EXISTENCE AND BIFURCATION OF PERIODIC SOLUTIONS IN SECOND ORDER NONLINEAR SYSTEMS: BROUWER EQUIVARIANT DEGREE METHOD

by

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This dissertation is dedicated to my family.

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EXISTENCE AND BIFURCATION OF PERIODIC SOLUTIONS

IN SECOND ORDER NONLINEAR SYSTEMS:

BROUWER EQUIVARIANT DEGREE METHOD

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We discuss the existence of periodic solutions in two types of symmetric second order nonlinear systems:

(a): Γ-symmetric autonomous system

$$\ddot{x}(t) = f(x(t)), x \in V = \mathbb{R}^n,$$

where V is an orthogonal Γ-representation and $f : V \to V$ is a Γ-equivariant map.

(b): The Γ-symmetric second order system of nonlinear difference equations:

$$\triangle^2 \mathbf{x}_{n-1} + f(n, \mathbf{x}_n) = 0, \quad \mathbf{x}_n \in \mathbb{R}^k, \; n \in \mathbb{Z},$$

where w is an orthogonal Γ-representation and $f : \mathbb{Z} \times W \to W$ a Γ-equivariant map.

Under some additional assumptions, we establish for (a) and (b) the existence of periodic solutions with fixed period. For (b), we also consider bifurcation problem for which we provide the topological classification of various symmetric types of solutions.

The applied method is the usage of the equivariant Brouwer degree to associate (a) and (b) with appropriate equivariant topological invariant. Several concrete examples provide illustrations for the abstract results.

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CHAPTER 1

INTRODUCTION

1.1 Motivation

Mathematical models of natural phenomena are often expressed as differential systems with symmetries. In particular, the problem of establishing the existence of periodic solutions in such systems is of immense importance to several applied areas. It is related to such events as occurrence of vibrations in mechanical systems (including structural vibrations), molecular motions, fluctuations in transmission lines (power lines, internet), machine chatters, oscillations in electronic circuits and neural networks, rises and falls of populations, reoccurrence of climate changes, internal organs rhythmic functions, etc.

Symmetric features of such models are expressed in the form of a spacial symmetry group Γ of the corresponding dynamical system. These symmetries have an impact on symmetric properties of the actual dynamics. In the context relevant to our discussion, this impact manifests itself as the so-called spatio-temporal patterns of periodic solutions. Developing mathematical tools allowing an effective classification of these patterns is a challenging problem, which is the main *motivation* for this dissertation.

1.2 Subject

The following two problems are the main focus of this dissertation:

(a): Existence, bifurcation and symmetric classifications of periodic solutions to second order autonomous symmetric systems of Ordinary Differential Equations(ODEs) of the type(in general, without variational structure)

$$\ddot{x}(t) = f(x(t)), \quad x(t) \in \mathbb{R}^k =: V, \tag{1.1}$$

where V is an orthogonal Γ-representation and $f : V \to V$ is a C^1-differentiable odd map commuting with the Γ-action on V.

(b): Existence and symmetric classifications of periodic and subharmonic solutions to periodic reversible second order symmetric difference systems (in general, without variational structure) of the type

$$\triangle^2 \mathbf{x}_{n-1} + f(n, \mathbf{x}_n) = 0, \quad n \in \mathbb{Z},\ \mathbf{x}_n \in \mathbb{R}^k =: V, \tag{1.2}$$

where f is a continuous odd map, being p-periodic ($p \in \mathbb{N}$), reversible (with respect to the variable n) and commuting with the Γ-action on V.

1.3 Method

For the equation (1.1), under the assumption that $f(x) = \nabla\varphi(x)$ (i.e. the system (1.1) is Newtonian), there is a vast collection of topological variational methods (for such systems with or without symmetries) to detect periodic solutions. Among others, let us mention the methods based on Lusternik-Schnirelman theory and Morse theory (including the classical one, Conley index and Floer homology, cf. [9; 13; 14]). On the other hand, some progress was achieved using the equivariant gradient degree theory (cf. [15; 16; 48; 49; 50; 52; 51]. However, in the case (1.1) doesn't admit variational structure, there are not so many methods designed for dealing with the existence of periodic solutions in second order autonomous systems with/without symmetries. In addition, the methods originated in equivariant singularity theory (due to their local character) cannot be applied either. As it is also well-known, the classical Brouwer degree does not detect periodic solutions in autonomous systems – it is insensitive to the S^1-symmetry related to the time-shift.

New important advances in the equivariant degree theory, which in the last 20 years has undergone systematic development, opened doors to effectively studying the problem (a) by means of the so-called Brouwer equivariant degree [7; 5]. To be more specific, the problem (a) can be reformulated as a G-equivariant fixed-point problem in an appropriate functional space, e.g. the Sobolev space $\mathscr{H} := H^2_{2\pi}(\mathbb{R}; V)$, with respect to the action of the group $G := \Gamma \times \mathbb{Z}_2 \times O(2)$ on $\mathscr{H}$ (where $O(2)$ acts on $\mathscr{H}$ by time-shifting and time reversion, while $\mathbb{Z}_2 = \{1, -1\}$ acts by simple multiplication and is related to the oddness of f). The proposed method also allows to treat the degenerate case, when (1.1) admits resonances/degeneracy at zero.

Although the research on difference equation (1.2) has a long history, up to the recent years there was a little progress achieved regarding the *multiple* periodic solutions in such systems. Discrete nature of difference systems doesn't allow a direct application of the classical methods such as critical point theory. For the first time, such methods were effectively developed in the second order difference systems with variational structure/discrete Hamiltonian systems. However, till very recently, there were no results obtained on multiple periodic solutions to difference systems (1.2) *without* assuming additional variational structure. In [2], the authors initiated applications of the equivariant Brouwer degree to address the multiplicity problem in system (1.2). Similarly to the case of the continous system (1.1), one can reformulate the problem of existence of subharmonic pm-periodic solutions for (1.2) as an equivariant fixed-point problem with respect to the action of the group $G := \Gamma \times \mathbb{Z}_2 \times D_m$, where D_m acts by shifts and reversion, while $\mathbb{Z}_2$ is related to the oddness of f.

Surprisingly, in spite of completely different character of the problems (a) and (b), they can be treated in an absolutely parallel way. The associated with these two problems equivariant topological invariants provide effective symmetric classification of periodic subharmonic solutions to both problems.

1.4 Results

Under the additional different version of Nagumo condition, the following results for both problems (a) and (b) have been established:

- a priori estimates for periodic/subharmonic solutions;
- computational formulae for the associated equivariant invariants;
- abstract existence results providing symmetric classification of periodic.subharmonic solutions.

All abstract results have been supported by concrete examples admitting dihedral and octahedral symmetries. In addition, for the problem (a) we succeeded to treat the system (1.1) with *degeneracy* and/or *resonances*, while for the problem (b) we obtained the *local* and *global* bifurcation result.

1.5 Overview

After the Introduction, the dissertation is organized as follows. In Chapter 2, we present the Preliminaries, where we include the necessary equivariant degree background. In Chapter 3, we study the system (1.1), and Chapter 4 is devoted to the problem (1.2).

CHAPTER 2

PRELIMINARIES

Throughout this section, we assume that G is a compact Lie group.

2.1 Equivariant Jargon

For a subgroup $H \leq G$ (which is always assumed to be closed), denote by $N(H)$ the normalizer of H in G, by $W(H) = N(H)/H$ the Weyl group of H in G, and by (H) the conjugacy class of H in G. The set $\Phi(G)$ of all conjugacy classes in G admits a partial order defined as follows: $(H) \leq (K)$ if and only if $gHg^{-1} \leq K$ for some $g \in G$. We will also put $\Phi_k(G) := \{(H) \in \Phi(G) : \dim W(H) = k\}$.

For a G-space X and $x \in X$, subgroup $G_x := \{g \in G : gx = x\}$ is called the isotropy of x and $G(x) := \{gx : g \in G\}$ is called the orbit of x. One can easily verify that $G(x) \cong G/G_x$. Denote by X/G the set of all orbits in X under the action of G equipped with the quotient topology, which is called the *orbit space* of X. Furthermore, we call conjugacy class (G_x) the orbit type of x in X and put $\Phi(G; X) := \{(H) \in \Phi(G) : H = G_x \text{ for some } x \in X\}$. Also, for a G-space X and a closed subgroup H of G, we adopt the following notations:

$$X_H := \{x \in X : G_x = H\},$$
$$X^H := \{x \in X : G_x \geq H\},$$

where X^H is called H-fixed-point subspace of X.

Let X and Y be two G-spaces. A continuous map $f : X \to Y$ is said to be equivariant if $f(gx) = gf(x)$ for all $x \in X$ and $g \in G$. If the G-action on Y is trivial, then f is called *invariant*, i.e. $f(gx) = f(x)$ for all $x \in X$ and $g \in G$. In a standard way, one can define a concept of G-homotopy. As is known (see, for instance, [7] and [12]), for any subgroup

$H \leq G$ and equivariant map $f : X \to Y$, the map $f^H : X^H \to Y^H$, where $f^H := f|_{X^H}$, is well-defined and $W(H)$-equivariant.

Let $L \leq H \leq G$. We put $N(L, H) := \{g \in G : gLg^{-1} \leq H\}$. Then, if (L), $(H) \in \Phi_0(G)$, the number

$$n(L, H) = |N(L, H)/N(H)|,$$

where $|X|$ stands for the cardinality of X, is well defined and finite (see, for example, [28]). In the case $(L), (H) \in \Phi_0(G), n(L, H)$ is equal to the number of subgroups $H' \leq G$ such that $H \sim H'$ and $L \leq H'$. For more information about the number $n(L, H)$ and its properties, we refer to [7].

2.2 Isotypical Decomposition of Finite-Dimensional Representations

As is well-known, any compact group admits only countably many non-equivalent real irreducible representations. Therefore, given a compact Lie group G, we always assume that a complete list of all real irreducible G-representations, denoted by $\mathcal{V}_i$, $i = 0, 1, \dots$, is available. Let V be a finite-dimensional real G-representation. Without loss of generality, V can be assumed to be orthogonal . Then, one can represent V as the direct sum

$$V = V_0 \oplus V_1 \oplus \cdots \oplus V_r, \tag{2.1}$$

which is called the G-isotypical decomposition of V , where the isotypical component V_i is modeled on the irreducible G-representation $\mathcal{V}_i$, $i = 0, 1, \dots, r$, i.e., V_i contains all the irreducible subrepresentations of V which are equivalent to $\mathcal{V}_i$. Notice that for a G-equivariant linear map $A : V \to V$, $A(V_i) \subset V_i$, $i = 0, 1, 2, \dots, r$. We will denote by $\sigma(A)$ the spectrum of A and for $\mu \in \sigma(A)$, $E(\mu)$ will stand for the generalized eigenspace corresponding to μ. Clearly, $E(\mu)$ is G-invariant. Then we can put

$$m_i(\mu) := \dim\left(V_i \cap E(\mu)\right) / \dim \mathcal{V}_i, \tag{2.2}$$

and will call the number $m_i(\mu)$ the $\mathcal{V}_i$-multiplicity of the eigenvalue μ.

Given an orthogonal G-representation V, denote by $\mathrm{GL}^G(V)$ the group of all G-equivariant linear invertible operators on V. Then, the isotypical decomposition (2.1) induces a decomposition of $\mathrm{GL}^G(V)$:

$$\mathrm{GL}^G(V) = \bigoplus_{i=0}^{r} \mathrm{GL}^G(V_i). \tag{2.3}$$

For every isotypical component V_i in (2.1), one has $\mathrm{GL}^G(V_i) \simeq \mathrm{GL}(m_i, \mathbb{F})$, where $m_i = \dim V_i / \dim \mathcal{V}_i$ and $\mathbb{F}$ is a finite-dimensional division algebra, i.e., $\mathbb{F} = \mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$, depending on the type of the irreducible representation $\mathcal{V}_i$.

2.3 Burnside Ring $A(G)$

Let G be a compact Lie group. Denote by $A(G) := \mathbb{Z}[\Phi_0(G)]$ the $\mathbb{Z}$-module generated by $(H) \in \Phi_0(G)$, i.e., an element $a \in A(G)$ is a finite sum

$$a = n_1(H_1) + \cdots + n_m(H_m),$$

where $n_i \in \mathbb{Z}$ and $(H_i) \in \Phi_0(G)$. In addition, one can define an operation of *multiplication* in $A(G)$ by

$$(H) \cdot (K) = \sum_{(L) \in \Phi_0(G)} n_L \, (L), \tag{2.4}$$

where the integer n_L represents the number of orbits of type (L) contained in the space $G/H \times G/K$ equipped with the diagonal G-action. In this way, $A(G)$ becomes a ring with the unity (G). The ring $A(G)$ is called the *Burnside ring* of G. By using the partial order on $\Phi_0(G)$, the multiplication table for $A(G)$ can be effectively computed using a simple recurrence formula:

$$n_L = \frac{n(L,H)\,|W(H)|\,n(L,K)\,|W(K)| - \sum_{(\widetilde{L})>(L)} n(L,\widetilde{L}) n_{\widetilde{L}} \left|W(\widetilde{L})\right|}{|W(L)|}. \tag{2.5}$$

2.4 G-Equivariant Brouwer Degree: Basic Properties and Recurrence Formula

Suppose that V is an orthogonal G-representation and $f : V \to V$ a G-equivariant map. Consider an open bounded G-invariant set Ω. Then the G-map f is called Ω-*admissible* if for all $x \in \partial\Omega$, we have $f(x) \neq 0$. In such a case, the pair (f, Ω) is called admissible G-pair (in V). The set of all possible G-pairs will be denoted by $\mathcal{M}^G$.

For this dissertation, we use a definition of the G-equivariant Brouwer degree based on its properties that can be used as a set of axioms, which uniquely determines this degree (see [7] for all the details):

Theorem 2.4.1. *There exists a unique map $G\text{-}\deg : \mathcal{M}^G \to A(G)$ called G-equivariant Brouwer degree(or simply G-degree), which assigns to every admissible G-pair (f, Ω) an element $G\text{-}\deg(f, \Omega) \in A(G)$,*

$$G\text{-}\deg(f, \Omega) = \sum_{(H)\in\Phi_0(G)} n_H(H) = n_{H_1}(H_1) + \cdots + n_{H_m}(H_m), \tag{2.6}$$

satisfying the following properties:

(G1) **(Existence)** *If $G\text{-}\deg(f, \Omega) \neq 0$, i.e., $n_{H_i} \neq 0$ for some i in (2.6), then there exists $x \in \Omega$ such that $f(x) = 0$ and $(G_x) \geq (H_i)$.*

(G2) **(Additivity)** *Let Ω_1 and Ω_2 be two disjoint open G-invariant subsets of Ω such that $f^{-1}(0) \cap \Omega \subset \Omega_1 \cup \Omega_2$. Then,*

$$G\text{-}\deg(f, \Omega) = G\text{-}\deg(f, \Omega_1) + G\text{-}\deg(f, \Omega_2).$$

(G3) **(Homotopy)** *If $h : [0, 1] \times V \to V$ is an Ω-admissible G-homotopy, then*

$$G\text{-}\deg(h_t, \Omega) = \text{constant}.$$

(G4) **(Normalization)** *Let Ω be a G-invariant open bounded neighborhood of 0 in V. Then,*

$$G\text{-}\deg(\mathrm{Id}, \Omega) = (G).$$

(G5) **(Multiplicativity)** *For any $(f_1, \Omega_1), (f_2, \Omega_2) \in \mathcal{M}^G$,*

$$G\text{-}\deg(f_1 \times f_2, \Omega_1 \times \Omega_2) = G\text{-}\deg(f_1, \Omega_1) \cdot G\text{-}\deg(f_2, \Omega_2),$$

where the multiplication '$\cdot$' is taken in the Burnside ring $A(G)$.

(G6) **(Suspension)** *If W is an orthogonal G-representation and $\mathscr{B}$ is an open bounded invariant neighborhood of $0 \in W$, then*

$$G\text{-}\deg(f \times \mathrm{Id}_W, \Omega \times \mathscr{B}) = G\text{-}\deg(f, \Omega).$$

(G7) **(Recurrence Formula)** *For an admissible G-pair (f, Ω), the G-degree (2.6) can be computed using the following recurrence formula*

$$n_H = \frac{\deg(f^H, \Omega^H) - \sum_{(K)>(H)} n_K \, n(H,K) \, |W(K)|}{|W(H)|}, \tag{2.7}$$

where $|X|$ stands for the number of elements in the set X and $\deg(f^H, \Omega^H)$ is the usual Brouwer degree of the map $f^H := f|_{V^H}$ on the set $\Omega^H \subset V^H$.

The G-equivariant Brouwer degree can be generalized to a G-equivariant Leray-Schauder degree in infinite-dimensional Banach spaces (see [7]).

2.5 Basic Degrees and Computational Formulae for G-Equivariant Brouwer Degree

Put $B(V) := \{x \in V : |x| < 1\}$. For each irreducible G-representation $\mathcal{V}_i$, $i = 0, 1, 2, \dots$, we define

$$\deg_{\mathcal{V}_i} := G\text{-}\deg(-\mathrm{Id}, B(\mathcal{V}_i)),$$

and will call $\deg_{\mathcal{V}_i}$ the basic G-degree in $\mathcal{V}_i$.

Consider a G-equivariant linear isomorphism $T : V \to V$ and assume that V has a G-isotypical decomposition (2.1). Then, by the Multiplicativity property (G5),

$$G\text{-}\deg(T, B(V)) = \prod_{i=0}^{r} G\text{-}\deg(T_i, B(V_i)) = \prod_{i=0}^{r} \prod_{\mu \in \sigma_-(T)} \left(\deg_{\mathcal{V}_i}\right)^{m_i(\mu)} \tag{2.8}$$

where $T_i = T|_{V_i}$ and $\sigma_-(T)$ denotes the real negative spectrum of T.

Notice that the basic degrees can be effectively computed. Indeed

$$\deg_{\mathcal{V}_i} = \sum_{(H) \in \Phi_0(G)} n_H(H),$$

where the coefficients n_H can be computed from the following recurrence formula

$$n_H = \frac{(-1)^{\dim \mathcal{V}_i^H} - \sum_{H<K} n_K \, n(H,K) \; |W(K)|}{|W(H)|}. \tag{2.9}$$

CHAPTER 3

EXISTENCE OF PERIODIC SOLUTIONS IN SECOND ORDER AUTONOMOUS SYSTEM[1]

3.1 Introduction

The purpose of this chapter is to study the autonomous second order system of ODEs (1.1) satisfying the standard Nagumo condition:

$$\exists_{M>0}\, \forall_{x\in V} \quad |x| > M \Longrightarrow x \bullet f(x) > 0. \tag{3.1}$$

We also assume that 0 is an isolated solution. By applying the $\Gamma \times \mathbb{Z}_2 \times O(2)$-equivariant Brouwer degree method, we establish for the system (1.1) the existence of multiple spatio-temporal periodic solutions with a fixed period $p > 0$, exhibiting various patterns.

The equivariant degree method (similar to the classical Brouwer degree methods) allows to detect non-constant p-periodic solutions with various types of spatio-temporal symmetries. It is important to notice that the system (1.1), even without any spacial symmetries Γ, still leads to an equivariant problem in a functional space. Indeed, since the system (1.1) is time-reversible, the space of periodic functions is an $O(2)$-representation and the related operators are $O(2)$-equivariant. The additional assumption that f is odd, implies that these operators are also $\mathbb{Z}_2$-equivariant, where $\mathbb{Z}_2 := \{1, -1\}$ acts simply by multiplication. With additional spacial symmetries Γ, we obtain that the associated with the system (1.1) operator equation is $\Gamma \times \mathbb{Z}_2 \times O(2)$-symmetric. By comparing the equivariant degrees near the zero solution with the equivariant degree on a large ball (which is determined using *apriori* bounds), we associate with the system (1.1) a $\Gamma\times\mathbb{Z}_2\times O(2)$-equivariant invariant ω, which contains the full

[1]© 2017 Journal of Convex and Nonlinear Analysis. Reprinted, with permission, from Wieslaw Krawcewize, Hao-pin Wu and Shi Yu, Periodic solutions in reversible second order autonomous systems with symmetries, Volume 18, Number 8, pp. 1393-1419, 2017.

equivariant topological information about the solution set for this equation. In particular, it allows us to obain the existence and multiplicity results on the orbits of periodic solutions to (1.1) and the equivariant topological classification of their spatio-temporal symmetries.

The equivariant degree methods were used to study the existence of solutions for BVPs in second order ODEs (see [3; 4]) and in the case of a reversible systems of FDEs, to study the equivariant bifurcation of periodic solutions in [8]. For the Newtonian systems with or without symmetries the equivariant degree methods were applied in [11; 15; 16; 48; 49; 50; 51; 52; 53] and for general Hamiltonian systems in [27; 45; 46]. We should also mention other related works, see [5; 18; 19; 20; 21; 22]. We refer to [5; 7; 24; 25] for more details about various equivariant degrees and their properties.

There is a similarity between the system (1.1) and Newtonian systems which were studied by A. Golebiewska, J. Fura, A. Ratajczak, W. Radzki, H. Ruan and S. Rybicki (cf. [15; 16; 27; 48; 50; 51; 52; 53]). The novelty in our approach is that we are taking advantage of the fact that (1.1) is time-reversible, so it admits the symmetry group $\Gamma \times O(2)$. As the main difficulty in using the G-equivariant degree method (especially the gradient equivariant degree) seems to be related to the sophistication of its definition and complicated nature of the associated with these degrees in algebraic structures, one needs to employ symbolic programming in G.A.P. This software allows exact symbolic computations of the equivariant invariants for several types of symmetry groups Γ. Therefore, supported by the computer programs, we are now able to compute the **full** $\Gamma \times \mathbb{Z}_2 \times O(2)$-equivariant invariant ω. To illustrate the power of the equivariant invariant ω, we consider a specific example of a system of second order ODEs with $\Gamma = D_6$-symmetries.

For example, by applying an H-fixed point reduction,

$$H := \{(1,1), (-1,-1), (1,\kappa), (-1,-\kappa)\} \leq \mathbb{Z}_2 \times O(2),$$

we are able to apply the equivariant degree method in the case the system (1.1) is resonant, i.e., the linearization $A := D_x f(0)$ at zero of (1.1) admits eigenvalues $-k^2$ with k being an even integer. This case also includes a possibility that 0 is not a regular point. In such a case, the reduction to the fixed point subspace allows us to apply the $\Gamma \times \mathbb{Z}_2$-equivariant Brouwer degree and to obtain the existence results for the system (1.1). The reduction to the H-fixed point space was used in previous works to show the existence of periodic solutions with the prescribed minimal period (cf. [38; 39; 40; 58] see also [32]). All the abstract results are supported by the concrete examples with $\Gamma = S_4$, for which we evaluate full equivariant invariants leading to the existence results.

3.2 Second Order Autonomous Systems

Assume that $p > 0$ is an arbitrary number. Let Γ be a finite group and $V = \mathbb{R}^n$ an orthogonal representation of Γ (Γ is acting on $\mathbb{R}^n$ by permuting the vector coordinates in $\mathbb{R}^n$). We are interested in the following second order autonomous system:

$$\begin{cases} \ddot{x}(t) = f(x(t)), \quad t \in \mathbb{R},\ x(t) \in V, \\ x(t) = x(t+p),\ \dot{x}(t) = \dot{x}(t+p), \end{cases} \tag{3.2}$$

where $f : V \to V$ is a C^1-function satisfying the following assumptions:

(A1) f is Γ-equivariant, i.e., $f(\gamma x) = \gamma f(x)$ for all $\gamma \in \Gamma$ and $x \in V$;

(A2) f is odd function, i.e., $f(-x) = -f(x)$, for all $x \in V$.

(A3) $\exists M > 0\ \forall x \in V\ \ |x| > M \Longrightarrow x \bullet f(x) > 0$.

By substituting $y(t) = x(\frac{pt}{2\pi})$, the system (3.2) is transformed to

$$\ddot{y}(t) = \left(\frac{p}{2\pi}\right)^2 \ddot{x}\left(\frac{pt}{2\pi}\right) = \left(\frac{p}{2\pi}\right)^2 f\left(x\left(\frac{pt}{2\pi}\right)\right)$$

which can be written as

$$\ddot{y}(t) = \alpha^2 f(y(t)), \tag{3.3}$$

where $\alpha = \frac{p}{2\pi}$. Notice that p-periodic solutions $x(t)$ to system (3.2) are one to one correspondence with 2π-periodic solution to system (3.3). Therefore, by replacing $\alpha^2 f$ by $\mathfrak{f}$ and y by x, we reduce the system (3.2) to

$$\begin{cases} \ddot{x}(t) = \mathfrak{f}(x(t)), \quad t \in \mathbb{R},\ x(t) \in V, \\ x(t) = x(t+2\pi),\ \dot{x}(t) = \dot{x}(t+2\pi), . \end{cases} \tag{3.4}$$

Clearly, the function $\mathfrak{f}$ is a C^1-function and satisfies conditions (A1)–(A3). In addition, for $k = 0, 1, 2, \dots,$ we introduce the following condition:

$(A4)_k$ The spectrum $\sigma(A)$ of the matrix $A := D\mathfrak{f}(0)$ does not contain $-k^2$.

Remark 3.2.1. Condition (A1) suggests a symmetric setting; condition (A2) excludes the non-zero constant solutions; condition (A3) provides apriori bounds at infinity; condition (A4) allows us to define equivariant degree around 0.

3.2.1 Sobolev Spaces of 2π-Periodic Functions

Let $\widetilde{\mathscr{H}}$ denote the second Sobolev space of 2π-periodic functions from $\mathbb{R}$ to V, i.e.,

$$\widetilde{\mathscr{H}} := H^2_{2\pi}(\mathbb{R}, V) = \left\{ x : \mathbb{R} \to V : x(0) = x(2\pi),\ x|_{[0,2\pi]} \in H^2([0, 2\pi]; V) \right\},$$

equipped with the inner product

$$\langle x, y \rangle := \int_0^{2\pi} (\ddot{x}(t) \bullet \ddot{y}(t) + \dot{x}(t) \bullet \dot{y}(t) + x(t) \bullet y(t))\, dt, \quad x, y \in \widetilde{\mathscr{H}},$$

and the associated norm

$$\|x\|_{\widetilde{\mathscr{H}}} := \left[\int_0^{2\pi} \left(|\ddot{x}(t)|^2 + |\dot{x}(t)|^2 + |x(t)|^2 \right) dt \right]^{\frac{1}{2}}, \quad x \in \widetilde{\mathscr{H}}.$$

Let $O(2)$ denote the group of 2×2-orthogonal matrices. Notice that $O(2) = SO(2) \cup SO(2)\kappa$, where $\kappa = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$. It is convenient to identify a rotation $\begin{bmatrix} \cos\tau & -\sin\tau \\ \sin\tau & \cos\tau \end{bmatrix} \in SO(2)$ with $e^{i\tau} \in S^1 \subset \mathbb{C}$. Notice that $\kappa e^{i\tau} = e^{-i\tau}\kappa$.

Put $\widetilde{G} = \Gamma \times \mathbb{Z}_2 \times O(2)$. Then the space $\widetilde{\mathscr{H}}$ is an orthogonal Hilbert representation of $\widetilde{G}$. Indeed, for $x \in \widetilde{\mathscr{H}}$, $\gamma \in \Gamma$ and $e^{i\tau} \in S^1$, we can define the group action as

$$(\gamma, \pm 1, e^{i\tau})x(t) = \pm\gamma x(t + \tau),$$
$$(\gamma, \pm 1, e^{i\tau}\kappa)x(t) = \pm\gamma x(-t + \tau),$$

and Γ acting on $V = \mathbb{R}^n$ by permuting the vector coordinates.

In a standard way we identify a 2π-periodic function $x : \mathbb{R} \to V$ with a function $\widetilde{x} : S^1 \to V$, so we can write $H^2(S^1, V)$ instead of $H^2([0, 2\pi]; V)$. Consider the $O(2)$-isotypical decomposition of $\widetilde{\mathscr{H}}$

$$\widetilde{\mathscr{H}} = \overline{\bigoplus_{k=0}^{\infty} \mathbb{V}_k}, \quad \mathbb{V}_k := \{u_k \cos(kt) + v_k \sin(kt) : u_k, v_k \in V\}. \tag{3.5}$$

Remark 3.2.2. It is clear that each $O(2)$-isotypical component $\mathbb{V}_k$, for $k = 1, 2, \dots,$ is modeled on the irreducible $O(2)$-representation $\mathcal{W}_k \simeq \mathbb{R}^2$, where $SO(2)$ acts by k-folded rotations, i.e., $\gamma z := \gamma^k \cdot z$, $\gamma \in S^1 \simeq SO(2)$, $z = x + iy = (x, y) \in \mathbb{R}^2$ and '$\cdot$' denotes the complex multiplication, and κ acts by complex conjugation. Then (3.5) is also a $(\mathbb{Z}_2 \times O(2))$-isotypical decomposition of $\widetilde{\mathscr{H}}$. Under the action of the group $G' := \mathbb{Z}_2 \times O(2)$, the constant functions $\overline{x} \in \mathbb{V}_0$ have isotropy $G'_{\overline{x}} = \mathbb{Z}_1 \times O(2)$, while for a non-zero function $x \in \mathbb{V}_k$, $k > 0$, the isotropy is $G'_x = \mathbb{Z}_2 \times_{\mathbb{Z}_2} D_{2k}$ or $G'_x = \mathbb{Z}_2 \times_{\mathbb{Z}_2} \mathbb{Z}_{2k}$ (cf. [11]), and consequently, $G'_x \not\subset \mathbb{Z}_1 \times O(2)$.

Definition 3.2.3. *An orbit type (H) in the space $\widetilde{\mathscr{H}}$ is called to be of* maximal type *if there exists $k > 0$ and $x \neq 0$, $x \in \mathbb{V}_k$ such that $H = \widetilde{G}_x$ and (H) is a maximal orbit type in $\Phi(\widetilde{G}, \mathbb{V}_k \setminus \{0\})$.*

Notice that an orbit type (H) of maximal type is not an orbit type of a constant function and since Γ acts on $V = \mathbb{R}^n$ by permuting the coordinates of vectors in V, it follows by Remark 3.2.2 that there is no constant function $\overline{x} \in \widetilde{\mathscr{H}}$ such that $H \leq \widetilde{G}_{\overline{x}}$.

3.2.2 Apriori Bounds for System (3.4)

Consider the following modification of system (3.4):

$$\begin{cases} \ddot{x}(t) = \lambda(\mathfrak{f}(x(t)) - x(t)) + x(t), \quad t \in \mathbb{R},\ x(t) \in V, \lambda \in [0,1] \\ x(t) = x(t + 2\pi),\ \dot{x}(t) = \dot{x}(t + 2\pi), \end{cases} \tag{3.6}$$

where $\mathfrak{f} : V \to V$ is a C^1-function satisfying conditions (A1)–(A3) and $(A4)_k$ for $k = 0, 1, 2, \dots$. We have the following lemma:

Lemma 3.2.4. *If $x(t)$ is a 2π-periodic function of class C^2 such that $\max_{t\in\mathbb{R}} \|x(t)\| > M$, then $x(t)$ is not a solution of (3.6) for $\lambda \in [0,1]$, where M is from condtion (A3).*

Proof. Assume for the contradiction that $x(t)$ is a solution to (3.1) with $\max_{t\in\mathbb{R}} \|x(t)\| > M$ (for $\lambda > 0$). Consider the function $\phi(t) := \frac{1}{2}\|x(t)\|^2$. Suppose that $\phi(t_0) = \max_{t\in\mathbb{R}} \phi(t)$, then $\phi'(t_0) = x(t_0) \bullet \dot{x}(t_0) = 0$ and $\phi''(t_0) = \dot{x}(t_0) \bullet \dot{x}(t_0) + \ddot{x}(t_0) \bullet x(t_0) \leq 0$. However, by condition (A3), $\phi''(t_0) = \dot{x}(t_0) \bullet \dot{x}(t_0) + \ddot{x}(t_0) \bullet x(t_0) = (\lambda(\mathfrak{f}(x(t_0)) - x(t_0)) + x(t_0)) \bullet x(t_0) + \dot{x}(t_0) \bullet \dot{x}(t_0) > (1-\lambda)x(t_0) \bullet x(t_0) + \lambda\mathfrak{f}(x(t_0)) \bullet x(t_0) > 0$, which is a contradiction. For $\lambda = 0$, the statement is trivial. □

Lemma 3.2.5. *There exists $\widetilde{M} > 0$ such that for every solution $x(t)$ to (3.6) and $\lambda \in [0,1]$, one has $\|x\|_{\widetilde{\mathscr{H}}} < \widetilde{M}$.*

Proof. By lemma 3.2.4, there exists an $M > 0$ such that any 2π-periodic solution $x(t)$ to (3.6) satisfies $|x(t)| < M$. Take $\Omega_M := \{x \in \widetilde{\mathscr{H}} : \ \|x\| < M, x(t) \in V\}$, since the function $\widetilde{\mathfrak{f}} : [0,1] \times V \longrightarrow V$ given by

$$\widetilde{\mathfrak{f}}(\lambda, x) = \lambda(\mathfrak{f}(x) - x) + x,\ x \in V,\ \lambda \in [0,1]$$

is continuous, and the set $[0, 1] \times \overline{\Omega_M}$ is compact, then there exists a constant $M_1 > 0$ such that for every solution to (3.6) one has

$$|\ddot{x}(t)| < M_1.$$

Finally, if $\dot{x}(\tau) = 0$ (for some $\tau \in [0, 2\pi]$), then for all $t > \tau$, we have

$$|\dot{x}(t)| = \left|\dot{x}(\tau) + \int_\tau^t \ddot{x}(s)ds\right| = \left|\int_\tau^t \ddot{x}(s)ds\right| \leq$$
$$\int_\tau^t |\ddot{x}(s)|\, ds \leq \int_0^{2\pi} |\ddot{x}(s)|\, ds \leq 2\pi M_1 =: M_2.$$

Similarly, the same inequality is true for $t \leq \tau$. Then clearly

$$\|x\|^2_{\widetilde{\mathscr{H}}} := \int_0^{2\pi} (\ddot{x}(t) \bullet \ddot{x}(t) + \dot{x}(t) \bullet \dot{x}(t) + x(t) \bullet x(t))dt \leq 2\pi(M_2^2 + M_1^2 + M^2) := \widetilde{M}^2.$$

□

3.2.3 Setting in Functional Spaces

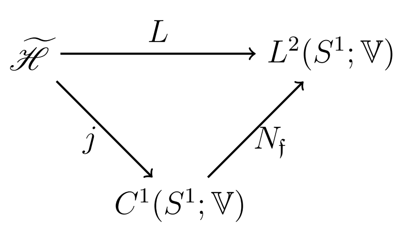
Define the operators:

$$L : \widetilde{\mathscr{H}} \longrightarrow L^2(S^1; \mathbb{V}), \quad Lx := \ddot{x} - x, \tag{3.7}$$

$$N_\mathfrak{f} : C^1(S^1; \mathbb{V}) \longrightarrow L^2(S^1; \mathbb{V}), \quad N_\mathfrak{f}(x)(t) := N(\mathfrak{f}(x(t))), \tag{3.8}$$

$$j : \widetilde{\mathscr{H}} \longrightarrow C^1(S^1; \mathbb{V}), \quad (jx)(t) := x(t) \tag{3.9}$$

We have the diagram



Then the system (3.6) is equivalent to

$$Lx = \lambda(N_\mathfrak{f}(j(x)) - j(x)),\ x \in \widetilde{\mathscr{H}},\ \lambda \in [0, 1]. \tag{3.10}$$

Since L is an isomorphism, the equation (3.10) can be reformulated as

$$\mathscr{F}_\lambda(x) := x - \lambda L^{-1}(N_{\mathfrak{f}}(j(x)) - j(x)) = 0,\ x \in \widetilde{\mathscr{H}} \tag{3.11}$$

and we put $\mathscr{F} := \mathscr{F}_1$.

Remark 3.2.6. *Notice that $\mathscr{F}_\lambda$ is a $\widetilde{G}$-equivariant completely continuous field.*

Proposition 3.2.7. *Let $\widetilde{M}$ be the constant provided by Lemma 3.2.5. Put $R := \widetilde{M} + 1$. Then any solution to $\mathscr{F}_\lambda = 0$ must belong to the set $\Omega_{\widetilde{M}} := \left\{x \in \widetilde{\mathscr{H}} :\ |x| < R, x \in V\right\}$. In particular, $\{\mathscr{F}_\lambda\}_{\lambda\in[0,1]}$ is an $\Omega_{\widetilde{M}}$-admissible homotopy between $\mathscr{F}_1 = \mathscr{F}$ and $\mathscr{F}_0 = \mathrm{Id}$.*

Notice that $x \equiv 0$ is a solution to the equation $\mathscr{F}_\lambda(x) = 0$. Put

$$A := D\mathfrak{f}(0) :\ \widetilde{\mathscr{H}} \longrightarrow \widetilde{\mathscr{H}},$$

then

$$\mathscr{A} := D\mathscr{F}(0) = \mathrm{Id} - L^{-1}(N_A(j) - j) : \widetilde{\mathscr{H}} \longrightarrow \widetilde{\mathscr{H}}. \tag{3.12}$$

Put $\mathscr{A}_k := \mathscr{A}|_{V_k}; k = 0, 1 \cdots$.

One can easily check that $\mathscr{A}$ is a Fredholm operator of index zero. Therefore, $\mathscr{A}$ is an isomorphism if and only if $0 \notin \sigma(\mathscr{A})$. Since $\mathscr{A}$ is $O(2)$-equivariant, it preserves the isotypical decomposition (3.5). The operator L on the isotypical components $\mathbb{V}_k$ is given by,

$$\begin{aligned} L(u_k\cos(kt) + v_k\sin(kt)) &= -k^2u_k\cos(kt) - k^2v_k\sin(kt) - u_k\cos(kt) - v_k\sin(kt) \\ &= (-k^2 - 1)(u_k\cos(kt) + v_k\sin(kt)), \end{aligned}$$

which implies

$$L^{-1}(u_k\cos(kt) + v_k\sin(kt)) = \frac{1}{-k^2 - 1}(u_k\cos(kt) + v_k\sin(kt)).$$

Therefore, we have

$$\mathscr{A} = \mathrm{Id} + \frac{1}{1+k^2}[A - \mathrm{Id}] : V_k \to V_k$$

and we obtain the following description of the spectrum of $\mathscr{A}$

$$\sigma(\mathscr{A}) = \left\{1 + \frac{1}{1+k^2}(\mu - 1) : \mu \in \sigma(A)\right\}, k = 0, 1, 2, \cdots . \tag{3.13}$$

Clearly, $\mathscr{A}$ is an isomorphism if and only if the condition (A4)$_k$ is satisfied for all $k = 0, 1, 2, \cdots$ (cf. condtion (A4)). We have the following lemma:

Lemma 3.2.8. *Assume that condition (A1)-(A3), (A4)$_k$ are satisfied and $\mathscr{A}$ is an isomorphism. Then, for a sufficiently small $\varepsilon > 0$, the map $\mathscr{F}$ (given by (3.11)) is Ω_ε-admissibly $\widetilde{G}$-equivariantly homotopic to $\mathscr{A} = D_x\mathscr{F}(0)$, where $\Omega_\varepsilon := \left\{x \in \widetilde{\mathscr{H}} : \|x\| < \varepsilon\right\}$,*

Proof. Define the deformation $\mathscr{F}_t(x) = (1-t)\mathscr{A}(x) + t\mathscr{F}(x),\ x \in \widetilde{\mathscr{H}},\ t \in [0,1]$. We claim that there exists a sufficiently small $\varepsilon > 0$ such that $\mathscr{F}_t(x)$ is Ω_ε-admissible homotopy. Indeed, assume for contradiction, there exist $\{x_n\} \subset \widetilde{\mathscr{H}},\ \{t_n\} \subset [0,1]$ such that $x_n \to 0,\ t_n \to t_0$ and

$$\mathscr{F}_{t_n}(x_n) = \mathscr{A}(x_n) - t_n(\mathscr{A}(x_n) - \mathscr{F}(x_n)) = 0.$$

Then

$$\frac{\mathscr{A}(x_n)}{\|x\|_{\widetilde{\mathscr{H}}}} = \frac{t_n(\mathscr{A}(x_n) - \mathscr{F}(x_n))}{\|x\|_{\widetilde{\mathscr{H}}}} \to 0,\ \text{as } n \to \infty.$$

Put $v_n := \frac{x_n}{\|x\|_{\widetilde{\mathscr{H}}}}$. Then (see (3.12)), we have

$$\mathscr{A}(v_n) = v_n - L^{-1}(N_A(j(v_n)) - j(v_n)) \to 0,\ \text{as } n \to \infty.$$

Since operator $L^{-1}(N_A(j) - j)$ is compact, then there exist y_0 and subsequence $\{v_{n_k}\}$ such that $L^{-1}(N_A(j(v_{n_k})) - j((v_{n_k})) \to y_0$, so $v_{n_k} \to y_0$ and $\|y_0\| = 1$ by the continuity of $\mathscr{A}$. Therefore, $\mathscr{A}(y_0) = 0$ which is impossible since $\mathscr{A}$ is an isomorphism. □

Corollary 3.2.9. *Under the assumptions (A1)-(A3) and $(A4)_k$ for all $k = 0, 1, 2, \dots,$ there exists a sufficiently small $\varepsilon > 0$ such that $\mathscr{F}$ is Ω_ε-admissible and*

$$\widetilde{G}\text{-}\deg(\mathscr{F}, \Omega_\varepsilon) = \widetilde{G}\text{-}\deg(\mathscr{A}, B(\widetilde{\mathscr{H}})).$$

3.2.4 Existence of 2π-Periodic Solutions to (3.4)

Consider the set $\Omega_{\widetilde{M}} := \left\{x \in \widetilde{\mathscr{H}} : |x| < \widetilde{M}\right\}$, where $\widetilde{M} > 0$ is given in Lemma 3.2.5. Then, $\Omega_{\widetilde{M}}$ contains all 2π-periodic solutions to (3.4), including the zero solution $x \equiv 0$. Notice that, by Lemma 3.2.8, $x \equiv 0$ is an isolated 2π-periodic solution to (3.4) and it is the only solution to $\mathscr{F}(x) = 0$ in Ω_ε. Therefore, all the non-zero 2π-periodic solutions to (3.4) belong to the set $\Omega := \Omega_{\widetilde{M}} \setminus \overline{\Omega_\varepsilon}$. We define the $\widetilde{G}$*-equivariant invariant* ω associated with the system (3.4) by

$$\omega := \widetilde{G}\text{-}\deg(\mathscr{F}, \Omega) = \widetilde{G}\text{-}\deg(\mathscr{F}, \Omega_{\widetilde{M}}) - \widetilde{G}\text{-}\deg(\mathscr{F}, \Omega_\epsilon), \tag{3.14}$$

which implies that ω provides an equivariant topological classification of the solution set $\mathscr{S} \subset \widetilde{\mathscr{H}}$ to $\mathscr{F}(x) = 0$ which consists of all non-zero 2π-periodic solutions to (3.4). More precisely, suppose that

$$\omega = n_1(H_1) + n_2(H_2) + \cdots + n_m(H_m), \quad n_j \neq 0,\ (H_j) \in A(\widetilde{G}),\ j = 1, 2, \dots, m, \tag{3.15}$$

then for each of the indices j, there exists a 2π-periodic solution $x \in \widetilde{\mathscr{H}}$ such that $\widetilde{G}_x \geq H_j$. In the case that $\widetilde{G}_x$ is a maximal orbit type in $\Phi(\widetilde{G}, \mathbb{V}_k \setminus \{0\})$ (see Remark 3.2.2), consider the following group homomorphism

$$\begin{cases} \psi_l : \Gamma \times \mathbb{Z}_2 \times O(2) \to \Gamma \times \mathbb{Z}_2 \times O(2) \\ \psi_l(h, \pm 1, g) = (h, \pm 1, \mu_l(g)) \end{cases},$$

where $\mu_l : O(2) \to O(2)/\mathbb{Z}_l \simeq O(2)$ is the natural l-folding homomorphism of $O(2)$ into itself. Then, $(\psi_l(\widetilde{G}_x)) = (H_j)$. We can also deduce that x is a $2\pi/l$-periodic function, which under the identification $[0, 2\pi/l]/\{0, 2\pi/l\} \simeq S^1$, has exactly the orbit type (H_j).

Let us summarize our conclusions in the following:

Theorem 3.2.10. *Consider the system* (3.4) *for which the assumptions (A1), (A2), (A3) and (A4)$_k$ for all* $k = 0, 1, 2, \dots$ *are satisfied. Then the equivariant invariant* ω*, given by* (3.14)*, is well defined. Additionally, suppose*

$$\omega = n_1(H_1) + n_2(H_2) + \cdots + n_m(H_m), \quad n_j \neq 0, \ (H_j) \in \Phi_0(\widetilde{G}).$$

Then for every $j = 1, 2, \dots, m$ *there exists a* $\widetilde{G}$*-orbit of* 2π*-periodic solutions* $\widetilde{G}(x)$ *to* (3.4) *such that* $\widetilde{G}_x \geq H_j$*. Moreover, if* H_j *is finite, then the solution* x *is non-constant, and if* (H_j) *is of maximal type, then the solution* x *has the extended orbit type* (H_j)*.*

Given a complete list of all irreducible orthogonal Γ-representations $\mathcal{V}_1$, $\mathcal{V}_2$, ..., $\mathcal{V}_r$ (which are assumed to be of real type for simplicity), the irreducible $(\Gamma \times \mathbb{Z}_2)$-representations can split into two parts

(1) $\mathbb{Z}_2$ acts trivially on $\mathcal{V}_i$; we denote these representations by $\mathcal{V}_0$, $\mathcal{V}_1$, ..., $\mathcal{V}_r$, and

(2) $-1 \in \mathbb{Z}_2$ acts by multiplication on $\mathcal{V}_i$; we denote these representations by $\mathcal{V}_0^-$, $\mathcal{V}_1^-$, ..., $\mathcal{V}_r^-$.

Assume that $\mathbb{Z}_2$ acts (non-trivially) on V by multiplication, then we have the following $(\Gamma \times \mathbb{Z}_2)$-isotypical decomposition of V

$$V = V_0 \oplus V_1^- \oplus \cdots \oplus V_r^-.$$

On the other hand, notice that $\mathbb{V}_0$ can be identified with V and for $k > 0$, the component $\mathbb{V}_k$ can be naturally identified with the complexification of V, i.e., $\mathbb{V}_k = \mathbb{C} \otimes_{\mathbb{R}} V$ where $O(2)$ acts on $\mathbb{C}$ by k-folding and κ by complex conjugation. That means $\mathbb{C}$, equipped with this

$O(2)$ representation which is denoted by $\mathcal{W}_k$. Then we have $\mathbb{V}_k = \mathcal{W}_k \otimes_{\mathbb{R}} V$(cf. [7]), which leads to the following $(\Gamma \times \mathbb{Z}_2 \times O(2))$-isotypical decomposition of $\mathbb{V}_k$ for $k > 0$

$$\mathbb{V}_k = V_{0,k}^- \oplus V_{1,k}^- \oplus \cdots \oplus V_{r,k}^-,$$

where the component $V_{j,k}^- = \mathcal{W}_k \otimes_{\mathbb{R}} V_j^-$ is modeled on the irreducible $\widetilde{G}$-representation $\mathcal{V}_{j,k}^- = \mathcal{W}_k \otimes_{\mathbb{R}} \mathcal{V}_j^-$.

By (3.13), the $\mathcal{V}_{j,k}^-$-multiplicity of the eigenvalue $\xi \in \sigma(A)$ is equal to the $\mathcal{V}_j$-multiplicity $m_j(\mu)$ of the eigenvalue μ. For $k = 0, 1, 2, \dots$ and $\mu \in \sigma(A)$, we define the following integers

$$\mathfrak{m}_-^{j,k}(\mu) := \begin{cases} m_j(\mu) & \text{if } \mu < -k^2, \\ 0 & \text{otherwise.} \end{cases} \tag{3.16}$$

Therefore(see (2.8)), we obtain

$$\widetilde{G}\text{-}\deg(\mathscr{A}, B(\widetilde{\mathscr{H}})) = \prod_k \prod_{j=1}^r \prod_{\mu \in \sigma(A)} \left(\deg_{\mathcal{V}_{j,k}^-}\right)^{\mathfrak{m}_-^{j,k}(\mu)}. \tag{3.17}$$

Notice that the product in (3.17) is finite (since there are only finitely many non-zero $\mathfrak{m}_-^{j,k}(\mu)$'s and $(\deg_{\mathcal{V}_{j,1,k}})^i = (\widetilde{G})$ for any even integer i. Consequently, the equivariant topological invariant ω, given by (3.14), can be effectively computed.

3.2.5 Example: D_n-Symmetric Odd Polygonal Map

Consider the space $V := \mathbb{R}^n$ with Γ acting on V by permuting the coordinates of V. For the purpose of presenting an example, let us introduce the notion of a r*-th dominated polynomial map*, which is a map $f : V \to V$, $f(x) = (p_1(x), p_2(x), ..., p_n(x))^T \in V$, where $p_k(x) = x_k^r Q_k(x) + q_k(x)$ for $k = 1, \dots, n$ ($Q(x)$ is a positive polynomial, i.e., $Q_k(x) > 0$ for all $x \in V$,) and $q_k(x)$ is a polynomial of degree smaller than r). If $r > 0$ is an odd integer, then the map f satisfies Nagumo condition (A3). Indeed, since we have for all $x \in V$

$$\begin{aligned} f(x) \bullet x &= \sum_{k=1}^{n} x_k^{r+1} Q_k(x) + \sum_{k=1}^{n} x_k q_k(x) \\ &\geq \sum_{k=1}^{n} x_k^{r+1} q_o - \sum_{k=1}^{n} |x_k q_k(x)| \\ &= q_o \|x\|_{r+1}^{r+1} - \sum_{k=1}^{n} |x_k q_k(x)|, \end{aligned}$$

where $q_o := \min\{|Q_k(x)| : x \in V,\, k = 1, 2, \dots, n\} > 0$. Clearly, there is a constant $c_1 > 0$ such that for all $x \in V$, $\|x\|_{r+1} \geq c_1 \|x\|$. In addition, we have for integers $\alpha_j \geq 0$, such that $\alpha_1 + \alpha_2 + \cdots + \alpha_n \leq r$,

$$\forall_{x\in V} \quad |x_1^{\alpha_1} x_2^{\alpha_2} \dots x_n^{\alpha_n}| \leq \|x\|^{\alpha_1} \|x\|^{\alpha_2} \dots \|x\|^{\alpha_n} = \|x\|^{\alpha_1+\alpha_2+\cdots+\alpha_n} \leq 1 + \|x\|^r.$$

Therefore, there exist constant c, C, $D > 0$ such that

$$\forall_{x\in V} \;\; f(x) \bullet x \geq c\|x\|^{r+1} - C\|x\|^r - D,$$

and it easily follows that the condition (A3) is satisfied. Notice that if f_1, $f_2 : V \to V$ are two functions satisfying (A3) then for any constants α, $\beta > 0$ the function $f = \alpha f_1 + \beta f_2$ satisfies (A3). Moreover, if f satisfies (A3) and $\varphi : V \to \mathbb{R}$ is a function such that $\varphi(x) \geq \delta > 0$ for all $x \in V$ then $\widetilde{f}(x) = \frac{f(x)}{\varphi(x)}$ and $\hat{f}(x) = \varphi(x) f(x)$ satisfies (A3).

Assume that for a certain odd $r \geq 3$, $\widetilde{f} : V \to V$ is a r-th dominated odd map, which is also Γ-equivariant, and take

$$\mathfrak{f}(x) = Ax + \widetilde{f}(x), \quad x \in V, \tag{3.18}$$

where $A : V \to V$ is a linear Γ-equivariant operator satisfying condition $(A4)_k$, for $k = 0, 1, 2, \dots,$ i.e., $-k^2 \notin \sigma(A)$. Then, we can compute the $\widetilde{G}$-equivariant invariant ω and classify 2π-periodic solutions to (3.4) (see also Theorem 3.2.10).

Consider $\Gamma = D_n$ acting on coordinates of $V = \mathbb{R}^n$ by permuting the coordinates in the same way as the vertices of a regular n-gon are permuted by D_n. To be more explicit, we can associate the element $\gamma = e^{\frac{2\pi i}{n}}$ with the permutation $(1, 2, \dots, n)$ and the reflection κ with the permutation $(2, n)(3, n-1)\dots(m, n-m+2)$, where $m = \left\lfloor \frac{n+1}{2} \right\rfloor$. On the other hand, consider

$$A := \begin{bmatrix} c & d & 0 & \dots & d \\ d & c & d & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ d & 0 & 0 & \dots & c \end{bmatrix},$$

then we have

$$\sigma(A) = \left\{ \mu_j = c + 2d\cos\frac{2\pi j}{n} : 0 \le j \le \left\lfloor \frac{n}{2} \right\rfloor \right\},$$

where each eigenvalue μ_j has $\mathcal{V}_j$-multiplicativity one (here $\mathcal{V}_j$ stands for an irreducible D_n-representation with $\mathbb{Z}_n$-action by j-folding).

Example 3.2.11. Let us take $n = 6$ and the linear D_6-equivariant operator

$$A := \begin{bmatrix} -1 & 1/3 & 0 & 0 & 0 & 1/3 \\ 1/3 & -1 & 1/3 & 0 & 0 & 0 \\ 0 & 1/3 & -1 & 1/3 & 0 & 0 \\ 0 & 0 & 1/3 & -1 & 1/3 & 0 \\ 0 & 0 & 0 & 1/3 & -1 & 1/3 \\ 1/3 & 0 & 0 & 0 & 1/3 & -1 \end{bmatrix}.$$

In this case, V admits the primary $D_6 \times \mathbb{Z}_2$- isotypical decomposition $V = \mathcal{V}_1^- \oplus \mathcal{V}_2^- \oplus \mathcal{V}_5^- \oplus \mathcal{V}_6^-$ (see Tables 3.11, 3.13 and 3.14) and $\sigma(\mathscr{A})$ are described as follows:

Since there is no negative eigenvalue for $k > 3$, by (3.17), we have

$$\widetilde{G}\text{-}\deg(\mathscr{A}, B(\widetilde{\mathscr{H}})) = \deg_{\mathcal{V}_{1,0}^-} \cdot \deg_{\mathcal{V}_{2,0}^-} \cdot \deg_{\mathcal{V}_{5,0}^-} \cdot \deg_{\mathcal{V}_{6,0}^-} \cdot \deg_{\mathcal{V}_{2,1}^-} \cdot \deg_{\mathcal{V}_{6,1}^-}$$

Table 3.1: Eigenvalues of $\mathscr{A}$

$\xi_{j,k} \in \sigma(\mathscr{A}\vert_{\mathcal{V}_{j,1,k}})$				
$k \setminus j$	1	2	5	6
0	$-1/3$	$-5/3$	$-2/3$	$-4/3$
1	$1/3$	$-1/3$	$1/6$	$-1/6$
2	$11/15$	$7/15$	$2/3$	$8/15$

and

$$
\begin{aligned}
\omega &= (\widetilde{G}) - \widetilde{G}\text{-}\deg(\mathscr{A}, B(\widetilde{\mathscr{H}}))\\
= \quad & (D_6^{\tilde{d}} \times O(2)) + (D_6 \times O(2)) - (\tilde{D}_3 \times O(2)) - (D_2^{\tilde{d}} \times O(2))\\
&+ (D_2^z \times O(2)) + (D_2^d \times O(2)) - (D_2 \times O(2)) - (\tilde{D}_1^z \times O(2))\\
&+ (\tilde{D}_1 \times O(2)) + (D_6^2 {}^{D_6^{\tilde{d}}}\times_{\mathbb{Z}_2} D_2) - (D_6 {}^{\tilde{D}_3}\times_{\mathbb{Z}_2} D_2) + (D_2^2 {}^{D_2^z}\times_{\mathbb{Z}_2} D_2)\\
&+ (D_2^2 {}^{D_2}\times_{\mathbb{Z}_2} D_2) - (D_2^z {}^{D_1^z}\times_{\mathbb{Z}_2} D_2) - (D_2^z {}^{\mathbb{Z}_2}\times_{\mathbb{Z}_2} D_2) - (D_2^d {}^{\tilde{D}_1^z}\times_{\mathbb{Z}_2} D_2)\\
&- (D_2^d {}^{\mathbb{Z}_2^z}\times_{\mathbb{Z}_2} D_2) - (D_2^d {}^{D_1}\times_{\mathbb{Z}_2} D_2) - (\tilde{D}_1^2 {}^{\tilde{D}_1}\times_{\mathbb{Z}_2} D_2) + (D_2 {}^{\tilde{D}_1}\times_{\mathbb{Z}_2} D_2)\\
&- (\mathbb{Z}_2^2 {}^{\mathbb{Z}_2}\times_{\mathbb{Z}_2} D_2) - (D_1^2 {}^{D_1^z}\times_{\mathbb{Z}_2} D_2) + 2(\tilde{D}_1^z {}^{\mathbb{Z}_1}\times_{\mathbb{Z}_2} D_2) + (\mathbb{Z}_2^z {}^{\mathbb{Z}_1}\times_{\mathbb{Z}_2} D_2)\\
&+ (\mathbb{Z}_1^2 {}^{\mathbb{Z}_1}\times_{\mathbb{Z}_2} D_2) + (D_1 {}^{\mathbb{Z}_1}\times_{\mathbb{Z}_2} D_2) - (D_6^{\tilde{d}} \times D_1) + (\tilde{D}_3 \times D_1)\\
&+ (D_2^{\tilde{d}} \times D_1) - (D_2^z \times D_1) + (\tilde{D}_1^z \times D_1) + (D_1^z \times D_1)\\
&- (\tilde{D}_1 \times D_1) + (\mathbb{Z}_2 \times D_1) - 2(\mathbb{Z}_1 \times D_1) + (D_6^2 {}^{\mathbb{Z}_2}\times_{D_6} D_6)\\
&- (D_6^{\tilde{d}} {}^{\mathbb{Z}_1}\times_{D_6} D_6) - (D_6 {}^{\mathbb{Z}_2}\times_{D_3} D_3) + (\tilde{D}_3 {}^{\mathbb{Z}_1}\times_{D_3} D_3) - (D_2^2 {}^{D_1^z}\times_{D_2}^{D_1^2} D_2)\\
&- (D_2^2 {}^{\tilde{D}_1}\times_{D_2}^{\tilde{D}_1^2} D_2) - 2(D_2^2 {}^{\mathbb{Z}_2}\times_{D_2}^{\mathbb{Z}_2^2} D_2) + (D_2^{\tilde{d}} {}^{\mathbb{Z}_1}\times_{D_2}^{\mathbb{Z}_2^z} D_2) + (D_2^z {}^{\mathbb{Z}_1}\times_{D_2}^{\tilde{D}_1^z} D_2)\\
&+ (D_2^d {}^{\mathbb{Z}_1}\times_{D_2}^{\tilde{D}_1^z} D_2) + (D_2^d {}^{\mathbb{Z}_1}\times_{D_2}^{\mathbb{Z}_2^z} D_2) + (D_2^d {}^{\mathbb{Z}_1}\times_{D_2}^{D_1} D_2) + (\tilde{D}_1^2 {}^{\mathbb{Z}_1}\times_{D_2}^{\mathbb{Z}_1^2} D_2)\\
&+ (\mathbb{Z}_2^2 {}^{\mathbb{Z}_1}\times_{D_2}^{\mathbb{Z}_1^2} D_2) + (D_1^2 {}^{\mathbb{Z}_1}\times_{D_2}^{\mathbb{Z}_1^2} D_2) + (D_2^z {}^{D_1^z}\times_{D_1} D_1) + (D_2^z {}^{\mathbb{Z}_2}\times_{D_1} D_1)\\
&+ (D_2 {}^{\mathbb{Z}_2}\times_{D_1} D_1) - 2(\tilde{D}_1^z {}^{\mathbb{Z}_1}\times_{D_1} D_1) - (D_1^z {}^{\mathbb{Z}_1}\times_{D_1} D_1) - (\mathbb{Z}_2^z {}^{\mathbb{Z}_1}\times_{D_1} D_1)\\
&- (\tilde{D}_1 {}^{\mathbb{Z}_1}\times_{D_1} D_1) - (D_1 {}^{\mathbb{Z}_1}\times_{D_1} D_1) - (\mathbb{Z}_2 {}^{\mathbb{Z}_1}\times_{D_1} D_1),
\end{aligned}
$$

where the orbit types in red are maximal orbit types (see Table 3.4). Hence, according to Theorem 3.2.10, system (3.4) in this case admits $\widetilde{G}$-orbits of 2π-periodic non-constant solutions which have orbit types $(D_6^2 {}^{D_6^{\tilde{d}}}\times_{\mathbb{Z}_2} D_2)$ and $(D_6^2 {}^{\mathbb{Z}_2}\times_{D_6} D_6)$.

3.3 Existence of 2π-Periodic Solutions to (3.4) at Resonance

In this section, we consider the system (3.4), where $f : V \to \mathbb{R}$ is a C^1-function satisfying the assumptions (A1), (A2), (A3) and $(A4)_k$, for only odd natural numbers k. In other words, it is possible that $0 \in \sigma(A)$ (which means that zero is a *degenerate* solution to (3.4)).

3.3.1 Reduction to Fixed-Point Subspace $\widetilde{\mathscr{H}}^H$

Since for any closed subgroup $H \subset \widetilde{G}$, one has $\mathscr{F}^H := \mathscr{F}|_{\widetilde{\mathscr{H}}^H} : \widetilde{\mathscr{H}}^H \to \widetilde{\mathscr{H}}^H$ and $\mathscr{A}^H := \mathscr{A}|_{\widetilde{\mathscr{H}}^H} : \widetilde{\mathscr{H}}^H \to \widetilde{\mathscr{H}}^H$ is $W(H)$-equivariant, it is a common practice to look for the $W(H)$-orbits of solutions for (3.4) in the subspace $\widetilde{\mathscr{H}}^H$. In other words, if for some $x \in \widetilde{\mathscr{H}}^H$, we have $\mathscr{F}^H(x) = 0$, then $\mathscr{F}(x) = 0$, i.e., x is a solution to (3.4).

Notice that, by $O(2)$-equivariant, if $x(t)$ is a solution to (3.4), so are $x(t+\gamma)$ and $x(-t)$. Also, by (A2), $-x(t)$ is a solution to (3.4) if and only if $x(t)$ is a solution. Consider the subgroup $H := \{e\} \times D_2^d \leq \Gamma \times \mathbb{Z}_2 \times O(2)$ where

$$H := D_2^d = \{(1,1), (-1,-1), (1,\kappa), (-1,-\kappa)\} \leq \mathbb{Z}_2 \times O(2).$$

Clearly, a function $x \in \widetilde{\mathscr{H}}$ belongs to $\widetilde{\mathscr{H}}^H$ if and only if $x(-t) = x(t)$ and $x(t+\pi) = -x(t)$, i.e., x is an even function which is invariant with respect to the $\mathbb{Z}_2$-action induced by the element $(-1,-1) \in \mathbb{Z}_2 \times O(2)$. Therefore, we will call it a $\mathbb{Z}_2$-*even function*.

Put $\mathscr{H} := \widetilde{\mathscr{H}}^H$. The $O(2)$-isotypical decomposition (3.5) of $\widetilde{\mathscr{H}}$ leads to the following decomposition

$$\mathscr{H} = \overline{\bigoplus_{k=0}^{\infty} \mathbb{V}_k^H}, \tag{3.19}$$

where

$$\mathbb{V}_k^H := \begin{cases} \{0\} & \text{if } k \text{ is even,} \\ \{u_k \cos(kt) : u_k \in V\} & \text{if } k \text{ is odd.} \end{cases}$$

Clearly, each $\mathbb{V}_k^H$ in the decomposition (3.19) is $N(H)$-invariant and therefore it is $W(H)$-invariant decomposition of $\mathscr{H}$. Notice that $N(H) = \Gamma \times \mathbb{Z}_2 \times D_2$, thus $W(H) = \Gamma \times \mathbb{Z}_2$. Therefore, each component $\mathbb{V}_k^H$ can be refined to the $\Gamma \times \mathbb{Z}_2$-isotypical decomposition

$$\mathbb{V}_k^H = (V_{0,k}^-)^H \oplus (V_{1,k}^-)^H \oplus \cdots \oplus (V_{r,k}^-)^H, \tag{3.20}$$

where $(V_{j,k}^-)^H$ is modeled on the irreducible $\Gamma \times \mathbb{Z}_2$-representation $\mathcal{V}_j^-$.

3.3.2 Existence of $\mathbb{Z}_2$-Even Periodic Solutions to (3.4)

By assumption $(A4)_k$, for k being an odd natural number, we have that $\mathscr{A}^H = D\mathscr{F}(0)^H : \mathscr{H} \to \mathscr{H}$ is an isomorphism. By exactly the same argument as in the proof of Lemma 3.2.8, we obtain that for sufficiently small $\varepsilon > 0$, the map $\mathscr{F}^H$ is Ω_ε^H-admissibly G-equivariantly homotopic to $\mathscr{A}^H$. On the other hand, by Lemma 3.2.5, $\mathscr{A}^H$ is also $\Omega_{\widetilde{M}}^H$-admissibly G-equivariantly homotopic to Id. Using the same notation as in the previous section, the equivariant topological invariant ω^H given by

$$\omega^H = G\text{-}\deg(\mathscr{F}^H, \Omega_{\widetilde{M}}^H) - G\text{-}\deg(\mathscr{F}^H, \Omega_\varepsilon^H) = (G) - G\text{-}\deg(\mathscr{A}, B(\mathscr{H}))$$

is well-defined and clearly,

$$\omega^H = G\text{-}\deg(\mathscr{F}^H, \Omega^H), \tag{3.21}$$

where $\Omega := \Omega_{\widetilde{M}} \setminus \Omega_\varepsilon$. On the other hand, we have

$$G\text{-}\deg\left(\mathscr{A}, B(\mathscr{H})\right) = \prod_l \prod_{j=1}^r \prod_{\mu \in \sigma(A)} \left(\deg_{\mathcal{V}_j^-}\right)^{\mathfrak{m}_-^{j,2l+1}(\mu)}, \tag{3.22}$$

where $\mathfrak{m}_-^{j,k}(\mu)$ is defined by (3.16). We have the following theorem for the resonance case.

Theorem 3.3.1. *Let $\mathfrak{f}: V \to V$ satisfy the conditions (A1), (A2), (A3) and $(A4)_k$ for k being an odd integer, and suppose that the equivariant invariant ω^H given by* (3.21) *is not equal to zero. Then, the system* (3.4) *admits a non-constant 2π-periodic solution. More precisely, if*

$$\omega^H = n_1(H_1) + n_2(H_2) + \cdots + n_k(H_k), \quad n_j \neq 0, \ \ j = 1, 2, \dots, k,$$

then for every (H_j), there exists a non-constant 2π-periodic solution $x(t)$ to (3.4) *such that $G_x \geq H_j$. In addition, if (H_j) is a maximal orbit type in $\Phi(G; \mathscr{H}) \setminus \{(G)\}$, then there exists a periodic solution to* (3.4) *with exactly isotropy group H_j.*

Proof. This result is a direct consequence of the existence property (G1) for the Brouwer equivariant G-degree. □

3.3.3 Example: S_4-Symmetric Odd Polygonal Map with Resonance

Example 3.3.2. In the setting similar to that in Subsection 3.2.5, consider $V := \mathbb{R}^4$, $\Gamma := S_4$ and linear S_4-equivariant operator

$$A := \begin{bmatrix} -3 & 1 & 1 & 1 \\ 1 & -3 & 1 & 1 \\ 1 & 1 & -3 & 1 \\ 1 & 1 & 1 & -3 \end{bmatrix}$$

Put

$$H := D_2^d \leq \mathbb{Z}_2 \times O(2)$$

In this case, V admits the primary isotypical decomposition $V = \mathcal{V}_4 \oplus \mathcal{V}_5$ and $\sigma(\mathscr{A})$ (see Tables 3.12, 3.13 and 3.15) are described as follows:

Since there is no negative eigenvalue for $k > 3$, by (3.22), we have

$$G\text{-}\deg(\mathscr{A}, B(\mathscr{H})) = \deg_{\mathcal{V}_{4,1}}$$

Table 3.2: Eigenvalues of $\mathscr{A}$

$\xi_{j,1,k} \in \sigma(\mathscr{A}\|_{\mathcal{V}_{j,1,k}})$		
$k \setminus j$	4	5
0	-4	0
1	$-3/2$	$1/2$
2	0	$4/5$
3	$1/2$	$9/10$

and

$$\begin{aligned}\omega^H &= (G) - G\text{-}\deg(\mathscr{A}, B(\mathscr{H}))\\ &= (D_4^d) + (D_3) + (D_2^d) - (\mathbb{Z}_2^z) - 2(D_1) + (\mathbb{Z}_1).\end{aligned}$$

(see Table 3.10). However, since ω^H does not contain any maximal orbit type in $\mathscr{H}$, there is no guarantee for the existence of solutions to system (3.4).

Table 3.3: Conjugacy Classes of Subgroups in $D_6 \times \mathbb{Z}_2$

ID	(H)	ID	(H)
1	$(\mathbb{Z}_1)$	17	(D_3)
2	$(\mathbb{Z}_2)$	18	$(\mathbb{Z}_6)$
3	(D_1)	19	$(\tilde{D}_3)$
4	$(\mathbb{Z}_1^2)$	20	$(\mathbb{Z}_3^2)$
5	$(\tilde{D}_1)$	21	(D_3^z)
6	$(\mathbb{Z}_2^z)$	22	$(\mathbb{Z}_6^z)$
7	(D_1^z)	23	$(\tilde{D}_3^z)$
8	$(\tilde{D}_1^z)$	24	(D_2^2)
9	$(\mathbb{Z}_3)$	25	(D_3^2)
10	(D_1^2)	26	$(\mathbb{Z}_6^2)$
11	$(\mathbb{Z}_2^2)$	27	(D_6)
12	(D_2)	28	$(\tilde{D}_3^2)$
13	$(\tilde{D}_1^2)$	29	(D_6^d)
14	(D_2^d)	30	(D_6^z)
15	(D_2^z)	31	$(D_6^{\tilde{d}})$
16	$(D_2^{\tilde{d}})$	32	(D_6^2)

Table 3.4: Conjugacy Classes of Subgroups in $D_6 \times \mathbb{Z}_2 \times O(2)$ (part 1)

ID	(S)	$\lvert W(S)\rvert$	ID	(S)	$\lvert W(S)\rvert$	ID	(S)	$\lvert W(S)\rvert$
1	$(\mathbb{Z}_1 \times \mathbb{Z}_n)$	∞	31	$(D_6^{\tilde{d}} \times \mathbb{Z}_n)$	∞	61	$(D_3 {}^{\mathbb{Z}_3}\times_{\mathbb{Z}_2} \mathbb{Z}_{2n})$	∞
2	$(\mathbb{Z}_2 \times \mathbb{Z}_n)$	∞	32	$(D_6^2 \times \mathbb{Z}_n)$	∞	62	$(\mathbb{Z}_6 {}^{\mathbb{Z}_3}\times_{\mathbb{Z}_2} \mathbb{Z}_{2n})$	∞
3	$(D_1 \times \mathbb{Z}_n)$	∞	33	$(\mathbb{Z}_2 {}^{\mathbb{Z}_1}\times_{\mathbb{Z}_2} \mathbb{Z}_{2n})$	∞	63	$(\tilde{D}_3 {}^{\mathbb{Z}_3}\times_{\mathbb{Z}_2} \mathbb{Z}_{2n})$	∞
4	$(\mathbb{Z}_1^2 \times \mathbb{Z}_n)$	∞	34	$(D_1 {}^{\mathbb{Z}_1}\times_{\mathbb{Z}_2} \mathbb{Z}_{2n})$	∞	64	$(\mathbb{Z}_3^2 {}^{\mathbb{Z}_3}\times_{\mathbb{Z}_2} \mathbb{Z}_{2n})$	∞
5	$(\tilde{D}_1 \times \mathbb{Z}_n)$	∞	35	$(\mathbb{Z}_1^2 {}^{\mathbb{Z}_1}\times_{\mathbb{Z}_2} \mathbb{Z}_{2n})$	∞	65	$(D_3^z {}^{\mathbb{Z}_3}\times_{\mathbb{Z}_2} \mathbb{Z}_{2n})$	∞
6	$(\mathbb{Z}_2^z \times \mathbb{Z}_n)$	∞	36	$(\tilde{D}_1 {}^{\mathbb{Z}_1}\times_{\mathbb{Z}_2} \mathbb{Z}_{2n})$	∞	66	$(\mathbb{Z}_6^z {}^{\mathbb{Z}_3}\times_{\mathbb{Z}_2} \mathbb{Z}_{2n})$	∞
7	$(D_1^z \times \mathbb{Z}_n)$	∞	37	$(\mathbb{Z}_2^z {}^{\mathbb{Z}_1}\times_{\mathbb{Z}_2} \mathbb{Z}_{2n})$	∞	67	$(\tilde{D}_3^z {}^{\mathbb{Z}_3}\times_{\mathbb{Z}_2} \mathbb{Z}_{2n})$	∞
8	$(\tilde{D}_1^z \times \mathbb{Z}_n)$	∞	38	$(D_1^z {}^{\mathbb{Z}_1}\times_{\mathbb{Z}_2} \mathbb{Z}_{2n})$	∞	68	$(D_2^2 {}^{D_1^2}\times_{\mathbb{Z}_2} \mathbb{Z}_{2n})$	∞
9	$(\mathbb{Z}_3 \times \mathbb{Z}_n)$	∞	39	$(\tilde{D}_1^z {}^{\mathbb{Z}_1}\times_{\mathbb{Z}_2} \mathbb{Z}_{2n})$	∞	69	$(D_2^2 {}^{\mathbb{Z}_2^2}\times_{\mathbb{Z}_2} \mathbb{Z}_{2n})$	∞
10	$(D_1^2 \times \mathbb{Z}_n)$	∞	40	$(D_1^2 {}^{D_1}\times_{\mathbb{Z}_2} \mathbb{Z}_{2n})$	∞	70	$(D_2^2 {}^{D_2}\times_{\mathbb{Z}_2} \mathbb{Z}_{2n})$	∞
11	$(\mathbb{Z}_2^2 \times \mathbb{Z}_n)$	∞	41	$(D_1^2 {}^{\mathbb{Z}_1^2}\times_{\mathbb{Z}_2} \mathbb{Z}_{2n})$	∞	71	$(D_2^2 {}^{\tilde{D}_1^2}\times_{\mathbb{Z}_2} \mathbb{Z}_{2n})$	∞
12	$(D_2 \times \mathbb{Z}_n)$	∞	42	$(D_1^2 {}^{D_1^z}\times_{\mathbb{Z}_2} \mathbb{Z}_{2n})$	∞	72	$(D_2^2 {}^{D_2^d}\times_{\mathbb{Z}_2} \mathbb{Z}_{2n})$	∞
13	$(\tilde{D}_1^2 \times \mathbb{Z}_n)$	∞	43	$(\mathbb{Z}_2^2 {}^{\mathbb{Z}_2}\times_{\mathbb{Z}_2} \mathbb{Z}_{2n})$	∞	73	$(D_2^2 {}^{D_2^z}\times_{\mathbb{Z}_2} \mathbb{Z}_{2n})$	∞
14	$(D_2^d \times \mathbb{Z}_n)$	∞	44	$(\mathbb{Z}_2^2 {}^{\mathbb{Z}_1^2}\times_{\mathbb{Z}_2} \mathbb{Z}_{2n})$	∞	74	$(D_2^2 {}^{D_2^{\tilde{d}}}\times_{\mathbb{Z}_2} \mathbb{Z}_{2n})$	∞
15	$(D_2^z \times \mathbb{Z}_n)$	∞	45	$(\mathbb{Z}_2^2 {}^{\mathbb{Z}_2^z}\times_{\mathbb{Z}_2} \mathbb{Z}_{2n})$	∞	75	$(D_3^2 {}^{D_3}\times_{\mathbb{Z}_2} \mathbb{Z}_{2n})$	∞
16	$(D_2^{\tilde{d}} \times \mathbb{Z}_n)$	∞	46	$(D_2 {}^{\mathbb{Z}_2}\times_{\mathbb{Z}_2} \mathbb{Z}_{2n})$	∞	76	$(D_3^2 {}^{\mathbb{Z}_3^2}\times_{\mathbb{Z}_2} \mathbb{Z}_{2n})$	∞
17	$(D_3 \times \mathbb{Z}_n)$	∞	47	$(D_2 {}^{D_1}\times_{\mathbb{Z}_2} \mathbb{Z}_{2n})$	∞	77	$(D_3^2 {}^{D_3^z}\times_{\mathbb{Z}_2} \mathbb{Z}_{2n})$	∞
18	$(\mathbb{Z}_6 \times \mathbb{Z}_n)$	∞	48	$(D_2 {}^{\tilde{D}_1}\times_{\mathbb{Z}_2} \mathbb{Z}_{2n})$	∞	78	$(\mathbb{Z}_6^2 {}^{\mathbb{Z}_6}\times_{\mathbb{Z}_2} \mathbb{Z}_{2n})$	∞
19	$(\tilde{D}_3 \times \mathbb{Z}_n)$	∞	49	$(\tilde{D}_1^2 {}^{\mathbb{Z}_1^2}\times_{\mathbb{Z}_2} \mathbb{Z}_{2n})$	∞	79	$(\mathbb{Z}_6^2 {}^{\mathbb{Z}_3^2}\times_{\mathbb{Z}_2} \mathbb{Z}_{2n})$	∞
20	$(\mathbb{Z}_3^2 \times \mathbb{Z}_n)$	∞	50	$(\tilde{D}_1^2 {}^{\tilde{D}_1}\times_{\mathbb{Z}_2} \mathbb{Z}_{2n})$	∞	80	$(\mathbb{Z}_6^2 {}^{\mathbb{Z}_6^z}\times_{\mathbb{Z}_2} \mathbb{Z}_{2n})$	∞
21	$(D_3^z \times \mathbb{Z}_n)$	∞	51	$(\tilde{D}_1^2 {}^{\tilde{D}_1^z}\times_{\mathbb{Z}_2} \mathbb{Z}_{2n})$	∞	81	$(D_6 {}^{D_3}\times_{\mathbb{Z}_2} \mathbb{Z}_{2n})$	∞
22	$(\mathbb{Z}_6^z \times \mathbb{Z}_n)$	∞	52	$(D_2^d {}^{D_1}\times_{\mathbb{Z}_2} \mathbb{Z}_{2n})$	∞	82	$(D_6 {}^{\mathbb{Z}_6}\times_{\mathbb{Z}_2} \mathbb{Z}_{2n})$	∞
23	$(\tilde{D}_3^z \times \mathbb{Z}_n)$	∞	53	$(D_2^d {}^{\mathbb{Z}_2^z}\times_{\mathbb{Z}_2} \mathbb{Z}_{2n})$	∞	83	$(D_6 {}^{\tilde{D}_3}\times_{\mathbb{Z}_2} \mathbb{Z}_{2n})$	∞
24	$(D_2^2 \times \mathbb{Z}_n)$	∞	54	$(D_2^d {}^{\tilde{D}_1^z}\times_{\mathbb{Z}_2} \mathbb{Z}_{2n})$	∞	84	$(\tilde{D}_3^2 {}^{\tilde{D}_3}\times_{\mathbb{Z}_2} \mathbb{Z}_{2n})$	∞
25	$(D_3^2 \times \mathbb{Z}_n)$	∞	55	$(D_2^z {}^{\mathbb{Z}_2}\times_{\mathbb{Z}_2} \mathbb{Z}_{2n})$	∞	85	$(\tilde{D}_3^2 {}^{\mathbb{Z}_3^2}\times_{\mathbb{Z}_2} \mathbb{Z}_{2n})$	∞
26	$(\mathbb{Z}_6^2 \times \mathbb{Z}_n)$	∞	56	$(D_2^z {}^{D_1^z}\times_{\mathbb{Z}_2} \mathbb{Z}_{2n})$	∞	86	$(\tilde{D}_3^2 {}^{\tilde{D}_3^z}\times_{\mathbb{Z}_2} \mathbb{Z}_{2n})$	∞
27	$(D_6 \times \mathbb{Z}_n)$	∞	57	$(D_2^z {}^{\tilde{D}_1^z}\times_{\mathbb{Z}_2} \mathbb{Z}_{2n})$	∞	87	$(D_6^d {}^{D_3}\times_{\mathbb{Z}_2} \mathbb{Z}_{2n})$	∞
28	$(\tilde{D}_3^2 \times \mathbb{Z}_n)$	∞	58	$(D_2^{\tilde{d}} {}^{\tilde{D}_1}\times_{\mathbb{Z}_2} \mathbb{Z}_{2n})$	∞	88	$(D_6^d {}^{\mathbb{Z}_6^z}\times_{\mathbb{Z}_2} \mathbb{Z}_{2n})$	∞
29	$(D_6^d \times \mathbb{Z}_n)$	∞	59	$(D_2^{\tilde{d}} {}^{\mathbb{Z}_2^z}\times_{\mathbb{Z}_2} \mathbb{Z}_{2n})$	∞	89	$(D_6^d {}^{\tilde{D}_3^z}\times_{\mathbb{Z}_2} \mathbb{Z}_{2n})$	∞
30	$(D_6^z \times \mathbb{Z}_n)$	∞	60	$(D_2^{\tilde{d}} {}^{D_1^z}\times_{\mathbb{Z}_2} \mathbb{Z}_{2n})$	∞	90	$(D_6^z {}^{\mathbb{Z}_6}\times_{\mathbb{Z}_2} \mathbb{Z}_{2n})$	∞

Table 3.5: Conjugacy Classes of Subgroups in $D_6 \times \mathbb{Z}_2 \times O(2)$ (part 2)

ID	(S)	$\|W(S)\|$	ID	(S)	$\|W(S)\|$	ID	(S)	$\|W(S)\|$
91	$(D_6^z {}^{D_3^z}\times_{\mathbb{Z}_2} \mathbb{Z}_{2n})$	∞	121	$(D_1^2 {}^{D_1}\times_{D_1} D_n)$	8	151	$(D_2^2 {}^{D_2}\times_{D_1} D_n)$	4
92	$(D_6^z {}^{\tilde{D}_3^z}\times_{\mathbb{Z}_2} \mathbb{Z}_{2n})$	∞	122	$(D_1^2 {}^{\mathbb{Z}_1^2}\times_{D_1} D_n)$	8	152	$(D_2^2 {}^{\tilde{D}_1^2}\times_{D_1} D_n)$	4
93	$(D_6^{\tilde{d}} {}^{\tilde{D}_3}\times_{\mathbb{Z}_2} \mathbb{Z}_{2n})$	∞	123	$(D_1^2 {}^{D_1^z}\times_{D_1} D_n)$	8	153	$(D_2^2 {}^{D_2^d}\times_{D_1} D_n)$	4
94	$(D_6^{\tilde{d}} {}^{D_3^z}\times_{\mathbb{Z}_2} \mathbb{Z}_{2n})$	∞	124	$(\mathbb{Z}_2^2 {}^{\mathbb{Z}_2}\times_{D_1} D_n)$	24	154	$(D_2^2 {}^{D_2^z}\times_{D_1} D_n)$	4
95	$(D_6^{\tilde{d}} {}^{\mathbb{Z}_6^z}\times_{\mathbb{Z}_2} \mathbb{Z}_{2n})$	∞	125	$(\mathbb{Z}_2^2 {}^{\mathbb{Z}_1^2}\times_{D_1} D_n)$	24	155	$(D_2^2 {}^{D_2^{\tilde{d}}}\times_{D_1} D_n)$	4
96	$(D_6^2 {}^{D_3^2}\times_{\mathbb{Z}_2} \mathbb{Z}_{2n})$	∞	126	$(\mathbb{Z}_2^2 {}^{\mathbb{Z}_2^z}\times_{D_1} D_n)$	24	156	$(D_3^2 {}^{D_3}\times_{D_1} D_n)$	8
97	$(D_6^2 {}^{\mathbb{Z}_6^2}\times_{\mathbb{Z}_2} \mathbb{Z}_{2n})$	∞	127	$(D_2 {}^{\mathbb{Z}_2}\times_{D_1} D_n)$	8	157	$(D_3^2 {}^{\mathbb{Z}_3^2}\times_{D_1} D_n)$	8
98	$(D_6^2 {}^{D_6}\times_{\mathbb{Z}_2} \mathbb{Z}_{2n})$	∞	128	$(D_2 {}^{D_1}\times_{D_1} D_n)$	8	158	$(D_3^2 {}^{D_3^z}\times_{D_1} D_n)$	8
99	$(D_6^2 {}^{\tilde{D}_3^2}\times_{\mathbb{Z}_2} \mathbb{Z}_{2n})$	∞	129	$(D_2 {}^{\tilde{D}_1}\times_{D_1} D_n)$	8	159	$(\mathbb{Z}_6^2 {}^{\mathbb{Z}_6}\times_{D_1} D_n)$	8
100	$(D_6^2 {}^{D_6^d}\times_{\mathbb{Z}_2} \mathbb{Z}_{2n})$	∞	130	$(\tilde{D}_1^2 {}^{\mathbb{Z}_1^2}\times_{D_1} D_n)$	8	160	$(\mathbb{Z}_6^2 {}^{\mathbb{Z}_3^2}\times_{D_1} D_n)$	8
101	$(D_6^2 {}^{D_6^z}\times_{\mathbb{Z}_2} \mathbb{Z}_{2n})$	∞	131	$(\tilde{D}_1^2 {}^{\tilde{D}_1}\times_{D_1} D_n)$	8	161	$(\mathbb{Z}_6^2 {}^{\mathbb{Z}_6^z}\times_{D_1} D_n)$	8
102	$(D_6^2 {}^{D_6^{\tilde{d}}}\times_{\mathbb{Z}_2} \mathbb{Z}_{2n})$	∞	132	$(\tilde{D}_1^2 {}^{\tilde{D}_1^z}\times_{D_1} D_n)$	8	162	$(D_6 {}^{D_3}\times_{D_1} D_n)$	8
103	$(\mathbb{Z}_3 {}^{\mathbb{Z}_1}\times_{\mathbb{Z}_3} \mathbb{Z}_{3n})$	∞	133	$(D_2^d {}^{D_1}\times_{D_1} D_n)$	8	163	$(D_6 {}^{\mathbb{Z}_6}\times_{D_1} D_n)$	8
104	$(\mathbb{Z}_6 {}^{\mathbb{Z}_2}\times_{\mathbb{Z}_3} \mathbb{Z}_{3n})$	∞	134	$(D_2^d {}^{\mathbb{Z}_2^z}\times_{D_1} D_n)$	8	164	$(D_6 {}^{\tilde{D}_3}\times_{D_1} D_n)$	8
105	$(\mathbb{Z}_3^2 {}^{\mathbb{Z}_1^2}\times_{\mathbb{Z}_3} \mathbb{Z}_{3n})$	∞	135	$(D_2^d {}^{\tilde{D}_1^z}\times_{D_1} D_n)$	8	165	$(\tilde{D}_3^2 {}^{\tilde{D}_3}\times_{D_1} D_n)$	8
106	$(\mathbb{Z}_6^z {}^{\mathbb{Z}_2^z}\times_{\mathbb{Z}_3} \mathbb{Z}_{3n})$	∞	136	$(D_2^z {}^{\mathbb{Z}_2}\times_{D_1} D_n)$	8	166	$(\tilde{D}_3^2 {}^{\mathbb{Z}_3^2}\times_{D_1} D_n)$	8
107	$(\mathbb{Z}_6^2 {}^{\mathbb{Z}_2^2}\times_{\mathbb{Z}_3} \mathbb{Z}_{3n})$	∞	137	$(D_2^z {}^{D_1^z}\times_{D_1} D_n)$	8	167	$(\tilde{D}_3^2 {}^{\tilde{D}_3^z}\times_{D_1} D_n)$	8
108	$(\mathbb{Z}_6 {}^{\mathbb{Z}_1}\times_{\mathbb{Z}_6} \mathbb{Z}_{6n})$	∞	138	$(D_2^z {}^{\tilde{D}_1^z}\times_{D_1} D_n)$	8	168	$(D_6^d {}^{D_3}\times_{D_1} D_n)$	8
109	$(\mathbb{Z}_3^2 {}^{\mathbb{Z}_1}\times_{\mathbb{Z}_6} \mathbb{Z}_{6n})$	∞	139	$(D_2^{\tilde{d}} {}^{\tilde{D}_1}\times_{D_1} D_n)$	8	169	$(D_6^d {}^{\mathbb{Z}_6^z}\times_{D_1} D_n)$	8
110	$(\mathbb{Z}_6^z {}^{\mathbb{Z}_1}\times_{\mathbb{Z}_6} \mathbb{Z}_{6n})$	∞	140	$(D_2^{\tilde{d}} {}^{\mathbb{Z}_2^z}\times_{D_1} D_n)$	8	170	$(D_6^d {}^{\tilde{D}_3^z}\times_{D_1} D_n)$	8
111	$(\mathbb{Z}_6^2 {}^{\mathbb{Z}_2}\times_{\mathbb{Z}_6} \mathbb{Z}_{6n})$	∞	141	$(D_2^{\tilde{d}} {}^{D_1^z}\times_{D_1} D_n)$	8	171	$(D_6^z {}^{\mathbb{Z}_6}\times_{D_1} D_n)$	8
112	$(\mathbb{Z}_6^2 {}^{\mathbb{Z}_1^2}\times_{\mathbb{Z}_6} \mathbb{Z}_{6n})$	∞	142	$(D_3 {}^{\mathbb{Z}_3}\times_{D_1} D_n)$	16	172	$(D_6^z {}^{D_3^z}\times_{D_1} D_n)$	8
113	$(\mathbb{Z}_6^2 {}^{\mathbb{Z}_2^z}\times_{\mathbb{Z}_6} \mathbb{Z}_{6n})$	∞	143	$(\mathbb{Z}_6 {}^{\mathbb{Z}_3}\times_{D_1} D_n)$	16	173	$(D_6^z {}^{\tilde{D}_3^z}\times_{D_1} D_n)$	8
114	$(\mathbb{Z}_2 {}^{\mathbb{Z}_1}\times_{D_1} D_n)$	48	144	$(\tilde{D}_3 {}^{\mathbb{Z}_3}\times_{D_1} D_n)$	16	174	$(D_6^{\tilde{d}} {}^{\tilde{D}_3}\times_{D_1} D_n)$	8
115	$(D_1 {}^{\mathbb{Z}_1}\times_{D_1} D_n)$	16	145	$(\mathbb{Z}_3^2 {}^{\mathbb{Z}_3}\times_{D_1} D_n)$	16	175	$(D_6^{\tilde{d}} {}^{D_3^z}\times_{D_1} D_n)$	8
116	$(\mathbb{Z}_1^2 {}^{\mathbb{Z}_1}\times_{D_1} D_n)$	48	146	$(D_3^z {}^{\mathbb{Z}_3}\times_{D_1} D_n)$	16	176	$(D_6^{\tilde{d}} {}^{\mathbb{Z}_6^z}\times_{D_1} D_n)$	8
117	$(\tilde{D}_1 {}^{\mathbb{Z}_1}\times_{D_1} D_n)$	16	147	$(\mathbb{Z}_6^z {}^{\mathbb{Z}_3}\times_{D_1} D_n)$	16	177	$(D_6^2 {}^{D_3^2}\times_{D_1} D_n)$	4
118	$(\mathbb{Z}_2^z {}^{\mathbb{Z}_1}\times_{D_1} D_n)$	48	148	$(\tilde{D}_3^z {}^{\mathbb{Z}_3}\times_{D_1} D_n)$	16	178	$(D_6^2 {}^{\mathbb{Z}_6^2}\times_{D_1} D_n)$	4
119	$(D_1^z {}^{\mathbb{Z}_1}\times_{D_1} D_n)$	16	149	$(D_2^2 {}^{D_1^2}\times_{D_1} D_n)$	4	179	$(D_6^2 {}^{D_6}\times_{D_1} D_n)$	4
120	$(\tilde{D}_1^z {}^{\mathbb{Z}_1}\times_{D_1} D_n)$	16	150	$(D_2^2 {}^{\mathbb{Z}_2^2}\times_{D_1} D_n)$	4	180	$(D_6^2 {}^{\tilde{D}_3^2}\times_{D_1} D_n)$	4

Table 3.6: Conjugacy Classes of Subgroups in $D_6 \times \mathbb{Z}_2 \times O(2)$ (part 3)

ID	(S)	$\|W(S)\|$	ID	(S)	$\|W(S)\|$	ID	(S)	$\|W(S)\|$
181	$(D_6^2 {}^{D_6^d}\times_{D_1} D_n)$	4	211	$(D_2^2 {}^{\mathbb{Z}_1^2}\times_{D_2}^{D_1^2} D_{2n})$	4	241	$(D_6^z {}^{\mathbb{Z}_3}\times_{D_2}^{\mathbb{Z}_6} D_{2n})$	8
182	$(D_6^2 {}^{D_6^z}\times_{D_1} D_n)$	4	212	$(D_2^2 {}^{\mathbb{Z}_1^2}\times_{D_2}^{\mathbb{Z}_2^2} D_{2n})$	4	242	$(D_6^z {}^{\mathbb{Z}_3}\times_{D_2}^{D_3^z} D_{2n})$	8
183	$(D_6^2 {}^{D_6^{\tilde{d}}}\times_{D_1} D_n)$	4	213	$(D_2^2 {}^{\mathbb{Z}_1^2}\times_{D_2}^{\tilde{D}_1^2} D_{2n})$	4	243	$(D_6^z {}^{\mathbb{Z}_3}\times_{D_2}^{\tilde{D}_3^z} D_{2n})$	8
184	$(D_1^2 {}^{\mathbb{Z}_1}\times_{D_2}^{D_1} D_{2n})$	8	214	$(D_2^2 {}^{\tilde{D}_1}\times_{D_2}^{D_2} D_{2n})$	4	244	$(D_6^{\tilde{d}} {}^{\mathbb{Z}_3}\times_{D_2}^{\tilde{D}_3} D_{2n})$	8
185	$(D_1^2 {}^{\mathbb{Z}_1}\times_{D_2}^{\mathbb{Z}_1^2} D_{2n})$	8	215	$(D_2^2 {}^{\tilde{D}_1}\times_{D_2}^{\tilde{D}_1^2} D_{2n})$	4	245	$(D_6^{\tilde{d}} {}^{\mathbb{Z}_3}\times_{D_2}^{D_3^z} D_{2n})$	8
186	$(D_1^2 {}^{\mathbb{Z}_1}\times_{D_2}^{D_1^z} D_{2n})$	8	216	$(D_2^2 {}^{\tilde{D}_1}\times_{D_2}^{D_2^{\tilde{d}}} D_{2n})$	4	246	$(D_6^{\tilde{d}} {}^{\mathbb{Z}_3}\times_{D_2}^{\mathbb{Z}_6^z} D_{2n})$	8
187	$(\mathbb{Z}_2^2 {}^{\mathbb{Z}_1}\times_{D_2}^{\mathbb{Z}_2} D_{2n})$	24	217	$(D_2^2 {}^{\mathbb{Z}_2^z}\times_{D_2}^{\mathbb{Z}_2^2} D_{2n})$	4	247	$(D_6^2 {}^{D_3}\times_{D_2}^{D_3^2} D_{2n})$	4
188	$(\mathbb{Z}_2^2 {}^{\mathbb{Z}_1}\times_{D_2}^{\mathbb{Z}_1^2} D_{2n})$	24	218	$(D_2^2 {}^{\mathbb{Z}_2^z}\times_{D_2}^{D_2^d} D_{2n})$	4	248	$(D_6^2 {}^{D_3}\times_{D_2}^{D_6} D_{2n})$	4
189	$(\mathbb{Z}_2^2 {}^{\mathbb{Z}_1}\times_{D_2}^{\mathbb{Z}_2^z} D_{2n})$	24	219	$(D_2^2 {}^{\mathbb{Z}_2^z}\times_{D_2}^{D_2^{\tilde{d}}} D_{2n})$	4	249	$(D_6^2 {}^{D_3}\times_{D_2}^{D_6^d} D_{2n})$	4
190	$(D_2 {}^{\mathbb{Z}_1}\times_{D_2}^{\mathbb{Z}_2} D_{2n})$	8	220	$(D_2^2 {}^{D_1^z}\times_{D_2}^{D_1^2} D_{2n})$	4	250	$(D_6^2 {}^{\mathbb{Z}_6}\times_{D_2}^{\mathbb{Z}_6^2} D_{2n})$	4
191	$(D_2 {}^{\mathbb{Z}_1}\times_{D_2}^{D_1} D_{2n})$	8	221	$(D_2^2 {}^{D_1^z}\times_{D_2}^{D_2^z} D_{2n})$	4	251	$(D_6^2 {}^{\mathbb{Z}_6}\times_{D_2}^{D_6} D_{2n})$	4
192	$(D_2 {}^{\mathbb{Z}_1}\times_{D_2}^{\tilde{D}_1} D_{2n})$	8	222	$(D_2^2 {}^{D_1^z}\times_{D_2}^{D_2^{\tilde{d}}} D_{2n})$	4	252	$(D_6^2 {}^{\mathbb{Z}_6}\times_{D_2}^{D_6^z} D_{2n})$	4
193	$(\tilde{D}_1^2 {}^{\mathbb{Z}_1}\times_{D_2}^{\mathbb{Z}_1^2} D_{2n})$	8	223	$(D_2^2 {}^{\tilde{D}_1^z}\times_{D_2}^{\tilde{D}_1^2} D_{2n})$	4	253	$(D_6^2 {}^{\tilde{D}_3}\times_{D_2}^{D_6} D_{2n})$	4
194	$(\tilde{D}_1^2 {}^{\mathbb{Z}_1}\times_{D_2}^{\tilde{D}_1} D_{2n})$	8	224	$(D_2^2 {}^{\tilde{D}_1^z}\times_{D_2}^{D_2^d} D_{2n})$	4	254	$(D_6^2 {}^{\tilde{D}_3}\times_{D_2}^{\tilde{D}_3^2} D_{2n})$	4
195	$(\tilde{D}_1^2 {}^{\mathbb{Z}_1}\times_{D_2}^{\tilde{D}_1^z} D_{2n})$	8	225	$(D_2^2 {}^{\tilde{D}_1^z}\times_{D_2}^{D_2^z} D_{2n})$	4	255	$(D_6^2 {}^{\tilde{D}_3}\times_{D_2}^{D_6^{\tilde{d}}} D_{2n})$	4
196	$(D_2^d {}^{\mathbb{Z}_1}\times_{D_2}^{D_1} D_{2n})$	8	226	$(D_3^2 {}^{\mathbb{Z}_3}\times_{D_2}^{D_3} D_{2n})$	8	256	$(D_6^2 {}^{\mathbb{Z}_3^2}\times_{D_2}^{D_3^2} D_{2n})$	4
197	$(D_2^d {}^{\mathbb{Z}_1}\times_{D_2}^{\mathbb{Z}_2^z} D_{2n})$	8	227	$(D_3^2 {}^{\mathbb{Z}_3}\times_{D_2}^{\mathbb{Z}_3^2} D_{2n})$	8	257	$(D_6^2 {}^{\mathbb{Z}_3^2}\times_{D_2}^{\mathbb{Z}_6^2} D_{2n})$	4
198	$(D_2^d {}^{\mathbb{Z}_1}\times_{D_2}^{\tilde{D}_1^z} D_{2n})$	8	228	$(D_3^2 {}^{\mathbb{Z}_3}\times_{D_2}^{D_3^z} D_{2n})$	8	258	$(D_6^2 {}^{\mathbb{Z}_3^2}\times_{D_2}^{\tilde{D}_3^2} D_{2n})$	4
199	$(D_2^z {}^{\mathbb{Z}_1}\times_{D_2}^{\mathbb{Z}_2} D_{2n})$	8	229	$(\mathbb{Z}_6^2 {}^{\mathbb{Z}_3}\times_{D_2}^{\mathbb{Z}_6} D_{2n})$	8	259	$(D_6^2 {}^{D_3^z}\times_{D_2}^{D_3^2} D_{2n})$	4
200	$(D_2^z {}^{\mathbb{Z}_1}\times_{D_2}^{D_1^z} D_{2n})$	8	230	$(\mathbb{Z}_6^2 {}^{\mathbb{Z}_3}\times_{D_2}^{\mathbb{Z}_3^2} D_{2n})$	8	260	$(D_6^2 {}^{D_3^z}\times_{D_2}^{D_6^z} D_{2n})$	4
201	$(D_2^z {}^{\mathbb{Z}_1}\times_{D_2}^{\tilde{D}_1^z} D_{2n})$	8	231	$(\mathbb{Z}_6^2 {}^{\mathbb{Z}_3}\times_{D_2}^{\mathbb{Z}_6^z} D_{2n})$	8	261	$(D_6^2 {}^{D_3^z}\times_{D_2}^{D_6^{\tilde{d}}} D_{2n})$	4
202	$(D_2^{\tilde{d}} {}^{\mathbb{Z}_1}\times_{D_2}^{\tilde{D}_1} D_{2n})$	8	232	$(D_6 {}^{\mathbb{Z}_3}\times_{D_2}^{D_3} D_{2n})$	8	262	$(D_6^2 {}^{\mathbb{Z}_6^z}\times_{D_2}^{\mathbb{Z}_6^2} D_{2n})$	4
203	$(D_2^{\tilde{d}} {}^{\mathbb{Z}_1}\times_{D_2}^{\mathbb{Z}_2^z} D_{2n})$	8	233	$(D_6 {}^{\mathbb{Z}_3}\times_{D_2}^{\mathbb{Z}_6} D_{2n})$	8	263	$(D_6^2 {}^{\mathbb{Z}_6^z}\times_{D_2}^{D_6^d} D_{2n})$	4
204	$(D_2^{\tilde{d}} {}^{\mathbb{Z}_1}\times_{D_2}^{D_1^z} D_{2n})$	8	234	$(D_6 {}^{\mathbb{Z}_3}\times_{D_2}^{\tilde{D}_3} D_{2n})$	8	264	$(D_6^2 {}^{\mathbb{Z}_6^z}\times_{D_2}^{D_6^{\tilde{d}}} D_{2n})$	4
205	$(D_2^2 {}^{\mathbb{Z}_2}\times_{D_2}^{\mathbb{Z}_2^2} D_{2n})$	4	235	$(\tilde{D}_3^2 {}^{\mathbb{Z}_3}\times_{D_2}^{\tilde{D}_3} D_{2n})$	8	265	$(D_6^2 {}^{\tilde{D}_3^z}\times_{D_2}^{\tilde{D}_3^2} D_{2n})$	4
206	$(D_2^2 {}^{\mathbb{Z}_2}\times_{D_2}^{D_2} D_{2n})$	4	236	$(\tilde{D}_3^2 {}^{\mathbb{Z}_3}\times_{D_2}^{\mathbb{Z}_3^2} D_{2n})$	8	266	$(D_6^2 {}^{\tilde{D}_3^z}\times_{D_2}^{D_6^d} D_{2n})$	4
207	$(D_2^2 {}^{\mathbb{Z}_2}\times_{D_2}^{D_2^z} D_{2n})$	4	237	$(\tilde{D}_3^2 {}^{\mathbb{Z}_3}\times_{D_2}^{\tilde{D}_3^z} D_{2n})$	8	267	$(D_6^2 {}^{\tilde{D}_3^z}\times_{D_2}^{D_6^z} D_{2n})$	4
208	$(D_2^2 {}^{D_1}\times_{D_2}^{D_1^2} D_{2n})$	4	238	$(D_6^d {}^{\mathbb{Z}_3}\times_{D_2}^{D_3} D_{2n})$	8	268	$(D_3 {}^{\mathbb{Z}_1}\times_{D_3} D_{3n})$	8
209	$(D_2^2 {}^{D_1}\times_{D_2}^{D_2} D_{2n})$	4	239	$(D_6^d {}^{\mathbb{Z}_3}\times_{D_2}^{\mathbb{Z}_6^z} D_{2n})$	8	269	$(\tilde{D}_3 {}^{\mathbb{Z}_1}\times_{D_3} D_{3n})$	8
210	$(D_2^2 {}^{D_1}\times_{D_2}^{D_2^d} D_{2n})$	4	240	$(D_6^d {}^{\mathbb{Z}_3}\times_{D_2}^{\tilde{D}_3^z} D_{2n})$	8	270	$(D_3^z {}^{\mathbb{Z}_1}\times_{D_3} D_{3n})$	8

Table 3.7: Conjugacy Classes of Subgroups in $D_6 \times \mathbb{Z}_2 \times O(2)$ (part 4)

ID	(S)	$\|W(S)\|$	ID	(S)	$\|W(S)\|$	ID	(S)	$\|W(S)\|$
271	$(\tilde{D}_3^z {}^{\mathbb{Z}_1}\times_{D_3} D_{3n})$	8	301	$(D_2^d \times D_n)$	4	331	$(\mathbb{Z}_2^2 {}^{\mathbb{Z}_1^2}\times_{\mathbb{Z}_2} D_{2n})$	12
272	$(D_3^2 {}^{\mathbb{Z}_1^2}\times_{D_3} D_{3n})$	4	302	$(D_2^z \times D_n)$	4	332	$(\mathbb{Z}_2^2 {}^{\mathbb{Z}_2^z}\times_{\mathbb{Z}_2} D_{2n})$	12
273	$(D_6 {}^{\mathbb{Z}_2}\times_{D_3} D_{3n})$	4	303	$(D_2^{\tilde{d}} \times D_n)$	4	333	$(D_2 {}^{\mathbb{Z}_2}\times_{\mathbb{Z}_2} D_{2n})$	4
274	$(\tilde{D}_3^2 {}^{\mathbb{Z}_1^2}\times_{D_3} D_{3n})$	4	304	$(D_3 \times D_n)$	8	334	$(D_2 {}^{D_1}\times_{\mathbb{Z}_2} D_{2n})$	4
275	$(D_6^d {}^{\mathbb{Z}_2^z}\times_{D_3} D_{3n})$	4	305	$(\mathbb{Z}_6 \times D_n)$	8	335	$(D_2 {}^{\tilde{D}_1}\times_{\mathbb{Z}_2} D_{2n})$	4
276	$(D_6^z {}^{\mathbb{Z}_2}\times_{D_3} D_{3n})$	4	306	$(\tilde{D}_3 \times D_n)$	8	336	$(\tilde{D}_1^2 {}^{\mathbb{Z}_1^2}\times_{\mathbb{Z}_2} D_{2n})$	4
277	$(D_6^{\tilde{d}} {}^{\mathbb{Z}_2^z}\times_{D_3} D_{3n})$	4	307	$(\mathbb{Z}_3^2 \times D_n)$	8	337	$(\tilde{D}_1^2 {}^{\tilde{D}_1}\times_{\mathbb{Z}_2} D_{2n})$	4
278	$(D_6^2 {}^{\mathbb{Z}_2^2}\times_{D_3} D_{3n})$	2	308	$(D_3^z \times D_n)$	8	338	$(\tilde{D}_1^2 {}^{\tilde{D}_1^z}\times_{\mathbb{Z}_2} D_{2n})$	4
279	$(D_3^2 {}^{\mathbb{Z}_1}\times_{D_6} D_{6n})$	4	309	$(\mathbb{Z}_6^z \times D_n)$	8	339	$(D_2^d {}^{D_1}\times_{\mathbb{Z}_2} D_{2n})$	4
280	$(D_6 {}^{\mathbb{Z}_1}\times_{D_6} D_{6n})$	4	310	$(\tilde{D}_3^z \times D_n)$	8	340	$(D_2^d {}^{\mathbb{Z}_2^z}\times_{\mathbb{Z}_2} D_{2n})$	4
281	$(\tilde{D}_3^2 {}^{\mathbb{Z}_1}\times_{D_6} D_{6n})$	4	311	$(D_2^2 \times D_n)$	2	341	$(D_2^d {}^{\tilde{D}_1^z}\times_{\mathbb{Z}_2} D_{2n})$	4
282	$(D_6^d {}^{\mathbb{Z}_1}\times_{D_6} D_{6n})$	4	312	$(D_3^2 \times D_n)$	4	342	$(D_2^z {}^{\mathbb{Z}_2}\times_{\mathbb{Z}_2} D_{2n})$	4
283	$(D_6^z {}^{\mathbb{Z}_1}\times_{D_6} D_{6n})$	4	313	$(\mathbb{Z}_6^2 \times D_n)$	4	343	$(D_2^z {}^{D_1^z}\times_{\mathbb{Z}_2} D_{2n})$	4
284	$(D_6^{\tilde{d}} {}^{\mathbb{Z}_1}\times_{D_6} D_{6n})$	4	314	$(D_6 \times D_n)$	4	344	$(D_2^z {}^{\tilde{D}_1^z}\times_{\mathbb{Z}_2} D_{2n})$	4
285	$(D_6^2 {}^{\mathbb{Z}_2}\times_{D_6} D_{6n})$	2	315	$(\tilde{D}_3^2 \times D_n)$	4	345	$(D_2^{\tilde{d}} {}^{\tilde{D}_1}\times_{\mathbb{Z}_2} D_{2n})$	4
286	$(D_6^2 {}^{\mathbb{Z}_1^2}\times_{D_6} D_{6n})$	2	316	$(D_6^d \times D_n)$	4	346	$(D_2^{\tilde{d}} {}^{\mathbb{Z}_2^z}\times_{\mathbb{Z}_2} D_{2n})$	4
287	$(D_6^2 {}^{\mathbb{Z}_2^z}\times_{D_6} D_{6n})$	2	317	$(D_6^z \times D_n)$	4	347	$(D_2^{\tilde{d}} {}^{D_1^z}\times_{\mathbb{Z}_2} D_{2n})$	4
288	$(\mathbb{Z}_1 \times D_n)$	48	318	$(D_6^{\tilde{d}} \times D_n)$	4	348	$(D_3 {}^{\mathbb{Z}_3}\times_{\mathbb{Z}_2} D_{2n})$	8
289	$(\mathbb{Z}_2 \times D_n)$	24	319	$(D_6^2 \times D_n)$	2	349	$(\mathbb{Z}_6 {}^{\mathbb{Z}_3}\times_{\mathbb{Z}_2} D_{2n})$	8
290	$(D_1 \times D_n)$	8	320	$(\mathbb{Z}_2 {}^{\mathbb{Z}_1}\times_{\mathbb{Z}_2} D_{2n})$	24	350	$(\tilde{D}_3 {}^{\mathbb{Z}_3}\times_{\mathbb{Z}_2} D_{2n})$	8
291	$(\mathbb{Z}_1^2 \times D_n)$	24	321	$(D_1 {}^{\mathbb{Z}_1}\times_{\mathbb{Z}_2} D_{2n})$	8	351	$(\mathbb{Z}_3^2 {}^{\mathbb{Z}_3}\times_{\mathbb{Z}_2} D_{2n})$	8
292	$(\tilde{D}_1 \times D_n)$	8	322	$(\mathbb{Z}_1^2 {}^{\mathbb{Z}_1}\times_{\mathbb{Z}_2} D_{2n})$	24	352	$(D_3^z {}^{\mathbb{Z}_3}\times_{\mathbb{Z}_2} D_{2n})$	8
293	$(\mathbb{Z}_2^z \times D_n)$	24	323	$(\tilde{D}_1 {}^{\mathbb{Z}_1}\times_{\mathbb{Z}_2} D_{2n})$	8	353	$(\mathbb{Z}_6^z {}^{\mathbb{Z}_3}\times_{\mathbb{Z}_2} D_{2n})$	8
294	$(D_1^z \times D_n)$	8	324	$(\mathbb{Z}_2^z {}^{\mathbb{Z}_1}\times_{\mathbb{Z}_2} D_{2n})$	24	354	$(\tilde{D}_3^z {}^{\mathbb{Z}_3}\times_{\mathbb{Z}_2} D_{2n})$	8
295	$(\tilde{D}_1^z \times D_n)$	8	325	$(D_1^z {}^{\mathbb{Z}_1}\times_{\mathbb{Z}_2} D_{2n})$	8	355	$(D_2^2 {}^{D_1^2}\times_{\mathbb{Z}_2} D_{2n})$	2
296	$(\mathbb{Z}_3 \times D_n)$	16	326	$(\tilde{D}_1^z {}^{\mathbb{Z}_1}\times_{\mathbb{Z}_2} D_{2n})$	8	356	$(D_2^2 {}^{\mathbb{Z}_2^2}\times_{\mathbb{Z}_2} D_{2n})$	2
297	$(D_1^2 \times D_n)$	4	327	$(D_1^2 {}^{D_1}\times_{\mathbb{Z}_2} D_{2n})$	4	357	$(D_2^2 {}^{D_2}\times_{\mathbb{Z}_2} D_{2n})$	2
298	$(\mathbb{Z}_2^2 \times D_n)$	12	328	$(D_1^2 {}^{\mathbb{Z}_1^2}\times_{\mathbb{Z}_2} D_{2n})$	4	358	$(D_2^2 {}^{\tilde{D}_1^2}\times_{\mathbb{Z}_2} D_{2n})$	2
299	$(D_2 \times D_n)$	4	329	$(D_1^2 {}^{D_1^z}\times_{\mathbb{Z}_2} D_{2n})$	4	359	$(D_2^2 {}^{D_2^d}\times_{\mathbb{Z}_2} D_{2n})$	2
300	$(\tilde{D}_1^2 \times D_n)$	4	330	$(\mathbb{Z}_2^2 {}^{\mathbb{Z}_2}\times_{\mathbb{Z}_2} D_{2n})$	12	360	$(D_2^2 {}^{D_2^z}\times_{\mathbb{Z}_2} D_{2n})$	2

Table 3.8: Conjugacy Classes of Subgroups in $D_6 \times \mathbb{Z}_2 \times O(2)$ (part 5)

ID	(S)	$\lvert W(S)\rvert$	ID	(S)	$\lvert W(S)\rvert$	ID	(S)	$\lvert W(S)\rvert$
361	$(D_2^2 {}^{D_2^{\tilde{d}}} \times_{\mathbb{Z}_2} D_{2n})$	2	391	$(\mathbb{Z}_2 \times SO(2))$	24	421	$(D_6^2 \times SO(2))$	2
362	$(D_3^2 {}^{D_3} \times_{\mathbb{Z}_2} D_{2n})$	4	392	$(D_1 \times SO(2))$	8	422	$(\mathbb{Z}_2 {}^{\mathbb{Z}_1} \times_{D_1} O(2))$	24
363	$(D_3^2 {}^{\mathbb{Z}_3^2} \times_{\mathbb{Z}_2} D_{2n})$	4	393	$(\mathbb{Z}_1^2 \times SO(2))$	24	423	$(D_1 {}^{\mathbb{Z}_1} \times_{D_1} O(2))$	8
364	$(D_3^2 {}^{D_3^z} \times_{\mathbb{Z}_2} D_{2n})$	4	394	$(\tilde{D}_1 \times SO(2))$	8	424	$(\mathbb{Z}_1^2 {}^{\mathbb{Z}_1} \times_{D_1} O(2))$	24
365	$(\mathbb{Z}_6^2 {}^{\mathbb{Z}_6} \times_{\mathbb{Z}_2} D_{2n})$	4	395	$(\mathbb{Z}_2^z \times SO(2))$	24	425	$(\tilde{D}_1 {}^{\mathbb{Z}_1} \times_{D_1} O(2))$	8
366	$(\mathbb{Z}_6^2 {}^{\mathbb{Z}_3^2} \times_{\mathbb{Z}_2} D_{2n})$	4	396	$(D_1^z \times SO(2))$	8	426	$(\mathbb{Z}_2^z {}^{\mathbb{Z}_1} \times_{D_1} O(2))$	24
367	$(\mathbb{Z}_6^2 {}^{\mathbb{Z}_6^z} \times_{\mathbb{Z}_2} D_{2n})$	4	397	$(\tilde{D}_1^z \times SO(2))$	8	427	$(D_1^z {}^{\mathbb{Z}_1} \times_{D_1} O(2))$	8
368	$(D_6 {}^{D_3} \times_{\mathbb{Z}_2} D_{2n})$	4	398	$(\mathbb{Z}_3 \times SO(2))$	16	428	$(\tilde{D}_1^z {}^{\mathbb{Z}_1} \times_{D_1} O(2))$	8
369	$(D_6 {}^{\mathbb{Z}_6} \times_{\mathbb{Z}_2} D_{2n})$	4	399	$(D_1^2 \times SO(2))$	4	429	$(D_1^2 {}^{D_1} \times_{D_1} O(2))$	4
370	$(D_6 {}^{\tilde{D}_3} \times_{\mathbb{Z}_2} D_{2n})$	4	400	$(\mathbb{Z}_2^2 \times SO(2))$	12	430	$(D_1^2 {}^{\mathbb{Z}_1^2} \times_{D_1} O(2))$	4
371	$(\tilde{D}_3^2 {}^{\tilde{D}_3} \times_{\mathbb{Z}_2} D_{2n})$	4	401	$(D_2 \times SO(2))$	4	431	$(D_1^2 {}^{D_1^z} \times_{D_1} O(2))$	4
372	$(\tilde{D}_3^2 {}^{\mathbb{Z}_3^2} \times_{\mathbb{Z}_2} D_{2n})$	4	402	$(\tilde{D}_1^2 \times SO(2))$	4	432	$(\mathbb{Z}_2^2 {}^{\mathbb{Z}_2} \times_{D_1} O(2))$	12
373	$(\tilde{D}_3^2 {}^{\tilde{D}_3^z} \times_{\mathbb{Z}_2} D_{2n})$	4	403	$(D_2^d \times SO(2))$	4	433	$(\mathbb{Z}_2^2 {}^{\mathbb{Z}_1^2} \times_{D_1} O(2))$	12
374	$(D_6^d {}^{D_3} \times_{\mathbb{Z}_2} D_{2n})$	4	404	$(D_2^z \times SO(2))$	4	434	$(\mathbb{Z}_2^2 {}^{\mathbb{Z}_2^z} \times_{D_1} O(2))$	12
375	$(D_6^d {}^{\mathbb{Z}_6^z} \times_{\mathbb{Z}_2} D_{2n})$	4	405	$(D_2^{\tilde{d}} \times SO(2))$	4	435	$(D_2 {}^{\mathbb{Z}_2} \times_{D_1} O(2))$	4
376	$(D_6^d {}^{\tilde{D}_3^z} \times_{\mathbb{Z}_2} D_{2n})$	4	406	$(D_3 \times SO(2))$	8	436	$(D_2 {}^{D_1} \times_{D_1} O(2))$	4
377	$(D_6^z {}^{\mathbb{Z}_6} \times_{\mathbb{Z}_2} D_{2n})$	4	407	$(\mathbb{Z}_6 \times SO(2))$	8	437	$(D_2 {}^{\tilde{D}_1} \times_{D_1} O(2))$	4
378	$(D_6^z {}^{D_3^z} \times_{\mathbb{Z}_2} D_{2n})$	4	408	$(\tilde{D}_3 \times SO(2))$	8	438	$(\tilde{D}_1^2 {}^{\mathbb{Z}_1^2} \times_{D_1} O(2))$	4
379	$(D_6^z {}^{\tilde{D}_3^z} \times_{\mathbb{Z}_2} D_{2n})$	4	409	$(\mathbb{Z}_3^2 \times SO(2))$	8	439	$(\tilde{D}_1^2 {}^{\tilde{D}_1} \times_{D_1} O(2))$	4
380	$(D_6^{\tilde{d}} {}^{\tilde{D}_3} \times_{\mathbb{Z}_2} D_{2n})$	4	410	$(D_3^z \times SO(2))$	8	440	$(\tilde{D}_1^2 {}^{\tilde{D}_1^z} \times_{D_1} O(2))$	4
381	$(D_6^{\tilde{d}} {}^{D_3^z} \times_{\mathbb{Z}_2} D_{2n})$	4	411	$(\mathbb{Z}_6^z \times SO(2))$	8	441	$(D_2^d {}^{D_1} \times_{D_1} O(2))$	4
382	$(D_6^{\tilde{d}} {}^{\mathbb{Z}_6^z} \times_{\mathbb{Z}_2} D_{2n})$	4	412	$(\tilde{D}_3^z \times SO(2))$	8	442	$(D_2^d {}^{\mathbb{Z}_2^z} \times_{D_1} O(2))$	4
383	$(D_6^2 {}^{D_3^2} \times_{\mathbb{Z}_2} D_{2n})$	2	413	$(D_2^2 \times SO(2))$	2	443	$(D_2^d {}^{\tilde{D}_1^z} \times_{D_1} O(2))$	4
384	$(D_6^2 {}^{\mathbb{Z}_6^2} \times_{\mathbb{Z}_2} D_{2n})$	2	414	$(D_3^2 \times SO(2))$	4	444	$(D_2^z {}^{\mathbb{Z}_2} \times_{D_1} O(2))$	4
385	$(D_6^2 {}^{D_6} \times_{\mathbb{Z}_2} D_{2n})$	2	415	$(\mathbb{Z}_6^2 \times SO(2))$	4	445	$(D_2^z {}^{D_1^z} \times_{D_1} O(2))$	4
386	$(D_6^2 {}^{\tilde{D}_3^2} \times_{\mathbb{Z}_2} D_{2n})$	2	416	$(D_6 \times SO(2))$	4	446	$(D_2^z {}^{\tilde{D}_1^z} \times_{D_1} O(2))$	4
387	$(D_6^2 {}^{D_6^d} \times_{\mathbb{Z}_2} D_{2n})$	2	417	$(\tilde{D}_3^2 \times SO(2))$	4	447	$(D_2^{\tilde{d}} {}^{\tilde{D}_1} \times_{D_1} O(2))$	4
388	$(D_6^2 {}^{D_6^z} \times_{\mathbb{Z}_2} D_{2n})$	2	418	$(D_6^d \times SO(2))$	4	448	$(D_2^{\tilde{d}} {}^{\mathbb{Z}_2^z} \times_{D_1} O(2))$	4
389	$(D_6^2 {}^{D_6^{\tilde{d}}} \times_{\mathbb{Z}_2} D_{2n})$	2	419	$(D_6^z \times SO(2))$	4	449	$(D_2^{\tilde{d}} {}^{D_1^z} \times_{D_1} O(2))$	4
390	$(\mathbb{Z}_1 \times SO(2))$	48	420	$(D_6^{\tilde{d}} \times SO(2))$	4	450	$(D_3 {}^{\mathbb{Z}_3} \times_{D_1} O(2))$	8

Table 3.9: Conjugacy Classes of Subgroups in $D_6 \times \mathbb{Z}_2 \times O(2)$ (part 6)

ID	(S)	$\|W(S)\|$	ID	(S)	$\|W(S)\|$	ID	(S)	$\|W(S)\|$
451	$(\mathbb{Z}_6 {}^{\mathbb{Z}_3}\times_{D_1} O(2))$	8	476	$(D_6^d {}^{D_3}\times_{D_1} O(2))$	4	501	$(D_1^2 \times O(2))$	2
452	$(\tilde{D}_3 {}^{\mathbb{Z}_3}\times_{D_1} O(2))$	8	477	$(D_6^d {}^{\mathbb{Z}_6^z}\times_{D_1} O(2))$	4	502	$(\mathbb{Z}_2^2 \times O(2))$	6
453	$(\mathbb{Z}_3^2 {}^{\mathbb{Z}_3}\times_{D_1} O(2))$	8	478	$(D_6^d {}^{\tilde{D}_3^z}\times_{D_1} O(2))$	4	503	$(D_2 \times O(2))$	2
454	$(D_3^z {}^{\mathbb{Z}_3}\times_{D_1} O(2))$	8	479	$(D_6^z {}^{\mathbb{Z}_6}\times_{D_1} O(2))$	4	504	$(\tilde{D}_1^2 \times O(2))$	2
455	$(\mathbb{Z}_6^z {}^{\mathbb{Z}_3}\times_{D_1} O(2))$	8	480	$(D_6^z {}^{D_3^z}\times_{D_1} O(2))$	4	505	$(D_2^d \times O(2))$	2
456	$(\tilde{D}_3^z {}^{\mathbb{Z}_3}\times_{D_1} O(2))$	8	481	$(D_6^z {}^{\tilde{D}_3^z}\times_{D_1} O(2))$	4	506	$(D_2^z \times O(2))$	2
457	$(D_2^2 {}^{D_1^2}\times_{D_1} O(2))$	2	482	$(D_6^{\tilde{d}} {}^{\tilde{D}_3}\times_{D_1} O(2))$	4	507	$(D_2^{\tilde{d}} \times O(2))$	2
458	$(D_2^2 {}^{\mathbb{Z}_2^2}\times_{D_1} O(2))$	2	483	$(D_6^{\tilde{d}} {}^{D_3^z}\times_{D_1} O(2))$	4	508	$(D_3 \times O(2))$	4
459	$(D_2^2 {}^{D_2}\times_{D_1} O(2))$	2	484	$(D_6^{\tilde{d}} {}^{\mathbb{Z}_6^z}\times_{D_1} O(2))$	4	509	$(\mathbb{Z}_6 \times O(2))$	4
460	$(D_2^2 {}^{\tilde{D}_1^2}\times_{D_1} O(2))$	2	485	$(D_6^2 {}^{D_3^2}\times_{D_1} O(2))$	2	510	$(\tilde{D}_3 \times O(2))$	4
461	$(D_2^2 {}^{D_2^d}\times_{D_1} O(2))$	2	486	$(D_6^2 {}^{\mathbb{Z}_6^2}\times_{D_1} O(2))$	2	511	$(\mathbb{Z}_3^2 \times O(2))$	4
462	$(D_2^2 {}^{D_2^z}\times_{D_1} O(2))$	2	487	$(D_6^2 {}^{D_6}\times_{D_1} O(2))$	2	512	$(D_3^z \times O(2))$	4
463	$(D_2^2 {}^{D_2^{\tilde{d}}}\times_{D_1} O(2))$	2	488	$(D_6^2 {}^{\tilde{D}_3^2}\times_{D_1} O(2))$	2	513	$(\mathbb{Z}_6^z \times O(2))$	4
464	$(D_3^2 {}^{D_3}\times_{D_1} O(2))$	4	489	$(D_6^2 {}^{D_6^d}\times_{D_1} O(2))$	2	514	$(\tilde{D}_3^z \times O(2))$	4
465	$(D_3^2 {}^{\mathbb{Z}_3^2}\times_{D_1} O(2))$	4	490	$(D_6^2 {}^{D_6^z}\times_{D_1} O(2))$	2	515	$(D_2^2 \times O(2))$	1
466	$(D_3^2 {}^{D_3^z}\times_{D_1} O(2))$	4	491	$(D_6^2 {}^{D_6^{\tilde{d}}}\times_{D_1} O(2))$	2	516	$(D_3^2 \times O(2))$	2
467	$(\mathbb{Z}_6^2 {}^{\mathbb{Z}_6}\times_{D_1} O(2))$	4	492	$(\mathbb{Z}_1 \times O(2))$	24	517	$(\mathbb{Z}_6^2 \times O(2))$	2
468	$(\mathbb{Z}_6^2 {}^{\mathbb{Z}_3^2}\times_{D_1} O(2))$	4	493	$(\mathbb{Z}_2 \times O(2))$	12	518	$(D_6 \times O(2))$	2
469	$(\mathbb{Z}_6^2 {}^{\mathbb{Z}_6^z}\times_{D_1} O(2))$	4	494	$(D_1 \times O(2))$	4	519	$(\tilde{D}_3^2 \times O(2))$	2
470	$(D_6 {}^{D_3}\times_{D_1} O(2))$	4	495	$(\mathbb{Z}_1^2 \times O(2))$	12	520	$(D_6^d \times O(2))$	2
471	$(D_6 {}^{\mathbb{Z}_6}\times_{D_1} O(2))$	4	496	$(\tilde{D}_1 \times O(2))$	4	521	$(D_6^z \times O(2))$	2
472	$(D_6 {}^{\tilde{D}_3}\times_{D_1} O(2))$	4	497	$(\mathbb{Z}_2^z \times O(2))$	12	522	$(D_6^{\tilde{d}} \times O(2))$	2
473	$(\tilde{D}_3^2 {}^{\tilde{D}_3}\times_{D_1} O(2))$	4	498	$(D_1^z \times O(2))$	4	523	$(D_6^2 \times O(2))$	1
474	$(\tilde{D}_3^2 {}^{\mathbb{Z}_3^2}\times_{D_1} O(2))$	4	499	$(\tilde{D}_1^z \times O(2))$	4			
475	$(\tilde{D}_3^2 {}^{\tilde{D}_3^z}\times_{D_1} O(2))$	4	500	$(\mathbb{Z}_3 \times O(2))$	8			

Table 3.10: Conjugacy Classes of Subgroups in $S_4 \times \mathbb{Z}_2$

ID	(H)	ID	(H)
1	$(\mathbb{Z}_1)$	18	(D_3)
2	$(\mathbb{Z}_2)$	19	(D_3^z)
3	(D_1)	20	(D_4)
4	$(\mathbb{Z}_1^2)$	21	(D_4^v)
5	$(\mathbb{Z}_2^z)$	22	$(\mathbb{Z}_4^2)$
6	(D_1^z)	23	(D_2^2)
7	$(\mathbb{Z}_3)$	24	(V_4^2)
8	(V_4)	25	(D_4^d)
9	$(\mathbb{Z}_2^2)$	26	(D_4^z)
10	(V_4^z)	27	(D_3^2)
11	$(\mathbb{Z}_4^z)$	28	(A_4)
12	(D_2^z)	29	(D_4^2)
13	$(\mathbb{Z}_4)$	30	(S_4)
14	(D_2)	31	(A_4^2)
15	(D_1^2)	32	(S_4^a)
16	(D_2^d)	33	(S_4^2)
17	$(\mathbb{Z}_3^2)$		

Table 3.11: Character Table of D_6

	(1)	(κ)	(r)	(r^2)	(κr)	(r^3)
X_1	1	1	1	1	1	1
X_2	1	−1	−1	1	1	−1
X_3	1	−1	1	1	−1	1
X_4	1	1	−1	1	−1	−1
X_5	2	.	1	−1	.	−2
X_6	2	.	−1	−1	.	2

Table 3.12: Character Table of S_4

	()	(12)	(12)(34)	(123)	(1234)
V_1	1	−1	1	1	−1
V_2	3	−1	−1	.	1
V_3	2	.	2	−1	.
V_4	3	1	−1	.	−1
V_5	1	1	1	1	1

Table 3.13: Character Table of $\mathbb{Z}_2$

	(1)	(−1)
Z_1	1	1
Z_2	1	−1

Table 3.14: Basic Degrees of $D_6 \times \mathbb{Z}_2 \times O(2)$ $(n > 1)$

k	j	$\deg_{\mathcal{V}_{j,1,k}}$
0	1	$(D_6^2 \times O(2)) - (D_6 \times O(2))$
0	2	$(D_6^2 \times O(2)) - (D_6^{\tilde{d}} \times O(2))$
0	3	$(D_6^2 \times O(2)) - (D_6^{z} \times O(2))$
0	4	$(D_6^2 \times O(2)) - (D_6^{d} \times O(2))$
0	5	$(D_6^2 \times O(2)) - (D_2^{\tilde{d}} \times O(2)) - (D_2^{d} \times O(2)) + (\mathbb{Z}_2^{z} \times O(2))$
0	6	$(D_6^2 \times O(2)) - (D_2^{z} \times O(2)) - (D_2 \times O(2)) + (\mathbb{Z}_2 \times O(2))$
n	1	$(D_6^2 \times O(2)) - (D_6^2 {}^{D_6}\times_{\mathbb{Z}_2} D_{2n})$
n	2	$(D_6^2 \times O(2)) - (D_6^2 {}^{D_6^{\tilde{d}}}\times_{\mathbb{Z}_2} D_{2n})$
n	3	$(D_6^2 \times O(2)) - (D_6^2 {}^{D_6^{z}}\times_{\mathbb{Z}_2} D_{2n})$
n	4	$(D_6^2 \times O(2)) - (D_6^2 {}^{D_6^{d}}\times_{\mathbb{Z}_2} D_{2n})$
n	5	$(D_6^2 \times O(2)) - (D_6^2 {}^{\mathbb{Z}_2^z}\times_{D_6} D_{6n}) - (D_2^2 {}^{D_2^{\tilde{d}}}\times_{\mathbb{Z}_2} D_{2n}) - (D_2^2 {}^{D_2^{d}}\times_{\mathbb{Z}_2} D_{2n}) + 2(D_2^2 {}^{\mathbb{Z}_2^z}\times_{D_2}^{\mathbb{Z}_2^2} D_{2n}) + (\mathbb{Z}_2^2 {}^{\mathbb{Z}_2^z}\times_{\mathbb{Z}_2} D_{2n})$
n	6	$(D_6^2 \times O(2)) - (D_6^2 {}^{\mathbb{Z}_2}\times_{D_6} D_{6n}) - (D_2^2 {}^{D_2}\times_{\mathbb{Z}_2} D_{2n}) - (D_2^2 {}^{D_2^{z}}\times_{\mathbb{Z}_2} D_{2n}) + (\mathbb{Z}_2^2 {}^{\mathbb{Z}_2}\times_{\mathbb{Z}_2} D_{2n}) + 2(D_2^2 {}^{\mathbb{Z}_2}\times_{D_2}^{\mathbb{Z}_2^2} D_{2n})$

Table 3.15: Basic Degrees of $S_4 \times \mathbb{Z}_2$

j	$\deg_{\mathcal{V}_{j,1}}$
1	$(S_4^2) - (S_4^a)$
2	$(S_4^2) - (D_4^z) - (D_3^z) - (D_2^d) + 2(D_1^z) + (\mathbb{Z}_2^z) - (\mathbb{Z}_1)$
3	$(S_4^2) - (D_4^v) - (D_4) + (V_4)$
4	$(S_4^2) - (D_4^d) - (D_3) - (D_2^d) + (\mathbb{Z}_2^z) + 2(D_1) - (\mathbb{Z}_1)$
5	$(S_4^2) - (S_4)$

CHAPTER 4

EXISTENCE OF PERIODIC SOLUTIONS IN SECOND ORDER DIFFERENCE SYSTEM

4.1 Introduction

In this chapter we propose a new method, rooted in the equivariant degree theory, to predict and classify possible occurrence of various kinds of subharmonic solutions in nonlinear second order difference equations. In spite of a relative simplicity of these problems (all the related symmetric systems are finite-dimensional), the existence and bifurcation of subharmonic periodic solutions in these systems attract a lot of interest, which is mostly motivated by their various applications. For the existence of subharmonic solutions, ground state solutions, homoclinic orbits and heteroclinic orbits in these systems has attracted a lot of interests (see e.g. [10],[37],[47],[59]) and economic dynamics (see e.g. [42],[54],[55]. In particular, the existence of standing waves in discrete nonlinear Schrödinger systems (in short, DNLS systems) can be easily reformulated as a second order difference equation.

In applications, the model is the system of coupled DNLS equations

$$i\dot{\mathbf{u}}_n = \mathbf{u}_{n+1} - 2\mathbf{u}_n + \mathbf{u}_{n-1} + F(n, \mathbf{u}_n), \qquad \mathbf{u}_n \in \mathbb{C}^k, \tag{4.1}$$

where $\mathbf{u} : \mathbb{C}^k \times \mathbb{Z} \to \mathbb{C}$, $\mathbf{u} = (u_1, \dots, u_k)$. Determining the existence of periodic solutions $\mathbf{u}_n$ in (4.1), i.e. solutions $(\mathbf{u}_n)$ satisfying conditions $\mathbf{u}_n = \mathbf{u}_{n+N}$ for some $N \in \mathbb{N}$, and any N constitutes an important problem for this model. For example, without coupling, the classical DNLS equation is given by $F(u) = \sigma(|u_1|^2 u_1, \dots, |u_k|^2 u_k)$ splits into two cases: focusing DNLS equation for $\sigma = 1$ and defocusing DNLS for $\sigma = -1$ which is the case of our interest, although some results can be obtained with similar methods in the focusing case as well.

The general nonlinearity $F(n, \mathbf{u}_n)$ includes coupling in the system of DNLS equations and describes the interaction between different components. Being motivated by these models, we assume that the function $F:\mathbb{Z}\times\mathbb{C}^n\to\mathbb{C}^n$ is continuous and is gauge invariant, i.e.,

$$F(n,e^{i\theta}\mathbf{u})=e^{i\theta}F(n,\mathbf{u}),\qquad \overline{F(n,\mathbf{u})}=F(n,\overline{\mathbf{u}}),\qquad \theta\in\mathbb{R}.$$

By substituting a standing wave of the form $\mathbf{u}_n(t)=e^{i\lambda t}\mathbf{x}_n$ into (4.1), one obtains the following system

$$\triangle^2\mathbf{x}_{n-1}+\lambda\mathbf{x}_n+F(n,\mathbf{x}_n)=0, \tag{4.2}$$

where $\mathbf{x}_n\in\mathbb{R}^k$ and $\triangle^2$ denotes the discrete Laplacian, i.e., $\triangle^2_{n-1}\mathbf{x}:=\mathbf{x}_{n+1}+\mathbf{x}_{n-1}-2\mathbf{x}_n$.

We assume, in addition, that the system of coupled DNLS is symmetric with respect to subgroup $\Gamma\leq S_k$, where the permutation group S_k acts on $V:=\mathbb{R}^k$ by permuting the coordinates of vectors $\mathbf{x}\in V$, i.e., if $\mathbf{x}=(x_1,x_2,\dots,x_k)$ then for $\sigma\in S_k$ we have

$$\sigma\mathbf{x}:=(x_{\sigma(1)},x_{\sigma(2)},\dots,x_{\sigma(k)})^T\in\mathbb{R}^k.$$

That means, $F(n,g\mathbf{x})=gF(n,\mathbf{x})$ for all $g\in\Gamma$. For instance, the system of two DNLS equations with a coupling that satisfies the symmetry condition with $\Gamma=\mathbb{Z}_2$ (with respect to the permutation of u_1 and u_2) has been proposed as a model of study for two waveguides involving two polarizations.

By setting

$$f(n,\mathbf{x}_n)=\lambda\mathbf{x}_n+F(n,\mathbf{x}_n),$$

the equation (4.2) can be written as the system of second order difference equations (SODE)

$$\triangle^2\mathbf{x}_{n-1}+f(n,\mathbf{x}_n)=0,\quad \mathbf{x}_n\in\mathbb{R}^k,\ n\in\mathbb{Z}, \tag{4.3}$$

Motivated by the models discussed above, we assume that $f:\mathbb{Z}\times\mathbb{R}^k\to\mathbb{R}^k$ satisfies the following conditions:

(A1) f is continuous;

(A2) $\exists\, p\in 2\mathbb{N}+1$ such that $f(n+p,\mathbf{x})=f(n,\mathbf{x})$ for all $n\in\mathbb{Z}$ and $\mathbf{x}\in\mathbb{R}^k$;

(A3) $f(-n,\mathbf{x}) = f(n,\mathbf{x})$ for all $n \in \mathbb{Z}$ and $\mathbf{x} \in \mathbb{R}^k$;

(A4) $f(n,-\mathbf{x}) = -f(n,\mathbf{x})$ for all $n \in \mathbb{Z}$ and $\mathbf{x} \in \mathbb{R}^k$.

(A5) $f(n,g\mathbf{x}) = gf(n,\mathbf{x})$ for all $g \in \Gamma$, $n \in \mathbb{Z}$ and $\mathbf{x} \in V$;

Under the conditions (A1)–(A5), the problem (4.3) is a Γ-symmetric reversible (with respect to n) SODE. For a given integer m, we are interested in establishing the existence of non-constant pm-periodic solutions to (4.3), i.e., we are looking for sequences $\mathbf{x}_n$ such that $\mathbf{x}_{n+pm} = \mathbf{x}_n$ for all n and satisfying (4.3). These solutions are commonly called *subharmonic solutions* to (4.3). Let us point out that we do not need to assume that (4.3) has variational structure (see [2] and references therein).

Let us point out that the non-equivariant Brouwer degree turned out to be insufficient to investigate the existence of the subharmonic solutions, especially to examine the existence and multiplicity of continuous branches of such solutions. The new methods are being developed using the equivariant versions of the Brouwer degree, which allow effective study of the existence and multiplicity of solutions. The same method is effectively applicable to study the existence of periodic solutions in large classes of various types of equations, including ODEs, PDEs, FDEs, Hamiltonian systems, etc, with or without additional symmetries (cf. [4; 5; 6; 7; 11; 16; 24; 27; 29; 34; 45; 53]). We should emphasize that the equivariant degree methods are supported by GAP programs (for algebraic computations) which make them easier to apply to specific problems involving complicated groups of symmetries.

The rest of the chapter is organized as follows: in Section 4.2 we assume $\lambda = 0$ and introduce the settings relevant for application of the equivariant Brouwer degree method, present an outline of this method, and carry on the computations needed for the evaluation of the related to the problem (4.2)) (under hypothesis (A1)-(A5)) equivariant degrees. In Section 4.3 we assume $\lambda = 0$ and study the case were f satisfies the Nagumo condition.

In Section 4.4 we consider periodic solutions using λ as the bifurcation parameter under hypothesis. A short user guide for **EquiDeg** GAP package is outline of the equivariant bifurcation results (local and global) in the relevant to this work context is mentioned in Appendix.

4.2 Application of the Equivariant Brouwer Degree Method

In this chapter and chapter 4.3, we assume $\lambda = 0$.

4.2.1 Operator Reformulation

In this section we introduce the setting for the application of the equivariant Brouwer degree to the system (4.3) in order to establish the existence of multiple subharmonic solutions (with various symmetry properties).

We denote by $x = (\mathbf{x}_n)$ a sequence of elements $\mathbf{x}_n \in \mathbb{R}^k$, indexed by integer numbers $n \in \mathbb{Z}$. For such a sequence we also write $(x)_n := \mathbf{x}_n$ and denote by $\triangle\mathbf{x}_n := \mathbf{x}_{n+1} - \mathbf{x}_n$ the difference operator. The discrete Laplace operator is defined by $\triangle^2\mathbf{x}_{n-1} := \triangle(\triangle\mathbf{x}_{n-1}) = \mathbf{x}_{n+1} + \mathbf{x}_{n-1} - 2\mathbf{x}_n$. We also define the space $\mathscr{H}$ of all pm-periodic sequences $x = (\mathbf{x}_n)$, $n \in \mathbb{Z}$, $\mathbf{x}_n \in V$, i.e.,

$$\mathscr{H} := \{x = (\mathbf{x}_n) : \mathbf{x}_{n+pm} = \mathbf{x}_n,\ n \in \mathbb{N}\} \simeq \mathbb{R}^{mpk}.$$

Define the nonlinear operator $\mathscr{F} : \mathscr{H} \to \mathscr{H}$ by

$$(\mathscr{F}(x))_n := \triangle^2\mathbf{x}_{n-1} + f(n, \mathbf{x}_n), \quad x \in \mathscr{H}. \tag{4.4}$$

Clearly, pm-periodic solutions x to (4.3) are exactly pm-periodic sequences x from $\mathscr{H}$ satisfying the equation $\mathscr{F}(x) = 0$.

The cyclic group $\mathbb{Z}_m := \{1, \zeta, \zeta^2, \dots, \zeta^{m-1}\}$, $\zeta = e^{\frac{i2\pi}{m}}$, acts on $\mathscr{H}$ by $(\zeta x)_n := \mathbf{x}_{n+p}$, $x \in \mathscr{H}$. Notice that $\mathscr{F}$ is $\mathbb{Z}_m$-equivariant, i.e., $\mathscr{F}(\zeta x) = \zeta\mathscr{F}(x)$, $x = (\mathbf{x}_n) \in \mathscr{H}$. The $\mathbb{Z}_m$-

action on $\mathscr{H}$ can be extended to D_m-action[1] by $(\kappa x)_n = \mathbf{x}_{-n}$. Also we define the $\mathbb{Z}_2$-action on $\mathscr{H}$ by $(\pm 1)(\mathbf{x}_n) := (\pm \mathbf{x}_n)$, $x = (\mathbf{x}_n) \in \mathscr{H}$. The Γ-action on $\mathscr{H}$ is given by $\gamma(\mathbf{x}_n) := (\gamma \mathbf{x}_n)$, $\gamma \in \Gamma$, $n \in \mathbb{Z}$. Put $G := \Gamma \times D_m \times \mathbb{Z}_2$. Since $\mathscr{F}$ is also Γ-equivariant, we have the following:

Proposition 4.2.1. *Suppose $f : \mathbb{Z} \times V \to V$ is a continuous map satisfying conditions* (A1)–(A5). *Then:*

(i) *the operator $\mathscr{F} : \mathscr{H} \to \mathscr{H}$ given by* (4.4) *is G-equivariant;*

(ii) *$x = (\mathbf{x}_n)$ is a pm-periodic solution to* (4.3) *if and only if x satisfies the equation*

$$\mathscr{F}(x) = 0, \quad x \in \mathscr{H}. \tag{4.5}$$

4.2.2 Equivariant Degree Method

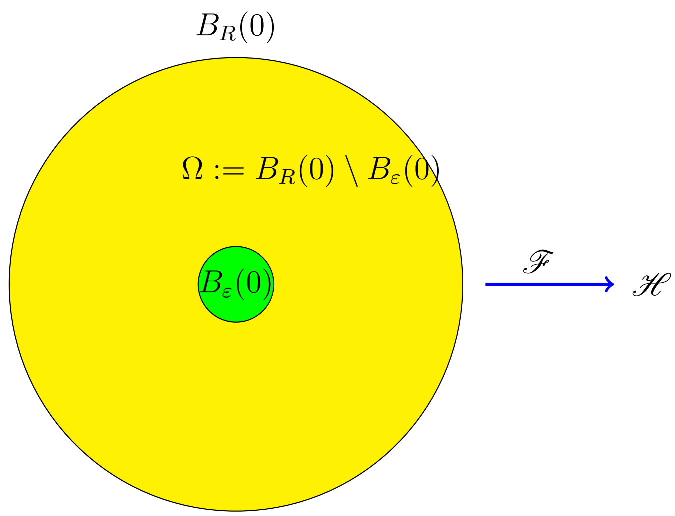
Let us describe how the G-equivariant degree can be used to show the existence of multiple pm-periodic solutions to (4.3). Clearly, the G-equivariance of $\mathscr{F}$ implies that $\mathscr{F}(0) = 0$, thus our task is to show the existence of other (nontrivial) non-constant periodic solutions.

Suppose that 0 is an isolated solution to (4.5), i.e., there exists $\varepsilon > 0$ such that $\mathscr{F}^{-1}(0) \cap \overline{B_\varepsilon(0)} = \{0\}$ and assume, in addition, that solutions to (4.3) admit a priori estimate, i.e., there exists a sufficiently large $R > 0$ such that $\mathscr{F}(x) \neq 0$ for $\|x\| \geq R$, meaning that all the non-zero solutions $x \in \mathscr{H}$ to the equation (4.5) are located in the set $\Omega := B_R(0) \setminus \overline{B_\varepsilon(0)}$. Then, by the additivity property of the G-equivariant degree, one has:

$$G\text{-deg}(\mathscr{F}, \Omega) := G\text{-deg}(\mathscr{F}, B_R(0)) - G\text{-deg}(\mathscr{F}, B_\varepsilon(0)). \tag{4.6}$$

---

[1] Here D_m stands for a dihedral group of a regular m-gone, i.e. $D_m = \mathbb{Z}_m \cup D_m\kappa$, where $\kappa z = \overline{z}$ for $z \in \mathbb{C}$.



Consequently, we immediately obtain the following existence result

Proposition 4.2.2. *Assume that f satisfies* (A1)—(A5) *and suppose that* 0 *is an isolated solution to* (4.5) *while solutions to* (4.5) *admit apriori estimate, i.e., there exist $\varepsilon > 0$ and $R > 0$ such that* 0 *is the only solution in $\overline{B_\varepsilon(0)}$ to* (4.5) *and there are no solutions to* (4.5) *in $\mathscr{H} \setminus B_R(0)$. Put $\Omega := B_R(0) \setminus \overline{B_\varepsilon(0)}$. Then, the degree*

$$G\text{-deg}(\mathscr{F}, \Omega) = G\text{-deg}(\mathscr{F}, B_R(0)) - G\text{-deg}(\mathscr{F}, B_\varepsilon(0)))$$

is well-defined. Moreover, if

$$G\text{-deg}(\mathscr{F}, \Omega) = n_1(H_1) + n_2(H_2) + \cdots + n_r(H_r), \quad n_j \neq 0, \ j = 1, 2, \dots, r. \tag{4.7}$$

then

(i) *for every $j = 1, 2, \dots, r$, there exists a pm-periodic solution $x \in \Omega$ to* (4.3) *such that $(G_x) \geq (H_j)$;*

(ii) *if (H_j) is a maximal orbit type in $\mathscr{H} \setminus \{0\}$, then the periodic solution x has exactly the symmetries (H_j), i.e., $(G_x) = (H_j)$.*

Notice that for a non-abelian group G (even in the non-symmetric case when $\Gamma = \{0\}$), there may exist multiple maximal orbit types (H) in $\mathscr{H} \setminus \{0\}$, thus the information extracted from G-deg$(\mathscr{F}, \Omega)$ could prove the existence of multiple G-orbits of pm-periodic solutions to (4.3).

4.2.3 Some Technical Computations

Let us consider the operator $\mathscr{L} : \mathscr{H} \to \mathscr{H}$, given by

$$(\mathscr{L}\mathbf{x})_n = \triangle^2 \mathbf{x}_{n-1} = \mathbf{x}_{n+1} + \mathbf{x}_{n-1} - 2\mathbf{x}_n;$$

where $\mathscr{L}$ has the following block $pm \times pm$ matrix form:

$$\mathscr{L} = \begin{bmatrix} -2\,\mathrm{Id} & \mathrm{Id} & 0 & \cdots & 0 & 0 & \mathrm{Id} \\ \mathrm{Id} & -2\,\mathrm{Id} & \mathrm{Id} & \cdots & 0 & 0 & 0 \\ 0 & \mathrm{Id} & -2\,\mathrm{Id} & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -2\,\mathrm{Id} & \mathrm{Id} & 0 \\ 0 & 0 & 0 & \cdots & \mathrm{Id} & -2\,\mathrm{Id} & \mathrm{Id} \\ \mathrm{Id} & 0 & 0 & \cdots & 0 & \mathrm{Id} & -2\,\mathrm{Id} \end{bmatrix}, \tag{4.8}$$

where Id is the $k \times k$ identity matrix.

Put

$$\mathfrak{s} := \left\lfloor \frac{m}{2} \right\rfloor, \quad N := pm \quad \text{and} \quad \mathfrak{r} := \left\lfloor \frac{N}{2} \right\rfloor. \tag{4.9}$$

The space $\mathscr{H}$ is both a D_m-representation and D_N-representation where

$$\zeta := e^{\frac{2\pi i}{m}} \in \mathbb{Z}_m \quad \text{and} \quad \xi := e^{\frac{2\pi i}{pm}}$$

act on vector $x = (\mathbf{x}_n)$ by

$$\zeta x := (\mathbf{x}_{n+p}) \quad \text{and} \quad \xi x = (\mathbf{x}_{n+1}),$$

and $\kappa x := (\mathbf{x}_{-n})$.

Spectrum of the Operator $\mathscr{L}$: Clearly, the operator $\mathscr{L}$ is equivariant with respect to the action of D_m and D_N. Consider the complexification of $\mathscr{H}$,

$$\mathscr{H}^c := \{(\mathbf{z}_1, \mathbf{z}_2, \dots, \mathbf{z}_N) : \mathbf{z}_l \in \mathbb{C}^k\},$$

and put

$$v_j := (\xi^j \mathbf{z}, \xi^{2j}\mathbf{z}, \cdots, \xi^{Nj}\mathbf{z}) \in \mathscr{H}^c, \quad \mathbf{z} \in V^c,\ j = 0, 1, \dots, N-1.$$

By direct computations we obtain

$$\mathscr{L}(v_j) = \big(-1 + 2\mathrm{Re}\,(\xi^j)\big)v_j = -4\sin^2\left(\frac{\pi j}{pm}\right) v_j =: \rho_j v_j, \quad j = 0, 1, 2, \dots, N-1,$$

which provides a full description of the complex eigenspace $E(\rho_j)$ of $\mathscr{L}$ in $\mathscr{H}^c$. Indeed, put

$$W_j := \left\{(\xi^j \mathbf{z}, \xi^{2j}\mathbf{z}, \cdots, \xi^{Nj}\mathbf{z}) : \mathbf{z} \in V^c\right\} \subset \mathscr{H}^c.$$

Notice, that $\xi \in \mathbb{Z}_N$ acts on W_j by complex multiplication by ξ^j. Indeed,

$$\begin{aligned}\xi(\xi^j \mathbf{z}, \xi^{2j}\mathbf{z}, \cdots \xi^{(N-1)j}\mathbf{z}, \xi^{Nj}\mathbf{z}) &= (\xi^{2j}\mathbf{z}, \xi^{3j}\mathbf{z}, \cdots \xi^{Nj}\mathbf{z}, , \xi^j \mathbf{z}) \\ &= \xi^j \cdot (\xi^j \mathbf{z}, \xi^{2j}\mathbf{z}, \cdots \xi^{(N-1)j}\mathbf{z}, \xi^{Nj}\mathbf{z}),\end{aligned}$$

where '$\cdot$' denotes the complex multiplication in $\mathscr{H}$. Then, since $\rho_j = \rho_{N-j}$, we immediately obtain:

Proposition 4.2.3. *The operator $\mathscr{L}$ on $\mathscr{H}^c$ has eigenvalues*

$$\rho_j = -4\sin^2\left(\frac{\pi j}{pm}\right), \qquad j = 0, ..., \mathfrak{r}. \tag{4.10}$$

The complex eigenspace $E(\rho_j)$ corresponding to eigenvalue ρ_j is described by:

- *if $j = 0$, then $E(\rho_0) = W_0$;*
- *if $j = N/2$, then $E(\rho_{N/2}) = W_{N/2}$;*

- *if* $0 < j < N/2$*, then* $E(\rho_j) = W_j \oplus W_{N-j}$*.*

Notice that κ *maps* W_j *onto* W_{N-j}*, so each complex eigenspace* $E(\rho_j)$ *is invariant with respect to the* D_N*-action.*

Denote by $\mathfrak{E}(\rho_j) \subset \mathscr{H}^c$ the real eigenspace correspoding to ρ_j, $0 \leq j \leq \mathfrak{r}$, by identifying $\mathscr{H}$ with the real part of $\mathscr{H} \oplus i\mathscr{H} = \mathscr{H}^c$, we can write

$$\mathscr{H} = \bigoplus_{j=0}^{N-1} \operatorname{Re} W_j.$$

Therefore, we obtain

$$\begin{aligned} \mathfrak{E}(\rho_0) &= \operatorname{Re} W_0, \\ \mathfrak{E}(\rho_j) &= \operatorname{Re} W_j \oplus \operatorname{Re} W_{N-j}, \quad 0 < j < N/2, \\ \mathfrak{E}(\rho_{N/2}) &= \operatorname{Re} W_{N/2}, \quad \text{if } N \text{ is even.} \end{aligned}$$

Moreover, since $\operatorname{Re} W_j$ is invariant with respect to the $\mathbb{Z}_N$-action and $\kappa(\operatorname{Re} W_j) = \operatorname{Re} W_{N-j}$, it follows that each of the spaces $\mathfrak{E}(\rho_j)$ is D_N-invariant.

The irreducible D_N-representations can be described as follows:

- if $j = 0$, then $\mathscr{V}_0 \simeq \mathbb{R}$ is the trivial D_N-representation,
- if $0 < j < N/2$, then $\mathscr{V}_0 \simeq \mathbb{C}$, where $\xi z = \xi^j \cdot z$, $\kappa z = \overline{z}$, $z \in \mathbb{C}$,
- if $j = N/2$, then $\mathscr{V}_{N/2} \simeq \mathbb{R}$ where $\xi x = -x$, $\kappa x = x$, $x \in \mathbb{R}$.

It is easy to observe, that all irreducible D_N-subrepresentations of $\mathfrak{E}(\rho_j)$ are equivalent to $\mathscr{V}_j$.

In particular, $\mathfrak{E}(\rho_j)$ are D_m-invariant. Let us identify the $D_m \times \mathbb{Z}_2$-isotypical type (and its isotypical dimension) of the eigenspaces $\mathfrak{E}(\rho_j)$. For this purpose, we introduce:

Definition 4.2.4. For an integer number j, we define $\alpha(j) \in \{0, 1, \dots, m-1\}$ by the condition $\alpha(j) \equiv j \pmod m$. Then for $0 \le j \le N$ we define

$$i(j) := \begin{cases} \alpha(j) & \text{if } \ \alpha(j) = 0, ..., \mathfrak{s}, \\ m - \alpha(j) & \text{if } \ \alpha(j) = \mathfrak{s}, ..., m-1. \end{cases} \tag{4.11}$$

The irreducible $D_m \times \mathbb{Z}_2$-representations (with the antipodal $\mathbb{Z}_2$-action), relevant for being considered in this section setting are:

- if $i = 0$, then $\mathcal{V}_0^- \simeq \mathbb{R}$ with the trivial D_m-action;
- if $0 < i < m/2$, then $\mathcal{V}_i^- \simeq \mathbb{R}^2 = \mathbb{C}$, where $\zeta z = \zeta^i \cdot z$, $\kappa z = \overline{z}$, $z \in \mathbb{C}$;
- if m is even, i.e. $\mathfrak{s} := \frac{m}{2}$, we have the irreducible $D_m \times \mathbb{Z}_2$-representation $\mathcal{V}_{\mathfrak{s}}^- \simeq \mathbb{R}$ with the D_m-action $\zeta x = -x$, $\kappa x = x$, $x \in \mathbb{R}$;
- we also have the representation $\mathcal{V}_{\mathfrak{s}+1}^- \simeq \mathbb{R}$ with the D_m-action $\zeta x = x$, $\kappa x = -x$, $x \in \mathbb{R}$;
- if m is even, we have the representation $\mathcal{V}_{\mathfrak{s}+2} \simeq \mathbb{R}$ with D_m-action $\zeta x = -x$, $\kappa x = -x$, $x \in \mathbb{R}$.

Remark 4.2.5. It is easy to see that the D_N-representation $\mathscr{V}_j$, restricted to the subgroup D_m and then extended to $D_m \times \mathbb{Z}_2$-representation, is irreducible if $0 < i(j) < m/2$ and is equivalent (as a real $D_m \times \mathbb{Z}_2$-representation) to the representation $\mathcal{V}_{i(j)}^-$. However, if $j \neq 0, N/2$ and $i(j) = 0, m/2$, then $\mathscr{V}_j$, considered as a $D_m \times \mathbb{Z}_2$-representation is reducible. Indeed, notice that for $j = lm$, $0 < l \le \frac{p-1}{2}$, we have the following character table

	(± 1)	$(\pm\zeta)$	...	$(\pm\zeta^l)$	...	$(\pm\zeta^{\mathfrak{s}})$	$(\pm\kappa)$	...
$\chi_{\mathscr{V}_j}$	± 2	2	...	2	...	2	0	...
$\chi_{\mathcal{V}_0^-}$	± 1	± 1	...	± 1	...	1	± 1	...
$\chi_{\mathcal{V}_{\mathfrak{s}+1}^-}$	± 1	± 1	...	± 1	...	± 1	∓ 1	...

thus $\mathscr{V}_j$ is equivalent to $\mathcal{V}_0^- \oplus \mathcal{V}_{\mathfrak{s}+1}^-$. On the other hand, if $j = lm + \mathfrak{s}$, $0 \le l < \frac{p-1}{2}$, $j \neq N/2$ but $i(j) = \frac{m}{2}$ (i.e. we assume that m- is even) then $\mathscr{V}_j$ is also reducible. Indeed, notice that we have the following character table for $D_m \times \mathbb{Z}_2$-representations

	(± 1)	$(\pm\zeta)$	...	$(\pm\zeta^l)$	...	$(\pm\zeta^{\mathfrak{s}})$	$(\pm\kappa)$	$(\pm\zeta\kappa)$
$\chi_{\mathscr{V}_j}$	± 2	∓ 2	...	$\pm(-1)^l 2$	...	± 2	0	0
$\chi_{\mathcal{V}_{\mathfrak{s}}^-}$	± 1	∓ 1	...	$\pm(-1)^l$	...	± 1	± 1	∓ 1
$\chi_{\mathcal{V}_{\mathfrak{s}+2}^-}$	± 1	∓ 1	...	$\pm(-1)^l$	...	± 1	∓ 1	± 1

which implies that the $D_m \times \mathbb{Z}_2$-representation $\mathscr{V}_j$ is equivalent to $\mathcal{V}_{\mathfrak{s}}^- \oplus \mathcal{V}_{\mathfrak{s}+1}^-$.

Put $\mathfrak{s}^* = \mathfrak{s} + 1$ if m is odd, and $\mathfrak{s} + 2$ if m is even. Then we deduce from Remark 4.2.5 the following:

Proposition 4.2.6. *For the operator $\mathscr{L}$ the eigenspaces $\mathfrak{E}(\rho_j)$ corresponding to the eigenvalues $\rho_j = -4\sin^2\left(\frac{\pi j}{pm}\right)$, $j = 0, ..., \mathfrak{r}$, we have the following $D_m \times \mathbb{Z}_2$-isotypical decompositions:*

- $\mathfrak{E}(\rho_0) \equiv \left(\mathcal{V}_0^-\right)^k$;
- *if $i(j) = 0$, $j > 0$, then* $\mathfrak{E}(\rho_j) \equiv \left(\mathcal{V}_0^- \oplus \mathcal{V}_{\mathfrak{s}+1}^-\right)^k$;
- *if $j = N/2$ (i.e. m is even), then* $\mathfrak{E}(\rho_j) \equiv (\mathcal{V}_{\mathfrak{s}}^-)^k$;
- *if $j \neq N/2$ and $i(j) = \frac{m}{2} = \mathfrak{s}$, then* $\mathfrak{E}(\rho_j) \equiv \left(\mathcal{V}_{\mathfrak{s}}^- \oplus \mathcal{V}_{\mathfrak{s}+2}^-\right)^k$;
- *if $0 < j < N/2$ is such that $i(j) \neq 0, m/2$, then* $\mathfrak{E}(\rho_j) \equiv \left(\mathcal{V}_{i(j)}^-\right)^k$.

Recall that V is assumed to be a Γ-representation (see assumption (A5)). Suppose that

$$V := V_0 \oplus V_1 \oplus \dots V_r,$$

is the Γ-isotypical decomposition of V, where the component V_l, $0 \le l \le r$, is modeled on an irreducible Γ-representation $\mathcal{W}_l$. Then the irreducible G-representations, relevant to our discussion, will be denoted by $\mathcal{V}_{i,l} := \mathcal{V}_i^- \otimes \mathcal{W}_l$, $i = 0, 1, \dots, \mathfrak{s}^*$, $l = 0, 1, \dots, r$. We also put

$$\mathfrak{m}_l = \mathfrak{m}_l(\mu_0) = \dim V_l / \dim \mathcal{W}_l, \quad l = 0, 1, 2, \dots .r$$

Clearly, for each eigenvalue ρ_j, $j = 0, 1, \dots, \mathfrak{r}$, the subspace $\mathfrak{E}(\rho_j)$ is G-invariant. The following Proposition describes the G-isotypical decompositions of eigenspaces $\mathfrak{E}(\rho_j)$.

Proposition 4.2.7. *For the operator $\mathscr{L}$ the eigenspaces $\mathfrak{E}(\rho_j)$ corresponding to the eigenvalues $\rho_j = -4\sin^2\left(\frac{\pi j}{pm}\right)$, $j = 0, ..., \mathfrak{r}$, we have the following G-isotypical decompositions:*

- $\mathfrak{E}(\rho_0) \equiv \bigoplus_{l=0}^{r} (\mathcal{V}_{0,l})^k$;
- *if* $i(j) = 0$, $j > 0$, *then* $\mathfrak{E}(\rho_j) \equiv \bigoplus_{l=0}^{r} (\mathcal{V}_{0,l} \oplus \mathcal{V}_{\mathfrak{s}+1,l})^k$;
- *if* $j = N/2$ *(i.e. m is even), then* $\mathfrak{E}(\rho_j) \equiv \bigoplus_{l=0}^{r} (\mathcal{V}_{\mathfrak{s},l})^k$;
- *if* $j \neq N/2$ *and* $i(j) = \frac{m}{2} = \mathfrak{s}$, *then* $\mathfrak{E}(\rho_j) \equiv \bigoplus_{l=0}^{r} (\mathcal{V}_{\mathfrak{s},l} \oplus \mathcal{V}_{\mathfrak{s}+2,l})^k$;
- *if* $0 < i(j) < m/2$, *then* $\mathfrak{E}(\rho_j) \equiv \bigoplus_{l=0}^{r} \left(\mathcal{V}_{i(j),l}\right)^k$.

By applying the computational formula, we obtain (since k is assumed to be odd):

$$G\text{-deg}(-\operatorname{Id}, B_1(0)) = \prod_{j=0}^{\mathfrak{r}} G\text{-deg}\big(-\operatorname{Id}, B(\mathfrak{E}(\rho_j))\big) \tag{4.12}$$

Then by Proposition 4.2.7 and the fact that k is assumed to be odd, we obtain the following result:

Proposition 4.2.8. *Under the above assumptions, we have*

$$G\text{-deg}(-\text{Id}, B(\mathfrak{E}(\rho_j)) = \begin{cases} \prod_{l=0}^{r} \deg_{\mathcal{V}_{0,l}} & \text{if } j = 0, \\ \prod_{l=0}^{r} \deg_{\mathcal{V}_{\mathfrak{s},l}} \cdot \deg_{\mathcal{V}_{\mathfrak{s}+1,l}} & \text{if } i(j) = 0 \text{ and } j \neq 0, \\ \prod_{l=0}^{r} \deg_{\mathcal{V}_{\mathfrak{s},l}} & \text{if } j = N/2 \text{ i.e. } m \text{ is even}, \\ \prod_{l=0}^{r} \deg_{\mathcal{V}_{\mathfrak{s},l}} \cdot \deg_{\mathcal{V}_{\mathfrak{s}+2,l}} & \text{if } j \neq N/2 \text{ and } i(j) = m/2, \\ \prod_{l=0}^{r} \deg_{\mathcal{V}_{i(j),l}} & \text{if } 0 < i(j) < m/2. \end{cases}$$

Then by Proposition 4.2.8, we have that if $0 < i(j) = i(j') < m/2$ then

$$G\text{-deg}(-\text{Id}, B(\mathfrak{E}(\rho_j)) = G\text{-deg}(-\text{Id}, B(\mathfrak{E}(\rho_{j'})).$$

Moreover, let us also point out that it is not true that the basic degrees are different for different representations. For example, we have the following $D_m \times \mathbb{Z}_2$-basic degrees $\deg_{\mathcal{V}_i^-}$ (for the relevant irreducible representations): for $0 \leq i \leq \lfloor \frac{m}{2} \rfloor$, $h := \gcd(m, i)$, $l := \frac{m}{h}$, one has:

if l is odd then

$$\deg_{\mathcal{V}_i^-} = (D_m \times \mathbb{Z}_2) - (D_h) - (D_h^z) + (\mathbb{Z}_h),$$

if $l \equiv 2 \pmod 4$ then

$$\deg_{\mathcal{V}_i^-} = (D_m \times \mathbb{Z}_2) - (D_{2h}^d) - (D_{2h}^{\hat{d}}) + (\mathbb{Z}_{2h}^d),$$

if $l \equiv 0 \pmod 4$ then

$$\deg_{\mathcal{V}_i^-} = (D_m \times \mathbb{Z}_2) - (D_{2h}^d) - (\widetilde{D}_{2h}^d) + (\mathbb{Z}_{2h}^d),$$

if m is even and $i = \mathfrak{s}$ then

$$\deg_{\mathcal{V}_\mathfrak{s}^-} = (D_m \times \mathbb{Z}_2) - (D_m^d),$$

if $i = 0$ then

$$\deg_{\mathcal{V}_0^-} = (D_m \times \mathbb{Z}_2) - (D_m),$$

if $i = \mathfrak{s}+1$ then

$$\deg_{\mathcal{V}^-_{\mathfrak{s}+1}} = (D_m \times \mathbb{Z}_2) - (D^z_m),$$

if m is even and $i = \mathfrak{s}+2$ then

$$\deg_{\mathcal{V}^-_{\mathfrak{s}+2}} = (D_m \times \mathbb{Z}_2) - (D^{\hat{d}}_m).$$

Notice that for $0 < i, i' \le \mathfrak{s}$, if $\gcd(m,i) = \gcd(m,i')$ then $\deg_{\mathcal{V}^-_i} = \deg_{\mathcal{V}^-_{i'}}$.

4.2.4 Maximal Orbit Types in the $D_m \times \mathbb{Z}_2$-Representation $\mathscr{H}$

Assume that $\Gamma = \{e\}$ is a trivial group. Then the maximal $D_m \times \mathbb{Z}_2$-orbit types in $\mathscr{H} \setminus \{0\}$ can be easily described. Indeed, we have the following

Proposition 4.2.9. *Assume that $\Gamma = \{e\}$ and*

$$m = p_1^{\varepsilon_1} p_2^{\varepsilon_2} \dots p_s^{\varepsilon_s}, \quad p_1 < p_2 < \dots < p_s \tag{4.13}$$

where p_ν are prime numbers and $\varepsilon_\nu \ge 1$, $\nu = 1, 2, \dots, s$. Then the $D_m \times \mathbb{Z}_2$-maximal orbit types in $\mathscr{H} \setminus \{0\}$ can be described as follows:

(a) (D_m) *and* (D^z_m) *are the maximal orbit types in the subspace of p-periodic sequences* $x = (\mathbf{a})$, $\mathbf{a} \in \mathbb{R}^k$,

(b) *If $p_\nu > 2$, $m_\nu := m/p_\nu$, where $\nu = 1, 2, \dots, s$, then $(D^z_{m_\nu})$ is a maximal orbit type in* $\mathscr{H} \setminus \{0\}$.

(b) *If m is even, then (D^d_m) is a maximal orbit type in $\mathscr{H} \setminus \{0\}$.*

(c) *If $4|m$ then there are two additional maximal orbit types $(D^d_{\frac{m}{2}})$ and $(\widetilde{D}^d_{\frac{m}{2}})$ in $\mathscr{H} \setminus \{0\}$.*

Proof. Notice that the $D_m \times \mathbb{Z}_2$-orbit types in $\mathscr{H}$ are the same as in the space

$$\mathcal{V}^-_0 \oplus \mathcal{V}^-_1 \oplus \dots \oplus \mathcal{V}^-_{\mathfrak{s}^*}, \quad \text{where} \quad \mathfrak{s}^* = \begin{cases} \mathfrak{s}+1 & \text{if } m \text{ is odd}, \\ \mathfrak{s}+2 & \text{if } m \text{ is even}. \end{cases}$$

The maximal orbit type $(D^z_{m_\nu})$, $m_\nu := \frac{m}{p_\nu}$, $p_\nu > 0$, occurs in $\mathcal{V}^-_{m_\nu}$. If m is even then the maximal orbit type (D^d_m) occurs in $\mathcal{V}^-_{\mathfrak{s}}$ and the orbit type $(D^{\hat{d}}_m)$ in $\mathcal{V}^-_{\mathfrak{s}+1}$. In the case m is divisible by 4, i.e. $p_1 = 2$ and $\varepsilon_1 \geq 2$, the maximal orbit types $(D^d_{\frac{m}{2}})$ and $(\widetilde{D}^d_{\frac{m}{2}})$ occur in $\mathcal{V}^-_{\frac{m}{4}}$.
□

4.3 Subharmonic Solutions in SODE Satisfying Nagumo Growth Condition

To present our result on the existence of subharmonic periodic solutions to (4.3), we introduce two additional assumptions on the function f:

(B1) for every $n = 1, ..., p$, there exists $\alpha > 4$, we have

$$\liminf_{\mathbf{x}\to 0} \frac{f(n,\mathbf{x}) \bullet \mathbf{x}}{|\mathbf{x}|^2} > \alpha.$$

(B2) there exists $M > 0$ such that for all $n = 1, ..., p$ and $\mathbf{x} \in V$, one has:

$$|\mathbf{x}| > M \quad \Rightarrow \quad \mathbf{x} \bullet f(n,\mathbf{x}) < 0.$$

The following two Lemmas are considered to be standard, however since we apply them in the context of difference equation, for the sake of completeness, we include the proofs.

Lemma 4.3.1. *Let $f : \mathbb{Z}\times V \to V$ satisfy* (A1)-(A5) *and* (B1). *Then there exists a constant $\varepsilon > 0$ such that if $x = (\mathbf{x}_n) \in \mathscr{H}$ is a non-zero solution to the difference system*

$$\triangle^2\mathbf{x}_{n-1} + (1-\lambda)\alpha\mathbf{x}_n + \lambda f(n,\mathbf{x}_n) = 0, \quad \mathbf{x}_n \in V,\ n \in \mathbb{Z},\ \lambda \in [0,1], \tag{4.14}$$

then $\|x\| > \varepsilon$.

Proof. The above statement is clearly true for $\lambda = 0$. By condition (B1), there exists a constant $\alpha > 4$ such that

$$\forall_{n\in\mathbb{Z}}\ \forall_{\mathbf{x}\in V}\ \exists_{\epsilon>0}\ \ 0 < |\mathbf{x}| \leq \varepsilon \ \Rightarrow \ \frac{f(n,\mathbf{x}) \bullet \mathbf{x}}{|\mathbf{x}|^2} \geq \alpha. \tag{4.15}$$

Assume for contradiction that there exists a solution $0 \neq x = (\mathbf{x}_n)$ to (4.14) such that $\|x\| \leq \varepsilon$, so for some $n_o \in \mathbb{Z}$, we have

$$0 < |\mathbf{x}_{n_o}| = \max\{|\mathbf{x}_n| : n \in \mathbb{Z}\} \leq \varepsilon.$$

Then, it follows from (4.15) that

$$\begin{aligned} 0 &= \frac{\triangle^2 \mathbf{x}_{n_o-1} \bullet \mathbf{x}_{n_o} + (1-\lambda)4\mathbf{x}_{n_o} \bullet \mathbf{x}_{n_o} + \lambda f(n_o, \mathbf{x}_{n_o}) \bullet \mathbf{x}_{n_o}}{|\mathbf{x}_{n_o}|^2} \\ &\geq \frac{-4|\mathbf{x}_{n_o}|^2 + 4|\mathbf{x}_{n_o}|^2 + \lambda\big(f(n_o, \mathbf{x}_{n_o}) \bullet \mathbf{x}_{n_o} - 4|\mathbf{x}_{n_o}|^2\big)}{|\mathbf{x}_{n_o}|^2} \\ &\geq \lambda\left(\frac{f(n_o, \mathbf{x}_{n_o}) \bullet \mathbf{x}_{n_o}}{|\mathbf{x}_{n_o}|^2} - 4\right) \geq \lambda(\alpha - 4) > 0, \end{aligned}$$

and consequently we get a contradiction. □

Lemma 4.3.2. *Let* $f : \mathbb{Z} \times V \to V$ *satisfy* (A1)–(A5) *and* (B2). *Then there exists a constant* $R > 0$ *such that if* $x = (\mathbf{x}_n) \in \mathscr{H}$ *is a solution to the difference system*

$$\triangle^2 \mathbf{x}_{n-1} - (1-\lambda)\mathbf{x}_n + \lambda f(n, \mathbf{x}_n) = 0, \quad \mathbf{x}_n \in V,\ n \in \mathbb{Z},\ \lambda \in [0,1], \tag{4.16}$$

then $\|x\| < R$.

Proof. Put $R := \sqrt{pmM^2 + 1}$ and assume, for contradiction, that $x = (\mathbf{x}_n) \in \mathscr{H}$ is a solution to (4.16) for some $\lambda \in [0,1]$ and $\|x\| \geq R$. Suppose that $|\mathbf{x}_{n_o}| = \max\{|\mathbf{x}_n| : n = 1, ..., pm]\}$, then clearly, $|\mathbf{x}_{n_o}| > M$, $\langle \mathbf{x}_{n_o}, \triangle^2 \mathbf{x}_{n_o-1}\rangle \leq 0$ and by assumption (B2), one has:

$$0 = \mathbf{x}_{n_o} \bullet \left(\triangle^2 \mathbf{x}_{n_o-1} - (1-\lambda)\mathbf{x}_{n_o} + \lambda f(n_o, \mathbf{x}_{n_o})\right) < -(1-\lambda)|\mathbf{x}_{n_o}|^2 \leq 0.$$

□

For any prime number $q \geq 2$ we put

$$\pi(q) := \begin{cases} 1 & \text{if } q = 2, \\ q & \text{if } \frac{q-1}{2} \text{ is odd}, \\ 0 & \text{if } \frac{q-1}{2} \text{ is even}. \end{cases} \tag{4.17}$$

Let us formulate our first result in a non-equivariant setting (i.e., with $\Gamma = \{e\}$ being a trivial group).

Theorem 4.3.3. *Let $\Gamma = \{e\}$ and assume that k is an odd number, $f : \mathbb{Z} \times V \to V$ satisfies the conditions* (A1)—(A5) and (B1)—(B2), *and $m = p_1^{\varepsilon_1} p_2^{\varepsilon_2} \dots p_s^{\varepsilon_s}$, $p_1 < p_2 < \dots < p_s$ prime numbers, $\varepsilon_\nu \geq 1$, $\nu = 1, 2, \dots, s$. Then:*

(i) *for every $p_\nu > 2$, with $\pi(p_\nu) = p_\nu$ ($\nu \in \{1, 2, \dots, s\}$), the system* (4.3) *admits a $D_m \times \mathbb{Z}_2$-orbit of pm-periodic solutions with symmetries exactly $(D^z_{m_\nu})$, $m_\nu := \frac{m}{p_\nu}$;*

(ii) *if m is even, i.e. $p_1 = 2$, then the system* (4.3) *admits a $D_m \times \mathbb{Z}_2$-orbit of pm-periodic solutions with symmetries exactly (D^d_m), or $(D^{\hat{d}}_m)$;*

(iii) *if m is divisible by* 4*, then the system* (4.3) *admits additional $D_m \times \mathbb{Z}_2$-orbits of pm-periodic solutions with symmetries exactly $(D^d_{\frac{m}{2}})$ and $(\widetilde{D}^d_{\frac{m}{2}})$.*

To summarize, the system (4.3) *admits at least $2(\pi(p_1) + \pi(p_2) + \dots + \pi(p_s))$ non-zero non-constant pm-periodic solutions, and in the case m is divisible by* 4*, at least $2(\pi(p_1) + \pi(p_2) + \dots + \pi(p_s)) + 8$ non-zero non-constant pm-periodic solutions.*

Proof. Consider the operator $\mathscr{F} : \mathscr{H} \to \mathscr{H}$ given by (4.4) and define the linear operators $\mathscr{A}, \mathscr{B} : \mathscr{H} \to \mathscr{H}$ by

$$(\mathscr{A} x)_n := \triangle^2 \mathbf{x}_{n-1} + 4\mathbf{x}_n,$$

$$(\mathscr{B} x)_n := \triangle^2 \mathbf{x}_{n-1} - \mathbf{x}_n,$$

where $x = (\mathbf{x}_n) \in \mathscr{H}$. Then, by Lemma 4.3.1, there exists $\varepsilon > 0$ such that $\mathscr{F}$ is $B_\varepsilon(0)$-admissibly G-homotopic to $\mathscr{A}$, and by Lemma 4.3.2, there exists $R > 0$ such that $\mathscr{F}$ is $B_R(0)$-admissibly G-homotopic to $\mathscr{B}$. Next, since all the eigenvalues (i.e. $-4\sin^2\left(\frac{\pi j}{pm}\right) + 4$) of $\mathscr{A}$ are positive, $\mathscr{A}$ is $B_\varepsilon(0)$-admissibly G-homotopic to Id, and since all the eigenvalues (i.e. $-4\sin^2\left(\frac{\pi j}{pm}\right) - 10$) of $\mathscr{B}$ are negative, $\widetilde{f}$ is $B_R(0)$-admissibly G-homotopic to $-\,$Id Put

$\Omega := B_R(0) \setminus \overline{B_\varepsilon(0)}$. Then, by the additivity property of the Brouwer G-equivariant degree we have

$$\begin{aligned} G\text{-deg}(\mathscr{F}, \Omega) &= G\text{-deg}(\mathscr{F}, B_R(0)) - G\text{-deg}(\mathscr{F}, B_\varepsilon(0)) \\ &= G\text{-deg}(\mathscr{B}, B_R(0)) - G\text{-deg}(\mathscr{A}, B_\varepsilon(0)) \\ &= G\text{-deg}(-\operatorname{Id}, B_R(0)) - G\text{-deg}(\operatorname{Id}, B_\varepsilon(0)). \end{aligned}$$

The G-equivariant degree $G\text{-deg}(-\operatorname{Id}, B_R(0))$ is given by (4.12) and $G\text{-deg}(-\operatorname{Id}, B(\mathfrak{E}(\rho_j)))$, $j = 0, 1, \dots, \mathfrak{r}$, are given in Proposition 4.2.8. In particular for $i(j) = m_\nu := \frac{m}{p_\nu}$, $p_\nu > 2$, we have that there are exactly $p^{\frac{p_\nu - 1}{2}}$ numbers $j \in \{0, 1, \dots, \mathfrak{r}\}$ such that $\gcd(i(j), m) = m_\nu$. Indeed, those numbers are: $lm + m_\nu$, $lm + 2m_\nu$, ..., $lm + \frac{p_\nu - 1}{2} m_\nu$, where $l = 0, 1, \dots, p-1$. Consequently, the basic degree $\deg_{\mathcal{V}^-_{m_\nu}}$ appears in the $G\text{-deg}(-\operatorname{Id}, B_R(0))$ exactly $N_\nu = kp^{\frac{p_\nu - 1}{2}}$ times. Since, $(\deg_{\mathcal{V}^-_{m_\nu}})^2 = (D_m \times \mathbb{Z}_2)$ and k is odd, it follows that

$$(\deg_{\mathcal{V}^-_{m_\nu}})^{N_\nu} = \begin{cases} \deg_{\mathcal{V}^-_{m_\nu}} & \text{if } \frac{p_\nu - 1}{2} \text{ is odd,} \\ (D_m \times \mathbb{Z}_2) & \text{if } \frac{p_\nu - 1}{2} \text{ is even.} \end{cases}$$

Since $(D^z_{m_\nu})$ is a maximal orbit type appearing in $\deg_{\mathcal{V}^-_{m_\nu}}$, the coefficient standing by $(D^z_{m_\nu})$ in $G\text{-deg}(\mathscr{F}, \Omega)$ is equal to ± 1 and the conclusion (i) follows.

Consider now the case where m is even, i.e. $p_1 = 2$. In this case we have $m_1 = \frac{m}{2} = \mathfrak{s}$, so $h_\nu := \gcd(m, \frac{m}{2}) = \frac{m}{2}$ and $l_1 := \frac{m}{m_1} = 2$, which implies that $l_\nu = 2 \bmod 2$. In such a case we have $\deg_{\mathcal{V}^-_{\mathfrak{s}}} = (D_m \times \mathbb{Z}_2) - (D^d_m)$. Since, by Proposition 4.2.8, if $j \neq N/2$ and $i(j) = \frac{m}{2}$, then

$$D_m \times \mathbb{Z}_2\text{-deg}(-\operatorname{Id}, \mathfrak{E}(\rho_j)) = \deg_{\mathcal{V}^-_{\mathfrak{s}}} \cdot \deg_{\mathcal{V}^-_{\mathfrak{s}+2}}.$$

By a similar argument as before, there are exactly $p-1$ numbers j such that $j \neq N/2$ and $i(j) = \frac{m}{2}$. Since $p-1$ is even, we obtain by Proposition 4.2.8,

$$\begin{aligned}\prod_{\substack{j\\ i(j)=\frac{m}{2}}} D_m \times \mathbb{Z}_2\text{-deg}(-\operatorname{Id}, \mathfrak{E}(\rho_j)) &= D_m \times \mathbb{Z}_2\text{-deg}(-\operatorname{Id}, \mathfrak{E}(\rho_{\frac{N}{2}}))\\
&\quad \cdot \prod_{\substack{j, j\neq \frac{N}{2}\\ i(j)=\frac{m}{2}}} D_m \times \mathbb{Z}_2\text{-deg}(-\operatorname{Id}, \mathfrak{E}(\rho_j))\\
&= \deg_{\mathcal{V}_{\mathfrak{s}}^-} \cdot \left(\deg_{\mathcal{V}_{\mathfrak{s}}^-} \cdot \deg_{\mathcal{V}_{\mathfrak{s}+2}^-}\right)^{p-1}\\
&= \deg_{\mathcal{V}_{\mathfrak{s}}^-},\end{aligned}$$

which implies that $\deg_{\mathcal{V}_{\mathfrak{s}}^-}$ can be factored out from the G-deg$(-\operatorname{Id}, B_R(0))$, i.e.

$$G\text{-deg}(-\operatorname{Id}, B_R(0)) = \big((D_m \times \mathbb{Z}_2) - (D_m^d)\big) \cdot \left((D_m \times \mathbb{Z}_2) + \sum_{(H)} a_H(H)\right)$$

where (H) is either strictly smaller or not comparable with (D_m^d). Therefore,

$$G\text{-deg}(\mathscr{F}, \Omega) = G\text{-deg}(-\operatorname{Id}, B_R(0)) - (D_m \times \mathbb{Z}_2) = -(D_m^d) + \sum_{(H)} b_H(H),$$

and since (H) are not comparable with (D_m^d) or strictly smaller , the (D_m^d)-coefficient in G-deg$(\mathscr{F}, \Omega)$ is -1. Consequently, by the existence property of the Brouwer $D_m \times \mathbb{Z}_2$-equivariant degree and the maximality of the orbit type (D_m^d), there exists an orbit of solutions to (4.3) with the orbit type (D_m^d).

In the case m is divisible by 4, we have for $i = \frac{m}{4}$, there are exactly p numbers $j \in \{0, 1, 2, \dots, \mathfrak{r}\}$ such that $i(j) = \frac{m}{4}$. Therefore, by Proposition 4.2.8, we have

$$\prod_{\substack{j\\ i(j)=\frac{m}{4}}} D_m \times \mathbb{Z}_2\text{-deg}(-\operatorname{Id}, \mathfrak{E}(\rho_j)) = (\deg_{\mathcal{V}_{\frac{m}{4}}})^p = \deg_{\mathcal{V}_{\frac{m}{4}}}.$$

On the other hand, we have

$$\deg_{\mathcal{V}_{\frac{m}{4}}} = (D_m \times \mathbb{Z}_2) - (D_{\frac{m}{2}}^d) - (\widetilde{D}_{\frac{m}{2}}^d) + (\mathbb{Z}_{\frac{m}{2}}^d),$$

where, by Proposition 4.2.9, $(D_m \times \mathbb{Z}_2)$ and $(D^d_{\frac{m}{2}})$ are maximal orbit types in $\mathscr{H} \setminus \{0\}$. By a similar argument as above, we obtain that there exist two orbits of solutions to (4.3) with the orbit types $(D_m \times \mathbb{Z}_2)$ and $(D^d_{\frac{m}{2}})$.

Finally, notice that for an element $x \in \mathscr{H}$, if $G_x = D^z_{m_\nu}$, then

$$|G(x)| = \left| \frac{D_m \times \mathbb{Z}_2}{D^z_{m_\nu}} \right| = \frac{2m \cdot 2}{2m_\nu} = 2p_\nu,$$

and similarly, for $G_x = D^d_m$ and $G_y = D^d_{\frac{m}{2}}$, we have

$$|G(x)| = \left| \frac{D_m \times \mathbb{Z}_2}{D^d_m} \right| = \frac{2m \cdot 2}{2m} = 2, \quad |G(x)| = \left| \frac{D_m \times \mathbb{Z}_2}{D^d_{\frac{m}{2}}} \right| = \frac{2m \cdot 2}{m} = 4,$$

where $|X|$ denotes the number of elements in the set X. Therefore, system (4.3) admits at least $2(\pi(p_1) + \pi(p_2) + \cdots + \pi(p_l))$ non-zero non-constant pm-periodic solutions and in the case m is divisible by 4 it has additional 8 pm-periodic solutions □

4.3.1 Existence Result for a Symmetric Difference Equation: GAP Assisted Computations

In this subsection, we illustrate how to use the **GAP system** and **EquiDeg package**, which is specially developed to assist the computations of the equivariant Brouwer degree. We refer to Appendix A for more details.

In order to illustrate the impact of additional symmetries (i.e., $\Gamma \neq \{e\}$) on the number of non-constant pm-periodic solutions to (4.3), we consider the case of the group D_3 acting on $V = \mathbb{R}^3$ (i.e., $k = 3$) by permutation of coordinates, i.e., for $\gamma = e^{\frac{2\pi}{3}}$ and $\kappa \in D_3$,

$$\gamma(x_1, x_2, x_3)^T = (x_3, x_1, x_2)^T, \quad \kappa(x_1, x_2, x_3)^T = (x_1, x_3, x_2), \tag{4.18}$$

where $\mathbf{x} = (x_1, x_2, x_3) \in V$.

Case: $\Gamma = D_3$**,** $m = 3$ **and** $p = 1$**:** Since $p = 1$, it actually means that (4.3) is autonomous. Thus in this case $\mathscr{H} = V \oplus V \oplus V = \mathbb{R}^9$ is equipped with the action of $G := D_3 \times D_3 \times \mathbb{Z}_2$ defined on generators

$$\kappa_1 := (\kappa, 1, 1), \quad \kappa_2 := (1, \kappa, 1), \quad (-\mathbf{1}) := (1, 1, -1),$$
$$\gamma_1 := (\gamma, 1, 1), \quad \gamma_2 := (1, \gamma, 1), \quad \gamma := e^{\frac{2\pi i}{3}},$$

as follows

$$\kappa_1((x_1, x_2, x_3), (y_1, y_2, y_3), (z_1, z_2, z_3)) := ((x_1, x_3, x_2), (y_1, y_3, y_2), (z_1, z_3, z_2)),$$
$$\kappa_2((x_1, x_2, x_3), (y_1, y_2, y_3), (z_1, z_2, z_3)) := ((x_1, x_2, x_3), (z_1, z_2, z_3), (y_1, y_2, y_3)),$$
$$\gamma_1((x_1, x_2, x_3), (y_1, y_2, y_3), (z_1, z_2, z_3)) := ((x_3, x_1, x_2), (y_3, y_1, y_2), (z_3, z_1, z_2)),$$
$$\gamma_2((x_1, x_2, x_3), (y_1, y_2, y_3), (z_1, z_2, z_3)) := ((z_1, z_2, z_3), (x_1, x_2, x_3), (y_1, y_2, y_3)),$$
$$(-\mathbf{1})(\mathbf{x}, \mathbf{y}, \mathbf{z}) := (-\mathbf{x}, -\mathbf{y}, -\mathbf{z})$$

Using GAP, one can easily find out that there are exactly 18 conjugacy classes of elements and 69 conjugacy classes of subgroups of G. While the the list of irreducible characters is indexed from 1 to 18, we shift the indices by 1 so that `irr_list[k+1]` in GAP represents $\chi_k = \chi_{\mathcal{V}_k}$ for $(k = 0, \dots, 17)$ in our notation.

The character $\chi_{\mathscr{H}}$ of the representation can be easily computed by inspection, i.e.,

	$(\pm\mathbf{1})$	$(\pm\kappa_2)$	$(\pm\boldsymbol{\gamma}_2)$	$(\pm\kappa_1)$	$(\pm\boldsymbol{\kappa}_1\kappa_2)$	$(\pm\kappa_1\gamma_2)$	$(\pm\gamma_1)$	$(\pm\gamma_1\kappa_2)$	$(\pm\gamma_1\gamma_2)$
$\chi_{\mathscr{H}}$	± 9	± 3	0	± 3	$\pm\mathbf{1}$	0	0	0	0

In such a case, the isotypical decomposition of $\mathscr{H}$ is described by a vector obtained in GAP:

$\mathcal{V}_0$	$\mathcal{V}_1$	$\mathcal{V}_2$	$\mathcal{V}_3$	$\mathcal{V}_4$	$\mathcal{V}_5$	$\mathcal{V}_6$	$\mathcal{V}_7$	$\mathcal{V}_8$	$\mathcal{V}_9$	$\mathcal{V}_{10}$	$\mathcal{V}_{11}$	$\mathcal{V}_{12}$	$\mathcal{V}_{13}$	$\mathcal{V}_{14}$	$\mathcal{V}_{15}$	$\mathcal{V}_{16}$	$\mathcal{V}_{17}$
0	0	0	0	1	0	0	0	0	1	0	0	0	1	0	0	0	1

That is, G-isotypical decomposition of the G-representation $\mathscr{H}$ is given by:

$$\mathscr{H} := \mathscr{H}_4 \oplus \mathscr{H}_9 \oplus \mathscr{H}_{13} \oplus \mathscr{H}_{17}$$

where $\mathscr{H}_j \simeq \mathcal{V}_j$, $j = 4, 9, 13, 17$.

Under the assumptions (A1)–(A5) and (B1)—(B2), without considering the additional symmetries i.e. $\Gamma = \{e\}$ (cf. (A5)), Theorem 4.3.3 guarantees the existence of at least one orbit of solutions with the orbit type (D_3^z), i.e., at least 6 non-constant different solutions to (4.3). However, with the additional assumptions that $\Gamma = D_3$, we have the following result:

Theorem 4.3.4. *Let $k = 3$, $\Gamma = D_3$ act on $V = \mathbb{R}^3$ by* (4.18)*, $m = 3$, $p = 1$ and $f : \mathbb{Z} \times V \to V$ be such that* (A1)—(A5) *and* (B1)—(B2) *are satisfied. Then system* (4.3) *admits:*

(a) at least one orbit of 3*-periodic solutions of type $(D_1 \times_{\mathbb{Z}_2}^{D_3} D_3^p)$, i.e., there are at least* 12 *different* 3*-periodic solutions of this type,*

(b) at least one orbit of 3*-periodic solutions of type $(D_3 \times D_3)$, i.e., there are at least* 2 *different* 3*-periodic solutions of this type,*

(c) at least one orbit of 3*-periodic solutions of type $(D_1 \times_{\mathbb{Z}_2}^{D_1^z} D_1^p)$, i.e., there are at least* 18 *different* 3*-periodic solutions of this type,*

(d) at least one orbit of 3*-periodic solutions of type $(D_3 \times D_1^z)$, i.e., there is are least* 18 *different* 3*-periodic solutions of this type,*

(e) at least one orbit of 3*-periodic solutions of type $(D_3 \times_{D_3} D_3^z)$, i.e., there are at least* 12 *different* 3*-periodic solutions of this type.*

Consequently, with the additional D_3-symmetries, the system (4.3) *admits at least* 50 *different 3-periodic solutions.*

Proof. We apply the same steps as in the proof of Theorem 4.3.3, i.e., we have for the operator $\mathscr{F} : \mathscr{H} \to \mathscr{H}$ is given by (4.4) that

$$G\text{-deg}(\mathscr{F}, \Omega) = \deg_{\mathcal{V}_4} \cdot \deg_{\mathcal{V}_9} \cdot \deg_{\mathcal{V}_{13}} \cdot \deg_{\mathcal{V}_{17}} -(G).$$

Using GAP, we obtain the following result:

$$G\text{-deg}(\mathscr{F},\Omega) = (H_3) - (H_4) + (H_{14}) - (H_{16}) + (H_{18}) + (H_{28}) \\ - (H_{31}) - (H_{43}) - (H_{44}) + (H_{45}) + (H_{51}) + (H_{67}).$$

In addition, the maximal orbit types in $\mathscr{H} \setminus \{0\}$ are (H_{14}), (H_{28}), (H_{45}), (H_{51}), and (H_{67}). These conjugacy classes (H_k) can be easily recognized in GAP, whose representative are described below:

$$H_{14} = \langle -\boldsymbol{\kappa}_2, \boldsymbol{\kappa}_1\boldsymbol{\kappa}_2\rangle, \quad H_{28} = \langle -\boldsymbol{\kappa}_1\boldsymbol{\kappa}_2, \boldsymbol{\gamma}_1\boldsymbol{\gamma}_2\rangle, \quad H_{45} = \langle -\boldsymbol{\kappa}_1, \boldsymbol{\kappa}_2, \boldsymbol{\gamma}_2\rangle, \\ H_{51} = \langle -\boldsymbol{\kappa}_2, \boldsymbol{\kappa}_1, \boldsymbol{\gamma}_1\rangle, \quad H_{61} = \langle \boldsymbol{\kappa}_1, \boldsymbol{\kappa}_2, \boldsymbol{\gamma}_1, \boldsymbol{\gamma}_2\rangle.$$

These subgroups, by simple inspection, can be easily described via amalgamated notation:

$$H_{14} = D_1 \times_{\mathbb{Z}_2}^{D_1^z} D_1^p, \quad H_{28} = D_3 \times_{D_3} D_3^z, \quad H_{45} = D_1 \times^{D_3} D_3^p, \\ H_{51} = D_3 \times D_1^z, \quad H_{61} = D_3 \times D_3.$$

Therefore, the statement follows. □

Case $m = 5$ **and** $p = 1$**:** In this example, we look for 5-periodic solutions in the same $D_3 \times \mathbb{Z}_2$-symmetric system (4.3), where we consider again the case of the group D_3 acting on $V = \mathbb{R}^3$ (i.e., $k = 3$) by permutation of coordinates like it was described above. In this case $\mathscr{H} = V \oplus V \oplus V \oplus V \oplus V = \mathbb{R}^{15}$ is equipped with the action of $G := D_5 \times D_3 \times \mathbb{Z}_2$ defined on generators

$$\boldsymbol{\kappa}_1 := (\kappa, 1, 1), \quad \boldsymbol{\kappa}_2 := (1, \kappa, 1), \quad (-\mathbf{1}) := (1, 1, -1), \\ \boldsymbol{\gamma}_1 := (\tau, 1, 1), \quad \boldsymbol{\gamma}_2 := (1, \gamma, 1), \quad \gamma := e^{\frac{2\pi i}{3}}, \ \tau = e^{\frac{2\pi i}{5}}.$$

Using the same GAP routine `ConjugacyClasses(G)` we find out that there are exactly 18 conjugacy classes of elements in G, then we list 64 conjugacy classes (H) of subgroups in G

is created by GAP command `ccs_list := ConjugacyClassesSubgroups(`G`)` and generate the characters of irreducible G-representations by `irr_list := Irr(`G`)`. Since this list is indexed from 1 to 24, where `irr_list[1]` stands for the trivial representation, as usual, we shift the indices $\chi_k = \chi_{\mathcal{V}_k}$:=`irr_list[`$k+1$`]`, in order to create our list of irreducible G-representations $\mathcal{V}_k$.

The character $\chi_{\mathscr{H}}$ of the representation can be easily computed by inspection, namely we have:

	$(\pm\mathbf{1})$	$(\pm\boldsymbol{\kappa}_2)$	$(\pm\boldsymbol{\gamma}_2)$	$(\pm\boldsymbol{\kappa}_1)$	$(\pm\boldsymbol{\kappa}_1\boldsymbol{\kappa}_2)$	$(\pm\boldsymbol{\kappa}_1\gamma_2)$	$(\pm\boldsymbol{\gamma}_1)$	$(\pm\boldsymbol{\gamma}_1\kappa_2)$	$(\pm\boldsymbol{\gamma}_1\gamma_2)$	$(\pm\boldsymbol{\gamma}_1^2)$	$(\pm\boldsymbol{\gamma}_1^2\boldsymbol{\kappa}_2)$	$(\pm\boldsymbol{\gamma}_1^2\boldsymbol{\gamma}_2^2)$
$\chi_{\mathscr{H}}$	±15	±5	0	±3	$\pm\mathbf{1}$	0	0	0	0	0	0	0

The isotypical decomposition of $\mathscr{H}$ can be obtained by applying the command `SolutionMat(irr_list,`$\chi_{\mathscr{H}}$`)`

$\mathcal{V}_0$	$\mathcal{V}_1$	$\mathcal{V}_2$	$\mathcal{V}_3$	$\mathcal{V}_4$	$\mathcal{V}_5$	$\mathcal{V}_6$	$\mathcal{V}_7$	$\mathcal{V}_8$	$\mathcal{V}_9$	$\mathcal{V}_{10}$	$\mathcal{V}_{11}$	$\mathcal{V}_{12}$	$\mathcal{V}_{13}$	$\mathcal{V}_{14}$	$\mathcal{V}_{15}$	$\mathcal{V}_{16}$	$\mathcal{V}_{17}$	$\mathcal{V}_{18}$	$\mathcal{V}_{19}$	$\mathcal{V}_{20}$	$\mathcal{V}_{21}$	$\mathcal{V}_{22}$	$\mathcal{V}_{23}$
0	0	0	0	1	0	0	0	0	1	0	0	0	0	1	1	0	0	0	0	0	0	1	1

Consequently, G-isotypical decomposition of the G-representation $\mathscr{H}$ (see Appendix, section 4.2.1) is given by:

$$\mathscr{H} := \mathscr{H}_4 \oplus \mathscr{H}_9 \oplus \mathscr{H}_{14} \oplus \mathscr{H}_{15} \oplus \mathscr{H}_{22} \oplus \mathscr{H}_{23}$$

where $\mathscr{H}_j \simeq \mathcal{V}_j$, $j = 4, 9, 14, 15, 22, 23$.

The maximal orbit types in $\mathscr{H} \setminus \{0\}$ are obtained by:

`MaximalOrbitTypes(`$\chi_{\mathscr{H}}$`)`

which are (H_{10}), (H_{37}), (H_{45}), and (H_{62}). These conjugacy classes (H_k) can be easily recognized by using commands `ccs_list[k][1]`,

$$H_{10} = \langle -\boldsymbol{\kappa}_1, \boldsymbol{\kappa}_1\boldsymbol{\kappa}_2\rangle = D_1 \times_{\mathbb{Z}_2}^{D_1^z} D_1^p, \quad H_{37} = \langle -\boldsymbol{\kappa}_1, \boldsymbol{\kappa}_2, \boldsymbol{\gamma}_2\rangle = D_1 \times_{\mathbb{Z}_2}^{D_3} D_3^p$$

$$H_{45} = \langle \boldsymbol{\gamma}_1, \boldsymbol{\kappa}_1, \boldsymbol{\kappa}_2\rangle = D_5 \times D_1^z, \quad H_{62} = \langle \boldsymbol{\gamma}_1, \boldsymbol{\kappa}_1, \boldsymbol{\gamma}_2, \boldsymbol{\kappa}_2\rangle = D_5 \times D_3.$$

The degree G-deg$(\mathscr{F}, \Omega)$ can be computed by one command

$$\texttt{basis[64]-BasicDegree(}\chi_{\mathscr{H}}\texttt{)}$$

which leads to

$$G\text{-deg}(\mathscr{F}, \Omega) = (H_{45}) + (H_{62}) - (H_{45}).$$

In this way, we obtain the following result:

Theorem 4.3.5. *Let* $k = 3$, $\Gamma = D_3$ *act on* $V = \mathbb{R}^3$ *by* (4.18), $m = 5$, $p = 1$ *and* $f : \mathbb{Z} \times V \to V$ *be such that* (A1)—(A5) *and* (B1)—(B2) *are satisfied. Then the system* (4.3) *admits:*

(a) *at least one orbit of* 5*-periodic solutions of type* $(D_5 \times D_1^z)$*, i.e., there are at least* 3 *different non-constant* 5*-periodic solutions of this type,*

(b) *at least one orbit of* 5*-periodic solutions of type* $(D_5 \times D_3)$*, i.e., there are at least* 2 *different (constant)* 5*-periodic solutions of this type,*

Consequently, with the additional D_3*-symmetries, the system* (4.3) *admits at least* 5 *different 5-periodic solutions.*

Notice that Theorem 4.3.3 (with D_3-action ignored) provides no non-constant solution to the system (4.3).

Case: $\Gamma = D_3$, $m = 4$ **and** $p = 1$**:** We use similar notation that was introduced in the previous examples. The character $\chi_{\mathscr{H}}$ of the representation $\mathscr{H}$ can be easily computed by inspection, namely we have:

	$(\pm\mathbf{1})$	$(\pm\boldsymbol{\kappa}_2)$	$(\pm\boldsymbol{\gamma}_2)$	$(\pm\boldsymbol{\kappa}_1)$	$(\pm\boldsymbol{\kappa}_1\boldsymbol{\kappa}_2)$	$(\pm\boldsymbol{\kappa}_1\boldsymbol{\gamma}_2)$	$(\pm\boldsymbol{\gamma}_1\boldsymbol{\kappa}_1)$	$(\pm\boldsymbol{\gamma}_1\boldsymbol{\kappa}_1\boldsymbol{\kappa}_2)$	$(\pm\boldsymbol{\gamma}_1)$	$(\pm\boldsymbol{\gamma}_1\boldsymbol{\gamma}_1)$	$(\pm\boldsymbol{\gamma}_1^2\boldsymbol{\kappa}_2)$	$(\pm\boldsymbol{\gamma}_1\boldsymbol{\gamma}_2)$	$\pm\boldsymbol{\gamma}_1^2$	$\pm\boldsymbol{\gamma}_1^2\kappa_2$	$\pm\boldsymbol{\gamma}_1^2\boldsymbol{\gamma}_2$
$\chi_{\mathscr{H}}$	± 12	± 4	0	± 6	$\pm\mathbf{2}$	0	0	0	0	0	0	0	0	0	0

Then, by using just one GAP command: `BasicDegree(`$\chi_{\mathscr{H}}$`)` we obtain that

$$
\begin{aligned}
G\text{-deg}(\mathscr{F},\Omega) &= (H_8)-(H_{12})-(H_{26})-(H_{32})+(H_{33})\\
&+(H_{35})+(H_{37})-(H_{43})+(H_{62})+(H_{84})+(H_{103})-(H_{104})\\
&-(H_{116})+(H_{117})-(H_{135})-(H_{144})-(H_{157})-(H_{169})+(H_{170})\\
&+(H_{173})-(H_{175})+(H_{177})+(H_{179})-(H_{200})+(H_{221})+(H_{229})
\end{aligned}
$$

Then the maximal orbit types in $\mathscr{H}$ can be easily identified by using the following GAP command `MaximalOrbitTypes(` $\chi_{\mathscr{H}}$`)`, which gives us the following list:

$$
\begin{aligned}
H_{103} &= \langle -\boldsymbol{\gamma}_1^2, \boldsymbol{\gamma}_1^2\boldsymbol{\kappa}_2, \boldsymbol{\gamma}_1^2\rangle, & |H_{103}| &= 8\\
H_{117} &= \langle -\boldsymbol{\gamma}_1^2, \boldsymbol{\gamma}_1^2\boldsymbol{\kappa}_2, \boldsymbol{\kappa}_1\rangle, & |H_{117}| &= 8\\
H_{170} &= \langle -\boldsymbol{\gamma}_1^2, -\boldsymbol{\gamma}_1^2, \boldsymbol{\gamma}_1\boldsymbol{\kappa}_2, \boldsymbol{\kappa}_1\rangle, & |H_{170}| &= 16\\
H_{173} &= \langle -\boldsymbol{\gamma}_1^2, \boldsymbol{\kappa}_1, \boldsymbol{\gamma}_1, -\boldsymbol{\kappa}_2\rangle, & |H_{173}| &= 16\\
H_{177} &= \langle -\boldsymbol{\gamma}_1^2, \boldsymbol{\kappa}_2, \boldsymbol{\gamma}_2, \boldsymbol{\gamma}_1\boldsymbol{\kappa}_1\rangle, & |H_{177}| &= 24\\
H_{179} &= \langle -\boldsymbol{\gamma}_1^2, -\boldsymbol{\gamma}_1^2, \boldsymbol{\kappa}_2, \boldsymbol{\kappa}_2, \boldsymbol{\gamma}_2, \boldsymbol{\gamma}_1\rangle, & |H_{179}| &= 24\\
H_{221} &= \langle \boldsymbol{\gamma}_1^2, \boldsymbol{\kappa}_1, \boldsymbol{\gamma}_1, \boldsymbol{\gamma}_2, \boldsymbol{\kappa}_2\rangle, & |H_{221}| &= 48\\
H_{229} &= \langle \boldsymbol{\gamma}_1^2, -\boldsymbol{\gamma}_1, \boldsymbol{\kappa}_2, \boldsymbol{\gamma}_2\boldsymbol{\kappa}_1\rangle, & |H_{229}| &= 48.
\end{aligned}
$$

Consequently, we have the following result

Theorem 4.3.6. *Let $k=3$, $\Gamma = D_3$ act on $V=\mathbb{R}^3$ by* (4.18)*, $m=4$, $p=1$ and $f:\mathbb{Z}\times V\to V$ be such that* (A1)—(A5) *and* (B1)—(B2) *are satisfied. Then the system* (4.3) *admits:*

(a) at least one orbit of 4-periodic solutions of type (H_{103}), i.e., there are at least 12 different 4-periodic solutions of this type,

(b) at least one orbit of 4-periodic solutions of type (H_{117}), i.e., there are at least 12 different 4-periodic solutions of this type,

(c) *at least one orbit of* 4*-periodic solutions of type* (H_{170})*, i.e., there are at least* 6 *different* 4*-periodic solutions of this type,*

(d) *at least one orbit of* 4*-periodic solutions of type* (H_{173})*, i.e., there are at least* 6 *different* 4*-periodic solutions of this type,*

(e) *at least one orbit of* 4*-periodic solutions of type* (H_{177})*, i.e., there are at least* 4 *different* 4*-periodic solutions of this type.*

(f) *at least one orbit of* 4*-periodic solutions of type* (H_{221})*, i.e., there are at least* 2 *different* 4*-periodic solutions of this type,*

(g) *at least one orbit of* 4*-periodic solutions of type* (H_{229})*, i.e., there are at least* 2 *different* 4*-periodic solutions of this type.*

Consequently, with the additional D_3*-symmetries, the system* (4.3) *admits at least* 48 *different 4-periodic solutions.*

4.4 Bifurcation of Subharmonic Solutions in Second Order Difference Equations

Let us consider the following parametrized system of second order difference equations

$$\triangle^2 \mathbf{x}_{n-1} + \lambda \mathbf{x}_n + f(n, \mathbf{x}_n) = 0 \tag{4.19}$$

where $\lambda \in \mathbb{R}$, $\mathbf{x}_n \in \mathbb{R}^k$ (here we assume that k is odd) and $f : \mathbb{Z} \times \mathbb{R}^k \to \mathbb{R}^k$ is continuous map satisfying (A1)-(A5). We will make the following additional assumptions

(C1) f is such that

$$D_{\mathbf{x}} f(n, 0) = A\mathbf{x},$$

where A is a symmetric $k \times k$-matrix.

(C2) there exist $\alpha > 2$, and $C > 0$

$$\forall_{n\in\mathbb{Z}}\,\forall_{\mathbf{x}\neq 0}\ ;\ f(n,\mathbf{x})\bullet\mathbf{x} < -C|\mathbf{x}|^{\alpha}.$$

In view of its connection to the coupled system of DNLS equations, the matrix A can be considered as the interaction matrix (or linearization of the interaction function), while the parameter λ is the frequency of the standing waves. We are interested in describing the local and global bifurcation of subharmonic pm-periodic solutions for (4.19). In order to keep the presentation relatively simple and avoid unnecessary complications that may occur in the case m is even, in this section we will assume that $m \geq 3$ **is an odd integer**.

For this purpose we introduce, as it was done earlier, the space $\mathscr{H}$ consisting of all pm-periodic sequences $x = (\mathbf{x}_n)$, $\mathbf{x}_n \in \mathbb{R}^k$ and define the map $\mathscr{F} : \mathbb{R}\times\mathscr{H} \to \mathscr{H}$ by

$$(\mathscr{F}(\lambda, x))_n = \triangle^2\mathbf{x}_{n-1} + \lambda\mathbf{x}_n + f(n,\mathbf{x}_n), \quad x = (\mathbf{x}_n) \in \mathscr{H}. \tag{4.20}$$

Under the assumptions (A1)–(A5) and (C1)–(C2), the map $\mathscr{F} : \mathbb{R}\oplus\mathscr{H} \to \mathscr{H}$ is continuous, $D_m \times \mathbb{Z}_2$-equivariant, $\mathscr{F}(\lambda, 0) = 0$, $D_{\mathbf{x}}\mathscr{F}(\lambda, 0)$ exists and

$$\left(D_{\mathbf{x}}\mathscr{F}(\lambda,0)(x)\right)_n = \triangle^2\mathbf{x}_{n-1} + \lambda\mathbf{x}_n + A\mathbf{x}_n, \quad x = (\mathbf{x}_n) \in \mathscr{H}.$$

We are looking for the bifurcation points $(\lambda_o, 0) \in \mathbb{R}\times\mathscr{H}$ for (4.20).

4.4.1 Application of the Equivariant Degree Method to Bifurcation Problems

Let us describe the equivariant degree method used here to study a general G-equivariant bifurcation problem

$$\mathscr{F}(\lambda, x) = 0, \quad (\lambda, x) \in \mathbb{R}\oplus\mathscr{H}, \tag{4.21}$$

(for a moment outside the settings speciffic to the second order difference equations and for an arbitrary group G). Assume that $a < b$ and let $\Omega \subset \mathscr{H}$ be an open bounded G-invariant subset. Assume that $\mathscr{F} : \mathbb{R}\oplus\mathscr{H} \to \mathscr{H}$ is a continuous G-equivariant map such that $(\mathscr{F}_a, \Omega)$,

$(\mathscr{F}_b, \Omega)$ are admissible G-pairs, where $\mathscr{F}_\lambda(x) := \mathscr{F}(\lambda, x)$, $\lambda \in \mathbb{R}$, $x \in \mathscr{H}$. A continuous G-invariant function $\varphi : \mathbb{R} \oplus \mathscr{H} \to \mathscr{H}$ is called a Ω-*complementing G-function* for $\mathscr{F}_\lambda$ at $\lambda = a$ and b, if

$$\begin{cases} \varphi(\lambda, x) < 0 & \text{if } \lambda = a,\ b,\ x \in \Omega \\ \varphi(\lambda, x) > 0 & \text{if } \lambda \in (a, b),\ x \in \partial\Omega. \end{cases} \tag{4.22}$$

In such a case we define the map $\mathfrak{F}_\varphi : \mathbb{R} \oplus \mathscr{H} \to \mathbb{R} \oplus \mathscr{H}$ by

$$\mathfrak{F}_\varphi(\lambda, x) = (\varphi(\lambda, x), \mathscr{F}(\lambda, x)), \quad \lambda \in \mathbb{R},\ x \in \mathscr{H}. \tag{4.23}$$

The following result is easy to prove, by adapting the proof of this well-known result in the non-equivariant case.

Theorem 4.4.1. *Suppose that $\mathscr{F} : \mathbb{R} \oplus \mathscr{H} \to \mathscr{H}$ is a continuous G-equivariant map such that $(\mathscr{F}_a, \Omega)$, $(\mathscr{F}_b, \Omega)$ are admissible G-pairs, and $\varphi : \mathbb{R} \oplus \mathscr{H} \to \mathscr{H}$ is an Ω-complementing G-function for $\mathscr{F}_\lambda$ at $\lambda = a$, b. Then $(\mathfrak{F}_\varphi, (a, b) \times \Omega)$ is an admissible G-pair, the G-equivariant degree $G\text{-deg}(\mathfrak{F}_\varphi, (a, b) \times \Omega)$ does not depend on the choice of the Ω-complementing G-function φ, and we have*

$$G\text{-deg}(\mathfrak{F}_\varphi, (a, b) \times \Omega) = G\text{-deg}(\mathscr{F}_a, \Omega) - G\text{-deg}(\mathscr{F}_b, \Omega).$$

As it is our case, we assume that $\mathscr{F} : \mathbb{R} \times \mathscr{H} \to \mathscr{H}$ satisfies $\mathscr{F}(\lambda, 0) = 0$ for all $\lambda \in \mathbb{R}$. Any pair $(\lambda, 0)$ satisfies (4.21), thus it is called a *trivial solution* to (4.21). All other solutions to (4.21), are called *nontrivial*. We denote by $\mathscr{S}$ the set of all nontrivial solutions to (4.21), i.e.,

$$\mathscr{S} := \{(\lambda, x) \in \mathbb{R} \times \mathscr{H} : \mathscr{F}(\lambda, x) = 0 \text{ and } x \neq 0\}.$$

Definition 4.4.2. *Let $\mathcal{C} \subset \mathscr{S}$ and $\mathscr{U} \subset \mathbb{R} \times \mathscr{H} \to \mathscr{H}$ be a G-invariant open subset. The set $\mathcal{C} \subset \mathscr{S}$ is called a branch of nontrivial solutions to (4.21) in $\mathscr{U}$ if $\overline{\mathcal{C}}$ is a connected component of $\overline{\mathscr{S}} \cap \overline{\mathscr{U}}$. Moreover, we say that the branch $\mathcal{C}$ bifurcates from a trivial solution $(\lambda_o, 0)$ if $(\lambda_o, 0) \in \overline{\mathcal{C}}$.*

The following well-known result in general topology is useful to detect bifurcation (cf. Theorem 3 on page 170 in [35]).

Theorem 4.4.3. (Kuratowski) *Let X be a metric space, A_0, $A_1 \subset X$ two disjoint closed sets in X, and K a compact set in X such that $K \cap A_0 \neq \emptyset \neq K \cap A_1$. If the set K does not contain a connected component K_o such that $K_o \cap A_0 \neq \emptyset \neq A_1 \cap K_o$, then there exist two disjoint open sets V_0, V_1 such that $A_0 \subset V_0$, $A_1 \subset V_1$ and $A_0 \cup A_1 \cup K \subset V_0 \cup V_1$.*

One can easily deduct an equivariant version of Theorem 4.4.3, where the space X is assumed to be G-space and all the involved sets are G-invariant. Using Theorem 4.4.3, we have the following.

Theorem 4.4.4. *Let $\mathscr{F} : \mathbb{R} \times \mathscr{H} \to \mathscr{H}$ be a continuous G-equivariant map and $a < \lambda_o < b$. Assume that*

(i) for all $\lambda \in \mathbb{R}$, $\mathscr{F}(\lambda, 0) = 0$,

(ii) for given λ_-, λ_+ such that $a \leq \lambda_- < \lambda_o < \lambda_+ \leq b$ there exists $\varepsilon > 0$ such that

$$\forall_{x \in \mathscr{H}} \;\; 0 < \|x\| \leq \varepsilon \;\; \Rightarrow \;\; \forall_{\lambda \in [a, \lambda_-] \cup [\lambda_+, b]} \;\; \mathscr{F}(\lambda, x) \neq 0,$$

(iii) for $\Omega := B_\delta(0)$, where $\delta > 0$ satisfies $\mathscr{F}(\lambda, x) \neq 0$ if $0 < \|x\| \leq \delta$ and $\lambda = a$, b, we have

$$G\text{-deg}(\mathscr{F}_a, \Omega) \neq G\text{-deg}(\mathscr{F}_b, \Omega).$$

Then, there exists a branch of nontrivial solutions $\mathcal{C}$ to (4.21) in $(a, b) \times \Omega$ bifurcating from $(\lambda_o, 0)$.

Proof. Let $\theta : \mathbb{R} \oplus \mathscr{H} \to \mathscr{H}$ be an arbitrary Ω-complementing G-function for $\mathscr{F}_\lambda$ at $\lambda = a$, b. By Theorem 4.4.1 we have

$$G\text{-deg}(\mathfrak{F}_\theta, (a, b) \times \Omega) = G\text{-deg}(\mathscr{F}_a, \Omega) - G\text{-deg}(\mathscr{F}_b, \Omega).$$

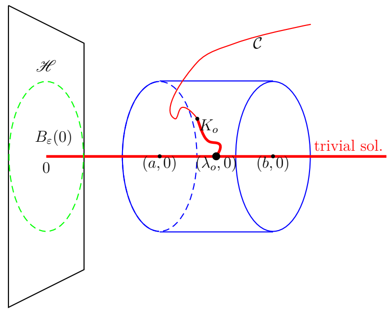


Figure 4.1: A component $\mathcal{C}$ bifurcating from $(\lambda_o, 0)$.

Choose $\rho > 0$, $\delta > \rho > 0$ and put $\theta(t,x) = \|x\| - \rho$. We claim that the set $\mathscr{F}^{-1}(0) \cap (a,b) \times \Omega$ contains a compact connected subset K_o such that $K_o \cap \{(\lambda, x) : a \le \lambda \le b,\ \|x\| = \rho\} \neq \emptyset$ and $(\lambda_o, 0) \in K_o$ (see the figure below). Then clearly, by definition a component $\mathcal{C}$ bifurcating from $(\lambda_o, 0)$ exists.

Suppose for the contradiction, that such a component K_o does not exist. Then put $K := \mathscr{F}^{-1}(0) \cap [a,b] \times \overline{\Omega}$, $A_0 := \{(\lambda_o, 0)\}$, $A_1 := \{(\lambda, x) : a \le \lambda \le b,\ \|x\| = \rho\}$. Then by the equivariant version of Theorem 4.4.3, there exist two G-invariant disjoint open sets V_0, V_1 such that $A_0 \subset V_0$, $A_1 \subset V_1$ and $A_0 \cup A_1 \cup K \subset V_0 \cup V_1$. Put $K_0 := K \cap V_0$ and $K_1 := K \cap V_1$. Clearly, K_0 and K_1 are G-invariant compact sets. Assume that $\mu : \mathbb{R} \times \mathscr{H} \to [0,1]$ is G-invariant (continuous) Urysohn function such that

$$\mu(\lambda, x) = \begin{cases} 1 & \text{if } (\lambda, x) \in K_0 \cup A_0, \\ 0 & \text{if } (\lambda, x) \in K_1 \cup A_1, \end{cases}$$

and define the complementing function $\phi : \mathbb{R} \times \mathscr{H} \to \mathbb{R}$ by

$$\phi(\lambda, x) := \|x\| - \mu(\lambda, x)\,\rho, \quad (\lambda, x) \in \mathbb{R} \times \mathscr{H}.$$

Therefore, since

$$G\text{-deg}(\mathfrak{F}_\phi, \Omega) = G\text{-deg}(\mathscr{F}_a, \Omega) - G\text{-deg}(\mathscr{F}_b, \Omega) \neq 0,$$

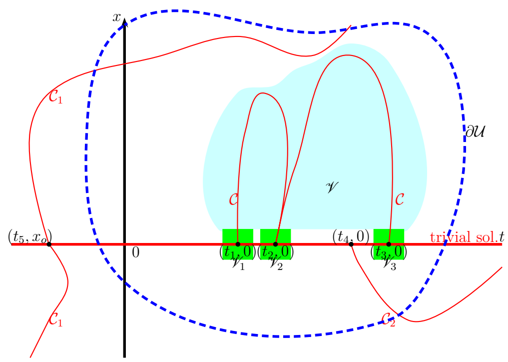


Figure 4.2: Branches of solutions: $\mathcal{C}_1$, $\mathcal{C}_2$ and $\mathcal{C}$, and bifurcation points.

there exists a solution $(\lambda^*, x^*) \in (a,b) \times \Omega$ to $\mathscr{F}(\lambda, x) = 0$ such that $\|x^*\| - \mu(\lambda^*, x^*)\rho = 0$. Clearly, $(\lambda^*, x^*) \in K_0$ or $(\lambda^*, x^*) \in K_1$. If $(\lambda^*, x^*) \in K_0$ then $\|x^*\| - \mu(\lambda^*, x^*)\rho = \|x^*\| - \rho = 0$, which implies that $(\lambda, x) \in K_1 \cap A_1$, which is a contradiction. Similarly, if $(\lambda^*, x^*) \in K_1$, then $\|x^*\| - \mu(\lambda^*, x^*)\rho = \|x^*\| = 0$, which implies that $(\lambda^*, x^*) = (\lambda_o, x_o) \in K_0 \cap A_0$, and this is also a contradiction. $\square$

Let us discuss the global bifurcation problem for (4.21) assuming that the derivative $D_x\mathscr{F}(\lambda, 0)$ exists for all $\lambda \in \mathbb{R}$ and the map $\lambda \mapsto D_x\mathscr{F}(\lambda, 0)$ is continuous. One can easily show that if $(\lambda_o, 0)$ is a bifurcation point for (4.21), then $D_x\mathscr{F}(\lambda_o, 0)$ is not an isomorphism. Denote by $\mathscr{B}$ the set of all $\lambda \in \mathbb{R}$ such that $(\lambda, 0)$ is a bifurcation point of (4.21) and put

$$\Lambda := \{\lambda \in \mathbb{R} : \det D_x\mathscr{F}(\lambda, 0) = 0\}. \tag{4.24}$$

Λ is called the set of *critical points* for (4.21). Clearly, $\mathscr{B} \subset \Lambda$.

Assume that the set Λ is discrete, $\lambda_o \in \Lambda$ and λ_-, λ_+ are such that

$$\lambda_- < \lambda_o < \lambda_+, \quad \text{and} \quad [\lambda_-, \lambda_+] \cap \Lambda = \{\lambda_o\}.$$

Then, we define the local *bifurcation invariant* $\omega_G(\lambda_o) \in A(G)$ by

$$\omega_G(\lambda_o) := G\text{-deg}(\mathscr{F}_{\lambda_-}, B_\varepsilon(0)) - G\text{-deg}(\mathscr{F}_{\lambda_+}, B_\varepsilon(0)), \tag{4.25}$$

where $\varepsilon > 0$ is assumed to be sufficiently a small number. Clearly, by the properties of the equivariant Brouwer degree and Theorem 4.4.1, the invariant $\omega_G(\lambda_o)$ does not depend on the choice of the numbers λ_-, λ_+, $\varepsilon > 0$.

The following local bifurcation result follows immediately from the properties of the equivariant Brouwer degree:

Theorem 4.4.5. *Suppose that $\mathscr{F} : \mathbb{R} \oplus \mathscr{H} \to \mathscr{H}$ is a continuous G-equivariant map such that $\mathscr{F}(\lambda, 0) = 0$ for all $\lambda \in \mathbb{R}$ and $D_x\mathscr{F}(\lambda, 0)$ exists and is continuous with respect to $\lambda \in \mathbb{R}$. We also assume that the set of critical points Λ for* (4.21) *(given by* (4.24)*) is discrete, and $\lambda_o \in \Lambda$. Then if $\omega_G(\lambda_o) \neq 0$, then $(\lambda_o, 0)$ is a bifurcation point for* (4.21). *To be more precise, if*

$$\omega_G(\lambda_o) = n_1(H_1) + n_2(H_2) + \cdots + n_m(H_m), \quad n_j \neq 0,\ j = 1, 2, \ldots, m,$$

then there exists a connected branch C bifurcating from $(\lambda_o, 0)$ containing nonzero solutions with orbit types at least (H_j), for each $j = 1, 2, \ldots, m$.

The following result, which in nonequivariant case was proved by P. Rabinowitz (cf. [43] [44]), is traditionally called *Rabinowitz's Alternative*:

Theorem 4.4.6. *Suppose that $\mathscr{F} : \mathbb{R} \oplus \mathscr{H} \to \mathscr{H}$ is a continuous G-equivariant map such that $\mathscr{F}(\lambda, 0) = 0$ for all $\lambda \in \mathbb{R}$ and $D_x\mathscr{F}(\lambda, 0)$ exists and is continuous with respect to $\lambda \in \mathbb{R}$. We also assume that the set of critical points Λ for* (4.21) *(given by* (4.24)*) is discrete, and consider an open bounded G-invariant set $\mathcal{U} \subset \mathbb{R} \oplus V$ such that $(\lambda_o, 0) \in \mathcal{U}$ for some $\lambda_o \in \Lambda$. For a connected component $\mathscr{C}$ of the set $\overline{\mathcal{U}} \cap \overline{\mathscr{S}}$ such that $(\lambda_o, 0) \in \mathscr{C}$ we have the following alternative:*

(i) either $\mathscr{C} \cap \partial U \neq \emptyset$,

(ii) *or* $\mathscr{C} \cap (\Lambda \times \{0\}) = \{(\lambda_1, 0), (\lambda_2, 0), \dots, (\lambda_n, 0)\}$ *for some* $n \in \mathbb{N}$ *(here* $\lambda_j \neq \lambda_k$ *for* $j \neq k$*) and*

$$\sum_{k=1}^{n} \omega_G(\lambda_k) = 0. \tag{4.26}$$

Proof. We will use the G-equivariant version of Kuratowski's Theorem 4.4.3. Suppose that $\mathscr{C} \cap \partial U = \emptyset$ and since Λ is discrete and U is bounded, we have the finite set $\mathscr{C} \cap (\Lambda \times \{0\}) = \{(\lambda_1, 0), (\lambda_2, 0), \dots, (\lambda_n, 0)\}$. Then, we assume that $X := \overline{\mathcal{U}}$, $A_0 := \{(\lambda_o, 0)\}$,

$$A_1 := \partial\mathcal{U} \cup \{(\lambda, 0) : \lambda \in \Lambda,\ (\lambda, 0) \in \mathcal{U},\ (\lambda, 0) \notin \mathscr{C}\}$$

and $K := \overline{\mathscr{S}} \cap \overline{\mathcal{U}}$. Since $\mathcal{U}$ is bounded, the set K is compact. All the above sets are G-invariant. which by equivariant version of Theorem 4.4.3 implies that there exist two disjoint G-invariant open in X sets $\mathscr{U}_0$ and $\mathscr{U}_1$ such that $A_0 \subset \mathscr{U}_0$, $A_1 \subset \mathscr{U}_1$ and $A_0 \cup A_1 \cup K \subset \mathscr{U}_0 \cup \mathscr{U}_1$. Since $\mathscr{U}_0 \cap \partial\mathcal{U} = \emptyset$, it follows that $\mathscr{U}_0$ is an open (and of course bounded) set in $\mathbb{R} \times V$ and $\mathscr{U}_0 \cap (\Lambda \times \{0\}) = \{(\lambda_1, 0), (\lambda_2, 0), \dots, (\lambda_n, 0)\}$. Then, there exist sufficiently small $\varepsilon > 0$ and $\delta > 0$ such that for every $k = 1, 2, \dots, n$, $\mathscr{V}_k := (\lambda_k - \delta, \lambda_k + \delta) \times B_\delta(0) \subset \mathcal{U}_0$ and $\mathscr{F}(\lambda_k \pm \delta, x) \neq 0$ for $0 < \|x\| \leq \varepsilon$. Then we define $\mathscr{V} := \mathscr{U}_0 \setminus (\mathbb{R} \times \overline{B_\varepsilon(0)})$ and put

$$\Omega := \mathcal{V} \cup \bigcup_{k=1}^{n} (\lambda_k - \delta, \lambda_k + \delta) \times \overline{B_\delta(0)}.$$

Clearly, Ω is a G-invariant open bounded set such that

$$\partial\Omega \cap \{(\lambda, x) : \mathscr{F}(\lambda, x) = 0\} = \{(\lambda_1, 0), (\lambda_2, 0), \dots, (\lambda_n, 0)\}$$

(see Figure 4.2). Then we define the following G-invariant complementing function $\theta : \overline{\Omega} \to \mathbb{R}$

$$\theta(\lambda, x) := \begin{cases} \|x\| - \frac{\varepsilon}{2} & \text{if } (\lambda, x) \in \bigcup_{k=1}^{n} \overline{\mathscr{V}_k}, \\ \frac{\varepsilon}{2} & \text{otherwise.} \end{cases}$$

The function θ can be extended G-invariantly to $\mathbb{R} \oplus \mathscr{H}$, so we can consider the complemented G-equivariant map $\mathfrak{F}_\theta : \mathbb{R} \oplus \mathscr{H} \to \mathbb{R} \oplus \mathscr{H}$ given by

$$\mathfrak{F}_\theta(\lambda, x) := (\theta(\lambda, x), \mathscr{F}(\lambda, x)), \quad (\lambda, x) \in \mathbb{R} \oplus \mathscr{H}.$$

One can easily verify that $\mathfrak{F}_\theta$ is Ω-admissible, therefore the G-equivariant degree $G\text{-deg}(\mathfrak{F}_\theta, \Omega)$ is well-defined. Since $\mathfrak{F}_\theta^{-1}(0) \cap \overline{\mathscr{V}} = \emptyset$, by excision and additivity properties of the G-equivariant degree we have

$$G\text{-deg}(\mathfrak{F}_\theta, \Omega) = G\text{-deg}\left(\mathfrak{F}_\theta, \bigcup_{k=1}^{n} \mathscr{V}_k\right) = \sum_{k=1}^{n} G\text{-deg}(\mathfrak{F}_\theta, \mathscr{V}_k) = \sum_{k=1}^{n} \omega_G(\lambda_k). \tag{4.27}$$

On the other hand, consider the homotopy

$$\mathfrak{H}(\tau, \lambda, x) = ((1-\tau)\theta(\lambda, x) - \tau, \mathscr{F}(\lambda, x)), \quad \tau \in [0,1],\ (\lambda, x) \in \mathbb{R} \oplus \mathscr{H}.$$

Since $\mathscr{F}(\lambda, x) \neq 0$ for $(\lambda, x) \in \partial\Omega \setminus \{(\lambda_k \pm \delta, 0) : k = 1, 2, \dots, n\}$ and $\theta(\lambda_k \pm \delta, 0) < 0$, the homotopy $\mathfrak{H}_\tau(\lambda, x) := H(\tau, \lambda, x)$ is Ω-admissible, thus

$$G\text{-deg}(\mathfrak{F}_\theta, \Omega) = G\text{-deg}(\mathfrak{H}_\tau, \Omega) = G\text{-deg}(\mathfrak{H}_1, \Omega), \quad \tau \in [0,1].$$

However, $\mathfrak{H}_1(\lambda, x) \neq 0$ for $(\lambda, x) \in \Omega$ thus $G\text{-deg}(H_1, \Omega) = 0$ and the statement follows from (4.27). □

4.4.2 Bifurcation of Subharmonic Solutions in SODE

The set $\mathscr{B}$ of the bifurcation points for (4.21) is contained in the critical set

$$\begin{aligned}\Lambda &:= \{\lambda : D_{\mathbf{x}}\mathscr{F}(\lambda, 0) \text{ is not an isomorphism}\} \\ &= \left\{\lambda : 4\sin^2 \frac{\pi j}{pm} - \mu,\ \mu \in \sigma(A),\ j = 0, 1, 2, \dots, \mathfrak{s}\right\},\end{aligned}$$

where $\mathfrak{s} = \lfloor \frac{pm}{2} \rfloor$. Put $\mathscr{A}(\lambda) := D_x\mathscr{F}(\lambda, 0)$. That means the critical set Λ is discrete in $\mathbb{R}$. Notice that

$$\lambda \in \Lambda \quad \Leftrightarrow \quad \lambda = 4\sin^2 \frac{\pi j}{pm} - \mu, \quad \text{for some } \mu \in \sigma(A),\ j = 0, 1, \dots, \mathfrak{s}.$$

Moreover,

$$\sigma(\mathscr{A}(\lambda)) = \left\{\xi_j(\mu) := -4\sin^2 \frac{\pi j}{pm} + \lambda + \mu,\ \mu \in \sigma(A),\ j = 0, 1, \dots, \mathfrak{s}\right\}.$$

Under the assumptions (A1)-(A5) and (C1)-(C2), for $G := \Gamma \times D_m \times \mathbb{Z}_2$ and for $\lambda_o \in \Lambda$ we define the bifurcation invariant $\omega_G(\lambda_o)$ by

$$\omega_G(\lambda_o) := G\text{-deg}(\mathscr{A}(\lambda_-), B_1(0)) - G\text{-deg}(\mathscr{A}(\lambda_+), B_1(0)),$$

where $\lambda_- < \lambda_o < \lambda_+$ are sufficiently close to λ_o numbers. It is possible that $\xi_j(\mu) = \xi_{j'}(\mu')$ for $j \neq j'$ and $\mu \neq \mu'$, however for the sake of simplicity, we exclude that case.[2] Then, the $\Gamma \times D_m \times \mathbb{Z}_2$-isotypical multiplicity of $\xi_j(\mu)$ is equal to the Γ-multiplicity of μ. In addition, if each of the eigenvalues μ has Γ-isotypical multiplicity one, we will call such value λ_o a *G-isotypically simple critical value.*

Using the topological invariants $\omega_G(\lambda_o)$ (by Theorem 4.4.5) we can determine the existence of the bifurcating branches of non-trivial solutions from $(\lambda_o, 0)$. Let's list all the elements of Λ as an increasing sequence:

$$\lambda_1 < \lambda_2 < \lambda_3 < \cdots < \lambda_{N-1} < \lambda_N.$$

Denote by $\{\mathcal{V}_j^-\}$, $0 \leq j \leq \mathfrak{s}$ all the $D_m \times \mathbb{Z}_2$-irreducible representations (which were described in section 4.2) and by $\mathcal{W}_l$, $0 \leq l \leq r$, the Γ-irreducible representations. Since for given $\lambda_k \in \Lambda$, the G-subrepresentation $E(\lambda_k) := \operatorname{Ker} \mathscr{A}(\lambda_k)$ is irreducible, i.e., there exist $0 \leq i(k) \leq \mathfrak{s}$ and $l(k)$ such that $E(\lambda_l) \simeq \mathcal{W}_{l(k)} \otimes \mathcal{V}_{i(k)}^- =: \mathcal{V}_{l(k),i(k)}$, we have

$$\begin{aligned}\omega_G(\lambda_o) &= \prod_{\lambda_k < \lambda_o} \left(\deg_{\mathcal{V}_{l(k),i(k)}}\right) - \prod_{\lambda_k \leq \lambda_o} \left(\deg_{\mathcal{V}_{l(k),i(k)}}\right) \\ &= \prod_{\lambda_k < \lambda_o} \left(\deg_{\mathcal{V}_{l(k),i(k)}}\right) \left((G) - \deg_{\mathcal{V}_{l(k_o),i(k_o)}}\right),\end{aligned}$$

where $\lambda_o = \lambda_{k_o}$ (i.e. it is k_o-th critical number).

Lemma 4.4.7. *Under the above assumptions, if* $\deg_{\mathcal{V}_{l(k_o),i(k_o)}} \neq (G)$ *then* $\omega_G(\lambda_o) \neq 0$.

[2] Let us point out, that in such a 'resonance' case we are still able to compute exactly all the bifurcation invariants for any concrete group G, so there is no essential difficulty in handling this situation.

Proof. Assume that $\omega_G(\lambda_o) = 0$. Since for each of basic G-degrees $\deg^2_{\mathcal{V}_{l(k),i(k)}} = (G)$, it follows that the element

$$a := \prod_{\lambda_l<\lambda_o} \left(\deg_{\mathcal{V}_{i(l),j(l)}}\right)$$

is invertible in $A(G)$, thus we have

$$\begin{aligned} 0 = a^{-1}\cdot 0 = a^{-1}\cdot \omega_G(\lambda_o) &= a^{-1}\cdot \prod_{\lambda_l<\lambda_o}\left(\deg_{\mathcal{V}_{l(k),i(k)}}\right)\left((G) - \deg_{\mathcal{V}_{l(k_o),i(k_o)}}\right)\\ &= (G)\cdot\left((G) - \deg_{\mathcal{V}_{l(k_o),i(k_o)}}\right) = (G) - \deg_{\mathcal{V}_{l(k_o),i(k_o)}}. \end{aligned}$$

Thus $\deg_{\mathcal{V}_{l(k_o),i(k_o)}} = (G)$. □

Remark 4.4.8. Let us point out that there exist several examples of irreducible G-representations $\mathcal{V}_j$ (see [36]) for which $\deg_{\mathcal{V}_j} = (G)$. In this moment we should mention that this does not contradict universality property of the G-equivariant degree, as this degree is computed up to one-suspension, i.e., the universality property is valid up to a suspension.

Lemma 4.4.9. *Under the above assumptions, if* $\deg_{\mathcal{V}_{l(k_o),i(k_o)}} = (G) + x(H) + b$, *where* (H) *is a maximal orbit type in* $\mathscr{H}\setminus\{0\}$, $x \neq 0$ *and* $b \in A(G)$ *are the other terms of* $\deg_{\mathcal{V}_{l(k_o),i(k_o)}}$, *then* $\omega_G(\lambda_o)$ *also has a nonzero coefficient standing by* (H), *i.e.,*

$$\omega_G(\lambda_o) = y(H) + c, \quad y \neq 0,$$

where $c \in A(G)$ *denotes other terms of* $\omega_G(\lambda_o)$. *Moreover, in such a case either* $y = \pm 1$ *or* ± 2.

Proof. Under the above assumptions, $(H) \in \Phi(G; \mathcal{V}_{l(k_o),i(k_o)})$ and

$$0 \neq x = \frac{(-1)^{\dim \mathcal{V}_{l(k_o),i(k_o)}} - 1}{|W(H)|},$$

which implies that

$$x = \begin{cases} -1 & \text{if } \ |W(H)| = 2, \\ -2 & \text{if } \ |W(H)| = 1, \end{cases}$$

and

$$(H) \cdot (H) = n_H(H) + \sum_{(L)<(H)} n_L(L),$$

where

$$n_H = \frac{|W(H)|\,|W(H)|}{|W(H)|} = |W(H)|.$$

There are two possibilities: (a) $\deg_{\mathcal{V}_{l(k_o),i(k_o)}}$ appears an even number of times in the product $\prod_{\lambda_k<\lambda_o} \left(\deg_{\mathcal{V}_{l(k),i(k)}}\right)$ or (b) it appears an odd number of times in this product. In the first case, using the relation $\left(\deg_{\mathcal{V}_{l(k),i(k)}}\right)^2 = (G)$ we can represent the invariant $\omega_G(\lambda_o)$ as the product

$$\omega_G(\lambda_o) = ((G) + d) \cdot (-x(H) - b) = -x(H) + e$$

where d, b, $e \in A(G)$ do not contain non-zero term (H). On the other hand, in the case (b), we have

$$\begin{aligned}
\omega_G(\lambda_o) &= ((G) + d) \cdot ((G) + x(H) + b) \cdot (-x(H) - b) \\
&= ((G) + d) \cdot (-x(H) - x^2(H)^2 + \boldsymbol{f}) \\
&= -x((G) + d) \cdot ((H) + x|W(H)| + \boldsymbol{h}) = -x(1 + x|W(H))(H) + \boldsymbol{k}
\end{aligned}$$

where $\boldsymbol{f}$, $\boldsymbol{h}$, $\boldsymbol{k} \in A(G)$ do not contain non-zero term (H). Notice that

$$-x(1 + x|W(H)|) = \begin{cases} -1 & \text{if } \ |W(H)| = 2, \\ -2 & \text{if } \ |W(H)| = 1. \end{cases}$$

So the result follows. $\square$

For convenience, we introduce the following notation: given an element $a \in A(H)$ we denote by $\mathfrak{n}_H(a) = n_H$, where $a = \sum_{(L)} n_L(L)$. Under the assumption that all the critical values $\lambda \in \Lambda$ are G-isotypically simple, we also put

$$\sigma_H := \{\lambda : \mathfrak{n}_H(\omega_G(\lambda)) \neq 0,\ \lambda \in \Lambda\}.$$

As a consequence, we obtain the following result:

Proposition 4.4.10. *Consider the system* (4.19) *and assume that the assumptions (A1)-(A5) and (C1)-(C2) are satisfied, let* $m > 2$ *be an integer number and* $G := \Gamma \times D_m \times \mathbb{Z}_2$. *In addition assume that every* $\lambda \in \Lambda$ *is* G*-isotypically simple. Then, for every* $\lambda_o := \lambda_{k_o} \in \Lambda$ *such that* $\deg_{\mathcal{V}_{l(k_o),i(k_o)}} \neq (G)$ *and* $i(k_o) \neq 0$, *we have* $\omega_G(\lambda_o) \neq 0$ *and there exists a branch of non-trivial* pm*-periodic solutions to* (4.19) *bifurcating from* $(\lambda_o, 0)$.

Motivated by Proposition 4.4.10, we make the additional assumption about the critical set Λ:

(*) Every $\lambda_k \in \Lambda$ is G-isotypically simple and $\deg_{\mathcal{V}_{l(k),i(k)}} \neq (G)$.

Theorem 4.4.11. *Consider the system* (4.19) *and assume that the assumptions (A1)-(A5) and (C1)-(C2) are satisfied, let* $m > 2$ *be an integer number and* $G := \Gamma \times D_m \times \mathbb{Z}_2$. *In addition assume that the condition (*) is satisfied and* (H) *is a maximal orbit type in* $\mathscr{H} \setminus \{0\}$ *such that* $|\sigma_H|$ *is an odd integer. Then there exists unbounded in* $\mathbb{R} \oplus \mathscr{H}$ *branch* $\mathcal{C}$ *of nontrivial* pm*-periodic solutions to* (4.19), *bifurcating from the trivial solutions with the orbit type exactly* (H). *Moreover, if in addition the assumption (C2) is satisfied, then for every sufficiently large* $\lambda > 0$ *we have* $(\{\lambda\} \times \mathscr{H}) \cap \mathcal{C} \neq \emptyset$.

Proof. Suppose that $\sigma_H = \{\lambda_1, \lambda_2, \dots \lambda_{2s+1}\}$, which implies it is non-empty. Suppose that $\lambda_o \in \sigma_H$, then $\mathfrak{n}_H \neq 0$ and therefore $\omega_H(\lambda_o) \neq (G)$, so there exists a branch $\mathcal{C}'$ of nontrivial pm-periodic solutions to (4.19) bifurcation from (4.19). If the connected branch $\mathcal{C}'$ is bounded, then there exists finitely many $\lambda_{k_i} \in \sigma_H$, $1 \leq k \leq t$, such that

$$\mathcal{C}' \cap (\Lambda \times \{0\}) = \{(\lambda_{k_1}, 0), (\lambda_{k_2}, 0), \dots, (\lambda_{k_s}, 0)\}$$

and by Theorem 4.4.6 we have

$$\sum_{i=1}^{t} \mathfrak{n}_H(\omega_G(\lambda_{k_i}) = 0.$$

However, by Lemma 4.4.9, either for every $i = 1, 2, \ldots, t$, we have $\mathfrak{n}_H(\omega_G(\lambda_{k_i}) = \pm 1$ or $\mathfrak{n}_H(\omega_G(\lambda_{k_i}) = \pm 2$, which implies that the number t is an even number. By repeating this argument, we find out, since $2s + 1$ is odd, that it is impossible that all the branches bifurcating from the points $(\lambda_i, 0)$, $1 \le i \le 2s + 1$ are bounded. Therefore, for at least one point $(\lambda_o, 0)$, $\lambda_o \in \sigma_H$, we have that the branch of non-trivial pm-periodic solutions bifurcating from $(\lambda_o, 0)$ is unbounded.

Suppose now that the assumption (C2) is satisfied. Then for $F_\lambda(n, \mathbf{x}) := \lambda\mathbf{x} + f(n, \mathbf{x})$, we have

$$F_\lambda(n, \mathbf{x}) \bullet \mathbf{x} = \lambda|\mathbf{x}|^2 + f(n, \mathbf{x}) \bullet x \le \lambda|\mathbf{x}|^2 - C|\mathbf{x}|^\alpha \le (\lambda - C|\mathbf{x}|^{\alpha-2})|\mathbf{x}|^2.$$

Then, for $M := (\lambda/C)^{\frac{1}{\alpha-2}}$,

$$\forall_{\mathbf{x}\in\mathscr{H}} \;\; |\mathbf{x}| > M \quad \Rightarrow \quad F_\lambda(n, \mathbf{x}) \bullet \mathbf{x} < 0.$$

On the other hand, if $\lambda \le 0$, then

$$\forall_{\mathbf{x}\neq 0} \;\; F_\lambda(n, \mathbf{x}) \bullet \mathbf{x} < 0.$$

This implies that for $\lambda \le 0$ there are no nonzero periodic solutions to (4.19) and for every every $\lambda > 0$ the condition (C2) is satisfied. Consequently, an unbounded branch $\mathcal{C}$ of non-trivial solutions, can only be extended in the direction of $\lambda \to \infty$, which concludes the proof.

□

Bifurcation Invariants and Unbounded Branches of Periodic Solutions for $m = 5$ and $\Gamma = D_3$: Let us consider $m = 5$, $k = 3$ and assume that $\Gamma = D_3$ acts on $V = \mathbb{R}^3$ by perturbing the coordinates of vectors $\mathbf{x} = (x_1, x_2, x_2)^T$. Then we put $G := D_5 \times D_3 \times \mathbb{Z}_2$ and

$$f(\mathbf{x}) := Ax + \begin{bmatrix} \varphi(x_1) \\ \varphi(x_2) \\ \varphi(x_3) \end{bmatrix}, \quad A = \begin{bmatrix} 4 & 2 & 2 \\ 2 & 4 & 2 \\ 2 & 2 & 4 \end{bmatrix}, \quad \varphi(t) := -t|t|^\beta,$$

where $0 < \beta < 1$, $t \in \mathbb{R}$ and $\mathbf{x} = (x_1, x_2, x_3)^T$. Then one can easily verify that all the conditions (A1)-(A5) and (C1)-(C2) are satisfied. We refer to subsection 4.3.1 (case $m = 5$ and $p = 1$) for the definitions of the space $\mathscr{H}$, its isotypical decomposition and other related properties. Notice that $\sigma(A) := \{\mu_0 := 8,\ \mu_1 = 2\}$.

The map $\mathscr{F} : \mathbb{R} \times \mathscr{H} \to \mathscr{H}$ is defined by (4.21), so the system (4.19) is equivalent to the following bifurcation equation

$$\mathscr{F}(\lambda, x) = 0, \quad \lambda \in \mathbb{R},\ x \in \mathscr{H},$$

for which we have the critical set

$$\begin{aligned}\Lambda &:= \left\{ \lambda_{ji} := 4\sin^2 \frac{\pi j}{5} - \mu_i : j = 0,1,2,\ i = 0,1 \right\} \\ &= \{\lambda_{00} = -8,\ \lambda_{10} \simeq -6.62,\ \lambda_{20} \simeq -4.49, \lambda_{01} \simeq -1, \lambda_{11} \simeq -0.62, \lambda_{21} \simeq 1.51\}\end{aligned}$$

The irreducible D_5 representations (relevant for this example) are denoted by $\mathcal{W}_0$, $\mathcal{W}_1$, $\mathcal{W}_2$, and irreducible $D_3 \times \mathbb{Z}_2$ representations (relevant for this example) are denoted by $\mathcal{W}_0^-$ and $\mathcal{W}_1^-$. One can easily verify (from the character table) that the G-isotypical component $\mathscr{H}_4$ is modeled on $\mathcal{V}_4 = \mathcal{W}_0 \otimes \mathcal{W}_0^-$, the G-isotypical component $\mathscr{H}_{15}$ is modeled on $\mathcal{V}_{15} = \mathcal{W}_1 \otimes \mathcal{W}_0^-$, the G-isotypical component $\mathscr{H}_{14}$ is modeled on $\mathcal{V}_{14} = \mathcal{W}_2 \otimes \mathcal{W}_0^-$, the G-isotypical component $\mathscr{H}_9$ is modeled on $\mathcal{V}_9 = \mathcal{W}_0 \otimes \mathcal{W}_1^-$, the G-isotypical component $\mathscr{H}_{22}$ is modeled on $\mathcal{V}_{22} = \mathcal{W}_1 \otimes \mathcal{W}_1^-$, and the G-isotypical component $\mathscr{H}_{23}$ is modeled on $\mathcal{V}_{23} = \mathcal{W}_2 \otimes \mathcal{W}_1^-$. Therefore, we have the following local bifurcation invariants

$$\begin{aligned}
\omega_G(\lambda_{00}) &= (G) - \deg_{\mathcal{V}_4} \\
\omega_G(\lambda_{10}) &= \deg_{\mathcal{V}_4} \cdot ((G) - \deg_{\mathcal{V}_{15}}) \\
\omega_G(\lambda_{20}) &= \deg_{\mathcal{V}_4} \cdot \deg_{\mathcal{V}_{15}} \cdot ((G) - \deg_{\mathcal{V}_{14}}) \\
\omega_G(\lambda_{01}) &= \deg_{\mathcal{V}_4} \cdot \deg_{\mathcal{V}_{15}} \cdot \deg_{\mathcal{V}_{14}} \cdot ((G) - \deg_{\mathcal{V}_9}) \\
\omega_G(\lambda_{11}) &= \deg_{\mathcal{V}_4} \cdot \deg_{\mathcal{V}_{15}} \cdot \deg_{\mathcal{V}_{14}} \cdot \deg_{\mathcal{V}_9} \cdot ((G) - \deg_{\mathcal{V}_{22}}) \\
\omega_G(\lambda_{21}) &= \deg_{\mathcal{V}_4} \cdot \deg_{\mathcal{V}_{15}} \cdot \deg_{\mathcal{V}_{14}} \cdot \deg_{\mathcal{V}_9} \cdot \deg_{\mathcal{V}_{22}} \cdot ((G) - \deg_{\mathcal{V}_{23}}),
\end{aligned}$$

which can be exactly computed using GAP routines, i.e.,

$$\omega_G(\lambda_{00}) = (H_{62}),$$

$$\omega_G(\lambda_{10}) = -(H_{34}) + (H_{37}),$$

$$\omega_G(\lambda_{20}) = (H_{34}) - (H_{37}),$$

$$\omega_G(\lambda_{01}) = -(H_{42}) + (H_{45}),$$

$$\omega_G(\lambda_{11}) = (H_1) - (H_5) - (H_7) + (H_{10}) + (H_{11}) - (H_{15}) + (H_{16}),$$

$$\omega_G(\lambda_{21}) = -(H_1) + (H_5) + (H_7) - (H_{10}) + (H_{11}) + (H_{15}) - (H_{16}).$$

Recall that (see subsection 4.3.1) the maximal orbit types in $\mathscr{H} \setminus \{0\}$ are (H_{10}), (H_{37}), (H_{45}), and (H_{62}), where

$$H_{10} = D_1 \times_{\mathbb{Z}_2}^{D_1^z} D_1^p, \quad H_{37} = D_1 \times_{\mathbb{Z}_2}^{D_3} D_3^p, \quad H_{45} = D_5 \times D_1^z, \quad H_{62} = D_5 \times D_3.$$

Consequently, there is an unbounded branch of solutions to (4.19) with the orbit type (H_{45}) bifurcating from $(\lambda_{01}, 0)$ and another unbounded branch of (constant) solutions with the orbit type (H_{62}) bifurcating from $(\lambda_{0,0}, 0)$. However, since

$$\omega(\lambda_{10}) + \omega(\lambda_{20}) = 0, \quad \omega(\lambda_{11}) + \omega(\lambda_{21}) = 0,$$

the branches with the maximal orbit type (H_{37}) bifurcating from $(\lambda_{10}, 0)$ and $(\lambda_{20}, 0)$, and with the maximal orbit type (H_{10}) bifurcating from $(\lambda_{11}, 0)$ and $(\lambda_{21}, 0)$ may be bounded. Let us examine the properties of the branch of solutions with the orbit type (H_{37}).

In order to identify the branch of solutions with the orbit type (H_{37})' use the H_{37}-fixed point reduction of the equation (4.19). Notice that

$$\mathscr{H}^{H_{37}} = \left\{(\mathbf{0}, \mathbf{y}, \mathbf{z}, -\mathbf{z}, -\mathbf{y}) : \mathbf{y} = (y, y, y)^T,\ \mathbf{z} = (z, z, z)^T,\ y,\ z \in \mathbb{R}\right\},$$

so the equation (4.19) can be reduced to the system

$$\begin{cases} z + (\lambda + 6)y - y|y|^\beta = 0, \\ y + (\lambda + 5)z - z|z|^\beta = 0, \end{cases} \tag{4.28}$$

which can be written as

$$\begin{cases} z = y\left(|y|^{\beta} - (\lambda+6)\right), \\ y = z\left(|z|^{\beta} - (\lambda+5)\right). \end{cases} \tag{4.29}$$

Notice that the bifurcation system (4.28) has the critical set composed of $\{\lambda_{10}, \lambda_{20}\}$ and it is easy to show that the system (4.29) has two pairs of $\mathbb{Z}_2$-orbits of non-zero solution for any sufficiently large λ, which implies that there exist unbounded branches with the orbit type (H_{37}). In fact, it is easy to see that the these two branches of solutions with the orbit type (H_{37}), bifurcating from the points $(\lambda_{10}, 0)$ and $(\lambda_{20}, 0)$ are unbounded and do not intersect.

Similarly, for the branch of solutions with the orbit type (H_{10}), we use the H_{10}-fixed point reduction of the equation (4.19). In this case

$$\mathscr{H}^{H_{10}} = \left\{(\mathbf{0}, \mathbf{y}, \mathbf{z}, -\mathbf{z}, -\mathbf{y}) : \mathbf{y} = (0, y, -y)^T,\ \mathbf{z} = (0, z, -z)^T,\ y,\ z \in \mathbb{R}\right\},$$

so the equation (4.19) can be reduced to the system

$$\begin{cases} z + \lambda y - y|y|^{\beta} = 0, \\ y + (\lambda - 1)z - z|z|^{\beta} = 0, \end{cases} \tag{4.30}$$

which can be written as

$$\begin{cases} z = y\left(|y|^{\beta} - \lambda)\right), \\ y = z\left(|z|^{\beta} - (\lambda-1)\right). \end{cases} \tag{4.31}$$

Similarly, as in the case of the (H_{37}) orbit type, we can show that system (4.30) has the critical set composed of $\{\lambda_{11}, \lambda_{21}\}$ and it is easy to show that the system (4.29) has two pairs of $\mathbb{Z}_2$-orbits of non-zero solution for any sufficiently large λ, which implies that there exist disjoint unbounded branches with the orbit type (H_{10}) – one bifurcating from the point $(\lambda_{10}, 0)$ and another one from $(\lambda_{20}, 0)$.

Conclusion: Even in the case the bifurcation invariant $\omega_G(\lambda_o)$ does not allow us to eliminate the relations (4.26), the existence of unbounded global branches bifurcating from $(\lambda_o, 0)$ still can occur.

APPENDIX

USING EQUIDEG PACKAGE IN GAP SYSTEM

Throughout this dissertation, the involved computations focus on the equivariant degree of linear maps. Therefore, in this appendix, we will briefly discuss aspects in GAP system ([60]) and EquiDeg package ([57]) which can assist such computations.

The software can be found on websites indicated in the reference, which contain links for downloading and instruction for installing. To use functions provided in EquiDeg package, it is required to load the package into GAP system first by command:

- gap> LoadPackage(”EquiDeg”);

(a) Group and direct product. In this dissertation, we need to deal with direct product of groups. This is done in GAP by calling the group library and then performing direct product. For example, the following commands create group $G := D_3 \times (D_3 \times \mathbb{Z}_2)$ in GAP:

- gap> G1 := pDihedralGroup(3);
- gap> G2 := pCyclicGroup(2);
- gap> G := DirectProduct(G1, G2);
- gap> G := DirectProduct(G1, G);

One can then compute *conjugacy classes* and *conjugacy classes of subgroups* for G.

- cc_list := ConjugacyClasses(G);
- ccs_list := ConjugacyClassesSubgroups(G);

(b) Character and isotypical decomposition. The following command computes characters of irreducible (complex) G-representations:

- gap> irr_list := Irr(G);

The order of character values coincides with the order of the conjugacy classes in `cc_list`. Keep that in mind, if the character of a G-representation is known, one can create a character object by the following command:

- gap> chi := ClassFunction(G, <list of character values>);

To find the G-isotypical decomposition of the state space, it suffices to use the following command:

- gap> decomp := SolutionMat(irr_list, chi);

This returns the multiplicity of each irreducible G-representation in the G-isotypical decomposition of the state space. Finally, in describing symmetries in a G-representation, it is essential to know what are its orbit types. This is done by the following commands:

- > orbt_list := OrbitTypes(chi); all orbit types of chi

- > max_orbts := MaximalOrbitTypes(chi); maximal ones

(c) Burnside ring and basic degree The most important functionality of EquiDeg package is its support of Burnside ring arithmetic. What follow are basic commands for Burnside ring:

- gap> A := BurnsideRing(G); Burnside ring A(G)

- gap> basis := Basis(A); the Z-module basis of A(G)

- gap> zero := Zero(A); additive identity of A(G)

- gap> one := One(A); multiplicative identity of A(G)

Note that operations '+' and '*' can be applied to elements in $A(G)$. In connection with equivariant degree theory, the following command computes basic degrees with respect to irreducible character `psi`:

- gap> bdeg := BasicDegree(psi);

Assuming the formula of equivariant degree in terms of basic degrees is known, one can compute the value with command like

- gap> eqdeg := bdeg1*bdeg2-bdeg3*bdeg4.

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