $B(t,M,L)$ on the left, we have that $B(t,M,L)\alpha(t,M,L)$ is equal to

$$e_2(t,M)L^2 + e_1(t,M)L + e_0(t,M) + e_{-1}(t,M)L^{-1} + e_{-2}(t,M)L^{-2} \tag{3.3}$$

where

$$\begin{aligned}
e_2(t,M) &= b_1(t,M)a_1(t,t^2M)\\
e_1(t,M) &= b_0(t,M)a_1(t,M) + b_1(t,M)a_0(t,t^2M)\\
e_0(t,M) &= b_0(t,M)a_0(t,M) + b_{-1}(t,M)a_1(t,t^{-2}M) + b_1(t,M)a_{-1}(t,t^2M)\\
e_{-1}(t,M) &= b_0(t,M)a_{-1}(t,M) + b_{-1}(t,M)a_0(t,t^{-2}M)\\
e_{-2}(t,M) &= b_{-1}(t,M)a_{-1}(t,t^{-2}M).
\end{aligned}$$

We will denote the polynomial 3.3 as $Q(t,M,L)$. Note that $Q(t,M,L)J_K(n) \in \mathcal{R}(M)$, so the conditions that remains are $\sigma(Q(t,M,L)) = Q(t,M,L)$, and $Q(t,M,L)$ is a Luarent polynomial in t,M,L. We state the conditions explicitly in the next subsection.

3.2.3 Conditions as a System of Equations

Hence the conditions that remains for $Q(t, M, L)$ to satisfy the conditions of lemma 2.3.4 are,

$$e_2(t, M) = e_{-2}(t, M^{-1}) \tag{3.4}$$

$$e_1(t, M) = e_{-1}(t, M^{-1}) \tag{3.5}$$

$$e_0(t, M) = e_0(t, M^{-1}) \tag{3.6}$$

and

$$e_2(1, M) = ae_2(M) \tag{3.7}$$

$$e_1(1, M) = ae_1(M) \tag{3.8}$$

$$e_0(1, M) = ae_0(M) \tag{3.9}$$

$$e_{-1}(1, M) = ae_{-1}(M) \tag{3.10}$$

$$e_{-2}(1, M) = ae_{-2}(M). \tag{3.11}$$

We will call the first set of conditions the symmetry conditions, and the second set the initial conditions. We also note that e_i must be polynomial, we call this condition the polynomial conditions.

These are conditions on e_i that in turn become conditions on b_i.

3.2.4 Solutions to Conditions

We have the following proposition,

Proposition 3.2.1. *A solution to the system of equations above is the following,*

$$\begin{aligned}
b_1(t, M) &= \frac{M^2t^2}{(-1+Mt)(1+Mt)(1+M^2t^2)} \\
b_0(t, M) &= -\frac{(-1+M)(1+M)(1+M^2)t^4(-M^2-M^4-M^6+t^4+M^8t^4-M^4t^8)}{M^2(M-t)(M+t)(-1+Mt)(1+Mt)(M^2+t^2)(1+M^2t^2)} \\
b_{-1}(t, M) &= \frac{M^2t^2}{(M-t)(M+t)(M^2+t^2)}.
\end{aligned}$$

Moreover, this solution makes $Q(t,M,L) = B(t,M,L)\alpha(t,M,L)$ the polynomial used in the proof of theorem 2.1.1.

Proof. By direct verification. □

3.3 Explicitly Solving Equations

We now give detailed method in solving such a system of equations.

3.3.1 Step 1: Applying Symmetric Conditions

We first choose the leading coefficients e_2 and e_{-2} to be the same as the A-polynomial squared. Namely we have

$$e_2(t,M) = 1 \quad \text{and} \quad e_{-2}(t,M) = 1$$

which gives,

$$\begin{aligned} b_1(t,M) &= \frac{1}{a_1(t,t^2M)} \\ b_{-1}(t,M) &= \frac{1}{a_{-1}(t,t^{-2}M)} \end{aligned}$$

We see that this takes care of equation 3.4.

We then need $e_1(t,M) = e_{-1}(t,M^{-1})$, which gives,

$$b_0(t,M^{-1}) = -b_0(t,M) \tag{3.12}$$

One then notes that $e_0(t,M) = e_0(t,M^{-1})$ is already satfied with the conditions stated above. Meaning conditions 3.5 and 3.6, become equation 3.12.

3.3.2 Step 2: Applying Polynomial Conditions

We now use the fact that e_i must be polynomial. This imposes additional conditions on b_i, but it is hard to work with a rational equation in terms of b_i. So what we do is try to work with a equation in terms of e_i. To do this, we express b_i in terms of e_i.

Using the explicit equations for b_1, b_{-1} we see that

$$e_1(t,M) = \frac{a_0(t,Mt^2) + a_1(t,M)a_1(t,Mt^2)b_0(t,M)}{a_1(t,Mt^2)}$$

which is not necessarily a polynomial. Hence the need of more conditions. Solving for b_0 we have,

$$b_0(t,M) = -\frac{a_0(t,t^2M) + a_1(t,t^2M)e_1(t,M)}{a_1(t,M)a_1(t,t^2M)}.$$

We then take this, and substitue this into the expression(3.3) of $e_0(t,M)$. Which then gives the following diophantine equation in a polynomial ring,

$$A_1(t,M)e_0(t,M) + B_1(t,M)e_1(t,M) = C_1(t,M) \tag{3.13}$$

where

$$\begin{aligned}
A_1(t,M) &= -a_1(t,M)a_1(t,Mt^2)a_{-1}(t,Mt^{-2}) \\
B_1(t,M) &= a_0(t,M)a_1(t,Mt^2)a_{-1}(t,Mt^{-2}) \\
C_1(t,M) &= -\Big(a_1(t,M)a_1(t,Mt^{-2})a_1(t,Mt^2) + a_0(t,M)a_0(t,Mt^2)a_{-1}(t,Mt^{-2}) \\
&\quad -a_1(t,M)a_{-1}(t,Mt^{-2})a_{-1}(t,Mt^2)\Big)
\end{aligned}$$

Note in arriving to this equation, we did not use $b_0(t,M^{-1}) = b_0(t,M)$, meaning we will have to implement this condition later.

3.3.3 Step 3: Checking if Equation has solution

Ideal Membership Problem

We see that the equation

$$A_1(t,M)e_0(t,M) + B_1(t,M)e_1(t,M) = C_1(t,M)$$

is only satisfied if $C_1(t,M)$ is in the ideal generated by $A_1(t,M)$ and $B_1(t,M)$. This type of problem is known as the Ideal membership problem. One way to solve the ideal membership problem is to use Groebner Basis.

We also note that this condition is satisfied when $t=-1$. Namely the equation

$$A_1(1,M)e_0(1,M) + B_1(1,M)e_1(1,M) = C_1(1,M)$$

holds true.

Proposition 3.3.1. *$A_1(t,M)$ and $B_1(t,M)$ are in the ideal of $C_1(t,M)$.*

Proof. The proof is done explicity in the next subsection using Groebner basis. □

Groebner Basis

We now use Groebner basis to indeed show that $A_1(t,M), B_1(t,M)$ is in the ideal of $C_1(t,M)$. For calculation speed purposes we make things strictly polynomial instead of Laurent polynomial. We do this by multiplying by factors of M and t. This gives,

$$\begin{aligned} A_2(t,M) &= M^{10}t^8 A_1(t,M) \\ B_2(t,M) &= M^{10}t^8 B_1(t,M) \\ C_2(t,M) &= M^{10}t^8 C_1(t,M). \end{aligned}$$

By direct calculation we can find the Groebner basis of $A_2(t,M), B_2(t,M)$. Namely we use the following command in Mathematica

$$\{gb, mat\} = GroebnerBasis`BasisAndConversionMatrix[A_2(t,M), B_2(t,M), t, M, \{\}].$$

Where $gb = \{g_1, g_2, g_3, g_4\}$ is the Groebner basis obtained from A_2, B_2, and mat is the matrix that allows us to write the Groebner basis in terms of A_2, B_2. Namely we have that

$$mat.\{A_2, B_2\} - gb = 0.$$

Explicitly, we have the following groebner basis for A_2 and B_2,

$$\begin{aligned}
g_1 &= -M^{14}(-1+M^8)(t^4+M^8t^4-M^4(1+t^8)) \\
g_2 &= -M^{10}(-1+M^8)t^2(t^4+M^8t^4-M^4(1+t^8)) \\
g_3 &= (-1+M)(1+M)(1+M^2)(M-t)(M+t)(-1+Mt)(1+Mt)(M^2+t^2)(1+M^2t^2) \\
&\quad \times(M^4+M^6+M^8+2M^{10}+3M^{12}+M^{14}+2M^{16}-t^4) \\
g_4 &= t^2(-M+t)(M+t)(-1+Mt)(1+Mt)(M^2+t^2)(1+M^2t^2) \\
&\quad \times(-M^4-M^6-M^8+M^{14}+M^{16}+t^4).
\end{aligned}$$

And for the *mat* matrix, we have m_{ij} as its entries with the values,

$$\begin{aligned}
m_{11} &= (-1+M)(1+M)(1+M^2)t^2 \\
&\quad \times(1-2M^2+M^4-2M^6+2M^8-M^{10}+M^{12}-M^14-M^4t^4+M^6t^4+M^{10}t^4) \\
m_{12} &= -M^4(M^6+M^{10}+t^4-M^2t^4-M^6t^4) \\
m_{21} &= -(-1+M)(1+M)(1+M^2) \\
&\quad \times(1-M^2-M^6-M^8-M^6t^4-M^{10}t^4+M^{12}t^4-M^8t^8) \\
m_{22} &= M^4t^2(-1+M^8-M^4t^4) \\
m_{31} &= (-1+M)(1+M)(1+M^2)t^2 \\
&\quad \times(-4+M^2-2M^4+3M^6-2M^8+M^{10}-2M^{12}+3M^4t^4+M^6t^4+2M^8t^4) \\
m_{32} &= 1+M^2+M^4+2M^6+3M^8+M^{10}+2M^{12}-3M^4t^4-M^6t^4-2M^8t^4 \\
m_{41} &= -2-M^2-M^4+2M^8+M^{10}-M^2t^4-M^4t^4+M^8t^4+2M^{10}t^4-M^{14}t^4 \\
&\quad -M^4t^8-M^6t^8+M^8t^8+M^{10}t^8 \\
m_{42} &= -(1+M^2)t^2(1-M^8+M^4t^4).
\end{aligned}$$

Now to check if C_2 is in the Ideal of A_2, B_2 we use

$$\{q,r\} = PolynomialReduce[C_2, gb, t, M].$$

One would see that $r = 0$, which means that

$$C_2 = q_1 g_1 + q_2 g_2 + q_3 g_3 + q_4 g_4,$$

and hence $C_2(t, M)$ is in the ideal generated by $A_2(t, M), B_2(t, M)$. The explicit values of q_i are,

$$\begin{aligned} q_1 &= M^{-4}t^{-10}(3 + M^2 + 2M^4 + t^4 - M^2t^4 - M^4t^4 - 2M^6t^4 - 4t^8 - 6M^2t^8 \\ &\quad -7M^4t^8 - 7M^6t^8 - 4M^8t^8 - 4M^{10}t^8 - 5t^{12} - 7M^2t^{12} - 6M^4t^{12} \\ &\quad -7M^6t^{12} - 2M^8t^{12} - 4t^{16} - 6M^2t^{16} - 3M^4t^{16} - 5M^6t^{16} - M^8t^{16} - 2M^{10}t^{16} \\ &\quad -t^{20} - 2M^2t^{20} - 2M^4t^{20} - 3M^6t^{20} - 2M^8t^{20} + M^4t^{24} + M^6t^{24} + M^8t^{24} + 2M^{10}t^{24}) \\ q_2 &= -M^{-4}t^{-8}(-1 - M^2 + 4t^4 + 8M^2t^4 + 4t^8 + 6M^2t^8 + 4t^{12} + 7M^2t^{12} - t^{20}) \\ q_3 &= M^{-4}t^{-10}(-1 + M^2 + M^6 + M^2t^4 - M^8t^4 - 2t^8 - M^4t^8 - M^6t^8 - 2M^8t^8 - M^{10}t^8 \\ &\quad -2M^{12}t^8 - 2t^{12} - M^4t^{12} - M^6t^{12} - 3M^8t^{12} - M^{10}t^{12} - 2t^{16} - 2M^4t^{16} \\ &\quad -2M^8t^{16} - M^{12}t^{16} - M^8t^{20} - M^{10}t^{20} + M^{12}t^{24}) \\ q_4 &= -(2(1 - t + t^2)(1 + t + t^2)(1 - t^2 + t^4)(M^{-4}t^{-4}). \end{aligned}$$

3.3.4 Step 5: Finding Solutions

In this step, we prove the following proposition.

Proposition 3.3.2. *A family of solutions to $A_1(t, M)x(t, M) + B_1(t, M)y(t, M) = C_1(t, M)$ is*

$$ge_0(t, M) = pe_0(t, M) - \frac{f(t, M)B_2(t, M)}{gcd(t, M)} \quad (3.14)$$

$$ge_1(t, M) = pe_1(t, M) + \frac{f(t, M)A_2(t, M)}{gcd(t, M)}. \quad (3.15)$$

where

$$
\begin{aligned}
pe_0(t,M) &= \frac{1}{M^4t^{10}}(-M^4t^2 - M^6t^2 - M^8t^2 + 2M^{12}t^2 + M^{14}t^2 + t^6 \\
&- M^2t^6 + 14M^4t^6 + 5M^6t^6 - 2M^8t^6 + 3M^{10}t^6 - 10M^{12}t^6 - 6M^{14}t^6 \\
&- M^{16}t^6 - M^{18}t^6 - 12t^{10} + 6M^2t^{10} + 18M^4t^{10} + 17M^6t^{10} + 6M^8t^{10} \\
&- 12M^{10}t^{10} - 14M^{12}t^{10} - 19M^{14}t^{10} + 4M^{16}t^{10} + 8M^{18}t^{10} - 13t^{14} \\
&+ 6M^2t^{14} + 35M^4t^{14} + 23M^6t^{14} + 2M^8t^{14} - 10M^{10}t^{14} - 26M^{12}t^{14} \\
&- 25M^{14}t^{14} + 4M^{16}t^{14} + 6M^{18}t^{14} - 12t^{18} + 6M^2t^{18} + 20M^4t^{18} + 17M^6t^{18} \\
&+ 3M^8t^{18} - 12M^{10}t^{18} - 15M^{12}t^{18} - 18M^{14}t^{18} + 4M^{16}t^{18} + 7M^{18}t^{18} \\
&- M^2t^{22} + 13M^4t^{22} + 7M^6t^{22} - 3M^8t^{22} - 10M^{12}t^{22} - 6M^{14}t^{22} + 2M^4t^{26} \\
&- M^6t^{26} - 2M^8t^{26} - M^{10}t^{26} + 2M^{12}t^{26} + 2M^{14}t^{26} - 2M^{16}t^{26} - 2M^8t^{30} \\
&\quad +2M^{12}t^{30}) \\
pe_1(t,M) &= \frac{1}{M^4t^{10}}(-1 + M^2t^4 + M^8t^4 + M^{10}t^4 + 2M^{12}t^4 + M^{14}t^4 + M^6t^8 \\
&- 14M^8t^8 - 7M^{10}t^8 - 12M^{12}t^8 - 8M^{14}t^8 + 13M^4t^{12} + 6M^6t^{12} - M^8t^{12} \\
&+ 2M^{10}t^{12} - 10M^{12}t^{12} - 6M^{14}t^{12} + 13M^4t^{16} + 7M^6t^{16} - 2M^8t^{16} \\
&- 11M^{12}t^{16} - 7M^{14}t^{16} + 12M^4t^{20} + 6M^6t^{20} + 10M^8t^{20} + 6M^{10}t^{20} \\
&+ M^6t^{24} + 2M^{12}t^{24} - 2M^8t^{28}).
\end{aligned}
$$

Proof. By direct verification. □

In the following subsection, we show explicitly the method used to obtain this family of solutions.

Diophantine Equation

We see that the equation

$$A_1(t,M)e_0(t,M) + B_1(t,M)e_1(t,M) = C_1(t,M)$$

is reminiscent of a Diophantine equation.

Recall for $a, b, c, x, y \in \mathbb{Z}$, the equation

$$ax + by = c$$

has the following solutions,

$$\begin{aligned} x &= x_0 + \frac{b}{gcd(a,b)}k \\ y &= y_0 - \frac{a}{gcd(a,b)}k \end{aligned}$$

where x_0, y_0 are particular solutions to the original equation, $k \in \mathbb{Z}$, and $gcd(a, b)$ is the greatest common divisor between a, b.

We will apply the same ideas to our problem, except our ring is the ring polynomials.

Particular Solution

To find a particular solution of e_i, we express $C_2(t, M)$ in terms of the Groebner basis, which we then in turn express the Groebner basis in terms of $A_2(t, M), B_2(t, M)$ using *mat*. Namely, the *mat* matrix tells that we have,

$$\begin{aligned} g_1 &= m_{11}A_2(t, M) + m_{12}B_2(t, M) \\ g_2 &= m_{21}A_2(t, M) + m_{22}B_2(t, M) \\ g_3 &= m_{31}A_2(t, M) + m_{32}B_2(t, M) \\ g_4 &= m_{41}A_2(t, M) + m_{42}B_2(t, M). \end{aligned}$$

Where m_{ij} are the values of the *mat* matrix. With this we have that ($q_i m_j k$ are two different quantities multiplying each other),

$$A_2e_0 + B_2e_1 = (q_1m_{11} + q_2m_{21} + q_3m_{31} + q_4m_{41})A_2 + (q_1m_{12} + q_2m_{22} + q_3m_{32} + q_4m_{42})B_2.$$

This implies that

$$
\begin{aligned}
e_0 &= q_1 m_{11} + q_2 m_{21} + q_3 m_{31} + q_4 m_{41} \\
e_1 &= q_1 m_{12} + q_2 m_{22} + q_3 m_{32} + q_4 m_{42}
\end{aligned}
$$

More explicitly, we have ($A_i e_j$ are two seperate quantities multiplying each other)

$$
\begin{aligned}
A_2 e_0 + B_2 e_1 &= \frac{1}{M^4 t^{10}} B_2(-1 + M^2 t^4 + M^8 t^4 + M^{10} t^4 + 2M^{12} t^4 + M^{14} t^4 + M^6 t^8 \\
&- 14M^8 t^8 - 7M^{10} t^8 - 12M^{12} t^8 - 8M^{14} t^8 + 13M^4 t^{12} + 6M^6 t^{12} - M^8 t^{12} \\
&+ 2M^{10} t^{12} - 10M^{12} t^{12} - 6M^{14} t^{12} + 13M^4 t^{16} + 7M^6 t^{16} - 2M^8 t^{16} \\
&- 11M^{12} t^1 6 - 7M^{14} t^{16} + 12M^4 t^{20} + 6M^6 t^{20} + 10M^8 t^{20} + 6M^{10} t^{20} \\
&+ M^6 t^{24} + 2M^{12} t^{24} - 2M^8 t^{28}) \\
&+ \frac{1}{M^4 t^{10}} A_2(-M^4 t^2 - M^6 t^2 - M^8 t^2 + 2M^{12} t^2 + M^{14} t^2 + t^6 \\
&- M^2 t^6 + 14M^4 t^6 + 5M^6 t^6 - 2M^8 t^6 + 3M^{10} t^6 - 10M^{12} t^6 - 6M^{14} t^6 \\
&- M^{16} t^6 - M^{18} t^6 - 12t^{10} + 6M^2 t^{10} + 18M^4 t^{10} + 17M^6 t^{10} + 6M^8 t^{10} \\
&- 12M^{10} t^{10} - 14M^{12} t^{10} - 19M^{14} t^{10} + 4M^{16} t^{10} + 8M^{18} t^{10} - 13t^{14} \\
&+ 6M^2 t^{14} + 35M^4 t^{14} + 23M^6 t^{14} + 2M^8 t^{14} - 10M^1 0t^{14} - 26M^{12} t^{14} \\
&- 25M^{14} t^{14} + 4M^{16} t^{14} + 6M^{18} t^{14} - 12t^{18} + 6M^2 t^{18} + 20M^4 t^{18} + 17M^6 t^{18} \\
&+ 3M^8 t^{18} - 12M^{10} t^{18} - 15M^{12} t^{18} - 18M^{14} t^{18} + 4M^{16} t^{18} + 7M^{18} t^{18} \\
&- M^2 t^{22} + 13M^4 t^{22} + 7M^6 t^{22} - 3M^8 t^{22} - 10M^{12} t^{22} - 6M^{14} t^{22} + 2M^4 t^{26} \\
&- M^6 t^{26} - 2M^8 t^{26} - M^{10} t^{26} + 2M^{12} t^{26} + 2M^{14} t^{26} - 2M^{16} t^{26} - 2M^8 t^{30} \\
&+ 2M^{12} t^{30}).
\end{aligned}
$$

which then gives the following particular solutions,

$$\begin{aligned}
pe_0(t,M) &= \frac{1}{M^4t^{10}}(-M^4t^2 - M^6t^2 - M^8t^2 + 2M^{12}t^2 + M^{14}t^2 + t^6 \\
&- M^2t^6 + 14M^4t^6 + 5M^6t^6 - 2M^8t^6 + 3M^{10}t^6 - 10M^{12}t^6 - 6M^{14}t^6 \\
&- M^{16}t^6 - M^{18}t^6 - 12t^{10} + 6M^2t^{10} + 18M^4t^{10} + 17M^6t^{10} + 6M^8t^{10} \\
&- 12M^{10}t^{10} - 14M^{12}t^{10} - 19M^{14}t^{10} + 4M^{16}t^{10} + 8M^{18}t^{10} - 13t^{14} \\
&+ 6M^2t^{14} + 35M^4t^{14} + 23M^6t^{14} + 2M^8t^{14} - 10M^{10}t^{14} - 26M^{12}t^{14} \\
&- 25M^{14}t^{14} + 4M^{16}t^{14} + 6M^{18}t^{14} - 12t^{18} + 6M^2t^{18} + 20M^4t^{18} + 17M^6t^{18} \\
&+ 3M^8t^{18} - 12M^{10}t^{18} - 15M^{12}t^{18} - 18M^{14}t^{18} + 4M^{16}t^{18} + 7M^{18}t^{18} \\
&- M^2t^{22} + 13M^4t^{22} + 7M^6t^{22} - 3M^8t^{22} - 10M^{12}t^{22} - 6M^{14}t^{22} + 2M^4t^{26} \\
&- M^6t^{26} - 2M^8t^{26} - M^{10}t^{26} + 2M^{12}t^{26} + 2M^{14}t^{26} - 2M^{16}t^{26} - 2M^8t^{30} \\
&+2M^{12}t^{30}) \\
pe_1(t,M) &= \frac{1}{M^4t^{10}}(-1 + M^2t^4 + M^8t^4 + M^{10}t^4 + 2M^{12}t^4 + M^{14}t^4 + M^6t^8 \\
&- 14M^8t^8 - 7M^{10}t^8 - 12M^{12}t^8 - 8M^{14}t^8 + 13M^4t^{12} + 6M^6t^{12} - M^8t^{12} \\
&+ 2M^{10}t^{12} - 10M^{12}t^{12} - 6M^{14}t^{12} + 13M^4t^{16} + 7M^6t^{16} - 2M^8t^{16} \\
&- 11M^{12}t^{16} - 7M^{14}t^{16} + 12M^4t^{20} + 6M^6t^{20} + 10M^8t^{20} + 6M^{10}t^{20} \\
&+ M^6t^{24} + 2M^{12}t^{24} - 2M^8t^{28}).
\end{aligned}$$

One can explicitly check that indeed,

$$A_1(t,M)pe_0(t,M) + B_1(t,M)pe_1(t,M) = C_1(t,M)$$

holds true.

Family of Solutions

This then gives us a family of solutions with the following form,

$$
\begin{aligned}
ge_0(t,M) &= pe_0(t,M) - \frac{f(t,M)B_2(t,M)}{gcd(t,M)} \\
ge_1(t,M) &= pe_1(t,M) + \frac{f(t,M)A_2(t,M)}{gcd(t,M)}.
\end{aligned}
$$

where $gcd(t,M)$ is the greatest common divisor between A_2, B_2, and f is a arbitrary polynomial that we can vary to adjust solutions. We now need to adjust f so that ge_0, ge_1 satisfy the conditions needed. Namely the condition of $b_0(t,M^{-1}) = -b_0(t,M)$.

3.3.5 Step 6: Imposing Conditions on Solutions

We now implement the condition of $b_0(t,M^{-1}) = -b_0(t,M)$. Using the fact that we can express b_0 in terms of e_1, we have

$$
\begin{aligned}
e_1(t,M) &= \frac{1}{M^4t^{10}(-1+M^4t^4)}(1+M^{12}-M^2t^4-M^4t^4-M^8t^4-M^{10}t^4-M^4t^8-M^8t^8 \\
&+ M^4t^{16}+M^8t^{16}+M^2t^{20}+M^4t^{20}+M^8t^{20}+M^{10}t^{20}-t^{24}-M^{12}t^{24}) \\
&+ \frac{(M^8t^{10}-M^4t^{14})e_1(t,M^{-1})}{M^4t^{10}(-1+M^4t^4)}.
\end{aligned}
$$

Now using our set of solutions 3.15 with its particular form, we have

$$
\begin{aligned}
f(t,M) &= \frac{1}{M^{14}t^8}\Big(-1-2M^2-M^4-M^6-M^{10}-M^{12}-2M^{14}-M^{16} \\
&+ 8t^4+12M^2t^4+6M^4t^4+12M^6t^4-2M^8t^4+12M^{10}t^4+6M^{12}t^4+12M^{14}t^4 \\
&+ 8M^{16}t^4+6t^8+10M^2t^8+6M^4t^8+13M^6t^8+13M^{10}t^8+6M^{12}t^8+10M^{14}t^8 \\
&+ 6M^{16}t^8+7t^{12}+11M^2t^{12}+6M^4t^{12}+12M^6t^{12}+12M^{10}t^{12}+6M^{12}t^{12} \\
&+ 11M^14t^{12}+7M^{16}t^{12}+M^4t^{16}+M^{12}t^{16}-2M^2t^{20}-2M^{14}t^{20}-M^2t^8f(t,M^{-1})\Big)
\end{aligned} \tag{3.16}
$$

We now use the fact that we know

$$f(t,M) = g(M) + (t^2-1)h(t,M)$$

where

$$g(M) = \frac{1 - M^2 + 35M^4 + 18M^6 + 29M^8 + 20M^{10}}{M^8},$$

and $h(t, M)$ is an unknown function. It should be noted that $g(m)$ comes from the fact that we know $ge_1(1, M) = ae_1(M)$. Now it remains to find $h(t, M)$ explicitly.

Collecting and factoring equation 3.16 with the change of variables $h(t, M) = M^{-6}k(t, M)$, we have

$$\begin{aligned}
- \quad & \frac{1}{M^{14}t^8}(1 + 2M^2 + M^4 + M^6 + M^{10} + M^{12} + 2M^{14} + M^{16} + t^2 + 2M^2t^2 + M^4t^2 \\
+ \quad & M^6t^2 + M^{10}t^2 + M^{12}t^2 + 2M^{14}t^2 + M^{16}t^2 - 7t^4 - 10M^2t^4 - 5M^4t^4 - 11M^6t^4 \\
+ \quad & 2M^8t^4 - 11M^{10}t^4 - 5M^{12}t^4 - 10M^{14}t^4 - 7M^{16}t^4 - 7t^6 - 10M^2t^6 - 5M^4t^6 - 11M^6t^6 \\
- \quad & 5M^{12}t^6 - 10M^{14}t^6 - 7M^{16}t^6 + 7t^8 + 9M^2t^8 + 7M^4t^8 + 12M^6t^8 + 12M^{10}t^8 + 7M^12t^8 \\
+ \quad & 9M^{14}t^8 + 7M^{16}t^8 + 7t^{10} + 9M^2t^{10} + 7M^4t^{10} + 12M^6t^{10} + 12M^{10}t^{10} + 7M^{12}t^{10} \\
+ \quad & 9M^{14}t^{10} + 7M^{16}t^{10} - 2M^2t^{12} + M^4t^{12} + M^{12}t^{12} - 2M^{14}t^{12} - 2M^2t^{14} + M^4t^{14} \\
+ \quad & M^{12}t^{14} - 2M^{14}t^{14} - 2M^2t^{16} - 2M^{14}t^{16} - 2M^2t^{18} - 2M^{14}t^{18} + 2M^8t^6 - 11M^{10}t^6) \\
& + \frac{k(t, M^{-1})}{M^6} + \frac{k(t, M)}{M^6} = 0
\end{aligned}$$

With this, we can now split up the constant term and can label one half as $k(t,M)$ and also multiply by M^6. This then gives us the following,

$$
\begin{aligned}
&- \frac{2t^4+2t^6}{t^8} - \frac{1+t^2-7t^4-7t^6+7t^8+7t^{10}}{M^8t^8} - \frac{M^8(1+t^2-7t^4-7t^6+7t^8+7t^{10})}{t^8} \\
&- \frac{1+t^2-11t^4-11t^6+12t^8+12t^{10}}{M^2t^8} - \frac{M^2(1+t^2-11t^4-11t^6+12t^8+12t^{10})}{t^8} \\
&- \frac{1+t^2-5t^4-5t^6+7t^8+7t^{10}+t^{12}+t^{14}}{M^4t^8} \\
&- \frac{M^4(1+t^2-5t^4-5t^6+7t^8+7t^{10}+t^{12}+t^{14})}{t^8} \\
&- \frac{2+2t^2-10t^4-10t^6+9t^8+9t^{10}-2t^{12}-2t^{14}-2t^{16}-2t^{18}}{M^6t^8} \\
&- \frac{M^6(2+2t^2-10t^4-10t^6+9t^8+9t^{10}-2t^{12}-2t^{14}-2t^{16}-2t^{18})}{t^8}.
\end{aligned}
$$

Noting the symmetry in the constant term, we can split the terms using looking at the positive and negative powers of M.

Which then gives the following,

$$
\begin{aligned}
k(t,M) &= -\frac{-t^4-t^6}{t^8} - \frac{M^2(-1-t^2+11t^4+11t^6-12t^8-12t^{10})}{t^8} \\
&- \frac{M^8(-1-t^2+7t^4+7t^6-7t^8-7t^{10})}{t^8} \\
&- \frac{M^4(-1-t^2+5t^4+5t^6-7t^8-7t^{10}-t^{12}-t^{14})}{t^8} \\
&- \frac{M^6(-2-2t^2+10t^4+10t^6-9t^8-9t^{10}+2t^{12}+2t^{14}+2t^{16}+2t^{18})}{t^8}
\end{aligned}
$$

with this we have an explicit solution to $f(t,M)$, which gives us explicit solutions to the b_i. Namely we have,

$$
\begin{aligned}
b_1(t,M) &= \frac{1}{a_1(t,t^2M)} \\
b_{-1}(t,M) &= \frac{1}{a_{-1}(t,t^{-2}M)} \\
b_0(t,M) &= -\frac{(-1+M)(1+M)(1+M^2)t^4(-M^2-M^4-M^6+t^4+M^8t^4-M^4t^8)}{M^2(M-t)(M+t)(-1+Mt)(1+Mt)(M^2+t^2)(1+M^2t^2)}.
\end{aligned}
$$

Recall that the polynomial we are constructing is

$$e_2(t,M)L^2 + e_1(t,M)L + e_0(t,M) + e_{-1}(t,M)L^{-1} + e_{-2}(t,M)L^{-2}$$

where

$$\begin{aligned}
e_2(t,M) &= b_1(t,M)a_1(t,t^2M)\\
e_1(t,M) &= b_0(t,M)a_1(t,M) + b_1(t,M)a_0(t,t^2M)\\
e_0(t,M) &= b_0(t,M)a_0(t,M) + b_{-1}(t,M)a_1(t,t^{-2}M) + b_1(t,M)a_{-1}(t,t^2M)\\
e_{-1}(t,M) &= b_0(t,M)a_{-1}(t,M) + b_{-1}(t,M)a_0(t,t^{-2}M)\\
e_{-2}(t,M) &= b_{-1}(t,M)a_{-1}(t,t^{-2}M)
\end{aligned}$$

Hence having now explicit expressions for b_i, we have that the following polynomial,

$$L^2 + L^{-2} + e_1(t,M)L + e_{-1}(t,M)L^{-1} + e_0(t,M)$$

where

$$\begin{aligned}
e_1(t,M) &= \frac{-1-t^{16}+M^4t^8(1+t^4)^2+M^2(t^4+t^{12})+M^6(t^8+t^{16})-M^8(t^{12}+t^{20})}{M^4t^{10}}\\
e_0(t,M) &= \frac{1}{M^8t^4}\Big(t^8+M^{16}t^8-M^4t^8(2+t^4)-M^{12}t^8(2+t^4)-M^2(t^4+t^8)\\
&\quad +M^6(t^4+t^8)+M^{10}(t^4+t^8)-M^{14}(t^4+t^8)+2M^8(1+t^4+2t^8+t^{12})\Big)\\
e_{-1}(t,M) &= \frac{M^4t^8(1+t^4)^2-t^{12}(1+t^8)+M^6(t^4+t^{12})-M^8(1+t^{16})+M^2(t^8+t^{16})}{M^4t^{10}}.
\end{aligned}$$

This completes the construction of the polynomial used in the proof of theorem 2.1.1.

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